CayleyPy Growth: Efficient growth computations for Cayley graphs

Alexander Chervov, Dmytro Fedoriaka, Ivan Koltsov, Anton Kostin, Elena Konstantinova .....

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Abstract

We propose efficient algorithms and implementations for growth computations in Cayley (and Schreier coset) graphs. Outperform GAP 100-1000 times in timing. We use our package to obtain several mathematical results and conjectures.....

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1 Introduction

1.1 Cayley graphs and the growth function.

Let us consider a finite (or possibly infinite) group G and its set of generators $S \subseteq G$, which can be chosen arbitrarily. The *Cayley graph* of the group G with respect to S is defined as a graph whose vertex set is G, and whose edges connect vertices g and gs for each $g \in G$ and $s \in S$.

The set S determines the structure of the Cayley graph. For example, if for each $s \in S$ we have $s^{-1} \in S$, then the resulting graph is undirected. Furthermore, the standard assumption $e \notin S$, where e is the identity element of G, guarantees that the Cayley graph contains no loops.

A more general construction, extending the notion of Cayley graphs, is given by *Schreier graphs*. Consider a group G and its subgroup $H \leq G$. Together with the set of generators $S \subseteq G$, we define a graph whose vertices are the right cosets $Hg = \{hg,\ h \in H\}$ for each $g \in G$. The set of edges is determined by the action

of the generators S on these cosets: precisely, the pair $\{Hg,\, Hg\,s\}$ forms an edge for each $g \in G$ and $s \in S$. Thus, it is easy to see that when $H = \{e\}$, the Schreier graph coincides exactly with the Cayley graph.

From now on, let us fix one vertex v_0 of one of the graphs considered above, and let V denote its vertex set. The *distance* $d(v, u)$ between any two vertices v, u of a graph is defined as the minimum number of edges in a path connecting them. Using this distance, we define the ball of radius d around the vertex v_0 as follows

$$B(v_0, d) = \{v \in V \mid d(v_0, v) \leq d\}.$$

The number of vertices in the ball $B(v_0, d)$ is called the *growth function* of the graph with respect to the vertex v_0, and is denoted by

$$N(d) = |B(v_0, d)|.$$

Another quantity derived from the graph distance is the *diameter* of the graph. It is defined as the greatest distance between any two vertices in the graph

$$\text{diam} = \max_{v,u \in V} d(v, u).$$

2 Review

For $n \geq 3$ except for $n = 5, 6, 8$, S_n is generated by an element of order 2 and an element of order 3. [49], [21], [19]

3 Description of the algorithm

3.1 Main ideas

3.2 Detailed description

4 Benchmarks, comparison with GAP

4.1 Timing

	LRX		Coxeter		Transpositions		Pancake		Reversals	
n	GAP	CayleyPy	GAP	CayleyPy	GAP	CayleyPy	GAP	CayleyPy	GAP	CayleyPy
6	1.97	0.02	0.01	0.04	0.03	0.02	0.01	0.02	0.04	0.02
7	2.04	0.03	0.12	0.17	0.34	0.03	0.08	0.02	0.41	0.05
8	2.66	0.04	1.32	0.06	3.92	0.07	0.80	0.04	4.24	0.08
9	8.75	0.10	13.99	0.15	52.98	0.42	8.90	0.15	54.45	0.45
10	75.72	0.24	172.38	0.59	737.42	4.25	112.37	1.01	762.54	4.37
11	884	2.46	2352	12	10891	70.39	1559	15	11138	78
12	12024	62	34470	193	-	2737	35387	477	-	4250
13	-	-	-	-	-	-	-	-	-	-

Coxeter CayleyPy				
n	CPU	GPU T4 x2	GPU P100	TPU v3-8
1	-	-	-	-
2	0.126	0.423	0.394	0.005
3	0.010	0.064	0.061	0.005
4	0.013	0.013	0.014	0.010
5	0.029	0.026	0.026	0.022
6	0.038	0.043	0.043	0.037
7	0.052	0.068	0.067	0.067
8	0.079	0.151	0.138	0.087
9	0.293	0.110	0.110	0.140
10	3.539	0.199	0.154	0.641
11	41.008	1.085	0.601	10.822
12	705.746	-	-	187.126
13	-	-	-	2613.550

GAP testing has been performed in the notebook.
CayleyPy transpositions testing has been performed in the notebook.
Coxeter testing has been performed in the notebook.

5 Maximal diameter generators / Babai like conjectures

In this section, we study the maximum diameter for Cayley graphs of the group S_n by the system of generators $S \subseteq S_n$. We would like to start with a review of the situation when the graph is oriented. That is, the system of generators does not have to contain inverse elements. We conducted separate experiments with sets of different numbers of generators, and it always turned out that the maximum diameter is achieved on sets S consisting of two generators. These experiments justify our approach: to study the maximum diameter, we considered all pairs of permutations. Thus, we find the diameters for $\frac{n! \times (n!-1)}{2} \sim n! \times n!$ pairs of permutations, removing those sets that do not generate S_n. This approach allows you to save all the permutations that achieve the maximum. But this method is not effective for large n. For $n > 6$, we used a different method. Note that the conjugation of all generators does not change the diameter of the Cayley graph. Therefore, one representative can be chosen for all equivalence classes of generators. And for a pair of generators, you can replace one generator with a fixed representative. Thus, the search will be $n! \times c$, where c is the number of equivalence classes of permutations from S_n (for example, for $n = 7 : c = 15$). This makes it much easier to iterate and is suitable for large n. Below is a table with the results for $n = 3...8$

Maximal diameter for Cayley graph of the group S_n (oriented graph)		
n	Maximal diameter	Example of a set of generators
3	3	[0,2,1],[2,1,0]
4	7	[1,3,2,0],[2,1,0,3]
5	14	[1,2,4,3,0],[2,3,0,1,4]
6	18	[4,2,1,5,0,3],[1,0,2,4,5,3]
7	34	[5,4,0,3,6,2,1],[1,0,3,2,5,4,6]
8	44	[0,7,5,6,4,2,1,3],[1,0,3,2,5,4,6,7]
9	61	[1,0,3,2,5,4,6,7,8],[0,2,1,6,7,8,4,3,5]

remark: in fact, for $n = 3, 4$, a complete search was carried out even for sets of $2, 3, 4, 5$ generators (the maximum was achieved on sets of 2 and 3 generators).
question: is the global maximum of the diameter necessarily reached on some set of two generators, that is, are the maxima we found global?

Further, we will consider similar undirected graphs. For each generator from the set, it is necessary that the element contained in the set be its inverse. We performed similar calculations, but only in this case we added inverse permutations to the set of generators. Since this set in this way is smaller than the set of all permutation sets, we expect to obtain smaller maximum diameters. Moreover, it can be noted that often the sets of generators on which the maximum is achieved contain elements that are the opposite of themselves. Thus, the number of elements in the set, on which the maximum is reached, remains small ($k = 2, 3$). The following is a table with the results for $n = 3...8$.

Maximal diameter for Cayley graph of the group S_n (undirected graph)		
n	Maximal diameter	Example of a set of generators
3	3	[0,2,1],[2,1,0]
4	6	[2,1,0,3],[3,0,2,1],[1,3,2,0]
5	10	[4,0,1,2,3],[1,2,3,4,0],[0,1,3,2,4]
6	15 (16)	[4,5,3,2,0,1],[2,5,0,3,1,4],[2,4,0,3,5,1] ([[0,1,2,3,5,4],[0,2,1,4,3,5],[1,0,3,2,5,4]])
7	28 (30)	[1,0,3,2,5,4,6],[2,6,5,3,1,0,4],[5,4,0,3,6,2,1] ([0,1,3,2,4,6,5],[0,4,6,5,1,3,2],[6,1,3,2,5,4,0])
8	33 (39)	[3,7,5,6,0,2,4,1],[4,7,5,0,6,2,3,1],[1,0,3,2,5,4,6,7] ([0,1,2,3,5,4,7,6],[0,1,3,2,6,7,4,5],[7,3,6,1,4,5,2,0])

remark: for $n = 6$, a set of permutations was found separately, giving a larger diameter (16). This set of permutations is a set of three involutions.

As we said earlier, all permutations of S_n are divided into equivalence classes. To understand the following notation, let us explain how these equivalence classes are obtained for the group S_n.
Theorem. Two substitutions of S_n are conjugate if and only if they have the same cyclic structure, that is, they have identical sets of cycle lengths.
Using the theorem, we can conclude that each equivalence class corresponds to the decomposition of n into the sum of natural numbers (where each term is the length of a cycle, when a permutation is decomposed into a product of cycles). Therefore, we will write the equivalence class of S_n as $[a_1, ..., a_k]$, where $a_1, ..., a_k \in \mathbb{N}$ and $a_1 + ... + a_k = n$.
The maximum diameter on a set of generators consisting of pairs of permutations is considered. There are many generators that achieve the maximum diameter. However, it can be observed that these are generally $1-2$ variants of pairs of permutations, where the equivalence classes are fixed in each pair. That is, $1-2$ pairs of equivalence classes are possible, for which there is an example of a pair of generators on which the maximum is achieved. Next, the results for $n = 3...9$.

