Mathematical Proofs for "Incentive Mechanism Design for Heterogeneous Crowdsourcing Using All-Pay Contests"

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Abstract—This document provides the mathematical proofs for [1].

1 Proof of Lemma 1

Proof. **Existence:**

The existence proof parallels [2] (Theorem 2) where the assumptions are:

(i) IPV model: the player types (e.g., values of prize) are independent and private;

(ii) Common support: all players share the same interval of types, $[\underline{v}, \bar{v}]$;

(iii) Properties of distribution: all c.d.f. F_i's are continuous over the closed interval $[\underline{v}, \bar{v}]$ and differentiable over the half-open interval $(\underline{v}, \bar{v}]$, and all p.d.f. f_i's are bounded away from zero over $(\underline{v}, \bar{v}]$;[1]

(iv) Mass at the lower extremity: either (a) there is no mass at $\underline{v}$, i.e., $F_i(\underline{v}) = 0, \forall i$, or (b) $F_i(\underline{v}) > 0$ and F_i is right-hand differentiable at $\underline{v}$ and $f_i(v)$ is bounded away from zero for all $v \in [\underline{v}, \bar{v}]$, $\forall i$.

Our model obeys all these assumptions. Moreover, we conjecture that the existence of equilibria in first-price auctions is reciprocal to the existence of equilibria in all-pay auctions, provided that all the assumptions are the same except for the auction institution. (However, monotonicity does not inherit this reciprocity and we need to reconcile a difference in utility functions; see next.)

Monotonicity:

The monotonicity proof follows [3] (Proposition 1), but to apply that result we need to reconcile a difference between first-price and all-pay auctions. In first-price auctions, the payoff of a bidder i is zero when his bid is unsuccessful, but in all-pay auctions, it is negative. Therefore, we will use a "modified" utility function, $\hat{u}_i$, by adding back the negative component (i.e., payment) to the original utility function, u_i, as

$$\hat{u}_i = u_i + p(b_i, v_i).$$

Furthermore, note that the utility referred to by [3] is actually the utility when a bidder *wins* the auction, not the *expected* utility that is commonly used and that involves a winning probability. Therefore, we end up using the following modified "winning" utility function:

$$\hat{u}_i^{win} = u_i^{win} + p(b_i, v_i),$$

where u_i^{win} is the original utility when agent i wins the contest, and is thus $V(v_i, Z_i(b_i)) - p(b_i, v_i)$. Hence, $\hat{u}_i^{win} = V(v_i, Z_i(b_i))$. Now, in order to apply the result of [3] we need to verify whether $\hat{u}_i^{win}$ satisfies the *weak supermodularity* which is defined as

$$\frac{\partial^2 u_i^{win}(b_i, v_i, v_{-i})}{\partial b_i \partial v_i} \geq 0, \quad \forall i, \forall \mathbf{v} = (v_i, v_{-i}).$$

It is reasonable to assume that the value function of a prize, $V(v, Z)$, satisfies $\frac{\partial V(v,Z)}{\partial Z} > 0$ (higher prize implies more value) and $\frac{\partial V(v,Z)}{\partial v} \geq 0$ (higher type is able to derive more value from the same prize). Further, we assume that $\frac{\partial^2 V(v,Z)}{\partial v \partial Z} \geq 0$ which means that, as prize increases, the value that a higher type is able to derive from the prize increases in a faster speed than the prize; in other words, if the prize increases linearly, then a higher type can gain value in a super-linear manner.[2] In addition, since $Z'(b) \geq 0$ which is self-explanatory (higher bids should deserve higher prize), we now have

$$\frac{\partial^2 V(v_i, Z_i(b_i))}{\partial b_i \partial v_i} = \frac{\partial^2 V(v_i, Z_i(b_i))}{\partial Z_i \partial v_i} \frac{\mathrm{d}Z_i}{\mathrm{d}b_i} \geq 0.$$

This means that u_i^{win}, which derives from our utility function, is weakly supermodular. The monotonicity of equilibrium thus follows from [3] (Proposition 1).

