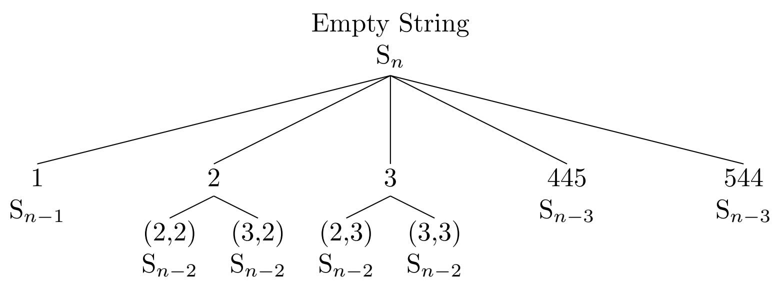
CS/MATH111 ASSIGNMENT 3

Problem 1: Strings of length n are composed of the following strings: 1, 22, 23, 32, 33, 445 and 544. Let S_n be the number of strings of length n that can be formed in this way. For example, for $n = 3$, we can form the following strings:

$$111, 122, 123, 132, 133, 221, 231, 321, 331, 445, 544$$

and thus $S_3 = 11$. (Note that $S_0 = 1$, because the empty string satisfies the condition.)

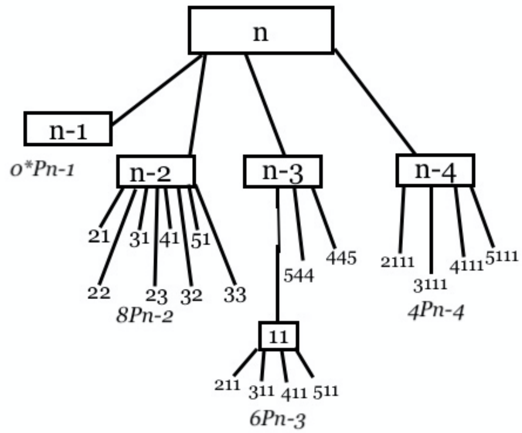
(a) Derive a recurrence relation for the numbers S_n. Justify it.



In order to find the recurrence, we need to satisfy all the conditions which is that the string contains substrings 1, 22, 23, 32, 33, 445,and 544. We start with an S_n or an empty string and then move on to the first case S_{n-1}
Only 1 satisfies the condition for S_{n-1} and so we end the recursive case there.
Next, we go to the 2nd character before the end and we find all the cases containing the sub-strings with 2 and 3
Finally, we go to the 3rd character before the end and we find all the cases containing the sub-strings with 544 and 455
We have accounted for all sub-strings that we have to put into our main string
Adding all cases, the recurrence relation is $S_n = S_{n-1} + 4 \cdot S_{n-2} + 2 \cdot S_{n-3}$ for $n \geq 3$

(b) **Extra credit**. Let P_n be the number of strings of length n that can be formed from the given strings, considering that four 1's cannot be next to each other. (The substring 1111 is not allowed.) Derive a recurrence relation for the numbers P_n. Justify it.

Initial Conditions are: $S_0 = 1$,$S_1 = 1$, $S_2 = 5$, $S_3 = 11$



Justification: For n-1, there are no options that can satisfy the condition so there are 0*P$_{n-1}$ options.
For n-2, the options are the given strings 22, 23, 32, 22 plus the conditions 21, 31, 41, and 51 because the strings can be concatenated. Therefore we have $8{*}P_{n-2}$. For n-3, we have the given strings 544, 445 and can concatenate 2, 3, 4, and 5 to the possible ending string 11. Therefore we have $6{*}P_{n-3}$. For n-4, we cannot have 1111 which is the condition we were given. Therefore we must concatenate to get the 2111, 3111, 4111, and 5111 given strings 544, 445 and can concatenate 2, 3, 4, and 5 to the possible ending string 11. Therefore we have $4 * P_{n-4}$

When we add them together we get the final Recurrence Equation:

$$\mathrm{P}_n = 8 * P_{n-2} + 6 * P_{n-3} + 4 * P_{n-4}$$

Problem 2: Solve the following recurrence equation:

$$
\begin{aligned}
&S_n = S_{n-1} + 4S_{n-2} + 2S_{n-3}\\
&S_0 = 1\\
&S_1 = 1\\
&S_2 = 5
\end{aligned}
$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

