Extension: Problem (b) and Joint Chance Constraints in Incentivizing Participatory Sensing

Tie Luo[*,†] and Chen-Khong Tham[†]

[*]Institute for Infocomm Research (I^{2}R), A*STAR, Singapore
[†]Department of Electrical and Computer Engineering, National University of Singapore
luot@i2r.a-star.edu.sg, eletck@nus.edu.sg

Intended for readers who are interested, this document extends [1] by

1. solving problem (b), which is a (simpler) variant of problem (a), by following the same line of thought as used for solving problem (a) in [1], and
2. giving further discussion on joint chance constraints, a variant of (and, in our case, less effective than) the individual chance constraints addressed in [1].

1 Solving Problem (b)

Let us reproduce problem (b) below:

$$\begin{aligned} \max\ & \sum_{i=1}^{N} \psi_i \log(1+\Psi\frac{q_i}{Q_i})/c_i. \\ \text{s.t.}\ \ & q_i \in [0, Q_i],\ i=1,...,N \\ & \textstyle\sum_{i=1}^{N} q_i \le Q_{tot}. \end{aligned} \tag{1}$$

Again, denote by $g(q_i)$ the i-th term of (1). The reciprocal of marginal weighted utility is $1/g'(q_i) = \frac{Q_i/\Psi+q_i}{\psi_i/c_i}$. Thus, the above optimization problem can be visualized as the iced-tank representation of problem (a) (Fig. 2 in [1]), except for the following changes: the bottom area of each tank changes to ψ_i/c_i, the ice volume of each tank changes to Q_i/Ψ, and the empty space of each tank changes to Q_i. Note that an important property here, is that the volume (and hence height) of ice and that of empty space of each tank, are proportional to each other (with a constant coefficient, Ψ). As a result, sorting the tanks in order of tank heights is equivalent to sorting the tanks in order of ice levels, which implies that a user who is given higher priority to be granted service quota (due to lower ice level) will surely be satisfied earlier (due to lower tank height). Therefore, the solution algorithm is simpler than ITF which handles irregular ice levels. The pseudo-code is given as Algorithm 1, which we call ITF-(b).

We give the following theoretical properties of problem (b) without proof, as they can be easily proven by following the proofs available in [1].

Theorem 1. *Proposition 1, Theorem 1, and Corollary 1 in [1] hold for problem (b) without change.*

Define a game π_b the same as defining π_a in Section IV-A of [1], except that the game rule ITF is replaced by ITF-(b).

Lemma 1. *The necessary and sufficient conditions for a Nash Equilibrium of game π_b are*

$$\begin{cases} \text{C1}:\ \sum_{i=1}^{N} Q_i = Q_{tot}, \\ \text{C2}:\ c_iQ_i/\psi_i = c_jQ_j/\psi_j,\ \forall i,j=1,...,N. \end{cases} \tag{2}$$

Algorithm 1 ITF-(b)

Input: $N, Q_{tot}, \Psi, \vec{Q} = \{Q_i\}, \vec{\psi} = \{\psi_i\}, \vec{c} = \{c_i\}$
Output: $\vec{q} = \{q_i\}$
1: **if** $\sum_{i=1}^{N} Q_i \leq Q_{tot}$ **then**
2:   **return** $\vec{q} \leftarrow \vec{Q}$
3: **end if**
4: $\overrightarrow{ice} \leftarrow \{ice_i = \frac{Q_i/\Psi}{\psi_i/c_i}\}$ sorted in ascending order
   $\overrightarrow{tank} \leftarrow \{tank_i = ice_i + \frac{Q_i}{\psi_i/c_i}\}$ sorted in ascending order
5: $\vec{q} \leftarrow \vec{0},\ left = 1,\ right = 1,\ ice_{N+1} \leftarrow \infty$
6: **while** $Q_{tot} > 0$ **do**
7:   ———— Find space to fill ————
8:   **while** $ice_{right+1} = ice_{left}$ **do**
9:     $right++$
10:   **end while**
11:   $w \leftarrow \sum_{i=left}^{right} \psi_i$ //bottom area
12:   $cap_1 \leftarrow ice_{right+1}$; $cap_2 \leftarrow tank_{left}$
13:   $h \leftarrow \min\{cap_1, cap_2\} - ice_{left}$ //height
14:   ———— Fill tanks ————
15:   **if** $w \cdot h < Q_{tot}$ **then**
16:     $Q_{tot} -= w \cdot h$
17:   **else** {the last iteration of filling}
18:     $h \leftarrow Q_{tot}/w$ //readjust height
19:     $Q_{tot} \leftarrow 0$
20:   **end if**
21:   **for** $i = left \rightarrow right$ **do**
22:     $ice_i += h$; $q_i += h \cdot \psi_i$
23:   **end for**
24:   ———— Remove full tanks ————
25:   **if** $cap_2 \leq cap_1$ or $cap_1 = \infty$ **then**
26:     **repeat**
27:       $left++$
28:     **until** $left > N$ or $tank_{left} > cap_2$
29:   **end if**
30: **end while**