Notation: $\leftarrow$ how many second-class representatives there are for any first-class representative
$\rightarrow$ how many first-class representatives there are for any second-class representative

Analysis of generators that give the maximum diameter for S_n(undirected graph)					
n	Max diam	Pairs of equivalence classes	Example	$\leftarrow$	$\rightarrow$
3	3				
4	7				
5	14	$[2+2+1]-[4+1]$	$[2,1,0,4,3],[1,2,3,0,4]$	8	4
6	18	$[6]-[2+1+1+1+1]$	$[1,2,3,4,5,0],[0,1,2,3,5,4]$	6	48
		$[3+2+1]-[2+2+2]$	$[1,2,0,4,3,5],[2,4,0,5,1,3]$	6	48
7	34	$[2+2+2+1]-[3+3+1]$	$[1,0,3,2,5,4,6],[4,6,2,0,3,1,5]$	96	28
8	44	$[4+2+1+1]-[2+2+2+1+1]$	$[1,2,3,0,5,4,6,7],[0,5,2,6,7,1,3,4]$	16	96
9	61	$[4+2+2+1]-[2+2+2+1+1+1]$	$[1,0,3,2,5,4,6,7,8],[0,2,1,6,7,8,4,3,5]$	32	288

Similar calculations for the minimum diameter

Analysis of generators that give the minimum diameter for S_n(undirected graph)					
n	Min diam	Pairs of equivalence classes	Example	$\leftarrow$	$\rightarrow$
3					
4					
5	10	$[2+2+1]-[3+2]$	$[1,0,3,2,4],[2,3,0,4,1]$	8	
		$[2+1+1+1]-[4+1]$	$[1,0,2,3,4],[0,2,3,4,1]$	12	
6	15	$[2+2+1+1]-[6]$	$[1,0,3,2,4,5],[1,2,4,5,3,0]$		
		$[2+2+1+1]-[3+2+1]$	$[1,0,3,2,4,5],[0,5,4,1,2,3]$		
7	20	$[2+2+2+1]-[7]$	$[1,0,3,2,5,4,6],[1,2,4,6,3,0,5]$		
		$[2+2+2+1]-[4+3]$	$[1,0,3,2,5,4,6],[1,2,0,6,3,4,5]$		
		$[2+2+1+1+1]-[6+1]$	$[1,0,3,2,4,5,6],[0,2,4,6,5,3,1]$		
		$[2+2+1+1+1]-[5+2]$	$[1,0,3,2,4,5,6],[2,4,0,5,3,6,1]$		
		$[2+2+1+1+1]-[4+3]$	$[1,0,3,2,4,5,6],[2,3,5,6,1,0,4]$		
		$[2+1+1+1+1+1]-[6+1]$	$[1,0,2,3,4,5,6],[0,2,4,5,6,1,3]$		
		$[2+1+1+1+1+1]-[5+2]$	$[1,0,2,3,4,5,6],[2,4,0,5,6,1,3]$		
		$[2+1+1+1+1+1]-[4+3]$	$[1,0,2,3,4,5,6],[2,4,5,1,3,6,0]$		
8	25	$[2+2+2+2]-[4+3+1]$	$[1,0,3,2,5,4,7,6],[0,2,3,4,1,6,7,5]$		
		$[2+2+2+1+1]-[7+1]$	$[1,0,3,2,5,4,6,7],[0,2,4,6,5,7,1,3]$		
		$[2+2+2+1+1]-[5+3]$			
		$[2+2+2+1+1]-[4+3+1]$	$[1,0,3,2,5,4,6,7],[0,2,3,4,1,7,5,6]$		
		$[2+2+2+1+1]-[5+2+1]$	$[1,0,3,2,5,4,6,7],[0,2,4,1,6,7,3,5]$		
		$[2+2+1+1+1+1]-[5+2+1]$	$[1,0,3,2,4,5,6,7],[0,4,5,6,2,7,3,1]$		
9	21				

It is easy to see that in the cases considered, one of the equivalence classes is always a product of transpositions. Therefore, we hypothesize that one of the equivalence classes is always the product of transpositions. This means that it is now sufficient to consider only the following enumeration of pairs of permutations: the first permutation is the product of transpositions and the second permutation is any one.

6 Generalized symmetric group

The generalized symmetric group is defined as the wreath product $S(m,n) := Z_m \wr S_n = (Z_m)^n \rtimes S_n$ of the cyclic group of order m and the symmetric group on n elements. Note that $S(1,n) = S_n$ and $S(2,n) = W(B_n) = W(C_n)$ is the signed symmetric group (also called the hyperoctahedral group), which is the Coxeter group of type $B_n = C_n$. For general m, one can represent elements of $S(m,n)$ as $n \times n$ matrices, where each entry is a m-th root of unity or 0 and there is exactly one nonzero entry in each row and each column.

Let $S \subset S_n$ be a subset of permutations that is closed under inverse. We construct two subsets $S^{\mathrm{fr}}(m,n), S^{\mathrm{all}}(m,n) \subset S(m,n)$ of the generalized symmetric group as follows: $S^{\mathrm{fr}}(m,n)$ is the closure under inverse of the set $\{((a_1,\ldots,a_n),\sigma))|\sigma \in S, a_i = \delta_j$ where $\sigma(i) = i$ for $i < j, \sigma(j) \neq j\}$ and $S^{\mathrm{all}}(m,n)$ is the closure under inverse of $\{((a_1,\ldots,a_n),\sigma))|\sigma \in S, a_i = 1$ if $\sigma(i) \neq i$ and $a_i = 0$ if $\sigma(i) = i\}$.

The subsets $S^{\mathrm{fr}}(m,n)$ and $S^{\mathrm{all}}(m,n)$, respectively, generate subgroups $G^{\mathrm{fr}}(m,n)$ and $G^{\mathrm{all}}(m,n)$ of $S(m,n)$. We are interested in the corresponding Cayley graphs.

6.1 Coxeter like generators

Let us consider a few of the simplest examples of $G^{\mathrm{fr}}(m,n)$ and $G^{\mathrm{all}}(m,n)$. First, let us look at the case when $S \subset S_n$ consists of the A_n type Coxeter generators (i.e. adjacent transpositions). For $m = 2$, both subgroups have explicit descriptions. Indeed, in this case, we are looking at subgroups of the signed symmetric group $S(2,n) = Z_2^n \rtimes S_n = \{1,-1\}^n \rtimes S_n$. There are three group homomorphisms $\varphi_1, \varphi_2, \varphi_2 : S(2,n) \to \{1,-1\}$ given by the product of the components of the projection onto $\{1,-1\}^n$, the sign of the projection onto S_n, and the product of the two homorphisms. respectively. Looking at the generating set, one can explicitly show that, in this case, $G^{\mathrm{fr}}(2,n) = \ker(\varphi_3)$, so it is a normal subgroup of $S(2,n)$ of index 2. Using the matrix description of $S(2,n)$, this is just the subgroup of matrices with determinant equal to 1. Similarly, one can check that the generators in $S^{\mathrm{all}}(2,n)$ satisfy the Coxeter relations for type A_n, so $G^{\mathrm{all}}(2,n) = S_n$ (this gives a different embedding of S_n into $S(2,n)$ than setting the Z_2^n component to be equal to the identity).

Starting with the same generating subset S, we can check that $G^{\mathrm{first}}(m,2) = Z_{2m}$, since this subgroup is generated by powers of $((1,0),(1,2)) \in S(m,2)$. Similarly, we can check that $G^{\mathrm{all}}(m,2)$ consists of elements of the form $((2k-1,2k-1),(1,2))$ and $((2k,2k),(1))$. This means, the group is Z_m when m is even and Z_{2m} when m is odd.

