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Assignment 3		**Matric:**	U1940558D
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Problem 0

c)
   i) The reason is to help him, when he gives the lecture, to speak like he would've spoken in the 60s.

   ii) The analysis of algorithm refers to the **quantitative** study of algorithms.

Problem 1

left-to-right minima: $\{9, 4, 1\}$
left-to-right maxima: $\{9, 14, 15\}$
right-to-left minima: $\{1, 2, 3, 7\}$
right-to-left maxima: $\{15, 7\}$

Problem 2

a) M is the number of right-to-left maxima of the corresponding permutation.

b) i) $N = 5$

- $k = 0$: 24
- $k = 1$: 50
- $k = 2$: 35
- $k = 3$: 10
- $k = 4$: 1

ii) The entry in row N and column k denotes the number of permutations of $\{1, 2, \dots, N\}$ with $A = k$, for $k \in \{0, 1, \dots, N-1\}$.

c) i) $h_N(u)$ is an ordinary generating function.

ii) Consider $\pi^N = \left(\pi_1^N, \pi_2^N, \dots, \pi_N^N\right) \in \mathcal{P}_N$. Consider the mapping,

$$\begin{aligned}
\pi^N &\to \left(\pi_1^N, \left(\pi_2^N - \mathbb{1}_{\pi_2^N > \pi_1^N}, \pi_3^N - \mathbb{1}_{\pi_3^N > \pi_1^N}, \dots, \pi_N^N - \mathbb{1}_{\pi_N^N > \pi_1^N}\right)\right) \\
&= \left(\pi_1^N, \pi^{N-1}\right) \\
&\in \{1, 2, \dots, N\} \times \mathcal{P}_{N-1}
\end{aligned}$$

which is a bijective mapping.

iii) We note that $h_2(u) = 1 + u$. Then, we have,

$$\begin{aligned}
h_N(u) &= \sum_{\pi^N \in \mathcal{P}_N} u^{A(\pi)} \\
&= \sum_{\pi^N \in \mathcal{P}_N} u^{A\left(\left(\pi_1^N, \pi_2^N, \dots, \pi_N^N\right)\right)} \\
&= \sum_{\pi_1^N \in [N]} \sum_{\pi^{N-1} \in \mathcal{P}_{N-1}} u^{A\left(\left(\pi_1^N, \pi_1^{N-1} + \mathbb{1}_{\pi_1^{N-1} > \pi_1^N}, \pi_2^{N-1} + \mathbb{1}_{\pi_2^{N-1} > \pi_1^N}, \dots, \pi_1^{N-1} + \pi_{N-1}^{N-1} + \mathbb{1}_{\pi_{N-1}^{N-1} > \pi_1^N}\right)\right)} \\
&= \sum_{\pi_1^N \in [N-1]} \sum_{\pi^{N-1} \in \mathcal{P}_{N-1}} u^{A\left(\pi^{N-1}\right)} + \sum_{\pi^{N-1} \in \mathcal{P}_{N-1}} u^{1 + A\left(\pi^{N-1}\right)} \\
&= (N-1)\, h_{N-1}(u) + u h_{N-1}(u) \\
&= (N - 1 + u)\, h_{N-1}(u) \\
&= \prod_{i=1}^{N-1} (i + u)
\end{aligned}$$

where the last equality comes from solving the recurrence relation in the second last equality.

We can now use $h_N(u)$ to solve for $g_N(u)$,

$$\begin{aligned} g_N(u) &= \sum_{\pi^N \in \mathcal{P}_N} \frac{1}{N!} u^{A(\pi)} \\ &= \frac{1}{N!} \sum_{\pi^N \in \mathcal{P}_N} u^{A(\pi)} \\ &= \frac{1}{N!} h_N(u) \\ &= \frac{1}{N!} \prod_{i=1}^{N-1} (i+u) \end{aligned}$$

d) i) $X = 1$

ii) Assume $G(z) = f(z)g(z)$ and that $f(1) = g(1) = 1$.

$$\begin{aligned} G'(1) &= f'(1)g(1) + f(1)g'(1) \\ G'(1) &= f'(1) + g'(1) \end{aligned}$$

iii)

$$\begin{aligned} g_N(z) &= \frac{(1+z)(2+z)\dots(N-1+z)}{N!} \\ \Rightarrow g_N^{'}(z) &= \frac{1}{N!} \sum_{i=1}^{N-1} \frac{(1+z)(2+z)\dots(N-1+z)}{i+z} \\ \Rightarrow g_N^{'}(1) &= \sum_{i=1}^{N-1} \frac{1}{i+1} \\ &= H_N - 1 \end{aligned}$$

iv) Since there exists an involution (which is, by nature, a bijection), $f : \mathcal{P}_N \to \mathcal{P}_N$, such that $\mathrm{lrm}(\pi) = M(f(\pi))$, $\pi \in \mathcal{P}_N$, we have

$$\begin{aligned} \mathbb{E}_{\pi \in \mathcal{P}_N}[lrm(\pi)] &= \mathbb{E}_{\pi \in \mathcal{P}_N}[M(f(\pi))] \\ &= \mathbb{E}_{\pi \in \mathcal{P}_N}[M(\pi)] \\ &= \mathbb{E}_{\pi \in \mathcal{P}_N}[1 + A(\pi)] \\ &= 1 + g_N^{'}(1) \\ &= H_N \end{aligned}$$

Problem 3

a) $\texttt{minimum\_updates} = 23$ and $\texttt{compares} = 105$.

b) Using the code snippet,

$$\begin{aligned} \texttt{compares} &= \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} 1 \\ &= \frac{N(N-1)}{2} \end{aligned}$$

c) Since $\texttt{minimum\_updates}$ is updated only when there is a change in the running minimum, and in the first iteration we iterate over the entire array, $\texttt{minimum\_updates} = \text{lrm}(\pi) - 1$.

d) Using the result from part iv) of part d) of Problem 2,

$$\begin{aligned} E_N &= \mathbb{E}_{\pi \in \mathcal{P}_N} [\texttt{minimum\_updates}] \\ &= \mathbb{E}_{\pi \in \mathcal{P}_N} [\text{lrm}(\pi) - 1] + \mathbb{E}_{\pi \in \mathcal{P}_{N-1}} [\texttt{minimum\_updates}] \\ &= H_N - 1 + E_{N-1} \\ \Rightarrow E_N - E_{N-1} &= H_N - 1 \end{aligned}$$

Now, since $E_1 = 0$,

$$\begin{aligned} \Rightarrow E_N &= \sum_{i=2}^{N} \sum_{j=2}^{i} \frac{1}{j} \\ &= \sum_{i=2}^{N} (H_i - 1) \\ &= \sum_{i=1}^{N} (H_i - 1) \\ &= \sum_{i=1}^{N} H_i - N \\ &= (N+1)(H_{N+1} - 1) - N \\ &= (N+1) H_N - 2N \end{aligned}$$

where we use the result from exercise 3.4. to get the second last equality.