Image Processing and Computer Vision

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Program

Lab #1: Python concepts for image processing

Intensity transformations

Lab #2: Basic image manipulation: channel processing, colormaps and cameras.

Lab #3: Image binarization and the error diffusion algorithm.

Lab #4: More on color and channel transformations (Equalization, entropy, steganography, encryption).

Fourier-based filtering

Lab #5: Fourier transforms and spatial filtering.

Lab #6: Point–spread functions and image restoration filters.

Lab #7: Computer tomography. Radon transforms and the Projection-Slide theorem.

Machine learning

Lab #8: K-means clustering.

Project #10: Automatic diagnostic using an X-ray images dataset: image classification using machine learning.

Schedule

February: 14, 18, 21, 25, 28
March: 4, 7, 11, 14, 18, 21, 25, 28
Abril: 1, 4, 8, 22, 25, 29
May: 2, 6, 9, 13, 16, 20

3 ECTS: 75 (up to 90) hours
Lectures: 48 hours
Personal work ~ 27 - 42 hours

Grading

- Midterm exam: 20 % (April 1st, 2022).
- Final exam: 40% (June 2nd, 2022; September 7th, 2022).
- Computational project: 40% (July 4th, 2022; September 1st, 2022.).

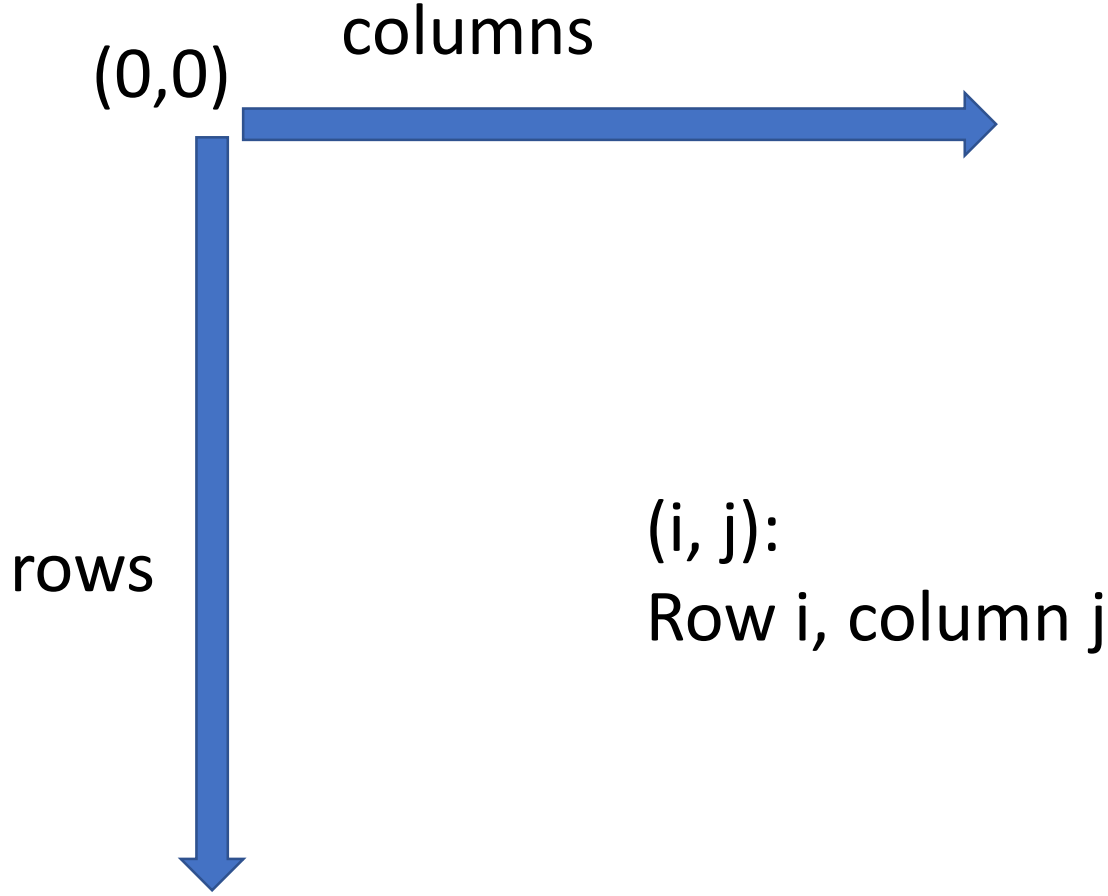
Case 1: Midterm exam (20%) + Final exam (40%) + Project (40%)

Case 2: Final exam (60%) + Project (40%)

Final grade = max(case1, case2)

Lab #2: Basic image manipulation: channel processing, colormaps and cameras.

Images: coordinate convention



Use of the : operator

- within an array, you can select part of it using begin: end: step
- if begin =0, end = M-1 and step = 1, then begin: end: step :
- if step = -1, the order of the array is inverted

Display

- Images are displayed using the RGB model
- Each channel (R, G or B) ranges from 0 (minimum) to 255 (maximum energy).
- A grey level image uses the three channels, but the information is the same for all of them

Memory. Let im be the stored image

MxN (gray level) images:

- np.uint8: The maximum value is 255
- np.double: The image is displayed taking im.max()

MxNx3 and MxNx4 (color images):

- np.uint8: The image is displayed as it is
- np.double: im **must** be normalized to 1. Otherwise, it will not be properly displayed
- Experiment with script `normalization_issues.py`

matplotlib.pyplot.imshow

```
matplotlib.pyplot.imshow(X, cmap=None, norm=None, aspect=None, interpolation=None,
alpha=None, vmin=None, vmax=None, origin=None, extent=None, shape=<deprecated parameter>,
filternorm=1, filterrad=4.0, imlim=<deprecated parameter>, resample=None, url=None, *,
data=None, **kwargs)
```
[source]

Display an image, i.e. data on a 2D regular raster.

Parameters:

X : array-like or PIL image

The image data. Supported array shapes are:

- (M, N): an image with scalar data. The data is visualized using a colormap.
- (M, N, 3): an image with RGB values (0-1 float or 0-255 int).
- (M, N, 4): an image with RGBA values (0-1 float or 0-255 int), i.e. including transparency.

Anastasiia is a given name

The first two dimensions (M, N) define the rows and columns of the image.

Out-of-range RGB(A) values are clipped.

cmap : str or `Colormap`, optional

The Colormap instance or registered colormap name used to map scalar

Quick search

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Normalization

- np.uint8 images do not require normalization. But the 255 limit cannot be exceeded after calculations
- In general, real-valued arrays are unbounded. But they should be normalized to 1 before they can be displayed with `plt.imshow()`. Use

```
im = im / im.max()
```

- Conversion from np.uint8 (imc) to np.float64 (imf) can be tricky. Very often, integer-valued images (imc) use the full np.uint8 range. But to be on the safe side use the following rule of thumb:

```
imf = imc / 255.
```

- Maximum values for integers: `255 = 2**8 - 1, 65535 = 2**16 -1.`

Comments Friday 25/02/2022

Objective of Image processing**:** to modify the image to adapt it to our interest. Image processing can be understood as a toolbox. It is, by no means, dark magic.

Very often, there are several ways to face the same problem. Some are better than others. Exercises are ‘open’. You can investigate other ways. Feel free to test alternatives.

In general, manipulation is performed at the channel level, e. g.: binarization is calculated on a 2D-array. If a color image is provided, brightness or one of the channels should be used.

Usually, if a technique needs to be applied to a color image, we calculate each channel independently and recombine them.

Gray level images from color images

R, G, or B represent 2D-arrays

- Average: M = (R + G + B) / 3

  Note that R + G + B > 255

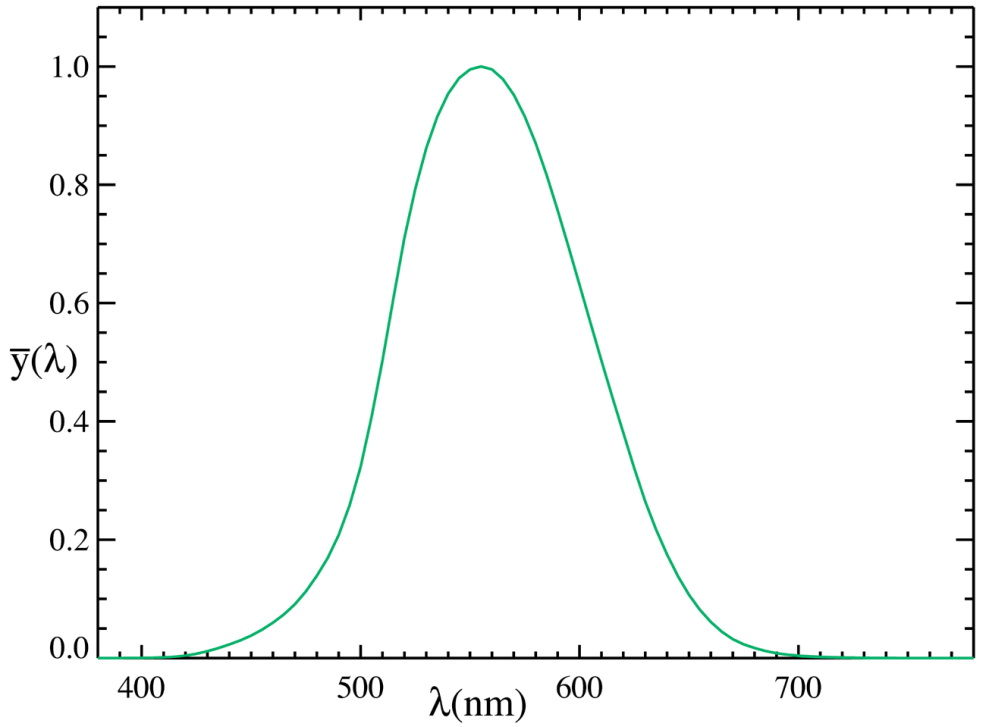
In general, calculations should be carried out using real numbers (Why?)

- Brightness (measure of Luminance):

L = 0.299 * R + 0.587 * G + 0.114 * B

L is the gray level image obtained from a color one, according to the human luminosity function (in well-lit conditions). L (Y) is related to JPEG

However, other definitions can be used https://en.wikipedia.org/wiki/YCbCr#JPEG_conversion (standards are painful problems)



1931 CIE photopic luminosity function. https://en.wikipedia.org/wiki/File:CIE_1931_Luminosity.png

Colormaps and Look-up tables (LUT).

- a **LUT** is a function that relates the processed gray level g' as a function of the original gray kevel g, i.e. gl' = f(gl). They are used to modify the contrast of an image.
- **Colormap**: an arbitrary color value is assigned to each g:

$$r = f_R(gl),\ g = f_G(gl),\ b = f_B(gl)$$

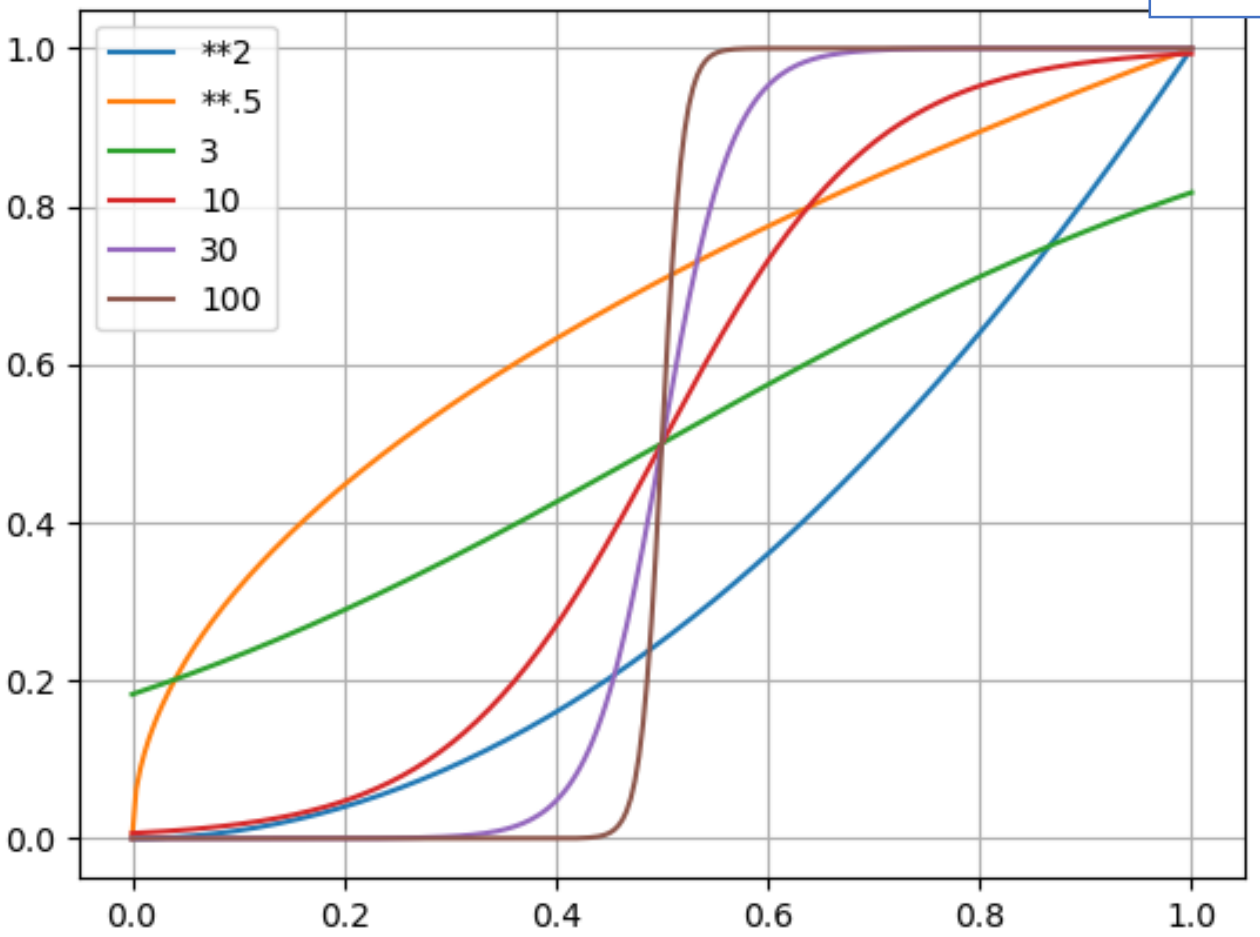
- LUTs might be used with color images as well (recall we usually process each channel independently):

$$r' = f_R(r),\ g' = f_G(g),\ b' = f_B(b).$$ Very often, $f_R = f_G = f_B$.

- Examples (change g by r, g or b when necessary)
  - Inversion of contrast: gl' = 255 – gl' (or 1 –gl')
  - Linear: gl' = m gl + n
  - Gamma: $gl' = gl^{\gamma}$. Cases: $0<\gamma<1$, $1<\gamma<\infty$ ($\gamma = 1$, g' = g)
  - Logistic function $gl' = gl_{max} / (1 + \exp(-k(gl-gl_0)))$
  - Binarization gl' = 255(1) if g > threshold, g'=0 otherwise.
- In general, calculations should be carried out using real numbers

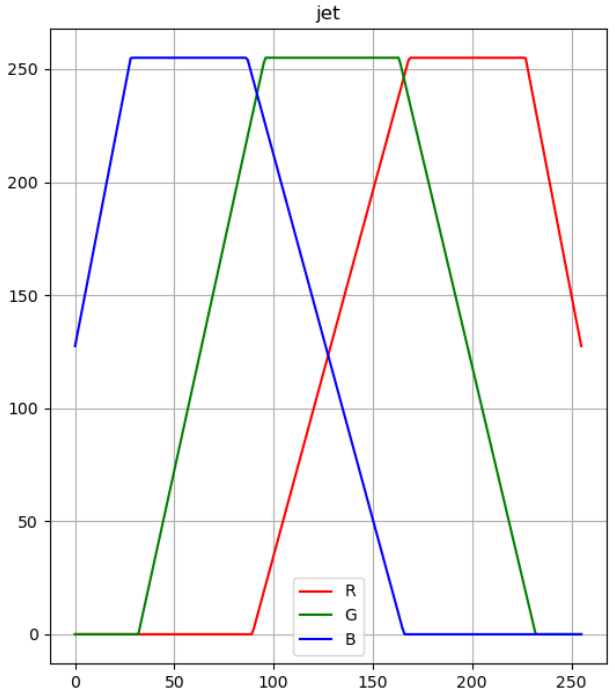
Look-up tables (LUT). Examples

```
from matplotlib import cm
jetv = cm.jet(range(256))
virv = cm.viridis(range(256))
```

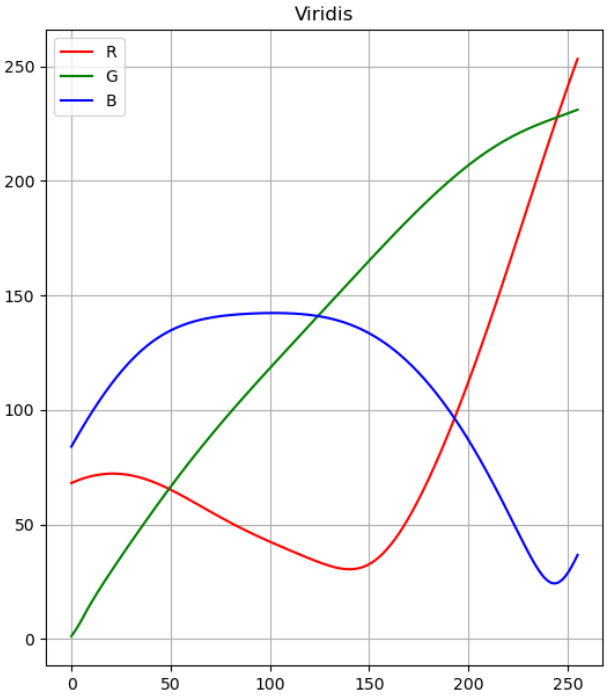


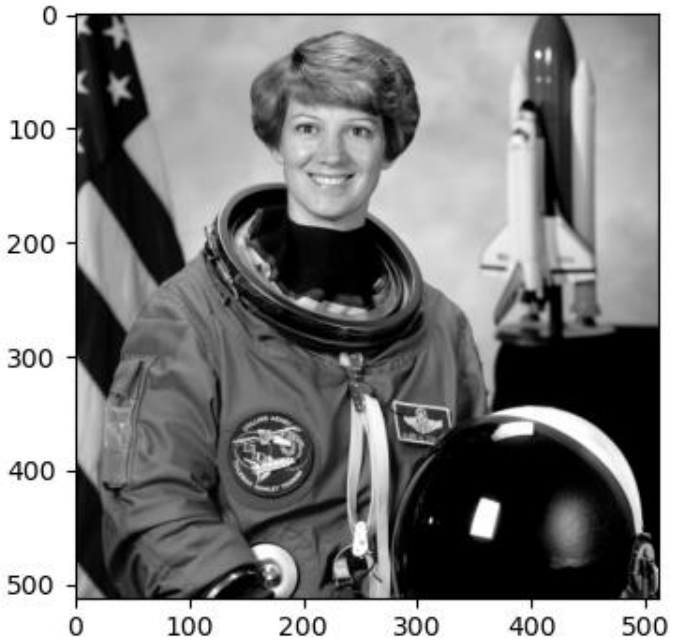
Gamma contrast & logistic function

Jet

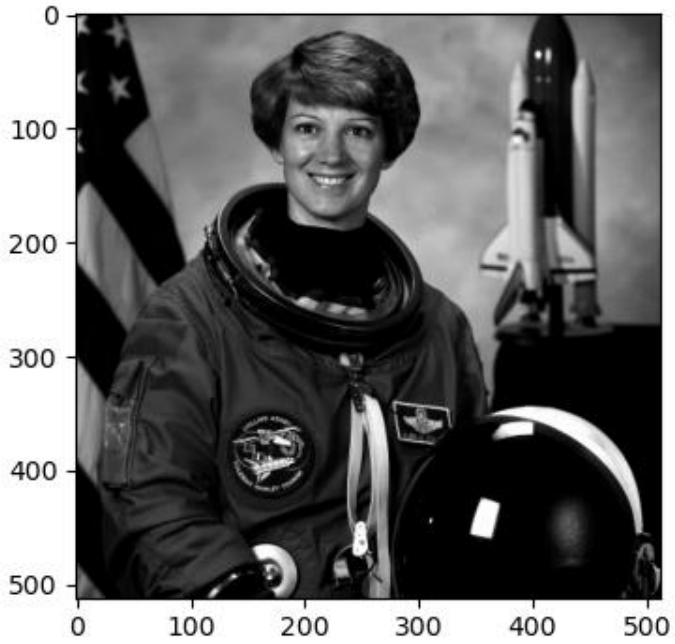


Viridis

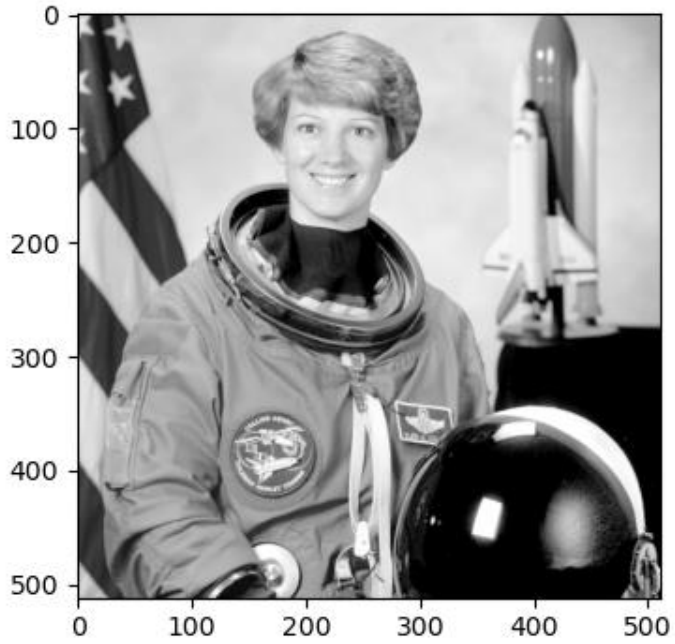




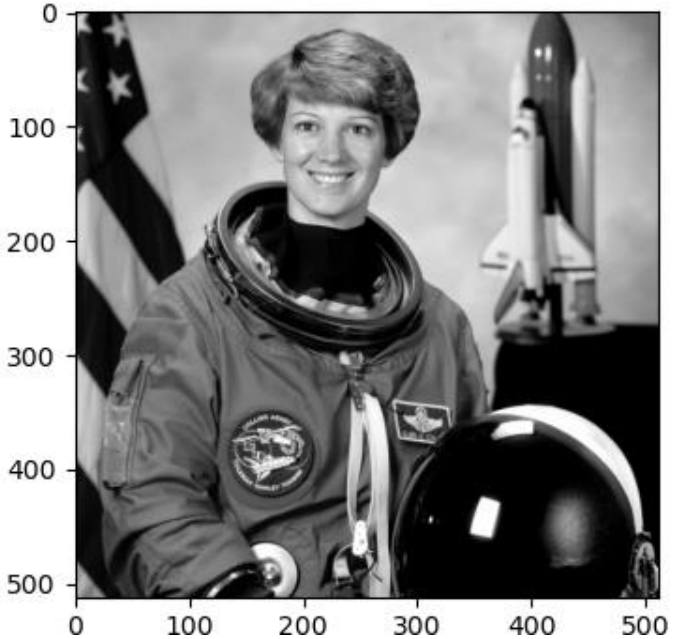
Original

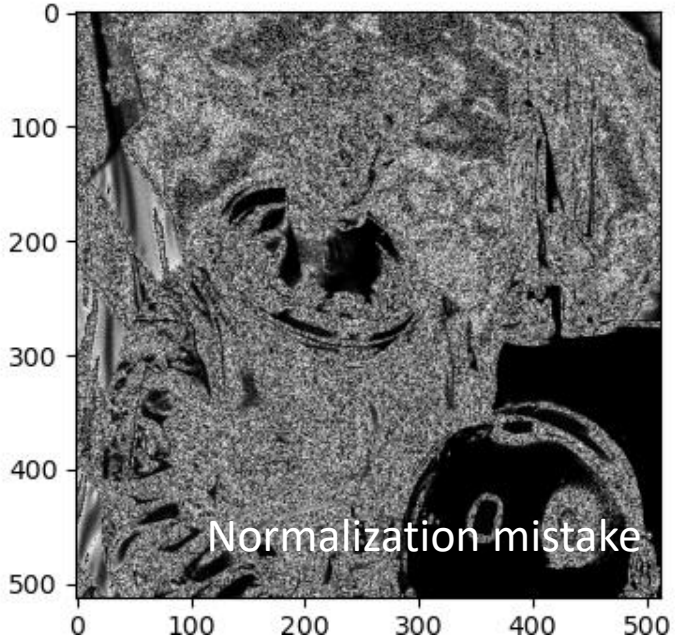


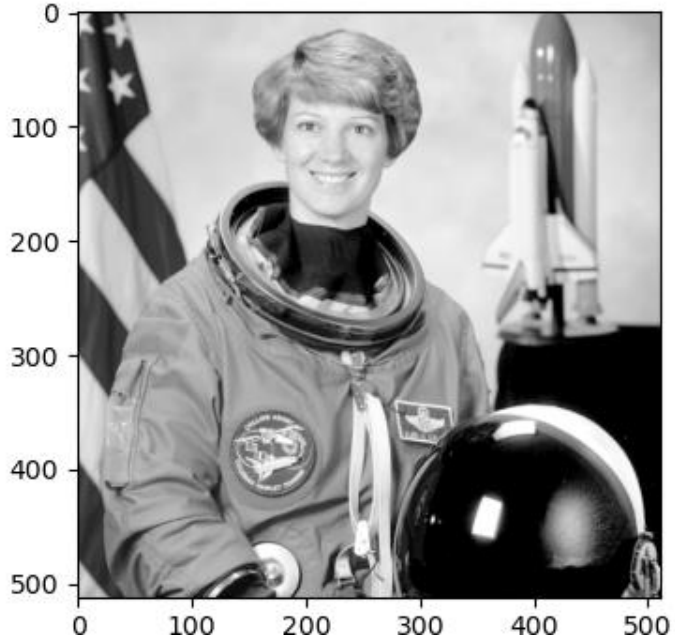
Gamma = 2



Gamma = 0.5







Lab 3: Binarization

Pros:

- Very useful LUT for gray level images.
- Weight reduction. Just 1 bit per pixel.
- Straightforward calculation: imb = im > th

Cons:

- Loss of visual information

Possible approaches:

- Determine optimum threshold (e.g.: Otsu method)
- Use of local thresholds (calculate the median of a neighborhood)
- Error diffusion (Floyd-Steinberg) algorithm

Threshold = 0.1

Threshold = 0.2

Threshold = 0.3

Threshold = 0.4

Threshold = 0.5

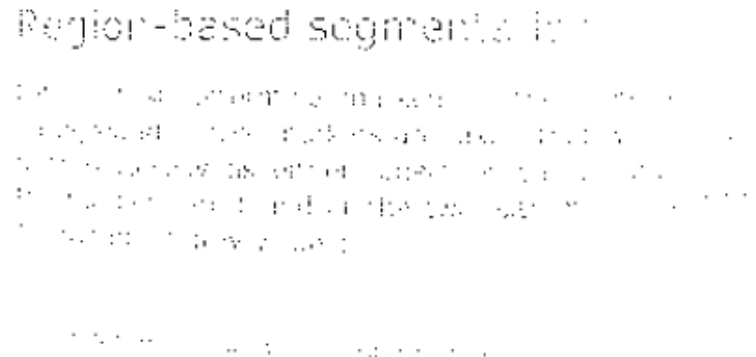
Threshold = 0.6

Threshold = 0.7

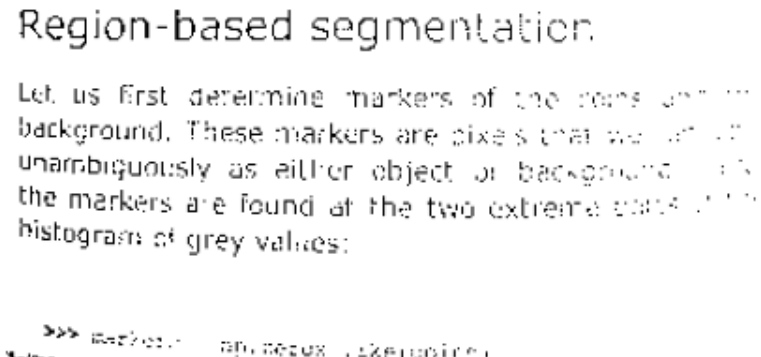
Threshold = 0.8

Threshold = 0.9

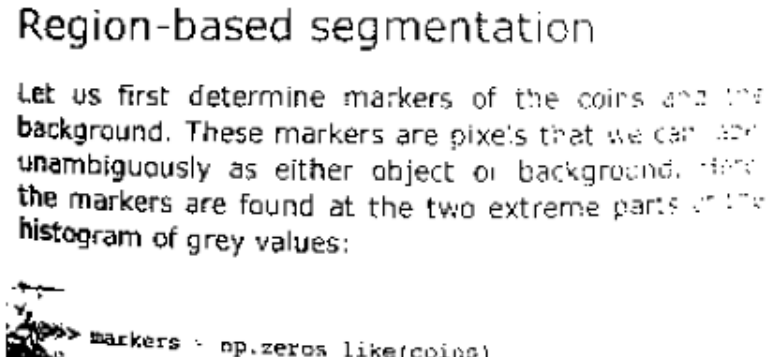
Threshold = 0.1



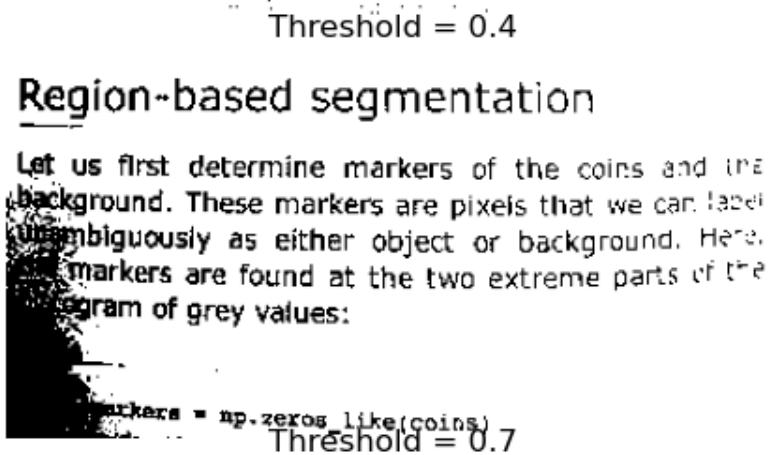
Threshold = 0.2



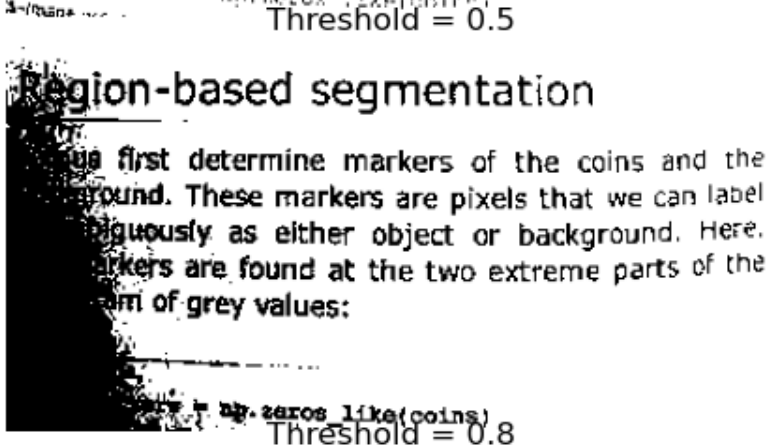
Threshold = 0.3



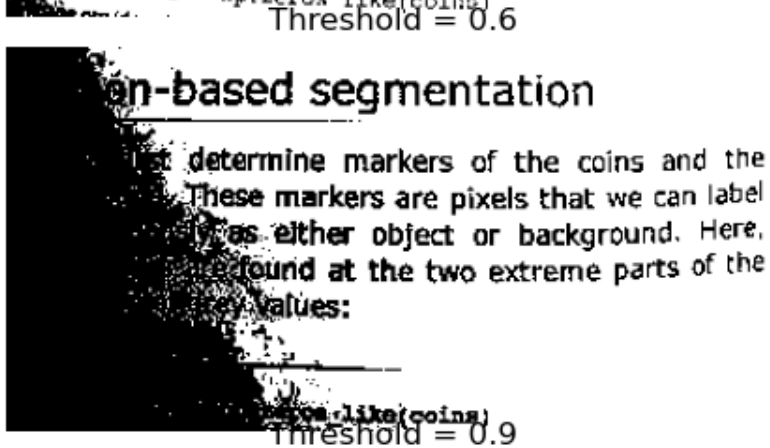
Threshold = 0.4



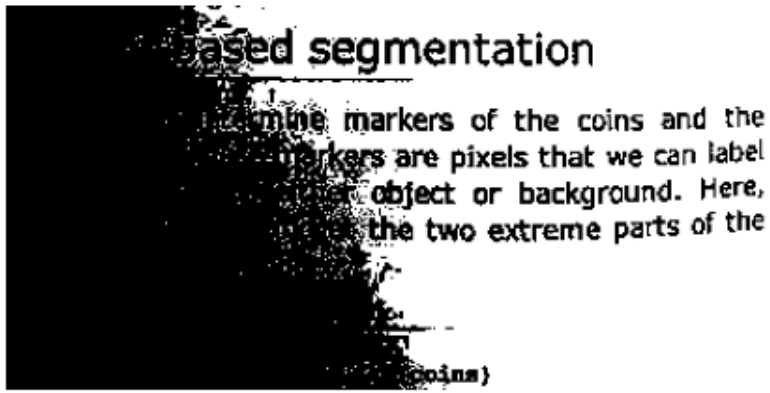
Threshold = 0.5



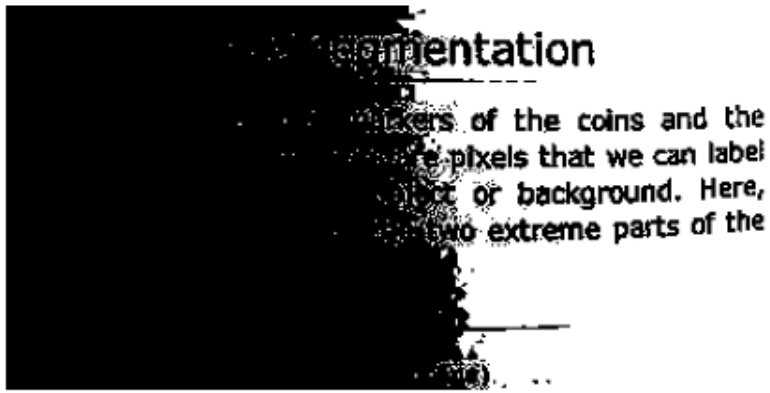
Threshold = 0.6



Threshold = 0.7



Threshold = 0.8



Threshold = 0.9

Adaptive threshold

- A single threshold for a complete image cannot be appropriate.
- Thresholds are calculated according to the statistics of the neighbor pixels window.
- Possible windows sizes: 3x3, 5x5, 7x7, … The processed pixels is set in the center.
- We calculated the median / mean / … of the neighborhood window. This value becomes the local threshold
- The local median according to a certain window can be calculated using `scipy.signal.medfilt`
- `scipy.signal.order_filtre` can be used to set local variable thresholds.

