Lecture 3

Temporal and spatio-temporal models

MSc in Epidemiology @ Imperial College London
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Temporal dependence

- Many data that we deal with in spatial analysis are actually both spatial and temporal in nature
- Similarly to spatial dependence, it is sometime necessary to model temporal dependence on data and parameters
- Examples:
  - the weekly or monthly number of cases of many diseases exhibit often a seasonal pattern as well as short term dependence
  - the underlying daily level of atmospheric pollutants, e.g. air particles, will show strong correlation over consecutive days because their lifetime lasts over several days
- Unlike space, the temporal data hold a natural order

Time series

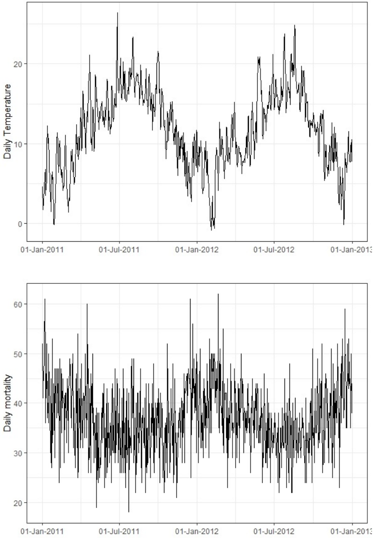
- A time series is a set of observations taken sequentially in time. Then, time series analysis deals with records that are collected over time.

- One distinguishing feature in time series is that the observations are usually dependent.

- Depending on different applications, data may be collected hourly, daily, weekly, monthly or yearly, and so on.

- We use notation such as $\{Z_t : t \in \mathcal{D}_t\}$. Henceforth, we assume that $\mathcal{D}_t = \{0, 1, \dots\}$ and we refer to $\{Z_t : t = 0, 1, \dots\}$ as a time series.

Plots of daily time series of temperature and cardiovascular mortality in Greater London, 2011-2012



Time series

- The natural (temporal) ordering in the time series creates an internal structure in the data, that shows, commonly, dependence in the observations, such that values in the present depend upon observations available in the past.
- This means that observations close together in time tend to be (auto)correlated (i.e. serially dependent).
- Time series analysis typically presents challenges, as it exhibits patterns and irregular fluctuations.
- The patterns can be specified as:
  - *Trend*, that is the most common time series feature to account for and refers to long-term change in the mean level;
  - *Seasonal variation*, which refers to periodic fluctuations which occur periodically within a year;
  - *Cyclic changes*, which are recurrent rise and fall that are not of fixed period and are over a period longer than one year.
- Irregular fluctuations are variations that are short in duration, following not regularity in the occurrence.

Stationarity

- In studying time series, a very important concept is given by stationary, that refers to the stability of the statistical properties of the process through time.

- A time series is said to be *completely* or *strongly stationary* if, for any sequence of times $t_1, t_2, \ldots, t_n$ and any lag h the probability distribution of the vector $(Z_{t_1}, \ldots, Z_{t_n})'$ is identical to the probability distribution of the vector $(Z_{t_1+h}, \ldots, Z_{t_n+h})'$ (i.e. all aspects of the process's behaviour are unaffected (unchanged) by a shift in time).

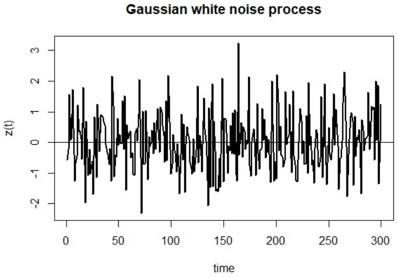
- A time series is said to be *second order* (or *weak*) *stationary*, if:
  - $E(Z_t) = \mu$, the mean is constant for all t
  - $var(Z_t) = \sigma^2$, the variance does not depend on t
  - $Cov(Z_t, Z_j) = C(j - t)$, i.e. the autocovariance depends only on the elapsed time between t and j and not their actual location.

Example: Gaussian White Noise Process

- The Gaussian white noise process is a stationary process

$$Z_t \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$$

```
# Simulation of for Gaussian white noise process
set.seed(123) # set random number seed
z <- rnorm(300) # generate iid normal random variables
ts.plot(z, main="", xlab="time", ylab="y(t)", col="black", lwd=2)
 abline(h=0)
```



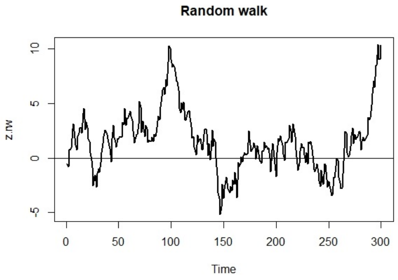
- $E[Z_t] = 0$ and $\mathrm{var}(Z_t) = \sigma^2$
- $\mathrm{cov}(Z_i, Z_j) = 0$ for $i \neq j$

Note: "iid" = independent and identically distributed

Example: Random Walk of Order 1 (RW1)

- The random walk describes how an observation directly depends upon one or more previous measurements plus a white noise
- It is an example of non-stationary process
- The random walk of order 1 (RW1) is defined as:
  - $Z_t = Z_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim (0, \sigma_\varepsilon^2)$
  - Z_0 is fixed

```
# Simulation of random walk process
set.seed(123)
e = rnorm(300)
z.rw = cumsum(e) # compute cumulative sums
ts.plot(z.rw, lwd=2, col="black", main="")
abline(h=0)
```



Example: Random Walk of Order 1 (RW1)

By recursively substitution starting from $t = 1$, it produces:

$$
\begin{aligned}
Z_1 &= Z_0 + \epsilon_1 \\
Z_2 &= Z_1 + \epsilon_2 = Z_0 + \epsilon_1 + \epsilon_2 \\
&\vdots \\
Z_t &= Z_0 + \epsilon_1 + \cdots + \epsilon_t \\
&= Z_0 + \sum_{j=1}^{t} \epsilon_j
\end{aligned}
$$

Hence, $E(Z_t) = Z_0$, which is independent of t, but $\text{Var}(Z_t) = \text{Var}\Big(\sum_{j=1}^{t} \epsilon_i\Big) = \sum_{j=1}^{t} \sigma_\epsilon^2 = \sigma_\epsilon^2 t$ depends on t, thus the random walk process $\{Z_t\}$ is not stationary

