1 Introduction

The van der Pol oscillator is a non-conservative oscillator with non-linear damping. Energy is dissipated at high amplitudes and generated at low amplitudes. As a result, there exists oscillations around a state at which energy generation and dissipation balance. The state towards which the oscillations converge is known as a *limit cycle*, which shall be formalized later in this paper.

Balthazar van der Pol was a pioneer in the field of radio and telecommunications. While he was working at Phillips, van der Pol discovered these stable oscillations. Van der Pol himself came across the system as he was building electronic circuit models of the human heart. Due to the unique nature of the van der Pol oscillator, it has become the cornerstone for studying systems with limit cycle oscillations. In fact, the van der Pol equation has become a staple model for oscillatory processes in not only physics, but also biology, sociology and even economics. For instance, it has been used to model the electrical potential across the cell membranes of neurons in the gastric mill circuit of lobsters [5, 9]. Additionally, Fitzhugh and Nagumo used the model to describe spike generation in giant squid axons [4, 7]. The equation has also been extended to the Burridge–Knopoff model which characterizes earthquake faults with viscous friction [3]. Thus, it stands to reason that we should develop a deep understanding of the van der Pol oscillator due to its widespread applications.

2 The Van der Pol Oscillator

The van der Pol oscillator is described by the equation

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$$

and equivalently the autonomous system,

$$\begin{aligned}
\dot{x} &= y - F(x) \coloneqq y - \epsilon\left(\frac{x^3}{3} - x\right) \\
\dot{y} &= -x
\end{aligned}$$

It differs from the systems we have studied in that it is non-conservative. That is, it is not volume-preserving except when $\epsilon = 0$. If $\epsilon = 0$, then the equation becomes $\ddot{x} + x = 0$, the simple harmonic oscillator, which is conservative. We formalize our analysis with some definitions.

Definition 1. *A* ***flow*** *is a mapping $\phi : U \times \mathbb{R} \to U$, for any set U that satisfies*

- $\phi(z, 0) = z \ \forall z \in U$
- $\phi(\phi(z, s), t) = \phi(z, s + t) \ \forall z \in U$ *and* $s, t \in \mathbb{R}$

Solutions to an autonomous system of the form $\dot{z} = v(z)$, where $z \in U$ and $v : U \to U$ will be a flow, namely $\phi(z,t)$ such that

$$\frac{d}{dt}\phi(z,t) = v(\phi(z,t))$$

Definition 2. *Given $z \in U$, the* ***trajectory*** *through z is defined to be*

$$\Gamma_z \equiv \{\phi(z,t) : t \in \mathbb{R}\}$$

A trajectory traces the motion of a single point through the flow. Thus, understanding the properties of trajectories is essential to our analysis.

Definition 3. *A trajectory Γ is called* ***closed*** *if it contains more than one point and there exists $T \in \mathbb{R}^+$ such that $\phi(z,T) = z$ $\forall z \in \Gamma$.*

Definition 4. *A* ***limit cycle*** *is an isolated closed trajectory. That is, there exists a neighborhood $N \subseteq U$ around this trajectory such that Γ_z is not closed for all $z \in N$.*

As a result of this definition of limit cycles, limit cycles can only occur in nonlinear systems. In a linear system, closed trajectories are neighboured by other closed trajectories. Consequently, neighboring trajectories of a limit cycle must spiral into or away from the limit cycle. We shall analyze limit cycles of this system when $\epsilon > 0$.

3 Results

We begin with some preliminary results before we start our analysis of the van der Pol oscillator.

