Algorithms & Data Structures Notes - SoSe 24

Igor Dimitrov

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Preface

This is a Quarto book.

To learn more about Quarto books visit https://quarto.org/docs/books.

Part I

Introduction

1 Program Run-time Analysis

1.1 Reccurence Relations

Consider a very simple reccurence relation:

$$T(n) := \begin{cases} 1 & n = 1 \\ n + T(n-1), & n > 1 \end{cases}$$

With **mathematical induction** we can formally show that $T(n)$ is quadratic. But there is a simpler & more intuitive way:

$$\begin{aligned}
T(n) &= n + T(n-1) && \text{(Def } T(\cdot)\text{)} \\
&= n + n - 1 + T(n-2) \\
&= \ldots && \text{(Repeat } n-2 \text{ times)} \\
&= n + n - 1 + \cdots + T(1) \\
&= n + n - 1 + \cdots + 1 && \text{(Def } T(1)\text{)} \\
&= \frac{n(n+1)}{2} && \text{(Gauss)} \\
&\in \mathcal{O}(n^2)
\end{aligned}$$

This method can be applied to the more complex divide-and-conquer reccurence relation from the lecture:

$$R(n) := \begin{cases} a, & n = 1 \\ c\dot{n} + d \cdot R(\frac{n}{b}), & n > 1 \end{cases}$$

Applying the above method we expand $R(\cdot)$ repetitively according to its definition until we reach the base case, rearranging terms when necessary:

$$
\begin{aligned}
R(n) &= c \cdot n + d \cdot R(\frac{n}{b}) && \text{(Def } R(\cdot)\text{)} \\
&= c \cdot n + d(c\frac{n}{b} + d \cdot R(\frac{n}{b^2})) \\
&= c \cdot n + d\Big(c\frac{n}{b} + d \cdot (c \cdot \frac{n}{b^2} + d \cdot R(\frac{n}{b^2}))\Big) \\
&= c \cdot n + d \cdot c\frac{n}{b} + d^2 c\frac{n}{b^2} + d^3 \cdot R(\frac{n}{b^3}) && \text{(Rearrange)} \\
&= c \cdot n \left(1 + \frac{d}{b} + \frac{d^2}{b^2}\right) + d^3 \cdot R(\frac{n}{b^3}) \\
&= \ldots && \text{(Repeat } k\text{-times)} \\
&= c \cdot n \left(1 + \frac{d}{b} + \cdots + \frac{d^{k-1}}{b^{k-1}}\right) + d^k \cdot R(\frac{n}{b^k}) \\
&= c \cdot n \sum_{i=0}^{k-1} \left(\frac{d}{b}\right)^i + d^k \cdot R(\frac{n}{b^k}) \\
&= c \cdot n \sum_{i=0}^{k-1} \left(\frac{d}{b}\right)^i + d^k \cdot R(1) && \text{(Ass } \tfrac{n}{b^k} = 1\text{)} \\
&= c \cdot n \sum_{i=0}^{k-1} \left(\frac{d}{b}\right)^i + a \cdot d^k && \text{(Def } R(1)\text{)}
\end{aligned}
$$

See lecture slides for the complexity analysis of final expression.

1.2 Master Theorem

For reccurence relations of the form:

$$
T(n) := \begin{cases} a, & n = 1 \\ b \cdot n + c \cdot T(\frac{n}{d}), & n > 1 \end{cases}
$$

Master theorem gives the solutions:

$$
T(n) = \begin{cases} \Theta(n), & c < d \\ \Theta(n \log(n)), & c = d \\ \Theta(n^{\log_b(d)}), & c > d \end{cases}
$$

Example: **Merge Sort**.

Complexity of merge sort satisfies the reccurence relation:

$$T(1) = 1$$
$$T(n) = \mathcal{O}(n) + 2 \cdot T(\frac{n}{2})$$

Thus with $c = 2 = d$ the second case of MT applies: $T(n) = \Theta(n \log n)$

1.3 Amortized Analysis

Part II

Data Structures

2 Lists

2.1 Sequences as Arrays and Lists

Many terms for same thing: sequence, field, list, stack, string, **file**… Yes, files are simply sequences of bytes!

three views on lists:

- **abstract**: $(2, 3, 5, 7)$
- **functionality**: stack, queue, etc… What operations does it support?
- **representation**: How is the list represented in a given programming model/language/paradigm?

2.2 Applications of Lists

- Storing and processing any kinds of data
- Concrete representation of abstract data types such as: set, graph, etc…

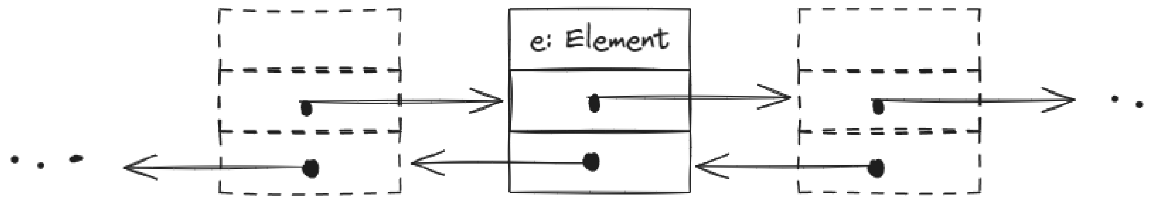
2.3 Linked and Doubly Linked Lists

	simply linked	doubly linked
lecture	`SList`	`List`
c++	`std::forward_list`	`std::list`

Doubly linked lists are usually **simpler** and require “only” double the space at most. Therefore their use is more widespread.

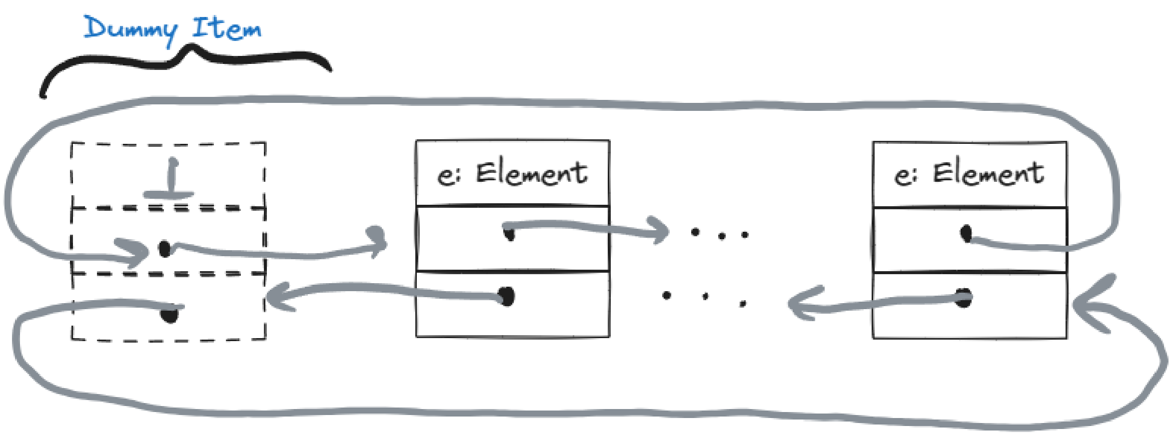
2.3.1 List Items

```
Class Item of T :=
    e: T //Data item of type T
    next: *Item //Pointer to Item
    prev: *Item //Pointer to Item
    invariant next->prev = this = prev->next
```



Problem: * predeccessor of first list element? * successor of last list element?