Even when the exact answers are unknown, we can still perform computations. Namely, let S once again by the set of Coxeter generators of S_n. The first few diameters of the Cayley graphs and the values (?) of the growth function for $G^{\mathrm{fr}}(3,n)$ are given in the table below.

n	Order	Diameter	Growth
1	1	0	1
2	6	3	1, 2, 2, 1
3	162	9	1, 4, 12, 26, 38, 36, 26, 14, 4, 1
4	1944	9	1, 6, 26, 91, 253, 508, 596, 355, 96, 12
5	29160	11	1, 8, 44, 196, 732, 2224, 5139, 8160, 7748, 3936, 916, 56
6	524880	15	1, 10, 66, 349, 1563, 5944, 18807, 47414, 89521, 120304, 113148, 75232, 35665, 12458, 3669, 729
7	11022480	21	1, 12, 92, 558, 2866, 12696, 48464, 156962, 418514, 886978, 1463483, 1884548, 1922083, 1584315, 1108675, 707398, 423861, 231846, 110079, 43740, 13122, 2187
8	264539520	28	1, 14, 122, 831, 4777, 23828, 104050, 396654, 1303998, 3622961, 8331402, 15688174, 24327620, 31464441, 34443624, 32819710, 28645158, 23943784, 19235009, 14602728, 10371507, 6850413, 4168422, 2311659, 1150362, 498636, 177147, 45927, 6561

Similarly, for $m = 4$ they are:

n	Order	Diameter	Growth
1	1	0	1
2	8	4	1, 2, 2, 2, 1
3	192	6	1, 4, 12, 28, 48, 64, 35
4	3072	9	1, 6, 26, 94, 276, 630, 936, 786, 297, 20
5	61440	12	1, 8, 44, 200, 772, 2496, 6281, 11850, 15963, 14358, 7480, 1808, 179
6	1474560	16	1, 10, 66, 354, 1624, 6434, 21426, 58822, 130468, 229378, 312080, 318594, 233826, 117308, 37459, 6380, 330
7	41287680	21	1, 12, 92, 564, 2952, 13484, 53463, 182958, 535474, 1325314, 2740738, 4688248, 6574551, 7509344, 6948297, 5200024, 3141994, 1525076, 596772, 190624, 49506, 8192
8	1321205760	28	1, 14, 122, 838, 4892, 25006, 112592, 447338, 1565876, 4806076, 12837766, 29614040, 58643862, 99403154, 144205985, 179555186, 192940456, 180355000, 147906472, 107581100, 70749954, 43060784, 24556834, 12969064, 6160564, 2555904, 884736, 229376, 32768

For $m = 5$ we get:

n	Order	Diameter	Growth
1	1	0	1
2	10	5	1, 2, 2, 2, 2, 1
3	750	15	1, 4, 12, 28, 50, 74, 98, 108, 108, 96, 74, 50, 28, 14, 4, 1
4	15000	15	1, 6, 26, 94, 279, 653, 1244, 2036, 2762, 2983, 2404, 1404, 696, 304, 88, 20
5	375000	16	1, 8, 44, 200, 776, 2536, 6976, 16258, 32456, 55240, 77396, 83146, 62186, 29074, 7599, 1038, 66
6	11250000	20	1, 10, 66, 354, 1629, 6495, 22576, 68400, 180981, 419184, 846869, 1465336, 2095793, 2369053, 2009982, 1195810, 455924, 99908, 11173, 450, 6
7	393750000	25	1, 12, 92, 564, 2958, 13570, 55148, 199558, 644094, 1854896, 4765416, 10886496, 21895721, 38138992, 56462764, 69747326, 70572112, 57303337, 36381314, 17387479, 5918762, 1330141, 176414, 12628, 204, 1

7 Generators families

7.1 Coxeter (neigbour transpositions / bubble sort / permutohedron / weak Bruhat order) generators

The Bubble-sort graph $BS_n = Cay(Sym_n, \mathcal{B})$, $n \geqslant 2$, is a Cayley graph over the symmetric group with the generating set $\mathcal{B} = \{(i\, i+1) \in Sym_n : 1 \leqslant i \leqslant n-1\}$ of all bubble-sort transpositions that swap neigbour elements of a permutation. It is a connected bipartite $(n-1)$–regular graph of order $n!$ and diameter $\binom{n}{2}$. Since this graph is bipartite it does not contain odd cycles but it contains all even l–cycles where $l = 4, 6, 8, \ldots, n!$ [43]. The generating set $\mathcal{B}$ plays an important role in computer science for generating all permutations [36, 45], in the theory of polytopes where the graph BS_n is known as 1-skeleton of the Permutahedron [62], and in the theory of reflection and Coxeter groups [34]. Transpositions from $\mathcal{B}$ are known as $(n-1)$ braids on n strands forming the braid group [8]. Growth function elements are given in the table.

n	Full	Diameter	Growth
1	Yes	0	1
2	Yes	1	1,1
3	Yes	3	1, 2, 2, 1
4	Yes	6	1, 3, 5, 6, 5, 3, 1
5	Yes	10	1, 4, 9, 15, 20, 22, 20, 15, 9, 4, 1
6	Yes	15	1, 5, 14, 29, 49, 71, 90, 101, 101, 90, 71, 49, 29, 14, 5, 1
7	Yes	21	1, 6, 20, 49, 98, 169, 259, 359, 455, 531, 573, 573, 531, 455, 359, 259, 169, 98, 49, 20, 6, 1
8	Yes	28	1, 7, 27, 76, 174, 343, 602, 961, 1415, 1940, 2493, 3017, 3450, 3736, 3836, 3736, 3450, 3017, 2493, 1940, 1415, 961, 602, 343, 174, 76, 27, 7, 1
9	Yes	36	1, 8, 35, 111, 285, 628, 1230, 2191, 3606, 5545, 8031, 11021, 14395, 17957, 21450, 24584, 27073, 28675, 29228, 28675, 27073, 24584, 21450, 17957, 14395, 11021, 8031, 5545, 3606, 2191, 1230, 628, 285, 111, 35, 8, 1
10	Yes	45	1, 9, 44, 155, 440, 1068, 2298, 4489, 8095, 13640, 21670, 32683, 47043, 64889, 86054, 110010, 135853, 162337, 187959, 211089, 230131, 243694, 250749, 250749, 243694, 230131, 211089, 187959, 162337, 135853, 110010, 86054, 64889, 47043, 32683, 21670, 13640, 8095, 4489, 2298, 1068, 440, 155, 44, 9, 1
11	Yes	55	1, 10, 54, 209, 649, 1717, 4015, 8504, 16599, 30239, 51909, 84591, 131625, 196470, 282369, 391939, 526724, 686763, 870233, 1073227, 1289718, 1511742, 1729808, 1933514, 2112319, 2256396, 2357475, 2409581, 2409581, 2357475,2256396, 2112319, 1933514, 1729808, 1511742, 1289718, 1073227, 870233, 686763, 526724, 391939, 282369, 196470, 131625, 84591, 51909, 30239, 16599, 8504, 4015, 1717, 649, 209, 54, 10, 1
12	Yes	66	1, 11, 65, 274, 923, 2640, 6655, 15159, 31758, 61997, 113906, 198497, 330121, 526581, 808896, 1200626, 1726701, 2411747, 3277965, 4342688, 5615807, 7097310, 8775209, 10624132, 12604826, 14664752, 16739858, 18757500, 20640357, 22311069, 23697232, 24736324, 25380120, 25598186, 25380120, 24736324, 23697232, 22311069, 20640357, 18757500, 16739858, 14664752, 12604826, 10624132, 8775209, 7097310, 5615807, 4342688, 3277965, 2411747, 1726701, 1200626, 808896, 526581, 330121, 198497, 113906, 61997, 31758, 15159, 6655, 2640, 923, 274, 65, 11, 1

7.2 Cyclic Coxeter (cyclic neighbor transpositions / modified bubble sort / cyclicly adjacent transpositions) generators

In the *modified Bubble-sort graph*, there is the additional generator $(1\,n)$ along with the set of all bubble-sort transpositions. This set of generators is also known as 'cyclicly adjacent transpositions' [35]. In this paper, it was shown that the most compact expression of a permutation in terms of cyclicly adjacent transpositions can be achieved in polynomial time. The modified Bubble-sort graph is a bipartite n–regular graph. In [63], it was proved that the diameter of this graph is $\lfloor n^2/4 \rfloor$ which means that any permutation can be sorted using at most this number of cyclicly adjacent transpositions. (See Ville Salo Mathoverflow 2020 for another proof.)

We have computed growth functions (i.e., layer sizes in a BFS from the identity) for $n = 3, 4, \ldots, 8, 14$ using the Cayleypy package [1]. Table 1 shows the growth vectors; Figure **??** illustrates for $n = 5, 6$ the empirical PDF versus a Gaussian fit. The growth functions exhibit an approximately Gaussian shape. From these layer sizes one extracts for each n:

- Total number of vertices: $n!$.
- Diameter $D_n = \lfloor n^2/4 \rfloor$.
- Mean distance from identity μ_n and standard deviation σ_n (based on the histogram of distances).

- Mode (most frequent distance).

Numerically for $3 \leq n \leq 8$ and $n = 14$, one finds:

n	$n!$	D_n	μ_n	σ_n	mode
3	6	2	1.1667	0.6872	1
4	24	4	2.1667	0.8975	2
5	120	6	3.5000	1.1328	4
6	720	9	5.1667	1.3964	5
7	5040	12	7.1667	1.6849	8
8	40320	16	9.5000	1.9958	10
14	87178291200	49	24.5000	3.6667	25

The "mode" may differ slightly from $\lfloor \mu_n \rfloor$ or $\lceil \mu_n \rceil$ but remains close to the center. A linear fit over $n = 3, \ldots, 8, 14$ gives

$$\mu_n \approx 1.75\, n \; - \; 4.75, \quad \sigma_n \approx 0.26\, n \; - \; 0.14,$$

indicating that the mean distance grows roughly linearly in n and the spread also grows moderately.

