Probability - Solved Problems

Probability (Course Textbook's Chp. 3)

Problem 1: Basic Set Operations and Probability
Let the universal set be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define two events:

$$A = \{2, 4, 6, 8, 10\}, \quad B = \{1, 2, 3, 4, 5\}.$$

(a) Find $A \cup B$, $A \cap B$, and A^c.

(b) If an element is randomly selected from U, compute the probabilities $P(A)$, $P(B)$, $P(A \cup B)$, and $P(A \cap B)$.

Solution

(a) The union $A \cup B$ includes all elements that are in A, or B, or in both. Since A and B share the elements 2 and 4, the union thus includes all of A and B's unique elements:

$$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}.$$

The intersection $A \cap B$ includes only the elements that are both in A and B:

$$A \cap B = \{2, 4\}.$$

The complement A^c includes all elements in the universal set U that are not in A:

$$A^c = \{1, 3, 5, 7, 9\}.$$

(b) We compute the probabilities based on the ratio of the number of favorable outcomes (number of elements in the event set) to the total number of outcomes (elements in the universal set U):

$$P(A) = \frac{|A|}{|U|} = \frac{5}{10} = 0.5, \quad P(B) = \frac{|B|}{|U|} = \frac{5}{10} = 0.5.$$

$$P(A \cup B) = \frac{|A \cup B|}{|U|} = \frac{8}{10} = 0.8.$$

$$P(A \cap B) = \frac{|A \cap B|}{|U|} = \frac{2}{10} = 0.2.$$

Problem 2: Conditional Probability and Independence
Consider two events C and D such that $P(C) = 0.6$, $P(D) = 0.5$, and $P(C \cap D) = 0.3$.

(a) Find $P(C \cup D)$.

(b) Calculate $P(C|D)$ and interpret it.

(c) Are the events C and D independent? Justify your answer.

Solution

(a) The probability of either event C or event D occurring, or both, can be computed using the general addition rule:

$$P(C \cup D) = P(C) + P(D) - P(C \cap D) = 0.6 + 0.5 - 0.3 = 0.8.$$

(b) Conditional probability $P(C|D)$ is the probability of C given D has already occurred. It is calculated by:

$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{0.3}{0.5} = 0.6.$$

This means if event D has occurred, the likelihood of C happening is 0.6.

(c) Two events are independent if the outcome or occurrence of one does not affect the outcome or occurrence of another. We can check independence by verifying:

$$P(C \cap D) = P(C)P(D).$$

Given:

$$P(C)P(D) = 0.6 \times 0.5 = 0.3.$$

Since $P(C \cap D) = P(C)P(D)$, C and D are independent.

Problem 3: Law of Total Probability and Bayes' Theorem

Let the sample space S be partitioned into three events B_1, B_2, B_3, where:

$$P(B_1) = 0.3, \quad P(B_2) = 0.4, \quad P(B_3) = 0.3.$$

Suppose an event A occurs with the following conditional probabilities:

$$P(A|B_1) = 0.5, \quad P(A|B_2) = 0.2, \quad P(A|B_3) = 0.8.$$

(a) Use the law of total probability to find $P(A)$.

(b) Given that A has occurred, use Bayes' theorem to find $P(B_1|A)$.

Solution

(a) The law of total probability states:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3).$$

Substituting the values:

$$P(A) = (0.5)(0.3) + (0.2)(0.4) + (0.8)(0.3) = 0.15 + 0.08 + 0.24 = 0.47.$$

(b) Using Bayes' theorem to find $P(B_1|A)$:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{(0.5)(0.3)}{0.47} = \frac{0.15}{0.47} \approx 0.319.$$

Problem 4: Disjoint Events and Probability Rules

Let E_1 and E_2 be disjoint events with $P(E_1) = 0.25$ and $P(E_2) = 0.35$.

(a) Find $P(E_1 \cup E_2)$.

(b) What is the probability of $E_1^c \cap E_2^c$?

Solution

(a) Since E_1 and E_2 are disjoint, we can use the addition rule for disjoint events:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = 0.25 + 0.35 = 0.6.$$

(b) The complement of $E_1^c \cap E_2^c$ is $E_1 \cup E_2$. Also, we can use the rule $P(A) = 1 - P(A^c)$, which can be rearranged as $P(A^c) = 1 - P(A)$. Therefore:

$$P(E_1^c \cap E_2^c) = 1 - P(E_1 \cup E_2) = 1 - 0.6 = 0.4.$$

Problem 5: Probability with Marbles

A jar contains 5 red marbles, 7 blue marbles, and 3 green marbles. One marble is randomly selected.

(a) What is the probability of selecting a blue marble?

(b) What is the probability of selecting a red or a green marble?

(c) If a marble is selected and it is not blue, what is the probability that it is red?

Solution

Let the total number of marbles be $5 + 7 + 3 = 15$.