Solution: Dummy Item with an empty data field as follows:



Advatanges of this solution:

- **Invariant** is always satisfied
- Exceptions are avoided, thus making the coding more:
    - simple
    - readable
    - faster
    - elegant

Disadvantages: a little more storage space.

2.3.2 The List Class

```
Class List of T :=
    dummy := (
        Null : T
        &dummy : *T // initially list is empty, therefore next points to
↪   the dummy itself
        &dummy : *T // initially list is empty, therefore prev points to
↪   the dummy itself
    ) : Item

    // returns the address of the dummy, which represents the head of
↪   the list
    Function head() : *Item :=
        return address of dummy

    // simple access functions
    // returns true iff list empty
    Function is_empty() : Bool :=
        return dummy.next == dummy

    // returns pointer to first Item of the list, given list is not
↪   empty
    Function first() : *Item :=
        assert (not is_empty())
        return dummy.next

    // returns pointer to last Item of the list, given list is not empty
    Function last() : *Item :=
        assert (not is_empty())
        return dummy.prev

    /* Splice is an all-purpose tool to cut out parts from a list
       Cut out (a, ... b) form this list and insert after t */
    Procedure splice(a, b, t : *Item) :=
        assert (
            b is not before a
            and
            t not between a and b
        )
        // Cut out (a, ... , b)
```

```
    a->prev->next := b->next
    b->next->prev := a->prev

    // insert (a, ... b) after t
    t->next->prev := b
    b->next := t->next
    t->next := a
    a->prev := t


// Moving items by utilising splice
//Move item a after item b
Procedure move_after(a, b: *Item) :=
    splice(a, a, b)

// Move item a to the front of the list
Procedure move_to_front(a: *Item) :=
    move_after(a, dummy)

Procedure move_to_back(a: *Item) :=
    move_after(b, last())

// Deleting items by moving them to a global freeList
// remove item a
Procedure remove(a: *Item) :=
    move_after(b, freeList.dummy)

// remove first item
Procedure pop_front() :=
    remove(first())

//remove last item
Procedure pop_back() :=
    remove(last())

// Inserting Elements
// Insert an item with value x after item a
Function insert_after(x : T, a : *Item) : *Item :=
    checkFreeList() //make sure freeList is non empty
    b := freeList.first() // obtain an item b to hold x
    move_after(b, a) // insert b after a
```

```
        b->e := x // set the data item value of b to x
        return b

    // Manipulating whole lists
    Procedure concat(L : List) :=
        splice(L.first(), L.last(), last()) //move whole of L after last
 ↪  element of this list

    Procedure clear()
        freeList.concat(this) //after this operation from from first to
 ↪  last element of this
                              // list are concatenated to the freeList,
 ↪  leaving only the
                              // dummy element in this list.

    Fuction get(i )
```

Splicing

The code for splicing of the `List` class:

```
/* Splice is an all-purpose tool to cut out parts from a list
    Cut out (a, ... b) form this list and insert after t */
Procedure splice(a, b, t : *Item) :=
    assert (
        b is not before a
        and
        t not between a and b
    )
    // Cut out (a, ... , b)
    a->prev->next := b->next
    b->next->prev := a->prev

    // insert (a, ... b) after t
    t->next->prev := b
    b->next := t->next
```

```
t->next := a
a->prev := t
```

- Dlist cut-out $(a, ..., b)$ (see Figure 2.1):

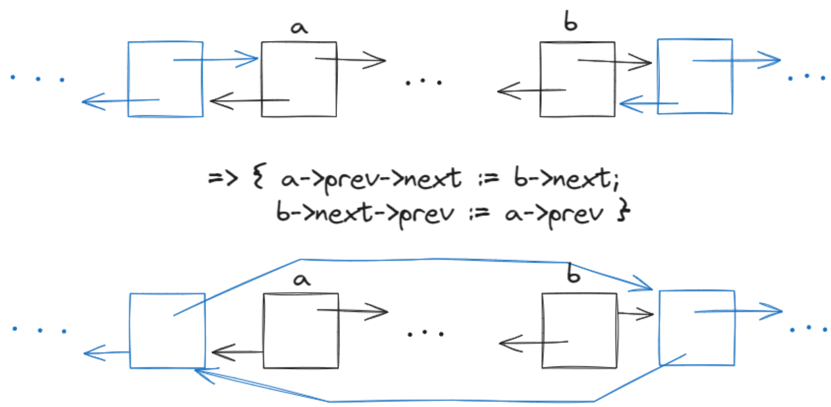


Figure 2.1: cutout

- Dlist insert $(a, ..., b)$ after t (see Figure 2.2):

Speicherverwaltung ./.FreeList

Methods (?):

- Niavely: allocate memory for each new element, deallocate memory after deleting each element:
  - advantage: simplicity
  - disadvantage: requires a good implementation of memory management: potentially very slow
- "global" `freeList` (e.g. `static` member in C++)
  - doubly linked list of all not used elements
  - transfer 'deleted' elements in `freeList`.
  - `checkFreeList` allocates, in case the list is empty

Real implementations: * naiv but with well implemented, efficient memory management * refined Free List Approach (class-agnostic, release) * implementation-specific.

