Beyond the “faster is slower” effect

I.M. Sticco, F.E. Cornes, G.A. Frank and C.O. Dorso

January 6, 2017

Table of contents

Force balance during a quasi-static situation

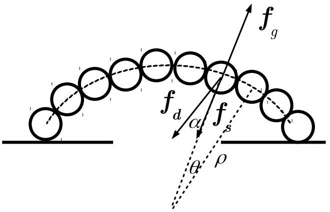

Figure: Snapshot of the blocking cluster. The picture shows the three forces actuating on the pedestrian.

- The two main hypothesis are: $f_s \sim v_d \sqrt{N}$ and $\dot{v} \simeq 0$ (mean field approximation and quasi-static regime, respectively).
- See PRE 88 052813 (2013) for details.

Quasi-static equation of motion

- Movement equation when only one pedestrian moves forward,

$$\frac{dv_i}{dt} + \left[\frac{1}{\tau} + \frac{2\kappa}{m}\, g(v_d, N)\right] v_i = \left(\frac{\cos\alpha}{\tau} + \frac{b}{m}\sqrt{N}\right) v_d \qquad (1)$$

For the stationary case,

$$v_i = \frac{\cos\alpha + \tau\, b\sqrt{N}/m}{1 + 2\kappa\tau\, g(v_d, N)/m}\, v_d \qquad (2)$$

- The force balance (on the tangential direction)

$$A\, e^{g/B} + \frac{m v_d}{\tau} \operatorname{sen}\alpha\, \cos(\theta/2) = \left(\frac{m v_d}{\tau} + b\, v_d\sqrt{N}\right) \operatorname{sen}(\theta/2) \qquad (3)$$

Main equations

$$\left\{\begin{array}{rcl} v_i & = & \dfrac{\cos\alpha + \tau\, b\,\sqrt{N}/m}{1+2\kappa\tau\, g/m}\, v_d \\ \\ g & = & B\,\ln\left[\dfrac{b\, v_d\,\sqrt{N}}{A}\,\mathrm{sen}\,\dfrac{\theta}{2} + \dfrac{m v_d}{\tau A}\left(\mathrm{sen}\,\dfrac{\theta}{2} - \mathrm{sen}\,\alpha\,\cos\dfrac{\theta}{2}\right)\right] \end{array}\right. \tag{4}$$

where the condition $g \geq 0$ can be fulfilled if

$$v_d \geq \frac{\tau A}{(\tau b\sqrt{N}+m)\,\mathrm{sen}\,\dfrac{\theta}{2} - m\,\mathrm{sen}\,\alpha\,\cos\dfrac{\theta}{2}} \tag{5}$$

Social force model parameters

- We roughly consider $f_s \simeq n\,m\,vd/\tau$ where $n = R/2r_i$ is the number of individuals on top of the arch. For a compact bulk, the packing density is $\pi/\sqrt{12}$, That is, the occupied surface $(N\pi\,r_i^2)$ divided the total surface $(\pi\,R^2/2)$. Thus,

$$\frac{\pi}{\sqrt{12}} = 0.91 = \frac{2N\pi\,r_i^2}{\pi\,R^2} = \frac{N}{2n^2} \qquad \Rightarrow \qquad n = \simeq 0.75\,\sqrt{N} \quad (6)$$

  This means that $b = 0.75\,m/\tau$

- The current parameters are (in the MKS units)

$$\begin{cases} \tau & = & 0.5\,\mathrm{s} \\ m & = & 70\,\mathrm{kg} \\ \kappa & = & 240000\,\mathrm{kg/(m.s)} \end{cases} \qquad (7)$$

  and $A = 2000\,\mathrm{N}$, $B = 0.08\,\mathrm{m}$.

The semi-circle example

The semi-circle corresponds to $\alpha = 0$ and $\theta = \pi/\eta$ (η is the number of pedestrians in the arch). We fix $\eta = 4$, and thus, $\mathrm{sen}\,(\theta/2) = \mathrm{sen}\,(\pi/2\eta) = 0.38$. The friction function g gives

$$g = \begin{cases} 0 & \text{if} \quad v_d \leq v_c \\ \\ B\,\ln\left[\left(1 + 0.75\sqrt{N}\right)\dfrac{m v_d}{\tau A}\,\mathrm{sen}\,\dfrac{\pi}{2\eta}\right] & \text{if} \quad v_d > v_c \end{cases} \qquad (8)$$

where $v_c \simeq 3.1\,\mathrm{m/s}$ for $N = 225$ and $\eta = 4$. Notice that $g \simeq 0.05\,\mathrm{m}$ for $v_d = 6\,\mathrm{m/s}$ and $g \simeq 0.15\,\mathrm{m}$ for $v_d = 20\,\mathrm{m/s}$.