Lemma 5. *(Gronwall's Lemma [1]) Let α, β, u be nonnegative real functions defined on $[0, M]$ where β, u are continuous, α is differentiable, and $\forall t \in [0, M]$*

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)ds$$

then

$$u(t) \leq \alpha(0)\exp\left(\int_0^t \beta(s)ds\right) + \int_0^t \alpha'(s)\left[\exp\left(\int_s^t \beta(r)dr\right)\right]ds \ , \ \forall t \in [0, M]$$

Proof. For $t \in [0, M]$, let $\gamma(t) = \int_0^t \beta(s)ds$ and $v(t) = \int_0^t \beta(s)u(s)ds$. By the Fundamental Theorem of Calculus, $\frac{d}{dt}\exp(-\gamma(t)) = -\beta(t)\exp(-\gamma(t))$ and

$$v'(t) = \beta(t)u(t) \leq \beta(t)(\alpha(t) + v(t))$$

Therefore,

$$\exp(-\gamma(t))v(t) = \int_0^t \exp(-\gamma(s))(v'(s) - \beta(s)v(s))ds \leq \int_0^t \exp(-\gamma(s))\alpha(s)\beta(s)ds$$

Furthermore, integration by parts yields

$$\int_0^t \exp(-\gamma(s))\alpha(s)\beta(s)ds = -\alpha(t)\exp(-\gamma(t)) + \alpha(0) + \int_0^t \alpha'(s)\exp(-\gamma(s))ds$$

Thus, it follows that

$$\begin{aligned} u(t) \le \alpha(t) + v(t) &= \exp(\gamma(t))\Big[\exp(-\gamma(t))\alpha(t) + \exp(-\gamma(t))v(t)\Big] \\ &\le \exp(\gamma(t))\Big[\alpha(0) + \int_0^t \alpha'(s)\exp(-\gamma(s))ds\Big] \\ &= \alpha(0)\exp\Big(\int_0^t \beta(s)ds\Big) + \int_0^t \alpha'(s)\Big[\exp\Big(\int_s^t \beta(r)dr\Big)\Big]ds \end{aligned}$$

□

Lemma 6. *(Averaging Lemma [10]) Let $U \subseteq \mathbb{R}^n$ be a compact set, ϵ be a real such that $0 < \epsilon \ll 1$. If x and z are solutions to*

$$\dot{x} = \epsilon f(x, t, \epsilon) \tag{†}$$

and

$$\dot{z} = \epsilon \overline{f}(z) := \epsilon \frac{1}{T}\int_0^T f(z, t, 0)dt \tag{‡}$$

respectively, where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^n$ is C^2 and periodic in t with period T, and satisfy initial conditions $x(0) = x_0$ and $z(0) = z_0$ where $|x_0 - z_0| = O(\epsilon)$, then $|x(t) - z(t)| = O(\epsilon)$ on a time scale $t \sim \frac{1}{\epsilon}$.

Proof. First we shall show that there is a C^2 change of coordinates $x = y + \epsilon w(y, t, \epsilon)$ under which (†) becomes

$$\dot{y} = \epsilon \overline{f}(y) + \epsilon^2 g(y, t, \epsilon) \tag{≀}$$

where g is periodic in t with period T. Define

$$\widetilde{f}(x, t, \epsilon) = f(x, t, \epsilon) - \overline{f}(x)$$

and consider the change of coordinates

$$x = y + \epsilon w(y, t, \epsilon)$$

where w is the anti-derivative of $\widetilde{f}$. That is, $\frac{\partial w}{\partial t} = \widetilde{f}(y, t, 0)$. Differentiating yields

$$\dot{x} = \dot{y} + \epsilon(\dot{y}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial t})$$

