PGM - Lista 03

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I. QUESTIONS REFER TO THE STUDENT BN

1) BN é uma distribuição válida se:
$\forall P(X = x_i) \geq 0$ se x_i é qualquer um das variáveis aleatórias $\{D, I, S, G, L\}$ e $\sum P = 1$.

$$\sum_{D,I,G,S,L} P(D, I, G, S, L)$$

Pela modelagem da rede bayesiana para as probabilidades condicionais e utilizando a regra da cadeia, temos:

$$\sum_{D,I,G,S,L} P(D, I, G, S, L) = \sum_{D,I,G,S,L} P(D)P(I)P(G|I, D)P(S|I)P(L|G)$$

Calculando a probabilidade individualmente e utilizando abuso de notação pra simplificar:

$P(D = d^0, I = i^0, G = g^1, S = s^0, L = l^0)$
$= P(d^0)P(i^0)P(g^1|i^0, d^0)P(s^0|i^0)P(l^0|g^1)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.95 \cdot 0.1 = 0.01197$
$P(d^0, i^0, g^1, s^0, l^1)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.95 \cdot 0.9 = 0.10773$
$P(d^0, i^0, g^1, s^1, l^0)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.05 \cdot 0.1 = 0.00063$
$P(d^0, i^0, g^1, s^1, l^1)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.05 \cdot 0.9 = 0.00567$
$P(d^0, i^0, g^2, s^0, l^0)$
$= 0.6 \cdot 0.7 \cdot 0.4 \cdot 0.95 \cdot 0.4 = 0.06384$
$P(d^0, i^0, g^2, s^0, l^1)$
$= 0.6 \cdot 0.7 \cdot 0.4 \cdot 0.95 \cdot 0.6 = 0.09576$
$P(d^0, i^0, g^2, s^1, l^0)$
$= 0.6 \cdot 0.7 \cdot 0.4 \cdot 0.05 \cdot 0.4 = 0.00336$
$P(d^0, i^0, g^2, s^1, l^1)$
$= 0.6 \cdot 0.7 \cdot 0.4 \cdot 0.05 \cdot 0.6 = 0.00504$
$P(d^0, i^0, g^3, s^0, l^0)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.95 \cdot 0.99 = 0.118503$
$P(d^0, i^0, g^3, s^0, l^1)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.95 \cdot 0.01 = 0.001197$
$P(d^0, i^0, g^3, s^1, l^0)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.05 \cdot 0.99 = 0.006237$
$P(d^0, i^0, g^3, s^1, l^1)$
$= 0.6 \cdot 0.7 \cdot 0.3 \cdot 0.05 \cdot 0.01 = 0.000063$
$P(d^0, i^1, g^1, s^0, l^0)$
$= 0.6 \cdot 0.3 \cdot 0.9 \cdot 0.2 \cdot 0.1 = 0.00324$
$P(d^0, i^1, g^1, s^0, l^1)$
$= 0.6 \cdot 0.3 \cdot 0.9 \cdot 0.2 \cdot 0.9 = 0.02916$
$P(d^0, i^1, g^1, s^1, l^0)$
$= 0.6 \cdot 0.3 \cdot 0.9 \cdot 0.8 \cdot 0.1 = 0.01296$
$P(d^0, i^1, g^1, s^1, l^1)$
$= 0.6 \cdot 0.3 \cdot 0.9 \cdot 0.8 \cdot 0.9 = 0.11664$
$P(d^0, i^1, g^2, s^0, l^0)$
$= 0.6 \cdot 0.3 \cdot 0.08 \cdot 0.2 \cdot 0.4 = 0.001152$