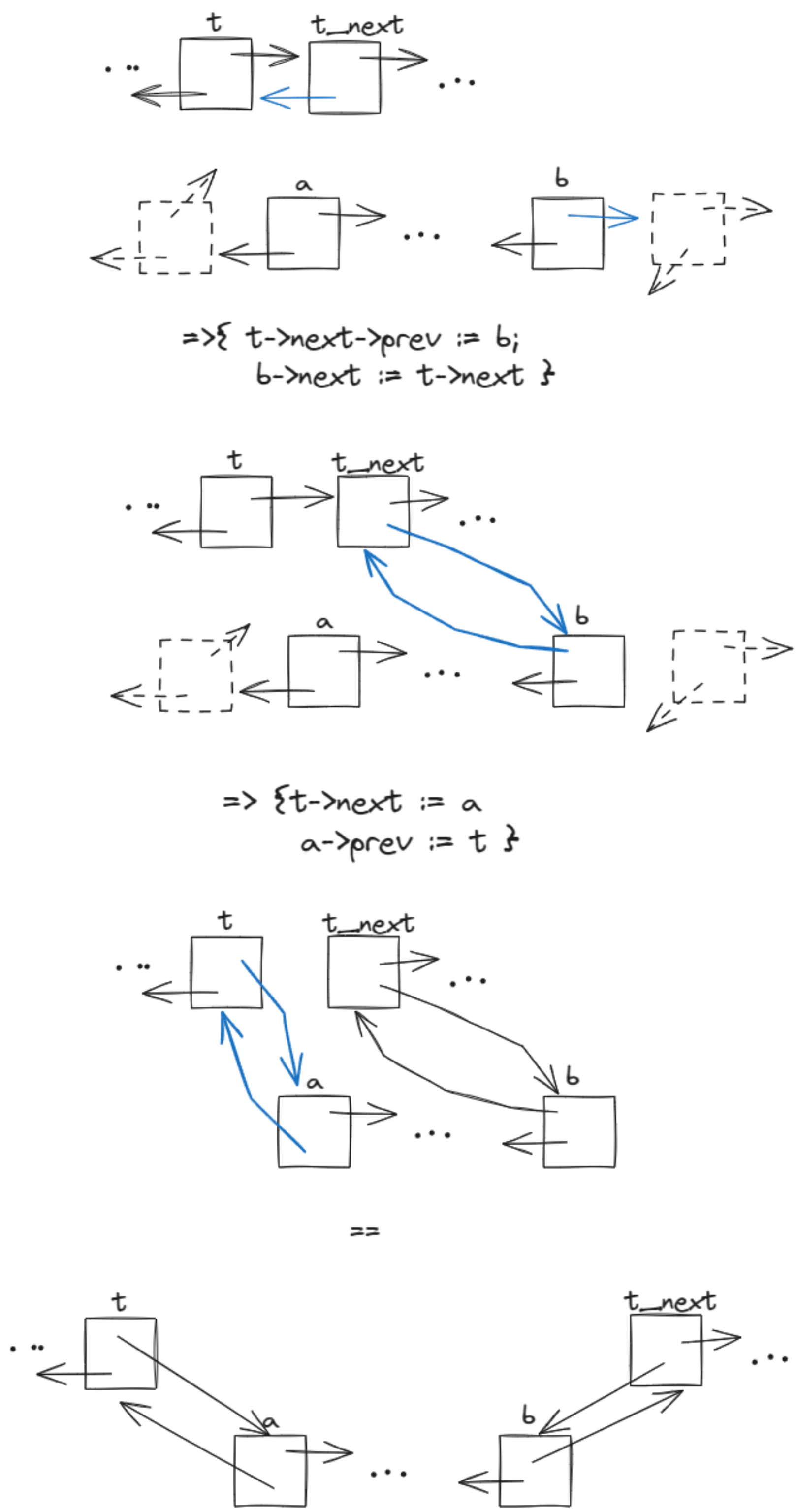


Figure 2.2: insert

Deleting Elements

Deleting elements realised by moving them to the global `freeList`:

```
Procedure remove(a: *Item) :=
    move_after(a, freeList.dummy) // item a is now a 'free' item.

Procedure pop_front() :=
    remove(first())

Procedure pop_back() :=
    remove(last())
```

Inserting Elements

Inserting elements into a list l also utilizes `freeList`, by fetching its first element an moving it into l.

```
Function insert_after(x : T, a : *Item) : *Item :=
    checkFreeList() //make sure freeList is non empty
    b := freeList.first() // obtain an item b to hold x
    move_after(b, a) // insert b after a
    b->e := x // set the data item value of b to x
    return b

Function insert_before(x : T, b : *Item) : *Item :=
    return insert_after(x, b->prev)

Procedure push_front(x : T) :=
    insert_after(x, dummy)

Procedure push_back(x : T) :=
    insert_after(x, last())
```

Manipulating whole Lists

```
// Manipulating whole lists
Procedure concat(L : List) :=
    splice(L.first(), L.last(), last()) //move whole of L after last
 ↪  element of this list

Procedure clear()
    freeList.concat(this) //after this operation from from first to last
 ↪  element of this
                            // list are concatenated to the freeList,
 ↪  leaving only the
                            // dummy element in this list.
```

This operations require **constant time** - indeendent of the list size!

3 Arrays

An array is a contigious sequence of memory cells.

3.1 Bounded Arrays

Bounded arrays have fixed size and are an efficient data structure.

- Size must be known during compile time and is fixed.
- Its memory location in the stack allows many compiler optimizations.

3.2 Unbounded Arrays

The size of an **unbounded array** can dynamically change during run-time. From the user POV it provides the same behaviour as a linked list.

It allows the operations:

- `pushBack(e: T)`: insert an element at the end of the array
- `popBack(e: T)`: remove an element at the end of the array

3.2.1 Memory Management

- `allocate(n)`: request a n contigious blocks of memory words and returns the address value of the first block. This we have the memory blocks:

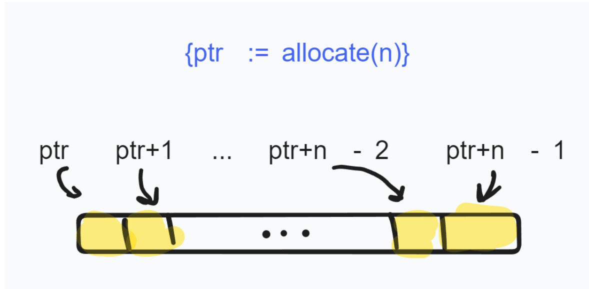


Figure 3.1: array memory allocation

where `ptr + i` addresses are determined by pointer arithmetic.

- `dispose(ptr)` marks the memory address value held in `ptr` as free, effectively deleting the object held there.

In general, the allocated memory can't grow dynamically during life time, since the immediate memory block after the last one might get unpredictably occupied $\Rightarrow$ If we need a new memory block of size $n' > n$, we must allocate a new block, copy the old block contents, and finally free it.

3.2.2 Implementation

First we consider a slow variant:

```
Class UArraySlow<T>:=
  c := 0 : Nat // capacity
  b : Array[0..c-1]<T> // the array itself

  pushBack(el : T) : void :=
    // c++
    // allocate new array on heap with new capacity
    // copy elements over from the old array
    // insert el at the last location

  popBack() : void :=
    // analagous
```

Problem: n `pushBack` operations require $1 + \cdots + n \in \mathcal{O}(n^2)$ time $\Rightarrow$ slow.