Therefore,

$$\begin{aligned}
\dot{y} &= (1+\epsilon\frac{\partial w}{\partial y})^{-1}(\dot{x}-\epsilon\frac{\partial w}{\partial t}) \\
&= (1-\epsilon\frac{\partial w}{\partial y}+O(\epsilon^2))(\epsilon\overline{f}(y+\epsilon w)+\epsilon\widetilde{f}(y+\epsilon w,t,\epsilon)-\epsilon\widetilde{f}(y,t,0)) \\
&= \epsilon(1-\epsilon\frac{\partial w}{\partial y}+O(\epsilon^2))(\overline{f}(y)+\epsilon w\frac{\partial \overline{f}}{\partial y}(y)+\epsilon w\frac{\partial \widetilde{f}}{\partial y}(y,t,0)+\epsilon\frac{\partial \widetilde{f}}{\partial \epsilon}(y,t,0)+O(\epsilon^2)) \\
&= \epsilon(1-\epsilon\frac{\partial w}{\partial y})(\overline{f}(y)+\epsilon w\frac{\partial f}{\partial y}(y,t,0)+\epsilon\frac{\partial \widetilde{f}}{\partial \epsilon}(y,t,0))+O(\epsilon^3) \\
&= \epsilon\overline{f}(y)+\epsilon^2\left(w\frac{\partial f}{\partial y}(y,t,0)+\frac{\partial \widetilde{f}}{\partial \epsilon}(y,t,0)-\frac{\partial w}{\partial y}\overline{f}(y)\right)+O(\epsilon^3) \\
&:= \epsilon\overline{f}(y)+\epsilon^2 g(y,t,\epsilon)
\end{aligned}$$

Since U is compact, $\overline{f}$ is uniformly continuous and g is bounded. Let L be the Lipschitz constant of $\overline{f}$ and B be the maximum absolute value of g, and $u = y - z$. Subtracting (‡) and (≀) and then integrating yields

$$u(t) = u(0) + \epsilon\int_0^t [\overline{f}(y(s)) - \overline{f}(z(s))]ds + \epsilon^2\int_0^t g(y(s),s,\epsilon)ds$$

Taking absolute values, and using the triangle inequality yields

$$|u(t)| = |u(0)| + \epsilon L\int_0^t |u(s)|ds + \epsilon^2 Bt$$

By Gronwall's Lemma,

$$|u(t)| \le |u(0)|\exp(\epsilon Lt) + \epsilon^2 B\int_0^t \exp(\epsilon L(t-s))ds \le \exp(\epsilon Lt)\left(|u(0)| + \frac{\epsilon B}{L}\right)$$

Therefore, if $|y(0)-z(0)| = |u(0)| = O(\epsilon)$, then for $t \in [0, \frac{1}{\epsilon L}]$, we have $|y(t)-z(t)| = |u(t)| = O(\epsilon)$. Recall, $|x(t)-y(t)| = \epsilon|w(y,t,\epsilon)| = O(\epsilon)$, so by the triangle inequality

$$|x(t)-z(t)| \le |x(t)-y(t)| + |y(t)-z(t)| = O(\epsilon)$$

as desired. □

Now we are ready to delve into our analysis of the van der Pol oscillator. First, we prove a fairly powerful result that characterizes the unique limit cycle of the van der Pol oscillator. Moreover, all other trajectories will approach this limit cycle. Then we seek to deepen our understanding by describing the behavior of this limit cycle for various values of ϵ.

Theorem 7. *(Lienard's Theorem [8]) The van der Pol oscillator has exactly one limit cycle and all other trajectories spiral into it.*

Proof. We refer to Figure 1 throughout this proof. Let $P_i = (x_i, y_i)$ for $i = 0, 1, 2, 3, 4$ where $P_0 \cdots P_4$ traces the trajectory Γ that starts at P_0. We know Γ follows this path since on the positive y-axis, $\dot{x} > 0, \dot{y} < 0$, so Γ spirals clockwise until it intersects the graph of $y = F(x)$ at P_2. At this point, $\dot{x} = 0, \dot{y} < 0$ so Γ travels below the graph of $y = F(x)$ after which $\dot{x} < 0, \dot{y} < 0$ so Γ curves left and keeps spiraling until it hits the negative y-axis at P_4. Notice the van der Pol equation is symmetric with respect to origin. That is, if $(x(t), y(t))$ describes a trajectory, so does $(-x(t), -y(t))$. Therefore, it follows that Γ is closed if and only if $y_4 = -y_0$ or equivalently, $u(0, y_0) = u(0, y_4)$ where $u(x, y) = \frac{1}{2}(x^2 + y^2)$. Let A be the arc $\overline{P_0P_4}$ of Γ. A is uniquely determined by $\alpha = x_2$ since trajectories do not cross. Thus, we can define

