Teoria da Informação - Lista 01B

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Review questions

1. (a) O valor esperado é a soma produto da probabilidade e o valor da variável aleatória.

$$E(X) = \sum_{s \in S} p(s)X(s)$$

É uma representação análoga ao centro de massa.

(b)
- $\{(1,1)\}$
  $X = 1 \Rightarrow p(X = 1) = \frac{1}{36}$
- $\{(2,1),(1,2),(2,2)\}$
  $X = 2 \Rightarrow p(X = 2) = \frac{3}{36}$
- $\{(3,1),(3,2),(1,3),(2,3),(3,3)\}$
  $X = 3 \Rightarrow p(X = 3) = \frac{5}{36}$
- $\{(k,1),(k,2),\ldots,(k,k)\}$
  $X = k \Rightarrow p(X = k) = \frac{2k-1}{36}$

Portanto o valor esperado é:

$$E(X) = \sum_{k=1} \frac{2k-1}{36} \cdot k = 1 \cdot \frac{1}{36} + \ldots + 6 \cdot \frac{11}{36} = \frac{161}{36} \approx 4.47$$

2. (a) O significado do ensaio de Bernoulli é a probabilidade ter k sucessos em n tentativas.

(b)

$$p(X = k) = C(n,k)p^k q^{n-k}$$

(c) Mostrando uma equivalência necessária para o desenvolvimento.

$$C(n,k) = \frac{n!}{(n-k)!k!} = \frac{n}{k} \cdot \frac{(n-1)!}{(n-k+1-1)(k-1)!} =$$
$$\frac{n}{k} \cdot \frac{(n-1)!}{((n-1)-(k-1))(k-1)!} = \frac{n}{k} \cdot C(n-1,k-1) \Rightarrow$$
$$C(n,k) = \frac{n}{k} \cdot C(n-1,k-1) \Rightarrow k \cdot C(n,k) = n \cdot C(n-1,k-1)$$

Lembrando que $q = 1-p$. Seja n o número de ensaios e k k número de sucessos. Lembrando que $p(X = k) = C(n,k) = p^k q^{n-k}$. Então:

$$E(X) = \sum_{k=1}^{n} X(k)p(X = k) =$$
$$\sum_{k=1}^{n} k \cdot C(n,k)p^k q^{n-k} = \sum_{k=1}^{n} n \cdot C(n-1,k-1)p^k q^{n-k} =$$
$$n \sum_{k=1}^{n} C(n-1,k-1)p^k q^{n-k} \underbrace{=}_{j=k-1} n \sum_{j=0}^{n-1} C(n-1,j)p^{j+1} q^{n-j+1} =$$
$$np \underbrace{\sum_{j=0}^{n-1} C(n-1,j)p^j q^{n-j+1}}_{(p+q)^{n-1}} = np(p+q)^{n-1} = np(p+(1-p))^{n-1} =$$
$$np(1)^{n-1} = np$$

3. (a) Linearidade da esperança é, dada duas variáveis aleatórias X e Y:

$$E(X+Y) = E(X) + E(Y)$$

4.

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})} =$$
$$\frac{\frac{1}{3} \cdot \frac{2}{3}}{\frac{1}{3} \cdot \frac{2}{3} + \frac{1}{4} \cdot \left(1 - \frac{2}{3}\right)} = \frac{\frac{1}{3} \cdot \frac{2}{3}}{\frac{1}{3} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3}} = \frac{8}{11}$$

5. Seja X uma variável aleatória no espaço S. Variância de X é:

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s) =$$
$$\sum_{s \in S} \left(X(s)^2 - 2X(s)E(X) + E(X)^2\right) p(s)$$

Utilizando a linearidade.

$$\sum_{s\in S} X(s)^2 p(s) - 2\sum_{s\in S} X(s)E(X)p(s) + \sum_{s\in S} E(X)^2 p(s) =$$
$$\underbrace{\sum_{s\in S}\left(X(s)\right)^2 p(s)}_{E(X^2)} - 2E(X)\underbrace{\sum_{s\in S} X(s)p(s)}_{E(X)} + E(X)^2\underbrace{\sum_{s\in S} p(s)}_{1} =$$
$$E(X^2) - 2(E(X))^2 + (E(X))^2 = E(X^2) - E(X)^2 \Rightarrow$$
$$V(X) = E(X^2) - E(X)^2$$

