**Information Theory** **DCC/ICEx/UFMG**
**Prof. Mário S. Alvim** **2017/02**

# SOLUTION OF HOMEWORK

Discrete Probability - Part 1/2
(Based on slide-set)

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**Necessary reading for this assignment:**

- *Slide-set of Lecture 01 - Discrete Probability*:
  - *An Introduction to Discrete Probability*
  - *Probability Theory*

**Note:** The exercises are labeled according to their level of difficulty: `[Easy]`, `[Medium]` or `[Hard]`. This labeling, however, is subjective: different people may disagree on the perceived level of difficulty of any given exercise. Don't be discouraged when facing a hard exercise, you may find a solution that is simpler than the one the instructor had in mind!

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**Review questions.**

1. (Rosen Review Question 7-1) `[Easy]`

   (a) Define the probability of an event when all outcomes are equally likely.

   **Instructor's solution:** Given a sample espace $S$ and an event $E \subseteq S$, the probability of $E$ is given by $p(E) = |E|/|S|$.

   (b) What is the probability that you select the six winning numbers in a lottery if the six different winning numbers are selected from the first 50 positive integers?

   **Instructor's solution:** The sample space $S$ is the set of all possible combinations of six different numbers taken from 50, which means that $|S| = C(50, 6) = 50!/(6!44!) = 15\,890\,700$.
   We are interested in the event $E$ that the right combination is selected, which means $|E| = 1$.
   Hence, we have $p(E) = |E|/|S| = 1/15\,890\,700$.

2. (Rosen Review Question 7-2) `[Easy]`

   (a) What conditions should be met by the probabilities assigned to the outcomes from a finite sample space?

   **Instructor's solution:** The probability distribution $p : S \to \mathbb{R}$ on a sample space $S$ must satisfy:
   i) $0 \leq p(s) \leq 1$, for all $s \in S$, and
   ii) $\sum_{s \in S} p(s) = 1$.

   (b) What probabilities should be assigned to the outcome of heads and the outcome of tails if heads comes up three times as often as tails?

**Instructor's solution:** The sample space is $S = \{H, T\}$, where $H$ represents heads and $T$ represents tails.
We are told that $p(H) = 3p(T)$. Since it must be the case that $p(H) + p(T) = 1$, we deduce that $3p(T) + p(T) = 1$, and hence $p(T) = 1/4$. From that we derive that $p(H) = 3/4$.

3. (Rosen Review Question 7-3) `[Easy]`

(a) Define the conditional probability of an event $E$ given an event $F$.

**Instructor's solution:** The probability of event $E$ given event $F$ is given by $p(E \mid F) = |E \cap F|/|F|$.

(b) Suppose $E$ is the event that when a die is rolled it comes up an even number, and $F$ is the event that when a die is rolled it comes up 1, 2, or 3. What is the probability of $F$ given $E$?

**Instructor's solution:** The sample space $S$ is the set of all outcomes of the roll of a die, i.e., $S = \{1, 2, 3, 4, 5, 6\}$, and hence $|S| = 6$.
We are interested in $p(F \mid E) = |E \cap F|/|E|$.
The event $E \cap F$ is the event of the die coming up both an even number and a number in $\{1, 2, 3\}$, hence $E \cap F = \{2\}$ and $|E \cap F| = 1$.
The event $E$ is the set $E = \{2, 4, 6\}$, hence $|E| = 3$.
Then we can calculate that $p(F \mid E) = |E \cap F|/|E| = 1/3$.

4. (Rosen Review Question 7-4) `[Easy]`

(a) When are two events $E$ and $F$ independent?

**Instructor's solution:** Two events $E$ and $F$ in a sample space $S$ are independent when $p(E) = p(E \mid F)$.
Equivalently, we can say they are independet when $p(E \cap F) = p(E)p(F)$.

(b) Suppose $E$ is the event that an even number appears when a fair die is rolled, and $F$ is the event that a 5 or 6 comes up. Are $E$ and $F$ independent?

