MH3500 Statistics

Tutorial 2 AY2022/23 Semester 2

Problem 2.1

Let X be a random variable with the following cumulative distribution function (CDF).

$$F(x) = \begin{cases} 1 - e^{-(x/\alpha)^{\beta}} & for\ x > 0 \\ 0 & otherwise \end{cases}$$

where α and $\beta > 0$ are constants. The distribution of X is called a **Weibull distribution**.

a) Find the PDF of the random variable X.
b) Let $Y = (X/\alpha)^{\beta}$. Identify the distribution of Y.

Problem 2.2

Let $X \sim Exp(\lambda)$. Determine the distributions of $Y_1 = \lambda X$ and $Y_2 = 2\lambda X$.

Problem 2.3

Let $X_1, \dots, X_5$ be i.i.d. $\sim N(0,1)$ and let $\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i$ be the sample mean. Let $Y \sim N(0,1)$ be another random variable which is independent from $X_1, \dots, X_5$. Find the distributions of the following random variables and justify your answers.

a) Random variable $W = \sum_{i=1}^{5} X_i^2$
b) Random variable $U = \sum_{i=1}^{5} (X_i - \bar{X})^2$
c) Random variable $2Y/\sqrt{U}$
d) Random variable $2(5\bar{X}^2 + Y^2)/U$

Problem 2.4

Let $X_1, \dots, X_{16}$ be i.i.d. $\sim N(0,1)$ and let $\bar{X}$ be the sample mean.

a) Find the constant c such that $P(|\bar{X}| < c) = 0.5$.
b) Find the mean and the variance of the sample variance S^2.

Problem 2.5

Let $X_1, \dots, X_n \sim \text{Poisson}(1)$ be an i.i.d. random sample. For each $n \in \{10, 100, 1000, 10000\}$, compute the following.

a) The exact probability $\Pr(\sum_{i=1}^{n} X_i \leq n)$ for each n.
b) The approximation of $\Pr(\sum_{i=1}^{n} X_i \leq n)$ obtained from the Central Limit Theorem (CLT).

Hint: note that $\sqrt{n}(\bar{X} - \mu)/\sigma$ converges in distribution to $N(0,1)$ for $n \to \infty$ where $\mu = E[X_1]$ and $\sigma = Var[X_1]$.