Information Theory **DCC/ICEx/UFMG**
Prof. Mário S. Alvim **2017/02**

SOLUTION OF HOMEWORK

Probability, Entropy, and Inference / More About Inference
(MacKay - Chapter 2 / Chapter 3)

Necessary reading for this assignment:

- *Information Theory, Inference, and Learning Algorithms* (MacKay):

  Chapter 2

  - Chapter 2.1: *Probabilities and ensembles*
  - Chapter 2.2: *The meaning of probability*
  - Chapter 2.3: *Forward probabilities and inverse probabilities*

  Chapter 3

  - Chapter 3.2: *The bent coin*
  - Chapter 3.3: *The bent coin and model comparison*
  - Chapter 3.4: *An example of legal evidence*

Note: The exercises are labeled according to their level of difficulty: `[Easy]`, `[Medium]` or `[Hard]`. This labeling, however, is subjective: different people may disagree on the perceived level of difficulty of any given exercise. Don't be discouraged when facing a hard exercise, you may find a solution that is simpler than the one the instructor had in mind!

Review questions.

1. `[Easy]` Answer formally the following questions:

   (a) Describe succinctly the two most common interpretations of probability: the *frequentist* interpretation, and the *Bayesian* interpretation.

   Instructor's solution: The frequentist interpretation sees probabilities as the relative frequencies of outcomes in random experiments. For instance, when we say that the (frequentist) probability of the outcome 3 in one roll of fair die is $1/6$, we mean that the relative frequency of the outcome 3, if many rolls are performed, is $1/6$.

   The Bayesian interpretation sees probabilities as degrees of belief in propositions that do not involve random variables. In general, the Bayesian interpretation sees the probability of a proposition as the extent to which it is believed that the proposition is true. For instance, when we say that the (Bayesian) probability of the proposition *"John Doe killed his wife with a knife"* is 0.8, we mean that we are 80% sure that John Doe killed his wife.

   (b) Define the concepts of *forward probability* and of *inverse probability.*

Instructor's solution: In a nutshell, forward probability is related to predict the probability of an outcome D of a process given a model with parameters θ and underlying hypothesis $\mathcal{H}$, whereas inverse probability is related to verify the likelihood that a given model has parameters θ given some observed data D and the underlying hypothesis $\mathcal{H}$.
Forward probability is related to the distribution $p(D \mid \theta, \mathcal{H})$, which tells us how to calculate the probability of an outcome given some hypothesis and parameters.
Inverse probability is related to the distribution $p(\theta \mid D, \mathcal{H})$, which tells us how likely some particular value for the parameters of the model are, given some hypothesis and some observed data. (The calculation of inverse probability depends on a prior distribution $p(D \mid \mathcal{H})$.)

(c) What is the difference between the terms *likelihood* and *probability*? In what situation should each of them be used?

Instructor's solution: Let $p(D \mid \theta, \mathcal{H})$ be a distribution where D are outcomes, θ are parameters of the model, and $\mathcal{H}$ are the underlying hypothesis.
If we fix the parameter θ, the formula $p(D \mid \theta, \mathcal{H})$ defines a probability distribution over each possible data D.
If we fix the data D, the formula $p(D \mid \theta, \mathcal{H})$ defines the likelihood of each choice of parameter θ.

Problems (Chapter 2).

2. (MacKay 2.30) `[Medium]`

Instructor's solution: Let *white* represent the outcome that a drawn ball is white, and *black* the outcome that a drawn ball is black.

Let b_1 be the first drawn ball, and b_2 be the second drawn ball.

Let n_{white} represent the number of white balls in the urn, and n_{black} represent the number of black balls in the urn.

The probability that the first ball is white is given by

$$p(b_1 = white) = \frac{n_{white}}{n_{white} + n_{black}}.$$

The probability that the second ball is white is given by

$$\begin{aligned} p(b_2 = white) = & \, p(b_1 = white, b_2 = white) + \\ & p(b_1 = black, b_2 = white) && \text{(by the sum rule)} \\ = & \, p(b_1 = white) \cdot p(b_2 = white \mid b_1 = white) + && \text{(by the product rule)} \\ & p(b_1 = black) \cdot p(b_2 = white \mid b_1 = black) \\ = & \, \frac{n_{white}}{n_{white} + n_{black}} \cdot \frac{n_{white} - 1}{n_{white} + n_{black} - 1} + \\ & \frac{n_{black}}{n_{white} + n_{black}} \cdot \frac{n_{white}}{n_{white} + n_{black} - 1} && \text{(calculating the frequentist probabilities)} \\ = & \, \frac{n_{white}(n_{white} + n_{black} - 1)}{(n_{white} + n_{black})(n_{white} + n_{black} - 1)} \\ = & \, \frac{n_{white}}{n_{white} + n_{black}}. \end{aligned}$$

Hence, we conclude that $p(b_1 = white) = p(b_2 = white) = n_{white}/(n_{white} + n_{black})$.