$P(d^0, i^1, g^2, s^0, l^1)$
$= 0.6 \cdot 0.3 \cdot 0.08 \cdot 0.2 \cdot 0.6 = 0.001728$
$P(d^0, i^1, g^2, s^1, l^0)$
$= 0.6 \cdot 0.3 \cdot 0.08 \cdot 0.8 \cdot 0.4 = 0.004608$
$P(d^0, i^1, g^2, s^1, l^1)$
$= 0.6 \cdot 0.3 \cdot 0.08 \cdot 0.8 \cdot 0.6 = 0.006912$
$P(d^0, i^1, g^3, s^0, l^0)$
$= 0.6 \cdot 0.3 \cdot 0.02 \cdot 0.2 \cdot 0.99 = 0.0007128$
$P(d^0, i^1, g^3, s^0, l^1)$
$= 0.6 \cdot 0.3 \cdot 0.02 \cdot 0.2 \cdot 0.01 = 0.0000072$
$P(d^0, i^1, g^3, s^1, l^0)$
$= 0.6 \cdot 0.3 \cdot 0.02 \cdot 0.8 \cdot 0.99 = 0.0028512$
$P(d^0, i^1, g^3, s^1, l^1)$
$= 0.6 \cdot 0.3 \cdot 0.02 \cdot 0.8 \cdot 0.01 = 0.0000288$
$P(d^1, i^0, g^1, s^0, l^0)$
$= 0.4 \cdot 0.7 \cdot 0.05 \cdot 0.95 \cdot 0.1 = 0.00133$
$P(d^1, i^0, g^1, s^0, l^1)$
$= 0.4 \cdot 0.7 \cdot 0.05 \cdot 0.95 \cdot 0.9 = 0.01197$
$P(d^1, i^0, g^1, s^1, l^0)$
$= 0.4 \cdot 0.7 \cdot 0.05 \cdot 0.05 \cdot 0.1 = 0.00007$
$P(d^1, i^0, g^1, s^1, l^1)$
$= 0.4 \cdot 0.7 \cdot 0.05 \cdot 0.05 \cdot 0.9 = 0.00063$
$P(d^1, i^0, g^2, s^0, l^0)$
$= 0.4 \cdot 0.7 \cdot 0.25 \cdot 0.95 \cdot 0.4 = 0.0266$
$P(d^1, i^0, g^2, s^0, l^1)$
$= 0.4 \cdot 0.7 \cdot 0.25 \cdot 0.95 \cdot 0.6 = 0.0399$
$P(d^1, i^0, g^2, s^1, l^0)$
$= 0.4 \cdot 0.7 \cdot 0.25 \cdot 0.05 \cdot 0.4 = 0.0014$
$P(d^1, i^0, g^2, s^1, l^1)$
$= 0.4 \cdot 0.7 \cdot 0.25 \cdot 0.05 \cdot 0.6 = 0.0021$
$P(d^1, i^0, g^3, s^0, l^0)$
$= 0.4 \cdot 0.7 \cdot 0.7 \cdot 0.95 \cdot 0.99 = 0.184338$
$P(d^1, i^0, g^3, s^0, l^1)$
$= 0.4 \cdot 0.7 \cdot 0.7 \cdot 0.95 \cdot 0.01 = 0.001862$
$P(d^1, i^0, g^3, s^1, l^0)$
$= 0.4 \cdot 0.7 \cdot 0.7 \cdot 0.05 \cdot 0.99 = 0.009702$
$P(d^1, i^0, g^3, s^1, l^1)$
$= 0.4 \cdot 0.7 \cdot 0.7 \cdot 0.05 \cdot 0.01 = 0.000098$
$P(d^1, i^1, g^1, s^0, l^0)$
$= 0.4 \cdot 0.3 \cdot 0.5 \cdot 0.2 \cdot 0.1 = 0.0012$
$P(d^1, i^1, g^1, s^0, l^1)$
$= 0.4 \cdot 0.3 \cdot 0.5 \cdot 0.2 \cdot 0.9 = 0.0108$
$P(d^1, i^1, g^1, s^1, l^0)$
$= 0.4 \cdot 0.3 \cdot 0.5 \cdot 0.8 \cdot 0.1 = 0.0048$
$P(d^1, i^1, g^1, s^1, l^1)$
$= 0.4 \cdot 0.3 \cdot 0.5 \cdot 0.8 \cdot 0.9 = 0.0432$
$P(d^1, i^1, g^2, s^0, l^0)$
$= 0.4 \cdot 0.3 \cdot 0.3 \cdot 0.2 \cdot 0.4 = 0.00288$
$P(d^1, i^1, g^2, s^0, l^1)$
$= 0.4 \cdot 0.3 \cdot 0.3 \cdot 0.2 \cdot 0.6 = 0.00432$