Solution:

Unbounded Arrays with Extra Memory

Idea: Request more memory than initial capacity. Reallocate memory only when array gets full or too empty:

Algorithm design principle: make common case fast.

```
Class UArray<T> :=
  c := 1 : Nat // capacity
  n := 0 : Nat // number of elements in the array

  //invariant n <= c < k*n || (n == 0 && c < 2)
  b : Array[0..c-1]<T>

  // Array access
  Operator [i : Nat] : T :=
    assert(0 <= i < n)
    return b[i]

  // accessor method for n
  Function size() : Nat := return n

  Procedure pushBack(e : T) :=
    if n == c :
      reallocate(2*n) // see definition below
    b[n] := e
    n++

  // reallocates a new memory with a given capacity c_new
  Procedure reallocate(c_new : Nat) :=
    c := c_new
    b_new := new Array[0..c_new - 1]<T>
    //copy elements over to new array
    for (i = 1 to n - 1) :
      b_new[i] := b[i]
    dispose(b)
    b := b_new

  Procedure popBack() :=
    // don't do anything for empty arrays
    assert n > 0
    n--
```

```
if 4*n <= c && n > 0 :
  reallocate(2*n)
```

4 Sorting and Priority Queues

4.1 Sorting Algorithms

4.1.1 Insertion Sort

```
def insertion_sort(a) :
    n = len(a)
    # i = 1
    # sorted a[0..i-1]
    for i in range(1, n) :
        # insert i in the right position
        j = i - 1
        el = a[i]
        while el < a[j] and j > 0 :
            a[j + 1] = a[j]
            j = j - 1
        # el >= a[j] or j == 0
        if el < a[j] : # j == 0
            a[1] = a[0]
            a[0] = el
        else : # el >= a[j]
            a[j + 1] = el
    return a
```

testing insertion sort for some inputs:

```
import numpy as np
for i in range (2, 8) :
    randarr = np.random.randint(1, 20, i)
    print("in:  ", randarr)
    print("out: ", insertion_sort(randarr))
```

```
in:   [10 17]
```

```
out:   [10 17]
in:    [ 8 11  1]
out:   [ 1  8 11]
in:    [ 8  3 13  2]
out:   [ 2  3  8 13]
in:    [ 6 11  1 11  2]
out:   [ 1  2  6 11 11]
in:    [ 5  5  3  7 12 14]
out:   [ 3  5  5  7 12 14]
in:    [18 16 14  3 19 10 16]
out:   [ 3 10 14 16 16 18 19]
```

Following illustrates the state after each insertion (ith iteration):

```python
def insertion_sort_print(a) :
    n = len(a)
    # i = 1
    # sorted a[0..i-1]
    for i in range(1, n) :
        # insert i in the right position
        j = i - 1
        el = a[i]
        while el < a[j] and j > 0 :
            a[j + 1] = a[j]
            j = j - 1
        # el >= a[j] or j == 0
        if el < a[j] : # j == 0
            a[1] = a[0]
            a[0] = el
        else : # el >= a[j]
            a[j + 1] = el
        print("after insertion ", i, ": ", a)
    # return a
```

```python
a = np.random.randint(-20, 20, 8)
print("input:                ", a)
insertion_sort_print(a)
```

```
input:                [-19  -6  11 -16  15 -11  12  -1]
after insertion  1 :  [-19  -6  11 -16  15 -11  12  -1]
after insertion  2 :  [-19  -6  11 -16  15 -11  12  -1]
```

```
after insertion  3 :  [-19 -16  -6  11  15 -11  12  -1]
after insertion  4 :  [-19 -16  -6  11  15 -11  12  -1]
after insertion  5 :  [-19 -16 -11  -6  11  15  12  -1]
after insertion  6 :  [-19 -16 -11  -6  11  12  15  -1]
after insertion  7 :  [-19 -16 -11  -6  -1  11  12  15]
```

4.1.2 Selection Sort

explanation:

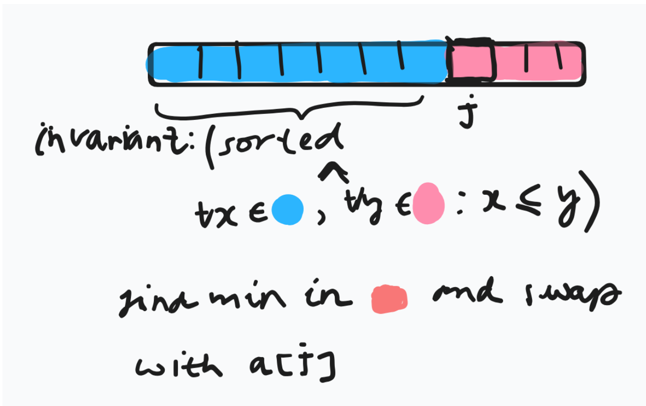


Figure 4.1: selection sort

python implementation:

```python
def selection_sort(a) :
    N = len(a)
    j = 0
    # sorted a[0..j-1] && a[0..j-1] <= a[j..N-1]
    while (j < N) :
        # find min a[j..N-1]
        k = j
        min = a[k]
        i = j + 1
        #a[k] == min == min(a[j .. i - 1])
        while (i < N) :
            if a[i] < min :
                min = a[i]
                k = i
            i = i + 1
```

```
        # k = j + find_min(a[j :])
        a[j], a[k] = a[k], a[j]
        j = j + 1
    return a
```

we test this on some random arrays:

```
for i in range (2, 8) :
    randarr = np.random.randint(-50, 50, i)
    print("in:  ", randarr)
    print("out: ", selection_sort(randarr))
```

```
in:   [36 15]
out:  [15 36]
in:   [-30  26 -11]
out:  [-30 -11  26]
in:   [ -4 -11 -33  10]
out:  [-33 -11  -4  10]
in:   [-46  30  48  15 -15]
out:  [-46 -15  15  30  48]
in:   [-22 -34  13 -17  27 -41]
out:  [-41 -34 -22 -17  13  27]
in:   [-17   2 -47  27  36 -21 -21]
out:  [-47 -21 -21 -17   2  27  36]
```

4.1.3 Bubble Sort

Let `a : Array[0..N-1]<Nat>`. The bubble operation pushes the largest element to the end of the array:

```
def bubble(a) :
    N = len(a)
    i = 0
    # a[i] == max(a[0..i])
    while i < N - 1:
        if a[i] > a[i + 1] :
            a[i], a[i+1] = a[i+1], a[i]
        i = i + 1
    # post-loop: i == N - 1
```

```python
    return a

bubble([-5, 10, 1, 3, 7, -2])
```

```
[-5, 1, 3, 7, -2, 10]
```