Uniqueness:

The uniqueness proof is analogous to [4] (Theorem 1) as a special case of possibly different type supports. Four assumptions need to be verified against: the first two are (i) and (iii) in the existence proof above (IPV and distribution function), the third is $F_i(\underline{c}) = 0$, i.e., there is no mass point at the lower extremity (the case with mass point, i.e., $F_i(\underline{c}) > 0$, and common support also admits a unique equilibrium, as proved in [2][3]). The fourth and last assumption is that

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1. That is, there exists some $\delta > 0$ such that $f_i(v) > \delta$ for all $v \in (\underline{v}, \bar{v}]$.

2. This is certainly feasible in practice. For example, a higher prize enables a stronger winner to invest in a wider portfolio with super-linear return, or to attract much larger attention from the media. In fact, one could draw an analogy here to the well-known Mathew effect, "the rich get richer and the poor get pooer'.

3. Uniqueness in the case with a mass point was also proved by [5] and [6]. However, [4] points out that both [7]—an early version of [5]—and [6] contain an error in their proofs.

there exists a $\delta > 0$ such that F_i is strictly *log-concave* over $(\underline{v}, \underline{v} + \delta)$.[4]

As our model obeys the first three assumptions, we only need to limit our c.d.f. F_i's to those that satisfy the log-concavity. This means that $\ln F_i$ must be strictly concave, or f_i/F_i is strictly decreasing. Nevertheless, this additional condition is *not* restrictive, as it is in fact common in economic theory (see [8] and [9]), and as an example, both uniform and exponential distributions are log-concave. Also, note that it only requires F_i to be "locally" log-concave, i.e., near the lower extremity $\underline{v}$ and not over the entire support.

Common (bid) support:

Given that the agent types have a common and nonnegative support, $[\underline{v}, \bar{v}]$, all the agent in our contest will bid in the range $[0, \bar{b}]$. This follows from combining Lemma 1 and 4 of [10] where the argument of the two lemmas holds for the n-player case. Alternatively, this can also be proved using Lemma 10 and 11 of [5] but with a few additional steps for verifying against the assumptions therein. $\square$

2 Proof of Lemma 2

Proof. In equilibrium, each agent takes the best response which maximizes his utility u_i (3), and hence b_i is also the solution to the optimization problem $\max_{b_i} u_i$. Thus we invoke the envelope theorem [11] on (3) with respect to v_i and obtain

$$\frac{\partial u_i}{\partial v_i} = V'_{v_i}(v_i, Z_i(b_i)) \prod_{j \neq i} F_j(v_j(b_i)) - p'_{v_i}(b_i, v_i)$$

$$\Rightarrow u_i(v_i) = u_i(\underline{v}) + \int_{\underline{v}}^{v_i} \Big[V'_{v_i}(\tilde{v}_i, Z_i(b_i)) \prod_{j \neq i} F_j(v_j(b_i)) - p'_{v_i}(b_i, \tilde{v}_i) \Big] \, \mathrm{d}\tilde{v}_i. \quad \text{(A.1)}$$

Since an agent with the lowest possible type never wins the auction, he will bid zero (i.e., exert no effort) in an all-pay auction (rather than bidding $b_i = \underline{v}$ as in first or second-price auctions). As a result, he reaps zero utility, i.e., $u_i(\underline{v}) = 0$. Thus, equating the r.h.s of (A.1) to that of (3) yields the result. $\square$

3 Proof of Corollary 1

Proof. Apply Lemma 2 with $V(v, Z) = h(v)Z$. Note that Lemma 2 is derived from $\max_{b_i} u_i$, or equivalently $\arg\max_{b_i} u_i$. When $V(v, Z) = h(v)Z$, it can be rewritten as $\arg\max_{b_i} \frac{u_i}{h(v_i)}$ for $v_i > 0$. By spelling this out, we have

$$\arg\max_{b_i} Z_i(b_i) \prod_{j \neq i} F_j(v_j(b_i)) - \hat{p}(b_i, v_i). \quad \text{(A.2)}$$

