Information Theory **DCC/ICEx/UFMG**
Prof. Mário S. Alvim **2017/02**

SOLUTION OF HOMEWORK

Communication over a Noisy Channel
(MacKay - Chapter 9)

Necessary reading for this assignment:

- *Information Theory, Inference, and Learning Algorithms* (MacKay):
  - Chapter 9.1: *The big picture*
  - Chapter 9.2: *Review of probability and information*
  - Chapter 9.3: *Noisy channels*
  - Chapter 9.4: *Inferring the input given the output*
  - Chapter 9.5: *Information conveyed by a channel*
  - Chapter 9.6: *The noisy-channel coding theorem*
  - Chapter 9.7: *Intuitive preview of proof*

Note: The exercises are labeled according to their level of difficulty: `[Easy]`, `[Medium]` or `[Hard]`. This labeling, however, is subjective: different people may disagree on the perceived level of difficulty of any given exercise. Don't be discouraged when facing a hard exercise, you may find a solution that is simpler than the one the instructor had in mind!

Review questions.

1. Answer formally the following questions:

   (a) Define what is a discrete memoryless channel.

   Instructor's solution: A discrete memoryless channel (DMC) Q is a triple $(\mathcal{A}_X, \mathcal{A}_Y, \mathcal{P}_{Y|X})$ where $\mathcal{A}_X$ is the input alphabet for the channel, $\mathcal{A}_Y$ is the output alphabet for the channel, and $\mathcal{P}_{Y|X}$ is a set of conditional probability distributions $p(y \mid x)$ (called the channel matrix) indicating the probability of the channel producing output $y \in \mathcal{A}_Y$ when the input is $x \in \mathcal{A}_X$.

   (b) Describe the problem of reliable communication over a noisy channel.

   Instructor's solution: Given a noisy channel Q from X to Y and an input ensemble X, the problem of reliable communication over the channel consists in finding a way to use the channel so that all bits transmitted through it can be reliably recovered.

   (c) Define the information conveyed by a channel in terms of mutual information. Explain what each term in the formula means.

Instructor's solution: The information conveyed by a channel from X to Y is the mutual information between the input ensemble X and the output ensemble Y: $I(X;Y) = H(X) - H(X \mid Y)$.

$H(X)$ is the initial uncertainty about the input ensemble X, and $H(X \mid Y)$ is the remaining uncertainty about the input ensemble X after the output produced by the channel has been observed. The decrease in uncertainty $I(X;Y) = H(X) - H(X \mid Y)$ is the information conveyed by the channel.

(d) What is the mathematical definition of the capacity of a channel? What is the operational definition of the capacity of a channel? What is the relation between both of them?

Instructor's solution: The mathematical definition of capacity of a channel from X to Y is the maximum mutual information between input and output in the channel over all possible probability distributions on the input ensemble: $C = \max_{\mathcal{P}_X} I(X;Y)$.

The operational definition of channel capacity is the maximum transmission rate you can achieve through a noisy channel if you want the transmission to be reliable (i.e., if you want the probability of incorrectly inferring the input from the output to be negligibly small).

The relation between the mathematical and operational definition of channel capacity is established by Shannon's channel coding theorem: both capacities are the same.

In other words, for any channel from X to Y, the maximum rate at which it is possible to transmit information through this channel with a negligible probability of error is given by the mathematical capacity $C = \max_{\mathcal{P}_X} I(X;Y)$.

Exercises.

2. (MacKay 9.2) `[Easy]`

   Instructor's solution: Given in the textbook.

3. (MacKay 9.4) `[Easy]`

   Instructor's solution: Given in the textbook.

4. (MacKay 9.7) `[Easy]`

   Instructor's solution: Given in the textbook.

5. (MacKay 9.8) `[Easy]`

   Instructor's solution: Given in the textbook.

6. (MacKay 9.12) `[Medium]`

   Instructor's solution: Given in the textbook.

7. (MacKay 9.13) `[Medium]`

Instructor's solution: Given in the textbook.