Source: Zivot E. and Wang J. (2006), Modeling Financial Time Series with S-PLUS, Springer, 2nd ed.
Cressie N. and Wikle C.K. (2011), Statistics for spatio-temporal data, Wiley

Back to modelling: spatial patterns

1. Data
   - Disease counts O_{ik} in area i and stratum k, aggregated over a time period, $i = 1, \ldots, N$, $k = 1, \ldots, K$
   - Population counts n_{ik} in area i and stratum k, aggregated over a time period
   - Expected numbers $E_i = \sum_k n_{ik} r_k$, where r_k reference rate for stratum (age, gender,...)
2. Spatial smoothing using BYM model

$$
\begin{aligned}
\mathrm{O}_i &\sim \mathrm{Poisson}(E_i \lambda_i); \quad i = 1, ..., N \\
\log \lambda_i &= \alpha + V_i + U_i \\
V_i &\sim \mathrm{Normal}(0, \sigma_v^2) \\
\mathbf{U} &\sim \mathrm{ICAR}(\mathbf{W}, \sigma_u^2) \rightarrow U_i | U_{j,\ j \neq i} \sim \mathrm{Normal}\left(\frac{\sum_j w_{ij} U_j}{\sum_j w_{ij}}, \frac{\sigma_u^2}{\sum_j w_{ij}} \right)
\end{aligned}
$$

$w_{ij} = 1$ if areas i and j are neighbours, 0 otherwise

Temporal patterns

1. Data
   - Disease counts O_{tk} in time period t and stratum k, aggregated over space, $t = 1, \ldots, T$ (equally-spaced time intervals), $k = 1, \ldots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)
2. Temporal trends

$$\begin{aligned} \mathrm{O}_t &\sim \text{Poisson}(E_t \lambda_t); \quad t = 1, ..., T \\ \log \lambda_t &= ??? \end{aligned}$$

Temporal patterns

1. Data
   - Disease counts O_{tk} in time period t and stratum k, aggregated over space, $t = 1, \dots, T$ (equally-spaced time intervals), $k = 1, \dots, K$
   - Population counts n_{tk} in time period t and stratum k, aggregated over space
   - Expected numbers $E_t = \sum_k n_{tk} r_k$, where r_k reference rate for stratum (age, gender,...)
2. Temporal trends

$$
\begin{array}{rcl}
\mathrm{O}_t & \sim & \text{Poisson}(E_t \lambda_t); \quad t = 1, ..., T \\
\log \lambda_t & = & \alpha + \beta t \quad \text{simple linear regression} \\
& = & \alpha + \gamma_t \quad \text{global temporal smoothing} \\
& & \gamma_t \sim \text{Normal}(0, \sigma_\gamma^2) \\
& = & \alpha + \xi_t \quad \text{local temporal smoothing} \\
& & \xi_t \sim \text{distribution ?} \\
& = & \alpha + \gamma_t + \xi_t \quad \text{global and local temporal smoothing}
\end{array}
$$

Autoregressive models

- Idea: predict an output based on the previous outputs
- Let $\theta = (\theta_1, \ldots, \theta_T)$ be a time ordered sequence of parameters
- An autoregressive Gaussian model for θ is defined by:
  - a time lag p
  - a set of coefficients $\{b_1, \ldots b_p\}$

  so that

$$\theta_t = b_1\theta_{t-1} + b_2\theta_{t-2} + \ldots + b_p\theta_{t-p} + \epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0, \sigma_\epsilon^2)$$

  equivalently

$$\theta_t|\theta_{t-1}, \theta_{t-2}, \ldots \theta_{t-p} \sim N\left(\sum_{j=1}^{p} b_j\theta_{t-j}, \sigma_\epsilon^2\right), \quad t = p+1, \ldots, T$$

RW(1) for θ

$$\theta_t = \theta_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathrm{N}(0, \sigma_\epsilon^2)$$

- This process as we saw before is non stationary (the variance depends on t)
- It only models the difference of levels on consecutive time points ($\theta_t - \theta_{t-1} = \epsilon_t$)

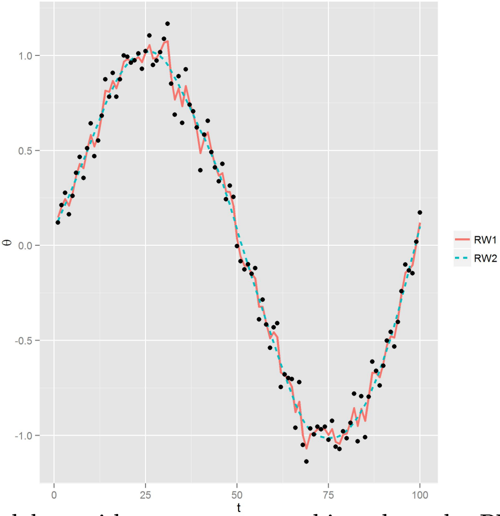
Random walk of order 2 (RW(2)) for θ

$$\theta_t = 2\theta_{t-1} - \theta_{t-2} + \epsilon_t$$

- This process is non stationary
- It only models a linear combination of levels on consecutive time points ($\theta_t - 2\theta_{t-1} + \theta_{t-2} = \epsilon_t$)

Comparison between RW(1) and RW(2)

Simulated data from a sine curve, then RW(1) and RW(2) models fitted



RW(2) model provides greater smoothing than the RW(1) model

Conditional distributions for a RW(1)

- The conditional distributions $p(\theta_t|\theta_{-t})$, where θ_{-t} represent the **vector of** θ's with θ_t removed, can be expressed as:

$$p(\theta_t|\theta_{-t},\sigma^2_\epsilon) = \begin{cases} N(\theta_{t+1},\sigma^2_\epsilon) & \text{for} \quad t=1 \\ N(\frac{\theta_{t-1}+\theta_{t+1}}{2},\frac{\sigma^2_\epsilon}{2}) & \text{for} \quad t=2,\dots,T-1 \\ N(\theta_{t-1},\sigma^2_\epsilon) & \text{for} \quad t=T \end{cases} \tag{1}$$

