Introduction to Statistical Methods in Political Science

Lecture 11: Large-Sample Inference for Means

Ignacio Urbina
Ph.D. Candidate in Political Science

Confidence Intervals for Means

Motivation: From Proportions to Means

Last week, we focused on inference for **population proportions (p)**.

- Confidence intervals and hypothesis tests for p using large-sample Z-procedures.

Now, we shift to **quantitative (continuous) data** where the parameter is the **population mean (μ)**.

- Examples: Average height, mean income, time.

Goal: Develop inference methods for μ using CLT and the sampling distribution of $\bar{x}$.

Foundation: The Central Limit Theorem (Recap)

For large samples ($n \geq 30$), the sampling distribution of $\bar{x}$ is approximately normal:

$$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

- Approximate normality holds regardless of the original population distribution.

Handling Unknown σ & The Z-Statistic

When σ is unknown, we estimate it using the sample standard deviation s (plug-in principle). The resulting test statistic:

$$Z = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Large-Sample CI for μ: Formula & Structure

Confidence Interval formula:

$$\bar{x} \pm z_{1-\frac{\alpha}{2}} \left(\frac{s}{\sqrt{n}} \right)$$

Structure:

- Point Estimate: $\bar{x}$
- Margin of Error: $z_{1-\frac{\alpha}{2}} \times \dfrac{s}{\sqrt{n}}$

Margin of Error for μ CI

Depends on:

1. Critical Value $z_{1-\frac{\alpha}{2}}$
2. Standard Error $SE = \frac{s}{\sqrt{n}}$

Thus, the Margin of Error (ME) is:

$$ME = z_{1-\frac{\alpha}{2}} \times SE$$

Interpretation of a Confidence Interval for μ

Example: 95% CI = (5.2, 6.8) Correct Interpretation:

- **"We are 95% confident that μ lies within (5.2, 6.8)."**

Example: CI for Average Commute Time (Problem Statement)

Problem: What is the average daily commute time for workers in a city?
Data: $n = 100$ workers, sample mean commute time $\bar{x} = 32.5$ minutes, sample standard deviation $s = 7.0$ minutes.
Task: Construct a 95% confidence interval for the true mean commute time.

Example: CI for Average Commute Time (Calculation)

Given:

- $n = 100$, $\bar{x} = 32.5$, $s = 7.0$
- 95% confidence level ($\alpha = 0.05$) $\Rightarrow z_{1-\frac{\alpha}{2}} = z_{0.975} = 1.96$

Standard error:

$$SE = \frac{7.0}{\sqrt{100}} = 0.7$$

Margin of error:

$$ME = 1.96 \times 0.7 = 1.372$$

Confidence Interval:

$$(32.5 - 1.372,\ 32.5 + 1.372) = (31.128,\ 33.872)$$

Foundation: Distribution of $\bar{x}_1 - \bar{x}_2$ (Recap)

For large independent samples:

$$\bar{x}_1 - \bar{x}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

Handling Unknown Variances & The Z-Statistic (Two Means)

Estimate population variances with sample variances:

$$s_1^2 \quad \text{and} \quad s_2^2$$

Standard error for $\bar{x}_1 - \bar{x}_2$:

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Use z-statistic for large samples.

Large-Sample CI for $\mu_1 - \mu_2$: Formula & Structure

Confidence Interval:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{1-\frac{\alpha}{2}} \times SE$$

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Interpretation of CI for $\mu_1 - \mu_2$

- Positive Interval: $\mu_1 > \mu_2$
- Interval Contains 0: No significant difference
- Negative Interval: $\mu_1 < \mu_2$

Example: CI for Comparing Study Methods (Problem Statement)

Problem: Do two different study methods lead to different average exam scores among university students?
Data: Method 1 ($n_1 = 50$, $\bar{x}_1 = 85.2$, $s_1 = 8.5$); Method 2 ($n_2 = 60$, $\bar{x}_2 = 81.5$, $s_2 = 9.1$).
Task: Construct a 99% confidence interval for $\mu_1 - \mu_2$.