Theorem 2. *The optimal strategy profile* $\vec{Q}^* = \{Q_i^*\}$ *where*

$$Q_i^* = \frac{\psi_i/c_i}{\sum_{l=1}^{N} \psi_l/c_l} Q_{tot} \tag{3}$$

is a unique Pareto-efficient Nash equilibrium, and achieves the global optimum for objective (1)*.*

Under uncertain service demands, the same CCP approach in [1] still applies. The constraint that involves the unknown random variable $\tilde{Q}_i$ will again be converted into a deterministic constraint as $q_i \leq \hat{Q}_i + \sigma_i z_{\alpha_i}$, and the solution algorithm, which we call ITF-CCP-(b), can be obtained by modifying ITF-(b) as follows:

- Input: replace $\vec{Q}$ by $\vec{\hat{Q}} = \{\hat{Q}_i\}$ and add $\vec{\sigma} = \{\sigma_i\}$.
- Line 4: replace ice_i and $tank_i$ with $ice_i = \frac{\hat{Q}_i/\Psi}{\psi_i/c_i}$ and $tank_i = ice_i + \frac{\hat{Q}_i + \sigma_i z_{\alpha_i}}{\psi_i/c_i}$, respectively.

2 Joint Chance Constraint

As a variant of individual chance constraints used in [1], a joint chance constraint is in the following form:

$$\Pr(q_i \le \tilde{Q}_i, \forall i = 1, ..., N) \ge 1 - \alpha. \tag{4}$$

As can be seen, it requires *all* the users to satisfy $q_i \le \tilde{Q}_i$ altogether in $1-\alpha$ of the time, or equivalently, that the probability that at least one user is granted $q_i > \tilde{Q}_i$ is less than α. This does not characterize our problem better than (in fact, not as well as) the individual chance constraints addressed in [1]. In addition, as the individual chance constraints associate an probability α_i to each user i, it can be leveraged to provide further user differentiation which (4) lacks. Nevertheless, we give further development and references for interested readers if any.

In general, the calculation of joint chance constraints involves dealing with multi-dimensional distributions. Fortunately, the $\tilde{Q}_i$'s in the participatory problem can be treated as independent of each other, and hence (4) can be broken down to

$$\prod_{i=1}^{N} \Pr(q_i \le \tilde{Q}_i) \ge 1 - \alpha \Longleftrightarrow \prod_{i=1}^{N} \operatorname{erfc}(\frac{q_i - \mu_i}{\sqrt{2}\sigma_i}) \ge 2 - 2\alpha \tag{5}$$

which assumes $\tilde{Q}_i \sim \mathcal{N}(\mu_i, \sigma_i)$. Unfortunately, an explicit form for the upper limit to q_i is not obtainable from (5), and one has to resort to a stochastic simulation-based approach. For this, the reader can refer to a stochastic simulation-based genetic algorithm proposed by [2], where the initialization process, selection, crossover, and mutation operations are the same as general genetic algorithms except that stochastic simulation must be employed to check the feasibility of new offspring (solution) and to handle stochastic objective and/or constraints.

References

[1] T. Luo and C.-K. Tham, "Fairness and Social Welfare in Incentivizing Participatory Sensing," *IEEE SECON 2012*, Seoul, Korea, June 18–21, 2012.

[2] K. Iwamura and B. Liu, "A Genetic Algorithm for Chance Constrained Programming," *Journal of Information and Optimization Sciences*, vol. 17, no. 2, pp. 409–422, 1996.