Lista 02 - PGM2017

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1. •

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\begin{aligned}
&P(D=d^0) = \sum_{k=0}^{1}\sum_{n=1}^{3} P(D=d^0, I=i^k, G=g^n)\\
&= 0.1260 + 0.1680 + 0.1260 + 0.2520 + 0.0224 + 0.0056\\
&= 0.7\\
&P(D=d^1) = 1 - P(D=d^0) = 0.3
\end{aligned}
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\begin{aligned}
P(I=i^0, D=d^0) &= 0.126 + 0.168 + 0.126 = 0.42\\
P(I=i^0, D=d^1) &= 0.009 + 0.045 + 0.126 = 0.18\\
P(I=i^1, D=d^0) &= 0.252 + 0.0224 + 0.0056 = 0.28\\
P(I=i^1, D=d^1) &= 0.06 + 0.036 + 0.024 = 0.12
\end{aligned}
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\begin{aligned}
P(I=i^0, D=d^0|G=g^2) &= \tfrac{P(I=i^0,D=d^0,G=g^2)}{P(G=g^2)} = \tfrac{0.168}{0.2714} = 0.619013\\
P(I=i^0, D=d^1|G=g^2) &= \tfrac{P(I=i^0,D=d^1,G=g^2)}{P(G=g^2)} = \tfrac{0.045}{0.2714} = 0.165807\\
P(I=i^1, D=d^0|G=g^2) &= \tfrac{P(I,D,G=g^2)}{P(G=g^2)} = \tfrac{0.0224}{0.2714} = 0.082535\\
P(I=i^1, D=d^1|G=g^2) &= \tfrac{P(I,D,G=g^2)}{P(G=g^2)} = \tfrac{0.036}{0.2714} = 0.132646
\end{aligned}
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\begin{aligned}
&P(I=i^0|G=g^2) =\\
&= \tfrac{P(I=i^0,D=d^0,G=g^2)+P(I=i^0,D=d^1,G=g^2)}{P(G=g^2)}\\
&= \tfrac{0.168+0.045}{0.2714} = 0.784819
\end{aligned}
$$

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\begin{aligned}
&P(I=i^1|G=g^2) =\\
&= \tfrac{0.0224+0.036}{0.2714} = 0.215181
\end{aligned}
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2.

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\begin{aligned}
&P(D=d|G=g^2) = P(I=i^0, D=d|G=g^2) + P(I=i^1, D=d|G=g^2)\\
&= \tfrac{P(I=i^0,D=d,G=g^2)}{P(G=g^2)} + \tfrac{P(I=i^1,D=d,G=g^2)}{P(G=g^2)} = \tfrac{P(I=i^0,D=d,G=g^2)+P(I=i^1,D=d,G=g^2)}{P(G=g^2)}\\
&= \tfrac{P(D=d,G=g^2)}{P(G=g^2)}
\end{aligned}
$$

O fator de normalização é $P(G=g^2)$

3. ESTE NÃO SEI FAZER.

   - $P(X = x, Y = y|Z = z) = P(x|z)P(y|z)$

$$P(X,Y,Z) = P(X,Y|Z)P(Z) = P(X|Z)P(Y|Z)P(Z)$$
$$\frac{P(X,Z)}{P(Z)} \cdot \frac{P(X,Y)}{P(Z)} \cdot P(Z)$$

   - $P(X = x|Y = y, Z = z) = P(X = x|Z = z)$
   - $P(Y = y|X = x, Z = z) = P(Y = y|Z = z)$
   - $P(X = x, Y = y, Z = z) \propto g(x,z)h(y,z)$

   Dado que os eventos de jogar a moeda são independentes e que os eventos de escolher uma das moedas possuem a mesma probabilidade. Temos:

4. - $P\left(X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_{10} = \text{Cara}|\text{Moeda} = A\right)$

$$P\left(X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_{10} = \text{Cara}|\text{Moeda} = A\right) =$$
$$= \prod_{i=1}^{10} P\left(X_i = \text{Cara}|\text{Moeda} = A\right) = (0.70)^{10}$$

   - $P\left(X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_{10} = \text{Cara}|\text{Moeda} = B\right)$

$$P\left(X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_{10} = \text{Cara}|\text{Moeda} = B\right) =$$
$$= \prod_{i=1}^{10} P\left(X_i = \text{Cara}|\text{Moeda} = B\right) = (0.10)^{10}$$

   - $P\left(X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_{10} = \text{Cara}\right)$

$$P\left(\text{Cara}\right) = P\left(\text{Moeda} = A\right) P\left(\text{Cara}|\text{Moeda} = A\right) + P\left(\text{Moeda} = B\right) P\left(\text{Cara}|\text{Moeda} = B\right)$$
$$= 0.7P\underbrace{\left(\text{Moeda} = A\right)}_{\frac{1}{2}} + 0.1P\underbrace{\left(\text{Moeda} = B\right)}_{\frac{1}{2}} = 0.4$$

$$P\left(X_1 = \text{Cara}, \ldots, X_{10} = \text{Cara}\right) = \prod_{i=1}^{10} P\left(X_i = \text{Cara}\right) = (0.4)^{10}$$

