The movement equation

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1 The problem

$$\begin{cases} m\dfrac{dv}{dt} = \beta v_d + \dfrac{m}{\tau}(v_d - v) - 2k\,(2r-d)\,v \\ 2r - d = B\ln\left(\dfrac{\beta v_d}{A}\right) \\ v(0) = v_0 \end{cases} \tag{1}$$

The equation can be expressed in the following reduced magnitudes

$$\begin{cases} v^* &=& v/v_d \\ t^* &=& t/\tau \\ z^* &=& (2r-d)/B \\ k^* &=& k\tau B/m \\ \beta^* &=& \beta\tau/m \end{cases} \tag{2}$$

The reduced equation in new variables read (the "*" has been omitted for clarity reasons)

$$\begin{cases} \dfrac{dv}{dt} + (1+2kz)\,v = 1+\beta \\ v(0) = v_0 \end{cases} \tag{3}$$

2 The velocity

$$\frac{dv}{(1+\beta) - (1+2kz)\,v} = dt \tag{4}$$

$$-\frac{1}{1+2kz}\ln[(1+\beta) - (1+2kz)\,v] + C = t \tag{5}$$

$$v = \frac{1+\beta}{1+2kz} - C\exp[-(1+2kz)t] \tag{6}$$

where the parameter C is related to the initial conditions as

$$C = \frac{1+\beta}{1+2kz} - v_0 \tag{7}$$

3 The position

The velocity is $v = dx/dt$ (in non-reduced variables). Then

$$v^* = \frac{v}{v_d} = \frac{1}{v_d}\frac{dx}{dt} = \frac{1}{v_d\tau}\frac{dx}{dt^*} \tag{8}$$

Thus, the reduced position can be defined as $x^* = x/v_d\tau$. The reduced position as a function of t^* is (the "*" has been omitted for clarity reasons)

$$x = D + \frac{1+\beta}{1+2kz}\,t + \frac{C}{1+2kz}\exp[-(1+2kz)t] \tag{9}$$

where the parameter D is related to the initial conditions as

$$D = x_0 + \frac{v_0}{1+2kz} - \frac{1+\beta}{(1+2kz)^2} \tag{10}$$

4 Relation between position-velocity

Recalling de differential relation (4) and the definition $v = dx/dt$ (in reduced units), both magnitudes are related by

$$\frac{v\,dv}{(1+\beta) - (1+2kz)\,v} = dx \tag{11}$$

Therefore,

$$-\frac{v}{1+2kz}-\frac{1+\beta}{(1+2kz)^2}\,\ln[(1+\beta)-(1+2kz)\,v]+C=x \quad (12)$$

where the argument of $\ln(\cdot)$ is supposed to be positive. Otherwise, the modulus of the argument should be computed.

We can further introduce the stationary value for the velocity v from eq. (3) as

$$v_\infty = \frac{1+\beta}{1+2kz} \quad (13)$$

(in reduced units). Thus, eq. (11) reads

$$v + v_\infty \, \ln[v_\infty - v] + D = -(1+2kz)\,x \quad (14)$$

Notice that at the initial condition

$$D = -(1+2kz)\,x_0 - v_0 - v_\infty \, \ln[v_\infty - v_0] \quad (15)$$

and the eq. (14) can be expressed in terms of the initial and ending condition as

$$v - v_0 + v_\infty \, \ln\left[\frac{v_\infty - v}{v_\infty - v_0}\right] = -(1+2kz)\,(x-x_0) \quad (16)$$

5 The numerical solution

The numerial solution can be obtained from eq. (11).

$$\frac{v_{i+1}-v_i}{h} \simeq \frac{1+\beta}{v_i} - (1+2kz) \quad (17)$$

for $v(0)=v_0$. The small increment h represents the step width of the position. That is, $x_{i+1}=x_i+h$. The code for computing the velocity is as follows (in Matlab/Octave)

```
i=1;
n=10000;

x0=0;
v0=1;     % could also be  v0=0;
xmax=1;
h=xmax/n;

while i<n,
   v(i)=v0+h*(b/v0-a);
   x(i)=x0+h;
   v0=v(i);
   x0=x(i);
end
```

where

$$\begin{cases} 1+\beta & = \; 1+\sqrt{225} = 16 \\ \\ 1+2kz & = \; 1+274\,z \end{cases} \quad (18)$$

Notice that z is the (reduced) compression parameter $(2r-d)/B$. No further condition has been achieved till now. For example, the balance between the repulsive force and the bulk social force $\beta\,v_d$ gives

$$\frac{2r-d}{B} = \ln(\beta v_d/A) \quad (19)$$

where

$$\beta v_d = \frac{\sqrt{N}\,m}{\tau A}\,v_d \simeq v_d \quad (20)$$

Thus, the z parameter should vary at least between $0 \leq z \leq 2$. Notice. however, that z becomes relevant beyond $1/274 \simeq 0.003$.