Econometria Espacial

Ciencia de Datos y Econometría Aplicada

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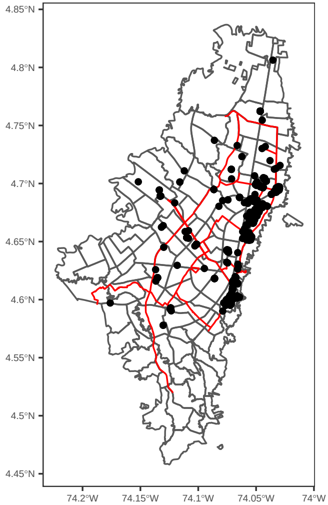
Agenda

Spatial Econometrics: Motivation

$$y = X\beta + u$$

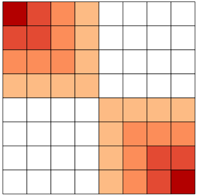
- Independence assumption between observation is no longer valid
- Attributes of observation i may influence the attributes of observation j.
- We will consider various alternatives to model spatial dependence

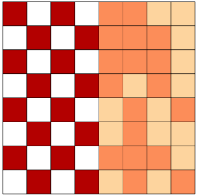


Spatial Econometrics: Motivation

$$y = X\beta + u$$

- Independence assumption between observation is no longer valid
- Attributes of observation i may influence the attributes of observation j.
- Positive Spatial correlation arises when units that are *close* to one another are more similar than units that are far apart
- Similarly spatial heterogeneity arises when some areas present more variability than others





Spatial Econometrics: Closeness

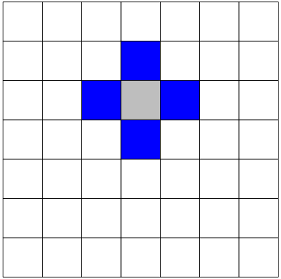
"Everything is related to everything else, but close things are more related than things that are far apart" (Tobler, 1979).

- One of the major differences between standard econometrics and standard spatial econometrics lies, in the fact that, in order to treat spatial data, we need to use two different sets of information

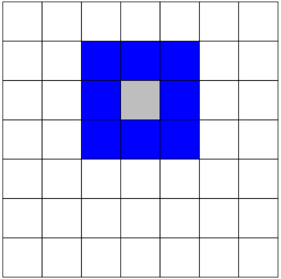
  1. Observed values of the economic variables
  2. Particular location where those variables are observed and to the various links of proximity between all spatial observations

Spatial Econometrics: Closeness

Rook criterion: two units are close to one another if they share a side



Queen criterion: two units are close if they share a side or an edge.



Spatial Econometrics: Weights Matrix

- At the heart of traditional spatial econometrics is the definition of the *weights matrix*:

$$W = \begin{pmatrix} w_{11} & \dots & \dots & w_{n1} \\ \vdots & w_{ij} & & \vdots \\ \vdots & & \ddots & \vdots \\ w_{n_1} & \dots & \dots & w_{nn} \end{pmatrix}_{n \times n} \tag{1}$$

with generic element:

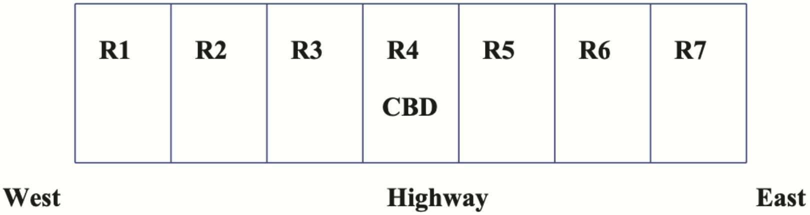
$$w_{ij} = \begin{cases} 1 & if\, j \in N\,(i) \\ 0 & o.w \end{cases} \tag{2}$$

$N(i)$ being the set of neighbors of location j. By convention, the diagonal elements are set to zero, i.e. $w_{ii} = 0$.