(a) The probability of selecting a blue marble is:

$$P(\text{blue}) = \frac{7}{15}.$$

(b) The probability of selecting a red or green marble (i.e., $P(\text{red} \cup \text{green})$) is:

$$P(\text{red} \cup \text{green}) = P(\text{red}) + P(\text{green}) = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}.$$

(c) The probability of selecting a red marble given that it is not blue (i.e., $P(\text{red}|\text{not blue})$) is calculated using conditional probability:

$$\begin{aligned} P(\text{red}|\text{not blue}) &= \frac{P(\text{red} \cap \text{not blue})}{P(\text{not blue})} \\ &= \frac{P(\text{red} \cap \{\text{red, green}\})}{P(\text{not blue})} \\ &= \frac{P(\text{red})}{P(\text{not blue})} \\ &= \frac{\frac{5}{15}}{\frac{8}{15}} = \frac{5}{8}. \end{aligned}$$

Problem 6: Rolling Two Dice

Two fair six-sided dice are rolled.

(a) What is the sample space for the sum of the two dice?

(b) What is the probability of getting a sum of 7?

(c) What is the probability of getting a sum of 7 or 11?

(d) If the outcome is greater than 7, what is the probability that the outcome is exactly 8?

Solution

(a) The sample space for rolling two dice consists of all pairs (x, y), where x and y are the results from rolling each die. There are $6 \times 6 = 36$ possible outcomes. The possible sums range from 2 (when both dice show 1) to 12 (when both dice show 6).
(b) The possible outcomes for a sum of 7 are: $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$. There are 6 outcomes, so:

$$P(\text{sum of 7}) = \frac{6}{36} = \frac{1}{6}.$$

(c) The possible outcomes for a sum of 11 are: $(5,6),(6,5)$. So:

$$P(\text{sum of 11}) = \frac{2}{36} = \frac{1}{18}.$$

Thus, the probability of getting a sum of 7 or 11 is:

$$P(\text{sum of 7 or 11}) = P(\text{sum of 7}) + P(\text{sum of 11}) = \frac{1}{6} + \frac{1}{18} = \frac{4}{18} = \frac{2}{9}.$$

(d) The condition is that the outcome is greater than 7. The possible sums greater than 7 are: 8, 9, 10, 11, and 12. The total number of favorable outcomes is:

$$(2,6),(3,5),(4,4),(5,3),(6,2) \quad (\text{sum of 8, 5 outcomes}),$$

$$(3,6),(4,5),(5,4),(6,3) \quad (\text{sum of 9, 4 outcomes}),$$

$$(4,6),(5,5),(6,4) \quad (\text{sum of 10, 3 outcomes}),$$

$$(5,6),(6,5) \quad (\text{sum of 11, 2 outcomes}),$$

$$(6,6) \quad (\text{sum of 12, 1 outcome}).$$

There are $5+4+3+2+1 = 15$ outcomes where the sum is greater than 7. The probability of getting a sum of exactly 8 given that the outcome is greater than 7 is:

$$\begin{aligned} P(\text{sum of 8} \mid \text{sum greater than 7}) &= \frac{P(\text{sum of 8})}{P(\text{sum greater than 7})} \\ &= \frac{5/36}{15/36} = \frac{5}{15} = \frac{1}{3}. \end{aligned}$$

Problem 7: Drawing Marbles with Replacement
A bag contains 4 red marbles and 6 white marbles. Two marbles are drawn randomly with replacement.

(a) What is the probability of drawing two red marbles?

(b) What is the probability of drawing one red and one white marble?

(c) Are the two events (first marble is red, second marble is white) independent? Justify your answer.

Solution

(a) Since the drawing is with replacement, the probability of drawing a red marble on each draw is independent. The probability of drawing two red marbles is:

$$\begin{aligned} P(\text{red and red}) &= P(\text{red on first draw}) \times P(\text{red on second draw}) \\ &= \frac{4}{10} \times \frac{4}{10} = \frac{16}{100} = 0.16. \end{aligned}$$

(b) The probability of drawing one red and one white marble (in any order) can be calculated using the multiplication rule:

$$P(\text{red then white}) = \frac{4}{10} \times \frac{6}{10} = \frac{24}{100} = 0.24,$$

$$P(\text{white then red}) = \frac{6}{10} \times \frac{4}{10} = \frac{24}{100} = 0.24.$$

Thus, the total probability of drawing one red and one white marble is:

$$\begin{aligned} P(\text{red and white}) &= P(\text{red then white}) + P(\text{white then red}) \\ &= 0.24 + 0.24 = 0.48. \end{aligned}$$

(c) The events "first marble is red" and "second marble is white" are independent because the probability of drawing the second marble is not affected by the first draw due to replacement. We check:

$$P(\text{second white} \mid \text{first red}) = P(\text{second white}) = \frac{6}{10}.$$

Since $P(\text{second white} \mid \text{first red}) = P(\text{second white})$, the events are independent.