Search the docs ...

Optimization and root finding (scipy.optimize)

Cython optimize zeros API

Signal processing (scipy.signal)

Sparse matrices (scipy.sparse)

Sparse linear algebra (scipy.sparse.linalg)

Compressed sparse graph routines (scipy.sparse.csgraph)

Spatial algorithms and data structures (scipy.spatial)

Distance computations (scipy.spatial.distance)

Special functions (scipy.special)

Statistical functions (scipy.stats)

Result classes

scipy.signal.medfilt2d

`scipy.signal.medfilt2d(input, kernel_size=3)` [source]

Median filter a 2-dimensional array.

Apply a median filter to the *input* array using a local window-size given by *kernel_size* (must be odd). The array is zero-padded automatically.

scipy.signal.medfilt

`scipy.signal.medfilt(volume, kernel_size=None)` [source]

Perform a median filter on an N-dimensional array.

Apply a median filter to the input array using a local window-size given by *kernel_size*. The array will automatically be zero-padded.

Parameters: **volume** : ***array_like***

An N-dimensional input array.

kernel_size : ***array_like, optional***

A scalar or an N-length list giving the size of the median filter window in each dimension. Elements of *kernel_size* should be odd. If *kernel_size* is a scalar, then this scalar is used as the size in each dimension. Default size is 3 for each dimension.

Returns: **out** : ***ndarray***

An array the same size as input containing the median filtered result.

Search the docs ...

scipy.signal.order_filter

`scipy.signal.order_filter(a, domain, rank)` [source]

Perform an order filter on an N-D array.

Perform an order filter on the array in. The domain argument acts as a mask centered over each pixel. The non-zero elements of domain are used to select elements surrounding each input pixel which are placed in a list. The list is sorted, and the output for that pixel is the element corresponding to rank in the sorted list.

Parameters: **a : *ndarray***

The N-dimensional input array.

domain : *array_like*

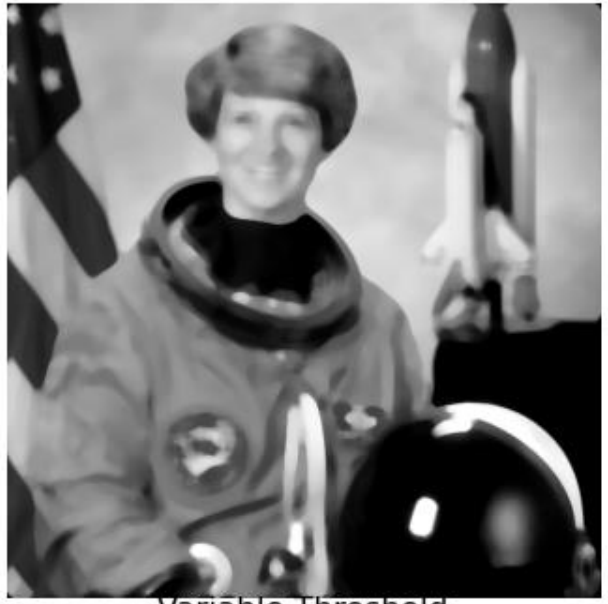
A mask array with the same number of dimensions as *a*. Each dimension should have an odd number of elements.

rank : *int*

A non-negative integer which selects the element from the sorted list (0 corresponds to the smallest element, 1 is the next smallest element, etc.)

Original

Median Threshold

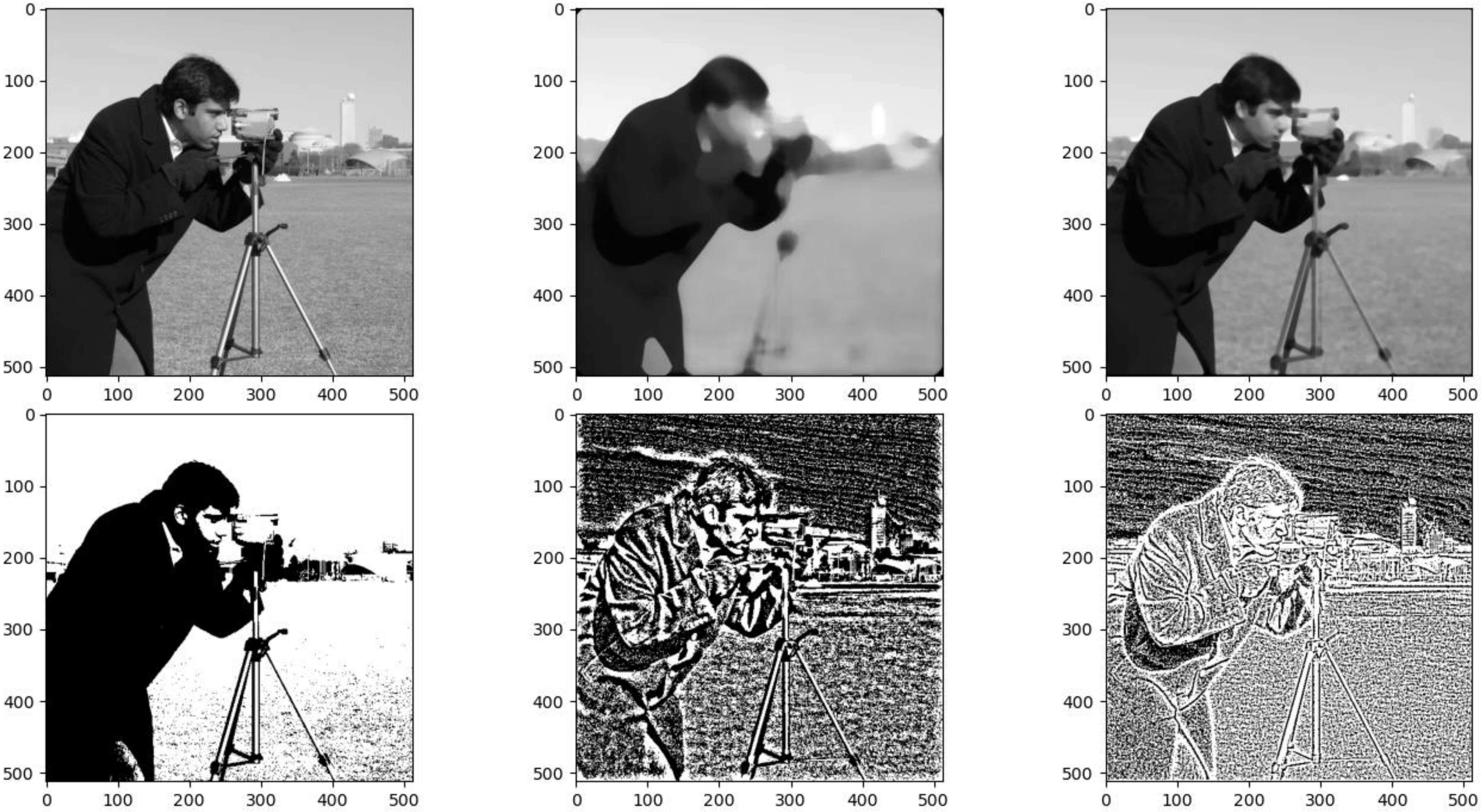


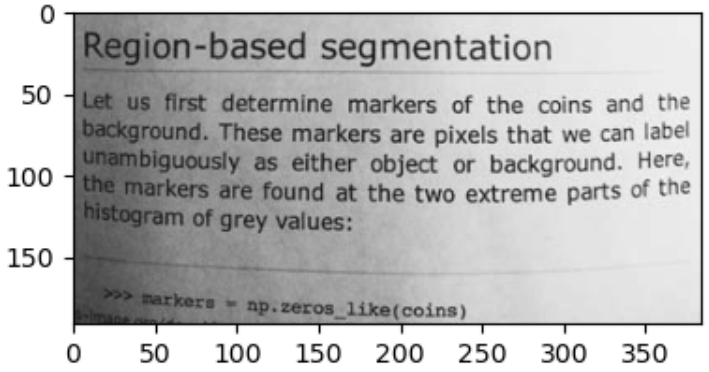
MT Binary

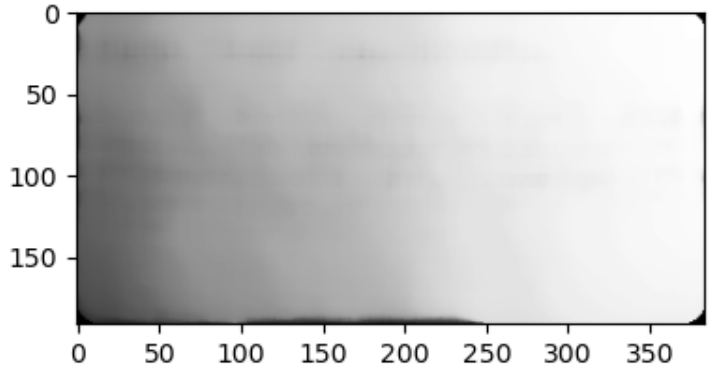
Threshold = 0.3

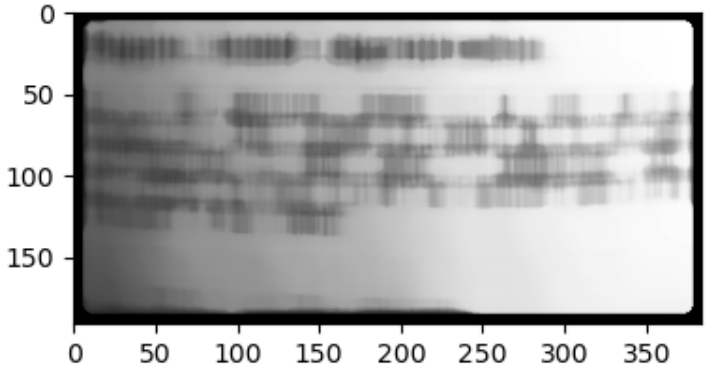
Variable Threshold

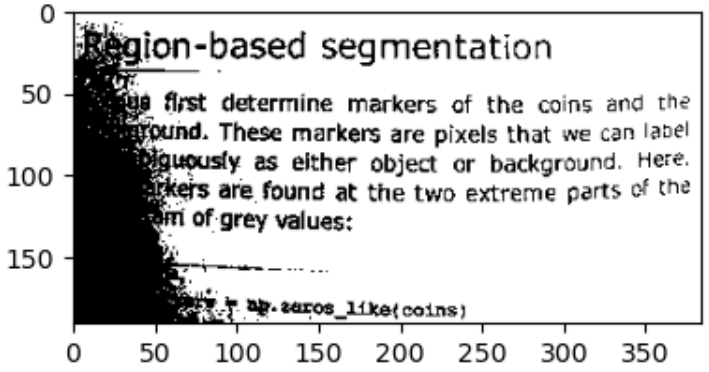
VT Binary

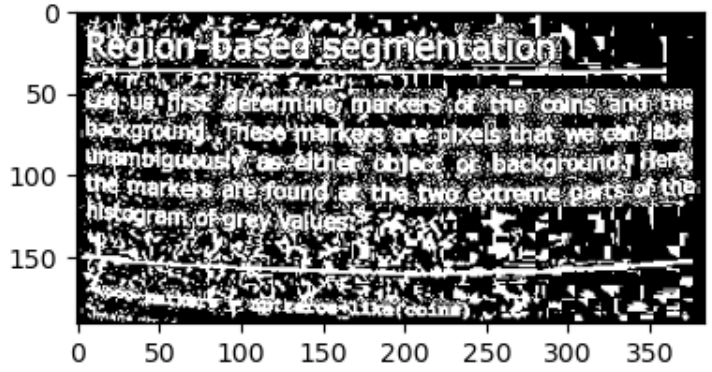

0
100
200
300
400
500


Region-based segmentation
Let us first determine markers of the coins and the background. These markers are pixels that we can label unambiguously as either object or background. Here, the markers are found at the two extreme parts of the histogram of grey values:
>>> markers = np.zeros_like(coins)






Region-based segmentation


Region-based segmentation

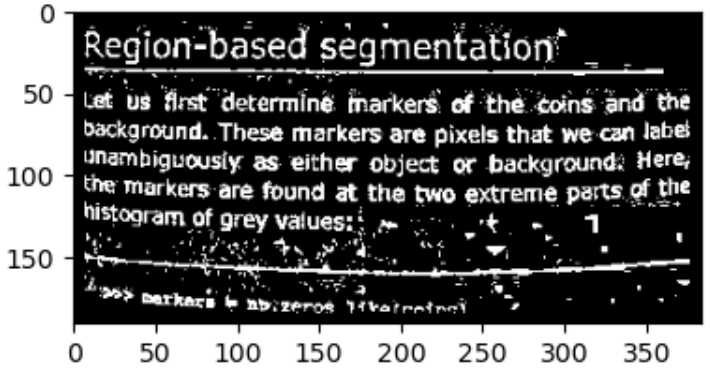

Region-based segmentation
Let us first determine markers of the coins and the background. These markers are pixels that we can label unambiguously as either object or background. Here, the markers are found at the two extreme parts of the histogram of grey values:

Image comparison

- Local calculation: window 8x8 for each pixel, then calculate the averaged SSIM
- n is assumed 8 bits, even for real valued images.
- Mutichannel: Similarity calculations are done independently for each channel; then the three values are averaged.

Several assessment methods are provided in
https://scikit-image.org/docs/dev/api/skimage.metrics.html

skimage.metrics.mean_squared_error

- Not normalized. Simple to calculate.
- MSE=0 means both images are identical

```
MSE=((im1-im2)**2).sum()/im1.size
```

skimage.metrics.structural_similarity
(also skimage.measure.compare_ssim)

- Is a perceptual quality measures, including connections to human visual neurobiology and perception, and direct validation of the index against human subject ratings. https://en.wikipedia.org/wiki/Structural_similarity
- Normalized $0 \leq SSIM \leq 1$
- SSIM=1 means both images are identical

$$SSIM = \frac{(2\mu_1\mu_2 + c_1)(2\sigma_{12} + c_2)}{(\mu_1^2\mu_2^2 + c_1^2)(\sigma_1^2 + \sigma_2^2 + c_2^2)}$$

$$c_1 = k_1(2^n - 1) \quad \text{with } k_1 = 0.01 \text{ and } n = 8 \text{ (bits)}$$

$$c_2 = k_2(2^n - 1) \quad \text{with } k_2 = 0.03 \text{ i and} = 8 \text{ (bits)}$$

Compare MSE and SSIM https://scikit-image.org/docs/stable/auto_examples/transform/plot_ssim.html

3.2. Error diffusion binarization (dithering).

Error diffusion is a half-toning method used in printing and displaying technologies. The binarized image has to be similar to the original gray-level one. The underlying idea is simple: thresholding residual errors are distributed among neighboring pixels.

The algorithm works as follows:

- The program processes an NxM gray-level image using a pixel-by-pixel approach. Starting from pixel [0,0], the algorithm scans every row starting from the first column. (Alternatively, in the snake approach, when pixel [0,M-1] is reached[1], scanning of pixel [1,M-1] follows and then the process continues until pixel [1,0] is reached.)

[1] Boundaries are tricky since column M-1 and row N-1 cannot be processed.

We ignore those pixels that cannot be calculated (values are set to zero)

Let p_{ij}, th and e be the pixel value, the threshold and the quantization error, respectively. Error e at pixel p_{ij} is:

$$e = \begin{cases} p_{ij} - \max\{\text{image}\} & \text{if } p_{ij} > th \\ p_{ij} & \text{if } p_{ij} < th \end{cases}$$

where $\max\{\text{image}\}$ is 255 (or 1.) depending on the numerical class the image belongs to (`uint8` or `float64`, respectively). The threshold, th, is usually set to 128 or 0.5. Of the different possibilities, the following two dithering kernels are widely used: (i) Floyd and Steinberg (FS) and (ii) Jarvis, Judice, and Ninke (JJN):

$$\mathbf{FS} = \frac{1}{16}\begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 7 \\ 3 & 5 & 1 \end{pmatrix} \qquad \mathbf{JJN} = \frac{1}{48}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 7 & 5 \\ 3 & 5 & 7 & 5 & 3 \\ 1 & 3 & 5 & 3 & 1 \end{pmatrix}.$$

Then, according to the FS kernel, the error diffusion transformation induced on the neighborhood of p_{11} is:

$$\begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} + e \cdot 7/16 \\ p_{20} + e \cdot 3/16 & p_{21} + e \cdot 5/16 & p_{22} + e \cdot 1/16 \end{pmatrix}.$$

Note that values p_{00}, p_{01}, p_{01}, p_{10} and p_{11} are not changed in this step.

Quantization error is diffused across the image and finally the global threshold th is applied.

mode : ***{'reflect', 'constant', 'nearest', 'mirror', 'wrap'}, optional***

The *mode* parameter determines how the input array is extended beyond its boundaries. Default is 'reflect'. Behavior for each valid value is as follows:

'reflect' (*d c b a | a b c d | d c b a*)

The input is extended by reflecting about the edge of the last pixel.

'constant' (*k k k k | a b c d | k k k k*)

The input is extended by filling all values beyond the edge with the same constant value, defined by the *cval* parameter.

'nearest' (*a a a a | a b c d | d d d d*)

The input is extended by replicating the last pixel.

'mirror' (*d c b | a b c d | c b a*)

The input is extended by reflecting about the center of the last pixel.

'wrap' (*a b c d | a b c d | a b c d*)

The input is extended by wrapping around to the opposite edge.

https://docs.scipy.org/doc/scipy/reference/generated/scipy.ndimage.convolve.html

Original

Luminance

Threshold = 0.5

Diffusion error image

FS algorithm

inverse HT

Indexed color

Color reduction: from 256x256x256 to 6x6x6 colors

- 6x6x6 color images can be stored in a single 8-bit channel because only 216 possible values are necessary.
- An index table is required to provide a four-column table that relates the integer values in the file (index) with the displayed color.
- this table can be understood as a LUT that describes the color-map used to transform the gray level distribution into the corresponding true color image.

Calculation

- Only several values are considered: (e.g.: 0, 43, 86, 129, 172, 215) imp = 43 * np.uint8(im / 43)). Other possibilities might be considered as well.
- **As a look-up table lut[im]**

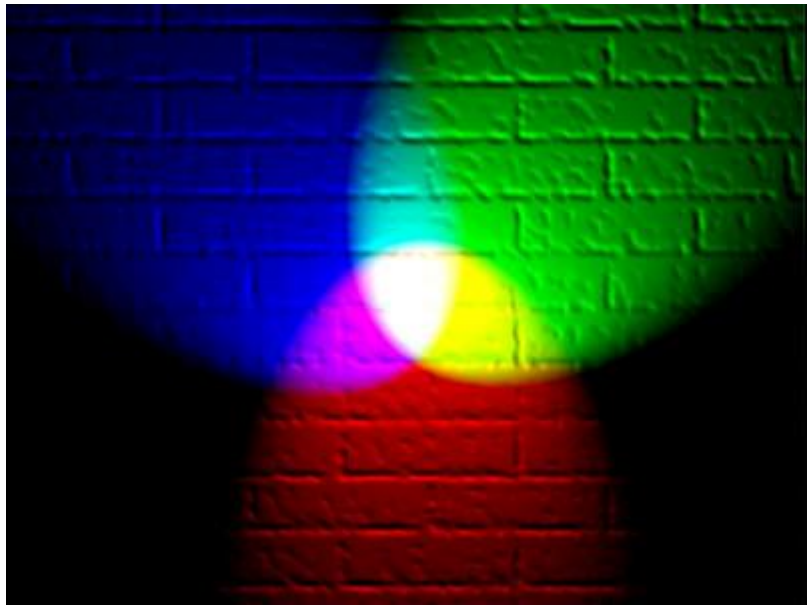
Indexed color table. Example.

File	Displayed colors		
stored gray level	R	G	B
0	0	0	0
1	43	0	0
2	0	43	0
3	0	0	43
4	86	0	0
…			
215	215	215	215
216	0	0	0
…			
255	0	0	0

Lab #4. More on color and intensity transformations.

1. RGB coordinates from spectrum data. The CIE 1931 color model.
2. Histogram equalization.
3. Image entropy.
4. Least significant bit steganography.
5. Visual encryption.

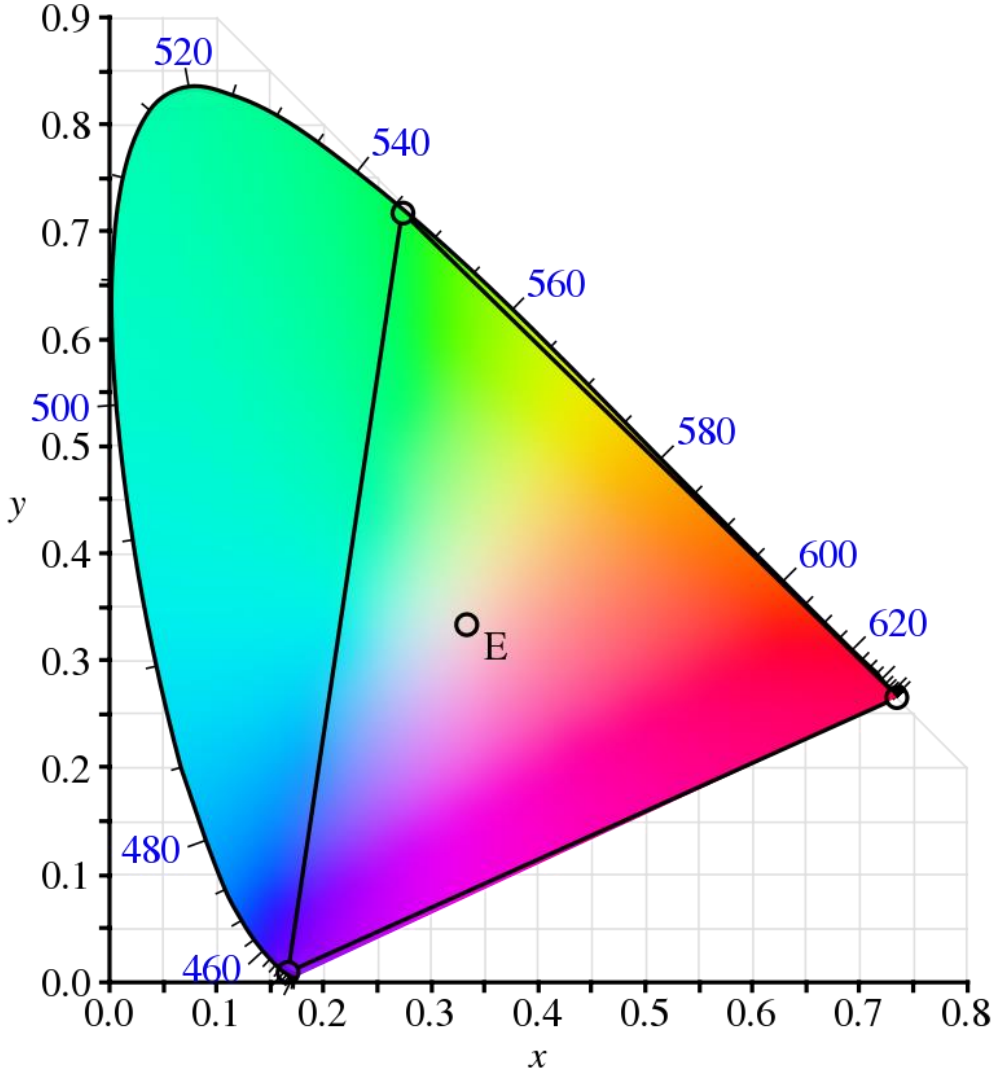
Color models: RGB, CIE 1931



https://en.wikipedia.org/wiki/RGB_color_model#/media/File:RGB_illumination.jpg

Each primary color ranges from 0 to 255
These values are related to the display intensity (e.g.: LED, LCD)

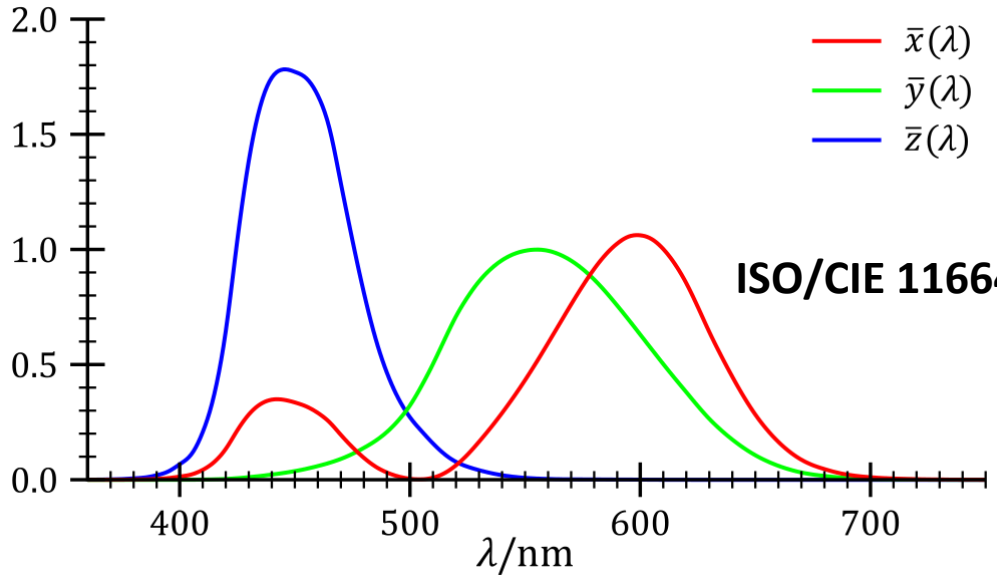
How RGB coordinates are related to physical measures?



https://en.wikipedia.org/wiki/CIE_1931_color_space#/media/File:CIE1931xy_CIERGB.svg

It is said that color is a property related to the wavelength λ (or the spectrum E(λ)). How are related E(λ) and RGB values?

The color matching functions x(λ), y(λ), and z(λ), relate the weight of the spectrum with the spectral sensitivity of the observer to calculate CIE coordinates X, Y and Z.



ISO/CIE 11664-4:2019

$$x = \int E(\lambda)x(\lambda)d\lambda$$
$$y = \int E(\lambda)y(\lambda)d\lambda$$
$$z = \int E(\lambda)z(\lambda)d\lambda$$

$$X = \frac{x}{x+y+z} \quad Y = \frac{y}{x+y+z} \quad Z = \frac{z}{x+y+z}$$

Use `scipy.integrate.simpson` (also `scipy.integrate.simps`)

Finally, XYZ and RGB are related by a linear transformation

$$\begin{pmatrix} R \\ G \\ B \end{pmatrix} = \text{np.uint8}\left[255 * \begin{pmatrix} 3.240479 & -1.537150 & -0.498535 \\ -0.969256 & 1.875992 & 0.041556 \\ 0.055648 & -0.204043 & 1.057311 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right]$$

Image histogram. Equalization.