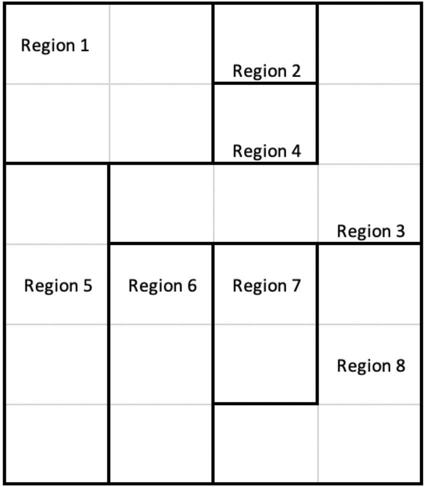
Spatial Econometrics: Weights Matrix

- The specification of the neighboring set ($N(i)$) is quite arbitrary and there's a wide range of suggestions in the literature.
  - Rook criterion
  - Queen criterion
  - Two observations are neighbors if they are within a certain distance, i.e., $j \in N(i)$ if $d_{ij} < d_{max}$ where d is the distance between location i and j.
  - Closest neighbor, ties can be solved randomly
  - More general matrices can also be specified by considering entries of w_{ij} as functions of geographical, economic or social distances between areas rather than simply characterized by dichotomous entries

W matrices are standardized to sum to one in each row



Some Examples of Weights Matrices



Testing for Spatial Dependence

- The problem with ignoring the spatial structure of the data implies that the OLS estimates in the non spatial model may be biased, inconsistent and/or inefficient, depending on what is the true underlying dependence (for more see Anselin and Bera (1998)).

$$e = y - X\hat{\beta}$$

Spatial Lag Model

- We can use the OLS reiduals to test for spatial correlation.
- The most basic one is Moran's I test (1950), a test statistics for the null of uncorrelation among regression residuals.

$$I = \left(\frac{e'We}{e'e}\right) \tag{3}$$

- where $e = y - X\beta$ is a vector of OLS residuals $\beta = (X'X)^{-1}X'y$, W is the row standardized spatial weights matrix
- Moran's I test was originally developed as a two-dimensional analog of Durbin-Watson's test

Spatial Lag Model

- We can think of situations where values observed at one location or region, say observation i, depend on the values of neighboring observations at nearby locations.

$$y_i = \rho_i y_j + X_i \beta + \epsilon_i \tag{4}$$
$$y_j = \rho_j y_i + X_j \beta + \epsilon_j \tag{5}$$

- This situation suggests a simultaneous data generating process, where the value taken by y_i depends on that of y_j and vice versa.

Spatial Lag Model

- Spatial lag dependence in a regression setting can be modeled similar to an autoregressive process in time series. Formally,

$$y = \rho W y + X\beta + u$$

- Wy induces a nonzero correlation with the error term, similar to the presence of an endogenous variable.
- Unlike to time series, Wy_i is always correlated with u
- OLS estimates in the non spatial model will be biased and inconsistent. (Anselin and Bera, 1998)
- In `R` the function `lagsarlm` uses MLE

Spatial Lag Model

The model is then

$$y = \rho W y + X\beta + u$$

with $|\rho| < 1$, we also assume that W is exogenous

If W is row standardized:

- Guarantees $|\rho| < 1$ (Anselin, 1982)
- $[0, 1]$ Weights
- Wy Average of neighboring values
- W is no longer symmetric $\sum_j w_{ij} \neq \sum_i w_{ji}$ (complicates computation)

Spatial Lag Model
Maximum Likelihood Estimator

Note that we can write

$$(I - \rho W)y = X\beta + u$$

- We can think this model as a way to correct for loss of information coming from spatial dependence.
- $(1 - \rho W)y$ is a spatially filtered dependent variable, i.e., the effect of spatial autocorrelation taken out

Spatial Lag Model

- In this case, endogeneity emerges because the spatially lagged value of y is correlated with the stochastic disturbance.
- One solution that emerged in the literature is MLE
- We need an extra assumption, i.e., $u \sim_{iid} N(0, \sigma^2 I)$.

Spatial Lag Model

Maximum Likelihood Estimator

The associated likelihood function is then

$$\mathcal{L}\left(\sigma^2, \rho, y\right) = \left(\frac{1}{\sqrt{2\pi}}\right)^n |\sigma^2\Omega|^{-\frac{1}{2}} exp\left\{-\frac{1}{2\sigma^2}(y-(I-\rho W)^{-1}X\beta)'\Omega^{-1}(y-(I-\rho W)^{-1}X\beta)\right\}$$

the log likelihood

$$l\left(\sigma^2, \rho, y\right) = constant - \frac{1}{2}ln|\sigma^2\Omega| - \frac{1}{2\sigma^2}(y-(I-\rho W)^{-1}X\beta)'\Omega^{-1}(y-(I-\rho W)^{-1}X\beta)$$

