MH3500 Statistics

Tutorial 5 AY2022/23 Semester 2

Problem 5.1

Let D_θ be the discrete distribution with the PMF $f(1|\theta) = \theta$, $f(2|\theta) = 1 - \theta$, and $f(x|\theta) = 0$ otherwise, where $\theta \in [0, 1]$ is an unknown parameter.

a) Find the maximum likelihood estimator (MLE) for θ, based on i.i.d. $X_1, \dots, X_n \sim D_\theta$.
[Hint: $f(x|\theta) = \theta^{2-x}(1-\theta)^{x-1}$ for $x = 1, 2$, and zero otherwise.

b) For $n = 3$, the observations $x_1 = 1,\ \ x_2 = 2,\ \ x_3 = 2$ are given. What is the MLE for θ based on these observations?

Problem 5.2

Let $X_1, \dots, X_n$ be an i.i.d. sample drawn from a $Gamma(2, \theta)$ distribution. The PDF of $Gamma(2, \theta)$ is given as

$$f(x|\theta) = \frac{1}{\theta^2} x e^{-x/\theta} \quad \text{for } \ x > 0$$

and zero otherwise, where $\theta \in (0, \infty)$ is an unknown parameter

a) Find the maximum likelihood estimator $\hat{\theta}$ (MLE) for θ based on $X_1, \dots, X_n$.
b) Compute the expected value and variance of $\hat{\theta}$.
c) For $n = 3$, the observations $x_1 = 120,\ \ x_2 = 130,\ \ x_3 = 128$ are given. Find the MLE for θ, with an estimated standard error, based on these observations.

Problem 5.3

Let $D_\theta, 0 \le \theta \le 1$, be the discrete distribution with the following PMF:

x	0	1	2	3
$f(x)$	$\frac{2}{3}\theta$	$\frac{1}{3}\theta$	$\frac{2}{3}(1-\theta)$	$\frac{1}{3}(1-\theta)$

and $f(x) = 0$ otherwise. Let $X_1, \dots, X_n$ be an i.i.d. random sample drawn from D_θ and let $\bar{X}$ denote the sample mean. We consider the following estimators for θ.

$$\hat{\theta}_1(n) = -\frac{1}{2}\bar{X}$$

$$\hat{\theta}_2(n) = \frac{1}{6}[7 - (X_1 + X_2 + X_3)]$$

$$\hat{\theta}_3(n) = \frac{1}{6}[7 - 3\bar{X}]$$

$$\hat{\theta}_4(n) = \frac{1}{16}\left(17 - \frac{3}{n}\sum_{i=1}^{n} X_i^2\right)$$

a) Which of these estimators are unbiased?
b) Which of these estimators satisfy $\lim_{n\to\infty} Var\left[\hat{\theta}_i(n)\right] = 0$?
c) For each of these estimators, compute the standard error and the mean squared error.
d) The following observations for $X_1, \dots, X_n$ are given with $n = 10$:

$$3, 0, 2, 1, 3, 2, 1, 0, 2, 1$$

For each unbiased estimator form above, substitute the observations into the estimator to obtain an estimation of θ . What do the respective estimated standard errors tell us about the accuracy of these estimations?