Objective: to take advantage of the full gray-level range.

- A very simple and effective processing suitable for (gray-level) images with unbalanced histograms.
- A perfectly balanced image displays a flat histogram (all gray levels are equiprobable). Accordingly, the cumulative histogram will be linear.
- Image equalization is the process to produce an image with a linear cumulative histogram.
- Images should be `np.uint8` (real values are also fine but consider a 256 steps quantization).

- Calculation of the histogram:
  `scipy.ndimage.histogram` or
  `scipy.ndimage.measurements.histogram`
- Calculation of the cumulative histogram: `numpy.cumsum`

Image histogram. Equalization.

- A histogram measures how many times (counts) a gray level appears on the image. The size of the image is M x N.

$$h[g] = \text{counts}[g]$$

- Probability

$$P[g] = \frac{\text{counts}[g]}{\text{M x N}}$$

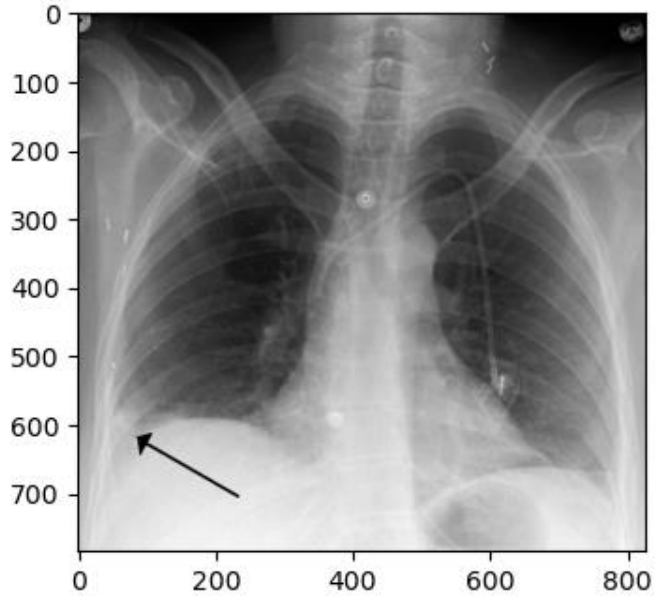
- Cumulative histogram

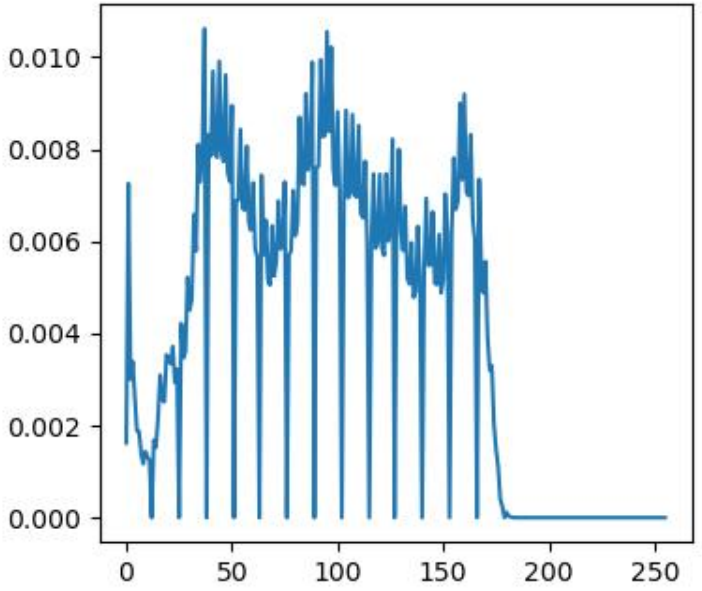
$$\text{ch}[g] = \sum_{i=0}^{g} \text{counts}[i] \qquad \text{ch}[g = 255] = \text{M x N}$$

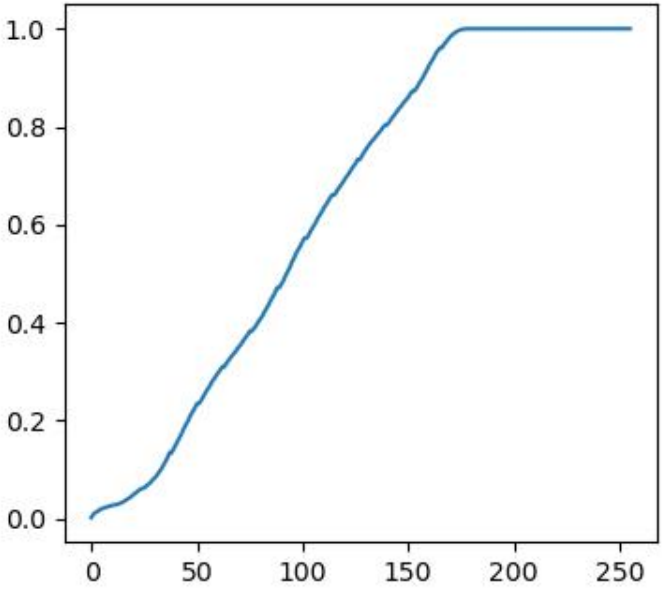
- Cumulative probability

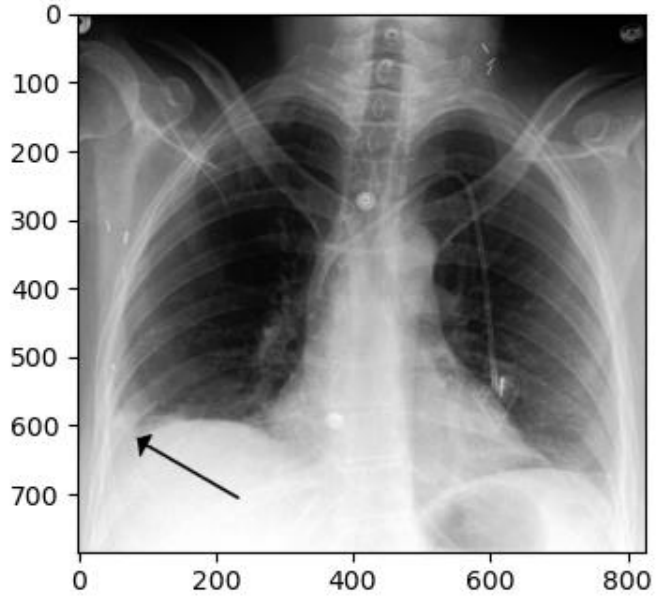
$$\text{cp}[g] = \frac{\sum_{i=0}^{g} \text{counts}[i]}{\text{M x N}} \qquad \text{cp}[g = 255] = 1$$

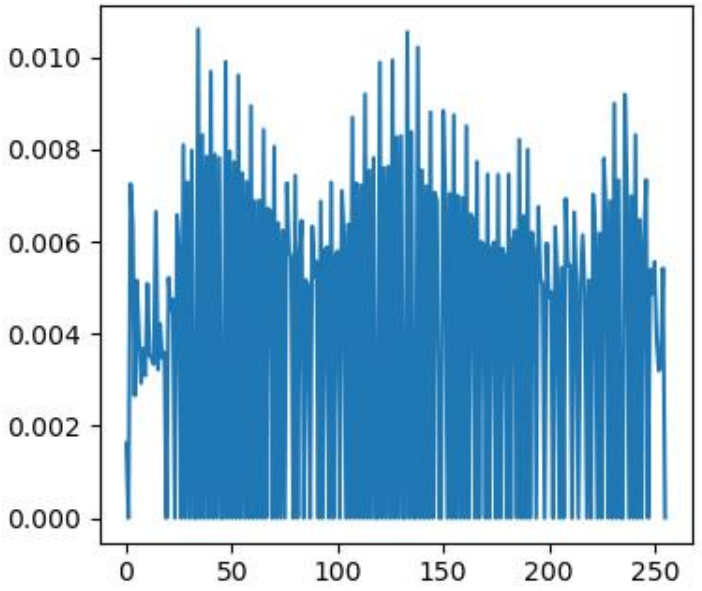
Image histogram. Equalization. Image credit: https://commons.wikimedia.org/wiki/File:Hamptonshump.PNG











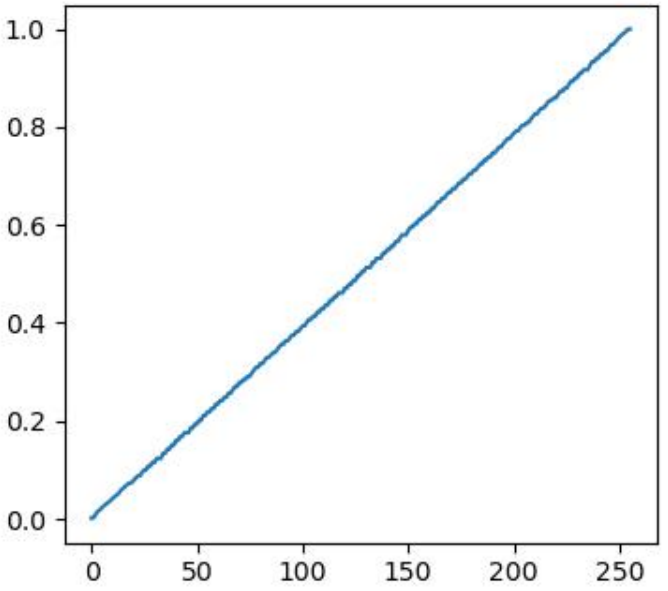
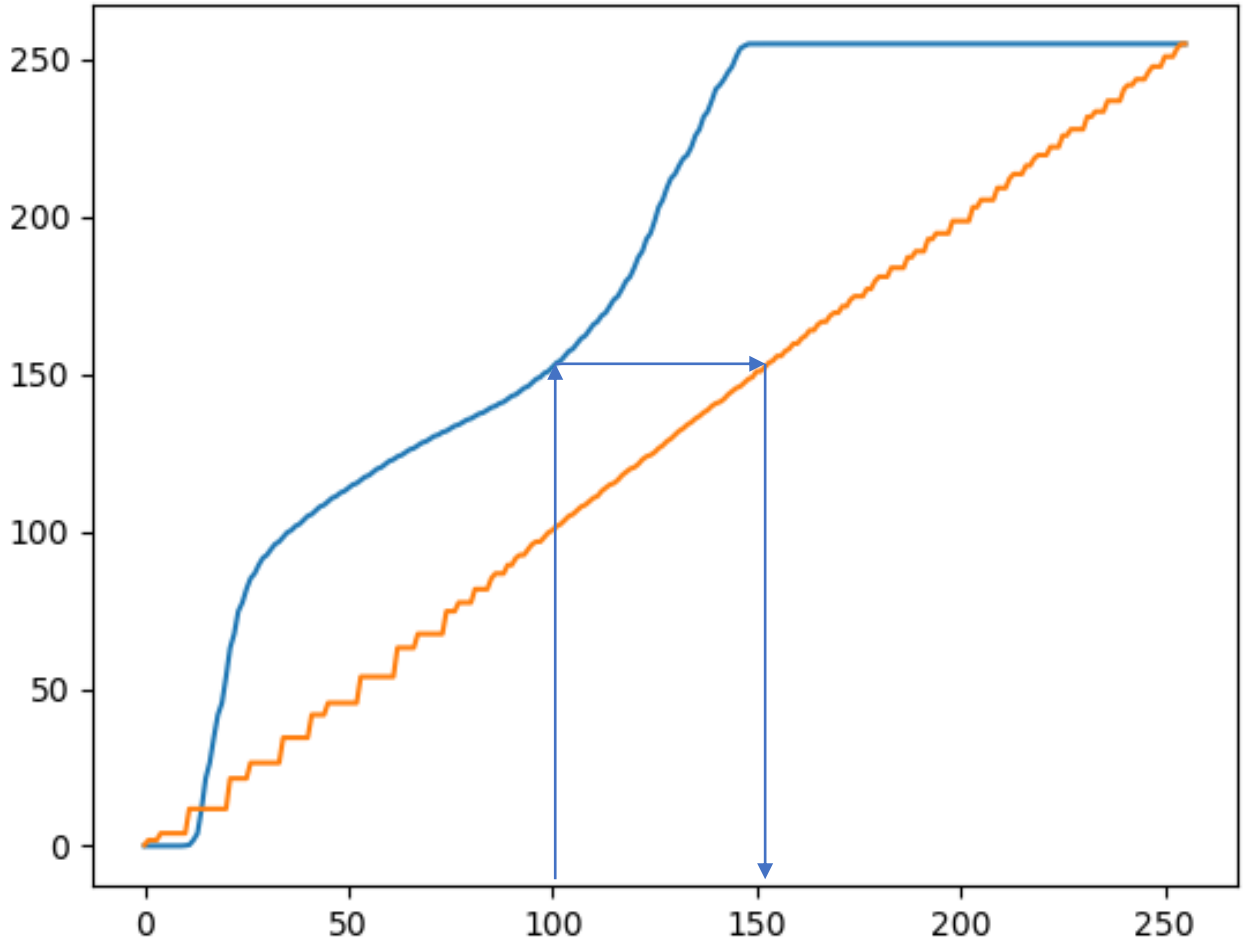


Image histogram. Equalization.

- Equalization formula $$eqim = np.uint8\left(\frac{255 * ch[im]}{M \times N}\right)$$



Rescale the values of the cumulative histogram in the range 0-255 (y-axis)

What does ch[im] mean?

- ch[] is a 1d array;
- but im is a 2D array

Image entropy

C. E. Shannon, "A mathematical theory of communication." *The Bell System Technical Journal* **27**, 379-423 (1948).

Shannon's entropy formula:

$$S = -\sum_{g=0}^{255} P[g] \log_2 P[g] \qquad \text{bits/pixel} \qquad 0 < S < 8$$

Images are assumed to be `np.uint8`

This definition is general and can be applied to any communication channel, e.g.: texts.

S provides an idea of the theoretical compression limit.

The **local** entropy can be calculated using `skimage.filters.rank.entropy`

For color images, the entropy should be calculated for every channel ($0 < S < 24$).

Cryptography and Steganography

- **Cryptography** is the science of writing in secret codes

  (from cryptos κρυπτός - hidden, secret)

- **Steganography** is the science of hiding information

  (from steganos στεγανός - covered, protected).

Whereas the goal of cryptography is to make data unreadable by a third party, the goal of steganography is to hide the data from a third party.

Applications

Digital watermarking (to prevent copyright infringement)

Data codification: time-stamps, serial numbers.

Steganography improves security if combined with cryptographic methods

Facts

"Color is often mistaken as a property of light when it really is a property of the brain. Our experience of color depends not only on the wavelength of the light rays that hit the retina, but also the context in which we perceive."

http://hypertextbook.com/facts/2006/JenniferLeong.shtml

According to Calkins, human beings can distinguish al least 100000 colors.

D. J. Calkins, "Mapping color perception to a physiological substrate" The Visual Neurosciences. The MIT Press, 1993.

https://mitpdev.mit.edu/library/erefs/chalupa/c064/section1.html

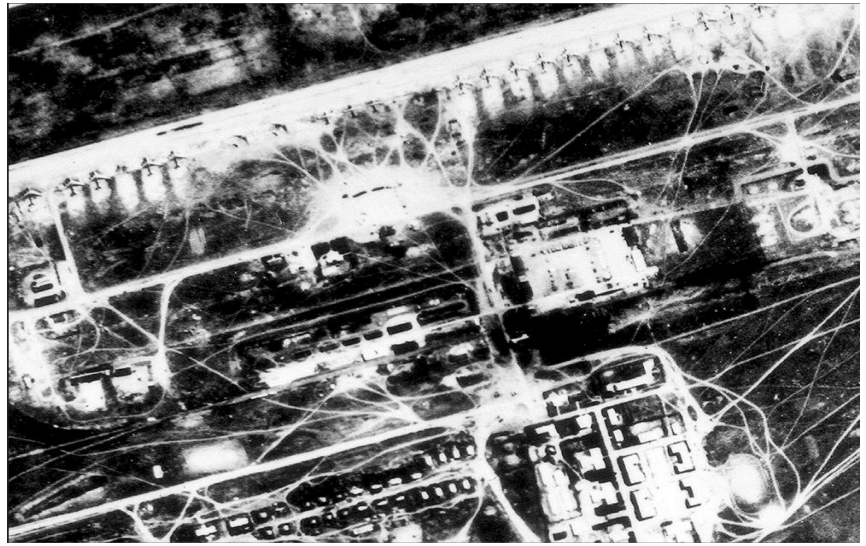
24-bit color images = 256 x256 x 256 = 16,777,216 colors

Steganography example

The secret image (cat) is hidden within the host image

Images from:
http://upload.wikimedia.org/wikipedia/commons/4/4e/StenographyOriginal.png
http://upload.wikimedia.org/wikipedia/commons/1/1b/StenographyRecovered.png

Another example



A satellite image of a Soviet strategic bomber base embedded in a Renoir painting

Images from:
www.jjtc.com/pub/r2026.pdf

Binary numbers

”*There are only 10 types of people in the world:*
Those who understand binary, and those who don't.”

Most significant bits

Least significant bits

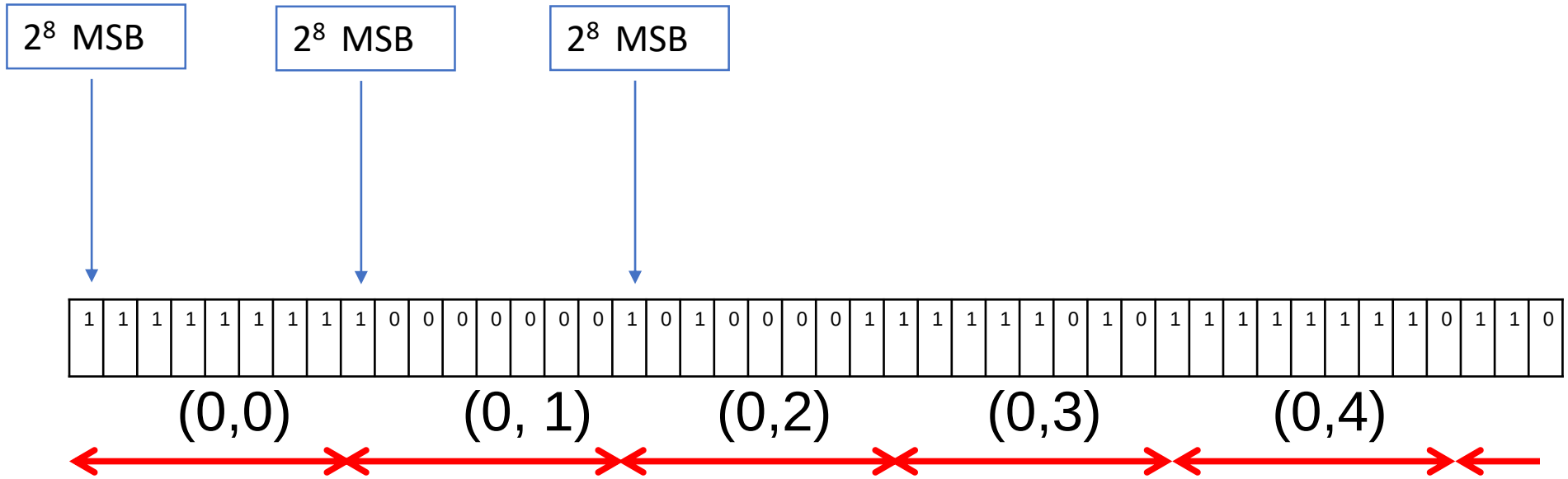
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1

- Example
- 161=128+32+1

1	0	1	0	0	0	0	1
128	0	32	0	0	0	0	1

Bitplanes

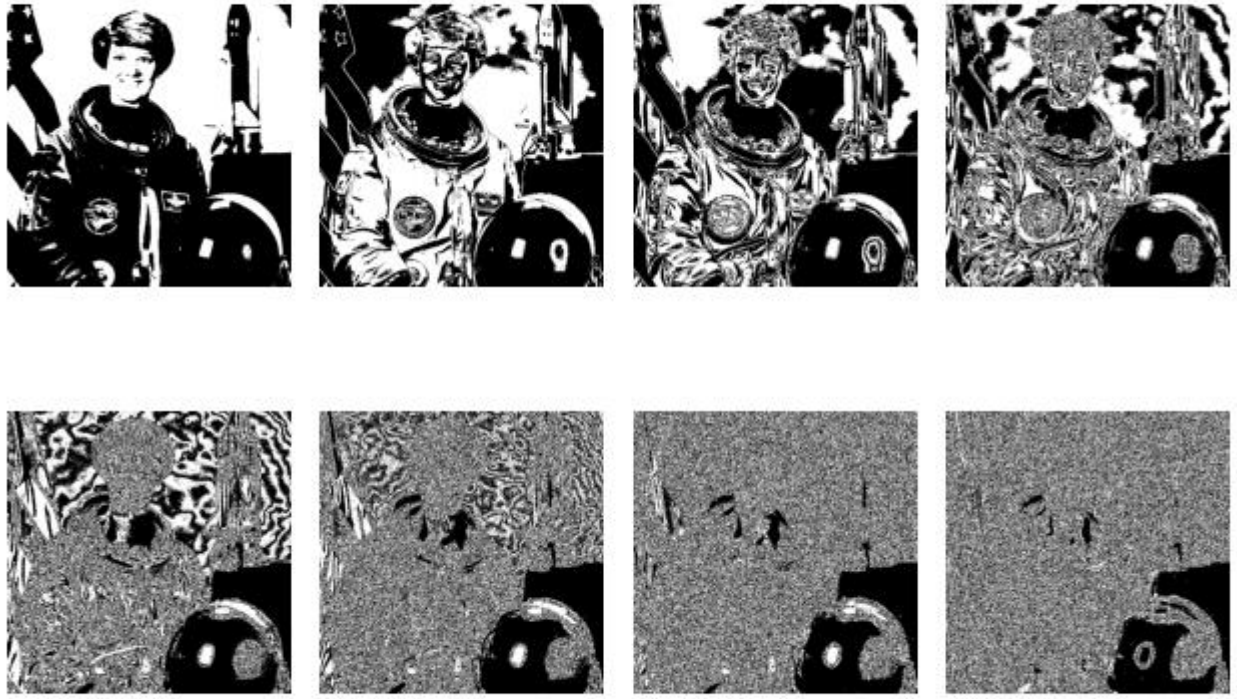
https://en.wikipedia.org/wiki/Bit_plane



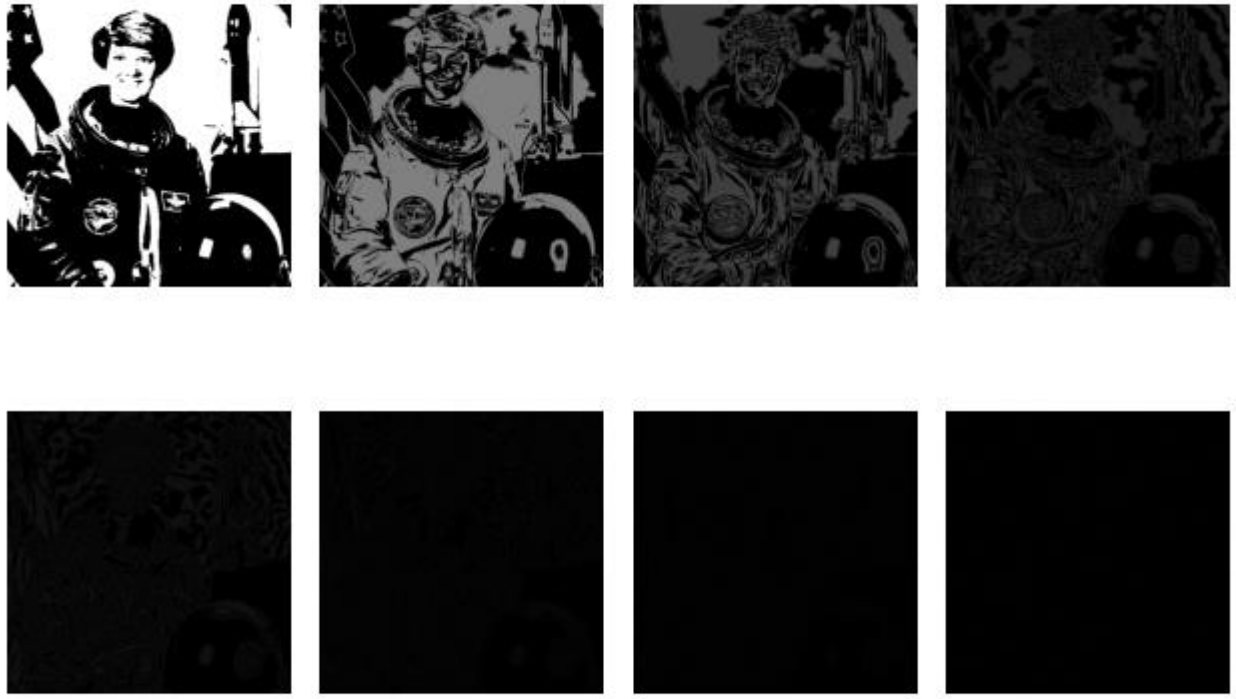
(0,0) MSB	(0,1) MSB	(0,2) MSB
(1,0) MSB	(1,1) MSB	…
(2,0) MSB	…	…

Image bitplanes

plt.imshow(…, cmap='gray')



plt.imshow(…, cmap='gray', vmax=255)



```
Graylevel_from_bitplanes =
128 * b0 + 64 * b1 + 32 * b2 + 16 * b3 + 8 * b4 + 4 * b5 +  2 * b6 + b7
```

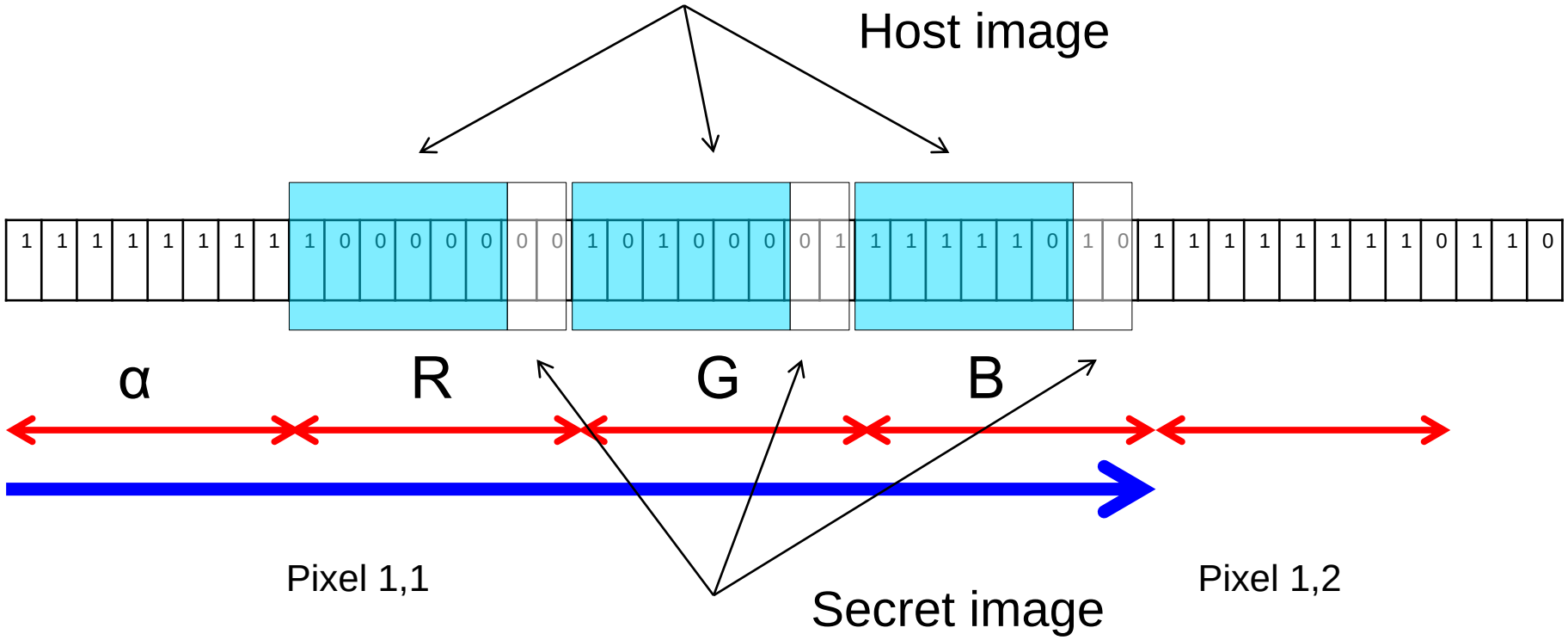
Least **Significant** Bits (LSB) steganography (I)

The 8-bit RGB model can display 256 x 256 x 256 = 16.777.216 colors (far more than required)

Using a 6-bit RGB model we would be able to display 64 x 64 x 64 = 262.144 color (<100000 tones, enough!)

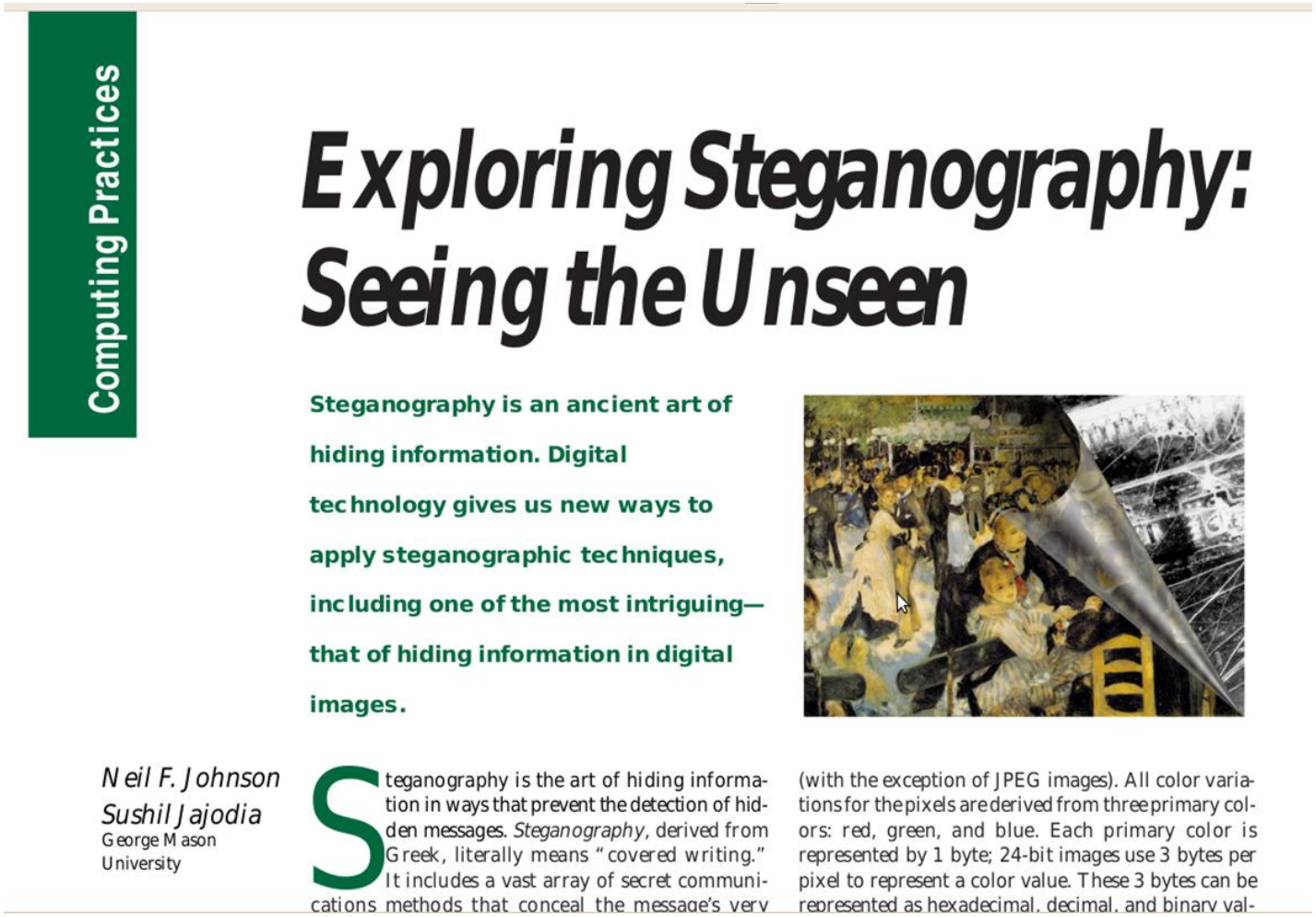
The remaining two bits are used to code the secret image using just 4 x 4 x 4 = 64 colors (less quality but not that bad!)