Therefore, when dealing with u_i, we can simultaneously treat $V(v, Z)$ as Z and $p(\cdot)$ as $\hat{p}(\cdot)$, thereby obtaining the result (5) from Lemma 2, where $Z'_{i v_i}(b_i) = 0$ due to the envelope theorem. $\square$

4. We have tailored this last assumption to our model. In detail, since our model essentially admits a reserve price of zero and adopts a common type support, two of the three "or" conditions (i–iii) postulated by [4, Theorem 1] are violated, and hence we must satisfy the remaining assumption (iii) therein which is the log-concavity stated here.

4 Proof of Theorem 1

Proof. We begin by expanding the principal's expected profit (2). First, the revenue portion can be expanded as

$$\mathbb{E}\Big[\sum_i b_i\Big] = \sum_i \int_{\underline{v}}^{\bar{v}} b_i(v_i) \, \mathrm{d}F_i(v_i).$$

Second, the prize portion can be expanded using the law of total expectation, as

$$\mathbb{E}\left[V(\lambda, Z_w(b_w))\right] = h(\lambda)\mathbb{E}\Big[\sum_i q_i Z_i(b_i(v_i))\Big]$$
$$= h(\lambda) \sum_i \int_{\underline{v}}^{\bar{v}} Z_i(b_i(v_i)) \prod_{j \neq i} F_j(v_j(b_i)) \, \mathrm{d}F_i(v_i).$$

Therefore,

$$\pi = \sum_i \int_{\underline{v}}^{\bar{v}} \Big[b_i(v_i) - h(\lambda) Z_i(b_i(v_i)) \prod_{j \neq i} F_j(v_j(b_i)) \Big] \, \mathrm{d}F_i(v_i). \quad \text{(A.3)}$$

With Corollary 1, substituting (5) into (A.3) yields

$$\pi = \sum_i \int_{\underline{v}}^{\bar{v}} \Big[b_i(v_i) - h(\lambda)\hat{p}(b_i, v_i) + h(\lambda) \int_{\underline{v}}^{v_i} \hat{p}'_{v_i}(b_i, \tilde{v}_i) \, \mathrm{d}\tilde{v}_i \Big] \, \mathrm{d}F_i. \quad \text{(A.4)}$$

Integrating the last term by parts,

$$\int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v_i} \hat{p}'_{v_i}(b_i(\tilde{v}_i), \tilde{v}_i) \, \mathrm{d}\tilde{v}_i \, \mathrm{d}F_i$$
$$= \int_{\underline{v}}^{\bar{v}} \hat{p}'_{v}(b_i(v_i), v_i) \, \mathrm{d}v_i - \int_{\underline{v}}^{\bar{v}} F_i(v_i) \hat{p}'_{v_i}(b_i(v_i), v_i) \, \mathrm{d}v_i$$
$$= \int_{\underline{v}}^{\bar{v}} \hat{p}'_{v_i}(b_i(v_i), v_i) \frac{1 - F_i}{f_i} \, \mathrm{d}F_i,$$

which gives (8) by substituting itself back into (A.4).

Completing the proof of (8) requires solving b_i. Consider the principal's optimization problem, $\max_{\mathbf{Z}} \pi$ (8). It is equivalent to $\max_{\mathbf{b}} \pi$ because the principal is using an optimal prize tuple $\mathbf{Z}$ to essentially induce the optimal effort vector $\mathbf{b}$ which, consequently, leads to the maximum profit. Furthermore, in (8) we have decoupled each agent i from other agents $j \neq i$. Therefore, maximizing π can be achieved by maximizing each individual integrand I_i over b_i, where

$$I_i := b_i(v_i) - h(\lambda)\hat{p}(b_i, v_i) + h(\lambda)\hat{p}'_{v_i}(b_i, v_i) \frac{1 - F_i}{f_i}. \quad \text{(A.5)}$$

Applying the first order condition to I_i with respect to b_i gives

$$\frac{\partial I_i}{\partial b_i} = 1 - h(\lambda)\hat{p}'_{b_i}(b_i, v_i) + h(\lambda)\hat{p}''_{b_i, v_i}(b_i, v_i) \frac{1 - F_i}{f_i} = 0,$$

which proves (7).