   - $P\left(X_{10} = \text{Cara}|X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_9 = \text{Cara}\right)$
     Como os eventos X_i são independentes entre si:

$$P\left(X_{10} = \text{Cara}|X_1 = \text{Cara}, \ldots, X_9 = \text{Cara}\right) = P\left(X_{10} = \text{Cara}\right) = 0.4$$

   - $P\left(X_{10} = \text{Cara}|\text{Moeda} = A, X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_9 = \text{Cara}\right)$
     Como os eventos X_i são independentes entre si:

$$P(X_{10} = \text{Cara}|\text{Moeda} = A, X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_9 = \text{Cara})$$
$$= P(X_{10} = \text{Cara}|\text{Moeda} = A) = 0.7$$

   - $P\left(X_{10} = \text{Cara}|\text{Moeda} = B, X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_9 = \text{Cara}\right)$
     Como os eventos X_i são independentes entre si:

$$P(X_{10} = \text{Cara}|\text{Moeda} = B, X_1 = \text{Cara}, X_2 = \text{Cara}, \ldots, X_9 = \text{Cara})$$
$$= P(X_{10} = \text{Cara}|\text{Moeda} = B) = 0.1$$

5. A princípio, temos:

$$\chi^2 = \sum\nolimits_{i,j} \frac{\left(N_{ij} - E_{ij}\right)^2}{E_{ij}}$$

e para facilitar a notação

$$N_{++} = \sum\nolimits_{i,j} N_{ij} \qquad \hat{p}_{+j} = \sum\nolimits_{i} \tfrac{N_{ij}}{N_{++}} \qquad \hat{p}_{i+} = \sum\nolimits_{j} \tfrac{N_{ij}}{N_{++}}$$
$$\hat{p}_{ij} = \tfrac{N_{ij}}{N_{++}} \qquad E_{ij} = N_{++}\hat{p}_{i+}\hat{p}_{+j}$$

- $\chi^2 = \sum\nolimits_{i,j} \dfrac{\left(N_{ij} - E_{ij}\right)^2}{E_{ij}} =$
  $= \sum\nolimits_{i,j} \dfrac{\left(N_{ij} - N_{++}\hat{p}_{i+}\hat{p}_{+j}\right)^2}{N_{++}\hat{p}_{i+}\hat{p}_{+j}} = \sum\nolimits_{i,j} \dfrac{\left(N_{++}\left(N_{ij}/N_{++} - \hat{p}_{i+}\hat{p}_{+j}\right)\right)^2}{N_{++}\hat{p}_{i+}\hat{p}_{+j}}$
  $= \sum\nolimits_{i,j} \dfrac{{N_{++}}^2}{N_{++}} \cdot \dfrac{\left(N_{ij}/N_{++} - \hat{p}_{i+}\hat{p}_{+j}\right)^2}{\hat{p}_{i+}\hat{p}_{+j}} = N_{++} \sum\nolimits_{i,j} \dfrac{\left(\hat{p}_{ij} - \hat{p}_{i+}\hat{p}_{+j}\right)^2}{\hat{p}_{i+}\hat{p}_{+j}}$
- $\chi^2 = \sum\nolimits_{i,j} \dfrac{\left(N_{ij} - E_{ij}\right)^2}{E_{ij}} =$

$$= \sum\nolimits_{i,j} \frac{{N_{ij}}^2 - 2N_{ij}E_{ij} + {E_{ij}}^2}{E_{ij}} = \sum\nolimits_{i,j} \frac{{N_{ij}}^2}{E_{ij}} + \sum\nolimits_{i,j} \frac{-2N_{ij}E_{ij}}{E_{ij}} + \sum\nolimits_{i,j} \frac{{E_{ij}}^2}{E_{ij}}$$
$$= \sum\nolimits_{i,j} \tfrac{{N_{ij}}^2}{E_{ij}} - 2\underbrace{\sum\nolimits_{i,j} N_{ij}}_{N_{++}} + \sum\nolimits_{i,j} E_{ij} = \sum\nolimits_{i,j} \tfrac{{N_{ij}}^2}{E_{ij}} - 2N_{++} + \sum\nolimits_{i,j} N_{++}\hat{p}_{i+}\hat{p}_{+j}$$
$$= \sum\nolimits_{i,j} \tfrac{{N_{ij}}^2}{E_{ij}} - 2N_{++} + N_{++}\underbrace{\sum\nolimits_{i,j} \hat{p}_{i+}\hat{p}_{+j}}_{1} = \sum\nolimits_{i,j} \tfrac{{N_{ij}}^2}{E_{ij}} - N_{++}$$

