MH4501 **Name:** Singh Jasraj
Assignment 3 **Matric:** U1940558D
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Problem 1

a) We can use the eigenvalue decomposition of Σ,

$$\Sigma = U\Lambda U^T = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 11 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix},$$

to see that

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} := U^T X = \begin{pmatrix} \frac{1}{2}(X_1 + X_2 + X_3 + X_4) \\ \frac{1}{2}(X_1 + X_2 - X_3 - X_4) \\ \frac{1}{2}(X_1 - X_2 + X_3 - X_4) \\ \frac{1}{2}(X_1 - X_2 - X_3 + X_4) \end{pmatrix}$$

b) With $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, we have

$$\min\left(\left\{ i \in \{1,2,3,4\} : \frac{\sum_{j=1}^{i} \lambda_j}{\sum_{j=1}^{4} \lambda_j} \geq 0.85 \right\}\right) = 3$$

The first 2 components explain 80% of the total variance and the first 3 components explain 95% of the total variance.

c) With $U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}$, we have

$$\rho(Y_3, X_1) = \frac{u_{31}\sqrt{\lambda_3}}{\sqrt{\sigma_{11}}} = \frac{\sqrt{3}}{2\sqrt{5}}$$
$$\rho(Y_3, X_2) = \frac{u_{32}\sqrt{\lambda_3}}{\sqrt{\sigma_{22}}} = \frac{-\sqrt{3}}{2\sqrt{5}}$$
$$\rho(Y_3, X_3) = \frac{u_{33}\sqrt{\lambda_3}}{\sqrt{\sigma_{33}}} = \frac{\sqrt{3}}{2\sqrt{5}}$$
$$\rho(Y_3, X_4) = \frac{u_{34}\sqrt{\lambda_3}}{\sqrt{\sigma_{44}}} = \frac{-\sqrt{3}}{2\sqrt{5}}$$

Problem 2

a) Using the eigenvalue decomposition of Σ, we have

$$\Sigma := VDV^T = \begin{pmatrix} 1.000198 & 0.627514 & 0.447809 \\ 0.627514 & 0.999702 & 0.348468 \\ 0.447809 & 0.348468 & 0.998335 \end{pmatrix}, \quad \text{where}$$

$$V = \begin{pmatrix} 0.625 & -0.219 & 0.749 \\ 0.593 & -0.491 & -0.638 \\ 0.507 & 0.843 & -0.177 \end{pmatrix}, \quad \text{and}$$

$$D = \begin{pmatrix} 1.96 & 0 & 0 \\ 0 & 0.68 & 0 \\ 0 & 0 & 0.36 \end{pmatrix}.$$

The approximation for the loading matrix using Principal Component Analysis is given as

$$\tilde{L} := V_1 D_1^{1/2} = \begin{pmatrix} 0.8750 \\ 0.8302 \\ 0.7098 \end{pmatrix}, \quad \text{where}$$

$$V_1 = \begin{pmatrix} 0.625 \\ 0.593 \\ 0.507 \end{pmatrix}, \quad \text{and}$$

$$D_1 = \left(\sqrt{1.96}\right).$$

Correspondingly, we calculate the approximation for matrix of specific variances using Principal Component Analysis as

$$\tilde{\Phi} := \operatorname{diag}\left(\Sigma - \tilde{L}\tilde{L}^T\right) = \begin{pmatrix} 0.234573 & 0 & 0 \\ 0 & 0.310470 & 0 \\ 0 & 0 & 0.494519 \end{pmatrix}$$

b) Communalities are given as

$$h_1^2 = 0.8750^2 = 0.765625$$
$$h_2^2 = 0.8302^2 = 0.689232$$
$$h_3^2 = 0.7098^2 = 0.503816$$

c)

$$\frac{\lambda_3}{\sigma_{11} + \sigma_{22} + \sigma_{33}} \frac{1.96}{3} = 0.6533$$

d)

$$\operatorname{Corr}(Z_1, F_1) = l_{11} = 0.8750$$
$$\operatorname{Corr}(Z_2, F_1) = l_{21} = 0.8302$$
$$\operatorname{Corr}(Z_3, F_1) = l_{31} = 0.7098$$

Problem 3

Consider the matrix of factor loadings, L, and the rotation matrix, G,

$$L = (l_{ij})_{5\times 2} = \begin{pmatrix} 0.115 & 0.755 \\ 0.322 & 0.788 \\ 0.182 & l_{32} \\ 1.000 & 0 \\ 0.683 & l_{52} \end{pmatrix},$$

$$G = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Now, we have that $LG = L^*$,

$$LG = \begin{pmatrix} 0.115\cos\theta + 0.755\sin\theta & -0.115\sin\theta + 0.755\cos\theta \\ 0.322\cos\theta + 0.788\sin\theta & -0.322\sin\theta + 0.788\cos\theta \\ 0.182\cos\theta + l_{32}\sin\theta & -0.182\sin\theta + l_{32}\cos\theta \\ \cos\theta & -\sin\theta \\ 0.683\cos\theta + l_{52}\sin\theta & -0.683\sin\theta + l_{52}\cos\theta \end{pmatrix} = \begin{pmatrix} l_{11}^* & 0.024 \\ l_{21}^* & 0.227 \\ l_{31}^* & 0.104 \\ 0.118 & l_{42}^* \\ 0.113 & l_{52}^* \end{pmatrix} = L^*$$