Least Significant Bits (LSB) steganography (II)



The bits of the secret can be placed in a non-natural order, i.e.:
R: 2 and 4, G:1 and 6, B: 3 and 5.
Since the host image is codified using the most significant bits (more intense), the secret image cannot be easily detected.

Further reading



Computing Practices

Exploring Steganography: Seeing the Unseen

Steganography is an ancient art of hiding information. Digital technology gives us new ways to apply steganographic techniques, including one of the most intriguing—that of hiding information in digital images.

Neil F. Johnson
Sushil Jajodia
George Mason University

Steganography is the art of hiding information in ways that prevent the detection of hidden messages. *Steganography*, derived from Greek, literally means "covered writing." It includes a vast array of secret communications methods that conceal the message's very

(with the exception of JPEG images). All color variations for the pixels are derived from three primary colors: red, green, and blue. Each primary color is represented by 1 byte; 24-bit images use 3 bytes per pixel to represent a color value. These 3 bytes can be represented as hexadecimal, decimal, and binary val-

Neil F. Johnson, Sushil Jajodia, "Exploring Steganography: Seeing the Unseen," Computer **31**, 26-34 (1998)

Programming tips:
Using np.unpackbits

Unpacks elements of a uint8 array into a binary-valued output array.

```
a = np.array([[180, 200],[30, 50]], dtype=np.uint8)

a_bin = np.unpackbits(a)

a_bin0 = np.unpackbits(a, axis=0)

a_bin1 = np.unpackbits(a, axis=1)
```

```
a
Out[19]:
array([[180, 200],
       [ 30,  50]], dtype=uint8)
```

```
a_bin
Out[20]:
array([1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1,
       1, 0, 0, 0, 1, 1, 0, 0, 1, 0], dtype=uint8)
```

```
a_bin0
Out[21]:
array([[1, 1],
       [0, 1],
       [1, 0],
       [1, 0],
       [0, 1],
       [1, 0],
       [0, 0],
       [0, 0],
       [0, 0],
       [0, 0],
       [0, 1],
       [1, 1],
       [1, 0],
       [1, 0],
       [1, 1],
       [0, 0]], dtype=uint8)
```

```
a_bin1
Out[23]:
array([[1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0],
       [0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0]], dtype=uint8)
```

```
a.shape
Out[37]: (2, 2)

a_bin0.shape
Out[38]: (16, 2)

a_bin1.shape
Out[39]: (2, 16)

a_bin.shape
Out[40]: (32,)
```

Programming tips:
Using np.unpackbits

An unconventional binarization method:

```
# Select bitplane 0
imhidden = data.camera()
imhi_b = 255 * np.unpackbits(imhidden, axis= 0)[0::8,  :]
```

```
# Select bitplane 7
bitplane7 = np.unpackbits(imhidden, axis= 0)[6::8,  :]
```

```
a_bin0
Out[21]:
array([[1,
        [0, 1],
        [1, 0],
        [1, 0],
        [0, 1],
        [1, 0],
        [0, 0],
        [0, 0],
        [0, 0],
        [0, 0],
        [0, 1],
        [1, 1],
        [1, 0],
        [1, 0],
        [1, 1],
        [0, 0]], dtype=uint8)
```

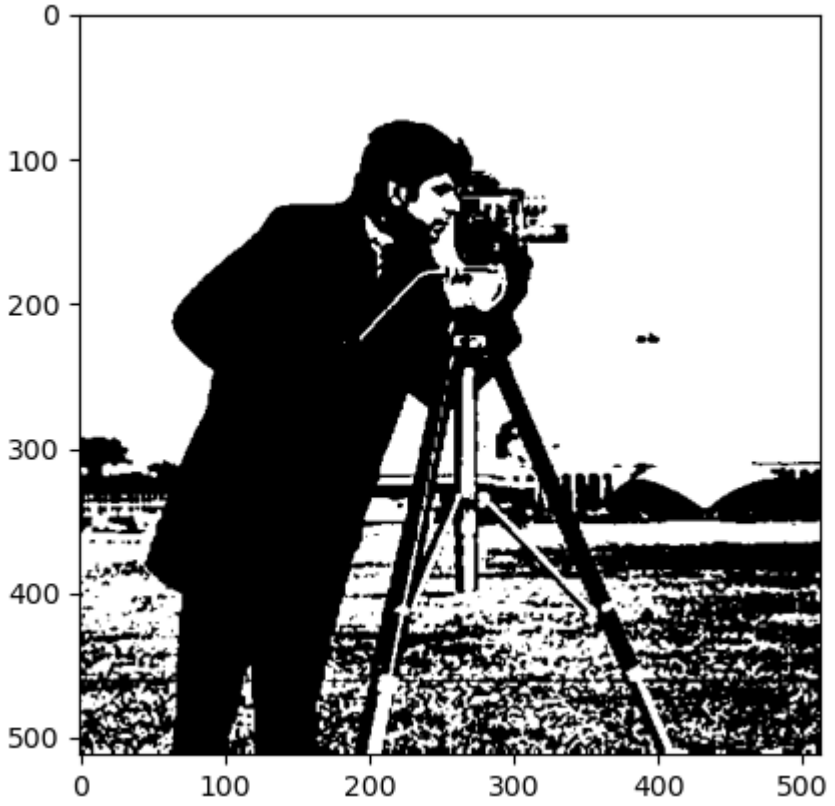
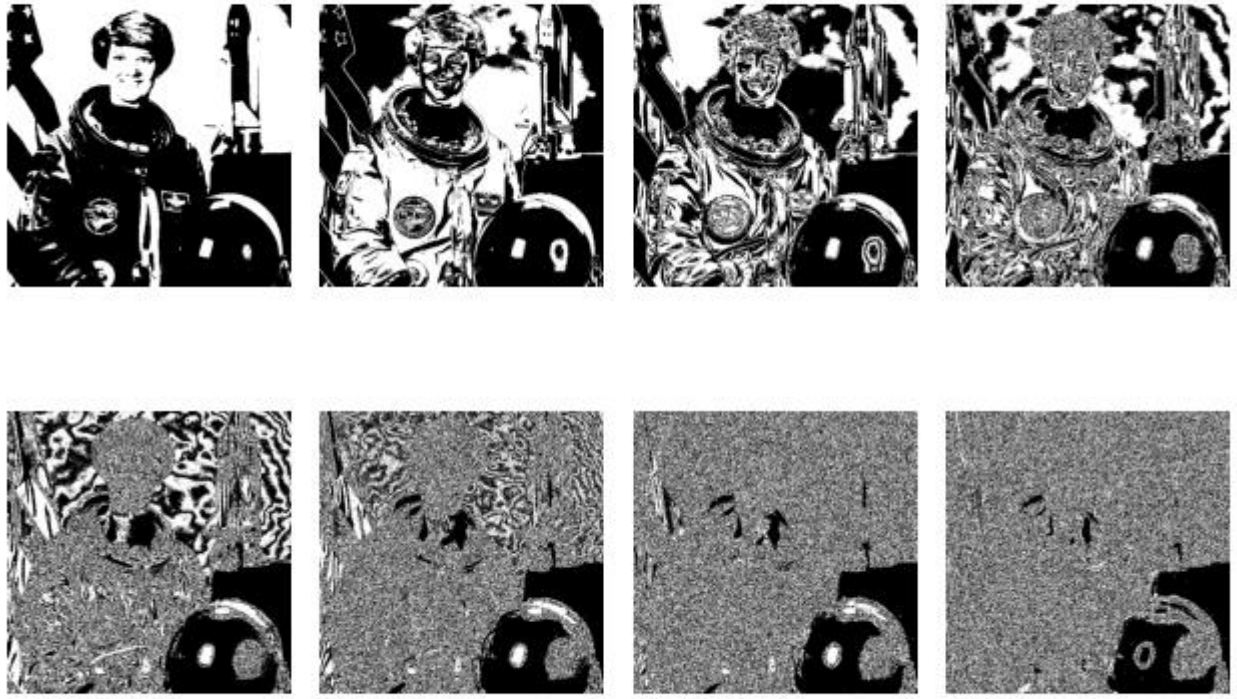
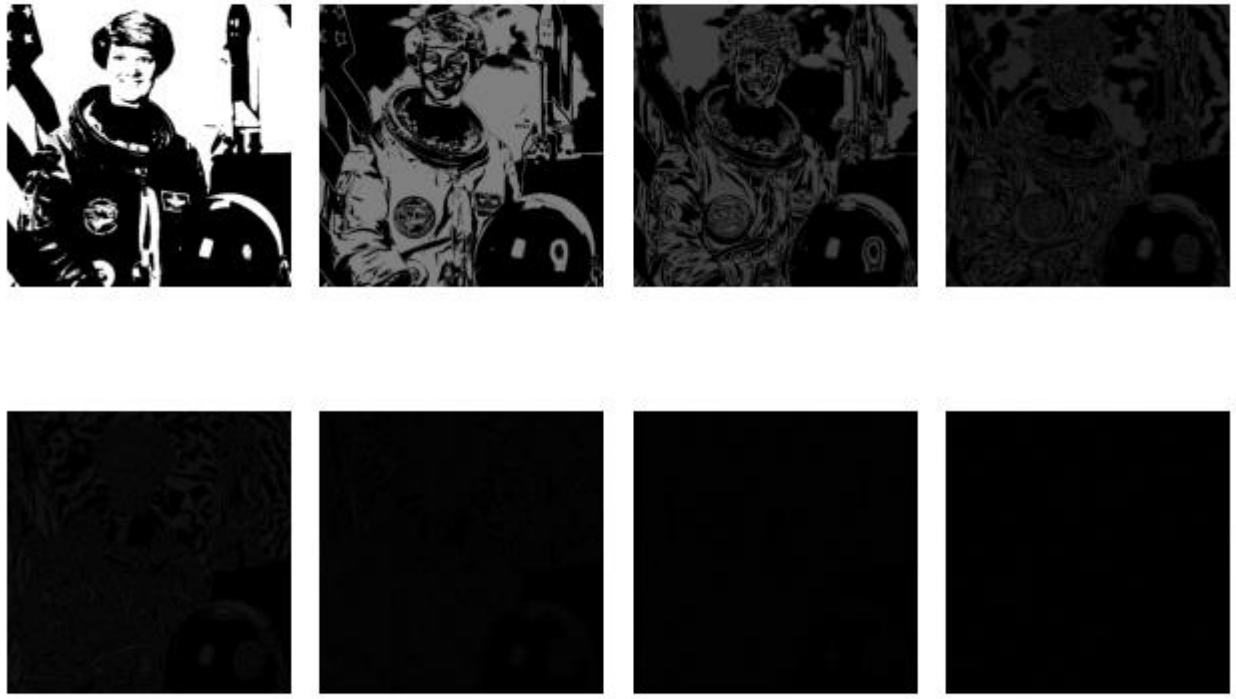


Image bitplanes

plt.imshow(…, cmap='gray')



plt.imshow(…, cmap='gray', vmax=255)



```
Graylevel_from_bitplanes =
128 * b0 + 64 * b1 + 32 * b2 + 16 * b3 + 8 * b4 + 4 * b5 +  2 * b6 + b7
```

Lab #5: Fourier transforms and spatial filtering.

Fourier transforms (FT) arranges the information in object (image) space into the frequency (Fourier) domain (space).

The image information in the Fourier space is organized in a very convenient way that enables manipulation of the frequency information (image detail).

However, FTs are for defined mathematical (analogic) functions:

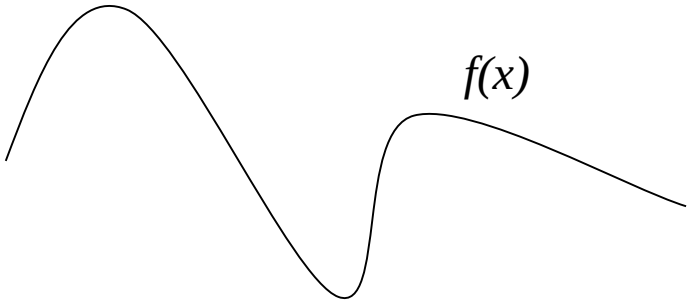
$$F\ u,v\ = \iint_{\mathbb{R}^2} f\ \ x,y\ \exp\ -2\pi i\ \ ux+vy\ \ \ dxdy$$

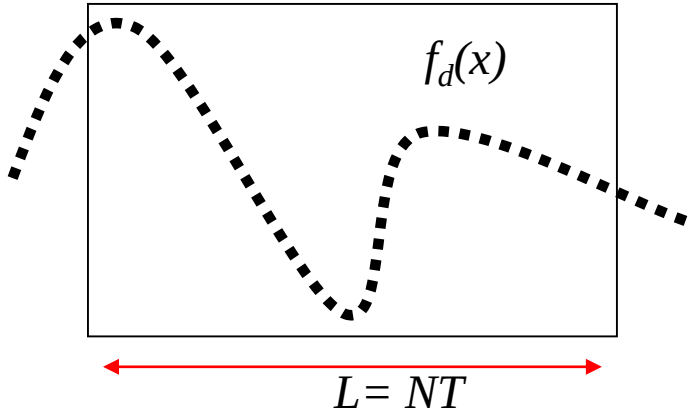
Can we extend this definition to sampled distributions?

The 2D Fast Fourier Transform algorithm enables us to perform this calculation

Digital Fourier Transforms?
Mathematical functions and sampled distributions

- $f(x)$ is a continuous signal (a real or complex valued function) and $F(u)$ its Fourier transform
- $f(x)$ is converted into a band-limited discrete signal $f_d(x)$ through a sampling process. L is the signal length, T the sampling rate and N the number of samples, $L=NT$.
- Relationship between $f_d(x)$ and $F_d(u)$?





Note an image can be understood as a 2D function f(x,y). Every row or column is a 1D function.

Digital Fourier Transforms Model

FT($f\ g$) = F * G and FT(f * g) = FG, convolution theorem
FT(comb(x; T)) = ($1/T$) comb(u; $1/T$)
FT(rect(x/L)) = (1/L) sinc(Lu)
If $f(x)$ is real-valued, $|F(u)|$ is symmetrical

$$f_d(x) = f(x)\sum_{m=-\infty}^{m=\infty}\delta(x-mT)\,\mathrm{rect}\left(\frac{x}{L}\right)$$

$$F_d(u) = F(u) * \frac{1}{T}\sum_{n=-\infty}^{n=\infty}\delta\left(u-\frac{n}{T}\right) * L\,\mathrm{sinc}\;\;Lu$$

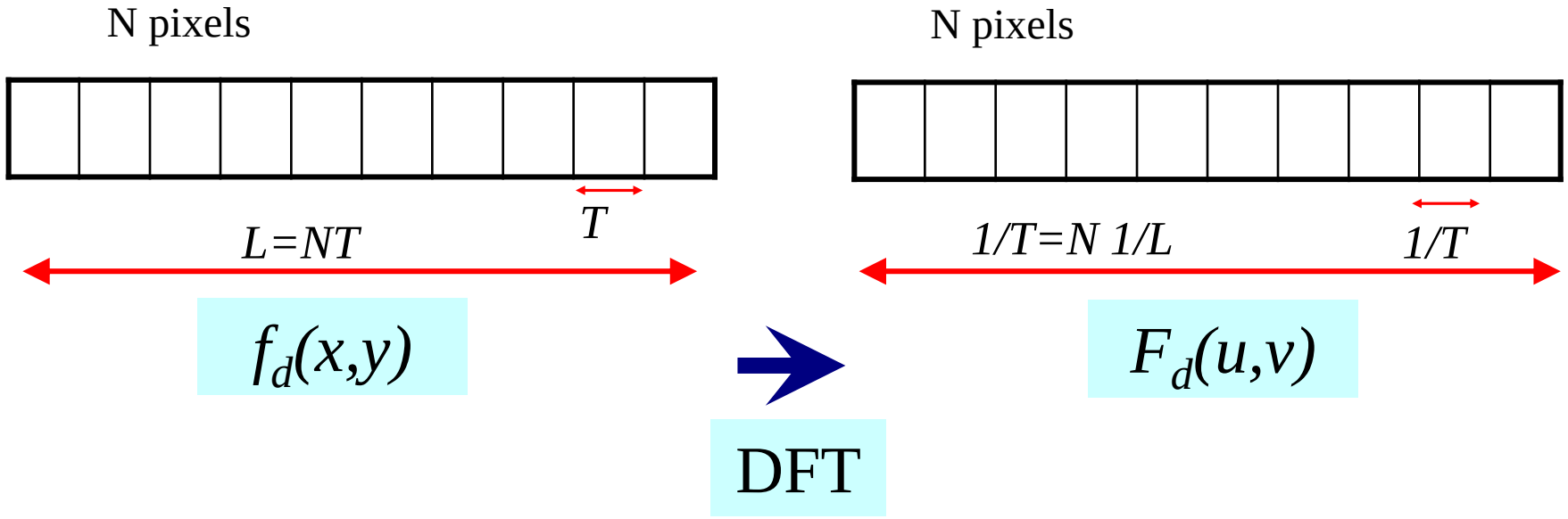
Not valid (not sampled)

$$f_d(x) = \left[f(x)\sum_{m=-\infty}^{m=\infty}\delta(x-mT)\,\mathrm{rect}\left(\frac{x}{L}\right)\right] * \sum_{s=-\infty}^{s=\infty}\delta(x-sL)$$

$$F_d(u) = \left[F(u) * \frac{1}{T}\sum_{n=-\infty}^{n=\infty}\delta\left(u-\frac{n}{T}\right) * L\,\mathrm{sinc}\;\;Lu\right]\frac{1}{L}\sum_{t=-\infty}^{t=\infty}\delta\left(u-\frac{t}{L}\right)$$

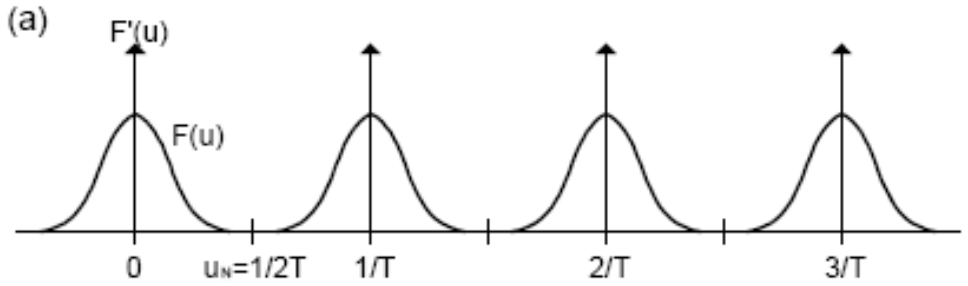
Both the image and the FT are replicated.

Frequencies range from -*1/(2T)* and *1/(2T)*

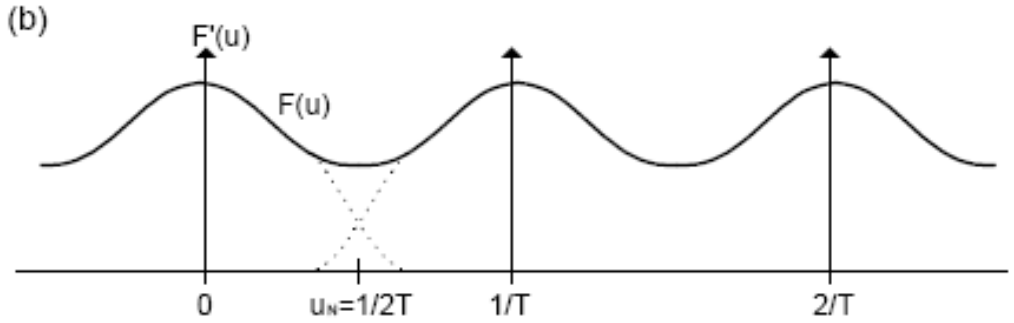


- If *L* is large, sinc*(Lu)* tends to *δ(u),* no leakage (blurring) present
- If *T* is small, no aliasing present
- Since *L=NT, N* must be large

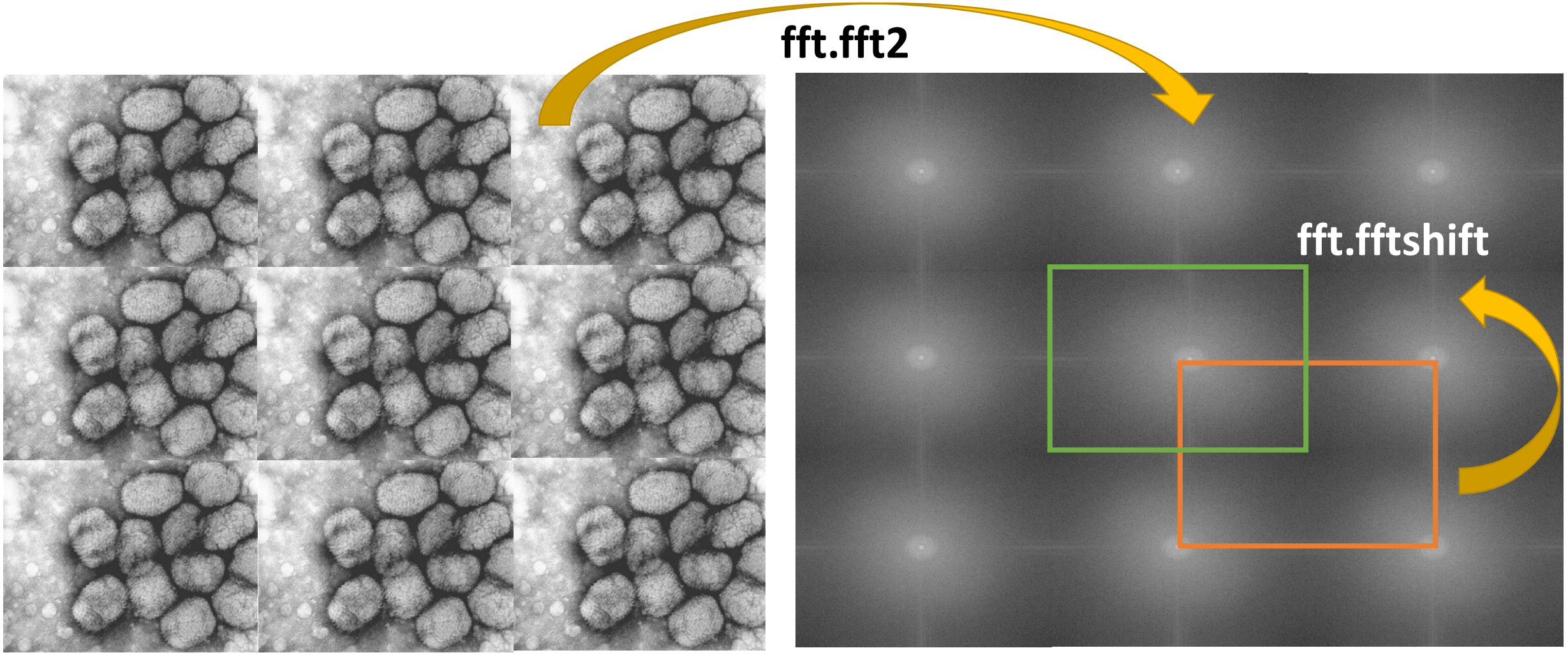
- $u_c=1/(2T)$ is the cut-off frequency or the Nyquist frequency (ie. the highest frequency that can be represented by the sampled signal - **Shannon sampling theorem**)



- Effect of aliasing



Multiple input images and multiple FT



Digital Fourier Transform (DFT) properties that should be considered:

DFT produce very similar results to the analogical FT if very general conditions are hold:

- Images are band limited and *zero valued outside the image limits. A **band-limited signal** is one whose Fourier transform, or spectral density has bounded support (https://en.wikipedia.org/wiki/Bandlimiting)*:
- Sampling frequency (pixels per length unit) is high enough (Large *L*, small *T*) and no aliasing is present.
- However, the convolution theorem should be used with care

  (circular convolution issues)

Shannon Sampling theorem:

- **Image space:** L is the signal length, T the sampling rate and N the number of samples, $L=NT$.
- **Fourier (frequency) space**: $1/T = N\ 1/L$.
- **Nyquist frequency (cut-off frequency):** $f_N = 1/2/T = N/2/L$

Remarks and questions:

The amplitude (modulus) of the FT of real signals (such as images) is symmetrical, whereas the phase is anti-symmetrical.

What is the role of the `fft.fftshift / fft.ifftshift` pair ?

Try not to use this functions and investigate what happens.

Provided that `f` is real, `f2` is complex. Why?

```
F = fft.fftshift(fft.fft2(f))
f2 = fft.ifft2(fft.ifftshift(fftshift(fft.fft2(f))))
```

Spatial filtering targets the amplitude of the Fourier transform. Do not attempt to modify the phase. Why? (exercise 5.3)

Recall to use the Shannon theorem to determine the length of the signal $1/T$ in Fourier space (only if necessary) (exercise 5.2)

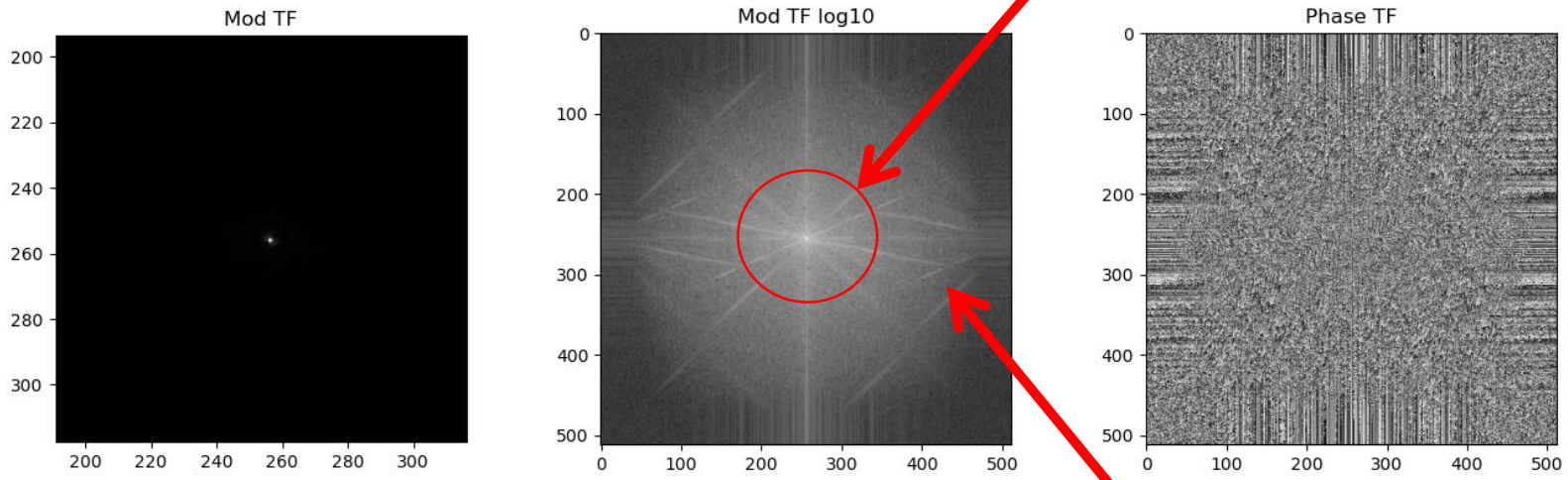
Color images: every channel with `fft.fft2` but not using a single `fft.fftn`

Fourier transforms: manipulation & interpretation

F(u, v) = FT[f(x,y)]

```
F = fft.fftshift(fft.fft2(f))
f = fft.ifft2(fft.ifftshift(fftshift(fft.fft2(f))))
```

```
np.real(F)
np.imag(F)
np.abs(F)
np.angle(F)
```




```
F = np.real(F) + 1j * np.imag(F)
F = np.abs(F)* np.exp(1j * np.angle(F))
```