$$\psi(\alpha) = \int_A du = u(0, y_4) - u(0, y_0)$$

Thus Γ is closed if and only if $\psi(\alpha) = 0$. Furthermore,

$$\begin{aligned} du = x \cdot dx + y \cdot dy &= x\left(\frac{y - F(x)}{-x}\right)dy + y \cdot dy = F(x)dy \\ &= x \cdot dx + y\left(\frac{-x}{y - F(x)}\right)dx = \frac{-xF(x)}{y - F(x)}dx \end{aligned}$$

$F(x) = 0$ has exactly one positive solution at $a = \sqrt{3}$. If $\alpha \leq a$, then along

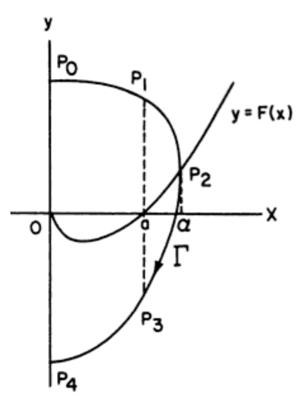

Figure 1: Retrieved From [8]

A, $F(x) < 0$ and $dy = -x \cdot dt < 0$ so $\psi(\alpha) > 0$. Now suppose $\alpha > a$. Let the trajectory intersect the line $x = a$ at P_1 and P_3. Divide A into three parts, $A_1 = \overline{P_0P_1}, A_2 = \overline{P_1P_3}, A_3 = \overline{P_3P_4}$ and for $i = 1, 2, 3$, define $\psi_i(\alpha) = \int_{A_i} du$. Let $\hat{\alpha} > \alpha$ and $\hat{A}_i$ be the corresponding arcs of the trajectory through $(\hat{\alpha}, F(\hat{\alpha}))$.

Along arcs A_1, A_3, $F(x) < 0$, $\frac{dx}{y-F(x)} = dt > 0$. Since trajectories do not cross, $\hat{A}_1$ lies above A_1 and $\hat{A}_3$ lies below A_3. The x-limits of integration are fixed at 0 and a for A_1 and $\hat{A}_1$. Thus, for all $x \in [0, a]$, $y < \hat{y}$, the y-coordinate of the corresponding point on $\hat{A}_1$.

$$\psi_1(\alpha) = \int_0^a \frac{-xF(x)}{y - F(x)} dx > \int_0^a \frac{-xF(x)}{\hat{y} - F(x)} dx = \psi_1(\hat{\alpha})$$

Similarly, along A_3, for all $x \in [0, a]$, $y > \hat{y}$ so

$$\psi_3(\alpha) = \int_a^0 \frac{-xF(x)}{y - F(x)} dx = \int_0^a \frac{-xF(x)}{F(x) - y} dx > \int_0^a \frac{-xF(x)}{F(x) - \hat{y}} dx = \psi_3(\hat{\alpha})$$

Additionally, $\hat{A}_2$ lies to the right of A_2. That is, $\hat{y_1} > y_1$ and $\hat{y_3} < y_3$. Therefore, along A_2, for all $y \in [y_3, y_1]$, $x < \hat{x}$, the x-coordinate of the corresponding point on $\hat{A}_2$. Since $F(x)$ is increasing and positive for $x > a$,

$$\psi_2(\alpha) = -\int_{y_3}^{y_1} F(x) dy > -\int_{y_3}^{y_1} F(\hat{x}) dy > -\int_{\hat{y_3}}^{\hat{y_1}} F(\hat{x}) d\hat{y} = \psi_2(\hat{\alpha})$$

Therefore, $\psi(\alpha) = \psi_1(\alpha) + \psi_2(\alpha) + \psi_3(\alpha)$ is strictly decreasing for $\alpha > a$. For sufficiently small $\delta > 0$,

$$\psi_2(\alpha) = -\int_{y_3}^{y_1} F(x) dy < -\int_{y_3+\delta}^{y_1-\delta} F(x) dy < -F(a+\delta) \int_{y_3+\delta}^{y_1-\delta} dy < -F(a+\delta) \cdot (y_1 - 2\delta)$$