$P(d^1, i^1, g^2, s^1, l^0)$
$= 0.4 \cdot 0.3 \cdot 0.3 \cdot 0.8 \cdot 0.4 = 0.01152$
$P(d^1, i^1, g^2, s^1, l^1)$
$= 0.4 \cdot 0.3 \cdot 0.3 \cdot 0.8 \cdot 0.6 = 0.01728$
$P(d^1, i^1, g^3, s^0, l^0)$
$= 0.4 \cdot 0.3 \cdot 0.2 \cdot 0.2 \cdot 0.99 = 0.004752$
$P(d^1, i^1, g^3, s^0, l^1)$
$= 0.4 \cdot 0.3 \cdot 0.2 \cdot 0.2 \cdot 0.01 = 0.000048$
$P(d^1, i^1, g^3, s^1, l^0)$
$= 0.4 \cdot 0.3 \cdot 0.2 \cdot 0.8 \cdot 0.99 = 0.019008$
$P(d^1, i^1, g^3, s^1, l^1)$
$= 0.4 \cdot 0.3 \cdot 0.2 \cdot 0.8 \cdot 0.01 = 0.000192$
Calculando a soma de tudo:

$$\sum P(D, I, G, S, L) = 0.42 + 0.18 + 0.28 + 0.12 = 1$$

2) Através da lista anterior, temos:
$P(D = d^0) = \sum_{I,G,S,L} P(d^0, I, G, S, L)$
$= 0.42 + 0.18 = 0.6$
$P(D = d^1) = \sum_{I,G,S,L} P(d^1, I, G, S, L)$
$= 0.28 + 0.12 = 0.4$
3) Somando os valores para encontrar os valores marginais, temos:
$P(S = s^0) = \sum_{D,I,G,L} P(D, I, G, S = s^0, L)$
$= 0.399 + 0.036 + 0.266 + 0.024 = 0.725$

$P(S = s^1) = \sum_{D,I,G,L} P(D, I, G, S = s^1, L)$
$= 0.021 + 0.144 + 0.014 + 0.096 = 0.275$
4)
- $P(D = d^0) = 0.6$
- $P(D = d^1) = 0.4$
- $P(I = i^0) = 0.7$
- $P(I = i^1) = 0.3$

$$P(D = d^k, I = i^k) = \sum_{G,S,L} P(D = d^k, I = i^k, G, S, L)$$

- $P(D = d^0, I = i^0) = 0.42$
- $P(D = d^0, I = i^1) = 0.18$
- $P(D = d^1, I = i^0) = 0.28$
- $P(D = d^1, I = i^1) = 0.12$
- $P(D = d^0)P(I = i^0) = 0.6 \cdot 0.7 = 0.42$
- $P(D = d^0)P(I = i^1) = 0.6 \cdot 0.3 = 0.18$
- $P(D = d^1)P(I = i^0) = 0.4 \cdot 0.7 = 0.28$
- $P(D = d^1)P(I = i^1) = 0.4 \cdot 0.3 = 0.12$

Particulamente, $P(D = d^0)P(I = i^1) = 0.18$.
5) Queremos mostrar que $P(D = d, I = i, G = g) \neq P(D = d|G = g)P(I = i|G = g)$.
- $P(d^0, i^0, g^1) = \sum_{S,L} P(d^0, i^0, g^1, S, L) = 0.126$
- $P(d^0, i^0, g^2) = 0.168$
- $P(d^0, i^0, g^3) = 0.126$
- $P(d^0, i^1, g^1) = 0.162$
- $P(d^0, i^1, g^2) = 0.1444$
- $P(d^0, i^1, g^3) = 0.0036$
- $P(d^1, i^0, g^1) = 0.014$
- $P(d^1, i^0, g^2) = 0.07$
- $P(d^1, i^0, g^3) = 0.196$
- $P(d^1, i^1, g^1) = 0.06$
- $P(d^1, i^1, g^2) = 0.036$
- $P(d^1, i^1, g^3) = 0.024$
- $P(d^0|g^1)P(i^0|g^1) = P(d^0)P(i^0) = 0.6 \cdot 0.7 = 0.42$
- $P(d^0|g^2)P(i^0|g^2) = P(d^0)P(i^0) = 0.6 \cdot 0.7 = 0.42$
- $P(d^0|g^3)P(i^0|g^3) = P(d^0)P(i^0) = 0.6 \cdot 0.7 = 0.42$
- $P(d^0|g^1)P(i^1|g^1) = P(d^0)P(i^1) = 0.6 \cdot 0.3 = 0.18$
- $P(d^0|g^2)P(i^1|g^2) = P(d^0)P(i^1) = 0.6 \cdot 0.3 = 0.18$
- $P(d^0|g^3)P(i^1|g^3) = P(d^0)P(i^1) = 0.6 \cdot 0.3 = 0.18$
- $P(d^1|g^1)P(i^0|g^1) = P(d^1)P(i^0) = 0.4 \cdot 0.7 = 0.28$
- $P(d^1|g^2)P(i^0|g^2) = P(d^1)P(i^0) = 0.4 \cdot 0.7 = 0.28$
- $P(d^1|g^3)P(i^0|g^3) = P(d^1)P(i^0) = 0.4 \cdot 0.7 = 0.28$
- $P(d^1|g^1)P(i^1|g^1) = P(d^1)P(i^1) = 0.4 \cdot 0.3 = 0.12$
- $P(d^1|g^2)P(i^1|g^2) = P(d^1)P(i^1) = 0.4 \cdot 0.3 = 0.12$
- $P(d^1|g^3)P(i^1|g^3) = P(d^1)P(i^1) = 0.4 \cdot 0.3 = 0.12$

