Information Theory **DCC/ICEx/UFMG**
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SOLUTION OF HOMEWORK

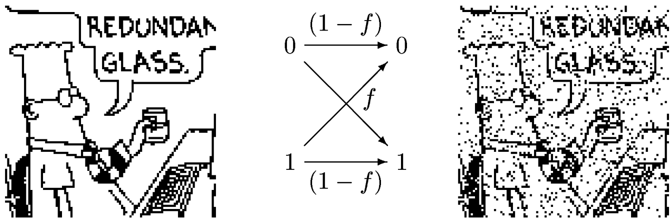
Communication through a noisy channel as inference
(In-Class Activity)

Note: The exercises are labeled according to their level of difficulty: `[Easy]`, `[Medium]` or `[Hard]`. This labeling, however, is subjective: different people may disagree on the perceived level of difficulty of any given exercise. Don't be discouraged when facing a hard exercise, you may find a solution that is simpler than the one the instructor had in mind!

Exercises.

1. **(Data transmission through a noisy channel.)**

   Consider a communication channel that can transmit bits 0 or 1 as depicted below. The channel is noisy and a fraction f of bits is flipped during transmission.



   More precisely, if x is the bit transmitted by the sender and y is the bit received by the receiver:

$$\begin{array}{ll} P(y=0 \mid x=0) = 1-f; & P(y=1 \mid x=0) = f; \\ P(y=0 \mid x=1) = f; & P(y=1 \mid x=1) = 1-f. \end{array}$$

   (a) `[Easy]` Consider the channel is used to transmit a black and white image of N pixels (each pixel is represented by one bit: 0 for a black pixel, 1 for a white pixel).
   What is the probability that the image is transmitted through this channel without any pixel being corrupted (i.e., without any bit being flipped)?

   Instructor's solution: The probability of no bit being corrupted is $(1-f)^N$.

   (b) `[Easy]` What is the expected number of corrupted bits during the transmission of N bits through this channel?

   Instructor's solution: The transmission of N bits can be modelled as a series of N Bernoulli trials with probability of failure f (here, we call a failure the event that the bit is flipped during transmission). Hence, the transmission of N bits follows a binomial distribution and the expected number of corrupted bits corresponds to the expected number of failures, which is $N \cdot f$.

(c) `[Easy]` To improve the quality of data transmission through this channel, assume we employ a repetition code R_4, so each pixel selected by the sender is repeated 4 times during transmission.[1] (E.g., if the pixel to be sent is $s = 0$, the sender transmits the sequence $t = 0000$, and if the pixel to be sent is $s = 1$, the sender transmits the sequence $t = 1111$.)

Since each one of all 4 bits in the sequence t can be flipped when passing through the channel, the receiver receives a sequence r that is not necessarily identical to the sequence t transmitted by the sender.

Recalling that the probability of any bit being flipped is f, find the probability $p(r \mid s)$ of the receiver getting message r if the sender selected pixel s to send, for the following received messages: $r_1 = 0010$, $r_2 = 0110$, and $r_3 = 1111$.

Instructor's solution: Table of probabilities $p(r \mid s)$:

$s \backslash r$	$r_1 = 0010$	$r_2 = 0110$	$r_3 = 1111$
0	$(1-f)^3 f$	$(1-f)^2 f^2$	f^4
1	$(1-f) f^3$	$(1-f)^2 f^2$	$(1-f)^4$

(d) `[Medium]` After receiving a sequence r of 4 bits, the receiver must infer what was the pixel s sent by the sender.

Formulate the problem of the receiver as a problem of inference, specifying what are the concurrent hypotheses, what is the available evidence, and what is the mathematical formula the receiver must employ to pick a most adequate hypothesis given the available evidence.

Instructor's solution: The receiver must test two alternative hypotheses:

- $s = 0$ is the hypothesis that the sender selected the pixel 0 for transmission, and
- $s = 1$ is the hypothesis that the sender selected the pixel 1 for transmission.

The receiver collects as evidence the received sequence r of 4 bits.

The receiver chooses among alternative hypotheses $s = 0$ and $s = 1$ by calculating the ratio

$$\frac{p(s=0 \mid r)}{p(s=1 \mid r)} = \frac{p(s=0)}{p(s=1)} \cdot \frac{p(r \mid s=0)}{p(r \mid s=1)}, \tag{1}$$

and chooses $s = 0$ if the ratio is greater than 1, chooses $s = 1$ if the ratio is smaller than 1, and is indifferent between the two hypotheses if the ratio is exactly 1.

In Equation (1), $p(s=0)/p(s=1)$ is the ratio of the a priori probability of the hypotheses, and $p(r|s=0)/p(r|s=1)$ is the contribution of the collected evidence r.

(e) `[Medium]` Assume the receiver has gotten the sequences $r_1 = 0010$, $r_2 = 0110$, and $r_3 = 1111$. Assuming the error rate of the channel is $f = 0.2$, evaluate how good of an evidence is each message (r_1, r_2, r_3) in favor of the sent pixel being $s = 0$.
(Justify your answer using the likelihood ratio $p(r \mid s=0)/p(r \mid s=1)$.)

Instructor's solution: For r_1 the evidence in favor of $s = 0$ is

$$\frac{p(r_1 \mid s=0)}{p(r_1 \mid s=1)} = \frac{(1-f)^3 f}{(1-f) f^3} = \frac{(1-f)^2}{f^2} = \frac{(0.8)^2}{(0.2)^2} = 16.$$

[1] To be precise, in the R_4 code each bit is repeated 3 times, since the first occurrence of the bit in a sequence is not itself a repetition. But let's not be too pedantic!