- Similar form with the spatial autoregression case (ICAR)

$$\text{Recall: } p(U_i|\boldsymbol{U}_{-i}) = \text{Normal}\left(\frac{\sum_j w_{ij}U_j}{\sum_j w_{ij}}, \frac{\sigma^2_u}{\sum_j w_{ij}}\right)$$

  - time neighbours of t are $t-1$ and $t+1$
  - if $t=1$ or $t=T$, only 1 neighbour: t_2 and t_{T-1}
  - weights: 1

Conditional distributions for a RW(2)

- The conditional distribution of θ_t involves $\theta_{t-2}, \theta_{t-1}, \theta_{t+1}$ and θ_{t+2}

$$p(\theta_t|\theta_{-t}, \sigma^2_\epsilon) = \begin{cases} N(2\theta_{t+1} - \theta_{t+2}, \sigma^2_\epsilon) & \text{for } t = 1 \\ N(\frac{2}{5}\theta_{t-1} + \frac{4}{5}\theta_{t+1} - \frac{1}{5}\theta_{t+2}, \sigma^2_\epsilon/5) & \text{for } t = 2 \\ N(-\frac{1}{6}\theta_{t-2} + \frac{2}{3}\theta_{t-1} + \frac{2}{3}\theta_{t+1} - \frac{1}{6}\theta_{t+2}, \sigma^2_\epsilon/6) & \text{for } t = 3, ..., T-2 \\ N(-\frac{1}{5}\theta_{t-2} + \frac{4}{5}\theta_{t-1} + \frac{2}{5}\theta_{t+1}, \sigma^2_\epsilon/5) & \text{for } t = T-1 \\ N(-\theta_{t-2} + 2\theta_{t-1}, \sigma^2_\epsilon) & \text{for } t = T \end{cases}$$

- Similarity with ICAR model

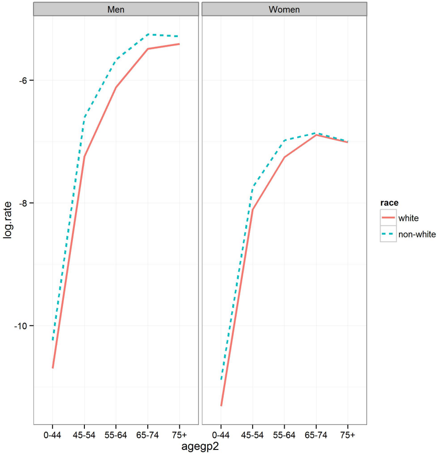
time	neighbours	weights
$t = 1$	$\theta_{t+1}, \theta_{t+2}$	2, -1
$t = 2$	$\theta_{t-1}, \theta_{t+1}, \theta_{t+2}$	2, 4, -1
$t = 3, ..., T-2$	$\theta_{t-2}, \theta_{t-1}, \theta_{t+1}, \theta_{t+2}$	-1, 4, 4, -1
$t = T-1$	$\theta_{t-2}, \theta_{t-1}, \theta_{t+1}$	-1, 4, 2
$t = T$	$\theta_{t-2}, \theta_{t-1}$	-1, 2

BUGS code for RW(1) model

Temporal model	**BUGS code**
For each time $t = 1, \ldots, T$	`for(t in 1:T) {`
$\mathrm{O}_t \sim \mathrm{Poisson}(\mu_t)$	`O[t] ~ dpois(mu[t])`
$\log \mu_t = \log E_t + \alpha + \xi_t$	`log(mu[t])<-log(E[t])+alpha+xi[t]`
$\log \lambda_t = \alpha + \xi_t$	`RR[t] <- exp(alpha + xi[t])`
	`}`
Temporal parameters	`for(t in 1:T) {`
Temporal $\mathrm{RR}_t = \exp(\xi_t)$	`residRRtm[t] <- exp(xi[t])`
	`}`
$\boldsymbol{\xi} \sim \mathrm{RW}(1)$	`xi[1:T]~car.normal(adj.tm[],weights.tm[],num.tm[],prec.xi)`
Temporal weights	`for(k in 1:sumximNeigh.tm) {weights.tm[k] <- 1 }`
Priors	`#Priors`
α vague prior	`alpha ~ dflat()`
σ^2_ξ vague prior	`prec.xi ~ dgamma(0.5,0.0005)`
	`#Parameters of interest`
Conditional temporal variance	`sigma2.xi <- 1/prec.xi`
Empirical temporal variance	`sigma2.xi.marginal <- sd(xi[]) * sd(xi[])`

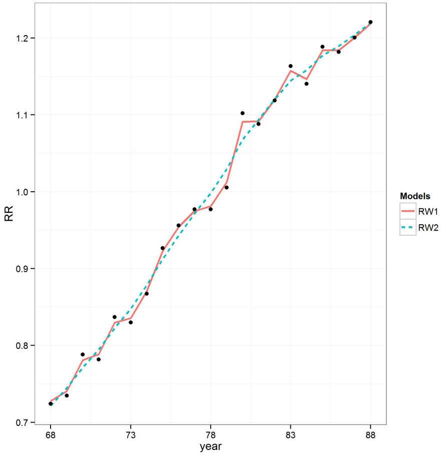
Ohio Lung cancer data

- Data on lung cancer in 88 counties of Ohio, 1968-1988
- Adjustments for gender, age, and race



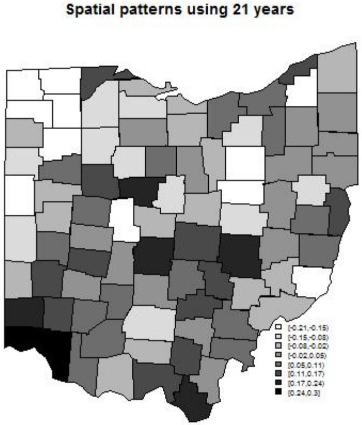
Reference mortality rates for each stratum r_k

Ohio Lung cancer temporal trends (no spatial dimension)



Smoothed temporal trends using RW(1) and RW(2) models (posterior means). Dots represent the SMRs

Ohio Lung cancer spatial patterns (no temporal dimension)