Table 1: Growth layer sizes for the cyclic Coxeter Cayley graph (cCoxeter) for $3 \leq n \leq 8$ and $n = 14$. Each "Growth" entry is the sequence of layer sizes from distance 0 up to the diameter.

n	Diameter D_n	Total vertices $n!$	Growth (layer sizes)
3	2	6	1, 3, 2
4	4	24	1, 4, 10, 8, 1
5	6	120	1, 5, 15, 35, 42, 20, 2
6	9	720	1, 6, 21, 56, 126, 197, 195, 100, 17, 1
7	12	5040	1, 7, 28, 84, 210, 462, 828, 1148, 1169, 777, 282, 42, 2
8	16	40320	1, 8, 36, 120, 330, 792, 1716, 3278, 5362, 7336, 8148, 6972, 4258, 1624, 308, 30, 1
14	49	87178291200	1, 14, 105, 560, 2380, 8568, 27132, 77520, 203490, 497420, 1144066, 2496144, 5200300, 10399573, 20044817, 37346764, 67385500, 117857285, 199872075, 328616470, 523576508, 807898884, 1206399194, 1741769068, 2428791549, 3266926299, 4232447674, 5272211438, 6301765938, 7210522410, 7875671024, 8184285564, 8059876006, 7486458696, 6521589932, 5291298298, 3964966720, 2715665810, 1678335828, 920955932, 439630193, 177894717, 59073300, 15497794, 3095274, 457459, 46501, 2912, 93, 1

Diameter analysis for large n. For $n = 14$, the diameter is confirmed to be exactly $D_{14} = \lfloor 14^2/4 \rfloor = \lfloor 196/4 \rfloor = 49$, validating the theoretical bound. The identity permutation (represented as `0x00789ABCD0123456` in hexadecimal notation) serves as the starting state for BFS traversal, confirming distance 0 from itself.

Spectral properties. We also computed the adjacency-spectrum for each n. As n increases, the spectrum remains symmetric around zero (since the graph is bipartite). For $n = 8$, the top part of the sorted eigenvalues (rounded to two decimals) begins as

$$8.00,\ 7.41\ (\text{mult.}14),\ 7.05\ (\text{mult.}40),\ 6.92\ (\text{mult.}14),\ 6.89\ (\text{mult.}56),\ \ldots$$

with the largest eigenvalue exactly $n = 8$ of multiplicity 1 and the distribution showing a clustering around values slightly below 8. This is consistent with the high regularity and combinatorial structure of the Cayley graph.

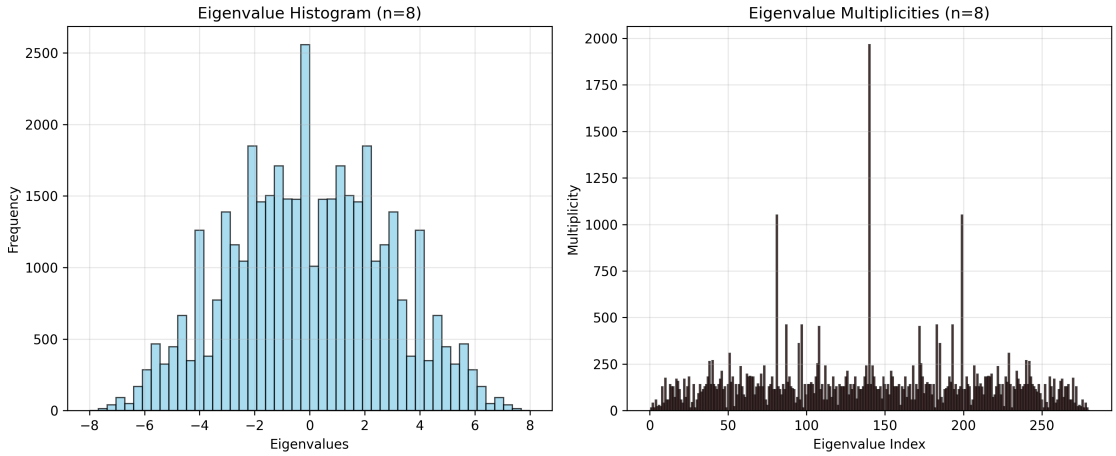


Figure 1: Eigenvalue spectrum histogram for the cyclic Coxeter graph at $n = 8$. The histogram shows multiplicities of eigenvalues; see main text for summary of top eigenvalues.

Combined growth and diameter analysis. Finally, Figure 2 displays two aspects side by side: on the left, the growth functions for $3 \leq n \leq 8$ in logarithmic scale; on the right, the comparison of actual diameter D_n, the theoretical $\lfloor n^2/4 \rfloor$, the mean distance μ_n, and the mode distance as functions of n. This reinforces that $D_n = \lfloor n^2/4 \rfloor$ exactly for $3 \leq n \leq 8, 14$, and shows the linear trend of μ_n and moderate growth of σ_n. The extension to $n = 14$ provides valuable confirmation of the scaling behavior for larger permutation groups.

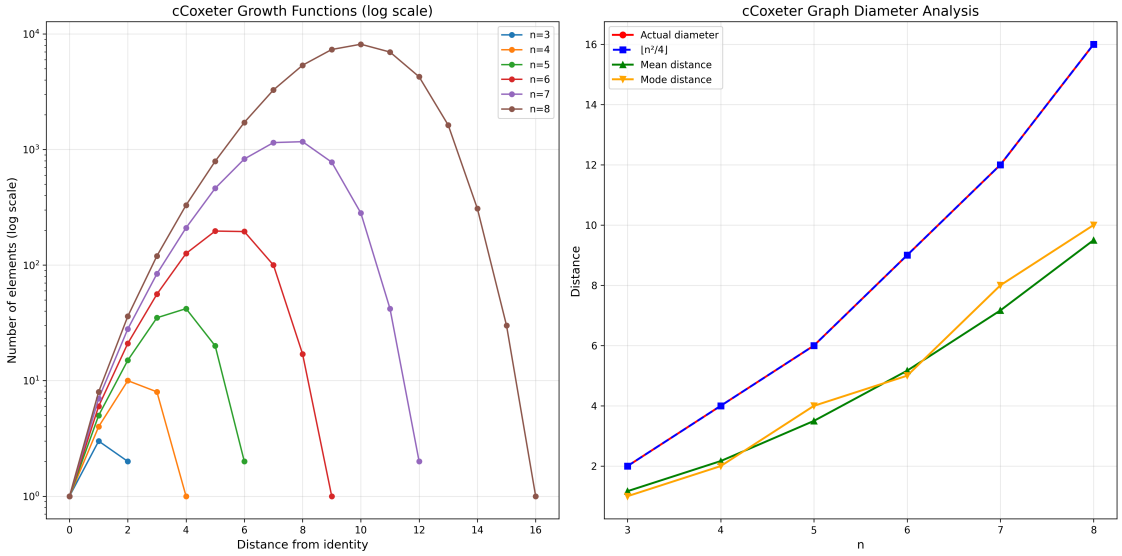


Figure 2: Left: log-scale growth functions for $n = 3, \ldots, 8, 14$. Right: diameter vs. n comparing actual diameter D_n, theoretical $\lfloor n^2/4 \rfloor$, mean distance μ_n, and mode distance.

