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Assignment 1		**Matric:**	U1940558D
30 August 2022		**Collaborators:**	Tan Hui Ru Kimberley

Problem 1

(a)

$$A(N) = A(n-1) + A(n-2) + 1, \text{ where } n \geq 2$$

This is a linear non-homogenous recurrence equation. We solve, first, for the homogenous part, and then for the particular solution.

- The characteristic equation for the homogenous part is $x^2 - x - 1 = 0$, which has roots ϕ and $-1/\phi$, where
$$\phi = \frac{1+\sqrt{5}}{2}$$
Hence, the homogenous solution is of the form
$$a_h(N) = X\phi^N + Y\left(\frac{-1}{\phi}\right)^N, \text{ where } X, Y \in \mathbb{R}$$
- The particular solution is of the form
$$a_p(N) = Z, \text{ where } Z \in \mathbb{R}$$

Combining the two, we get the general solution of the form,

$$\begin{aligned} A(N) &= A_h(N) + A_p(N) \\ &= X\phi^N + Y\left(\frac{-1}{\phi}\right)^N + Z \end{aligned}$$

We can solve for the constants with the initial conditions,

$$\begin{aligned} A(0) &= 1 \\ A(1) &= 1 \\ A(2) &= 3 \end{aligned}$$

and get the general solution of the form,

$$\begin{aligned} A(N) &= \left(1 + \frac{1}{\sqrt{5}}\right)\phi^N + \left(1 - \frac{1}{\sqrt{5}}\right)\left(\frac{-1}{\phi}\right)^N - 1 \\ &= \left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^N + \left(1 - \frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^N - 1 \\ &= \frac{2}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{N+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{N+1}\right) - 1 \\ &= \frac{2}{\sqrt{5}}\left(\phi^{N+1} - \left(\frac{-1}{\phi}\right)^{N+1}\right) - 1 \end{aligned}$$

(b) Let $b : \mathbb{N}_0 \to \mathbb{N}_0$ be defined such that $b(n)$ is the number of additional accesses when $i = n$ in the loop, so that

$$B(N) = \sum_{n=0}^{N} b(n)$$

Clearly,

$$b(n) = \begin{cases} 1, & n = 0 \text{ or } n = 1 \\ 3, & n \geq 2 \end{cases}$$
$$\Rightarrow B(N) = \begin{cases} 1, & N = 0 \\ 3n - 1, & N \geq 1 \end{cases}$$

(c)

$$\begin{aligned}
\lim_{N\to\infty} \frac{B(N)}{A(N)} &= \lim_{N\to\infty} \frac{3N-1}{\frac{2}{\sqrt{5}}\left(\phi^{N+1} - \left(\frac{-1}{\phi}\right)^{N+1}\right) - 1} \\
&\leq \lim_{N\to\infty} \frac{3N-1}{\frac{2}{\sqrt{5}}\left(\phi^{N+1} - \left(\frac{1}{\phi}\right)^{N+1}\right) - 1} \\
&= \frac{3\sqrt{5}}{2\ln\phi} \lim_{N\to\infty} \frac{1}{\phi^{N+1} + \left(\frac{1}{\phi}\right)^{N+1}} \\
&\leq \frac{3\sqrt{5}}{2\ln\phi} \lim_{N\to\infty} \left(\frac{1}{\phi}\right)^{N+1} \\
&= 0
\end{aligned}$$

Also, since $A(N)$ and $B(N)$ are clearly positive (from their recurrence construction),

$$\frac{B(N)}{A(N)} > 0,\ \forall N \in \mathbb{N}_0$$

From here, we can use squeeze theorem to conclude that,

$$\lim_{N\to\infty} \frac{B(N)}{A(N)} = 0$$

(d) Since the time to access an array and time to call a routine are operations that don't scale with the parameter N, and are bounded, we can express running times of routines A and B, R_A and R_B, respectively, as

$$R_A = \Theta(A)$$
$$R_B = \Theta(B)$$

where $f = \Theta(g)$ means that f can be bounded by affine transforms of g. This includes the running time for arithmetic operations like expression comparison and addition. We can write,

$$a_1 A + m_1 \leq R_A \leq a_2 A + m_2$$
$$b_1 B + n_1 \leq R_B \leq b_2 B + n_2$$

where $a_i, b_i, m_i, n_i \in \mathbb{R}^+$, $i \in \{1, 2\}$. Here, a_i is the cost of calling a subroutine, b_i is the cost of accessing an array, and m_i and n_i are the total costs of arithmetic and other auxiliary operations in the two routines