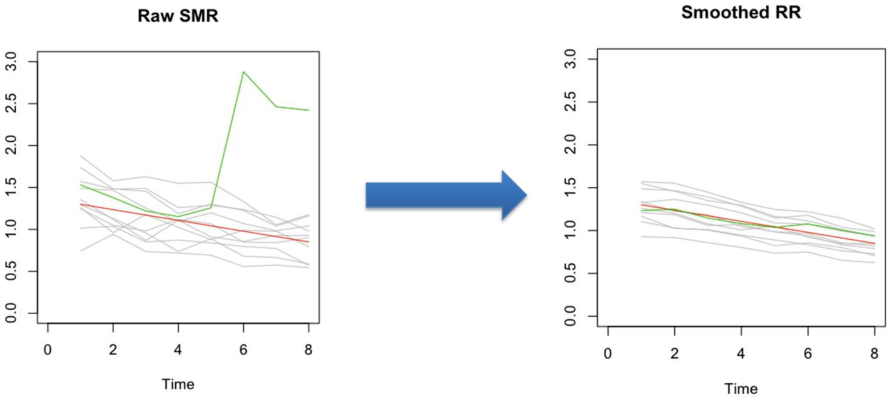


Smoothed RRs using the BYM model (posterior means)

Disease mapping: Extending space to space-time

- Disease mapping is usually carried out on aggregated data over a time period
- Rather than suppressing the time dimension, it can be interesting to use models that combine the space and time dimension
- The stability (or not) of the spatial pattern can aid interpretation
- The specific space-time components of the model can potentially pinpoint unusual/emerging hazards
- Data O_{it} and E_{it}: the observed and expected number of cases in area i at time t
  $E_{it} = \sum_k n_{itk} r_k$, where r_k reference rate for stratum (age, gender,...)

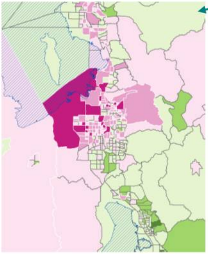
Temporal smoothing

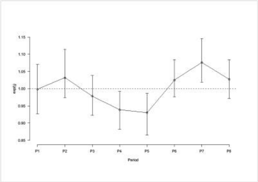


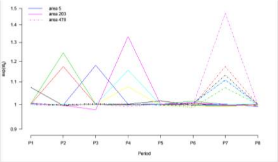
Schematic representation

Noise model: Poisson/Binomial

Latent structure: Space + Time + Interactions



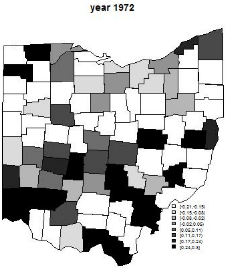


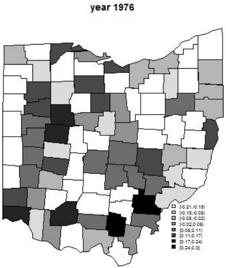


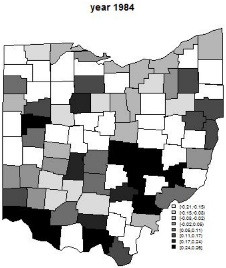
joint Bayesian estimation

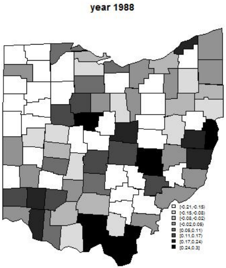
Inference

Ohio Lung cancer - temporal SMRs



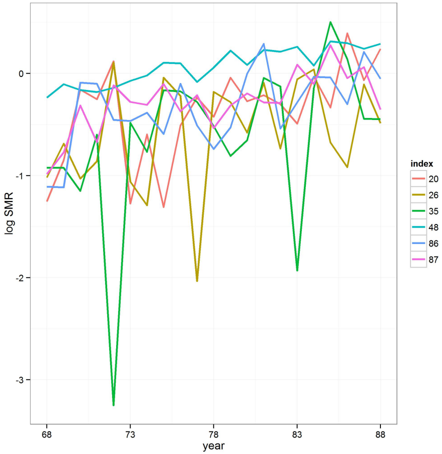






Ohio Lung cancer - temporal SMRs for selected counties

Log SMR time trends in 6 areas of Ohio



Increasing trends across all areas but some patterns are more variable than others

Simple log-linear temporal model

Model 1

$$
\begin{aligned}
\mathrm{O}_{it} &\sim \mathrm{Poisson}(\mathrm{E}_{it}\lambda_{it}) \\
\log \lambda_{it} &= \alpha + \beta * t + U_i + V_i
\end{aligned}
$$

where

- α overall log RR in Ohio over the 21-year period
- $V_i \sim \mathrm{Normal}(0, \sigma_v^2)$ spatially unstructured RE
- $\mathbf{U} \sim \mathrm{ICAR}(\mathbf{W}, \sigma_u^2)$ spatially structured RE
- $\exp(\beta)$ is the change in the RR associated with a 1-year increase in time

$\Rightarrow$ Assumes a linear temporal trend

Log-linear temporal model with unstructured temporal RE

Model 2

$$\begin{aligned} \mathrm{O}_{it} &\sim \mathrm{Poisson}(\mathrm{E}_{it}\lambda_{it}) \\ \log \lambda_{it} &= \alpha + \beta * t + U_i + V_i + \gamma_t \end{aligned}$$

where

- α overall log RR in Ohio over the 21-year period
- $V_i \sim \mathrm{Normal}(0, \sigma_v^2)$ spatially unstructured RE
- $\mathbf{U} \sim \mathrm{ICAR}(\mathbf{W}, \sigma_u^2)$ spatially structured RE
- $\exp(\beta)$ is the change in the RR associated with a 1-year increase in time
- $\gamma_t \sim \mathrm{Normal}(0, \sigma_\gamma^2)$ temporally unstructured RE

Ohio lung cancer - comparison models 1 and 2

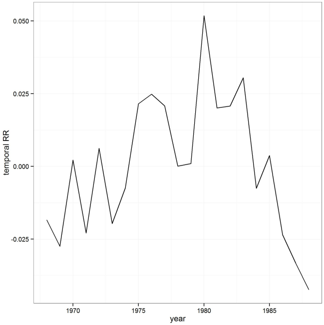
- Posterior mean and 95% CI for Model 1

  $\exp(\beta) = 1.027(1.026–1.028)$

- Posterior mean and 95% CI for Model 2

  $\exp(\beta) = 1.027(1.026–1.030)$

Unstructured temporal RE (γ) model 2



Some temporal structure which suggests that the linear term is not capturing the temporal trend well

