MH4501
Assignment 1
2 February 2022

Name: Singh Jasraj
Matric: U1940558D
Collaborators: N.A.

Problem 1

a)

$$
\begin{aligned}
\overline{x}_{n+1} &= \frac{1}{n+1}\sum_{i=1}^{n+1} x_i \\
&= \overline{x}_n - \frac{\sum_{i=1}^{n} x_i}{n\left(n+1\right)} + \frac{x_{n+1}}{n+1} \\
&= \overline{x}_n + \frac{x_{n+1} - \overline{x}_n}{n+1} \\
&= \overline{x}_n + \frac{d_{n+1}}{n+1} \\
Q_{n+1} &= \sum_{i=1}^{n+1} \left(x_i - \overline{x}_{n+1}\right)\left(x_i - \overline{x}_{n+1}\right)^T \\
&= \sum_{i=1}^{n+1} \left(x_i - \overline{x}_n - \frac{d_{n+1}}{n+1}\right)\left(x_i - \overline{x}_n - \frac{d_{n+1}}{n+1}\right)^T \\
&= Q_n - \frac{1}{n+1}\left(\cancelto{0}{\sum_{i=1}^{n} (x_i - \overline{x}_n)} d_{n+1}^T + d_{n+1}\cancelto{0}{\sum_{i=1}^{n} (x_i - \overline{x}_n)^T}\right) \\
&\quad + \left(\frac{1}{n+1}\right)^2 \sum_{i=1}^{n} d_{n+1}d_{n+1}^T + \left(1 - \frac{1}{n+1}\right)^2 d_{n+1}d_{n+1}^T \\
&= Q_n + \left(1 - \frac{1}{n+1}\right) d_{n+1}d_{n+1}^T
\end{aligned}
$$

□

b) We start by noting that,

$$
\begin{aligned}
d_n &= x_n - \overline{x}_{n-1} \\
\Rightarrow \left(n-1\right) d_n &= \left(n-1\right) x_n - \sum_{i=1}^{n-1} x_i \\
&= n x_n - \sum_{i=1}^{n} x_i \\
&= n\left(x_n - \overline{x}_n\right) \\
&= n e_n
\end{aligned}
$$

Now, we use part (a) and substitute in the result above,

$$\begin{aligned}
\overline{x}_{n-1} &= \overline{x}_n - \frac{d_n}{n} \\
&= \overline{x}_n - \frac{e_n}{n-1} \\
Q_{n-1} &= Q_n - \left(1 - \frac{1}{n}\right) d_n d_n^T \\
&= Q_n - \left(1 + \frac{1}{n-1}\right) e_n e_n^T
\end{aligned}$$

□

Problem 2

Let $Y = (Y_1, \ldots, Y_n) \in \mathbb{R}^{n\times 1}$, $A = (a_{ij})_{n\times n}$, and $\Sigma = (\sigma_{ij})_{n\times n}$. Then,

$$\begin{aligned}
\mathbb{E}\left[Y^T A Y\right] &= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} Y_i a_{ij} Y_j\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\mathbb{E}\left[Y_i Y_j\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(\sigma_{ij} + \mu_i\mu_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}\sigma_{ij} + \mu_i a_{ij}\mu_j\right) \\
&= \text{tr}\left(A\Sigma\right) + \mu^T A \mu
\end{aligned}$$

□

Problem 3

a) With $B = U\Sigma U^T$, where U is orthogonal, i.e., $UU^T = I_p$, and $\Sigma = \text{diag}_{p\times p}\left(\lambda_1, \ldots, \lambda_p\right)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$,

$$\max_{x\neq 0}\frac{x^T B x}{x^T x} = \max_{x\neq 0}\frac{x^T U\Sigma U^T x}{x^T U U^T x}$$

Since U is invertible, it is full rank, and so,

$$\begin{aligned}
\max_{x\neq 0}\frac{x^T U\Sigma U^T x}{x^T U U^T x} &= \max_{y\neq 0}\frac{y^T\Sigma y}{y^T y} \\
&= \max_{y\neq 0}\frac{\lambda_1 y_1^2 + \ldots + \lambda_p y_p^2}{y_1^2 + \ldots + y_p^2} \\
&\leq \max_{y\neq 0}\frac{\lambda_1 y_1^2 + \ldots + \lambda_1 y_p^2}{y_1^2 + \ldots + y_p^2} \\
&= \lambda_1
\end{aligned}$$

and this value is achieved for

$$y = \begin{pmatrix}1 & 0 & \ldots & 0\end{pmatrix}^T$$

The corresponding value of x is given as

$$\begin{aligned} y &= U^T x \\ \Rightarrow x &= Uy = u_1 \end{aligned}$$

$\square$

b)

$$\begin{aligned} \det(A - \lambda I_3) &= \det\left(\begin{pmatrix} 13-\lambda & -4 & 2 \\ -4 & 13-\lambda & -2 \\ 2 & -2 & 10-\lambda \end{pmatrix}\right) \\ &= 13\lambda\left[(13-\lambda)(10-\lambda) - 4\right] + 4\left[-4(10-\lambda) + 4\right] + 2\left[8 - 2(13-\lambda)\right] \\ &= -(\lambda - 9)^2(\lambda - 18) \end{aligned}$$