Using this, we have that $\theta = 1.452520$ or $\theta = 4.830664$. Using that l_{32} and l_{52} are positive, we have

$$\begin{aligned} \theta &= \arccos(0.118) = 1.452520 \\ l_{32} &= \frac{0.104 + 0.182\sin\theta}{\cos\theta} = 2.412953 \\ l_{52} &= \frac{0.113 - 0.683\cos\theta}{\sin\theta} = 0.032634 \end{aligned}$$

This gives

$$LG = \begin{pmatrix} 0.763295 & -0.024 \\ 0.820490 & -0.227 \\ 2.417571 & 0.104 \\ 0.118 & -0.993013 \\ 0.113 & -0.674377 \end{pmatrix}$$

But $(LG)_{12} \neq l_{12}^*$ and $(LG)_{22} \neq l_{22}^*$. Furthermore, $l_{42}^* < 0$ and $l_{52}^* < 0$. So, we try with a rotation plus reflection matrix,

$$G = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Now, we do the same calculations as before with this new matrix,

$$LG = \begin{pmatrix} 0.115\cos\theta + 0.755\sin\theta & 0.115\sin\theta - 0.755\cos\theta \\ 0.322\cos\theta + 0.788\sin\theta & 0.322\sin\theta - 0.788\cos\theta \\ 0.182\cos\theta + l_{32}\sin\theta & 0.182\sin\theta - l_{32}\cos\theta \\ \cos\theta & \sin\theta \\ 0.683\cos\theta + l_{52}\sin\theta & 0.683\sin\theta - l_{52}\cos\theta \end{pmatrix} = \begin{pmatrix} l_{11}^* & 0.024 \\ l_{21}^* & 0.227 \\ l_{31}^* & 0.104 \\ 0.118 & l_{42}^* \\ 0.113 & l_{52}^* \end{pmatrix} = L^*$$

Using this, we have that $\theta = 1.452520$ or $\theta = 4.830664$. Using that l_{32} and l_{52} are positive, we have

$$\begin{aligned}
\theta &= \arccos(0.118) = 1.452520 \\
l_{32} &= \frac{-0.104 + 0.182\sin\theta}{\cos\theta} = 0.650241 \\
l_{52} &= \frac{0.113 - 0.683\cos\theta}{\sin\theta} = 0.032634
\end{aligned}$$

This gives

$$LG = \begin{pmatrix} 0.763295 & 0.024 \\ 0.820490 & 0.227 \\ 0.667174 & 0.104 \\ 0.118 & 0.993013 \\ 0.113 & 0.674377 \end{pmatrix}$$

and we can compare this with L^* to get

$$\begin{aligned}
l_{11}^* &= 0.763295 \\
l_{21}^* &= 0.820490 \\
l_{31}^* &= 0.667174 \\
l_{42}^* &= 0.993013 \\
l_{52}^* &= 0.674377
\end{aligned}$$

Since multiplication with an orthogonal matrix does not change the proportion of cumulative variance explained,

$$\begin{aligned}
p_1 &= 0.352 \\
p_2 &= 0.653
\end{aligned}$$

Now, we know that communality remains unchanged after rotation. Hence, we have

$$\begin{aligned}
\tilde{h}_1^2 &= \tilde{h}_1^{*2} = l_{11}^2 + l_{12}^2 = 0.583250 \\
\tilde{h}_2^2 &= \tilde{h}_2^{*2} = l_{21}^2 + l_{22}^2 = 0.724628 \\
\tilde{h}_3^2 &= \tilde{h}_3^{*2} = l_{31}^2 + l_{32}^2 = 0.455937 \\
\tilde{h}_4^2 &= \tilde{h}_4^{*2} = l_{41}^2 + l_{42}^2 = 1 \\
\tilde{h}_5^2 &= \tilde{h}_5^{*2} = l_{51}^2 + l_{52}^2 = 0.467553
\end{aligned}$$

Also, we know that specific variance remains unchanged after rotation. Hence, we have

$$\begin{aligned}
\tilde{\phi}_1^2 &= \tilde{\phi}_1^{*2} = 1 - \tilde{h}_1^2 = 0.416750 \\
\tilde{\phi}_2^2 &= \tilde{\phi}_2^{*2} = 1 - \tilde{h}_2^2 = 0.275372 \\
\tilde{\phi}_3^2 &= \tilde{\phi}_3^{*2} = 1 - \tilde{h}_3^2 = 0.544062 \\
\tilde{\phi}_4^2 &= \tilde{\phi}_4^{*2} = 1 - \tilde{h}_4^2 = 0 \\
\tilde{\phi}_5^2 &= \tilde{\phi}_5^{*2} = 1 - \tilde{h}_5^2 = 0.532446
\end{aligned}$$

Problem 4

a)

```
# read the csv
iris = read.csv('iris.csv', sep=',', header=TRUE)
# remove the species column
newiris = iris[, (names(iris) != 'Species')]
```

b)

```
# perform pca
irpc = prcomp(newiris, scale=TRUE)
# print result
print(irpc)
```

c) The principal components stored in `irpc` are the same as the vectors outputted by the command `eigen(cor(newiris))`. This demonstrates that the principal components of the data are given by the eigenvectors of its correlation matrix.

d) The first two principal components are given as

$$PC1 = \begin{pmatrix} 0.521065 \\ -0.269347 \\ 0.580413 \\ 0.564856 \end{pmatrix}$$

$$PC2 = \begin{pmatrix} 0.377417 \\ 0.923295 \\ 0.024491 \\ 0.066941 \end{pmatrix}$$

Fraction of variance explained by the first two principle components,

$$\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} = \frac{2.918497 + 0.914030}{2.918497 + 0.914030 + 0.146756 + 0.020714} = \frac{3.832528}{4} = 0.958132$$