7.3 All transpositions (and strong Bruhat order)

The transposition graph $T_n = Cay(\mathrm{Sym}_n, T), n \geqslant 2$, is the Cayley graph over the symmetric group generated by all transpositions from the set $T = \{(i\,j) \in Sym_n,\ 1 \leqslant i < j \leqslant n\},\ |T| = \binom{n}{2}$. The distance in this graph is defined as the minimal number of transpositions that transform one permutation into another. The diameter is at most $(n-1)$ since this number of transpositions suffices to transform any permutation of n elements into another. On the other hand, transforming the identity permutation into an n-cycle requires $(n-1)$ transpositions. The graph is bipartite since the endpoints of every edge consist of an even and odd permutation. The cardinalities of the metric spheres $|S_i|, 1 \leqslant i \leqslant n-1$, of radius i centered at the identity permutation are the same as the signless Stirling numbers of the first kind [48, 60]. These numbers are presented as *Growth* in the table below.

n	Diameter	Growth
1	0	1
2	1	1,1
3	2	1, 3, 2
4	3	1, 6, 11, 6
5	4	1, 10, 35, 50, 24
6	5	1, 15, 85, 225, 274, 120
7	6	1, 21, 175, 735, 1624, 1764, 720
8	7	1, 28, 322, 1960, 6769, 13132, 13068, 5040
9	8	1, 36, 546, 4536, 22449, 67284, 118124, 109584, 40320
10	9	1, 45, 870, 9450, 63273, 269325, 723680, 1172700, 1026576, 362880
11	10	1, 55, 1320, 18150, 157773, 902055, 3416930, 8409500, 12753576, 10628640, 3628800
12	11	1, 66, 1925, 32670, 357423, 2637558, 13339535, 45995730, 105258076, 150917976, 120543840, 39916800
13	12	1, 78, 2717, 55770, 749463, 6926634, 44990231, 206070150, 657206836, 1414014888, 1931559552, 1486442880, 479001600
14	13	1, 91, 3731, 91091, 1474473, 16669653, 135036473, 790943153, 3336118786, 9957703756, 20313753096, 26596717056, 19802759040, 6227020800

Table 2: All transpositions eigenvalues for different n (format: value^{multiplicity})

n	Eigenvalues (value ^{multiplicity})
3	$-3.0^1,\quad 0.0^4,\quad 3.0^1$
4	$-6.0^1,\quad -2.0^9,\quad 0.0^4,\quad 2.0^9,\quad 6.0^1$
5	$-10.0^1,\quad -5.0^{16},\quad -2.0^{25},\quad 0.0^{36},\quad 2.0^{25},\quad 5.0^{16},\quad 10.0^1$
6	$-15.0^1,\quad -9.0^{25},\quad -5.0^{81},\quad -3.0^{125},\quad 0.0^{256},$ $3.0^{125},\quad 5.0^{81},\quad 9.0^{25},\quad 15.0^1$
7	$-21.0^1,\quad -14.0^{36},\quad -9.0^{196},\quad -7.0^{225},\quad -6.0^{196},$ $-3.0^{1225},\quad -1.0^{441},\quad 0.0^{400},\quad 1.0^{441},\quad 3.0^{1225},$ $6.0^{196},\quad 7.0^{225},\quad 9.0^{196},\quad 14.0^{36},\quad 21.0^1$
8	$-28.0^1,\quad -20.0^{49},\quad -14.0^{400},\quad -12.0^{441},\quad -10.0^{784},$ $-8.0^{196},\quad -7.0^{4096},\quad -4.0^{6125},\quad -2.0^{3136},\quad 0.0^{9864},$ $2.0^{3136},\quad 4.0^{6125},\quad 7.0^{4096},\quad 8.0^{196},\quad 10.0^{784},$ $12.0^{441},\quad 14.0^{400},\quad 20.0^{49},\quad 28.0^1$

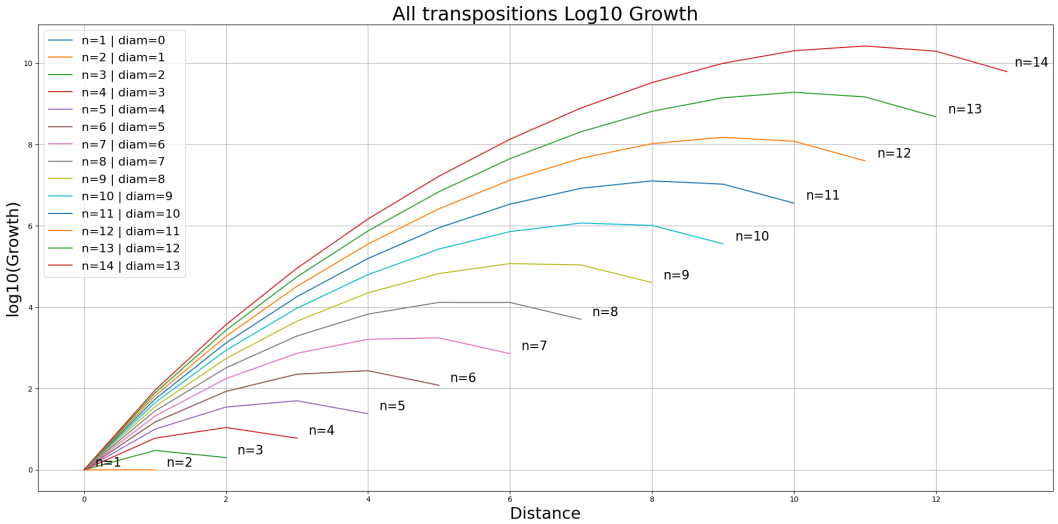


Figure 3: All transpositions Growth

Dima: (Growth functions are given by https://oeis.org/A094638 and can be computed efficiently, because these are Stirling numbers of first kind. So it doesn't really make sense to compute these with CayleyPy, but we can extend the table up to N=15 for completeness. We also need a reference to a paper that proves that growth function elements are equal to Stirling numbers, although OEIS entry claims this explicitly).

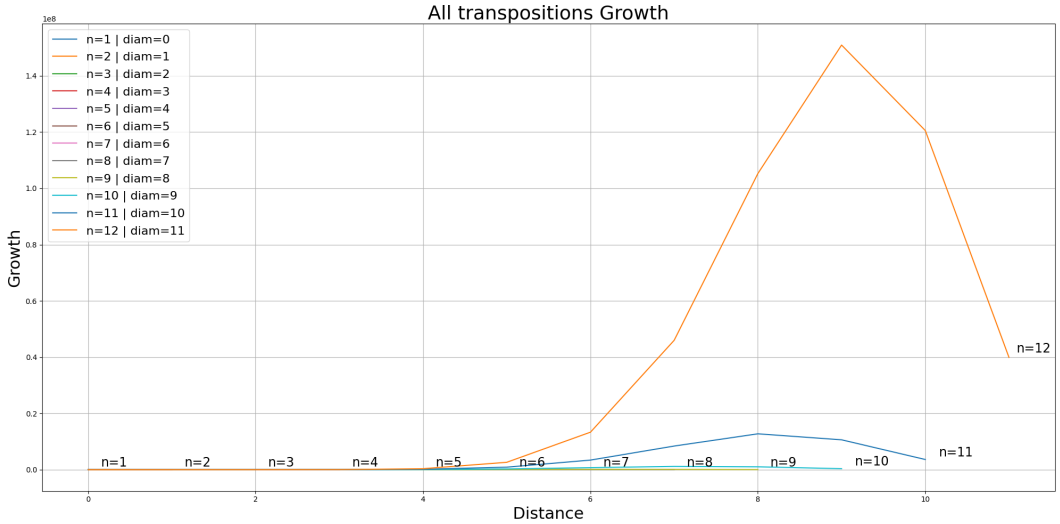


Figure 4: All transpositions Growth

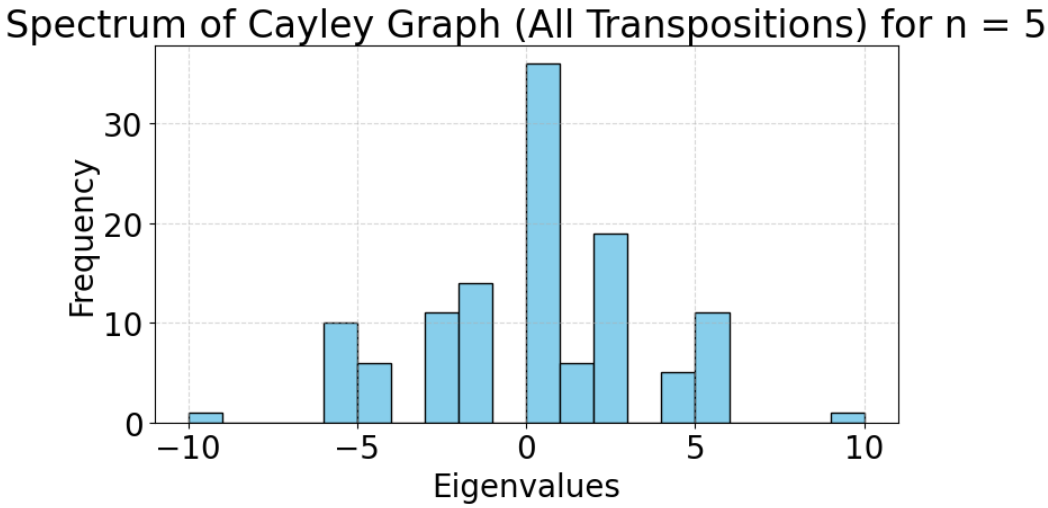


Figure 5: Eigenvalue spectrum histogram for the all transpositions at $n = 4$. The histogram shows multiplicities of eigenvalues