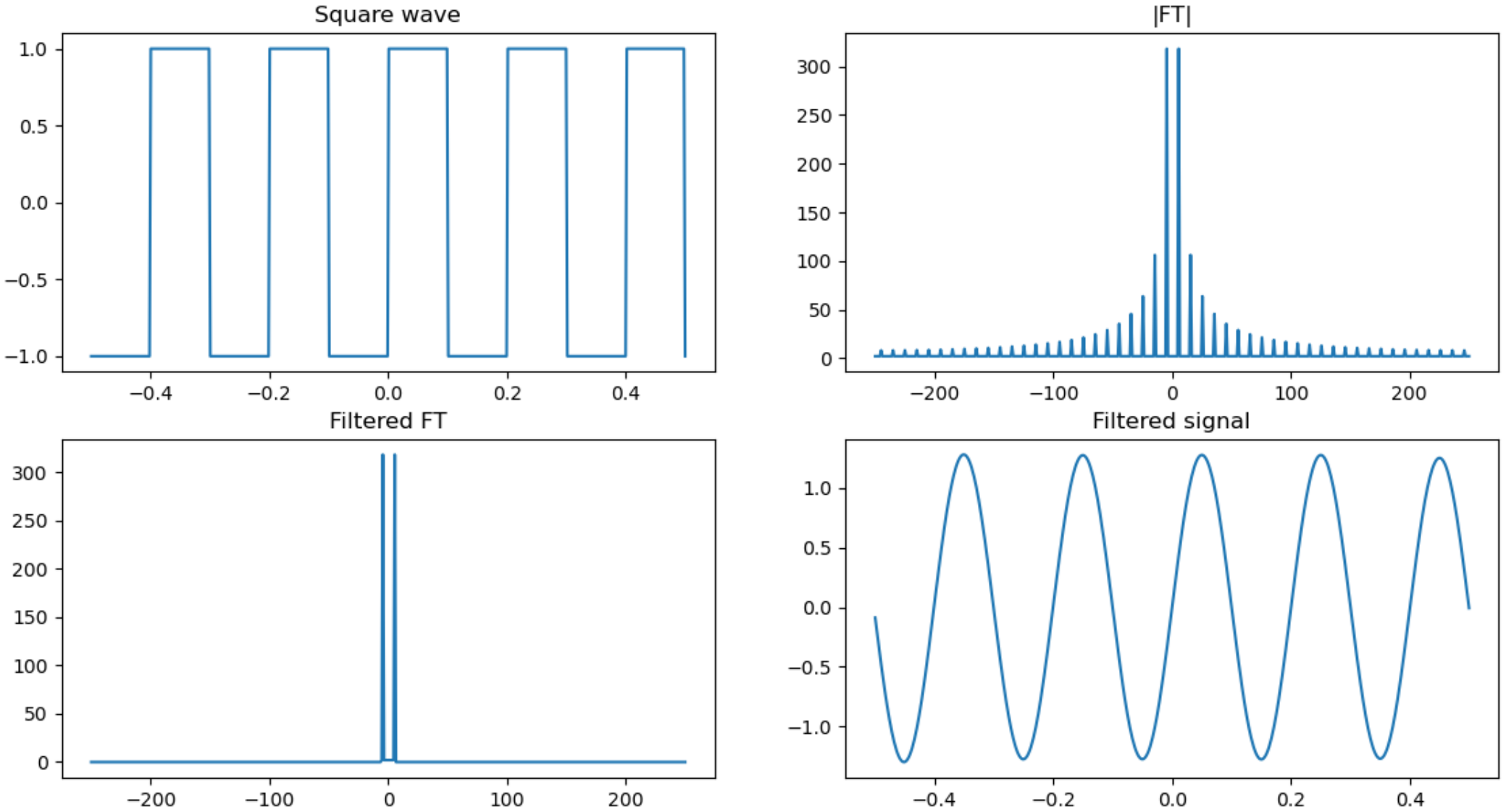
```
Log LUT: np.log10(np.abs(F)+1)
```

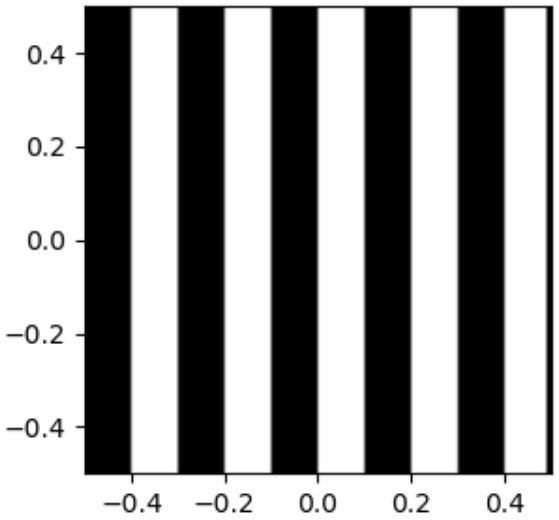
Low frequency area
High energy
Information describes by slow harmonics
(less details, less entropy)

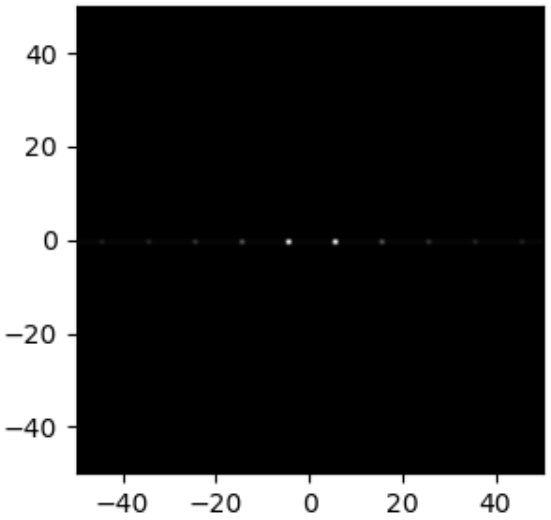
High frequency area
Low energy
Information described by fast harmonics
(image details, more entropy)

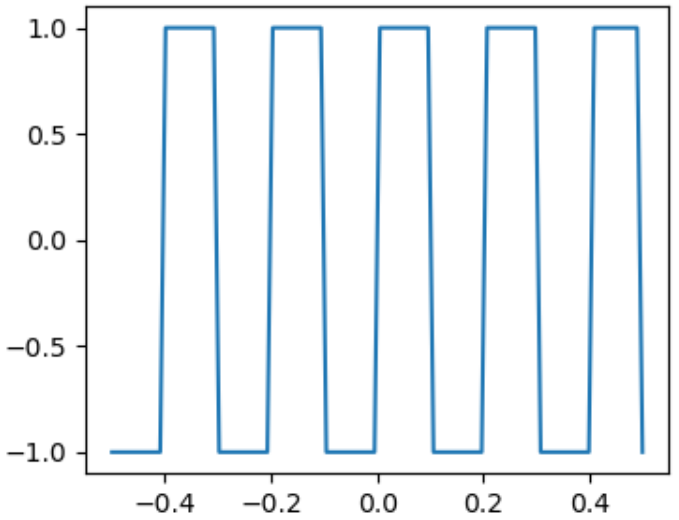
Why are we interested in Fourier transform?

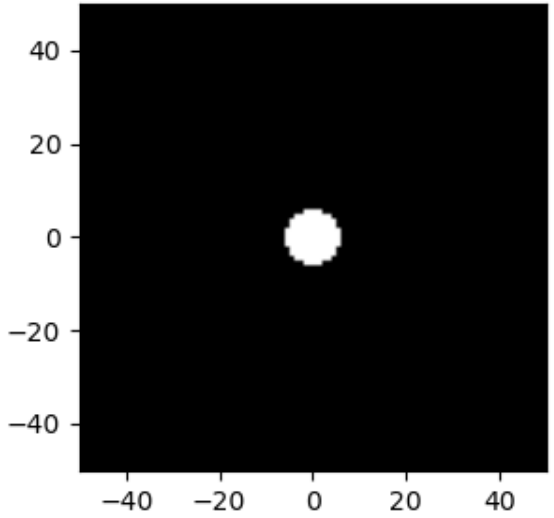
- Fourier transforms can be understood as the generalization of Fourier series for non-periodical functions.
- Let s($2\pi f_0 x$) a periodical function with frequency f_0 .According to the Fourier series theory, s() can be described as the superposition of harmonic functions (sin(), cos()) with frequencies with integer multiples of f_0 : f_0 , $2f_0$, $3f_0$ ...
- If $s()$ is not periodical, then f_0 varies in a continuous way
- The modulus of the Fourier transform provide information of the weight of each harmonic whereas the phase describes the position (shift) of the corresponding harmonic.
- Removing certain frequencies and FT^{-1}, we can alter the information of the image

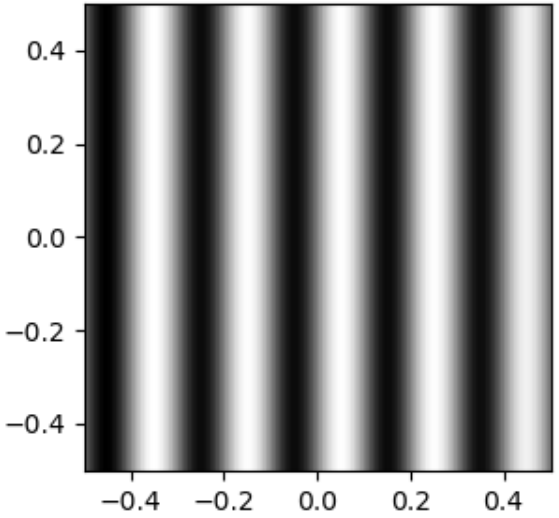

Square wave
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4
|FT|
300
250
200
150
100
50
0
−200
−100
0
100
200
Filtered FT
300
250
200
150
100
50
0
−200
−100
0
100
200
Filtered signal
1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4

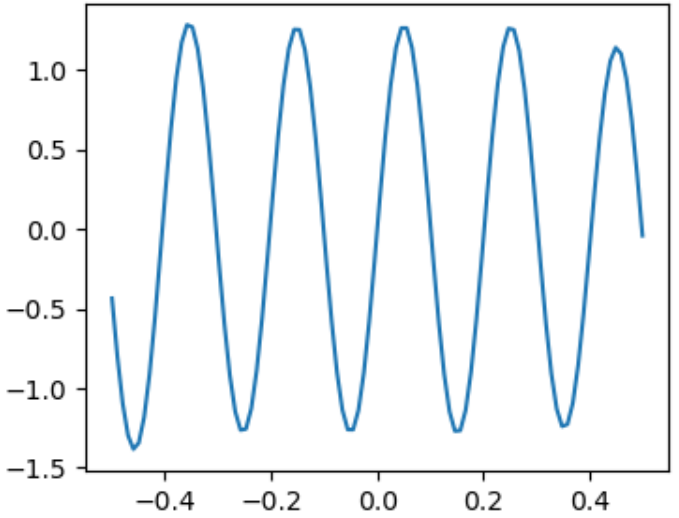

0.4
0.2
0.0
−0.2
−0.4
−0.4
−0.2
0.0
0.2
0.4


40
20
0
−20
−40
−40
−20
0
20
40


1.0
0.5
0.0
−0.5
−1.0
−0.4
−0.2
0.0
0.2
0.4

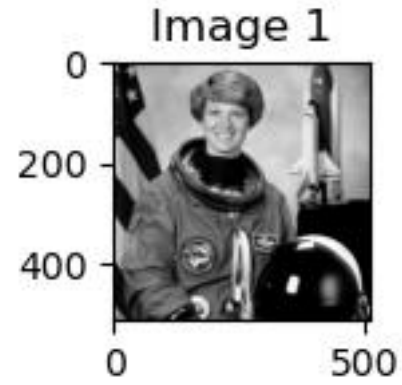

40
20
0
−20
−40
−40
−20
0
20
40

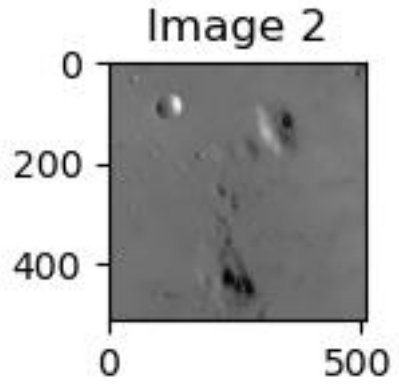

0.4
0.2
0.0
−0.2
−0.4
−0.4
−0.2
0.0
0.2
0.4

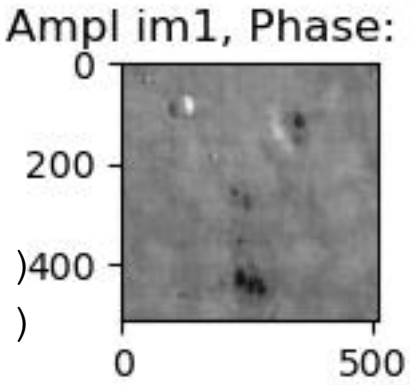

1.0
0.5
0.0
−0.5
−1.0
−1.5
−0.4
−0.2
0.0
0.2
0.4

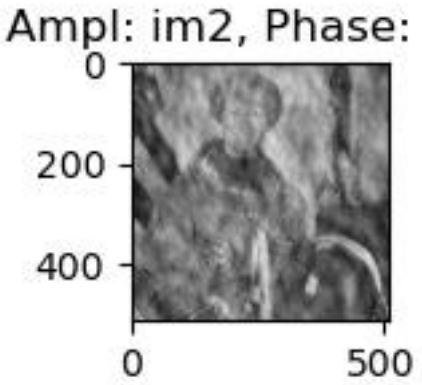
Oppenheim and Lim Experiment
Importance of phase in signals

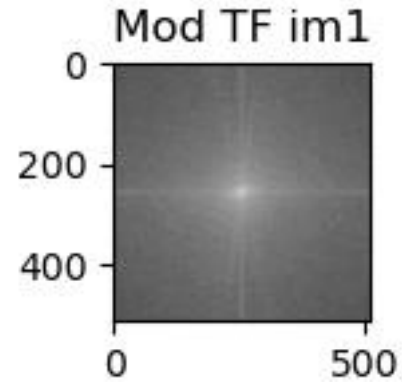
```
im1 = data.astronaut()[:, :, 1]
im2 = data.moon()

tf1 = fft.fftshift(fft.fft2(im1))
tf2 = fft.fftshift(fft.fft2(im2))

tf12 = np.abs(tf1) * np.exp(1j * np.angle(tf2))
tf21 = np.abs(tf2) * np.exp(1j * np.angle(tf1))

im12 = np.real(fft.ifft2(fft.ifftshift(tf12)))
im21 = np.real(fft.ifft2(fft.ifftshift(tf21)))
```

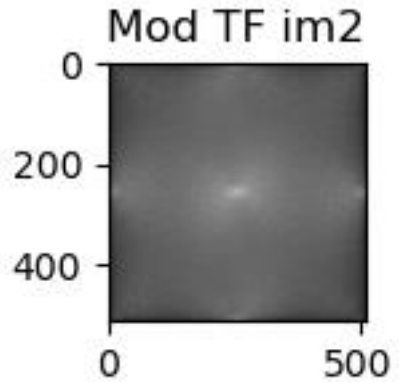












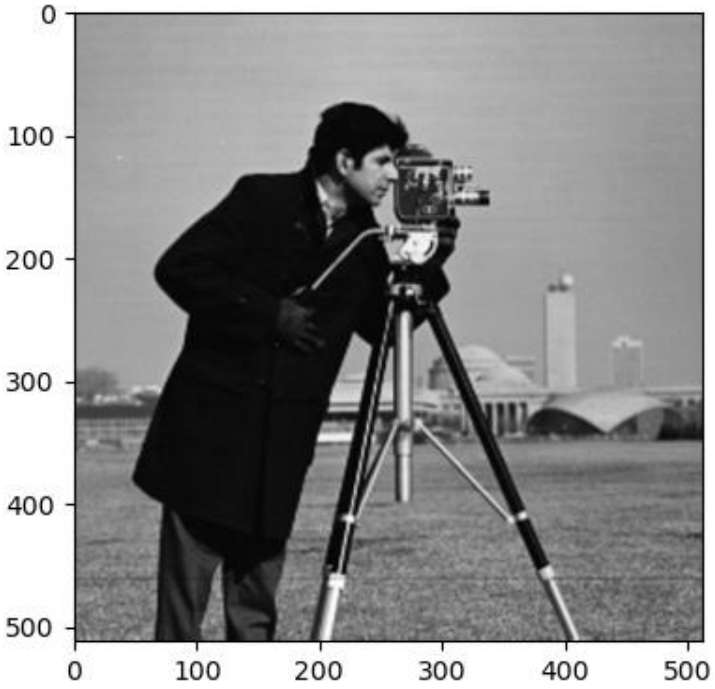
Phase Rules!

The phase characterizes the image.

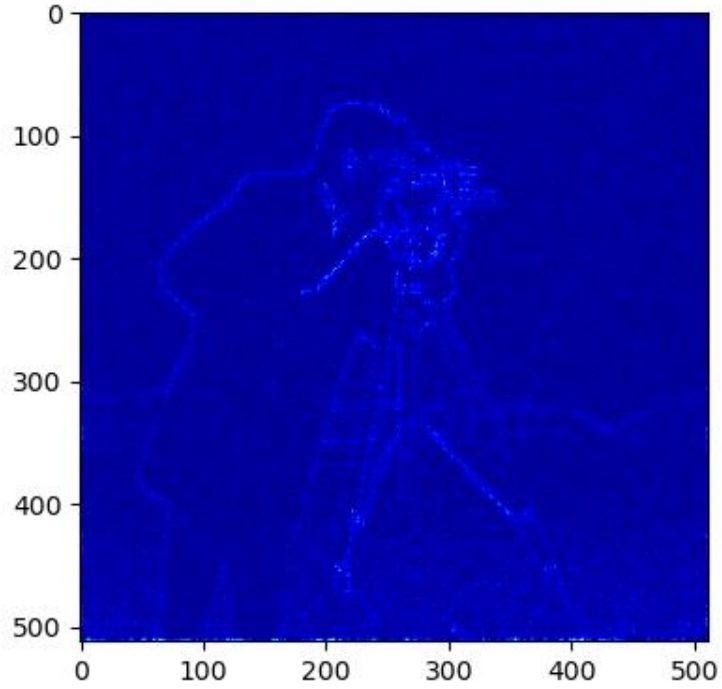
Filters should modify the amplitude

An example of high pass filtering: Phase-only reconstruction

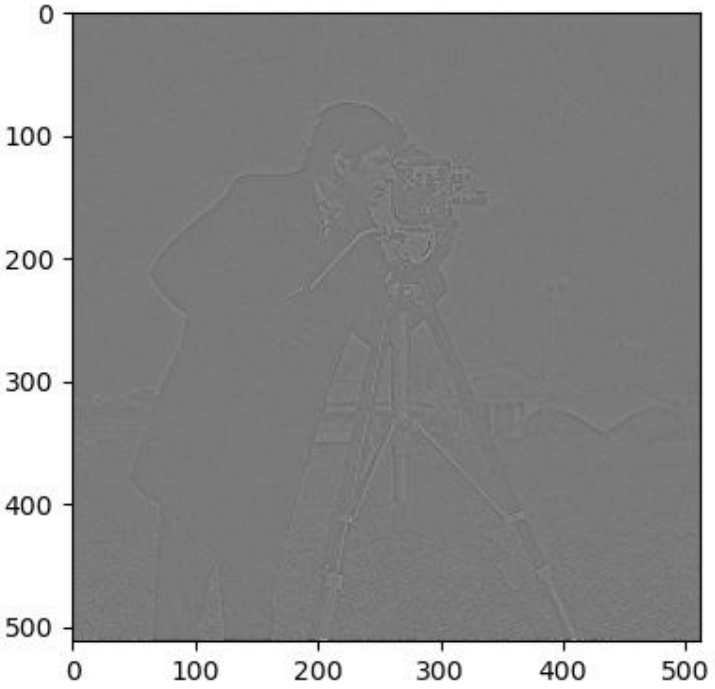
```
im = data.camera()
tfim = fft.fftshift(fft.fft2(im))
tfpo = tfim / np.abs(tfim)
poim = fft.ifft2(fft.fftshift(tfpo))
```



np.abs



np.real



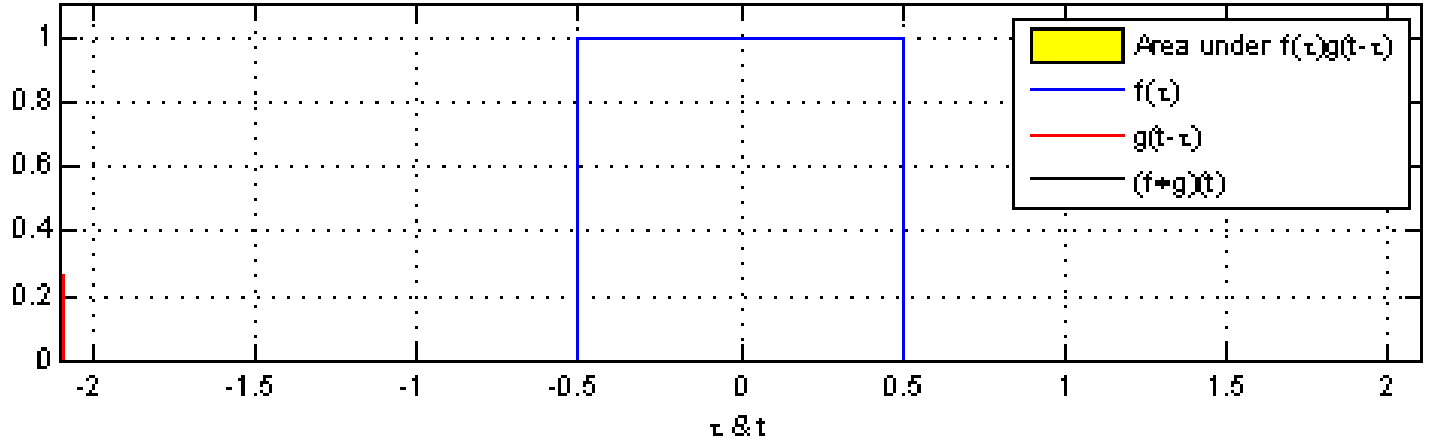
Convolution

$$[f * g](x) = \int_{\mathbb{R}} f(u)g(x-u)du$$

$$[f * g](x,y) = \int_{\mathbb{R}^2} f(u,v)g(x-t,y-v)dt$$

$$[f(x-a) * \delta(x)] = \delta(x-a)$$

Computationally intensive: one integral for every point (x, y)



This image is still in the pdf version

Image credit: Convolution_of_box_signal_with_itself.gif: Brian Amberg derivative work: Tinos (talk) - Convolution_of_box_signal_with_itself.gif, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=11003835

Convolution theorem: $$f * g = \mathrm{FT}^{-1}\left[\mathrm{FT}[f]\mathrm{FT}[g]\right]$$

Spatial filtering using the convolution theorem

```
F = fft.fftshift(fft.fft2(f))

U, V = np.meshgrid(np.linspace(-1, 1, NP),
                   np.linspace(-1, 1, NP))
```

Note arbitrary limits -1, 1

Low pass filter:

```
radius = 0.2 # e.g.
d = np.sqrt(U * U + V * V)
Lp = d < radius
```

Gaussian filter:

```
Gf = np.exp(-A(u * u + v * v))# set A
```

Laplacian filter:

```
Lf = u * u + v * v
```

Convolution

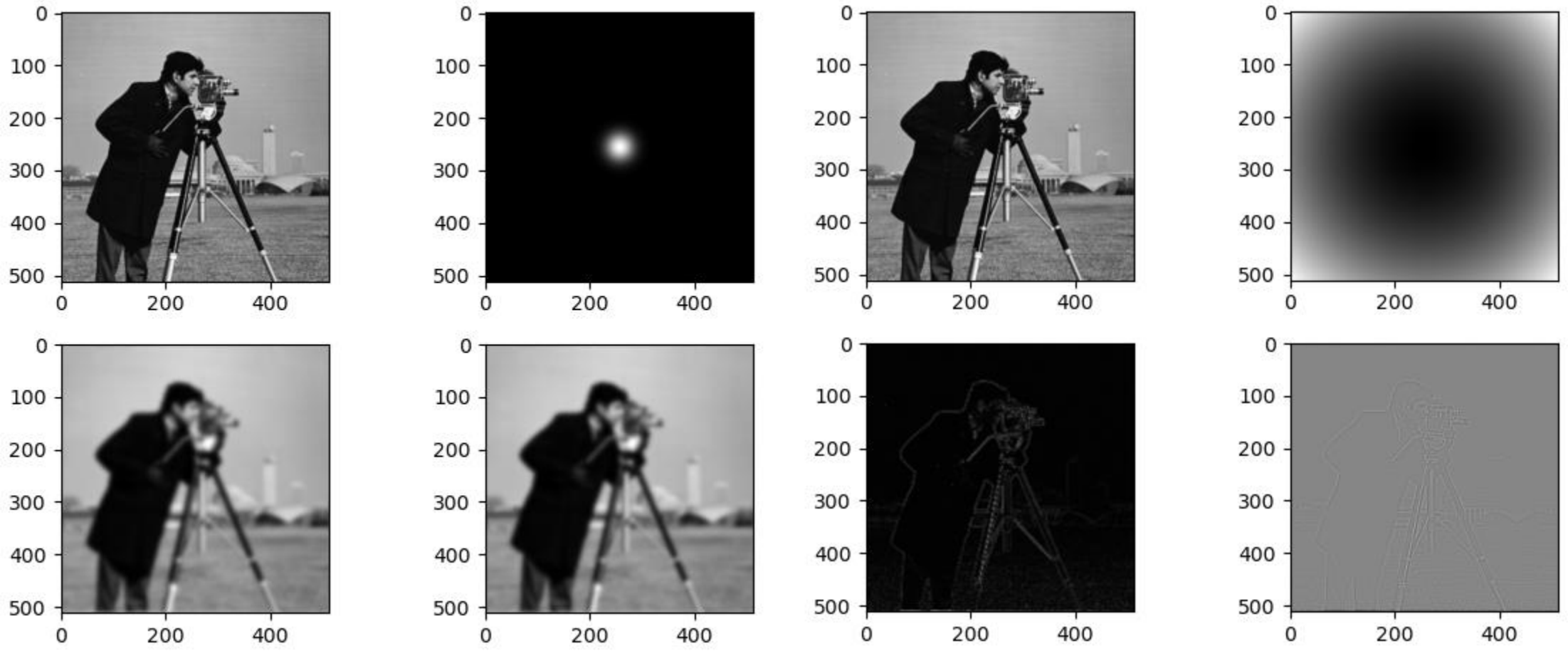
```
C = fft.ifft2(fft.ifftshift(F * Lp))
C = fft.ifft2(fft.ifftshift(F * Gf))
C = fft.ifft2(fft.ifftshift(F * Lf))
```

Gaussian and Laplacian filters
Examples of low- and high-pass filters

Equivalent to calculate the second derivatives of the image (Laplacian operator)

Gaussian filter A=100

Laplacian filter



Low pass filters remove high frequency information = details

- Additive noise is high frequency. It can be removed with a low pass filter
- Nevertheless, this is a trade-off problem: after a low-pass filtering both noise and image information are removed as well.

High pass filters enhance details i.e., fast contrast changes.

- Close to the DC term (frequency (0,0)), the Fourier transform accumulates most of its energy. After a high-pass filter, the image looks mainly black.
- The real part looks grayish. Values are real but with positive and negative values.
- Zero is in-between. If displayed with imshow, zero is shown around gray level 128.

Convolution in object space

$$f * g = \mathrm{FT}^{-1}\left[\mathrm{FT}[f]\mathrm{FT}[g]\right]$$

$$[f * g](x,y) = \int_{\mathbb{R}^2} f(u,v)g(x-t,y-v)dt$$

- So far, we have considered the convolution theorem, but we can filter images in object space as well.
- Statement: # rows x # columns of f and g are expected to be the same.
- Well, not really.

- The information of the filter is concentrated in the center of the array: 3x3, 5x5, et cetera. The rest of the array is zero. The contribution of the pixels with zero values to the convolution is zero

```
filt = = np.ones([5, 5])
Out[4]:
array([[1., 1., 1., 1., 1.],
       [1., 1., 1., 1., 1.],
       [1., 1., 1., 1., 1.],
       [1., 1., 1., 1., 1.],
       [1., 1., 1., 1., 1.]])
```

```
im[0:12, 0:12]
Out[6]:
array([[147, 103,  58,  51,  76, 100, 121, 135, 141, 134, 115,  90],
       [171, 141, 114, 107, 128, 144, 158, 165, 167, 150, 124,  86],
       [194, 178, 165, 157, 165, 180, 185, 186, 176, 154, 115,  69],
       [213, 206, 198, 191, 194, 198, 202, 194, 174, 142, 101,  45],
       [223, 220, 215, 210, 211, 206, 206, 187, 161, 123,  76,  25],
       [229, 228, 224, 218, 216, 210, 198, 175, 143, 101,  49,  13],
       [229, 230, 226, 224, 215, 205, 184, 159, 121,  73,  30,  19],
       [226, 226, 223, 219, 212, 200, 182, 149, 102,  49,  26,  22],
       [223, 223, 220, 216, 211, 200, 179, 145,  95,  41,  29,  21],
       [221, 222, 218, 213, 210, 201, 178, 151, 110,  63,  31,  22],
       [222, 221, 219, 217, 211, 203, 188, 165, 138,  98,  57,  27],
       [219, 222, 220, 219, 212, 209, 197, 181, 159, 131,  90,  45]],
      dtype=uint8)
```

```
[[147, 103,  58,  51,  76, 100, 121, 135, 141, 134, 115,  90],
 [171, 141, 114, 107, 128, 144, 158, 165, 167, 150, 124,  86],
 [194, 178, 165, 157, 165, 180, 185, 186, 176, 154, 115,  69],
 [213, 206, 198, 191, 194, 198, 202, 194, 174, 142, 101,  45],
 [223, 220, 215, 210 x 1, 211 x 1, 206 x 1, 206 x 1, 187 x 1, 161, 123,  76,  25],
 [229, 228, 224, 218 x 1, 216 x 1, 210 x 1, 198 x 1, 175 x 1, 143, 101,  49,  13],
 [229, 230, 226, 224 x 1, 215 x 1, 205 x 1, 184 x 1, 159 x 1, 121,  73,  30,  19],
 [226, 226, 223, 219 x 1, 212 x 1, 200 x 1, 182 x 1, 149 x 1, 102,  49,  26,  22],
 [223, 223, 220, 216 x 1, 211 x 1, 200 x 1, 179 x 1, 145 x 1 ,  95,  41,  29,  21],
 [221, 222, 218, 213, 210, 201, 178, 151, 110,  63,  31,  22],
 [222, 221, 219, 217, 211, 203, 188, 165, 138,  98,  57,  27],
 [219, 222, 220, 219, 212, 209, 197, 181, 159, 131,  90,  45]],
```

Pixel under consideration

Convolution:
210 x 1 + 211 x 1 + 206 x 1 + … + 179 x 1 + 145 x 1 >> 255 !!!

scipy.ndimage.convolve

scipy.ndimage.convolve(*input*, *weights*, *output=None*, *mode='reflect'*, *cval=0.0*, *origin=0*)

Multidimensional convolution. [source]

The array is convolved with the given kernel.