3. (MacKay 2.37) `[Medium]`

Instructor's solution: Let us denote by $S_1 = T$ the event that the statement made by the first inhabitant of the island is true, and by $S_1 = F$ the event that the statement made by this first inhabitant is false.

Let us denote by $S_2 = \mathit{YES}$ the event that the answer given by the second inhabitant (when asked about the truth of the statement S_1) is "yes", and by $S_2 = \mathit{NO}$ the event that the answer given to the same question by the second inhabitant is "no".

We know that the second inhabitant answered "yes" to the question, and we want to calculate the probability of the first statement being actually true. That is, we want to calculate $p(S_1 = T \mid S_2 = \mathit{YES})$.

By the Bayes Theorem,

$$\begin{aligned} p(S_1 = T \mid S_2 = \mathit{YES}) &= \frac{p(S_1 = T)p(S_2 = \mathit{YES} \mid S_1 = T)}{p(S_2 = \mathit{YES})} \\ &= \frac{p(S_1 = T)p(S_2 = \mathit{YES} \mid S_1 = T)}{p(S_1 = T)p(S_2 = \mathit{YES} \mid S_1 = T) + p(S_1 = F)p(S_2 = \mathit{YES} \mid S_1 = F)}. \end{aligned}$$

We know that

- $p(S_1 = T) = 1/3$, since the first inhabitant tells the truth only one third of the time;
- $p(S_1 = F) = 2/3$, since the first inhabitant lies two-thirds of the time;
- $p(S_2 = \mathit{YES} \mid S_1 = T) = 1/3$, because the second inhabitant tells the truth one third of the time, and therefore there is a one third chance that the second inhabitant would answer the question with a "yes" if the first inhabitant actually told the truth; and
- $p(S_2 = \mathit{YES} \mid S_1 = F) = 2/3$, because the second inhabitant lies two-thirds of the time, and therefore there is a two-thirds chance that the second inhabitant would answer the question with a "yes" if the first inhabitant lied.

Plugging the above quantities into the formula for $p(S_1 = T \mid S_2 = \mathit{YES})$ we get

$$\begin{aligned} p(S_1 = T \mid S_2 = \mathit{YES}) &= \frac{p(S_1 = T)p(S_2 = \mathit{YES} \mid S_1 = T)}{p(S_1 = T)p(S_2 = \mathit{YES} \mid S_1 = T) + p(S_1 = F)p(S_2 = \mathit{YES} \mid S_1 = F)} \\ &= \frac{1/3 \cdot 1/3}{1/3 \cdot 1/3 + 2/3 \cdot 2/3} \\ &= \frac{1}{5}. \end{aligned}$$

Hence, the probability that the first statement was indeed true, given that the second answer was a "yes", is 1/5.

Problems (Chapter 3).

4. (MacKay 3.11) `[Medium]`

   Instructor's solution: Given in the textbook.

5. (MacKay 3.12) `[Medium]`

Instructor's solution: Let us call $C = black$ the event that the original counter in the bag is black, and $C = white$ the event that the original counter in the bag is white.

Let us call $D_1 = white$ the event that the first counter drawn from the bag is white, and $D_1 = black$ the event that the first counter drawn from the bag is black.

Similarly, let us call $D_2 = white$ and $D_2 = black$ the events that the second counter drawn from the bag is, respectively, a white counter or a black counter.

We want to calculate $p(D_2 = white \mid D_1 = white)$. Note that this value depends on whether the original counter in the bag was white or black. To solve the problem we could just apply the sum rule to obtain $p(D_2 = white \mid D_1 = white) = p(D_2 = white, C = white \mid D_1 = white) + p(D_2 = white, C = black \mid D_1 = white)$, and proceed from there. Although this is a valid approach, it would demand quite a few tedious calculations.

Our work can be greatly simplified if we note that event $D_2 = white$ is equivalent to hypothesis $C = white$ being true, and event $D_2 = black$ is equivalent to hypothesis $C = black$ being true, since the second counter drawn from the bag must be the same color as the original counter in the bag. Hence we have that $p(D_2 = white \mid D_1 = white) = p(C = white \mid D_1 = white)$ and we can calculate

$$\begin{aligned}
&p(D_2 = white \mid D_1 = white) \\
=\,&p(C = white \mid D_1 = white) \\
=\,&\frac{p(C = white)p(D_1 = white \mid C = white)}{p(C = white)p(D_1 = white \mid C = white) + p(C = black)p(D_1 = white \mid C = black)} \quad \text{(by Bayes' Theorem)} \\
=\,&\frac{1/2 \cdot 1}{1/2 \cdot 1 + 1/2 \cdot 1/2} \\
=\,&\frac{2}{3}.
\end{aligned}$$

That is, the probability of drawing a white counter after a white counter was drawn is 2/3.