$x^3 = x^2 + 4 \cdot x + 2$	We get out characteristic polynomial first
$0 = x^3 - x^2 - 4 \cdot x - 2$	We get possible roots 1,-1,2 and -2 by diving the lowest coefficient
$0 = -1^3 - -1^2 + 4 - 2$	Only -1 is the only root that works
$0 = 0$	

We then do synthetic division with the root we just found

$$
\begin{array}{r@{}l}
 & \quad X^2 - 2X - 2\\ \cline{2-2}
X+1\big) & \quad X^3 \; - X^2 - 4X - 2\\
 & -X^3 \; - X^2\\ \cline{2-2}
 & \qquad -2X^2 - 4X\\
 & \qquad \;\; 2X^2 + 2X\\ \cline{2-2}
 & \qquad\qquad -2X - 2\\
 & \qquad\qquad \;\; 2X + 2\\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}
$$

We use quadratic formula to find the other roots $x = \frac{-2 \pm \sqrt{2^2 + 4 * 1 * 2}}{2 * 1}$
The final roots we get are 1,$1 - \sqrt{3}$ and $1 + \sqrt{3}$
Our general form equation is $S_n = (-1)^n \cdot \alpha_1 + (1 + \sqrt{3})^n \cdot \alpha_2 + (1 - \sqrt{3})^n \cdot \alpha_3$
Setting up our initial conditions and systems of linear equations

$$
\begin{cases}
+1 = \alpha_1 + \alpha_2 + \alpha_3\\
+1 = -\alpha_1 + (1 + \sqrt{3})\alpha_2 + (1 - \sqrt{3})\alpha_3\\
+5 = \alpha_1 + (1 + \sqrt{3})^2\alpha_2 + (1 - \sqrt{3})^2\alpha_3
\end{cases}
$$

We get roots $\alpha_1 = 1$,$\alpha_2 = \frac{1}{\sqrt{3}}$,$\alpha_3 = -\frac{1}{\sqrt{3}}$
The final solution is $S_n = (-1)^n + (1 + \sqrt{3})^n \cdot \frac{1}{\sqrt{3}} - (1 - \sqrt{3})^n \cdot \frac{1}{\sqrt{3}}$

Problem 3: Solve the following recurrence equation:

$$\begin{aligned} D_n &= 4D_{n-1} - 8D_{n-3} + 2n + 3 \\ D_0 &= 0 \\ D_1 &= 1 \\ D_2 &= 1 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Homogeneous Equation: $D_n^{(h)} = 4D_{n-1} - 8D_{n-3}$

Characteristic Polynomial: $x^3 - x^2 - 8 = 0$

To find possible roots 2, we use Rational Root Test along with Descartes' Rule of Signs. We use synthetic division with possible root 2:

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 0 & 8 \\ \hline & & 2 & -4 & -8 \\ \hline & 1 & -2 & -4 & 0 \end{array}$$

This gives us:

$$(x-2)(x^2-2x-4) = 0$$

Using Quadratic Formula:

$$x = \frac{2 \pm \sqrt{2^2 - 4(-4)}}{2} = \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

All roots are: 2, $1 \pm \sqrt{5}$
Therefore, General Solution of Homogeneous Equation:

$$D_n^{(h)} = (2)^n \cdot \alpha_1 + (1+\sqrt{5})^n \cdot \alpha_2 + (1-\sqrt{5})^n \cdot \alpha_3$$

Now we compute Particular Solution:

$$D_n = 2n + 3$$
$$D_n^{(p)} = P_1 n + P_0$$

We plug into our equation as follows:

$$\begin{aligned} P_1 n + P_0 &= 4(P_1(n-1) + P_0) - 8(P_1(n-3) + P_0) + 2n + 3 \\ P_1 n + P_0 &= 4P_1 n - 4P_1 + 4P_0 - 8P_1 n + 24P_1 - 8P_0 + 2n + 3 \\ P_1 n + P_0 &= -4P_1 n + 20P_1 - 4P_0 + 2n + 3 \end{aligned}$$

$$-5P_1 n + 20P_1 - 5P_0 + 2n + 3 = 0$$

We let n = 0 and let n = 1 to get system of equation:

$$\begin{cases} 15P_1 - 5P_0 + 5 = 0 \\ 20P_1 - 5P_0 + 3 = 0 \end{cases}$$

This gives solutions $P_1 = \frac{2}{5}$ and $P_0 = \frac{11}{5}$.
Therefore, General Solution of Non-Homogeneous Equation:

$$D_n^{(p)} = (\frac{2}{5})n + (\frac{11}{5})$$

Since $D_n = D_n^{(h)} + D_n^{(p)}$

$$D_n = (2)^n \cdot \alpha_1 + (1+\sqrt{5})^n \cdot \alpha_2 + (1-\sqrt{5})^n \cdot \alpha_3 + (\frac{2}{5})n + (\frac{11}{5})$$