To verify that I_i has an unique maximizer, we examine

$$\frac{\partial^2 I_i}{\partial {b_i}^2} = -h(\lambda)\hat{p}''_{b_i^2}(b_i, v_i) + h(\lambda)\hat{p}'''_{b_i^2 v_i}(b_i, v_i) \frac{1 - F_i}{f_i}.$$

Since $\hat{p} = p/h(v)$, and $v > 0$ is treated as constant due to the use of envelope theorem, our assumptions on $p(\cdot)$ also hold for $\hat{p}(\cdot)$, i.e., $\hat{p}''_{b_i^2} > 0$ and $\hat{p}'''_{b_i^2 v_i} \le 0$. Since $h(\lambda) > 0$ for $\lambda > 0$, therefore $I_i'' < 0$. Thus I_i is strictly concave, and hence b_i as given by (7) exists and is unique.

Finally, to prove the optimal prize tuple (6), given that b_i is solved, we rearrange (5) and change the variables thereof from v_i to b_i. The lower limit of the integral is 0 because $b_i(\underline{v}) = 0$ as the lowest-type agent will bid zero in an all-pay auction (cf. proof of Lemma 2). □

5 Proof of Proposition 1

Proof. Notice that the expression under maximization in (A.2) is $u_i/h(v_i)$. Thus it follows from (5) that

$$\frac{u_i}{h(v_i)} = -\int_{\underline{v}}^{v_i} \hat{p}'_{v_i}(b_i, \tilde{v}_i)\,\mathrm{d}\tilde{v}_i$$
$$\Rightarrow u_i = -h(v_i)\int_{\underline{v}}^{v_i} \frac{p'_{v_i}(b_i,\tilde{v}_i)h(\tilde{v}_i) - p(b_i,\tilde{v}_i)h'(\tilde{v}_i)}{h^2(\tilde{v}_i)}\,\mathrm{d}\tilde{v}_i.$$

According to Lemma 1, the equilibrium is strictly monotone and type $\underline{v}$ will bid zero. Therefore, $b_i(v_i) > 0$ for any $v_i > \underline{v}$. Since $p(0, v) = 0$ and $p'_b(b, v) > 0$, thus $p(b, v) > 0$ for any $b > 0$. Similarly, since $h'(v) > 0$ and $h(0) = 0$ (Section 4.2), $h(v) > 0$ for any $v > 0$. In addition, we know that $p'_v(b, v) \le 0$. Therefore, $u_i \ge 0$, which proves IR, and the equality holds iff $v_i = \underline{v}$ (where $\underline{v} \ge 0$). Since an agent of type $\underline{v}$ will choose not to participate ($b_i = 0$), any agent who exerts nonzero effort reaps a strictly positive payoff. □

6 Proof of Proposition 2

Proof. The existence and uniqueness are due to Lemma 1.[5] To solve for the equilibrium strategy $\mathbf{b} = (b_1, b_2)$, first write agent i's utility below, where we recall that $v_i(\cdot) := \beta_i^{-1}(\cdot)$,

$$u_1 = F_2(v_2(b_1))v_1 Z - p(b_1),$$
$$u_2 = F_1(v_1(b_2))v_2 Z - p(b_2).$$

To maximize u_i, applying the first-order condition yields

$$\partial u_1/\partial b_1 = F_2'(v_2(b_1))v_2'(b_1)v_1 Z - p'(b_1) = 0, \quad \text{(A.6)}$$
$$\partial u_2/\partial b_2 = F_1'(v_1(b_2))v_1'(b_2)v_2 Z - p'(b_2) = 0. \quad \text{(A.7)}$$