Spatial Lag Model

Maximum Likelihood Estimator

so returning to the log likelihood we have that the log likelihood is

$$
\begin{aligned}
l\left(\sigma^2, \rho, y\right) = {} & constant - \frac{n}{2} ln\left(\sigma^2\right) + ln\left(\left|(I-\rho W)\right|\right) \\
& - \frac{1}{2\sigma^2}(y-(I-\rho W)^{-1}X\beta)'(I-\rho W)'(I-\rho W)(y-(I-\rho W)^{-1}X\beta)
\end{aligned} \tag{6}
$$

then

$$
\begin{aligned}
l\left(\sigma^2, \rho, y\right) = {} & constant - \frac{n}{2} ln\left(\sigma^2\right) \\
& - \frac{1}{2\sigma^2}((I-\rho W)y - X\beta)'((I-\rho W) - X\beta) \\
& + ln\left(\left|(I-\rho W)\right|\right)
\end{aligned} \tag{7}
$$

Spatial Lag Model

Maximum Likelihood Estimator

- The determinant $|(I - \rho W)|$ is quite complicated because in contrast to the time series, where it is a triangular matrix, here it is a full matrix.
- However, Ord (1975) showed that it can be expressed as a function of the eigenvalues ω_i

$$|(I - \rho W)| = \Pi_{i=1}^{n}(1 - \rho\omega_i)$$

So the log likelihood is simplified to

$$\begin{aligned} l\left(\sigma^2, \rho, y\right) = {} & constant - \frac{n}{2} ln\left(\sigma^2\right) \\ & - \frac{1}{2\sigma^2}((I - \rho W)y - X\beta)'((I - \rho W)y - X\beta) \\ & + \sum ln(1 - \rho\omega_i) \end{aligned} \tag{8}$$

Spatial Lag Model
Maximum Likelihood Estimator

Applying FOC, the ML estimates for β and σ^2 are:

$$\beta_{MLE} = (X'X)^{-1}X'(I - \rho W)y$$

$$\sigma^2_{MLE} = \frac{1}{n}(y - \rho Xy - X\beta_{MLE})'(y - \rho Xy - X\beta_{MLE})$$

- Conditional on ρ these estimates are simply OLS applied to the spatially filtered dependent variable and explanatory variables X.

Spatial Lag Model

Maximum Likelihood Estimator

- Substituting these in the log likelihood we have a concentrated log-likelihood as a nonlinear function of a single parameter ρ

$$l(\rho) = -\frac{n}{2} ln\left(\frac{1}{n}(e_0 - \rho e_L)'(e_0 - \rho e_L)\right) + \sum ln(1 - \rho\omega_i) \tag{9}$$

- where e_0 are the residuals in a regression of y on X and
- e_L of a regression of Wy on X.

- This expression can be maximized numerically to obtain the estimators for the unknown parameters ρ.

Spatial Lag Model

Two-Stage Least Squares estimators

- An alternative to MLE we can us 2SLS to eliminate endogeneity.
- Key is to identify proper instruments
    - Need to be uncorrelated with the error term
    - Correlated with WY

Spatial Lag Model

Two-Stage Least Squares estimators

Consider the following

$$E(y) = (I - \rho W)^{-1} X\beta$$

now, since $|\rho| < 1$ we can use Neumann series property to expand the inverse matrix as

$$(I - \rho W)^{-1} = I + \rho W + \rho^2 W^2 + \rho^3 W^3 + ...$$

hence

$$E(y) = (I + \rho W + \rho^2 W^2 + \rho^3 W^3 + ...) X\beta$$

$$= X\beta + \rho W X\beta + \rho^2 W^2 X\beta + \rho^3 W^3 X\beta + ...$$

so we can express $E(y)$ as a function of X, WX, W^2X,...

Spatial Lag Model
Two-Stage Least Squares estimators

We can use the first three elements of the expansion as instruments. Let's define H as the matrix with our instruments

$$H = [X, WX, W^2X]$$

Spatial Error Model (SEM)

An OVB motivation

- True GDP

$$y = X\beta + Z\theta$$

- but Z is not observed and $Z \perp X$
- we estimate

$$y = X\beta + \epsilon$$

- if Z has a spatial autoregressive process

$$Z = \rho WZ + r$$

Spatial Error Model (SEM)

An OVB motivation

- Then

$$y = X\beta + (I - \rho W)^{-1}(\theta r)$$

- calling $(\theta r) = u$

$$y = X\beta + (I - \rho W)^{-1} u$$

- β will be unbiased but inefficient

Spatial Error Model (SEM)

The model is now

$$y = X\beta + u$$

with

$$u = \rho X u + \epsilon$$

with $|\rho| < 1$, we also assume that W is exogenous We can estimate this by MLE or FGLS