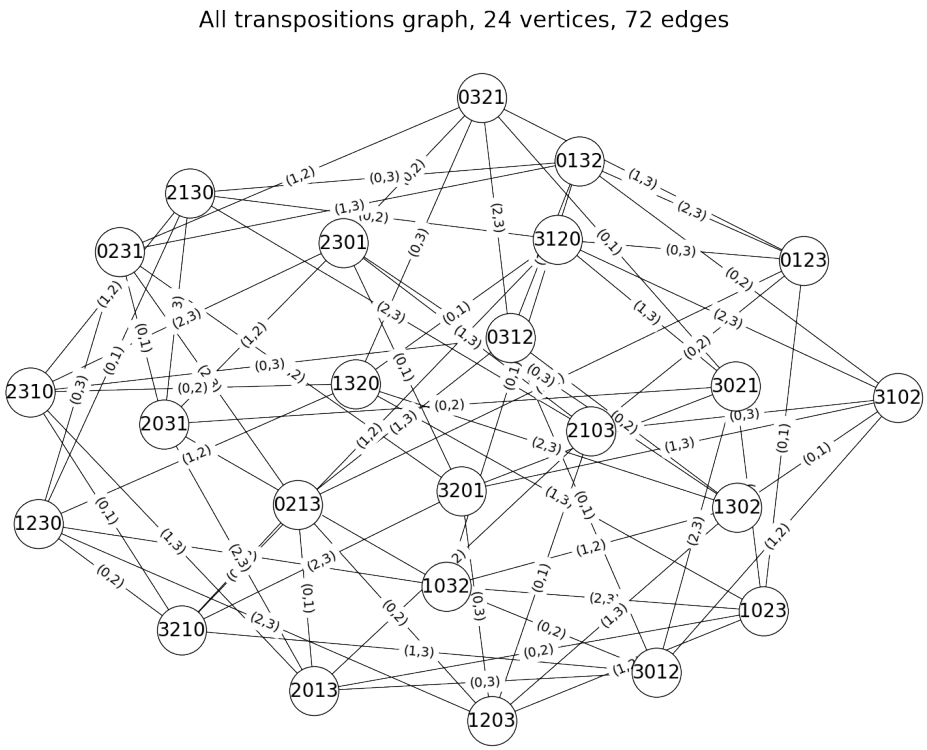


Figure 6: All transpositions $n = 4$ Graph

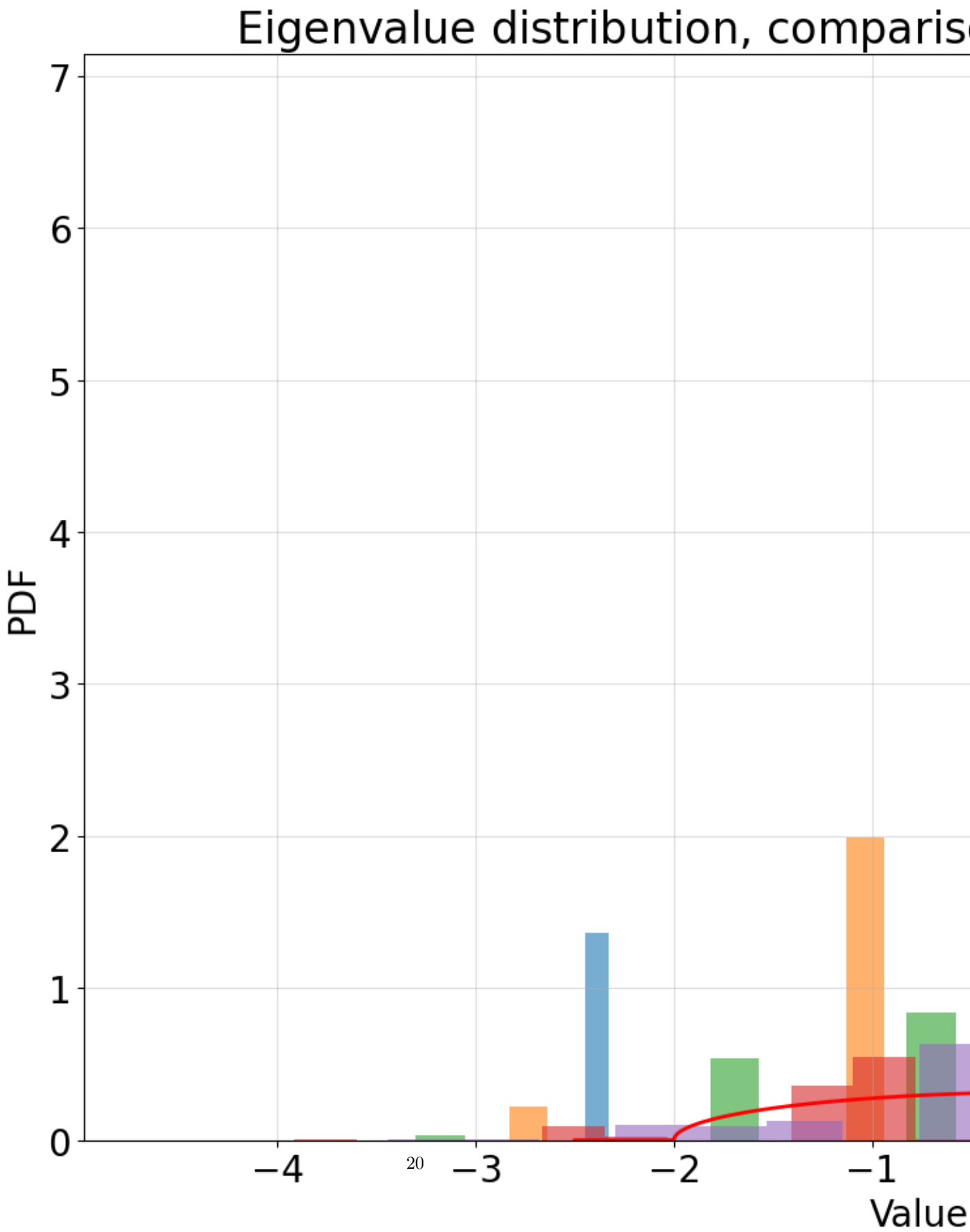


Figure 7: All transpositions Eigenvalue distribution compare to Winger law

7.4 Star transpositions: $(1, k)$

7.5 Generalized star transpositions:???

The class of graphs proposed in [32] (definition 16, page 177). The diameter is $n-1+max(\lfloor k/2 \rfloor; \lfloor (n-k)/2 \rfloor)$ (cited in [32] as "J.-F. Sacle, Diameter of some Cayley graphs, private communication, 1997.")

Change. Copy-pasted from [32]. The generalized star graph GST(n, k) is defined in Section 3.2. GST(n, k) is the Cayley graph of the group 6n generated by the k(n-k) transpositions S = (i, j), 1 :S i :S k, k+ 1 :S j :S n . The associated transposition graph is the bipartite complete graph Kk,n-k· GST(4, 2) is shown in Figure 12 and the associated transposition graph in Figure 13. The graph GST(n, k) is rotational if n = 4 and k = 2 or if k and n - k are relatively prime (see Proposition 17). Diam(GST(k, n)) = n- 1 + max(l J, l n;k j) [128].

Definition 16 Let k and n be two integers such that 1 k < n. The generalized star graph GST(n, k) is the Cayley graph Cay(Gr(S), S), where Sis the set of the transpositions (i,j),i E 1, ... ,k,j E k+1, ... ,n. The transposition graph T(S) associated to GST(n, k) is the complete bipartite graph Kk,n-k· In the case k = 1, GST(n, 1) is isomorphic to the star graph ST(n) and GST(2, 1) to the graph K2. Notice that GST(4, 2), which is depicted in Figure 12, is isomorphic to the modified bubble sort graph M BS(4), since in both cases T(S) is a cycle of length 4. The graph GST(n, k) is related to the arrangement graph A(n, k) studied in [44]. The graph A(n,k) is the graph with vertex set the set of arrangements of k elements chosen out of n elements in which an edge connects two vertices if and only if they correspond to arrangements differing in exactly one position. Thus A(n, k) is obtained from GST(n, k) by contracting into one vertex all the vertices of GST(n, k) which are permutations of 6n which fix i, 1 i k (and deleting loops and multiple edges). In other words, A(n, k) is the left coset graph of 6n with respect to the subgroup H and the subset S, where H = a E 6n : Supp(a) = k + 1, ... ,n.