Simple additive space-time structure version 1

This model assumes that the space time variations can be captured by the superimposition of a BYM spatial model and a structured time trend

Model 3

$$\begin{aligned} \mathrm{O}_{it} &\sim \mathrm{Poisson}(\mathrm{E}_{it}\lambda_{it}) \\ \log \lambda_{it} &= \alpha + U_i + V_i + \xi_t \end{aligned}$$

- $V_i \sim \mathrm{Normal}(0, \sigma_v^2)$ spatially unstructured RE
- $\mathbf{U} \sim \mathrm{ICAR}(\mathbf{W}, \sigma_u^2)$ spatially structured RE
- $\xi_t \sim \mathrm{RW}(1)$ temporally structured RE with variance parameter σ_ξ^2
- There are 3 hyperparameters
  - σ_v^2, the marginal variance of V_i, the unstructured spatial component
  - σ_u^2, the conditional variance of the spatial ICAR process U_i
  - σ_ξ^2, the conditional variance of the RW(1) modelling the time trend

Simple additive space-time structure version 2

Model 4

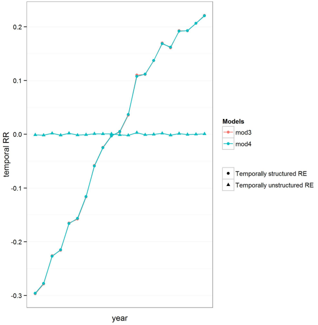
$$
\begin{aligned}
\mathrm{O}_{it} &\sim \mathrm{Poisson}(\mathrm{E}_{it}\lambda_{it}) \\
\log \lambda_{it} &= \alpha + U_i + V_i + \xi_t + \gamma_t
\end{aligned}
$$

where

- α overall log RR in Ohio over the 21-year period
- $V_i \sim \mathrm{Normal}(0, \sigma_v^2)$ spatially unstructured RE
- $\mathbf{U} \sim \mathrm{ICAR}(\mathbf{W}, \sigma_u^2)$ spatially structured RE
- $\xi_t \sim \mathrm{RW}(1)$ temporally structured RE with variance parameter σ_ξ^2
- $\gamma_t \sim \mathrm{Normal}(0, \sigma_\gamma^2)$ temporally unstructured RE

Ohio lung cancer - comparison models 3 and 4

Posterior means of the temporal RE
Model 3 = ξ_t, model 4 = $\xi_t + \gamma_t$



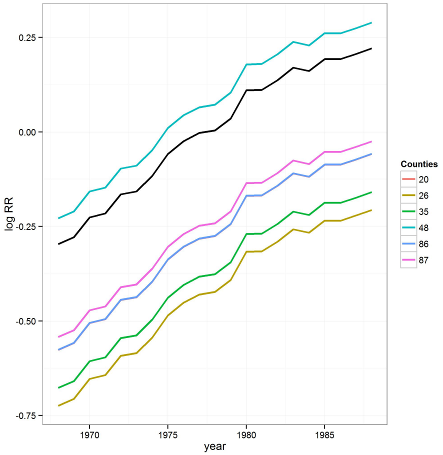
No difference between the 2 temporally structured RE

Temporally unstructured RE γ_t almost 0

$\Rightarrow$ does not contribute to the RR

Ohio lung cancer - area-specific temporal RR

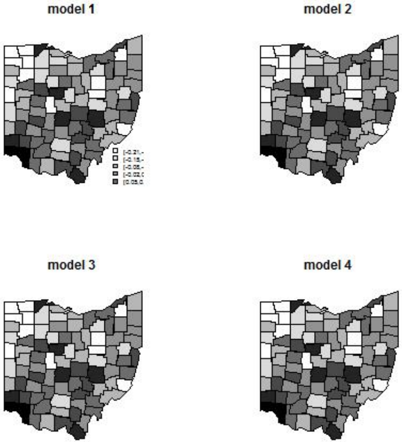
Posterior means of the temporal RE ξ_t (model 3)



(we plot $\boldsymbol{\xi_t} + \boldsymbol{U_i} + \boldsymbol{V_i}$ for 6 random areas i)

Ohio lung cancer - spatial patterns (all models)

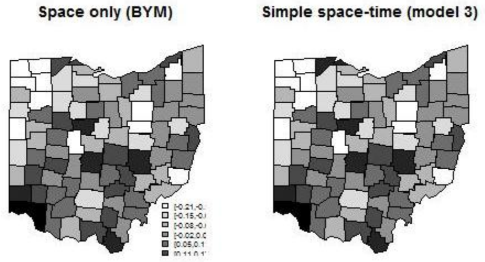
Posterior means of the spatial RE $(\boldsymbol{U_i} + \boldsymbol{V_i})$ for the 4 different models



$\Rightarrow$ Similar patterns with the 4 models

Ohio lung cancer - space vs space-time models

Posterior means of the spatial RE $(\boldsymbol{U_i} + \boldsymbol{V_i})$



$\Rightarrow$ Similar patterns with the 2 approaches

Space-time model with exchangeable interactions

Model 5

$$\begin{aligned} \mathrm{O}_{it} &\sim \mathrm{Poisson}(\mathrm{E}_{it}\lambda_{it}) \\ \log \lambda_{it} &= \alpha + \theta_i + \xi_t + \nu_{it} \\ \theta_i &= V_i + U_i \text{ (BYM = ICAR + HET)} \\ \xi_t &\sim \mathrm{RW}(1) \\ \nu_{it} &\sim \mathrm{Normal}(0, \sigma^2_\nu) \end{aligned}$$

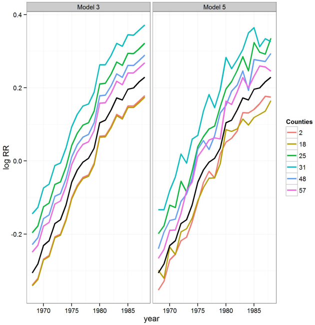
The simple additive model can be extended by including **space-time interactions parameters**, ν_{it}, modelled as exchangeable random effects, that capture departure from the additive structure.