Parameters: **input** : ***array_like***

The input array.

weights : ***array_like***

Array of weights, same number of dimensions as input

output : ***array or dtype, optional***

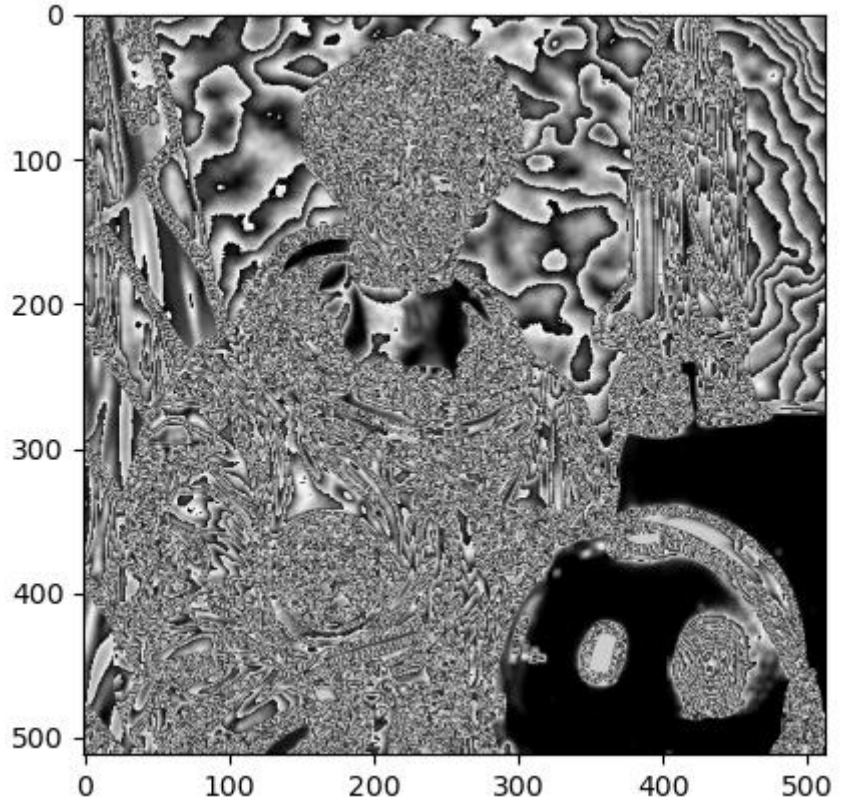
The array in which to place the output, or the dtype of the returned array. By default an array of the same dtype as input will be created.

mode : ***{'reflect', 'constant', 'nearest', 'mirror', 'wrap'}, optional***

The *mode* parameter determines how the input array is extended beyond its boundaries. Default is 'reflect'. Behavior for each valid value is as follows:

'reflect' (*d c b a | a b c d | d c b a*)

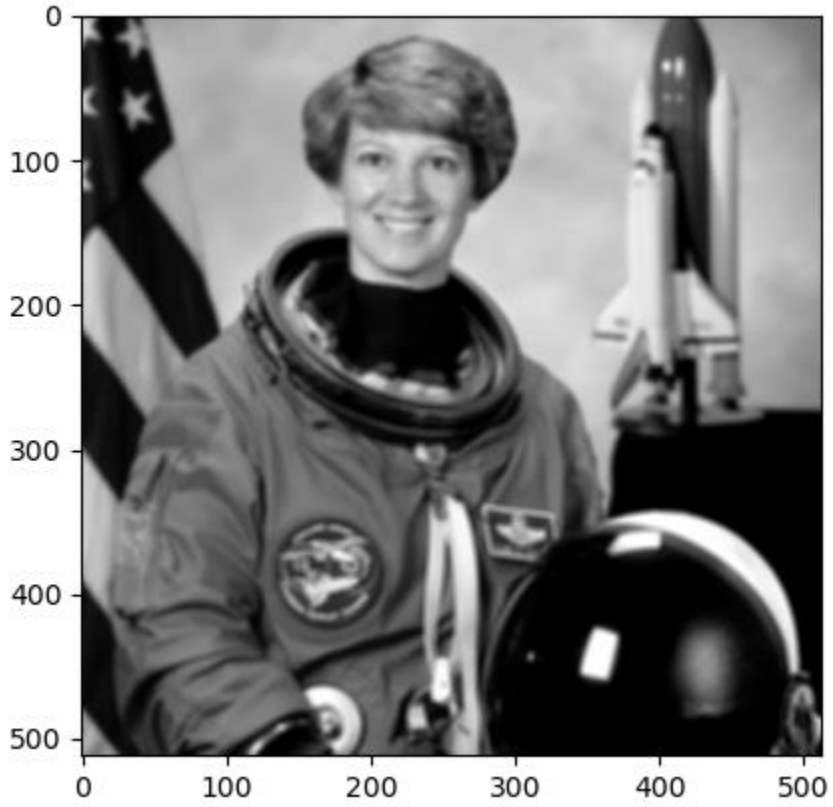
Nan	Type	Size	Value
conv	Array of uint8	(512, 512)	Min: 0 Max: 255
filt	Array of float64	(5, 5)	Min: 1.0 Max: 1.0
im	Array of uint8	(512, 512)	Min: 0 Max: 255




```
im = data.astronaut()[:, :, 1]
filt = np.ones([5, 5])
conv = ndimage.convolve(im, filt)
plt.figure()
plt.imshow(conv, cmap='gray')
```

```
im = np.double(data.astronaut()[:, :, 1])
filt = np.ones([5, 5])
conv = ndimage.convolve(im, filt)
plt.figure()
plt.imshow(conv, cmap='gray')
```

Nan	Type	Size	Value
conv	Array of float64	(512, 512)	Min: 0.0 Max: 6356.0
filt	Array of float64	(5, 5)	Min: 1.0 Max: 1.0
im	Array of float64	(512, 512)	Min: 0.0 Max: 255.0



The resulting image looks slightly defocused

How are filters in image and Fourier space related?

- Promote a n x n filter to a N x N array (padding)
- Calculate the Fourier transform
- Compare how they look

```
filt = np.array(
     [[-1., -1., -1.],
      [-1., 8., -1.],
      [-1., -1., -1.]])
```

Image space Laplacian filter.

Wait. What?

```
filt512 = np.zeros([512, 512])
filt512[255:258, 255:258] = filt
```

Cast a 3x3 array into a 512x512 array

```
ftfilt = fft.fftshift(fft.fft2(filt512))

u, v = np.meshgrid(np.linspace(-1, 1, 512), np.linspace(-1, 1, 512))
lap = u**2 + v**2
```

```
filt = np.array(
      [[0., -1., 0.],
       [-1., 4., -1.],
       [0., -1., 0.]])

filt512 = np.zeros([512, 512])
filt512[255:258, 255:258] = filt

ftfilt = fft.fftshift(fft.fft2(filt512))

u, v = np.meshgrid(np.linspace(-1, 1, 512),
                   np.linspace(-1, 1, 512))
lap = u**2 + v**2

ftim = fft.fftshift(fft.fft2(im))
res = np.abs(fft.ifft2(fft.ifftshift(ftim * lap)))

conv = np.abs(ndimage.convolve(im, filt))
```

Fourier Space

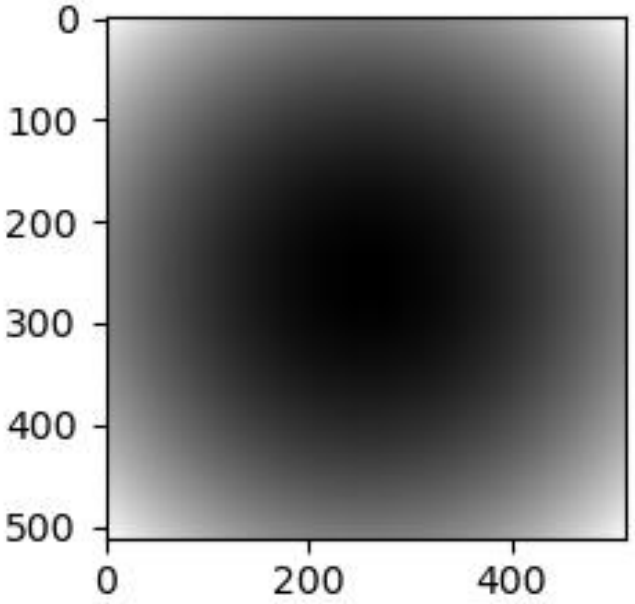
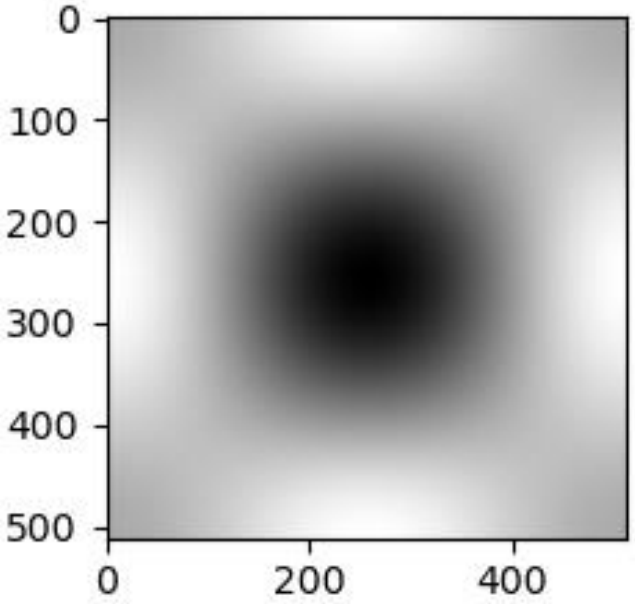
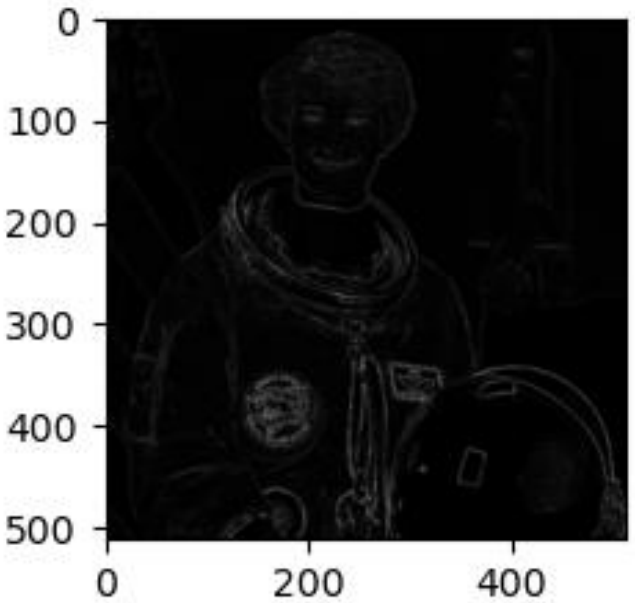
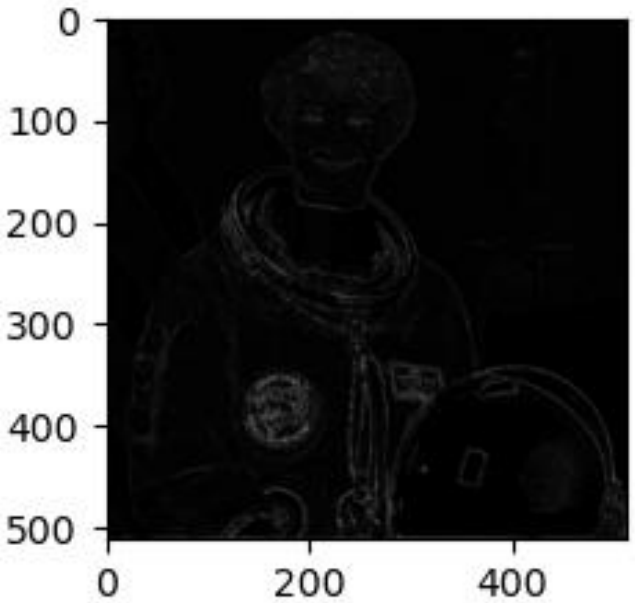


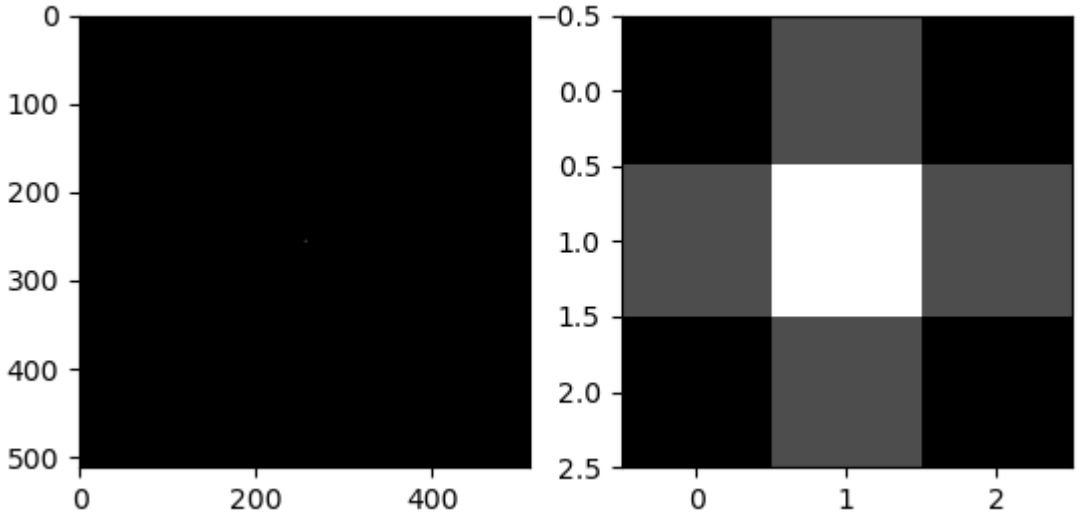
Image Space







```
lap = u**2 + v**2
tflap = np.real(fft.ifftshift(fft.fft2(lap)))
plt.figure()
plt.subplot(1,2,1)
plt.imshow(tflap, cmap='gray')
plt.subplot(1,2,2)
plt.imshow(tflap[255:258, 255:258], cmap='gray')

print(4 * tflap[255:258, 255:258] / tflap.max())
```




```
print(4 * tflap[255:258, 255:258] / tflap.max())
[[-9.17640509e-18  1.21582832e+00 -2.60796478e-19]
 [ 1.21582832e+00  4.00000000e+00  1.21582832e+00]
 [-1.04229685e-17  1.21582832e+00  2.60796478e-19]]
```

Signs are not properly recovered, because the definition of the Lapacian filter skips this information:

Lap = u**2 + v **2

Laplacian-like (edge extraction) filters

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

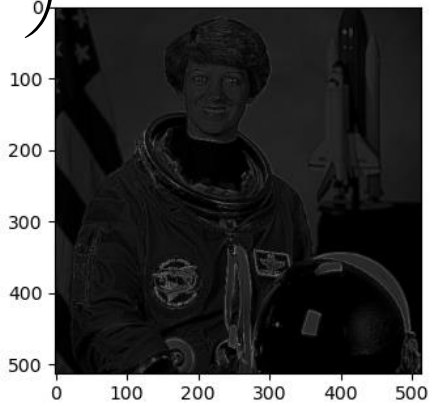
Average and gaussian like filters

$$\frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \frac{1}{16}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Edge enhancer filters

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



How to calculate the Laplacian of an image

(i) Fourier Space

$f, F = FT(f)$

$FT(\Delta f)$?

$FT\left(\frac{\partial f}{\partial x}\right) \propto uF \qquad FT\left(\frac{\partial f}{\partial y}\right) \propto vF$

$FT\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \propto (u^2+v^2)F$

↑ Laplacian filter

How to calculate the Laplacian of an image
(ii) Image Space

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

How to calculate derivatives with sampled information?

$$x - x_0 = 1 \quad \Rightarrow$$

$$f'_i = f_i - f_{i-1}$$

$$f_{i+1} - f_i$$

Ambiguous definition:
Left and right derivatives

Second derivative:

$$f''_i = f'_i - f'_{i-1}$$

$$= f_{i+1} - f_i - f_i + f_{i-1} =$$

$$-(-f_{i+1} + 2f_i - f_{i-1})$$

Laplacian operator for sampled functions

$$\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f$$

$$f''_{i,j} = -(-f_{i+1,j} + 2f_{ij} - f_{i-1,j})$$

$$-(-f_{i,j+1} + 2f_{ij} - f_{i,j-1}) =$$

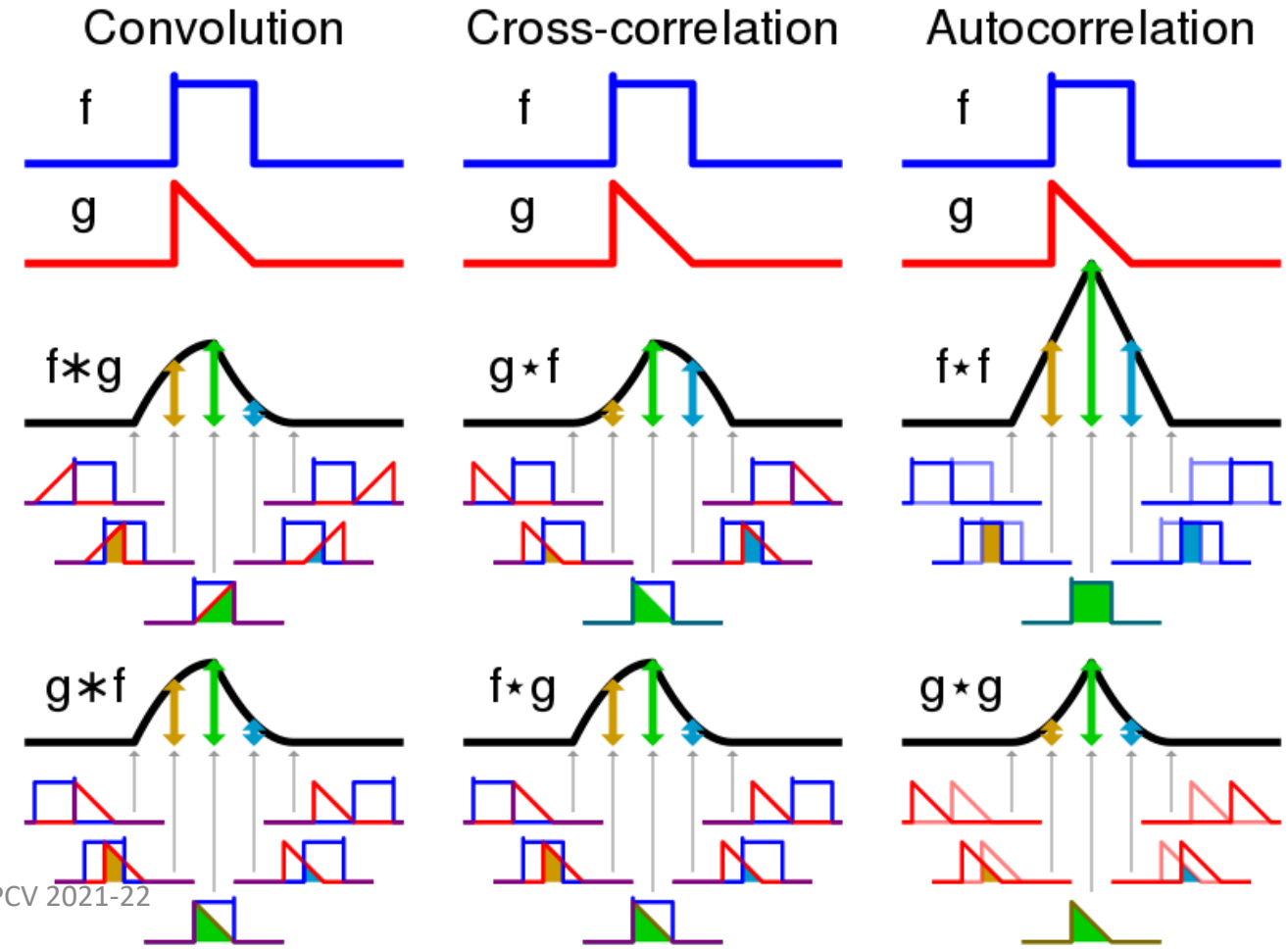
$$\cdot \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix} * f_{ij}$$

Calculation of the relative shift of two images

Cross-correlation between f ($NBR_{pre\text{-}fire}$) and g ($NBR_{post\text{-}fire}$).

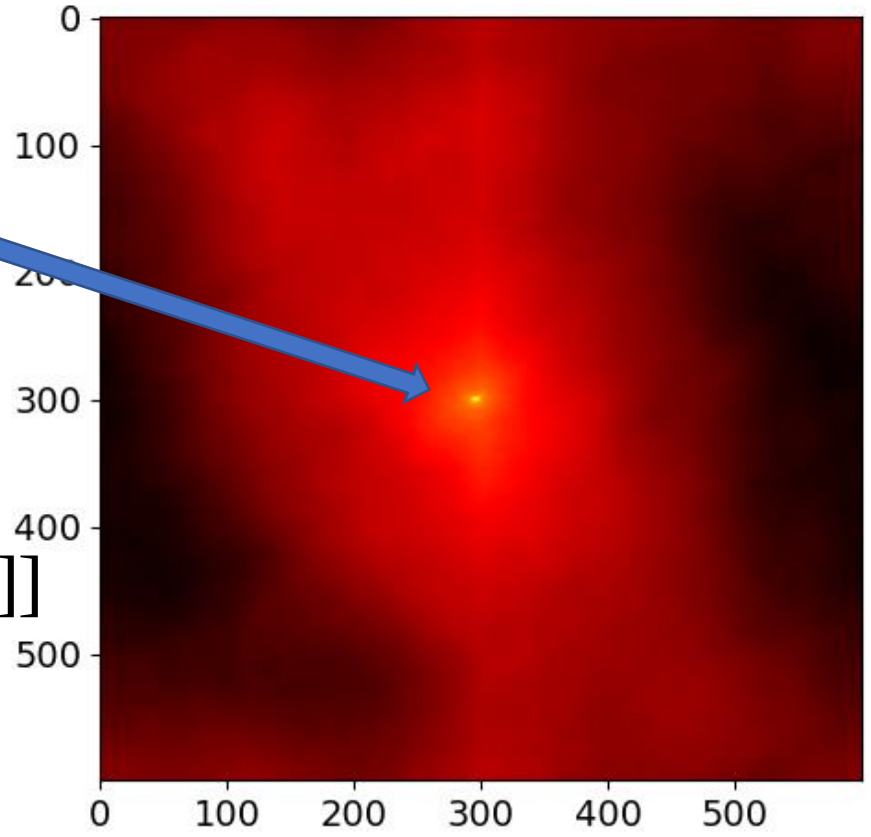
$$[f \otimes g](a,b) = \int f^*(x,y)g(x+a,y+b)dxdy$$

The image shift is determined by detecting the position of the **cross-correlation maximum** with respect to the center of the images.



https://upload.wikimedia.org/wikipedia/commons/thumb/2/21/Comparison_convolution_correlation.svg/800px-Comparison_convolution_correlation.svg.png

The correlation maximum is slightly shifted from the center. Its position determines the displacement .



Using Fourier transforms:

$$\mathrm{FT}[f \otimes g] = \mathrm{FT}[f]^{*}\mathrm{FT}[g] \Rightarrow f \otimes g = \mathrm{FT}^{-1}[\mathrm{FT}[f]^{*}\mathrm{FT}[g]]$$

Use
`numpy.fft.fft2` / `numpy.fft.ifft2` and
`numpy.fft.fftshift` / `numpy.fft.ifftshift`

```
corr = np.abs(fft.ifft2(fft.fft2(im1)) * np.conjugate(fft.fft2(im2)))

Or better:
corr = np.abs(fft.ifftshift(fft.ifft2(fft.fftshift(fft.fft2(im1)) *
       np.conjugate(fft.fftshift(fft.fft2(im2))))))
```

Circular convolutions

Circular convolution

From Wikipedia, the free encyclopedia

Circular convolution, also known as **cyclic convolution**, is a special case of **periodic convolution**, which is the convolution of two periodic functions that have the same period. Periodic convolution arises, for example, in the context of the discrete-time Fourier transform (DTFT). In particular, the DTFT of the product of two discrete sequences is the periodic convolution of the DTFTs of the individual sequences. And each DTFT is a periodic summation of a continuous Fourier transform function (see DTFT § Definition). Although DTFTs are usually continuous functions of frequency, the concepts of periodic and circular convolution are also directly applicable to discrete sequences of data. In that context, circular convolution plays an important role in maximizing the efficiency of a certain kind of common filtering operation.

Excerpt from https://en.wikipedia.org/wiki/Circular_convolution

This effect can be simple removed by using an extra `fft.ifftshift`
The problem is dependent on the extent of the FT of the images involved.

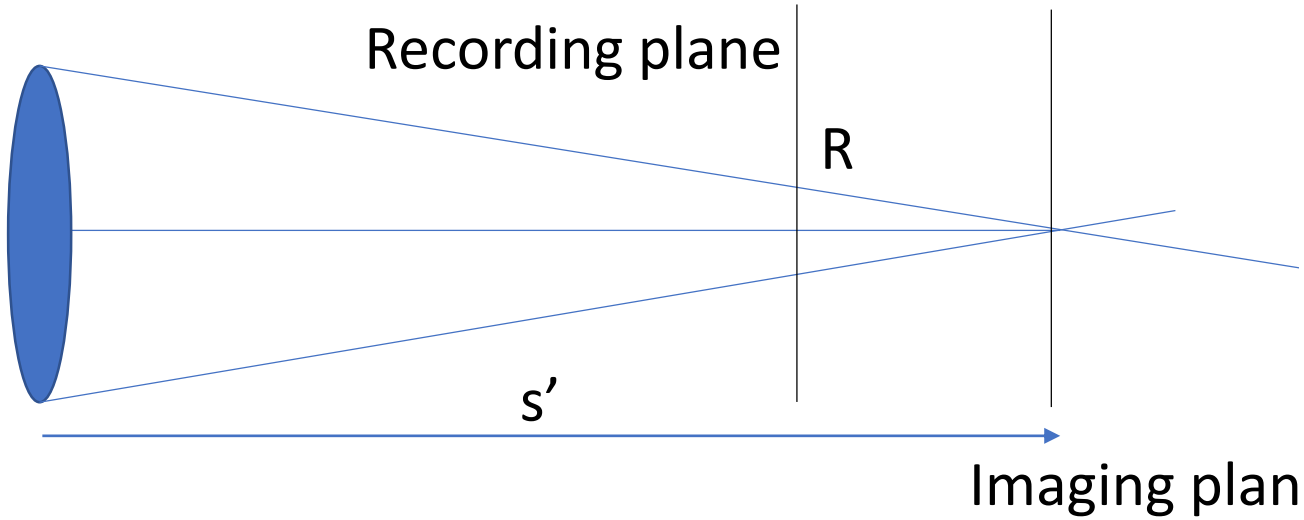
Lab #6: Point–spread functions and image restoration filters.

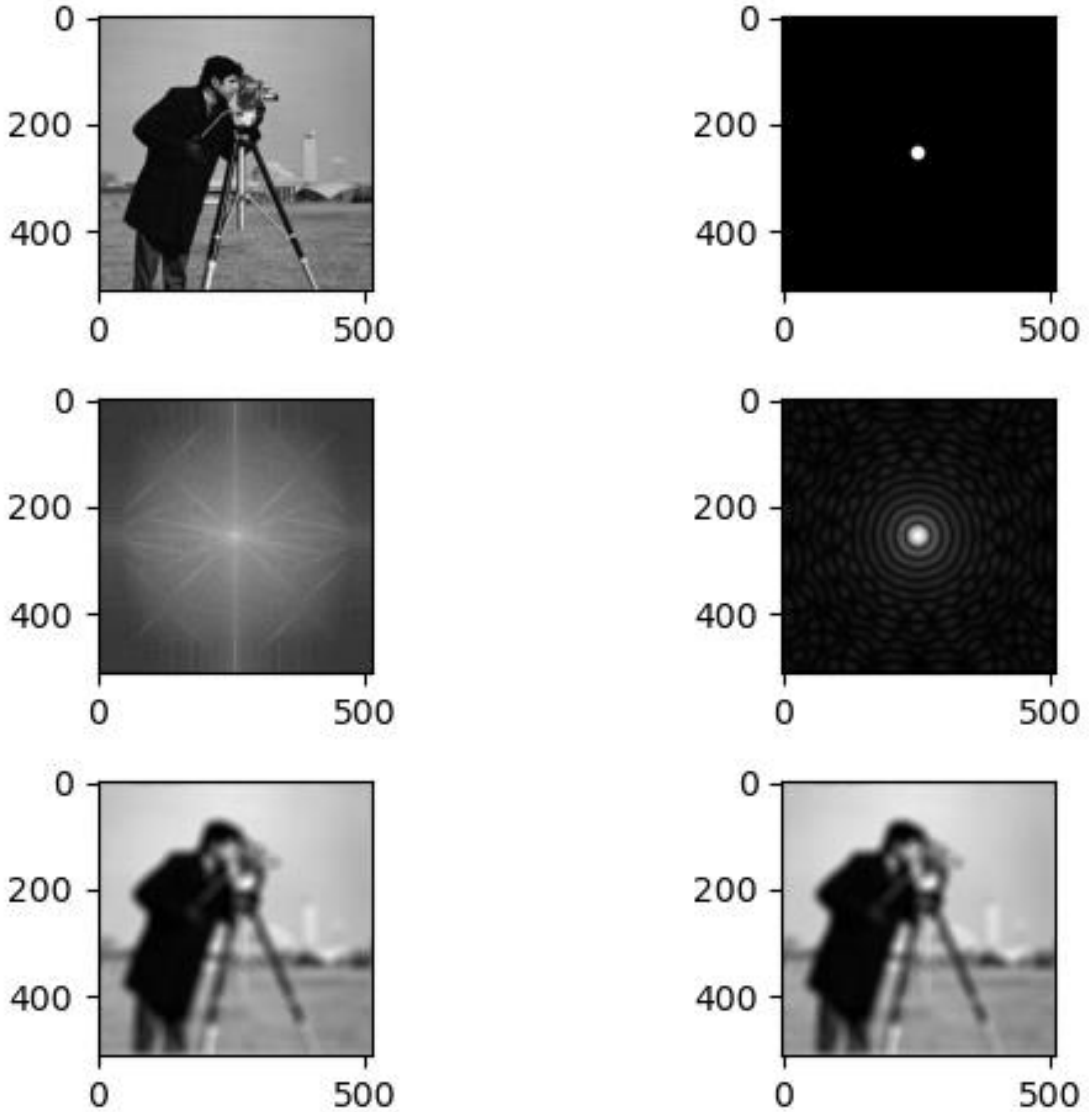
1. Geometrical model for defocus

$$\left[f(x,y) * \mathrm{circ}\frac{r}{R} \right] =$$

$$\mathrm{FT}^{-1}\left[F(u,v)\frac{R}{\rho} J_1(2\pi R\rho) \right]$$

$$r = \sqrt{x^2 + y^2} \qquad \rho = \sqrt{u^2 + v^2}$$





A defocused imaging system behaves as a physical low pass filter

2. Image formation model in diffracted-limited system

Perfect optical system: the image of a single point (in object space) is another point (in image space)

Microscope objective lens Tube lens

$$r_A = 1.22\frac{\lambda s'}{2R}$$

Despite this definition, optical systems are always affected by diffraction due the limited extent of lenses.

The Airy disc (with radius r_A) describes the *actual point* (point-spread function, PSF) for a perfect system

$$\left[f * \mathrm{FT}_{\lambda s}\left[\mathrm{circ}\frac{r}{R} \right] \right] = \mathrm{FT}^{-1}\left[\mathrm{FT}\left[f\left(\frac{x}{\beta}, \frac{y}{\beta} \right) \right] \frac{R}{\rho} J_1\left(\frac{2\pi R\rho}{\lambda s} \right) \right]$$

$$r = \sqrt{x^2 + y^2} \qquad \rho = \sqrt{u^2 + v^2} \qquad \beta = s'/s \text{ geometric magnification}$$

In an **diffracted limited system** the PSF is approximated as

$$\mathrm{PSF}(\rho) = \mathrm{TF}_{\lambda s}\left[\mathrm{circ}\left(\frac{r}{R}\right)\right]$$

Where the exit pupil is a circle with radius *R*); *s* is the distance between the exit pupil and the image plane and λ is the wavelength of the illuminating source. Note that the coordinates of the PSF are scaled according to the product λs.

In a **geometrically defocused** system, $\mathrm{PSF}(r) = \mathrm{circ}\left(\frac{r}{R}\right)$

Then, the recorded image $d(x,y)$ *is* modeled as the convolution between the ideal, perfect geometrical image $i(x,y)$ and the PSF.