7.6 LRX (transposition and long cycle)

n	Full	Diameter	Growth
1	Yes	0	1
2	Yes	1	1,1
3	Yes	2	1, 3, 2
4	Yes	6	1, 3, 5, 6, 5, 3, 1
5	Yes	10	1, 3, 6, 10, 16, 24, 29, 21, 6, 3, 1
6	Yes	15	1, 3, 6, 11, 20, 35, 55, 81, 109, 128, 126, 95, 40, 6, 3, 1
7	Yes	21	1, 3, 6, 12, 22, 42, 73, 124, 203, 303, 425, 559, 678, 713, 746, 611, 355, 122, 28, 10, 3, 1
8	Yes	28	1, 3, 6, 12, 23, 44, 80, 142, 247, 411, 662, 1019, 1481, 2059, 2745, 3465, 4126, 4633, 4913, 4777, 4163, 3079, 1612, 488, 94, 25, 6, 3, 1
9	Yes	36	1, 3, 6, 12, 24, 46, 85, 156, 283, 492, 845, 1415, 2258, 3476, 5188, 7448, 10303, 13816, 17827, 22311, 26810, 31091, 34664, 36935, 37516, 35651, 31354, 23197, 13382, 4680, 1212, 268, 88, 25, 8, 3, 1
10	Yes	45	1, 3, 6, 12, 24, 47, 87, 161, 297, 528, 927, 1611, 2726, 4492, 7184, 11109, 16751, 24624, 35105, 48718, 66154, 87373, 111996, 140388, 171657, 204213, 236429, 266276, 291271, 308831, 316158, 310824, 290837, 254374, 199563, 129134, 61718, 20467, 5183, 1116, 302, 92, 21, 6, 3, 1
11	Yes	55	1, 3, 6, 12, 24, 48, 89, 166, 309, 560, 1001, 1776, 3097, 5314, 8937, 14618, 23293, 36288, 55172, 81871, 118838, 169112, 235115, 319961, 427229, 559464, 716979, 903686, 1116773, 1352909, 1609076, 1880395, 2148532, 2408704, 2650204, 2844421, 2980448, 3049721, 3017424, 2878200, 2626374, 2235627, 1702643, 1044515, 475424, 155851, 42082, 10262, 2835, 937, 328, 100, 32, 10, 3, 1
12	Yes	66	1, 3, 6, 12, 24, 48, 90, 168, 314, 572, 1033, 1850, 3263, 5701, 9863, 16759, 27866, 45291, 72099, 112521, 171929, 257112, 377904, 545894, 773038, 1075171, 1473589, 1986076, 2630262, 3432644, 4416186, 5590709, 6971232, 8572605, 10385539, 12397678, 14595522, 16933499, 19356689, 21817886, 24222746, 26460935, 28458315, 30093680, 31217047, 31732310, 31542112, 30496145, 28508698, 25523039, 21432297, 16273946, 10527747, 5424324, 2143159, 658412, 170062, 42140, 10995, 3283, 1066, 361, 100, 23, 6, 3, 1
13	Yes	78	1, 3, 6, 12, 24, 48, 91, 170, 319, 584, 1063, 1920, 3421, 6052, 10634, 18473, 31652, 53352, 88140, 142883, 227466, 355271, 544412, 820735, 1216936, 1773910, 2544596, 3598674, 5013032, 6880807, 9320848, 12458079, 16429644, 21396997, 27535909, 34994952, 43951613, 54575165, 66975698, 81275047, 97554526, 115780301, 135902073, 157819493, 181229759, 205792009, 231141361, 256581125, 281453793, 305079428, 326430281, 344503519, 358540900, 367339920, 369741101, 365143802, 352536210, 330855376, 299705540, 258322962, 205546603, 144249635, 84147086, 39002166, 14317977, 4377538, 1218565, 328146, 90104, 26947, 9121, 3190, 1094, 363, 107, 28, 8, 3, 1
14	Yes	91	1, 3, 6, 12, 24, 48, 91, 171, 321, 589, 1075, 1950, 3491, 6210, 10986, 19263, 33514, 57769, 98159, 163941, 269338, 435449, 691859, 1080005, 1659436, 2512942, 3746333, 5498978, 7963527, 11386406, 16058844, 22353988, 30759714, 41839872, 56238352, 74757850, 98340264, 127996579, 164875626, 210268514, 265518647, 332061277, 411414715, 504958674, 614075484, 740272276, 884534812, 1047413235, 1229737728, 1431700846, 1652211880, 1890258403, 2144499030, 2411711632, 2688028475, 2969737924, 3251282972, 3525714941, 3786625399, 4026316754, 4235611752, 4406074739, 4529134791, 4595116230, 4595217203, 4521988761, 4367186737, 4123660173, 3785800928, 3347488055, 2805032872, 2167875057, 1486173810, 866316389, 416461908, 164327289, 54789051, 16347103, 4650644, 1305284, 367812, 107719, 34044, 11050, 3538, 1133, 379, 110, 25, 6, 3, 1
15	Yes	105	1, 3, 6, 12, 24, 48, 91, 172, 323, 594, 1087, 1980, 3559, 6364, 11330, 20011, 35132, 61335, 106143, 181594, 306453, 509504, 834420, 1345558, 2135163, 3337420, 5142639, 7812283, 11695713, 17286665, 25229139, 36363923, 51771066, 72900592, 101511185, 139820033, 190618894, 257291118, 343863221, 455239280, 597311896, 776589036, 1000841491, 1279049937, 1621165281, 2037842426, 2541402938, 3145237130, 3862647756, 4708060748, 5697063087, 6844203240, 8163027584, 9667949176, 11371551670, 13282011654, 15406247875, 17749303282, 20308424658, 23075401334, 26039774747, 29181833677, 32471184117, 35873841937, 39348546964, 42839034667, 46283550887, 49618145202, 52763506137, 55632559853, 58144169577, 60206715350, 61720093294, 62597509480, 62752416893, 62090019200, 60533684463, 58016158900, 54461924959, 49810775619, 44008688607, 37005873994, 28891225171, 20206897420, 12177570805, 6149567825, 2599370506, 944809925, 308529158, 94885710, 28497869, 8488126, 2539803, 791999, 264350, 93627, 33270, 11668, 4030, 1333, 425, 128, 37, 10, 3, 1

n	Diameter	Growth
1	0	1

n	Full	Diameter	Growth
16	No	120[1]	1, 3, 6, 12, 24, 48, 91, 172, 324, 596, 1092, 1992, 3589, 6432, 11484, 20355, 35881, 62975, 109909, 190552, 327513, 556338, 932643, 1543216, 2518151, 4047906, 6413758, 10029116, 15472319, 23541477, 35363771, 52510599, 77046169, 111701646, 160203409, 227455054, 319592852, 444506094, 612530084, 836536158, 1132149102
17	No	136 *conjectural*	3, 6, 12, 24, 48, 91, 172, 325, 598, 1097, 2004, 3619, 6500, 11636, 20695, 36622, 64566, 113305, 197928, 343997, 593523, 1014167, 1713828, 2862022, 4719086, 4719086, 12322523, 19525250, 30545435, 47171938, 71948269, 108484824, 161715604, 238341059, 347565044, 501807681, 717289814, 1015374627, 1424409540, 1980624511
18	No	No	3, 6, 12, 24, 48, 91, 172, 325 , 599, 1099, 2009, 3631, 6530, 11704, 20847, 36962, 65307, 114897, 201349, 351633, 611434, 1056943, 1813030, 3080558, 5179815, 8612622, 14146837, 22944379, 36763863, 58204766, 91018529, 140613990, 214832327, 324657341, 485209391, 717543731, 1050968217, 1524593337
19	No	No	3, 6, 12, 24, 48, 91, 172, 325, 600, 1101, 2014, 3643,6560, 11772, 20999, 37300, 66044, 116481, 204713, 358730, 626606, 1090461, 1889028, 3251855, 5553160, 9397426, 15744674, 26090871, 42737694, 69204377, 110786443, 175301582, 274190126, 424174740 649262217, 983257468, 1473766991, 2187812570
20	No	No	3, 6, 12, 24, 48, 91, 172, 325, 600, 1102, 2016, 3648, 6572, 11802, 21067, 37452, 66382, 117218, 206297, 362095, 633731, 1105961, 1924552, 3336573, 5754830, 9860484, 16764306, 28252288, 47148975, 77856800, 127186480, 205551832, 328568960, 519386825, 812291725, 1257401793, 1926394302

7.6.1 LRX Cosets with k-repeats

We computed growth and diameters for $3 \leq n \leq 12$ and repeats of elements whose numbers range from k to n-1. Then we plotted the dependence of the diameter on the number k.