The code of this function is used inside `bubble_sort()`:

```python
def bubble_sort(a) :
    N = len(a)
    j = N
    swapped = False
    while True : # emulate do while loop
        # sorted a[j .. N - 1] and a[0..j-1] <= a[j .. N - 1]
        while j > 0 :
            i = 0
            while i < j - 1 :
                if a[i] > a[i + 1] :
                    a[i], a[i + 1] = a[i + 1], a[i]
                i = i + 1
            j = j - 1
        if not swapped : break # if no swaps performed at all, array
         ↪  already
                                  # sorted
    return a
```

We test on some arrays:

```python
for i in range (2, 8) :
    randarr = np.random.randint(-50, 50, i)
    print("in:  ", randarr)
    print("out: ", bubble_sort(randarr))
```

```
in:   [ 24 -10]
out:  [-10  24]
in:   [  2 -13  41]
out:  [-13   2  41]
in:   [-26 -45  -2  -8]
out:  [-45 -26  -8  -2]
in:   [-17 -36 -25 -12 -22]
```

```
out:  [-36 -25 -22 -17 -12]
in:   [16 29 24 38 37 14]
out:  [14 16 24 29 37 38]
in:   [-28 -18  10  14 -32  11 -14]
out:  [-32 -28 -18 -14  10  11  14]
```

Visual expalantion:

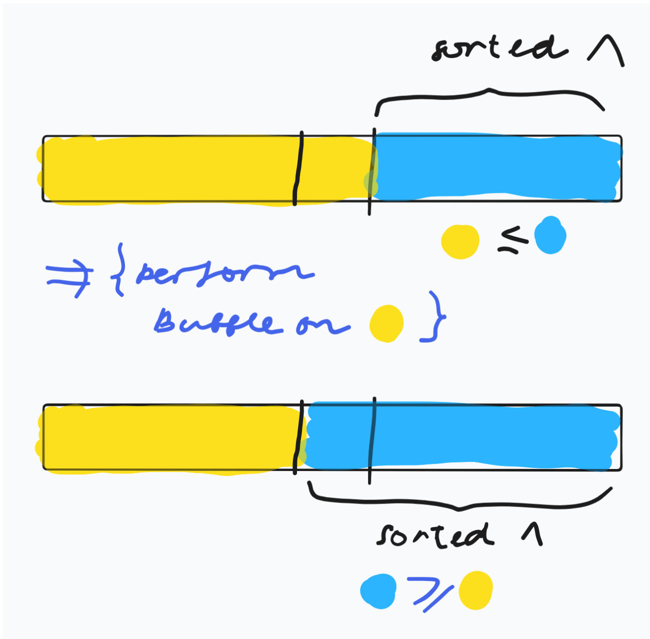


Figure 4.2: bubble sort

4.1.4 Merge Sort

Given by the following python implementation:

```python
def merge(a, b) :
    # assert: a and b are sorted
    c = []
    n1 = len(a)
    n2 = len(b)
    k1 = 0
    k2 = 0
    i = 0
```

```
    # invariant: merged a[0..k1 - 1] with b[0..k2 - 2]
    while k1 < n1 and k2 < n2 :
        if a[k1] <= b[k2] :
            c.append(a[k1])
            k1 = k1 + 1
        else :
            c.append(b[k2])
            k2 = k2 + 1
    # k1 >= n1 or k2 >= n2
    if k1 == n1 :
        while k2 < n2 :
            c.append(b[k2])
            k2 = k2 + 1
    if k2 == n2 :
        while k1 < n1 :
            c.append(a[k1])
            k1 = k1 + 1
    return c

def merge_sort(a) :
    if len(a) == 1 : return a[0:1]
    n = len(a)
    a1 = a[0 : n // 2]
    a2 = a[n // 2 : ]
    return  merge(merge_sort(a1), merge_sort(a2))
```

We test on some arrays:

```
for i in range (2, 8) :
    randarr = np.random.randint(-20, 20, i)
    print("in:  ", randarr)
    print("out: ", merge_sort(randarr))
```

```
in:   [ 3 10]
out:  [3, 10]
in:   [ 1  9 19]
out:  [1, 9, 19]
in:   [-11  -9  -2   1]
out:  [-11, -9, -2, 1]
in:   [-16 -18 -10   4  -4]
out:  [-18, -16, -10, -4, 4]
```

```
in:   [ 10 -20 -11  -1   1 -12]
out:  [-20, -12, -11, -1, 1, 10]
in:   [ -9 -12 -16  13  -2  19 -14]
out:  [-16, -14, -12, -9, -2, 13, 19]
```

4.1.5 Quick Sort

Naively:

```
def quicksort(s) :
    if len(s) <= 1 : return s
    p = s[len(s) // 2]
    a = []
    b = []
    c = []
    for i in range(0, len(s)) :
        if s[i] < p : a.append(s[i])
    for i in range(0, len(s)) :
        if s[i] == p : b.append(s[i])
    for i in range(0, len(s)) :
        if s[i] > p : c.append(s[i])
    return quicksort(a) + b + quicksort(c)
```

testing this naive implementation for some arrays:

```
for i in range (2, 8) :
    randarr = np.random.randint(-10, 20, i)
    print("in:  ", randarr)
    print("out: ", quicksort(randarr))
```

```
in:   [ 1 14]
out:  [1, 14]
in:   [ 6 15  5]
out:  [5, 6, 15]
in:   [-1  2  3  6]
out:  [-1, 2, 3, 6]
in:   [ 11  13  16   7 -10]
out:  [-10, 7, 11, 13, 16]
in:   [-5 15 -4  6 -6 -2]
out:  [-6, -5, -4, -2, 6, 15]
in:   [-10  12   7  11   7  -8   6]
```

```
out:  [-10, -8, 6, 7, 7, 11, 12]
```

Quicksort Refinements

pseudocode:

```
Procedure qSort(a : Array<T>; l, r : Nat) :=
    while r - l + 1 > n0 :
        j := pick_pivot_pos(a, l, r)
        swap(a[l], a[j]) // pivot is at the first position
        p := a[l]  // p is the value of the pivot
        i := l; j := r
        do
            while a[i] < p : i++; //skip over the elements
            while a[j] > p : j--; // already in the correct subarray
            if i <= j :
                swap(a[i], a[j])
                i++
                j--
        while i <= j
        qSort(a, l, j)
        qSort(a, i, r)
```

cpp implementation including testing for `{3, 6, 8, 1, 0, 7, 2, 4, 5, 9}`

```cpp
#include <iostream>

void qSort(int* a, int l, int r)
{
    if (r <= l) return;
    int p = a[l]; // first element is pivot
    int i = l;
    int j = r;
    do {
        while (a[i] < p) i++;
        while (a[j] > p) j--;
        if (i <= j) {// partitioning is not complete
            int temp = a[i];
            a[i] = a[j];
            a[j] = temp;
            i++;
```

```
            j--;
        }
    } while (i <= j);
    // i > j
    qSort(a, l, j);
    qSort(a, i, r);
}

int main(int argc, const char** argv) {
    int a[] = {3, 6, 8, 1, 0, 7, 2, 4, 5, 9};
    for (int i = 0; i < 9; i++)
        std::cout << a[i] << ", ";
    std::cout << a[9] << std::endl;
    return 0;
}
```

4.1.6 Bucket Sort

So far in our model we assumed no information on keys; we didn't know whether they are numbers, strings or any other data type. The only requirement was that any two keys where comparable. Our **comparison based** sorting algorithms relied solely on comparing any two keys. Theoretically it can be shown that the lower bound for such algorithms is $\Omega n \log n$.

