PL: a GAP 4 package for piecewise-linear topology and mathematical physics

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1 Loading the package

Extract the package `PL` in your `pkg` directory and then write

```
gap> LoadPackage("PL");
```

or, alternatively, put the line `LoadPackage("PL");` in your `.gaprc` file or the like.

2 PL ball complexes and how they are represented in `PL`

2.1 Definition of PL ball complex

Piecewise-linear, or simply PL, ball complexes are, at least at this moment, the central objects with which our package `PL` deals.

First, a ball complex (see e.g. [2]) is, simply speaking, a kind of cell complex but such where all *closed* cells (= balls) are *embedded*. In particular, their boundaries are genuine spheres, not crumpled/folded. The formal definition of PL ball complex reads:

> A PL ball complex is a pair (X, U), where X is a compact Euclidean polyhedron and U is a covering of X by closed PL-balls such that the following axioms are satisfied:
>
> - the relative interiors of balls from U form a partition of X,
> - the boundary of each ball from U is a union of balls from U.

We also call PL ball complexes "polytopes", for brevity, hence prefix "Pol" in the names of some of our functions.

2.2 Representation of a PL ball complex

A PL-ball complex is defined up to PL-homeomorphism only by the combinatorics of adjunctions of its balls. Due to this, we represent them combinatorially in the following way.

First, we assume that all vertices in the complex are numbered from 1 to their total number N_0. Hence, in this sense, the 0-skeleton of the complex is described. Next, assuming that the k-skeleton is already given, which implies (in particular) the numeration of all k-cells, we describe the $(k+1)$-skeleton as the list of all $(k+1)$-cells, each of which, in its turn, is the set of numbers of k-cells in its boundary. Then we compose the list of length n, where n is the dimension of the complex, whose elements are lists of 1-, ..., n-cells.

Thus, a three-dimensional ball B^3 can be represented by the following PL ball complex with two vertices 1 and 2:

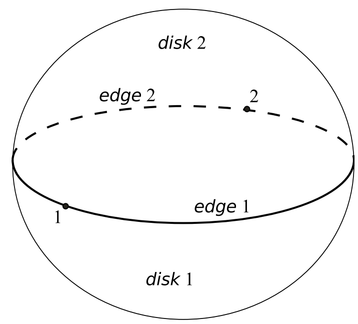

Figure 1: Three-dimensional ball as a PL ball complex

```
[
  [ [1,2], [1,2] ], # two one-dimensional simplexes, each with
                    # ends 1 and 2, of which the first is referred to
                    # in the next line as 1, the second - as 2;
  [ [1,2], [1,2] ], # two disks - bigons - bounded each by
                    # one-dimensional simplexes 1 and 2;
  [ [1,2] ]         # the three-ball bounded by bigons 1 and 2
]
```

This is depicted in Figure 1.

Actually, we add a list of vertices with their names or something like that in the beginning of the above ball complex representation. For instance, our function `ballAB(`n`)` calls them `"A"` and `"B"`. So, our GAP representation of the ball in Figure 1 is the following record:

```
gap> ballAB(3);
rec( vertices := [ "A", "B" ],
  faces := [ [ [ 1, 2 ], [ 1, 2 ] ], [ [ 1, 2 ], [ 1, 2 ] ], [ [ 1, 2 ] ] ]
 )
```

3 Available functions

The descriptions of available functions can be found, at this moment, in `.gd` files which are in the package's `lib` subdirectory. Gradually, they will appear, in a more detailed form, in the coming versions of this manual.

4 Example of a test calculation

We will check that if we take a 6-dimensional sphere $A = S^6$ and remove from it a tubular neighborhood of a non-knotted 4-dimensional sphere $B = S^4$, then the fundamental group af the resulting manifold is $\mathbb{Z}$.