Example: CI for Comparing Study Methods (Calculation)

Given:

- $n_1 = 50$, $\bar{x}_1 = 85.2$, $s_1 = 8.5$
- $n_2 = 60$, $\bar{x}_2 = 81.5$, $s_2 = 9.1$
- 99% confidence level $(\alpha = 0.01) \Rightarrow z_{1-\frac{\alpha}{2}} = z_{0.995} = 2.576$

Standard error:

$$SE = \sqrt{\frac{8.5^2}{50} + \frac{9.1^2}{60}} \approx 1.68$$

Margin of error:

$$ME = 2.576 \times 1.68 \approx 4.33$$

Confidence Interval:

$$(85.2 - 81.5) \pm 4.33 = 3.7 \pm 4.33$$

$$(-0.63,\ 8.03)$$

Hypothesis Testing for Means

Motivation: Hypothesis Tests for Means

Apply hypothesis testing to population **means (μ)**:

- Steps: Hypotheses, α, Test Statistic, Decision, Conclusion

Example: Average Hours of Sleep (Problem Statement)

Problem: Do college students sleep an average of 7 hours per night?
Data: $n = 64$, $\bar{x} = 6.8$ hours, $s = 0.6$ hours.
Task: Test $H_0 : \mu = 7$ against $H_a : \mu \neq 7$ at $\alpha = 0.05$.

Example: Average Hours of Sleep (Calculation)

Calculate standard error:

$$SE = \frac{0.6}{\sqrt{64}} = 0.075$$

Test statistic:

$$Z = \frac{6.8 - 7.0}{0.075} = -2.67$$

Critical value for a test with $\alpha = 0.05$: ± 1.96.
Since $|-2.67| > 1.96$, reject H_0.

Example: New Drug Effectiveness (Problem Statement)

Problem: Is a new drug more effective at reducing blood pressure than the current standard drug?
Data: Drug 1 ($n_1 = 120$, $\bar{x}_1 = 15.5$, $s_1 = 5.0$); Drug 2 ($n_2 = 100$, $\bar{x}_2 = 13.8$, $s_2 = 4.5$).
Task: Test $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_a : \mu_1 - \mu_2 > 0$ at $\alpha = 0.01$.

Example: New Drug Effectiveness (Calculation)

Calculate standard error:

$$SE = \sqrt{\frac{5.0^2}{120} + \frac{4.5^2}{100}} \approx 0.641$$

Test statistic:

$$Z = \frac{15.5 - 13.8}{0.641} \approx 2.65$$

Critical value: $z_{0.99} \approx 2.33$. Since $2.65 > 2.33$, reject H_0.

Hypothesis Tests with Nonzero Null Values

Previously, we tested $H_0 : \mu = 0$ or $H_0 : \mu_1 - \mu_2 = 0$.
Now, we allow the null hypothesis to state a nonzero value:

- $H_0 : \mu = \mu_0$ where $\mu_0 \neq 0$
- $H_0 : \mu_1 - \mu_2 = \Delta_0$ where $\Delta_0 \neq 0$

Interpretation: We are testing whether the population mean or the difference between two means equals a specific number.

Adjusted Z-Statistic Formulas

One Mean:

$$Z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where μ_0 is the null hypothesized mean (not necessarily 0).

Two Means:

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

where Δ_0 is the hypothesized difference (often 0, but not always).

Note: Subtract μ_0 or Δ_0 when forming the numerator.

Example: Average Calories Consumed (Problem Statement)

Problem: Do individuals consume an average of 2500 calories per day?
Data: $n = 49$, $\bar{x} = 2550$ calories, $s = 150$ calories.
Task: Test $H_0 : \mu = 2500$ against $H_a : \mu \neq 2500$ at $\alpha = 0.05$.

Example: Average Calories Consumed (Calculation)

Calculate standard error:

$$SE = \frac{150}{\sqrt{49}} = 21.43$$

Test statistic:

$$Z = \frac{2550 - 2500}{21.43} \approx 2.33$$

Critical value: ± 1.96.
Since $2.33 > 1.96$, reject H_0.

Hypothesis Tests for Two Means: Nonzero Difference

In many situations, we want to test whether the difference between two population means equals a value other than zero.
Examples:

- Testing if a new product improves scores by 5 points compared to an old product.
- Checking if two treatments differ by a clinically significant margin (e.g., 2 mmHg in blood pressure).

Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0$ where $\Delta_0 \neq 0$.

Adjusted Z-Statistic: Two Means

For large independent samples, the test statistic becomes:

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

where:

- Δ_0 is the hypothesized difference between means.
- $SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ is the standard error.

Key change: Subtract Δ_0 from the observed difference $(\bar{x}_1 - \bar{x}_2)$ in the numerator.