Since $y_1 > y_2$ and $y_2 = F(x_2) = F(\alpha) \to \infty$ as $\alpha \to \infty$, it follows that $\psi_2(\alpha) \to -\infty$ as $\alpha \to \infty$. Hence,

$$\lim_{\alpha \to \infty} \psi(\alpha) = -\infty$$

Since $\psi(\alpha)$ is positive for $\alpha \in [0, a]$ and strictly decreasing for $\alpha > a$ and approaches $-\infty$ as $a \to \infty$. It follows that $\psi(\alpha) = 0$ has exactly one positive solution $\alpha_0 \in (a, \infty)$. Therefore, the van der Pol equation has exactly one limit cycle Γ_0 which goes through the point $(\alpha_0, F(\alpha_0))$. Furthermore, since $\psi(\alpha) > 0$ for $\alpha < \alpha_0$ and $\psi(\alpha) < 0$ for $\alpha > \alpha_0$, it follows that all other trajectories will approach Γ_0, as desired. □

Theorem 8. *If $0 < \epsilon \ll 1$, then the unique limit cycle of the van der Pol oscillator lies within a neighborhood of distance $O(\epsilon)$ from the circle centered at the origin with radius 2.*

Proof. We first introduce a change of coordinates

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ -\sin(t) & -\cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is a counter-clockwise rotation by t radians followed by a reflection over the x-axis, and thus clearly invertible. Notice that

$$u\cos(t) - v\sin(t) = (x \cdot \cos(t) - y \cdot \sin(t))\cos(t) - (-x \cdot \sin(t) - y \cdot \cos(t))\sin(t) = x$$

Thus, the system becomes

$$\dot{u} = -\epsilon\cos(t)\left[\frac{(u\cos(t) - v\sin(t))^3}{3} - (u\cos(t) - v\sin(t))\right]$$
$$\dot{v} = \epsilon\sin(t)\left[\frac{(u\cos(t) - v\sin(t))^3}{3} - (u\cos(t) - v\sin(t))\right]$$

The Averaging Lemma tells us that we can solve the averaged differential system of equations instead and the solution will only be $O(\epsilon)$ away from the solution of the original system. Since we chose $\epsilon \ll 1$, this analysis suffices for our purposes. Integrating both equations with respect to t from 0 to 2π yields,

$$\dot{u} = \epsilon\frac{u}{2}\left[1 - \frac{u^2+v^2}{4}\right]$$
$$\dot{v} = \epsilon\frac{v}{2}\left[1 - \frac{u^2+v^2}{4}\right]$$

Changing to polar coordinates yields

$$\dot{r} = \frac{d}{dt}\sqrt{u^2+v^2} = \frac{u\dot{u}+v\dot{v}}{\sqrt{u^2+v^2}} = \frac{\epsilon(u^2+v^2)}{2\sqrt{u^2+v^2}}\left[1 - \frac{u^2+v^2}{4}\right] = \frac{\epsilon r}{2}\left(1 - \frac{r^2}{4}\right)$$
$$\dot{\theta} = \frac{d}{dt}\arctan\left(\frac{v}{u}\right) = \frac{u^2}{u^2+v^2} \cdot \frac{u\dot{v} - v\dot{u}}{u^2} = 0$$

In particular, $\dot{r} > 0$ when $r < 2$ and $\dot{r} < 0$ when $r > 2$. Thus, this averaged system has an attracting circle of fixed points on the circle of radius 2 centered at the origin. By Theorem 7, the van der Pol oscillator has a unique limit cycle. By Lemma 6, it clearly follows that the limit cycle of the van der Pol oscillator when $0 < \epsilon \ll 1$ lies within a neighborhood of distance at most $O(\epsilon)$ away from this circle, as desired. □

While not exact, Theorem 8 gives a rather good approximation for limit cycle of the van der Pol oscillator when $\epsilon \ll 1$. We can analyze the behavior of the van der Pol oscillator for other values of ϵ as well. A numerical approximation of the limit cycle of the van der Pol oscillator with $\epsilon = 1$ is shown in Figure 2.