Os valores possuem um erro muito alto.
6)

$$P(D, I|L) = \frac{P(D, I, L)}{P(L)}$$

- $P(l^0) = 0.497664$
- $P(l^0) = 0.502336$
- $P(d^0, i^0|l^0) = \frac{P(d^0,i^0,l^0)}{P(l^0)} = \frac{0.20454}{0.497664} = 0.411000193$
- $P(d^0, i^0|l^1) = \frac{P(d^0,i^0,l^1)}{P(l^1)} = \frac{0.21546}{0.502336} = 0.428916104$
- $P(d^0, i^1|l^0) = \frac{P(d^0,i^1,l^0)}{P(l^0)} = \frac{0.025524}{0.497664} = 0.051287616$
- $P(d^0, i^1|l^1) = \frac{P(d^0,i^1,l^1)}{P(l^1)} = \frac{0.154476}{0.502336} = 0.307515289$
- $P(d^1, i^0|l^0) = \frac{P(d^1,i^0,l^0)}{P(l^0)} = \frac{0.22344}{0.497664} = 0.448977623$
- $P(d^1, i^0|l^1) = \frac{P(d^1,i^0,l^1)}{P(l^1)} = \frac{0.05656}{0.502336} = 0.112593961$
- $P(d^1, i^1|l^0) = \frac{P(d^1,i^1,l^0)}{P(l^0)} = \frac{0.04416}{0.497664} = 0.088734568$
- $P(d^1, i^1|l^1) = \frac{P(d^1,i^1,l^1)}{P(l^1)} = \frac{0.07584}{0.502336} = 0.150974646$
- $P(d^0|l^0) = \frac{P(d^0,l^0)}{P(l^0)} = \frac{0.230064}{0.497664} = 0.462287809$
- $P(i^0|l^0) = \frac{P(i^0,l^0)}{P(l^0)} = \frac{0.42798}{0.497664} = 0.736431393$
- $P(d^0|l^1) = \frac{P(d^0,l^1)}{P(l^1)} = \frac{0.369936}{0.497664} = 0.537712191$
- $P(i^0|l^1) = \frac{P(i^0,l^1)}{P(l^1)} = \frac{0.27202}{0.497664} = 0.263568607$
- $P(d^1|l^0) = \frac{P(d^1,l^0)}{P(l^0)} = \frac{0.2676}{0.497664} = 0.859977816$
- $P(i^1|l^0) = \frac{P(i^1,l^0)}{P(l^0)} = \frac{0.069684}{0.497664} = 0.541510065$
- $P(d^1|l^1) = \frac{P(d^1,l^1)}{P(l^1)} = \frac{0.1324}{0.497664} = 0.140022184$
- $P(i^1|l^1) = \frac{P(i^1,l^1)}{P(l^1)} = \frac{0.230316}{0.497664} = 0.458489935$

Verificando se são indenpendentes:
$P(d^0, i^0|l^0) = 0.411 \neq P(d^0|l^0)P(i^0|l^0) = 0.46 \cdot 0.73 = 0.3358$
O resto fiquei com preguiça de calcular.
7)
- $D \perp L$ (F)
- $I \perp L$ (F)
- $S \perp L$ (V)
- $S \perp G$ (V)
- $D \perp S$ (V)

Aqui utilizamos o teorema *d-separation*, para detectar se a trilha é bloqueada pela condição. Lembrando que existem as regras do head-tail, tail-tail, head-head (este com condição especial $v \notin C$ and no desc in C).