For r_2 the evidence in favor of $s = 0$ is

$$\frac{p(r_2 \mid s=0)}{p(r_2 \mid s=1)} = \frac{(1-f)^2 f^2}{(1-f)^2 f^2} = \frac{(0.8)^2(0.2)^2}{(0.8)^2(0.2)^2} = 1.$$

For r_3 the evidence in favor of $s = 0$ is

$$\frac{p(r_3 \mid s=0)}{p(r_3 \mid s=1)} = \frac{(f)^4}{(1-f)^4} = \frac{(0.2)^4}{(0.8)^4} = \frac{1}{256}.$$

Hence, r_1 favors hypothesis $s = 0$ by a factor of 16, r_2 is neutral and does not favor neither hypothesis, and r_3 favors hypothesis $s = 1$ by a factor of 256.

(f) `[Medium]` Assume the channel will be used to transmit an image in which 20% of pixels are black (bit 0), and 80% of pixels are white (bit 1).
If the error rate of the channel is $f = 0.2$, determine the receiver's best inference about the pixel sent (black or white) for each one of the following received sequences: $r_1 = 0010$, $r_2 = 0110$, and $r_3 = 1111$.

Instructor's solution: Inference for r_1:

$$\frac{p(s=0 \mid r_1)}{p(s=1 \mid r_1)} = \frac{p(s=0)}{p(s=1)} \cdot \frac{p(r_1 \mid s=0)}{p(r_1 \mid s=1)} = \frac{0.2}{0.8} \cdot 16 = 4.$$

Inference for r_2:

$$\frac{p(s=0 \mid r_2)}{p(s=1 \mid r_2)} = \frac{p(s=0)}{p(s=1)} \cdot \frac{p(r_2 \mid s=0)}{p(r_2 \mid s=1)} = \frac{0.2}{0.8} \cdot 1 = \frac{1}{4}.$$

Inference for r_3:

$$\frac{p(s=0 \mid r_3)}{p(s=1 \mid r_3)} = \frac{p(s=0)}{p(s=1)} \cdot \frac{p(r_3 \mid s=0)}{p(r_3 \mid s=1)} = \frac{0.2}{0.8} \cdot \frac{1}{256} = \frac{1}{1\,024}.$$

Hence, when getting r_2 or r_3 the receiver must infer that the pixel sent was $s = 1$, and when getting r_1 the receiver must infer that the pixel sent was $s = 0$.

2. **(Channel breakdown!)**

Consider a noisy channel just like the one in the previous question, and that we want transmit bits through it using the repetition code R_4.

(a) `[Medium]` Unfortunately, the communication channel is of horrible quality and presents an error rate of $f = 1$ (i.e., every single bit is flipped during transmission).
In this case, among all 16 sequences r the receiver may get (0000, 0001, 0010, ..., 1111), which ones are evidence in favor of the sent pixel being $s = 0$? Which ones are evidence in favor of the sent pixel being $s = 1$?

Instructor's solution: For $r = 0000$ we have

$$\frac{p(r \mid s=0)}{p(r \mid s=1)} = \frac{0}{1} = 0.$$

For $r = 1111$ we have

$$\frac{p(r \mid s=0)}{p(r \mid s=1)} = \frac{1}{0} = \infty.$$

For all other values of r we have

$$\frac{p(r \mid s = 0)}{p(r \mid s = 1)} = \frac{0}{0},$$

which is undefined.

That means that $r = 0000$ is evidence in favor of $s = 1$, $r = 1111$ is evidence in favor of $s = 0$, and all other values of r cannot actually occur.

(b) `[Medium]` Someone tried to fix the channel, and now it is working with an error rate of $f = 0.5$ (that is, half of the bits are flipped during transmission).

In this case, among all 16 sequences r the receiver may get (0000, 0001, 0010, ..., 1111), which ones are evidence in favor of the sent pixel being $s = 0$? Which ones are evidence in favor of the sent pixel being $s = 1$?

Instructor's solution: For all values of r we have

$$\frac{p(r \mid s = 0)}{p(r \mid s = 1)} = \frac{(0.5)^4}{(0.5)^4} = 1,$$

therefore no value of r is evidence for neither $s = 0$ nor $s = 1$.

(c) `[Medium]` Based on your previous answers, which channel is better for the receiver: the one that flips every bit ($f = 1$), or the one that flips half of the bits ($f = 0.5$)? In other words, in this particular case, have you actually improved the quality of transmission by reducing the error rate?

Justify your answer by arguing that the channel you chose allows for more reliable inference.

Instructor's solution: The channel with error rate $f = 1$ is better than the channel with error rate $f = 0.5$. In fact, the channel having $f = 1$ allows for the receiver to infer with certainty whether the sent pixel was $s = 0$ or $s = 1$ (since every single bit is flipped, the receiver has no uncertainty at all about what pixel was sent: it is enough to flip the bit back to infer the intention of the sender). On the other hand, the channel having $f = 0.5$ never produces evidence in favor of neither of the hypotheses $s = 0$ or $s = 1$, so this channel is useless in helping the receiver picking among the hypotheses (in this case, the receiver must rely only on the a priori ratio of the probabilities of the hypotheses).