With this we can rephrase Albert's statement as

> "Well, yes, $A(N) > B(N)$ when N $\geq$ 5. However, $A(N)$ and $B(N)$ measure the number of calls and accesses, respectively, and not the actual running time. It is possible that the upper bound for the cost of calling a routine, a_2, is much lower than the lower bound for the cost of accessing an array, b_1. So, even though the number of accesses is small, the actual running time is very long. It is possible that $R_A(N) < R_B(N), \forall N \in \mathbb{N}$."

Now, if we compare the running time of the algorithms in the limit,

$$\lim_{N\to\infty} \frac{b_1 B(N) + n_1}{a_2 A(N) + m_2} \leq \lim_{N\to\infty} \frac{R_B(N)}{R_A(N)} \leq \lim_{N\to\infty} \frac{b_2 B(N) + n_2}{a_1 A(N) + m_1}$$
$$0 \leq \lim_{N\to\infty} \frac{R_B(N)}{R_A(N)} \leq 0$$

Therefore, Albert's statement is wrong - since the cost of each of the discussed operations is bounded and independent of the parameter N, asymptotically, Routine A would be slower than Routine B. Hence, his statement cannot hold $\forall N \in \mathbb{N}$.

(e)

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= F_0 + F_1 \\
F_3 &= F_1 + F_2 \\
&\vdots \\
F_N &= F_{N-1} + F_{N-2} \\
&\vdots \\
\Rightarrow G(z) &= \sum_{i\in\mathbb{N}_0} F_i z^i = z + F_0 z^2 + \sum_{i\in\mathbb{N}} F_i (z^{i+1} + z^{i+2}) \\
&= z + G(z)z + G(z)z^2 \\
\Rightarrow G(z) &= \frac{z}{1 - z - z^2}
\end{aligned}$$

(f) To get the sum of coefficients, we can do

$$\begin{aligned}
[z^N]\frac{1}{1-z}G(z) &= [z^N]\frac{z}{(1-z)(1-z-z^2)} \\
&= \sum_{i=0}^{N} F_i
\end{aligned}$$

Also, we have

$$
\begin{aligned}
[z^N]\frac{G(z) - F_0 - F_1 z}{z^2} &= F_{N+2} \\
[z^N]\left(\frac{G(z) - F_0 - F_1 z}{z^2} - \frac{1}{1-z}\right) &= F_{N+2} - 1
\end{aligned}
$$

The OGF for $F_{N+2} - 1$ is

$$
\begin{aligned}
\frac{G(z) - F_0 - F_1 z}{z^2} - \frac{1}{1-z} &= \frac{1+z}{1-z-z^2} - \frac{1}{1-z} \\
&= \frac{z}{(1-z)(1-z-z^2)}
\end{aligned}
$$

Since the OGF for the two sequences are equal, $\forall N \in \mathbb{N}_0$ we have

$$
F_0 + F_1 + \ldots + F_N = F_{N+2} - 1
$$

(g) Modification of Routine B to calculate the sum would use $M(N) = N + 1 + B(N) = 4N$ queries from the array. We say that a Routine X makes *significantly fewer* queries as compared to Routine B if the asymptotic ratio of these numbers is less than 1, i.e,

$$
\lim_{n\to\infty} \frac{X(N)}{M(N)} < 1
$$

We define this new Routine X, using part (f), as

```
def X(n):
    F = [0 for i in range (n+3)]
    access = 0
    for i in range(1, n+3):
        if i == 1:
            F[i] = 1
            access += 1
        else:
            F[i] = F[i-1] + F[i-2]
            access += 3
    return F[n+2] - 1
```

Solving for $X(N)$, we get $X(N) = 3N + 5$, and this implies,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{X(N)}{M(N)} &= \lim_{n\to\infty} \frac{3N+5}{4N} \\
&= \frac{3}{4} \\
&< 1
\end{aligned}
$$