First, look at our realization of S^6:

```
gap> A := sphereAB(6);
rec( vertices := [ "A", "B" ],
  faces := [ [ [ 1, 2 ], [ 1, 2 ] ], [ [ 1, 2 ], [ 1, 2 ] ],
      [ [ 1, 2 ], [ 1, 2 ] ], [ [ 1, 2 ], [ 1, 2 ] ], [ [ 1, 2 ], [ 1, 2 ] ],
      [ [ 1, 2 ], [ 1, 2 ] ] ] )
```

Then we prepare B in the following way:

```
gap> B := rec( vertices := [1,2], faces := [ [1,2],[1,2],[1,2],[1,2] ] );
rec( vertices := [ 1, 2 ],
  faces := [ [ 1, 2 ], [ 1, 2 ], [ 1, 2 ], [ 1, 2 ] ] )
```

Note the difference with A: here we provide *numbers* of vertices and faces from those *already listed* in A.

Now "polytope minus polytope", or, to be exact, polytope A minus tubular neighborhood of B. The end of the result is on page 11, just go there and find some interesting continuation:

```
gap> C := PolMinusPol( A, B );
rec( vertices := [ "A", "B", 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,
      16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32,
      33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49,
      50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64 ],
  faces :=
    [ [ [ 3, 34 ], [ 4, 35 ], [ 5, 36 ], [ 6, 37 ], [ 7, 38 ], [ 8, 39 ],
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```

Ah yes, syms just above stays for "symmetries", which may be defined for some manifolds. This is related to our further plans such as building the Kummer surface and some "exotic" manifolds using factorizations w.r.t. some symmetries of other manifolds.

But now, is there any way to simplify the above record? Sure, here it is:

```
gap> C := PolSimplify1( C );
rec( vertices := [ "A", 34, 51, 60 ],
  faces := [ [ [ 1, 2 ], [ 3, 4 ], [ 1, 2 ], [ 3, 4 ], [ 2, 3 ], [ 2, 3 ],
          [ 1, 4 ], [ 1, 4 ] ],
      [ [ 2, 4 ], [ 1, 3 ], [ 1, 3 ], [ 2, 4 ], [ 1, 2, 5, 7 ],
          [ 1, 2, 6, 8 ], [ 3, 4, 5, 7 ], [ 3, 4, 6, 8 ] ],
      [ [ 1, 4 ], [ 2, 3 ], [ 1, 4 ], [ 2, 3 ], [ 1, 2, 5, 7 ],
          [ 1, 2, 6, 8 ], [ 3, 4, 5, 7 ], [ 3, 4, 6, 8 ] ],
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          [ 1, 2, 6, 8 ], [ 3, 4, 5, 7 ], [ 3, 4, 6, 8 ] ],
      [ [ 2, 3 ], [ 1, 4 ], [ 1, 2, 5, 7 ], [ 1, 2, 6, 8 ], [ 3, 4, 5, 7 ],
          [ 3, 4, 6, 8 ] ], [ [ 1, 2, 3, 5 ], [ 1, 2, 4, 6 ] ] ],
  syms := [  ] )
```

Now it looks quite handy, and we calculate the fundamental group. It will appear as the component FG in the following record (the other component imoldgen is not described here at this moment; it is needed for the further purposes of the authors):

```
gap> FundGroup( C );
rec( FG := <fp group of size infinity on the generators [ f6 ]>,
  imoldgen := [ <identity ...>, <identity ...>, <identity ...>,
      <identity ...>, <identity ...>, f6, <identity ...>, f6 ] )
```

So, there is one generator and size infinity, as must be.

5 Some more remarks and invitation to join us

- This all is written primarily to make some calculations in fermionic finite-dimensional topological quantum field theories, such as in [1].
- These theories are based upon piecewise-linear topology, which is surely interesting also to people with other specific research interests.

- So, do join us and become a co-author of our package!

References

[1] I.G. Korepanov, arXiv:0911.1395

[2] N. Mnev, arXiv:math/0609257v3