Coherent illumination: $d(x,y) = i(x,y) * \mathrm{PSF}(x,y)$

Incoherent illumination: $d(x,y) = i(x,y) * \left|\mathrm{PSF}(x,y)\right|^2$

Reconstruction filters

$$D = \mathrm{FT}\left[d\right] \qquad H = \mathrm{FT}\left[\mathrm{PSF}\right]$$

Inverse filter

$$d_r = \mathrm{TF}^{-1}\left[DF_{\mathrm{I}}\right] = \mathrm{TF}^{-1}\left[D\frac{1}{H+k}\right]$$

Pseudo-analytical solution

Wiener filter

$$d_r = \mathrm{TF}^{-1}\left[DF_{\mathrm{W}}\right] = \mathrm{TF}^{-1}\left[D\frac{H^*}{|H|^2+k}\right]$$

Filter that produces the minimal distance $\|d_r - d\|$

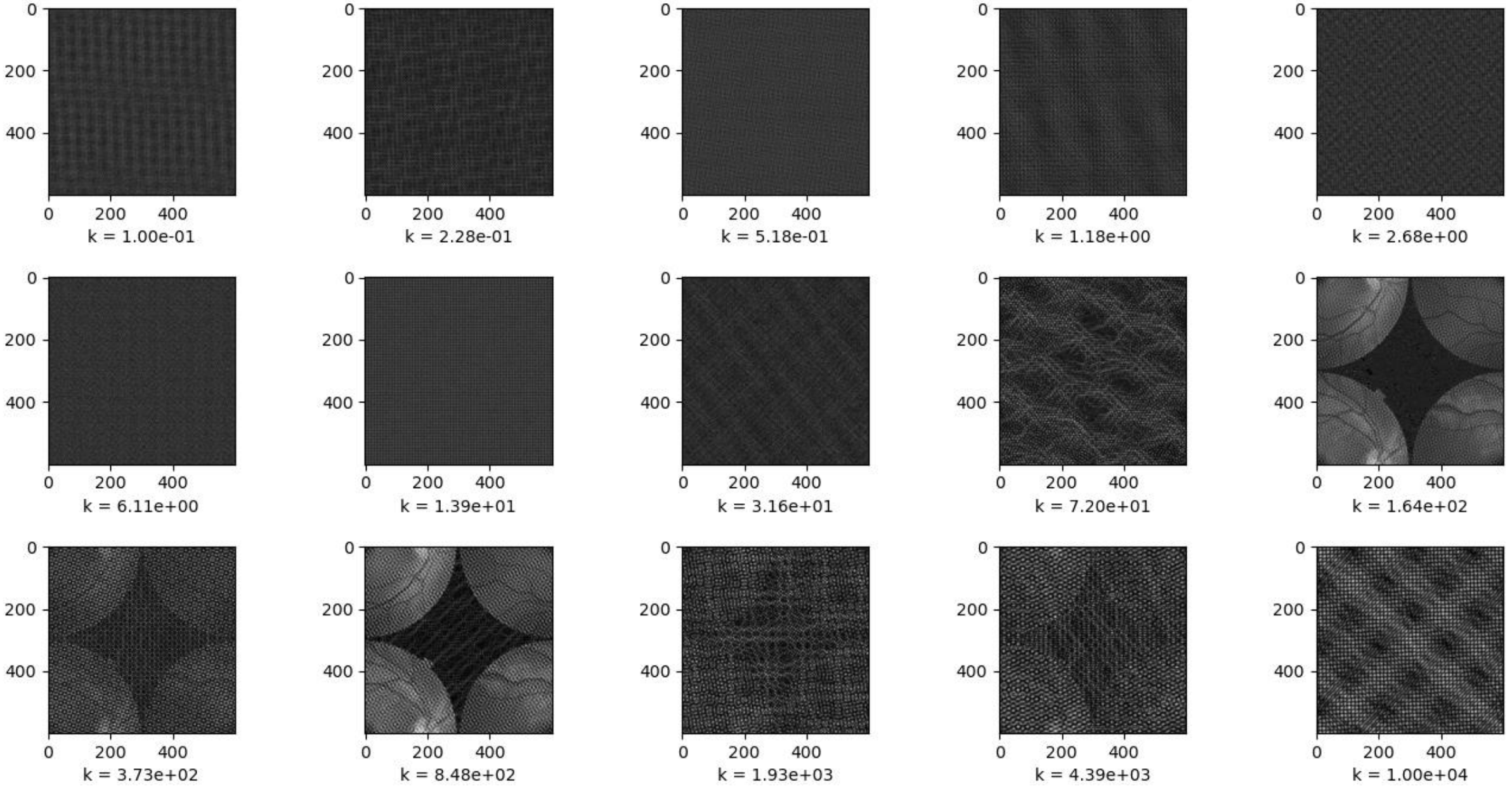
Least-squares filter

$$d_r = \mathrm{TF}^{-1}\left[DF_{\mathrm{LS}}\right] = \mathrm{TF}^{-1}\left[D\frac{H^*}{|H|^2+k\left(u^2+v^2\right)}\right]$$

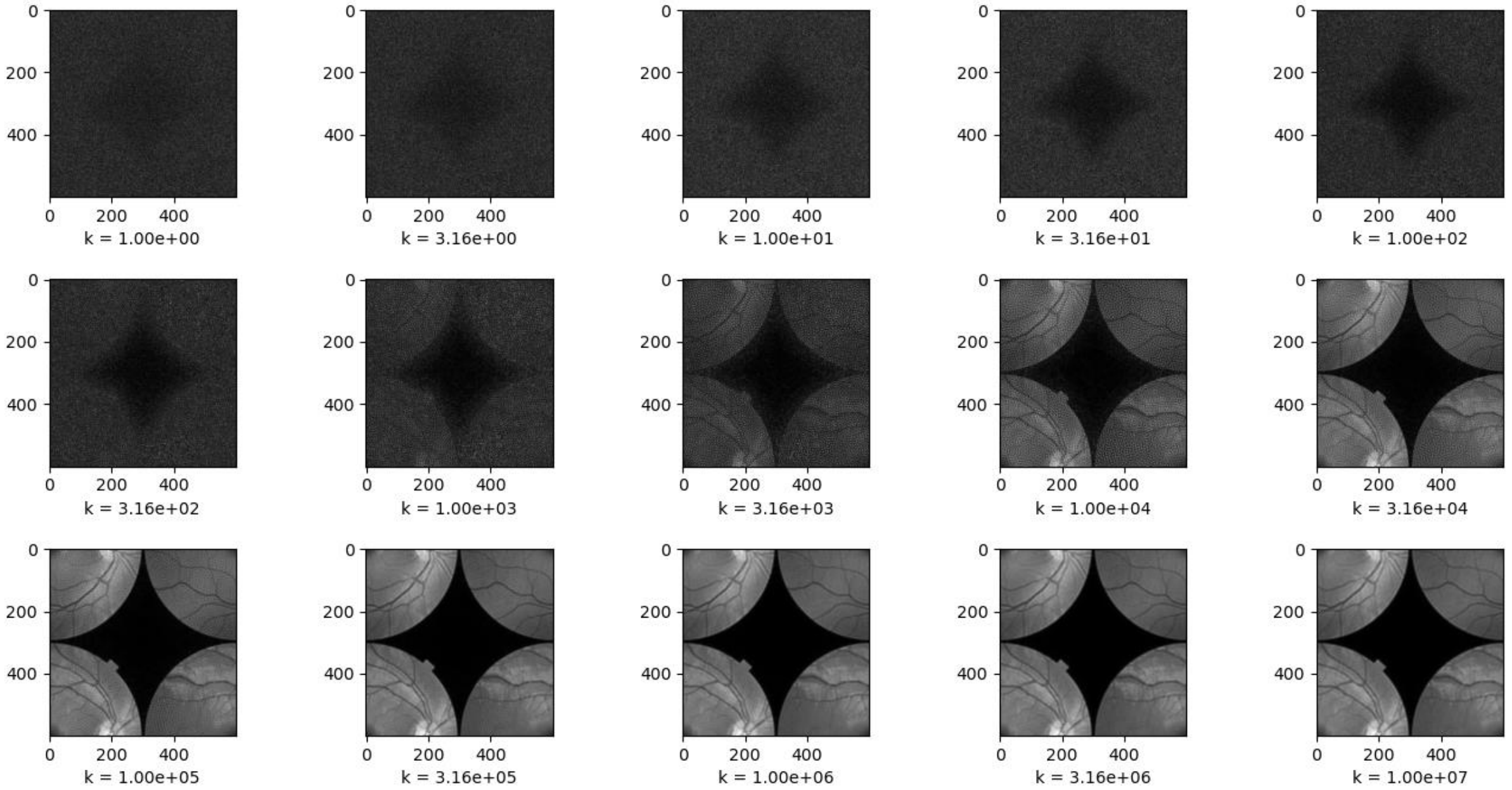
Lucy- Richardson

$$d_{k+1} = \left| d_k \left[\frac{d}{d_k * |\mathrm{PSF}|^2} \right] * |\mathrm{PSF}|^2 \right| \qquad \text{with } d_0 = d$$

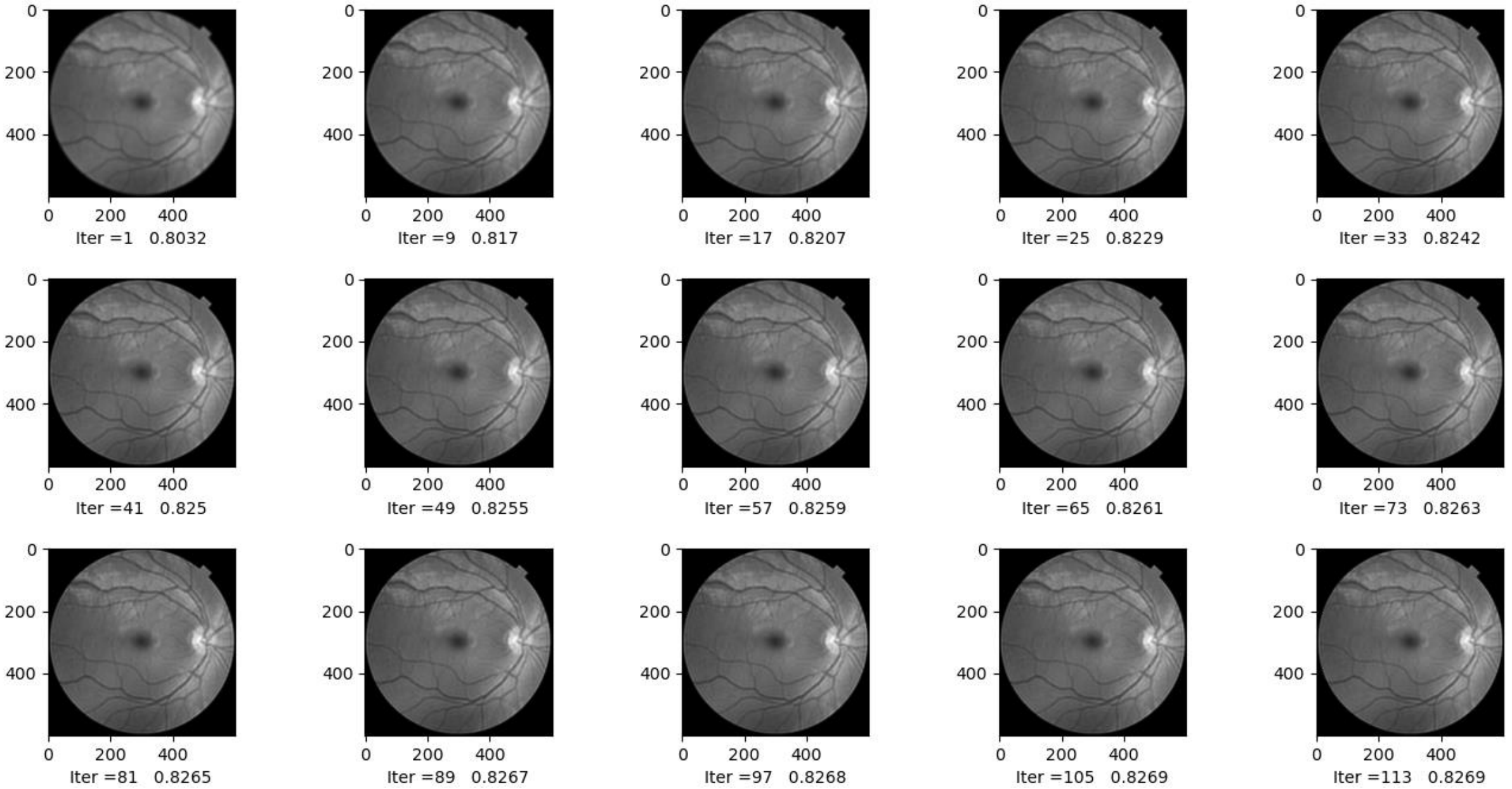
e.g.: $d_1 = \left| d_0 \left[\frac{d_0}{d_0 * \mathrm{circ}\left(r/R\right)} \right] * \mathrm{circ}\left(r/R\right) \right|$ (PSF - geometrically defocused system)



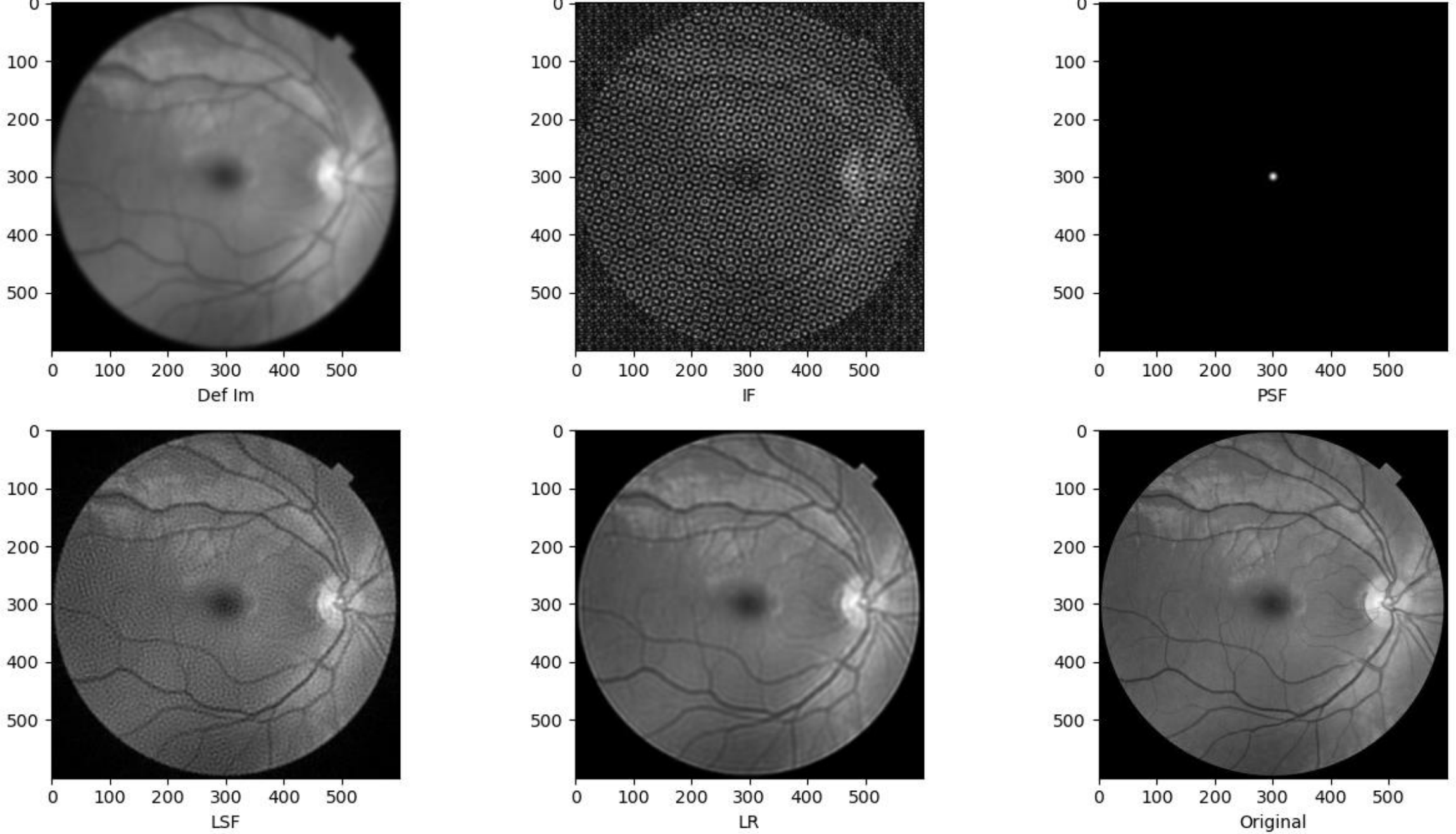
Inverse filter (without the extra `fft.ifftshift`)



Least squares filter (without the extra `fft.ifftshift`)



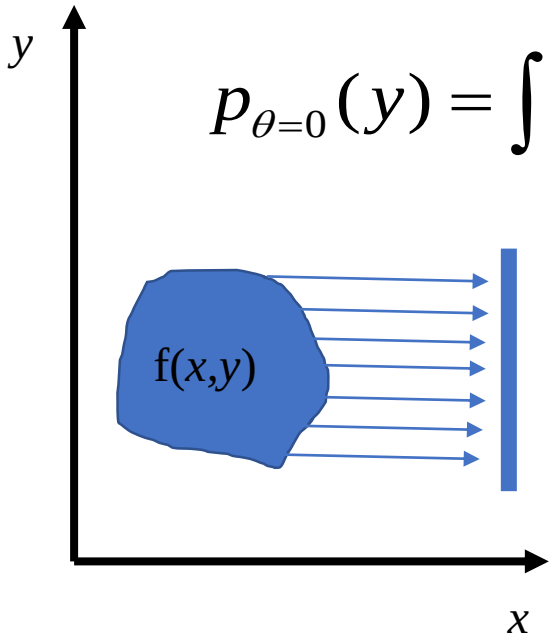
Lucy-Richardson filter


Def Im
IF
PSF
LSF
LR
Original

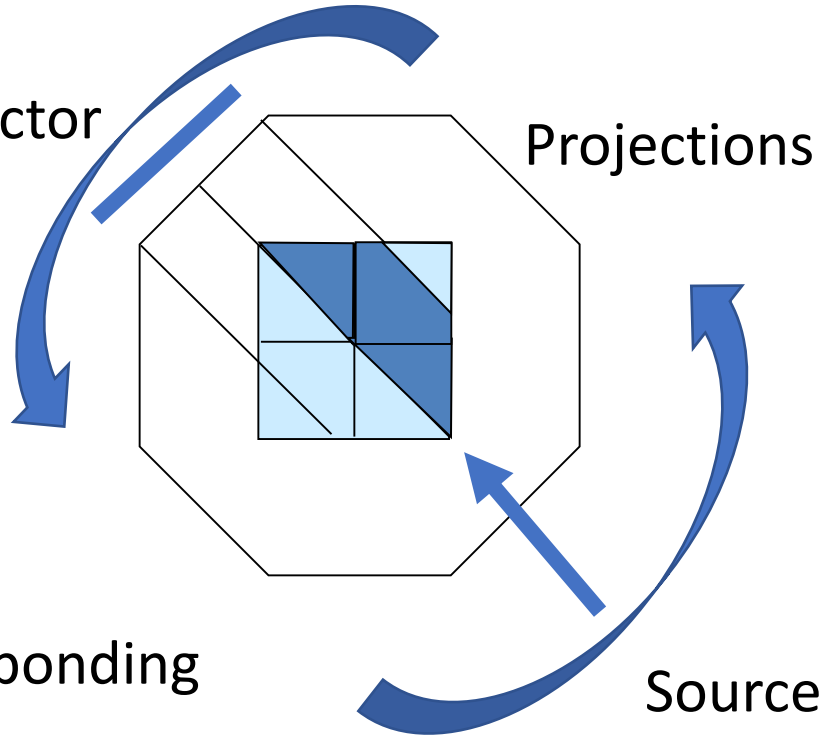
Lab #7: Radon transforms and the Projection-Slide Theorem.

A 2D object $f(x,y)$ can be determined from a set of 1D projection.
This problem is related with computer tomography: a rotating (X-ray) source + detectors measure the project

We calculate all possible projections $p_\theta(y)$
this is equivalent to rotate the object
while the detector is parallel to axis y.



$$p_{\theta=0}(y) = \int f(x,y)dx$$



We introduce the corresponding Fourier transforms:

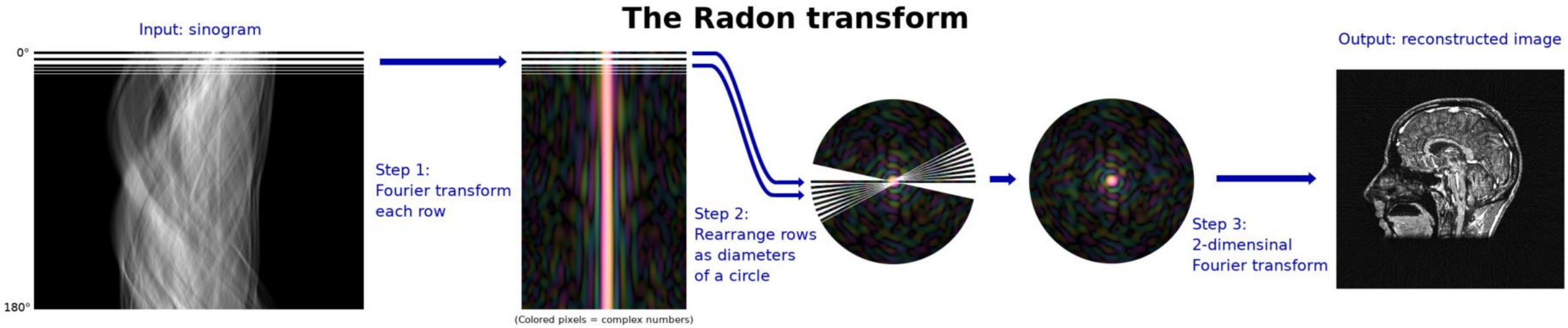
$$P_\theta(v) = \mathrm{FT}\left[p_\theta(y)\right]$$

The set of projections $p_{\theta=0}(y) = \int \mathrm{f}(x, y)dx$ is used to produce the so-called sinogram

Some noise (Poison) can be added to simulate the effect of real detectors.

In summary, the forward Radon transform is calculated just by rotating the object a certain angle (form 0 to 180º) and calculating the projection on the x direction.

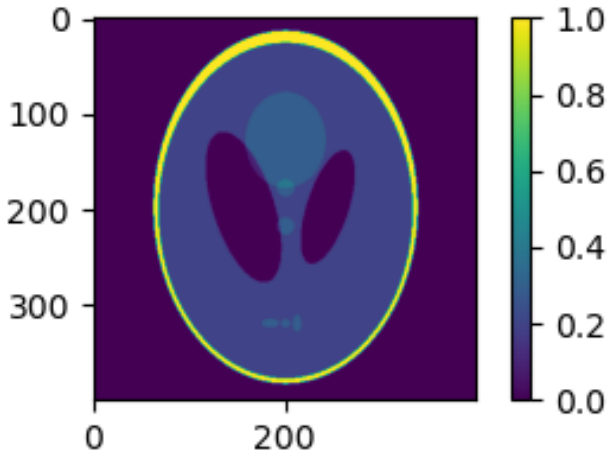
The problem arises when trying to calculate the inverse Radon transform (from projections to the 2D object).

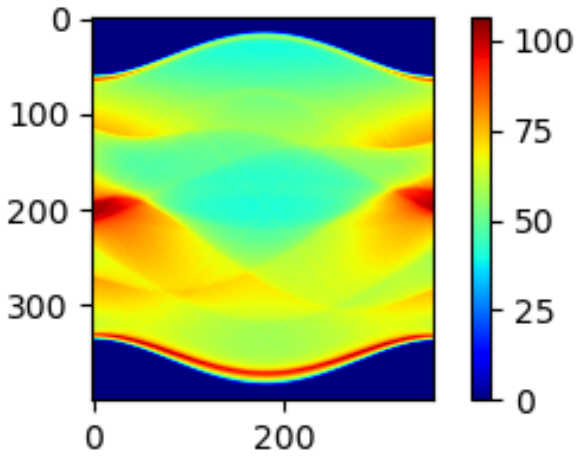


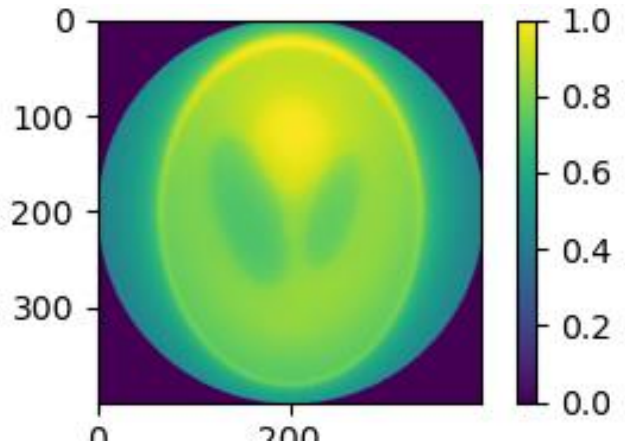
By Peter Selinger - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=72653784

Filtered Back Projection algorithm

1. Calculate the 1D Fourier transform of every slice of the sinogram $P_\theta(v) = \mathrm{FT}[p_\theta(y)]$
2. Rearrange the Fourier transforms according a polar geometry
3. Calculate the 2D inverse Fourier transform.







Forward RT

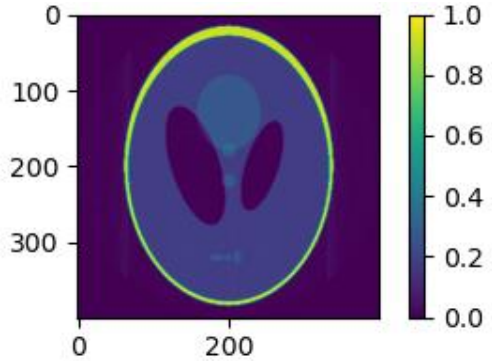
Inverse RT

The result is very poor. Why?

In the calculation of the IRT we missed the Jacobian term for the integration, i.e.:

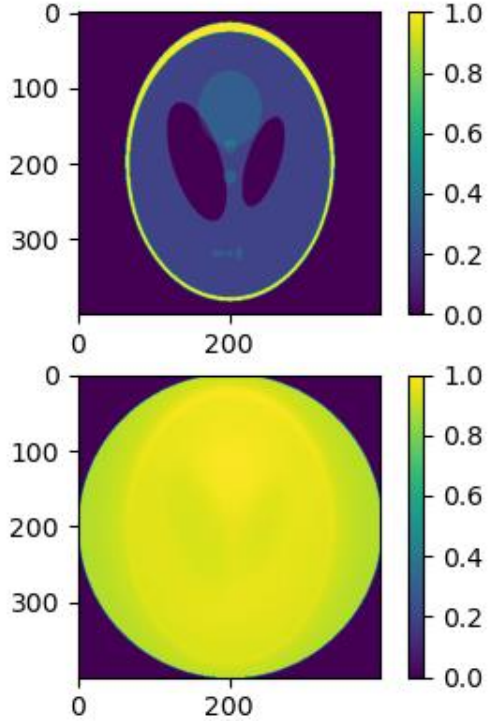
$$dxdy = r\, drd\theta$$

We have to multiply the distribution obtained from
the FT of the sinogram by a term r (a.k.a., the ramp kernel).
Now, reconstruction is undistinguishable from the original.



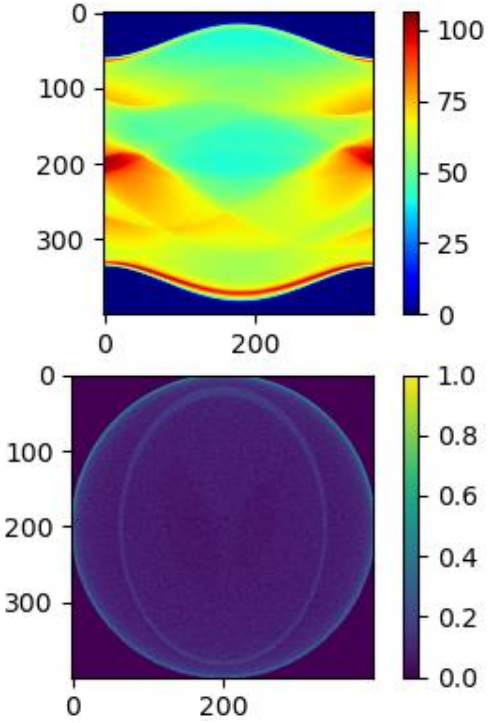
In realistic conditions, projections are affected by noise. Multiplying the FT term by r enhances high frequency information and thus, the effect of noise.
Accordingly, other filters [pass band] have been suggested.
See https://scikit-image.org/docs/dev/_images/sphx_glr_plot_radon_transform_002.png

Original Shepp-Logan phantom



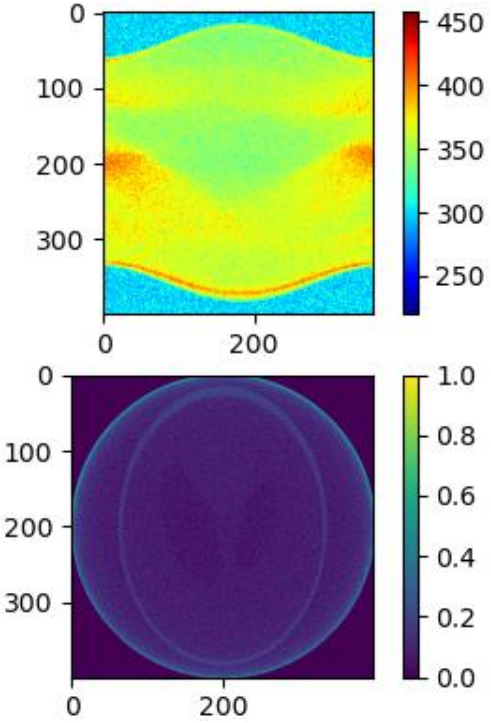
No filter

Sinogram



Ramp kernel

Sinogram + noise

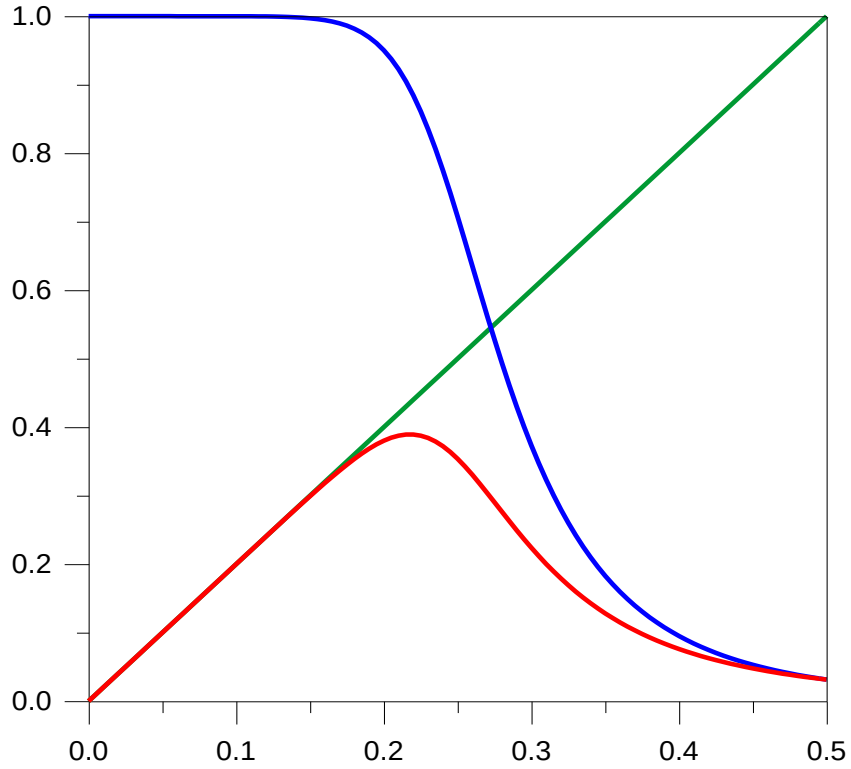


Shepp-Logan

Usually, projections are affected by clutter.