$\Rightarrow$ ν_{it} RE allow each area to follow a different time trend

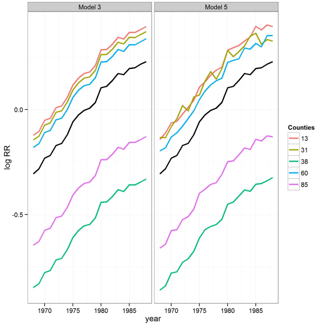
Interpretation of the interactions in model 5

- The interactions ν_{it} allow to highlight unusual temporal trends
- Rules based on the posterior probabilities $\text{Proba}(\nu_{it} > 0)$ for at least 1 time point t
- Ohio lung cancer: 6 counties with unusual temporal trends

'unusual' temporal trends



'common' temporal trends



BUGS code for model 3

Space-time model	**BUGS code**
For each area $i = 1, ..., N$	`for(i in 1:N) {`
For each time $t = 1, ..., T$	`for(t in 1:T) {`
$\text{O}_{it} \sim \text{Poisson}(\mu_{it})$	`O[i,t] ~ dpois(mu[i,t])`
$\log \mu_{it} = \log E_{it} + \alpha + V_i + U_i + \xi_t$	`log(mu[i,t])<-log(E[i,t])+alpha+V[i]+U[i]+xi[t]`
$\log \lambda_{it} = \alpha + V_i + U_i + \xi_t$	`RR[i,t] <- exp(alpha + V[i] + U[i]+ xi[t])`
	`}}`
Spatial parameters	`for(i in 1:N) {`
$V_i \sim \text{Normal}(0, \sigma_v^2)$	`V[i] ~ dnorm(0, prec.v)`
Spatial $\text{RR}_i = \exp(V_i + U_i)$	`residRRsp[i] <- exp(V[i] + U[i])`
	`}`
$\mathbf{U} \sim \text{CAR}(\mathbf{W}, \sigma_u^2)$	`U[1:N]~car.normal(adj[],weights[],num[],prec.u)`
Spatial weights	`for(k in 1:sumximNeigh) {weights[k] <- 1}`
Temporal parameters	`for(t in 1:T) {`
Temporal $\text{RR}_t = \exp(\xi_t)$	`residRRtm[t] <- exp(xi[t])`
	`}`
$\boldsymbol{\xi} \sim \text{RW}(1)$	`xi[1:T]~car.normal(adj.tm[],weights.tm[],num.tm[],prec.xi)`
Temporal weights	`for(k in 1:sumximNeigh.tm) {weights.tm[k] <- 1 }`

BUGS code for model 3 - continued

Priors	`#Priors`
α vague prior	`alpha ~ dflat()`
σ_v^2 vague prior	`prec.v ~ dgamma(0.5,0.0005)`
σ_u^2 vague prior	`prec.u ~ dgamma(0.5,0.0005)`
σ_ξ^2 vague prior	`prec.xi ~ dgamma(0.5,0.0005)`
	`#Parameters of interest`
Unstructured spatial variance	`sigma2.v <- 1/prec.v`
Conditional structured spatial variance	`sigma2.u <- 1/prec.u`
Empirical structured spatial variance	`sigma2.u.marginal <- sd(U[]) * sd(U[])`
Spatial fraction	`frac.spatial<-sigma2.u.marginal/(sigma2.v+sigma2.u.marginal)`
Conditional temporal variance	`sigma2.xi <- 1/prec.xi`
Empirical temporal variance	`sigma2.xi.marginal <- sd(xi[]) * sd(xi[])`

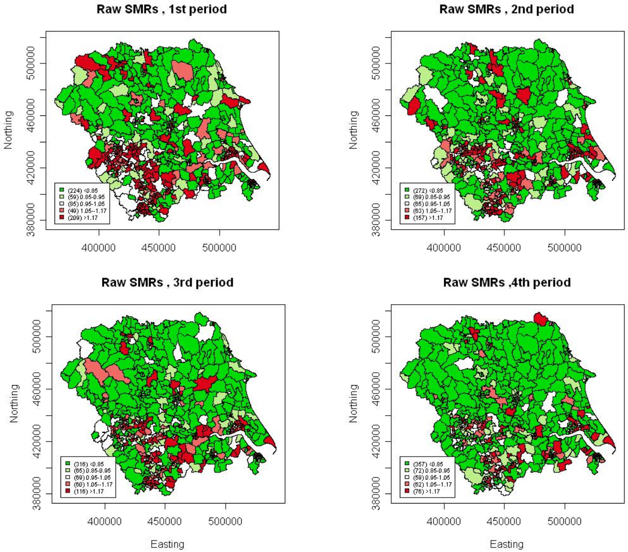
BUGS code for model 5

```
for(i in 1:N) {
 for (t in 1:T) {
    O[i,t] ~ dpois(mu[i,t])
    log(mu[i,t]) <- log(E[i,t]) + alpha + V[i] + U[i] + xi[t] + nu[i,t]
    nu[i,t] ~ dnorm(0, prev.nu)       #space-time interactions
    }}

for(i in 1:N) {
  V[i] ~ dnorm(0, prec.v)             # unstructured effects
  residRRsp[i] <- exp(V[i]+U[i])}    # spatial RR

 U[1:N] ~ car.normal(adj[], weights[], num[], prec.u) # spatially corr. effects
 for(k in 1:sumNumNeigh) { weights[k] <- 1 }

 xi[1:T] ~ car.normal(adj.xi[], weights.xi[], num.xi[], prec.xi) #RW(1) for time effects
 for(k in 1:sumNumNeigh.xi) { weights.xi[k] <- 1 }
 for (t in 1:T) {RRtemporal[t] <- exp(xi[t]) } #temporal RR

 alpha ~ dflat()                          # vague prior
 mean.RR <- exp(alpha)                    # overall mean risk
 prec.v ~ dgamma(0.5, 0.0005)
 sigma2.v <- 1/prec.v                     # variance of unstructured effects
 prec.u ~ dgamma(0.5, 0.0005)
 sigma2.u <- 1/prec.u                     # conditional variance of spatial effects
 sigma2.u.marginal <- sd(U[]) * sd(U[])   # empirical marginal var. of spatial effects
 prec.xi ~ dgamma(0.5, 0.0005)
 sigma2.xi <- 1/prec.xi                   # conditional variance of temporal effects
 sigma2.xi.marginal <- sd(xi[])*sd(xi[])  # empirical marginal var. of temporal effects
 prec.nu ~ dgamma(0.5, 0.0005)
 sigma2.nu <- 1/prec.nu                   # variance of space-time interactions
```

