Algorithms & Data Structures Notes - SoSe 24

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Preface

This is a Quarto book.

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Part I

Introduction

1 Program Run-time Analysis

1.1 Reccurence Relations

Consider a very simple reccurence relation:

$$T(n) := \begin{cases} 1 & n = 1 \\ n + T(n-1), & n > 1 \end{cases}$$

With **mathematical induction** we can formally show that $T(n)$ is quadratic. But there is a simpler & more intuitive way:

$$\begin{aligned}
T(n) &= n + T(n-1) && \text{(Def } T(\cdot)\text{)} \\
&= n + n - 1 + T(n-2) \\
&= \ldots && \text{(Repeat } n-2 \text{ times)} \\
&= n + n - 1 + \cdots + T(1) \\
&= n + n - 1 + \cdots + 1 && \text{(Def } T(1)\text{)} \\
&= \frac{n(n+1)}{2} && \text{(Gauss)} \\
&\in \mathcal{O}(n^2)
\end{aligned}$$

This method can be applied to the more complex divide-and-conquer reccurence relation from the lecture:

$$R(n) := \begin{cases} a, & n = 1 \\ c\dot{n} + d \cdot R(\frac{n}{b}), & n > 1 \end{cases}$$

Applying the above method we expand $R(\cdot)$ repetitively according to its definition until we reach the base case, rearranging terms when necessary:

$$
\begin{aligned}
R(n) &= c \cdot n + d \cdot R(\frac{n}{b}) && \text{(Def } R(\cdot)\text{)} \\
&= c \cdot n + d(c\frac{n}{b} + d \cdot R(\frac{n}{b^2})) \\
&= c \cdot n + d\Big(c\frac{n}{b} + d \cdot \big(c \cdot \frac{n}{b^2} + d \cdot R(\frac{n}{b^2})\big)\Big) \\
&= c \cdot n + d \cdot c\frac{n}{b} + d^2 c\frac{n}{b^2} + d^3 \cdot R(\frac{n}{b^3}) && \text{(Rearrange)} \\
&= c \cdot n \left(1 + \frac{d}{b} + \frac{d^2}{b^2}\right) + d^3 \cdot R(\frac{n}{b^3}) \\
&= \ldots && \text{(Repeat } k\text{-times)} \\
&= c \cdot n \left(1 + \frac{d}{b} + \cdots + \frac{d^{k-1}}{b^{k-1}}\right) + d^k \cdot R(\frac{n}{b^k}) \\
&= c \cdot n \sum_{i=0}^{k-1} \left(\frac{d}{b}\right)^i + d^k \cdot R(\frac{n}{b^k}) \\
&= c \cdot n \sum_{i=0}^{k-1} \left(\frac{d}{b}\right)^i + d^k \cdot R(1) && \text{(Ass } \tfrac{n}{b^k} = 1\text{)} \\
&= c \cdot n \sum_{i=0}^{k-1} \left(\frac{d}{b}\right)^i + a \cdot d^k && \text{(Def } R(1)\text{)}
\end{aligned}
$$

See lecture slides for the complexity analysis of final expression.