The semi-circle example

- The regime for $g = 0$ and $v_d < v_c$,

$$v_i = (1 + 0.75\sqrt{N})\, v_d \simeq 0.75\sqrt{N}\, v_d \tag{9}$$

The stationary velocity on v_d and $\sqrt{N}$.

- The regime for $g = 0$ and $v_d = v_c$,

$$v_i = (1 + 0.75\sqrt{N})\, v_c = \frac{\tau A}{m\,\mathrm{sen}\,(\pi/2\eta)} \tag{10}$$

Notice that the velocity at the critical desired velocity does not depend on the bulk (say, $\sqrt{N}$), but on the number of blocking pedestrians η.

The semi-circle example

- The regime for $g \neq 0$ and $v_d > v_c$,

$$v_i = \frac{(1 + 0.75\,\sqrt{N})\,v_d}{1 + \dfrac{2\kappa\tau\,B}{m}\,\ln\left[\left(1 + 0.75\,\sqrt{N}\right)\dfrac{mv_d}{\tau A}\,\mathrm{sen}\,\dfrac{\pi}{2\eta}\right]} \qquad (11)$$

This expression is similar to $f(x) = x/[1 + a\ln(c\,x)]$. Thus,

$$f'(x) = \frac{1}{1 + a\ln(c\,x)}\left[1 - \frac{a}{1 + a\ln(c\,x)}\right] \qquad (12)$$

where $a = 2\kappa\tau\,B/m \simeq 275$ and

$$c = (1 + 0.75\,\sqrt{N})\frac{m}{\tau A}\,\mathrm{sen}\,\frac{\pi}{2\eta} \simeq 0.32\,\mathrm{s/m} \qquad (13)$$

The semi-circle example

Notice that $f'(x) = 0$ if $a\ln(cx) = a - 1$. This means that $\ln(cx) \simeq 1$, or, $x = v_d = e/c = 8.4\,\mathrm{m/s}$. At this point, the velocity is

$$v_i = \frac{(1 + 0.75\sqrt{N})\, v_d}{\dfrac{2\kappa\tau\, B}{m}} = \frac{(1 + 0.75\sqrt{N})\, v_d}{275} \tag{14}$$

The slowing down depends on the friction coefficient κ. However, the current desired velocity e/c is always 2.7182... times the critical one v_c. According to this model, the slowing down will occur at $v_d = e\, v_c$.

The semi-circle example

We can envisage two forces in Eq. (14). Re-writting this expression, we arrive to

$$\underbrace{2\kappa\,B\,v_i}_{\text{granular}} = \underbrace{\frac{m\,v_d}{\tau}}_{\text{desired}} + \underbrace{\frac{0.75\,m}{\tau}\,\sqrt{N}\,v_d}_{\text{social}} \tag{15}$$

We conclude that at $v_d = 2.71\,v_c$ the three forces get balanced. The value of the granular force corresponds to $g = B = 0.08\,\mathrm{m}$. Perhaps the meaning of this is that the the current condition corresponds to acknowledging a “bottleneck” of width $0.6\,\mathrm{m} - 2B = 0.44\,\mathrm{m}$.

The semi-circle example

- The regime for $g \neq 0$ and $v_d \gg v_c$,

The breakup

The quasi-static approach is the starting point for the leaving process immediately after the *blocking cluster* breaks. Then,

$$\frac{dv_i}{dt} + \frac{v_i}{\tau} = \frac{v_d \cos\alpha_i}{\tau} + b\, v_d \sqrt{N} \tag{16}$$

This gives

$$v_i(t) = v(0)\, e^{-t/\tau} + v_d(\cos\alpha_i + \tau\, b\, \sqrt{N})\,(1 - e^{-t/\tau}) \tag{17}$$