We also give a brief informal analysis the van der Pol oscillator when $\epsilon \gg 1$. We make the substitution $y \to \epsilon y$ and the van der Pol system becomes

$$\dot{x} = \epsilon\left(y + x - \frac{x^3}{3}\right)$$
$$\dot{y} = -\frac{x}{\epsilon}$$

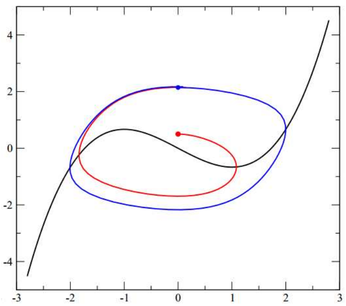

Figure 2: $\epsilon = 1$. Retrieved from [2]

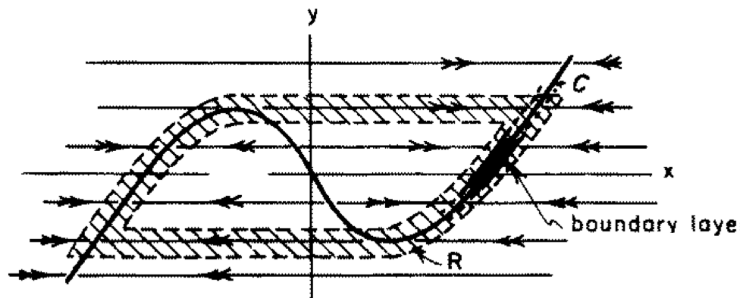

Figure 3: $\epsilon \gg 1$. Retrieved from [6]

Hence, if $|y + x - \frac{x^3}{3}| \gg \frac{1}{\epsilon^2}$, then $|\dot{x}| \gg |\dot{y}|$ and so trajectories that lie away from $\mathcal{C}$, the curve $y = \frac{x^3}{3} - x$, are approximately horizontal lines. If $y + x - \frac{x^3}{3} = O(\frac{1}{\epsilon^2})$, then $|\dot{x}| = O(\frac{1}{\epsilon}) = |\dot{y}|$ and so after entering this boundary layer, the trajectories turns sharply and follows $\mathcal{C}$ until it reaches either $(1, -\frac{2}{3})$ or $(-1, \frac{2}{3})$. That is, it is possible to find an annular region $\mathcal{R}$ from which trajectories do not exit. By Theorem 7, there is a unique limit cycle. Therefore, it follows that this limit cycles lies within the region $\mathcal{R}$.

4 Concluding Remarks

This concludes our analysis of the van der Pol oscillator. We have made much progress towards a greater understanding of the van der Pol oscillator. We showed that there exists a unique limit cycle and all other trajectories in the phase approach this limit cycle. We also characterized the behavior of this limit cycle under certain assumptions about ϵ. This provides a great foundation for explaining similar oscillations in nature.

Even though our understanding of the van der Pol oscillator has improved significantly, there is still more to be discovered. While we have roughly characterized the behavior of the limit cycle of the van der Pol oscillator for very small and very large values of ϵ, further analysis could be done for moderate values of ϵ. How are the behaviors for small ϵ and large ϵ connected? A thorough understanding of the van der Pol oscillator when $\epsilon \sim 1$ could be the bridge between the theories for large and small ϵ. Our analysis has also been of the unforced oscillator. Further study could be dedicated to the forced van der Pol oscillator

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = A\sin(\omega t)$$

where $A\sin(\omega t)$ represents a driving force. As we have seen, the van der Pol oscillator has a wide array of applications in a variety of subject areas. Additional analysis should be done to see if more applications can be uncovered.

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