Now if we extend our model

```
# sorts keys in range [0, 100)
def KSort(s) :
    # initialize orray of length 100 with empty buckets
    b = []
    for i in range(100) : b.append([])
    # place elements in buckets
    for el in s : b[el].append(el)
    # array holding results
    res = []
    # append elements in buckets to res
    for i in range(100) :
        for el in b[i] : res.append(el)
    return res

# testing:
```

```python
for i in range (2, 8) :
    randarr = np.random.randint(0, 100, i)
    print("in:  ", randarr)
    print("out: ", KSort(randarr))
```

```
in:   [29 92]
out:  [29, 92]
in:   [17 33 21]
out:  [17, 21, 33]
in:   [26  6 47 27]
out:  [6, 26, 27, 47]
in:   [32 70 36 23 38]
out:  [23, 32, 36, 38, 70]
in:   [64 79 20 92 40 14]
out:  [14, 20, 40, 64, 79, 92]
in:   [ 3 66 79 68 84 96 31]
out:  [3, 31, 66, 68, 79, 84, 96]
```

Radix Sort

Employing a clever trick we can significantly increase the range of keys. In bucket sort we perform the sorting on the key itself. In radix sort we iteratively perform bucket sort on the digits of the keys, starting from the least significant digit. This works especially because bucket sort is a stable sorting algorithm.

This way we can sort keys in range $10^d - 1$. We have 10 buckets. Different bases can be chosen.

We slightly modify previous bucket sort, where a `key` function is passed as an argument, with $d = 5$

```python
# sorts keys in range [0, 10)
def KSort2(s, key) :
    # initialize orray of length 100 with empty buckets
    b = []
    for i in range(10) : b.append([])
    # place elements in buckets
    for el in s : b[key(el)].append(el)
    # array holding results
    res = []
    # append elements in buckets to res
```

```python
    for i in range(10) :
        for el in b[i] : res.append(el)
    return res

# sorts keys in range [0, 10^5)
def LSDRadixSort(a) :
    for i in range(5) :
        a = KSort2(a, lambda x : (x // 10**i) % 10)
    return a

# testing
for i in range(5, 10) :
    randarr = np.random.randint(0, 10**5, i)
    print("input: ", randarr)
    print("output: ", LSDRadixSort(randarr))
```

```
input:  [91041  6457 42555 63414 33369]
output:  [6457, 33369, 42555, 63414, 91041]
input:  [85794 75770 82221 78571 55638  7828]
output:  [7828, 55638, 75770, 78571, 82221, 85794]
input:  [62663 87858 46620 13037 76723 87920 32504]
output:  [13037, 32504, 46620, 62663, 76723, 87858, 87920]
input:  [55954 58982  3414 81198 34609  3932 21178 70228]
output:  [3414, 3932, 21178, 34609, 55954, 58982, 70228, 81198]
input:  [17636 79022 12189 99140 97898 64085 52606 22526 84953]
output:  [12189, 17636, 22526, 52606, 64085, 79022, 84953, 97898, 99140]
```

4.2 Priority Queues and Heap Data Structure

A set M of Elements $e : T$ with Keys supporting two operations:

- `insert(e)`: Insert e into M.
- `delete_min()`: remove the min element from M and return it.

4.2.1 Applications

- Greedy algorithms (selecting the optimal local optimal solution)
- Simulation of discrete events
- branch-and-bound search
- time forward processing.

4.2.2 Binary Heaps

Heap Property:

- For any leaf $a \in M$ a is a heap.
- Let T_1, T_2 be heaps. If $a \leq x$, $\forall x \in T_1, T_2$, then $T_1 \circ a \circ T_2$ is also a heap.

Complete Binary Tree:

- A **complete** binary tree is a binary tree in which ever lebel, except possibly the last, is completely filled, and all nodes in the last level are as far left as possible.

Heap:

- A **heap** is a complete binary tree that satisfies the heap property:
- A heap can be succinctly represented as an array:

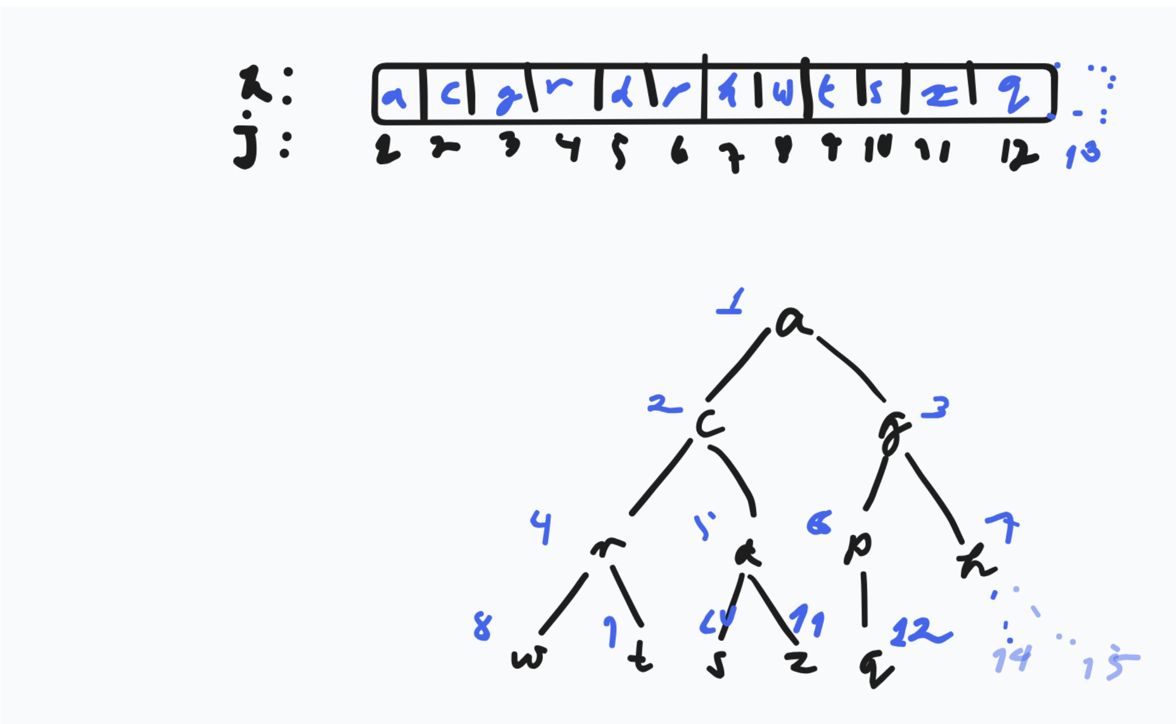


Figure 4.3: heap

- Array `h[1..n]`
- for any given node with the number `j`:
  - left child: `2*j`