**Instructor's solution:** The sample space $S$ is the set of all outcomes of the roll of a die, i.e., $S = \{1, 2, 3, 4, 5, 6\}$, and hence $|S| = 6$.
The event $E$ is the set $E = \{2, 4, 6\}$, hence $p(E) = |E|/|S| = 3/6 = 1/2$.
The event $F$ is the set $F = \{5, 6\}$, hence $p(F) = |F|/|S| = 2/6 = 1/3$.
The event $E \cap F$ is the set $\{6\}$, hence $p(E \cap F) = |E \cap F|/|S| = 1/6 = 1/6$.
We can verify that $p(E)p(F) = 1/2 \cdot 1/3 = 1/6$ is equal to $p(E \cap F) = 1/6$, hence $E$ and $F$ are independent events.

5. (Rosen Review Question 7-5) `[Easy]`

(a) What is a random variable?

**Instructor's solution:** A random variable is a function $X : S \to \mathbb{R}$ from the sample space $S$ of an experiment to the reals.

A random variable maps each possible result of an experiment to a real number.

(b) What are the possible values assigned by the random variable $X$ that assigns to a roll of two dice the larger number that appears on the two dice?

**Instructor's solution:** The possible values of $X$ are $1, 2, 3, 4, 5, 6$ .

**Exercises.**

6. (Rosen 7.1-5) `[Easy]` What is the probability that the sum of the numbers on two dice is even when they are rolled?

**Instructor's solution:** The sample space of rolling two dice can be represented by the following table, where the entry $(i, j)$ is the result where the first die ($d_1$) comes up $i$ and the second die ($d_2$) comes up $j$.

| $d_1/d_2$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 112 |

Of the 36 possible results, 18 are an even sum, hence the probability of obtaining an even sum in the rolling of two dice is $18/36 = 1/2$.

7. (Rosen 7.1-21) `[Easy]` What is the probability that a fair die never comes up an even number when it is rolled six times?

**Instructor's solution:** The sample space is the set of all possible 6-tuples of integers between 1 and 6, where the $i^{th}$ element in the tuple $(1 \leq i \leq 6)$ represents the result of the $i^{th}$ roll of the die.

The size of the sample space is the total of 6-tuples that can be formed, that is, $|S| = 6^6 = 46\,656$.

Let $E$ be the event that no even number comes up, that means that each of the 6 positions in the tuple is either 1, 3, or 5. That means that $|E| = 3^6 = 729$.

Hence, we can calculate $p(E) = |E|/|S| = 729/46\,656 = 1/64$.

8. (Rosen 7.1-37) `[Medium]` Which is more likely: rolling a total of 9 when two dice are rolled or rolling a total of 9 when three dice are rolled?

**Instructor's solution:** Let's approach each case separately.

- If two dice are rolled the sample space $S_2$ consists in all tuples $(i, j)$ where $i$ is the result of the first die and $j$ is the result of the second die. Because $1 \leq i, j \leq 6$, $|S_2| = 6 \cdot 6 = 36$.
  We are interested in the event $E_2$ that the sum of $i + j$ in a tuple $(i, j)$ equals 9. We can see that $E_2 = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$, hence $p_2(E_2) = |E_2|/|S_2| = 4/36 = 1/9$.

- If three dice are rolled the sample space $S_3$ consists in all tuples $(i, j, k)$ where $i$ is the result of the first die, $j$ is the result of the second die, and $k$ is the result of the third die. Because $1 \leq i, j, k \leq 6$, $|S_3| = 6 \cdot 6 \cdot 6 = 216$.

  We are interested in the event $E_3$ that the sum of $i + j + k$ in a tuple $(i, j, k)$ is equal 9.

  We can see that $E_3 = \{(1, 2, 6), (1, 3, 5), (1, 4, 4), (1, 5, 3), (1, 6, 2), (2, 1, 6), (2, 2, 5), (2, 3, 4), (2, 4, 3), (2, 5, 2), (2, 6, 1), (3, 1, 5), (3, 2, 4), (3, 3, 3), (3, 4, 2), (3, 5, 1), (4, 1, 4), (4, 2, 3), (4, 3, 2), (4, 4, 1), (5, 1, 3), (5, 2, 2), (5, 3, 1), (6, 1, 2), (6, 2, 1)\}$, hence $p_3(E_3) = |E_3|/|S_3| = 25/216$.

Since $p_2(E_2) = 1/9 = 216/1\,944 < 225/1\,944 = 25/216 = p_3(E_3)$, we conclude that it is more likely to obtain a sum of 9 by rolling 3 dice than by rolling 2.

9. (Rosen 7.2-1) `[Easy]` What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?

