Long line topological space

Igor Kan Zakhidov

University of Toronto Scarborough

igor.zakhidov@mail.utoronto.ca

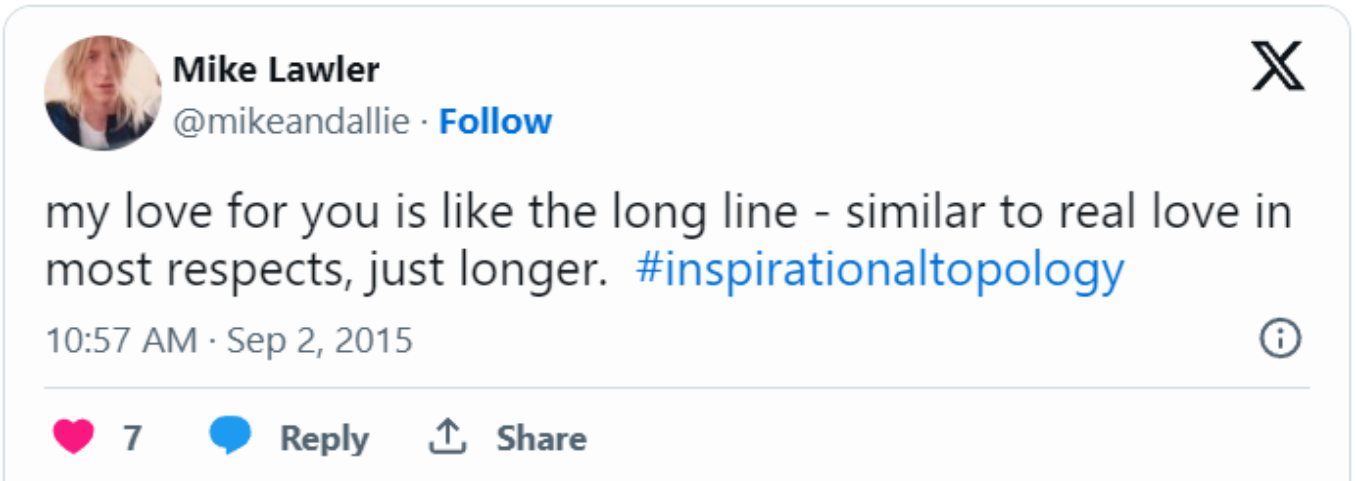
December 5, 2023

Overview

Motivation

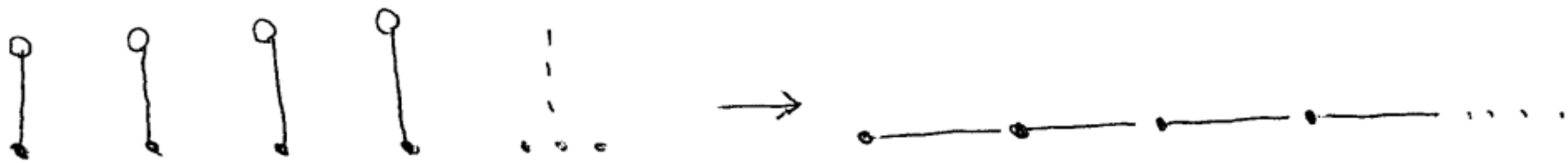
Roots of Unity

A Few of My Favorite Spaces: The Long Line



Motivation

The whole is greater than the sum of its parts.



Construction 1

Definition (Closed Long Ray)

$$L := \omega_1 \times [0, 1)$$

Definition (Open Long Ray)

$$L^o := L \setminus (0, 0)$$

Definition (Long Line)

$$\mathbb{L} := L^o \sqcup L = (L^o \times 0) \cup (L \times 1)$$

Well-ordering

Definition (Well-ordered set)

A set A with an order relation $<$ is said to be well-ordered if every nonempty subset of A has a smallest element.

Properties

- Any subset of A is well-ordered in the restricted order relation.
- If A and B are well-ordered sets, then AB is well-ordered in the dictionary order.

Theorem (Zermelo well-ordering)

If A is a set, there exists an order relation on A that is a well-ordering.

Corollary

There exists an uncountable well-ordered set.

Well-ordering

Lexicographic ordering

Let A and B be linearly ordered sets. A linearly ordered set $A \times B$ has the lexicographic (or dictionary) ordering if for any $(a, b), (a', b') \in A \times B$, $(a, b) < (a', b')$ if and only if either $a < a'$ or else $a = a'$ and $b < b'$
Lexicographic ordering is linear on $A \times B$

Definition (Section of A by Ω or minimal uncountable well-ordered set)

$$S_\Omega := x \in A | x < \alpha \in A$$

where A - well-ordered set having a largest element Ω, such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.

Theorem (S_Ω)

If A is a countable subset of S_Ω, then A has an upper bound in S_Ω.

Construction 2

Definition (Long Line)

$$\mathbb{L} := S_\Omega \times [0, 1)$$

Properties of the long line

- is a topological 1-manifold.
- is connected.
- is not Lindelöf.
- is not second countable.
- is not metrizable.
- is not paracompact.
- is not normal.
- is not path-connected.*
- is not compact.*
- is pseudocompact.*
- is sequentially compact.*

More properties and applications

Example (Pseudocompact 1-manifold)

There is no 1-manifold X containing $\mathbb{L}$ as a proper subspace.

$\mathbb{L}$ has no closed copy of the closed Euclidean n-ball.

If M is a pseudocompact n-manifold, then M cannot be properly extended to an n-manifold.

Example (Nonmetrizable topological n-manifolds)

For each $n \geq 2$, $\mathbb{L}^n$ is a nonmetrizable topological n-manifold.

References


General topology.


Long line in nLab.


Long line (topology).


MATH205C Notes: Long Line.


WHEN SPACE DOES NOT HAVE DISTANCE: What is the Long Line in Math and Other Examples (Version 2.0).


Anderson, H.
Long Line's Fundamental Group Reworded - General topology.


Dahmen, R. and Lukács, G.
Long colimits of topological groups I: Continuous maps and homeomorphisms.
270:106938.


Dugundji, J.
Topology.
Brown.


Glickenstein, D.
Topology Lectures –Integration workshop 2018.


Hews, J.
The Geometric Viewpoint — The Long Line.


Kim, J.
TOPOLOGY OF LONG LINE AND INSEPARABLE.


Koch, R.
MATH431 Notes: The Long Line.


Lamb, E.
A Few of My Favorite Spaces: The Long Line.


Lubarsky, R.
Topological spaces with nowhere locally constant functions to the reals.


Munkres, J. R.
Topology.
Prentice Hall, Inc, 2nd ed edition.


Shulman, M.
Topology - I, WOMP2004.


Steen, L. A. and Seebach, J. A.
Counterexamples in Topology.
Dover Publications.

The End