6. (a)

$$
\begin{aligned}
&V\left(\sum_{i=1}^{n} X_i\right)\\
&= E\left(\left(\sum_{i=1}^{n} X_i\right)^2\right) - E\left(\sum_{i=1}^{n} X_i\right)^2\\
&= V\left(X_1 + \sum_{i=2}^{n} X_i\right)\\
&= E\left(\left(X_1 + \sum_{i=2}^{n} X_i\right)^2\right) - E\left(X_1 + \sum_{i=2}^{n} X_i\right)^2\\
&= E\left({X_1}^2 + 2X_1\sum_{i=2}^{n} X_i + \left(\sum_{i=2}^{n} X_i\right)^2\right)\\
&\quad - \left(E\left(X_1\right) + E\left(\sum_{i=2}^{n} X_i\right)\right)^2\\
&= E({X_1}^2) + \underbrace{E\left(2X_1\sum_{i=2}^{n} X_i\right)}_{2E(X_1)E\left(\sum_{i=2}^{n} X_i\right)} + E\left(\left(\sum_{i=2}^{n} X_i\right)^2\right)\\
&\quad - \left(E(X_1)^2 + 2E\left(X_1\right)E\left(\sum_{i=2}^{n} X_i\right) + \left(E\left(\sum_{i=2}^{n} X_i\right)\right)^2\right)\\
&= E({X_1}^2) + 2E\left(X_1\right)E\left(\sum_{i=2}^{n} X_i\right) + E\left(\left(\sum_{i=2}^{n} X_i\right)^2\right)\\
&\quad - E(X_1)^2 - 2E\left(X_1\right)E\left(\sum_{i=2}^{n} X_i\right) - \left(E\left(\sum_{i=2}^{n} X_i\right)\right)^2\\
&= E({X_1}^2) + E\left(\left(\sum_{i=2}^{n} X_i\right)^2\right)\\
&\quad - E(X_1)^2 - \left(E\left(\sum_{i=2}^{n} X_i\right)\right)^2\\
&= \underbrace{E({X_1}^2) - E(X_1)^2}_{V(X_1)}\\
&\quad + \underbrace{E\left(\left(\sum_{i=2}^{n} X_i\right)^2\right) - \left(E\left(\sum_{i=2}^{n} X_i\right)\right)^2}_{V\left(\sum_{i=2}^{n} X_i\right)}
\end{aligned}
$$

Se refizermos a operação para as somas restantes $\sum_{i=2}^{n} X_i$, teremos:

$$V\left(\sum_{i=1}^{n} X_i\right) = V(X_1) + V(X_2) + \dots V(X_n) = \sum_{i=1}^{n} V(X_i)$$

Exercises

7. Caixa 1: 2 brancas e 3 azuis.

   Caixa 2: 4 brancas e 1 azuis.

   B: pegar bola branca

   A: pegar bola azul

   X_1: pegar na caixa 1

   X_2: pegar na caixa 2

$$p(X_1|A) = \frac{p(A|X_1).p(X_1)}{p(A|X_1)p(X_1) + p(A|X_2)p(X_2)} = \frac{\frac{3}{5} \cdot \frac{1}{2}}{\frac{3}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2}} = \frac{3}{3+1} = \frac{3}{4}$$

8. De princípio, definimos:

   - H: HIV positivo
   - $\overline{H}$: HIV negativo
   - T: teste indica HIV positivo
   - $\overline{T}$: teste indica HIV positivo

   Pelo enunciado, temos:

$$\begin{aligned}
&p(H) = 0.08 \\
&p(\overline{H}) = 1 - p(H) = 1 - 0.08 = 0.92 \\
&p(T|H) = 0.98 \\
&p(\overline{T}|H) = 1 - p(T|H) = 0.02 \\
&p(T|\overline{H}) = 0.03 \\
&p(\overline{T}|\overline{H}) = 1 - p(T|\overline{H}) = 0.97
\end{aligned}$$

   (a) Utilizando o Teorema de Bayes.

$$\begin{aligned}
p(H|T) &= \frac{p(T|H)p(H)}{p(T|H)p(H) + p(T|\overline{H})p(\overline{H})} \\
&= \frac{0.98 \cdot 0.08}{0.98 \cdot 0.08 + 0.03 \cdot 0.92} \approx 0.739
\end{aligned}$$

(b)

$$p(\overline{H}|T) = 1 - p(H|T) \approx 1 - 0.739 = 0.261$$

(c) Utilizando o Teorema de Bayes.

$$p(H|\overline{T}) = \frac{p(\overline{T}|H)p(H)}{p(\overline{T}|H)p(H)+p(\overline{T}|\overline{H})p(\overline{H})}$$
$$= \frac{0.02 \cdot 0.08}{0.02 \cdot 0.08 + 0.97 \cdot 0.92} \approx 0.002$$

(d)

$$p(\overline{H}|\overline{T}) = 1 - p(H|\overline{T}) \approx 1 - 0.002 = 0.998$$