Ex2: Space-time variations of male lung cancer in Yorkshire, Richardson *et al.* (2006)

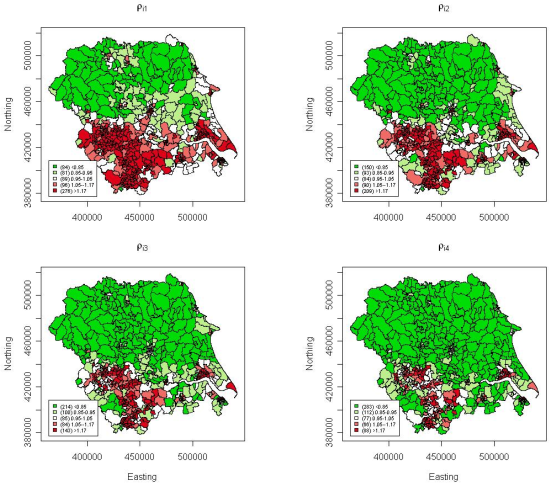
- Data
  - Male lung cancer incidence in Yorkshire
  - Spatial resolution: wards (626), between 0 and 20 new cases per year with mean around 4
  - Time periods: 1981-1999 $\rightarrow$ 4 periods
  - Expected counts based on sex-age incidence rates for the region and the total period 1981-1999
- Questions of interest
  - Study the persistence of spatial patterns over time (interpreted as associated with stable risk factors, environmental effects, distribution of health care access)
  - Highlight unusual patterns
- Model 5

Raw SMRs of lung cancer per period



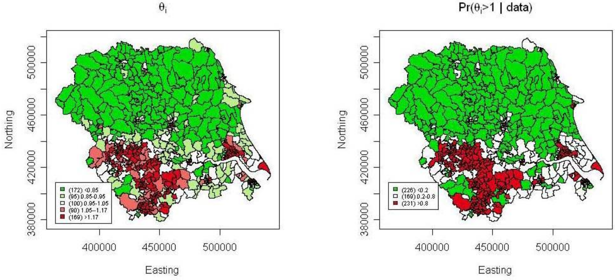
As time progresses, overall lower SMRs are seen for male lung cancer

Smoothed RRs of lung cancer per period (Model 5)



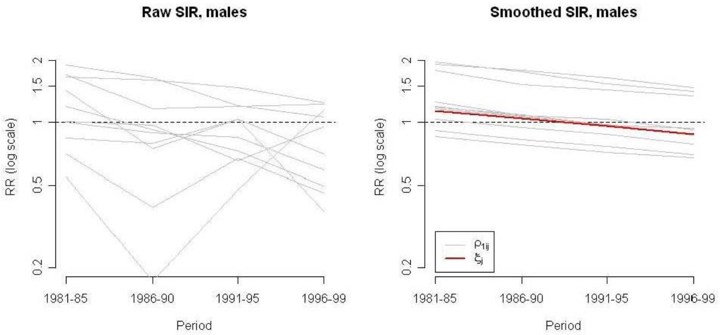
- Evidence of spatial heterogeneity with higher RRs in in the southwest corner and around Hull
- RRs maps are progressively becoming 'greener'

Spatial pattern $\exp(V_i + U_i)$ - model 5



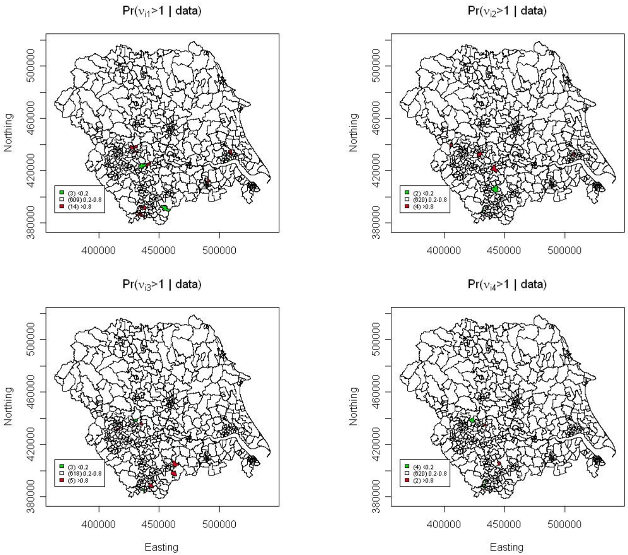
The pattern of higher RRs in urban areas around Sheffield, Leeds, Bradford and Hull is marked

Temporal trend ξ_t - model 5



A steady decrease of incidence over the period

Interactions ν_{it} - model 5



In this case, no evidence of significant space-time interaction patterns

Ex3: Modelling daily air pollution data at multiple sites

Shaddick and Wakefield (2002)

- Daily measure of pollutant (PM_{10}, CO, NO, SO_2) at 8 monitoring sites within London over the period 1994-1997

- Provide a daily exposure measure for use in studies investigating the health effects of pollution after accounting for
  - pollutant dependence: combustion processes (diesel)
  - covariates, e.g. temperature
  - temporal pattern: varying atmospheric lifetimes $\rightarrow$ strong daily dependence
  - using information from the 8 sites together with their spatial distribution
  - measurement error

Ex3: Model framework for daily air pollution data

- Bayesian hierarchical model
- Pollutants modelled as a function of the true underlying level θ_t with measurement error
- Incorporate covariate information, e.g. temperature. The vector of covariates will be denoted by X_t
- Underlying level θ_t is a function of the previous day's level, θ_{t-1}
- Missing values treated as unknown parameters within the Bayesian framework and can be estimated