  - right child 2*j + 1
  - parent: bottom(j/2)

Pseudocode:

```
Class BinaryHeapPQ(capacity: Nat)<T> :=
    h : Array[1..capacity]<T>
    size := 0 : Nat // current amount of elements

    // Heap-property
    // invariant: h[bottom(j/2) <= h(j)], for all j == 2..n

    Function min() :=
        assert size > 0 // heap non-emtpy
        return h[1]

    Procedure insert(e : T) :=
        assert size < capacity
        size++
        h[size] := e
        siftUp(size)

    Procedure siftUp(i : Nat) :=
        // assert Heap-property violated at most at position i
        if i == 1 or h[bottom(i / 2)] <= h[i] then return
        swap(h[i], h[bottom(i/2)])
        siftUp(bottom(i/2))

    Procedure popMin : T :=
        result = h[1] : T
        h[1] := h[size]
        size--
        siftDown(1)
        return result

    Procedure siftDown (i : Nat) :=
        // assert: Heap property is at most at position 2*i or 2*i + 1
 ↪  violated
        if 2i > n then return // i is a leaf

        // select the appropriate child
        if 2*i + 1 > n or h[2*i] <= h[2*i + 1] :
```

```
        //no right child exists or left child is smaller than right
            m := 2*i
        else : m := 2*i + 1
        if h[i] > h[m] :
            swap(h[i], h[m])
            siftDown(m)

    Procedure buildHeap(a[1..n]<T>) :=
        h := a
        buildRecursive(1)

    Procedure buildHeapRecursive(i : Nat) :=
        if 4*i <= size : // children are not leaves
            buildHeapRecursive(2*i) // assert: heap property holds for left subtree
            buildHeaprecursive(2*i + 1) // assert: heap property holds for right subtree
        siftDown(i) //assert Heap property holds for subtree starting at i

    //alternatively
    Procucure buildHeapBackwards :=
        for i := n/2 downto 1 :
            siftDown(i)

    Procedure heapSort(a[1..n]<T>) :=
        buildHeap(a) // O(n)
        for i := n downto 2 do :
            h[i] := deleteMin(); // O(log(n))
```

Heap Insert

```
Procedure insert(e : T) :=
    assert size < capacity
    size++
    h[n] := e
    siftUp(n)

Procedure siftUp(i : Nat) :=
```

```
    // assert Heap-property violated at most at position i
    if i == 1 or h[bottom(i / 2)] <= h[i] then return
    swap(h[i], h[bottom(i/2)])
    siftUp(bottom(i/2))
```

Illustration of heap insert:

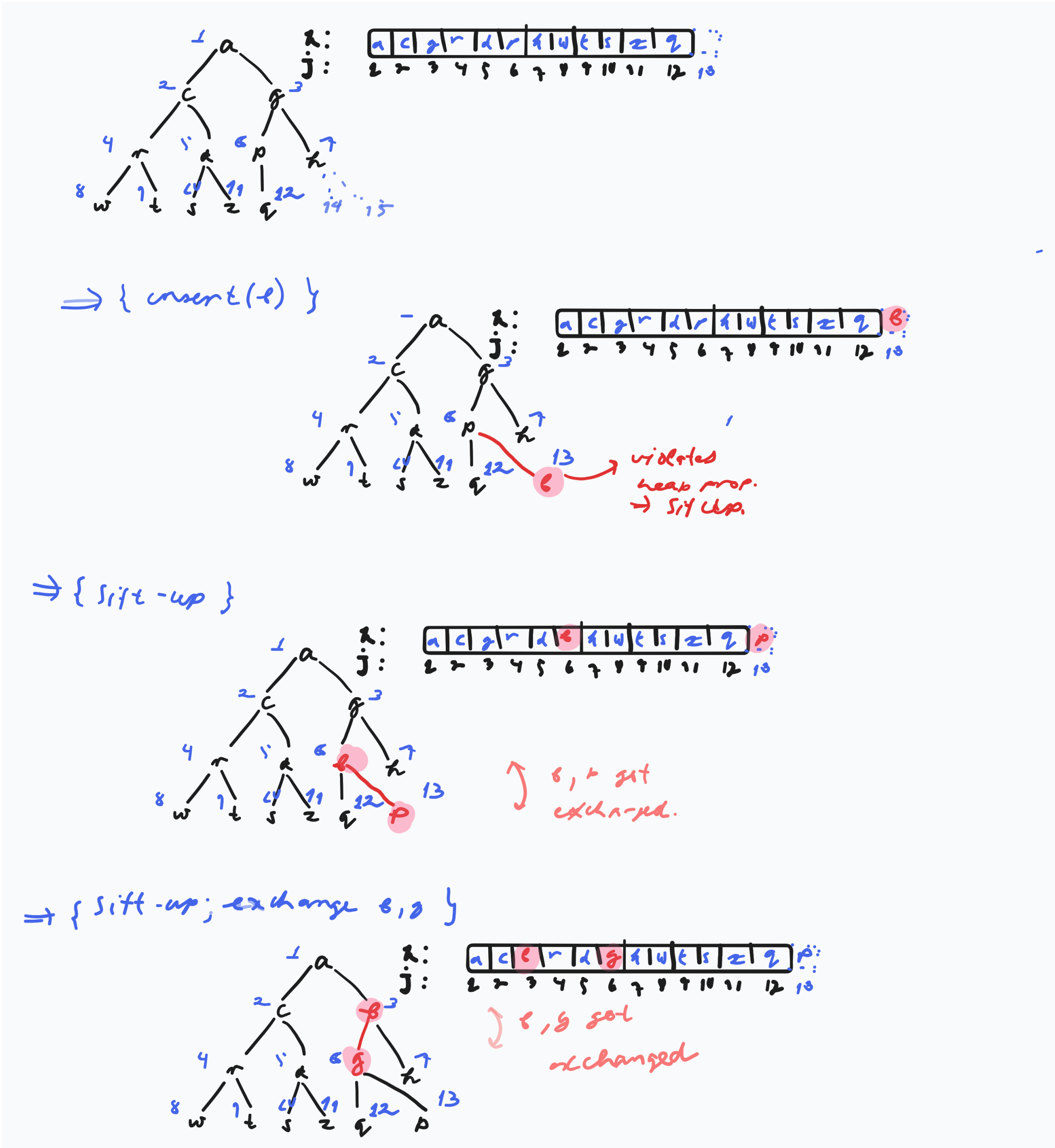


Figure 4.4: heap insert

Heap Pop Min (or Delete Min)

```
 Procedure popMin : T :=
        result = h[1] : T
        h[1] := h[n]
        n--
        siftDown(1)
        return result

Procedure siftDown (i : Nat)
    // assert: Heap property is at most at position 2*i or 2*i + 1
 ↪  violated
    if 2i > n then return // i is a leaf

    // select the appropriate child
    if 2*i + 1 > n or h[2*i] <= h[2*i + 1] :
    //no right child exists or left child is smaller than right
        m := 2*i
   if h[i] > h[m] :
            swap(h[i], h[m])
            siftDown(m)   else : m := 2*i + 1
```

Illustration of pop min:

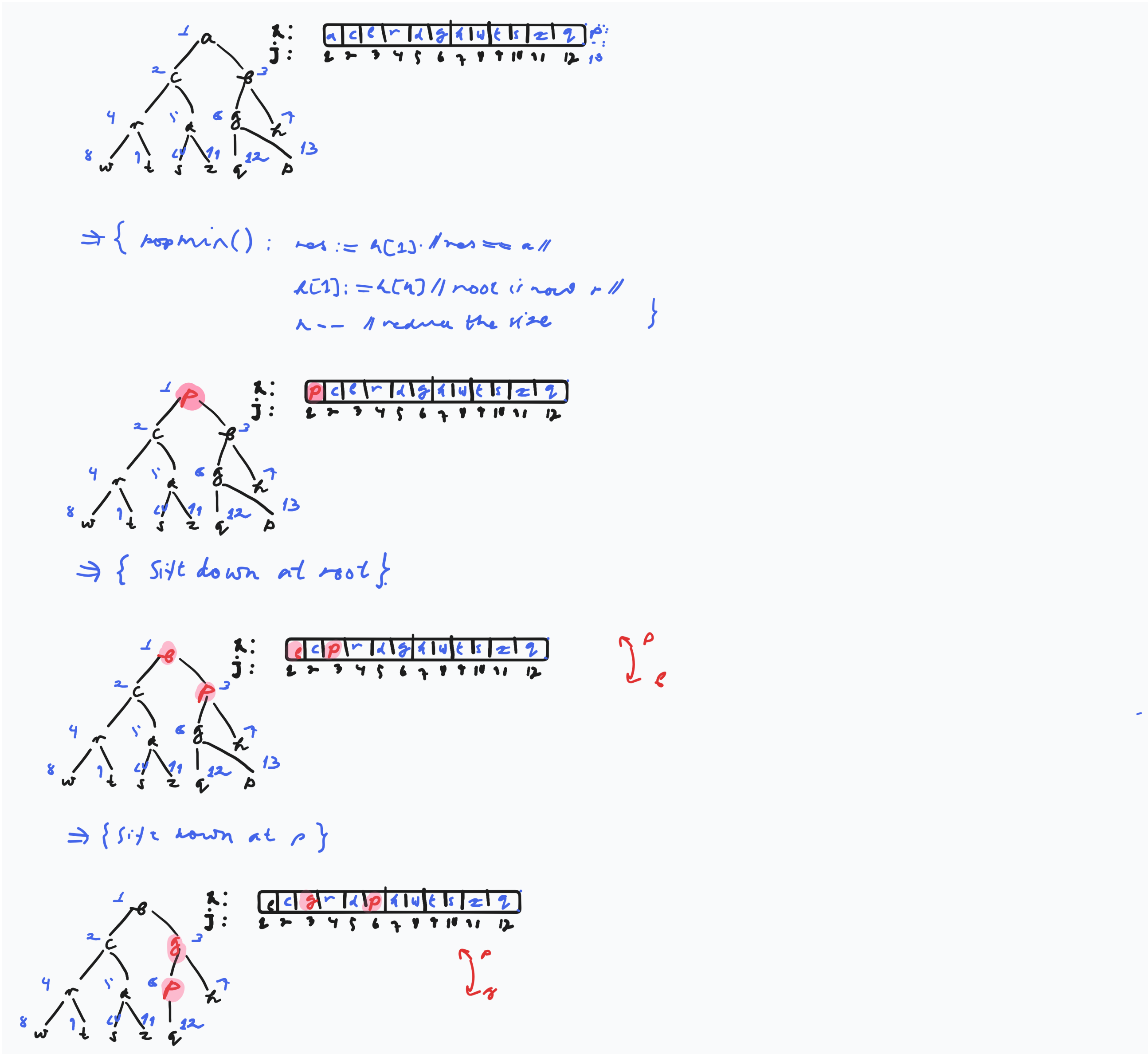


Figure 4.5: heap pop min

Construction of a Binary Heap

- Given are n numbers. Construct a heap from these numbers
- **Naive Solution**: n calls to `insert()` $\Rightarrow \mathcal{O}(n \log(n))$
  - Problem: If numbers are given in an array, we can't perform the construction in

place.
  - It is slow

- we can do faster and in place in $\mathcal{O}(n)$ time.

Pseudocode for recursive implementation:

```
 Procedure buildHeap(a[1..n] : T) :=
        h := a
        buildRecursive(1)

Procedure buildHeapRecursive(i : Nat) :=
    if 4*i <= n : // children are not leaves
        buildHeapRecursive(2*i) // assert: heap property holds for left
 ↪  subtree
        buildHeaprecursive(2*i + 1) // assert: heap property holds for
 ↪  right subtree
    siftDown(i) //assert Heap property holds for subtree starting at i
```

A simpler iterative one-liner:

```
 Procudure buildHeapBackwards :=
        for i := n/2 downto 1 :
            siftDown(i)
```

$\lfloor i/2 \rfloor$ is the last non-leaf node.

Time complexity of these binary heap construction algorithms is $\boldsymbol{\mathcal{O}}(n)$.

Heapsort

```
Procedure heapSort(a[1..n]<T>) :=
        buildHeap(a) // O(n)
        for i := n downto 2 do :
            h[i] := deleteMin(); // O(log(n))
```

Sorts in decreasing order in $\mathcal{O}(n\log(n))$, by removing the minimal element and writing the return value to the end of the array in place.