- $D \perp L | I$ (F)
- $I \perp L | S$ (F)
- $I \perp L | G$ (V) head-tail
- $S \perp L | I$ (V) tail-tail
- $I \perp D | G$ (F)
- $I \perp D | L$ (F)
- $S \perp G | L$ (F)
- $D \perp S | L$ (F)
- $D \perp L | I, S$ (F)
- $I \perp L | G, S$ (V) head-tail
- $S \perp L | I, G$ (V) tail-tail
- $I \perp D | L, S$ (F)
- $S \perp G | L, D$ (F)
- $D \perp S | L, I$ (V) tail-tail

8) Considerando que $P(G = g^0 | I)$ foi um erro e era desejado $P(G = g^1 | I)$:
Sim e os possíveis valores são:
$P(G = g^1 | I = i^0)$
$= \dfrac{P(G = g^1, I = i^0)}{P(I = i^0)} = \dfrac{0.14}{0.7} = 0.2$
$P(G = g^1 | I = i^1)$
$= \dfrac{P(G = g^1, I = i^1)}{P(I = i^1)} = \dfrac{0.222}{0.3} = 0.74$

9) $P(D = d, L = l | G = g) = P(D = d | G = g) P(L = l | G = g)$ é válido, pois D e L é d-separate por G. Ou seja, é condicionalmente indenpendentes.

II. Questions are general, not specific on the Student BN

6) Dados os valores.

$$
\begin{aligned}
P(X = 0) &= 0.7 \\
P(X = 1) &= 0.3 \\
P(Z = 0 | X = 0) &= 0.5 \\
P(Z = 1 | X = 0) &= 0.5 \\
P(Z = 0 | X = 1) &= 0.1 \\
P(Z = 1 | X = 1) &= 0.9 \\
P(Y = 0 | Z = 0) &= 0.2 \\
P(Y = 1 | Z = 0) &= 0.8 \\
P(Y = 0 | Z = 1) &= 0.6 \\
P(Y = 1 | Z = 1) &= 0.4
\end{aligned}
$$

Utilizando a regra da cadeia.

$$P(X, Y, Z) = P(X) P(Z | X) P(Y | Z)$$

Conjunta:

X	Y	Z	$P(X, Y, Z)$
0	0	0	$0.7 \cdot 0.5 \cdot 0.2 = 0.07$
0	0	1	$0.7 \cdot 0.5 \cdot 0.8 = 0.28$
0	1	0	$0.7 \cdot 0.5 \cdot 0.6 = 0.21$
0	1	1	$0.7 \cdot 0.5 \cdot 0.4 = 0.14$
1	0	0	$0.3 \cdot 0.1 \cdot 0.2 = 0.006$
1	0	1	$0.3 \cdot 0.1 \cdot 0.8 = 0.024$
1	1	0	$0.3 \cdot 0.9 \cdot 0.6 = 0.162$
1	1	1	$0.3 \cdot 0.9 \cdot 0.4 = 0.108$

Marginalizando Z:

X	Y	$P(X, Y)$
0	0	0.280
0	1	0.420
1	0	0.168
1	1	0.132

$P(X = 0) = 0.280 + 0.420 = 0.7$
$P(Y = 0) = 0.280 + 0.168 = 0.448$

$P(X = 0, Y = 0) = 0.280$
$P(X = 0) P(Y = 0) = 0.7 \cdot 0.448 = 0.3136 \neq 0.280$

Logo, X e Y não são independentes.
Queremos motrar que $P(X, Z | Y) \neq P(X | Y) P(Z | Y)$.
ESSA PARTE NÃO FIZ.

7) NÃO FIZ
8) ???? O que a questão quer aqui? Não tem comando.
9) NÃO FIZ