In (A.7), treat b_2 as a parameter and substitute it by b_1, and meanwhile notice that $v_2 = v_2(b_2)$. Then we have

$$F_1'(v_1(b_1))v_1'(b_1)v_2(b_1)Z = p'(b_1). \quad \text{(A.8)}$$

Define $k(v_1) := v_2(b_1(v_1)) = \beta_2^{-1}(b_1(v_1))$, in the spirit of [10]. Thus

$$k'(v_1) = v_2'(b_1(v_1))b_1'(v_1). \quad \text{(A.9)}$$

The first term on the r.h.s. equals, according to (A.6),

$$v_2'(b_1(v_1)) = \frac{p'(b_1)}{F_2'(v_2(b_1(v_1)))v_1 Z} = \frac{p'(b_1)}{F_2'(k(v_1))v_1 Z}.$$

The second term can be rewritten firstly using the theorem of derivative of inverse function, and secondly (A.8), as follows:

$$b_1'(v_1) = \frac{1}{v_1'(b_1(v_1))} = \frac{F_1'(v_1)v_2(b_1)Z}{p'(b_1)} = \frac{F_1'(v_1)k(v_1)Z}{p'(b_1)}. \quad \text{(A.10)}$$

Therefore, (A.9) equals, by replacing v_1 with v,

$$k'(v) = \frac{F_1'(v)k(v)}{F_2'(k(v))v}. \quad \text{(A.11)}$$

Agent 1's equilibrium strategy can now be solved via (A.10):

$$p'(b_1)b_1'(v_1) = p'_{v_1}(b_1(v_1)) = F_1'(v_1)k(v_1)Z$$
$$\Rightarrow b_1(v_1) = p^{-1}\Big(Z\int_{k^{-1}(\underline{v})}^{v_1} F_1'(v)k(v)\,\mathrm{d}v\Big)$$

where $k(v)$ is determined by (A.11). Using $k^{-1}(\underline{v})$ instead of $\underline{v}$ as the lower limit of integral is to ensure $k(v)$ to be differentiable (cf. (A.9)) as $k(\cdot)$ essentially maps the support of v_1 to that of v_2. In addition, using $\underline{v}$ in $k^{-1}(\cdot)$ is because the equilibrium strategy is monotone increasing (cf. Lemma 1).

Agent 2's equilibrium strategy is then solved by the definition of $k(\cdot)$, as

$$\beta_2(k(v_1)) = b_1(v_1) \quad \Rightarrow \quad b_2(v_2) = b_1(k^{-1}(v_2)).$$

The boundary condition $k(\bar{v}) = \bar{v}$ can be proved using Lemma 1 as follows. Since the common support of equilibrium bids is $[0, \bar{b}]$ and the strategy is monotone increasing, $b_1(\bar{v}) = \bar{b}$. Furthermore, the inverse function of the strategy is also monotone increasing, and hence $\beta_2^{-1}(\bar{b}) = \bar{v}$. Therefore, it follows from the definition of $k(v)$ that $k(\bar{v}) = \beta_2^{-1}(b_1(\bar{v})) = \bar{v}$. □

7 Proof of Proposition 3

Proof. The utility of an agent of type v is

$$u = vZF^{n-1}(v) - p(b).$$

To maximize u, applying the first-order condition with respect to b, and noting that the inner v is actually $v(b)$, give

$$vZ\frac{\mathrm{d}F^{n-1}(v)}{\mathrm{d}v}\frac{1}{b'(v)} - p'(b) = 0$$
$$\Rightarrow p'(b)b'(v) = p'_v(b(v)) = vZ\frac{\mathrm{d}F^{n-1}(v)}{\mathrm{d}v}$$
$$\Rightarrow p(b(v)) = Z\int_{\underline{v}}^{v} t\,\mathrm{d}F^{n-1} = ZtF^{n-1}|_{\underline{v}}^{v} - Z\int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t$$
$$\Rightarrow b(v) = p^{-1}\Big(vZF^{n-1}(v) - Z\int_{\underline{v}}^{v} F^{n-1}(t)\,\mathrm{d}t\Big).$$

□

5. Alternatively, the existence can be attributed to [10, Theorem 1] and the uniqueness to [5, Proposition 1].

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