Example: Comparing Manufacturing Processes (Problem Statement)

Problem: Is there a difference of 5 units in mean output between two manufacturing processes?
Data: Process 1 ($n_1 = 40$, $\bar{x}_1 = 105$, $s_1 = 10$); Process 2 ($n_2 = 50$, $\bar{x}_2 = 98$, $s_2 = 12$).
Task: Test $H_0 : \mu_1 - \mu_2 = 5$ against $H_a : \mu_1 - \mu_2 \neq 5$ at $\alpha = 0.05$.

Example: Comparing Manufacturing Processes (Calculation)

Calculate standard error:

$$SE = \sqrt{\frac{10^2}{40} + \frac{12^2}{50}} \approx 2.32$$

Test statistic:

$$Z = \frac{(105 - 98) - 5}{2.32} \approx 0.43$$

Critical value: ± 1.96.
Since $0.43 < 1.96$, fail to reject H_0.

Large Sample Inference with Paired Samples

Paired Samples: When and Why?

Paired data arise when observations are naturally matched:

- Before-and-after measurements (e.g., pre- and post-treatment)
- Measurements on the same individual under two conditions. Hence, before and after data *are not independent*.

Key idea: Reduce the two measurements to a single difference score:

$$d_i = x_{i,1} - x_{i,2}$$

Then apply inference methods to the sample of differences. Key assumption: *Independence across individuals*.

Paired Samples: CI and Hypothesis Test

Confidence Interval for μ_d

For large samples of paired differences ($n \geq 30$), apply CLT:

$$\bar{d} \sim N\left(\mu_d, \frac{s_d^2}{n}\right)$$

CI formula:

$$\bar{d} \pm z_{1-\frac{\alpha}{2}} \times \frac{s_d}{\sqrt{n}}$$

- $\bar{d}$: Mean of differences
- s_d: Std. dev. of differences
- $z_{1-\frac{\alpha}{2}}$: Critical value

Hypothesis Test for μ_d

Test the null hypothesis:

$$H_0 : \mu_d = \mu_{d,0} \quad \text{vs.} \quad H_a : \mu_d \neq \mu_{d,0}$$

Z-statistic:

$$Z = \frac{\bar{d} - \mu_{d,0}}{s_d/\sqrt{n}}$$

- Compare Z to critical value or use p-value
- Reject H_0 if evidence is strong

Example: Effect of a Sleep Intervention

Problem: Does a cognitive-behavioral sleep intervention improve sleep duration among adults with mild insomnia?
Study Design: A sample of $n = 200$ individuals participated in a 4-week cognitive-behavioral therapy (CBT) program targeting sleep hygiene, routines, and stress reduction. Each participant recorded their average nightly sleep duration *before and after* the program.

Data: Mean difference $\bar{d} = 0.75$ hours, $s_d = 1.2$ hours, $n = 200$.
Task: Construct a 95% confidence interval for the mean increase in sleep duration, μ_d.

$$SE = \frac{1.2}{\sqrt{200}} \approx 0.085, \quad z_{0.975} = 1.96$$

$$ME = 1.96 \times 0.085 \approx 0.17, \quad CI = (0.75 \pm 0.17) = (0.58,\ 0.92)$$

Conclusion: We are 95% confident the CBT program increased sleep duration by 0.58 to 0.92 hours per night.

Summary: Key Formulas and Concepts

Summary Table

Scenario	Confidence Interval (CI)	Hypothesis Test (Z-Statistic)	Notes
One Mean (μ)	$\bar{x} \pm z_{1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$	$Z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	μ_0 can be 0 or nonzero
Two Means ($\mu_1 - \mu_2$)	$(\bar{x}_1 - \bar{x}_2) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$	$Z = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$	Δ_0 can be 0 or nonzero
Paired Samples (μ_d)	$\bar{d} \pm z_{1-\frac{\alpha}{2}} \frac{s_d}{\sqrt{n}}$	$Z = \frac{\bar{d} - \mu_{d,0}}{s_d/\sqrt{n}}$	Analyze differences $d_i = x_{i,1} - x_{i,2}$.

Important:

- Subtract μ_0, Δ_0, or $\mu_{d,0}$ when performing a hypothesis test.
- $\mu_d = 0$ equals no difference for the average individual difference.
- For confidence intervals, center at the sample statistic.
- $z_{1-\frac{\alpha}{2}}$ depends on the confidence level (e.g., 1.96 for 95%).