Since noise can be considered as high-frequency information, it is amplified by the Jacobian term.

For this reason, in addition to the ramp kernel (green line), filters that remove high-frequency contributions while keeping low-frequency information are used (blue curve). The resulting filter is shown in red.



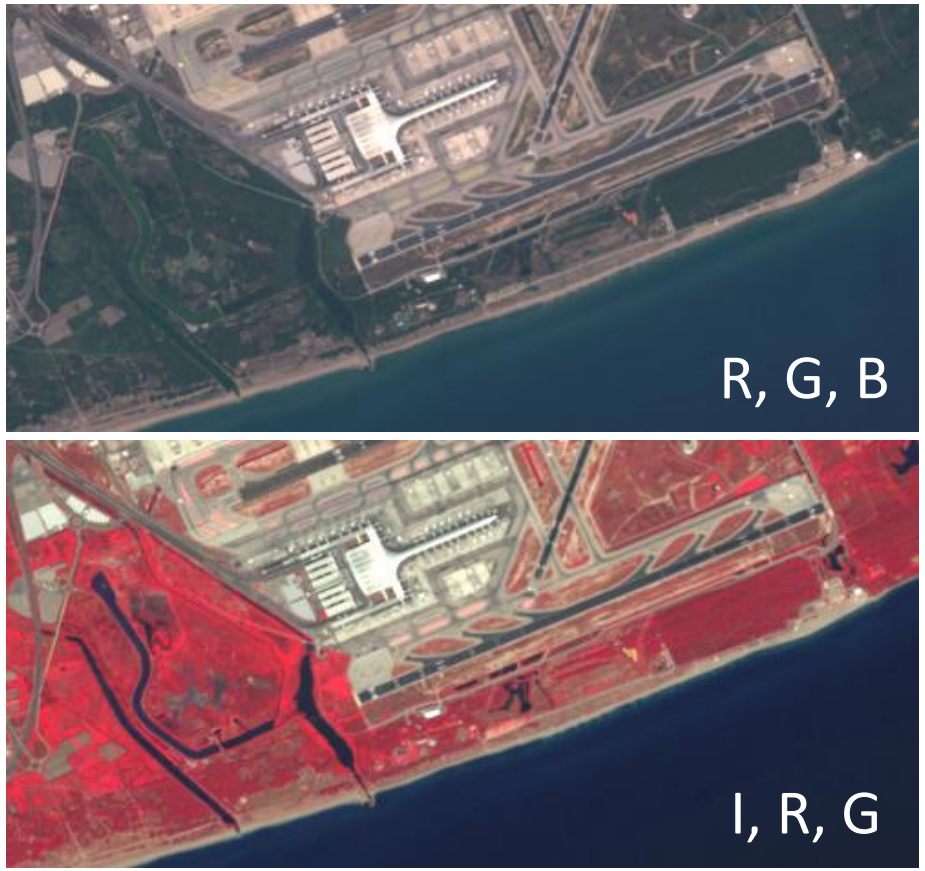
Other widely used kernels

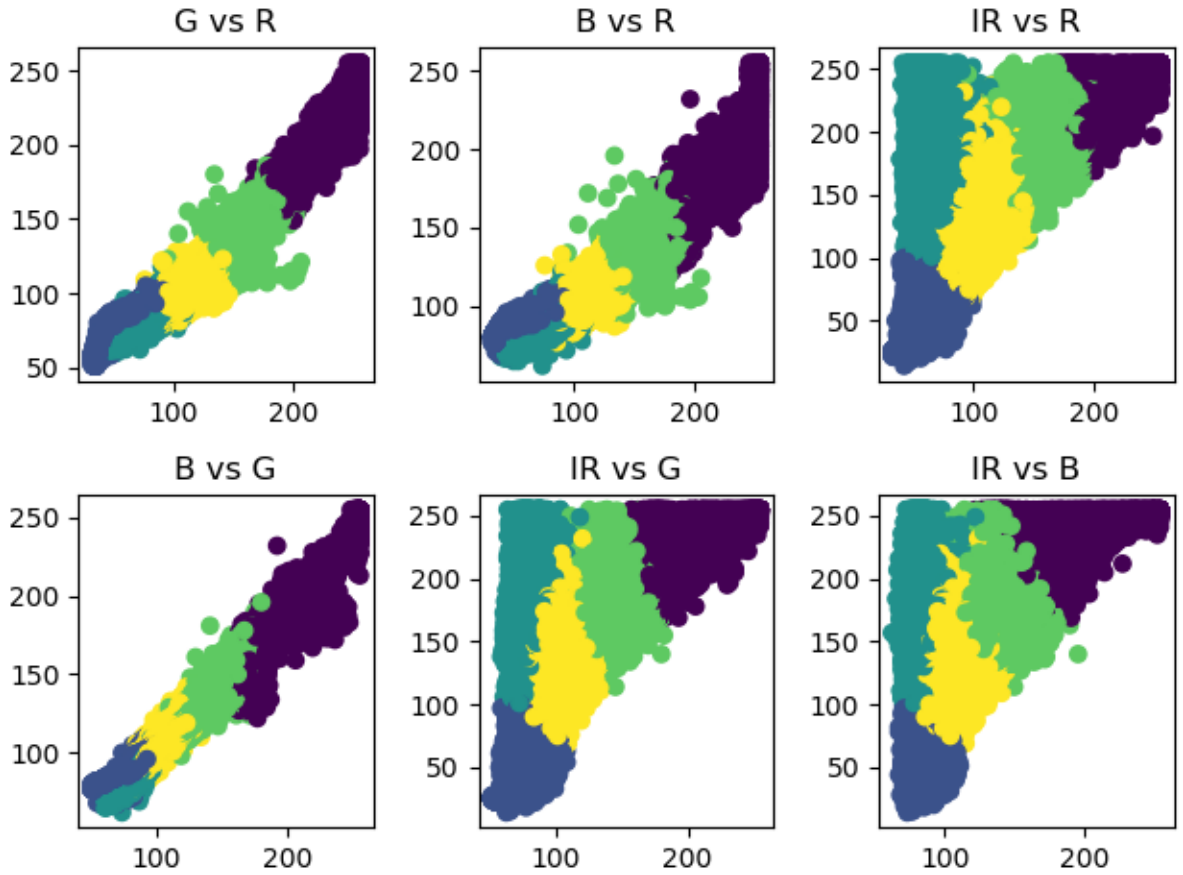
1. Shepp-Logan $k(w) = \text{sinc}\,\frac{w}{2w_c}$

2. cosine / Hanning / Hamming $k(w) = C_1 + C_2 \cos\frac{\pi w}{w_c}$

Lab #8: K-means clustering

Goal: Segmentation of an image according to the color (I, R, G and B).

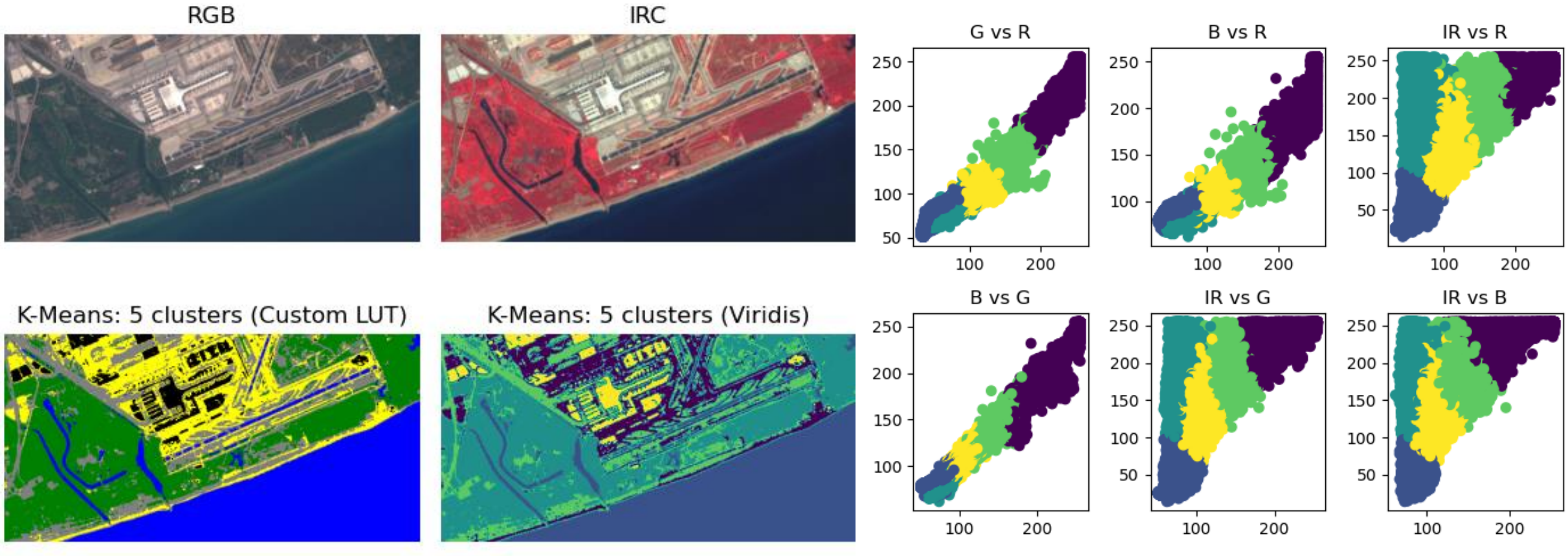




http://auriga.icgc.cat/descarregues2/dl.php?t=sen2rgb8bv10tf0f04s1_201807_0.zip&f=04&l=cat
http://auriga.icgc.cat/descarregues2/dl.php?t=sen2irc8bv10tf0f04s1_201807_0.zip&f=04&l=cat

The number of clusters is selected by the programmer (e.g.: 5).

K-means clustering



Infrared-red-green: the red channel is replaced with near infrared. This combination is often used to detect vegetation (because of it is highly reflective in near IR).

- **Machine learning** algorithms build a model based on sample data, known as training data, in order to make predictions or decisions. https://en.wikipedia.org/wiki/Machine_learning

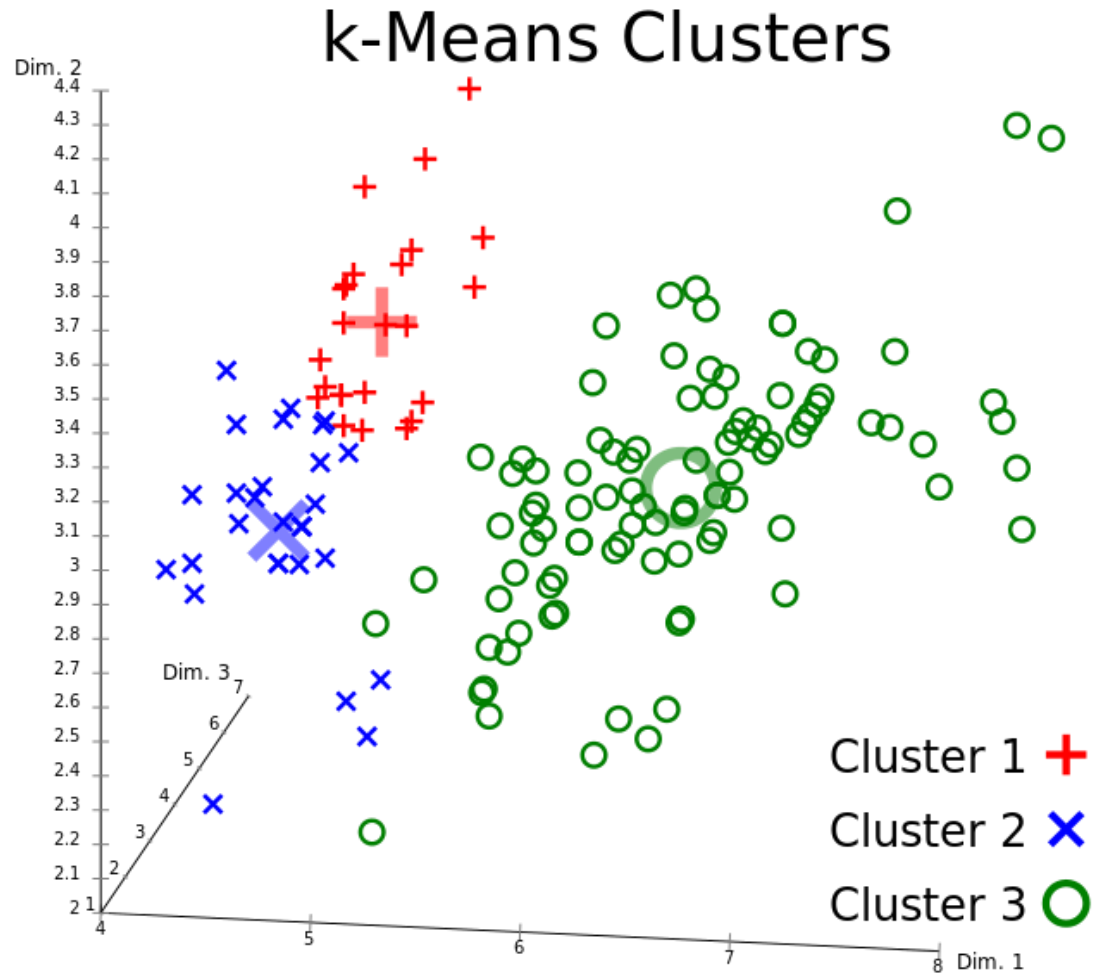
- The K-means algorithm clusters data by trying to separate samples in K groups, minimizing a criterion known as the ***within-cluster sum-of-squares***.
- **This algorithm requires the number of clusters to be specified**.
- It is an unlabeled, **not-supervised method**.
- It scales well to large number of samples.

- The K-means algorithm divides a set of N samples **X** (points or pixels of the image) into K disjoint clusters **C**, each described by the mean μ of the samples in the cluster. These means are commonly called the "cluster centroids"
- Note that the coordinates of the centroids are not, in general, points from **X.**

https://scikit-learn.org/stable/modules/clustering.html#k-means

K-means workflow

1. Determine how many clusters K are required
2. Set, at random, the position of the K centroids.
3. Calculate the Euclidian distance between the N points of the dataset and the K centroids (***within-cluster sum-of-squares)***. The point belongs to the cluster to the centroid is closer centroid.
4. A new centroid is determined using the set of points that belongs to each cluster. Repeat 3 until no changes in the position of the centroid is detected.

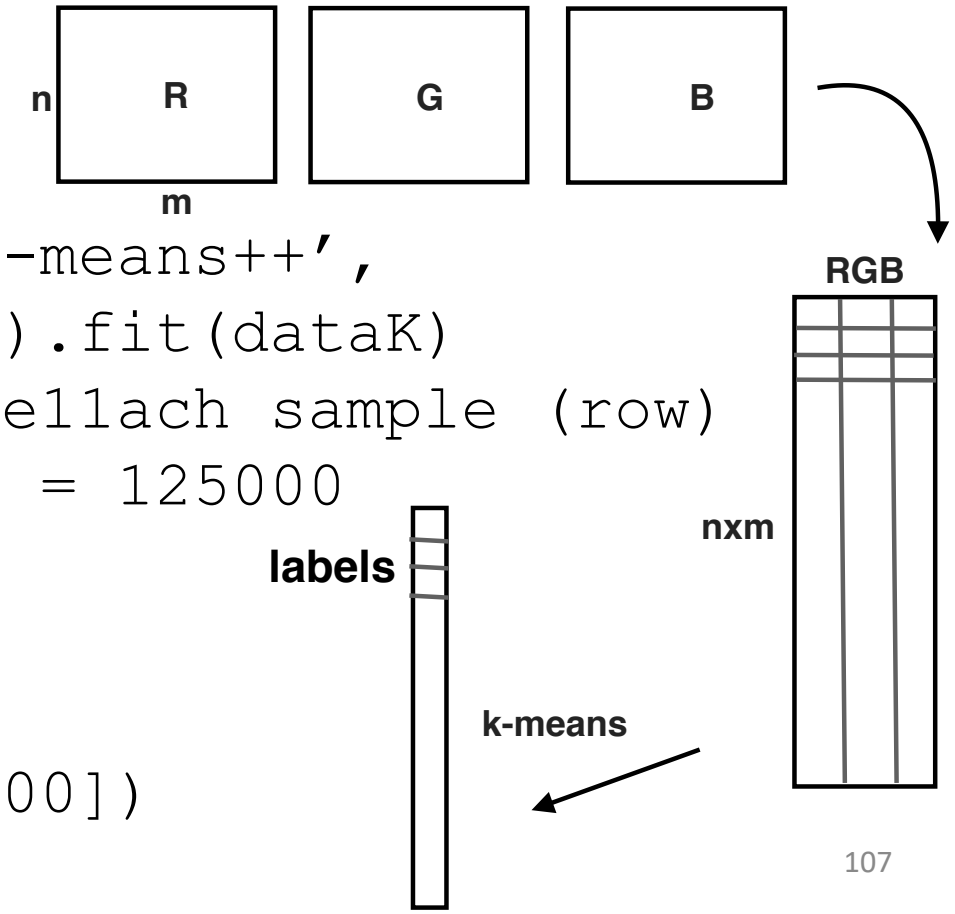


https://en.wikipedia.org/wiki/K-means_clustering#/media/File:Iris_Flowers_Clustering_kMeans.svg

How to arrange the data to use K-means

- `dataK` is an array containing N rows describing the samples and M columns describing the dimensionality of the problem
- In our problem the images are: 250 x 500 píxels, 4 channels.
  `dataK` is a 2D array: 250 x 500 = 125000 (rows) and 4 (columns)
- use `.flatten()`, `.reshape()` or `np.reshape()`

```
from sklearn.cluster import KMeans
# system training
kmn = KMeans(n_clusters=5, init='k-means++',
                    random_state=0).fit(dataK)
# a label (0 to 4) is assigned to e11ach sample (row)
labels = kmn.predict(dataK) # size = 125000
# centroids
centroids = kmn.cluster_centers_
# from 1d-array to 2d-array
imRes = np.reshape(labels, [250, 500])
```



Custom colors

```
Import skimage.color as color
newImage = color.label2rgb(imRes, colors=['yellow', 'blue',
                                          'green', 'gray', 'black'])
```

How to produce a scatterplot

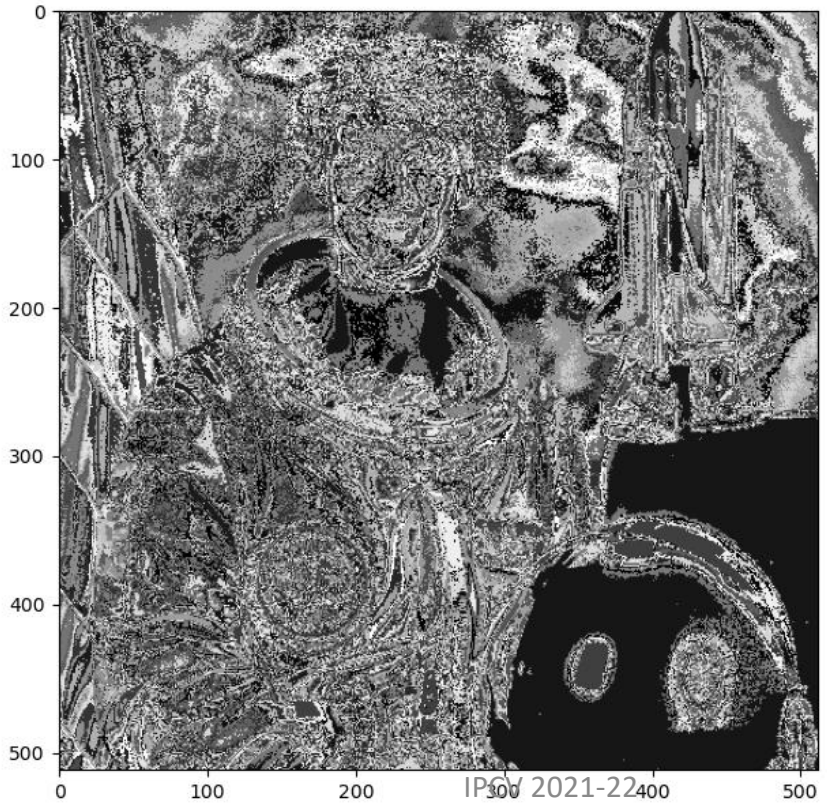
```
# a plt.scatter ≠ plt.plot
labels = kmn.predict(dataK)
plt.scatter(dataK[:,0], dataK[:,1], c = labels, cmap='jet')
```

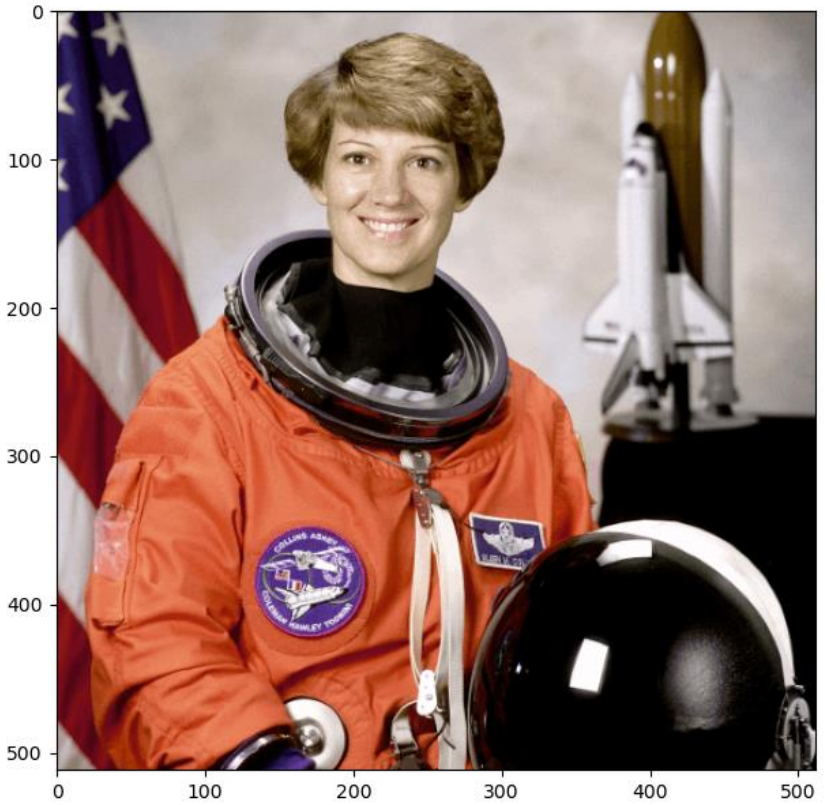
Revisiting indexed color

Goal: to produce a 216-level indexed image using K-means

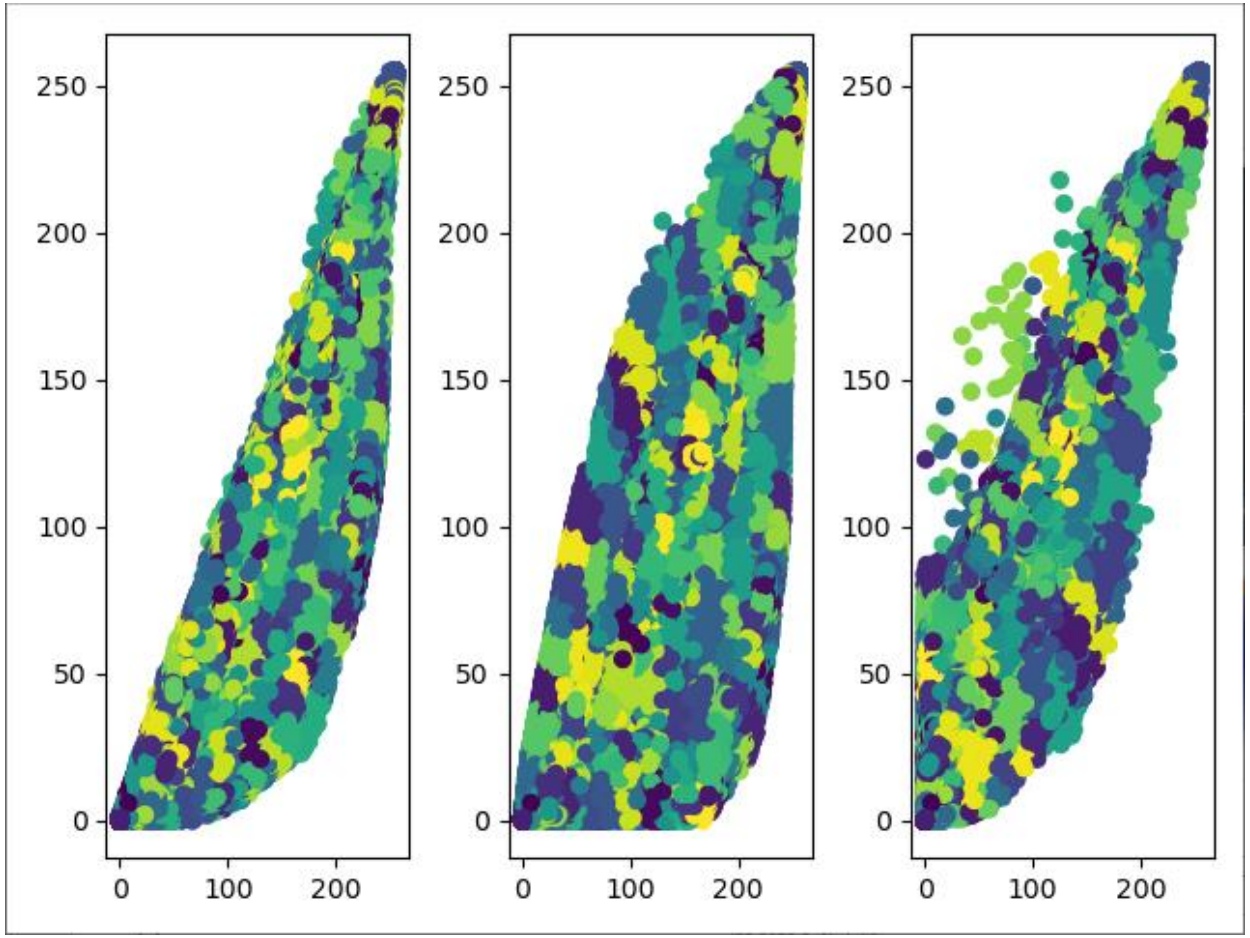
`sklearn.utils.shuffle` is great to speed up the process

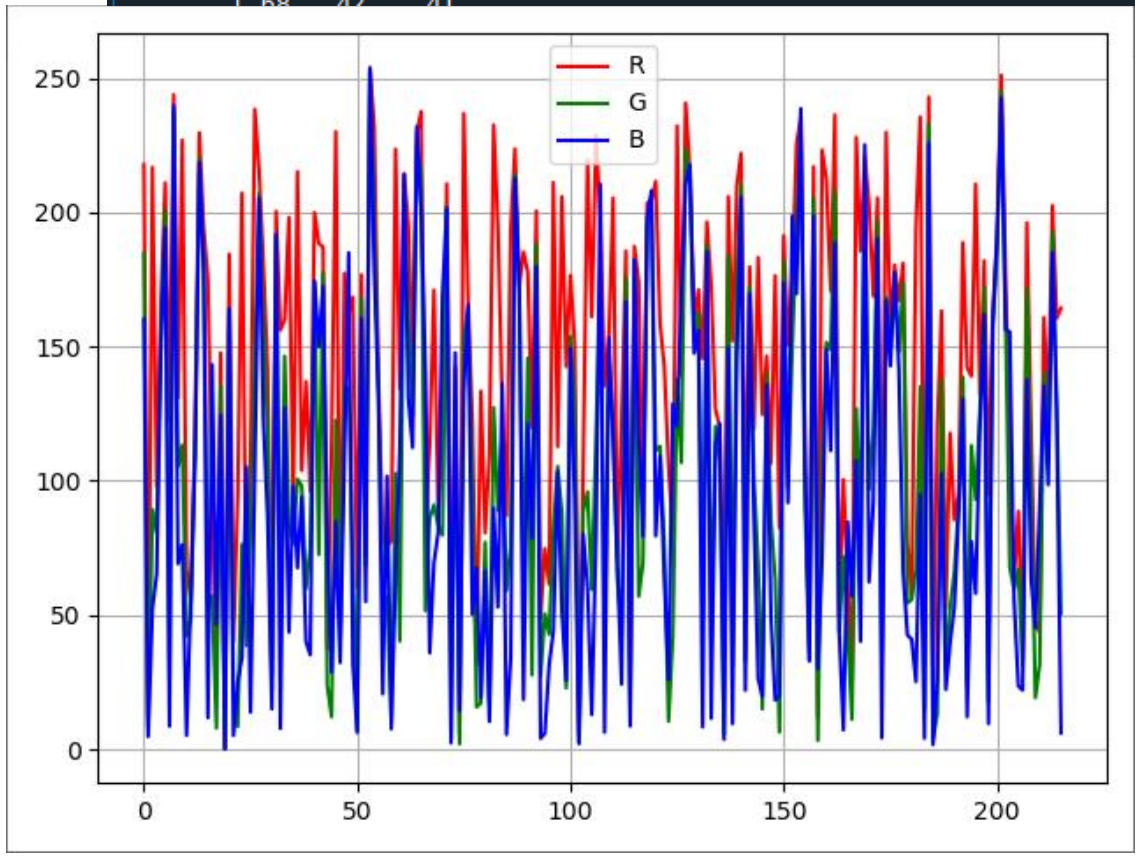
`.cluster_centers_` provides de coordinates of the centroids. This array might be considered as the colormap that transforms the index of the 2d array into a color image.





```
In [10]: index = np.linspace(0, 215, 216, dtype=np.uint8)
    ...: lut = np.uint8(kmeans_res.cluster_centers_[index])

In [11]: lut
Out[11]:
array([[217, 184, 160],
       [ 11,   8,   4],
       [216,  89,  52],
       [ 97,  80,  65],
       [167, 157, 149],
       [210, 200, 194],
       [122,  25,   8],
       [243, 237, 239],
       [130, 105,  69],
       [226, 113,  76],
       [ 68,  42,   4]
```





K-means example with PET, clusters = 6