Now we find α_1, α_2, and α_3 using given initial conditions.

$$\begin{aligned}
D_0 &= (2)^0 \cdot \alpha_1 + (1+\sqrt{5})^0 \cdot \alpha_2 + (1-\sqrt{5})^0 \cdot \alpha_3 + (\frac{2}{5})(0) + (\frac{11}{5}) = 0 \\
D_1 &= (2)^1 \cdot \alpha_1 + (1+\sqrt{5})^1 \cdot \alpha_2 + (1-\sqrt{5})^1 \cdot \alpha_3 + (\frac{2}{5})(1) + (\frac{11}{5}) = 1 \\
D_2 &= (2)^2 \cdot \alpha_1 + (1+\sqrt{5})^2 \cdot \alpha_2 + (1-\sqrt{5})^2 \cdot \alpha_3 + (\frac{2}{5})(2) + (\frac{11}{5}) = 1
\end{aligned}$$

$$\begin{cases}
\alpha_1 + \alpha_2 + \alpha_3 & = -\frac{11}{5} \\
2\alpha_1 + (1+\sqrt{5})\alpha_2 + (1-\sqrt{5})\alpha_3 & = -\frac{8}{5} \\
4\alpha_1 + (6+2\sqrt{5})\alpha_2 + (6-2\sqrt{5})\alpha_3 & = -2
\end{cases}$$

Using Gauss Jordan Elimination:

$$\begin{pmatrix}
1 & 1 & 1 & -\frac{11}{5} \\
2 & 1+\sqrt{5} & 1-\sqrt{5} & -\frac{8}{5} \\
4 & 6+2\sqrt{5} & 6-2\sqrt{5} & -2
\end{pmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{pmatrix}
4 & 6+2\sqrt{5} & 6-2\sqrt{5} & -2 \\
2 & 1+\sqrt{5} & 1-\sqrt{5} & -\frac{8}{5} \\
1 & 1 & 1 & -\frac{11}{5}
\end{pmatrix}$$

$$R_2 - \frac{1}{2} \cdot R_1 \to R_2$$

$$\begin{pmatrix}
4 & 6+2\sqrt{5} & 6-2\sqrt{5} & -2 \\
0 & -2 & -2 & -\frac{3}{5} \\
1 & 1 & 1 & -\frac{11}{5}
\end{pmatrix}$$

$$R_3 - \frac{1}{4} \cdot\ R_1 \to R_3$$

$$\begin{pmatrix}
4 & 6+2\sqrt{5} & 6-2\sqrt{5} & -2 \\
0 & -2 & -2 & -\frac{3}{5} \\
0 & 0 & \sqrt{5} & \frac{3\sqrt{5}-31}{20}
\end{pmatrix}$$

$$\frac{1}{\sqrt{5}} \cdot\ R_3 \to R_3 \;\&\; R_2 + 2 \cdot\ R_3 \to R_2$$

$$\begin{pmatrix}
4 & 6+2\sqrt{5} & 6-2\sqrt{5} & -2 \\
0 & -2 & 0 & \frac{-3\sqrt{5}-31}{10\sqrt{5}} \\
0 & 0 & 1 & \frac{3\sqrt{5}-31}{20\sqrt{5}}
\end{pmatrix}$$

$$R_1 - \left(6 - 2\sqrt{5}\right) \cdot\ R_3 \to R_1 \ \& \ -\tfrac{1}{2} \cdot\ R_2 \to R_2$$

$$\begin{pmatrix} 4 & 6+2\sqrt{5} & 0 & \frac{-30\sqrt{5}+54}{5\sqrt{5}} \\ 0 & -2 & 0 & \frac{-3\sqrt{5}-31}{10\sqrt{5}} \\ 0 & 0 & 1 & \frac{3\sqrt{5}-31}{20\sqrt{5}} \end{pmatrix}$$

$$R_1 - \left(6 + 2\sqrt{5}\right) \cdot\ R_2 \to R_1 \ \& \tfrac{1}{4} \cdot\ R_1 \to R_1$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{3\sqrt{5}+31}{20\sqrt{5}} \\ 0 & 0 & 1 & \frac{3\sqrt{5}-31}{20\sqrt{5}} \end{pmatrix}$$