Since the eigenvalues of A are positive, it is a positive-definite matrix, and we can use the result from part (a). The normalized eigenvector corresponding to $\lambda_1 = 18$ is

$$u_1 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

and hence,

$$\begin{aligned} \max_{x \neq 0} \frac{x^T A x}{x^T x} &= \frac{u_1^T A u_1}{u_1^T u_1} \\ &= \lambda_1 \\ &= 18 \end{aligned}$$

Problem 4

a)

- Case 1: $x < -1$

$$\begin{aligned} \{X_2 \leq x\} &= \{X_1 \leq x\} \\ \Rightarrow \mathbb{P}(X_2 \leq x) &= \mathbb{P}(X_1 \leq x) \end{aligned}$$

- Case 2: $-1 \leq x \leq 1$

$$\begin{aligned} \{X_2 \leq x\} &= \{X_2 < -1\} \cup \{-1 \leq X_2 \leq x\} \\ &= \{X_1 < -1\} \cup \{x \leq X_1 \leq 1\} \\ \Rightarrow \mathbb{P}(X_2 \leq x) &= \mathbb{P}(X_1 < -1) + \mathbb{P}(x \leq X_1 \leq 1) \\ &= \mathbb{P}(X_1 < -1) + \mathbb{P}(-1 \leq X_1 \leq x) \text{ by symmetry of } X_1 \text{ about } 0 \\ &= \mathbb{P}(X_1 \leq x) \end{aligned}$$

- Case 3: $x > 1$

$$\begin{aligned} \{X_2 > x\} &= \{X_1 > x\} \\ \Rightarrow \mathbb{P}(X_2 > x) &= \mathbb{P}(X_1 > x) \\ \Rightarrow \mathbb{P}(X_2 \leq x) = 1 - \mathbb{P}(X_2 > x) &= 1 - \mathbb{P}(X_1 > x) = \mathbb{P}(X_1 \leq x) \end{aligned}$$

$\square$

b) Assume (X_1, X_2) has a bivariate normal distribution. Then, by definition, all linear combination of X_1 and X_2 would follow a univariate normal distribution. Consider $Y = X_1 - X_2$. We have,

$$\begin{aligned}
\mathbb{P}\left(Y=0\right) &= \mathbb{P}\left(X_2 - X_1 = 0\right) \\
&= \mathbb{P}\left(X_1 < -1\right) + \mathbb{P}\left(X_1 > 1\right) \\
&= 2\Phi\left(-1\right) \\
&\approx 0.3174 \\
&\neq 0
\end{aligned}$$

implying that Y does not have a standard normal distribution, since point probabilities everywhere would be 0 in such a case. Hence, our assumption that (X_1, X_2) has a bivariate normal distribution is contradicted.

□

Problem 5

a) Since X is a multivariate normal, any linear combination of the components is normally distributed. Consider $A = \begin{pmatrix} 3 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{1\times 3}$. We want to find the distribution of $Y = AX$,

$$\begin{aligned}
\mu_Y &= A\mu \\
&= 13 \\
\Sigma_Y &= A\Sigma A^T \\
&= 9 \\
\Rightarrow Y &\sim \mathcal{N}\left(13, 9\right)
\end{aligned}$$

b) Consider $a = \begin{pmatrix} a_1 & a_3 \end{pmatrix}^T$, and define,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_3 \end{pmatrix} \in \mathbb{R}^{2\times 3}$$

Define

$$\begin{aligned}
Y &= AX \\
&= \begin{pmatrix} X_2 \\ X_2 - a^T \begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \end{pmatrix}
\end{aligned}$$

Obviously, Y has a bivariate normal distribution, and

$$\begin{aligned}
\Sigma_Y &= A\Sigma A^T \\
&= \begin{pmatrix} 3 & 3 - a_1 - 2a_3 \\ 3 - a_1 - 2a_3 & a_1^2 - 2a_1 + 2a_3^2 - 4a_3 + 2a_1a_3 + 3 \end{pmatrix}
\end{aligned}$$

For the components of Y to be independent, its covariance matrix must be diagonal. Consequently, we can choose a such that $a_1 + 2a_3 = 3$. One example could be $a = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$, in which case,

$$\begin{aligned}\mu_Y &= A\mu \\ &= \begin{pmatrix}-3 \\ -6\end{pmatrix} \\ \Sigma_Y &= A\Sigma A^T \\ &= \begin{pmatrix}3 & 0 \\ 0 & 2\end{pmatrix}\end{aligned}$$