Information Theory **DCC/ICEx/UFMG**
Prof. Mário S. Alvim **2017/02**

HOMEWORK

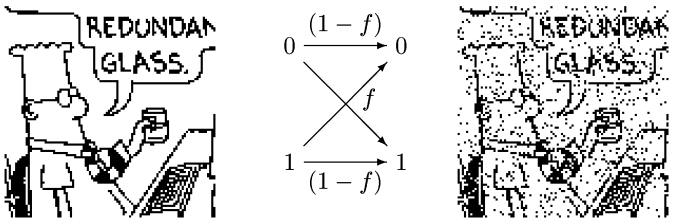
Communication through a noisy channel as inference
(In-Class Activity)

Note: The exercises are labeled according to their level of difficulty: `[Easy]`, `[Medium]` or `[Hard]`. This labeling, however, is subjective: different people may disagree on the perceived level of difficulty of any given exercise. Don't be discouraged when facing a hard exercise, you may find a solution that is simpler than the one the instructor had in mind!

Exercises.

1. **(Data transmission through a noisy channel.)**

   Consider a communication channel that can transmit bits 0 or 1 as depicted below. The channel is noisy and a fraction f of bits is flipped during transmission.



   More precisely, if x is the bit transmitted by the sender and y is the bit received by the receiver:

$$P(y = 0 \mid x = 0) = 1 - f; \qquad P(y = 1 \mid x = 0) = f;$$
$$P(y = 0 \mid x = 1) = f; \qquad P(y = 1 \mid x = 1) = 1 - f.$$

   (a) `[Easy]` Consider the channel is used to transmit a black and white image of N pixels (each pixel is represented by one bit: 0 for a black pixel, 1 for a white pixel).

   What is the probability that the image is transmitted through this channel without any pixel being corrupted (i.e., without any bit being flipped)?

   (b) `[Easy]` What is the expected number of corrupted bits during the transmission of N bits through this channel?

   (c) `[Easy]` To improve the quality of data transmission through this channel, assume we employ a repetition code R_4, so each pixel selected by the sender is repeated 4 times during transmission.[1] (E.g., if the pixel to be sent is $s = 0$, the sender transmits the sequence $t = 0000$, and if the pixel to be sent is $s = 1$, the sender transmits the sequence $t = 1111$.)

   Since each one of all 4 bits in the sequence t can be flipped when passing through the channel, the receiver receives a sequence r that is not necessarily identical to the sequence t transmitted by the sender.

   Recalling that the probability of any bit being flipped is f, find the probability $p(r \mid s)$ of the receiver getting message r if the sender selected pixel s to send, for the following received messages: $r_1 = 0010$, $r_2 = 0110$, and $r_3 = 1111$.

[1] To be precise, in the R_4 code each bit is repeated 3 times, since the first occurrence of the bit in a sequence is not itself a repetition. But let's not be too pedantic!

(d) `[Medium]` After receiving a sequence r of 4 bits, the receiver must infer what was the pixel s sent by the sender.

Formulate the problem of the receiver as a problem of inference, specifying what are the concurrent hypotheses, what is the available evidence, and what is the mathematical formula the receiver must employ to pick a most adequate hypothesis given the available evidence.

(e) `[Medium]` Assume the receiver has gotten the sequences $r_1 = 0010$, $r_2 = 0110$, and $r_3 = 1111$.

Assuming the error rate of the channel is $f = 0.2$, evaluate how good of an evidence is each message (r_1, r_2, r_3) in favor of the sent pixel being $s = 0$.

(Justify your answer using the likelihood ratio $p(r \mid s = 0)/p(r \mid s = 1)$.)

(f) `[Medium]` Assume the channel will be used to transmit an image in which 20% of pixels are black (bit 0), and 80% of pixels are white (bit 1).

If the error rate of the channel is $f = 0.2$, determine the receiver's best inference about the pixel sent (black or white) for each one of the following received sequences: $r_1 = 0010$, $r_2 = 0110$, and $r_3 = 1111$.

2. **(Channel breakdown!)**

Consider a noisy channel just like the one in the previous question, and that we want transmit bits through it using the repetition code R_4.

(a) `[Medium]` Unfortunately, the communication channel is of horrible quality and presents an error rate of $f = 1$ (i.e., every single bit is flipped during transmission).

In this case, among all 16 sequences r the receiver may get (0000, 0001, 0010, ..., 1111), which ones are evidence in favor of the sent pixel being $s = 0$? Which ones are evidence in favor of the sent pixel being $s = 1$?

(b) `[Medium]` Someone tried to fix the channel, and now it is working with an error rate of $f = 0.5$ (that is, half of the bits are flipped during transmission).

In this case, among all 16 sequences r the receiver may get (0000, 0001, 0010, ..., 1111), which ones are evidence in favor of the sent pixel being $s = 0$? Which ones are evidence in favor of the sent pixel being $s = 1$?

(c) `[Medium]` Based on your previous answers, which channel is better for the receiver: the one that flips every bit ($f = 1$), or the one that flips half of the bits ($f = 0.5$)? In other words, in this particular case, have you actually improved the quality of transmission by reducing the error rate?

Justify your answer by arguing that the channel you chose allows for more reliable inference.