This means $\alpha_1 = -\frac{5}{2}$, $\alpha_2 = \frac{3\sqrt{5}+31}{20\sqrt{5}}$, and $\alpha_3 = \frac{3\sqrt{5}-31}{20\sqrt{5}}$

Final Solution:

$$D_n = (-\frac{5}{2})(2)^n + (\frac{3\sqrt{5}+31}{20\sqrt{5}})(1+\sqrt{5})^n + (\frac{3\sqrt{5}-31}{20\sqrt{5}})(1-\sqrt{5})^n + (\frac{2}{5})n + (\frac{11}{5})$$

Problem 4: Solve the following recurrence equation:

$$\begin{aligned} A_n &= A_{n-1} + 2A_{n-2} + 3^n \\ A_0 &= 0 \\ A_1 &= 4 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Homogeneous Equation: $A_n^{(h)} = A_{n-1} + 2A_{n-2}$

Characteristic Polynomial:

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

This means possible roots are -1 and 2
Therefore, General Solution of Homogeneous Equation:

$$A_n^{(h)} = (-1)^n \cdot \alpha_1 + (2)^n \cdot \alpha_2$$

Now we compute Particular Solution:

$$A_n = 3^n$$
$$A_n^{(p)} = P_0 3^n$$

We plug into our equation as follows:

$$P_0 3^n = P_0 3^{n-1} + 2P_0 3^{n-2} + 3^n$$
$$P_0 3^n = 3^{n-2}(P_0 3 + 2P_0 + 3^2)$$
$$P_0 3^2 = 3P_0 + 2P_0 + 9$$
$$9P_0 = 5P_0 + 9$$
$$4P_0 = 9 \to P_0 = \frac{9}{4}$$

Therefore, General Solution of Non-Homogeneous Equation:

$$A_n^{(p)} = (\frac{9}{4})3^n$$

Since $A_n = A_n^{(h)} + A_n^{(p)}$

$$A_n = (-1)^n \cdot \alpha_1 + (2)^n \cdot \alpha_2 + (\frac{9}{4})3^n$$

Now we find α_1, α_2, and α_3 using given initial conditions.

$$A_0 = \alpha_1 + \alpha_2 + (\frac{9}{4}) = 0$$
$$A_1 = 2\alpha_1 - \alpha_2 + 3(\frac{9}{4}) = 4$$

We can convert to matrix form and by performing elementary row operations we get the matrix into Reduced Row Echelon Form:

$$\begin{pmatrix} 1 & 1 & -\frac{9}{4} \\ 2 & -1 & -\frac{11}{4} \end{pmatrix}$$

$$R_2 \leftrightarrow R_1$$

$$\begin{pmatrix} 2 & -1 & -\frac{11}{4} \\ 1 & 1 & -\frac{9}{4} \end{pmatrix}$$

$$R_2 - \tfrac{1}{2} \cdot\ R_1 \to R_2$$

$$\begin{pmatrix} 2 & -1 & -\frac{11}{4} \\ 0 & \frac{3}{2} & -\frac{7}{8} \end{pmatrix}$$

$$\tfrac{2}{3} \cdot\ R_2 \to R_2$$

$$\begin{pmatrix} 2 & -1 & -\frac{11}{4} \\ 0 & 1 & -\frac{7}{12} \end{pmatrix}$$

$$R_1 + 1 \cdot\ R_2 \to R_1$$

$$\begin{pmatrix} 2 & 0 & -\frac{10}{3} \\ 0 & 1 & -\frac{7}{12} \end{pmatrix}$$

$$\tfrac{1}{2} \cdot\ R_1 \to R_1$$

$$\begin{pmatrix} 1 & 0 & -\frac{5}{3} \\ 0 & 1 & -\frac{7}{12} \end{pmatrix}$$

This means $\alpha_1 = -\frac{5}{3}$ and $\alpha_2 = -\frac{7}{12}$

Final Solution:

$$A_n = (-1)^n(-\frac{5}{3}) + (2)^n(-\frac{7}{12}) + (\frac{9}{4})3^n$$

Submission. To submit the homework, you need to upload the pdf file into gradescope and iLearn.