   **Instructor's solution:** The sample space is $S = \{H, T\}$, where $H$ stands for heads and $T$ for tails. We know that $p(H) = 3p(T)$ and that $p(H) + p(T) = 1$. Hence we have a system of 2 equations and two variables that when solved lead to $p(H) = 3/4$ and $p(T) = 1/4$.

10. (Rosen 7.2-11) `[Hard]` Suppose that $E$ and $F$ are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

    **Instructor's solution:** First, note that since $E \subseteq E \cup F$, we have that $p(E) \leq p(E \cup F)$, and hence $p(E \cup F) \geq 0.7$.

    Second, note that $p(E \cup F) = p(E) + p(F) - p(E \cap F)$, and we can write

$$\begin{aligned}
p(E \cap F) &= p(E) + p(F) - p(E \cup F) \\
&= 0.7 + 0.5 - p(E \cup F) \\
&= 1.2 - p(E \cup F) \\
&\geq 1.2 - 1 && \text{(since } p(E \cup F) \leq 1\text{)} \\
&= 0.2.
\end{aligned}$$

11. (Rosen 7.2-17) `[Hard]` If $E$ and $F$ are independent events, prove or disprove that $\overline{E}$ and $F$ are necessarily independent events.

    **Instructor's solution:** We will prove that if $E$ and $F$ are independent events, $\overline{E}$ and $F$ are necessarily independent events.

    We want to show that $p(\overline{E} \cap F) = p(\overline{E})p(F)$, under the assumption that $E$ and $F$ are independent.

$$\begin{aligned}
p(\overline{E} \cap F) &= p(F)p(\overline{E} \mid F) && \text{(by def. of conditional probability)} \\
&= p(F)(1 - p(E \mid F)) && \text{(since } p(E \mid F) + p(\overline{E} \mid F) = 1\text{)} \\
&= p(F)(1 - p(E)) && \text{(since } E \text{ and } F \text{ are independent, } p(E \mid F) = p(E)\text{)} \\
&= p(F)p(\overline{E}) && (p(\overline{E}) = 1 - p(E))
\end{aligned}$$

12. (Rosen 7.2-25) `[Medium]` What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)

    **Instructor's solution:** The sample space $S$ is the set of all binary strings of size 4, that is, $|S| = 2^4 = 16$.

    Let $E$ be the event that the string starts with a 1, and $F$ be the event that the string has two consecutive zeros. We are interested in calculating $p(F \mid E) = |E \cap F|/|E|$.

    Let $E$ be the event that the string starts with a 1, hence $|E| = 2^3 = 8$ strings (all possibilities for the last 3 bits, since the first one is fixed).

    The event $E \cap F$ is the event that a string starts with a 1 and has two consecutive 0s. It is easy to see that $E \cap F = \{1000, 1001, 1100\}$.

    Hence we can calculate $p(F \mid E) = |E \cap F|/|E| = 3/8$.

13. (Rosen 7.2-27(b)) `[Medium]` Let $E$ and $F$ be the events that a family of 4 children has children of both sexes and has at most one boy, respectively. Are $E$ and $F$ independent?

    **Instructor's solution:** $E$ and $F$ are independent iff $p(E \cap F) = p(E)p(F)$.

    The sample space $S$ is the set of all possible ways a family can have 4 kids according to sex, hence $|S| = 2^4 = 16$.

    Let $B$ represent a boy and $G$ represent a girl.

    Let $E$ be the event that the family has children of both sexes. It means that $E = S - \{BBBB, GGGG\}$, hence $|E| = 16 - 2 = 14$ and $p(E) = 14/16 = 7/8$.

    Let $F$ be the event that the family has at most one boy. It means that $F = \{GGGG, BGGG, GBGG, GGBG, GGGB\}$, hence $|F| = 5$ and $p(F) = 5/16$.

    The event $E \cap F$ is that in which the family has at most one boy, and children of both sexes, hence $E \cap F = \{BGGG, GBGG, GGBG, GGGB\}$, leading to $|E \cap F| = 4$ and $p(E \cap F) = 4/16 = 1/4$

    Now we can verify that $p(E)p(F) = 7/8 \cdot 5/16 = 70/256$, which is different from $p(E \cap F) = 1/4$, and hence $E$ and $F$ are not independent.