Ex3: Single monitoring site

- **Stage One: Observed Data Model**

$$Y_t = X_t'\beta_1 + \theta_t + v_t,$$

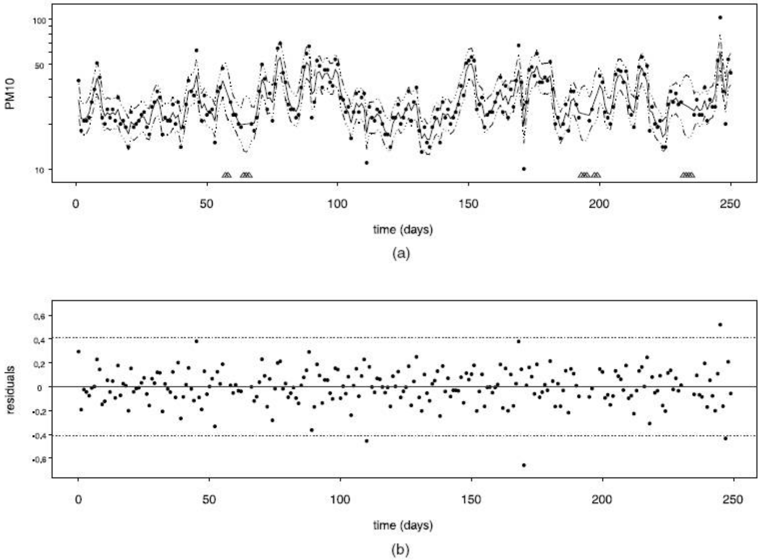
  - Y_t observed level of pollutant on day t
  - θ_t true underlying level
  - v_t referred to as *measurement error*, i.i.d. as $N(0, \tau_v)$
  - X covariates

- **Stage Two: Temporal Model RW(1)**

$$\theta_t = \theta_{t-1} + \epsilon_t, \quad \epsilon_t \text{ i.i.d. as } N(0, \sigma_\epsilon^2)$$

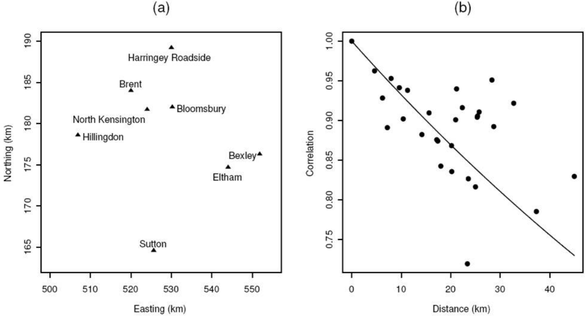
- **Stage Three: Hyperpriors**
  - $\beta_1 \sim N(c, C)$, where c is a $q_1 \times 1$ vector and C a $q_1 \times q_1$ variance-covariance matrix
  - $\tau_v \sim Ga(a_v, b_v)$, $\sigma_\epsilon^2 \sim Ga(a_\epsilon, b_\epsilon)$

Time series of 250 days of observed (Y_t) and estimated (θ) levels (together with their differences) of PM_{10} at Bloomsbury



Ex3: Locations of monitoring sites and correlations of PM_{10} with distance

- Daily pollution measured at 8 different sites in London
  - Temporal dependence - atmospheric lifetimes and relationship with meteorological conditions
  - Spatial dependencies - distance between sites and site type



- Measurements from sites that are close to each other are more highly correlated than those further away

Single pollutant, multiple monitoring site

- S monitoring sites measuring a single pollutant
- The underlying autoregressive structure remains constant across sites with a constant adjustment in the mean level for site s by an amount m_s, $s = 1, ..., S$
- **Stage One: Observed Data Model**

$$Y_{st} = X'_{st}\beta_1 + X'_s\beta_2 + m_s + \theta_t + v_{st}$$

  - θ_t true underlying level
  - v_{st} i.i.d. as $N(0, \tau_{vs})$
  - X covariates varying by site and day, or site only
- **Stage Two (a), Temporal Model:**

$$\theta_t = \theta_{t-1} + \epsilon_t, \quad \epsilon_t \text{ i.i.d. as } N(0, \sigma_\epsilon^2)$$

Spatial model (stage 2)

- The random effects $m = (m_1, ..., m_S)'$ arise from the multivariate normal distribution

$$m \sim MVN(0_S, \sigma_m^2 \Sigma_m)$$

  - 0_S $S \times 1$ vector of zeros
  - σ_m^2 the between-site variance
  - Σ_m correlation matrix

- Correlation between sites s and s' assumed to be a function of the distance between them

$$f(d_{ss'}, \phi) = \exp\left(-\phi d_{ss'}\right), \phi > 0 \text{ strength of the correlation}$$

where ϕ is the decay parameter that represents the rate of decline of spatial correlation among sites over distance.

Ex3: Site effects

	Median	2.5%	97.5%
Bexley	-0.0696	-0.0785	-0.0607
Bloomsbury	0.1341	0.1257	0.1426
Brent	-0.1210	-0.1294	-0.1125
Eltham	-0.1105	-0.1205	-0.1005
Harringey	0.1098	0.0999	0.1195
Hillingdon	0.0132	-0.0032	0.0300
North Kensington	0.0030	-0.0031	0.0090
Sutton	0.0410	0.0250	0.0572
σ_m	0.1019	0.0668	0.1794
ϕ	0.05675	0.02158	0.09778

- Increased levels at the Bloomsbury site which is in the centre of London, compared to the decreased ones at Bexley which is on the outskirts of the city.
- The estimated median of the distribution of ϕ corresponds to a correlation falling to 0.85 at 30 km , which is still very high.

Conclusions

- Increase quality of datasets that are both spatially and temporally indexed

- Advanced methods to deal with this type of data

- Allow to interpret the stability (or not) of the spatial patterns

Practical 3

- Bayesian small area disease risk models for respiratory hospital admission in Greater Glasgow and Clyde health board, 2007-2011.
  - spatial model
  - space-time model

References and further reading

Richardson S., Abellan JJ, Best N. (2006). Bayesian spatio-temporal analysis of joint patterns of male and female lung cancer risks in Yorkshire (UK). *Stat Methods Med Res* **15**:385-407

Knorr-Held, L. (2000) Bayesian modelling of inseparable space-time variation in disease risk. *Statistics in Medicine* **19**:2555-2567

Shaddick, G. and Wakefield, JC. (2002) Modelling multiple pollutants at multiple sites. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* **51**:351-372

Book cited in this lecture: Zivot E. and Wang J. (2006), Modeling Financial Time Series with S-PLUS, Springer, 2nd ed. Cressie N. and Wikle C.K. (2011), Statistics for spatio-temporal data, Wiley