6. V ou F.

- $P\left(A \cap B\right) \leq \min\{P(A), P(B)\}$
  Já que $(A \cap B) \subset A$:

$$\begin{array}{l} P\left(A \cap B\right) \leq P\left(A\right) \\ P\left(A\right) \cdot P\left(B|A\right) \leq P\left(A\right) \\ \text{[se,} P\left(A\right) > 0] \\ \frac{1}{P(A)} \cdot P\left(A\right) \cdot P\left(B|A\right) \leq \frac{1}{P(A)} \cdot P\left(A\right) \\ P\left(B|A\right) \leq 1 \end{array}$$

  O que é verdade pois toda probabilidade é menor que 1.
  Análogo para $(A \cap B) \subset B$:

$$\begin{array}{l} P\left(A \cap B\right) \leq P\left(B\right) \\ P\left(B\right) \cdot P\left(A|B\right) \leq P\left(B\right) \\ \text{[se,} P\left(B\right) > 0] \\ \frac{1}{P(B)} \cdot P\left(B\right) \cdot P\left(A|B\right) \leq \frac{1}{P(B)} \cdot P\left(B\right) \\ P\left(A|B\right) \leq 1 \end{array}$$

  Como é verdade para os dois, será verdade também para o mínimo dos dois.

- $P\left(A|C\right) = \sum_{i=1}^{k} P\left(A|C, B_i\right) P\left(B_i|C\right)$
  Lembrando que $A \subset B$ e todas as operações são válidas pelas hipóteses dada.

$$\begin{array}{l} P\left(A|C\right) = P\left(A \cap B|C\right) = P\left(A \cap \left(\bigcup_{i=1}^{k} B_i\right)|C\right) \\ = P\left(\bigcup_{i=1}^{k} \left(A \cap B_i\right)|C\right) = \sum_{i=1}^{k} P\left(A \cap B_i|C\right) \\ = \sum_{i=1}^{k} P\left(B_i|C\right) P\left(A|C, B_i\right) = \sum_{i=1}^{k} P\left(A|C, B_i\right) P\left(B_i|C\right) \end{array}$$

- $P\left(A \cap B\right) + P\left(A^c \cap B\right) = 1$
  Sabemos que:

$$\begin{array}{l} (A \cap B) \cap (A^c \cap B) = A \cap A^c \cap B \cap B = \emptyset \cap B = \emptyset \\ (A \cap B) \cup (A^c \cap B) = B \cap (A \cup A^c) = B \end{array}$$

  Temos:

$$\begin{array}{l} P\left(A \cap B\right) + P\left(A^c \cap B\right) - 0 = P\left(A \cap B\right) + P\left(A^c \cap B\right) - P\left(\emptyset\right) \\ = \underbrace{P\left(A \cap B\right) + P\left(A^c \cap B\right) - P\left(\left(A \cap B\right) \cap \left(A^c \cap B\right)\right)}_{P((A \cap B) \cup (A^c \cap B))} \\ = P\left(\left(A \cap B\right) \cup \left(A^c \cap B\right)\right) = P\left(B\right) \end{array}$$

  Logo, não é sempre verdade que $P(B) = 1$.

- $P\left(A|B\right) + P\left(A^c|B\right) = 1$
  Utilizando um dos resultados no exercício anterior, temos:

$$\begin{array}{l} P\left(A|B\right) + P\left(A^c|B\right) = \dfrac{P\left(A \cap B\right)}{P\left(B\right)} + \dfrac{P\left(A^c \cap B\right)}{P\left(B\right)} \\ = \frac{P(A \cap B) + P(A^c \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \end{array}$$

- $P(A|B) + P(A|B^c) = 1$
  Se A e B forem independemte, A e B^c, também será.
  Utilizando a hipótese que A e B são independentes.

  $$P(A|B) + P(A|B^c) = P(A) + P(A) = 2P(A)$$

  Será igual a um apenas se $P(A) = \dfrac{1}{2}$. Como pode pertencer a $0 \leq P(A) \leq 1$, logo é falso para $P(A) \neq \dfrac{1}{2}$.
- $P(A \cap B) + P(A^c \cap B) = P(B)$

  $$(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = B$$

  Como os conjuntos são disjuntos, temos:

  $$P(B) = P((A \cap B) \cup (A^c \cap B)) = P(A \cap B) + P(A^c \cap B)$$

- $P(A|B) > P(A) \Rightarrow P(A^c|B) < P(A^c)$

  $$\begin{array}{l} P(A|B) > P(A) \\ 1 - P(A^c|B) > 1 - P(A^c) \\ -P(A^c|B) > -P(A^c) \\ P(A^c|B) < P(A^c) \end{array}$$

- $P(A|B) > P(A^c|B) \Rightarrow P(A|B) > \dfrac{1}{2}$

  $$\begin{array}{l} P(A|B) > P(A^c|B) \\ P(A|B) > 1 - P(A|B) \\ P(A|B) + (P(A|B)) > 1 - P(A|B) + (P(A|B)) \\ 2P(A|B) > 1 \\ P(A|B) > \frac{1}{2} \end{array}$$