n	k	Diameter	Growth
3	0	2	1 3 2
3	2	1	1 2
4	0	6	1 3 5 6 5 3 1
4	2	3	1 3 5 3
4	3	2	1 2 1
5	0	10	1, 3, 6, 10, 16, 24, 29, 21, 6, 3, 1
5	2	7	1, 3, 6, 9, 12, 14, 12, 3
5	3	5	1, 3, 6, 6, 3, 1
5	4	2	1, 2, 2
6	0	15	1, 3, 6, 11, 20, 35, 55, 81, 109, 128, 126, 95, 40, 6, 3, 1
6	2	12	1, 3, 6, 10, 17, 29, 41, 55, 65, 64, 50, 15, 4
6	3	9	1, 3, 6, 10, 15, 20, 22, 22, 16, 5
6	4	6	1, 3, 6, 7, 7, 5, 1
6	5	3	1, 2, 2, 1
7	0	21	1, 3, 6, 12, 22, 42, 73, 124, 203, 303, 425, 559, 678, 713, 746, 611, 355, 122, 28, 10, 3, 1
7	2	19	1, 3, 6, 11, 19, 36, 60, 96, 149, 210, 281, 345, 384, 361, 315, 171, 53, 13, 5, 1
7	3	15	1, 3, 6, 11, 18, 32, 47, 68, 94, 116, 130, 136, 108, 50, 18, 2
7	4	11	1, 3, 6, 11, 16, 24, 30, 34, 37, 30, 17, 1
7	5	7	1, 3, 6, 8, 8, 8, 6, 2
7	6	3	1, 2, 2, 2
8	0	28	1 3 6 12 23 44 80 142 247 411 662 1019 1481 2059 2745 3465 4126 4633 4913 4777 4163 3079 1612 488 94 25 6 3 1
8	2	25	1 3 6 11 20 38 67 116 196 314 488 726 1021 1365 1739 2107 2370 2496 2437 2100 1566 723 181 51 15 3
8	3	21	1 3 6 11 19 34 58 95 150 223 320 442 576 702 799 856 843 718 507 257 81 19
8	4	17	1 3 6 11 19 33 50 73 104 137 174 205 223 230 195 147 58 11
8	5	13	1 3 6 11 17 25 34 42 48 49 45 36 17 2
8	6	8	1 3 6 8 9 10 9 7 3
8	7	4	1 2 2 2 1
9	0	36	1 3 6 12 24 46 85 156 283 492 845 1415 2258 3476 5188 7448 10303 13816 17827 22311 26810 31091 34664 36935 37516 35651 31354 23197 13382 4680 1212 268 88 25 8 3 1
9	2	33	1 3 6 11 21 40 72 129 232 394 658 1068 1660 2504 3634 5062 6829 8926 11125 13480 15658 17458 18670 18949 18073 15631 11661 6294 2352 621 154 49 12 3
9	3	29	1 3 6 11 20 36 63 111 191 307 489 754 1115 1598 2198 2896 3677 4516 5263 5920 6325 6453 6165 5267 3944 2116 768 212 50 5
9	4	25	1 3 6 11 20 35 59 97 156 234 345 494 669 879 1107 1321 1496 1638 1669 1600 1416 1092 559 164 43 6
9	5	20	1 3 6 11 20 34 52 78 115 153 201 253 297 335 360 338 311 249 147 51 9
9	6	14	1 3 6 11 18 26 36 47 58 62 66 66 54 37 13
9	7	10	1 3 6 8 10 11 10 11 8 3 1
9	8	4	1 2 2 2 2
10	0	45	1 3 6 12 24 47 87 161 297 528 927 1611 2726 4492 7184 11109 16751 24624 35105 48718 66154 87373 111996 140388 171657 204213 236429 266276 291271 308831 316158 310824 290837 254374 199563 129134 61718 20467 5183 1116 302 92 21 6 3 1
10	2	42	1 3 6 11 21 41 74 134 245 431 745 1267 2092 3378 5298 8024 11857 17091 23863 32460 43240 55769 69928 86059 102549 118696 134384 147372 155938 159658 157396 146957 127970 99765 62609 27469 8689 2218 483 145 49 13 2
10	3	38	1 3 6 11 20 37 65 116 208 354 590 966 1531 2373 3584 5232 7431 10279 13812 18021 22927 28352 33978 39622 44929 49452 52633 53922 52985 49535 43010 33258 21146 9895 3272 906 259 66 13
10	4	33	1 3 6 11 20 36 61 106 184 297 471 732 1095 1602 2287 3152 4208 5478 6882 8393 9951 11385 12553 13536 13928 13689 12967 11209 8706 5398 2036 630 168 19
10	5	29	1 3 6 11 20 36 60 98 158 239 352 510 704 941 1236 1561 1885 2210 2519 2735 2854 2888 2757 2427 1925 1300 614 158 29 3
10	6	23	1 3 6 11 20 35 53 80 120 164 216 280 340 404 463 497 514 497 462 382 297 157 34 4
10	7	17	1 3 6 11 18 27 37 49 63 72 79 86 82 74 62 37 12 1
10	8	11	1 3 6 8 10 12 12 12 11 9 5 1
10	9	5	1 2 2 2 2 1
11	0	55	1 3 6 12 24 48 89 166 309 560 1001 1776 3097 5314 8937 14618 23293 36288 55172 81871 118838 169112 235115 319961 427229 559464 716979 903686 1116773 1352909 1609076 1880395 2148532 2408704 2650204 2844421 2980448 3049721 3017424 2878200 2626374 2235627 1702643 1044515 475424 155851 42082 10262

7.6.2 Generalized LRX(n,k)

LRX(n,k) is a family of LRX graphs in which the transposition $X = (0, 1)$ of the first two positions is replaced by the transposition $X = (0, k)$ of the first and k-th positions.

The "sym?" column indicates whether the last BFS element belongs to the dihedral symmetries.

n	k	Diameter	Size $/n!$	sym?	Growth
4	1	6	1	+	1, 3, 5, 6, 5, 3, 1
5	1	10	1	+	1, 3, 6, 10, 16, 24, 29, 21, 6, 3, 1
6	1	15	1	+	1, 3, 6, 11, 20, 35, 55, 81, 109, 128, 126, 95, 40, 6, 3, 1
7	1	21	1	+	1, 3, 6, 12, 22, 42, 73, 124, 203, 303, 425, 559, 678, 713, 746, 611, 355, 122, 28, 10, 3, 1
8	1	28	1	+	1, 3, 6, 12, 23, 44, 80, 142, 247, 411, 662, 1019, 1481, 2059, 2745, 3465, 4126, 4633, 4913, 4777, 4163, 3079, 1612, 488, 94, 25, 6, 3, 1
9	1	36	1	+	1, 3, 6, 12, 24, 46, 85, 156, 283, 492, 845, 1415, 2258, 3476, 5188, 7448, 10303, 13816, 17827, 22311, 26810, 31091, 34664, 36935, 37516, 35651, 31354, 23197, 13382, 4680, 1212, 268, 88, 25, 8, 3, 1
10	1	45	1	+	1, 3, 6, 12, 24, 47, 87, 161, 297, 528, 927, 1611, 2726, 4492, 7184, 11109, 16751, 24624, 35105, 48718, 66154, 87373, 111996, 140388, 171657, 204213, 236429, 266276, 291271, 308831, 316158, 310824, 290837, 254374, 199563, 129134, 61718, 20467, 5183, 1116, 302, 92, 21, 6, 3, 1
11	1	55	1	+	1, 3, 6, 12, 24, 48, 89, 166, 309, 560, 1001, 1776, 3097, 5314, 8937, 14618, 23293, 36288, 55172, 81871, 118838, 169112, 235115, 319961, 427229, 559464, 716979, 903686, 1116773, 1352909, 1609076, 1880395, 2148532, 2408704, 2650204, 2844421, 2980448, 3049721, 3017424, 2878200, 2626374, 2235627, 1702643, 1044515, 475424, 155851, 42082, 10262, 2835, 937, 328, 100, 32, 10, 3, 1
12	1	66	1		1, 3, 6, 12, 24, 48, 90, 168, 314, 572, 1033, 1850, 3263, 5701, 9863, 16759, 27866, 45291, 72099, 112521, 171929, 257112, 377904, 545894, 773038, 1075171, 1473589, 1986076, 2630262, 3432644, 4416186, 5590709, 6971232, 8572605, 10385539, 12397678, 14595522, 16933499, 19356689, 21817886, 24222746, 26460935, 28458315, 30093680, 31217047, 31732310, 31542112, 30496145, 28508698, 25523039, 21432297, 16273946, 10527747, 5424324, 2143159, 658412, 170062, 42140, 10995, 3283, 1066, 361, 100, 23, 6, 3, 1
13	1	78	1		1, 3, 6, 12, 24, 48, 91, 170, 319, 584, 1063, 1920, 3421, 6052, 10634, 18473, 31652, 53352, 88140, 142883, 227466, 355271, 544412, 820735, 1216936, 1773910, 2544596, 3598674, 5013032, 6880807, 9320848, 12458079, 16429644, 21396997, 27535909, 34994952, 43951613, 54575165, 66975698, 81275047, 97554526, 115780301, 135902073, 157819493, 181229759, 205792009, 231141361, 256581125, 281453793, 305079428, 326430281, 344503519, 358540900, 367339920, 369741101, 365143802, 352536210, 330855376, 299705540, 258322962, 205546603, 144249635, 84147086, 39002166, 14317977, 4377538, 1218565, 328146, 90104, 26947, 9121, 3190, 1094, 363, 107, 28, 8, 3, 1
14	1	91	?	+	1, 3, 6, 12, 24, 48, 91, 171, 321, 589, 1075, 1950, 3491, 6210, 10986, 19263, 33514, 57769, 98159, 163941, 269338, 435449, 691859, 1080005, 1659436, 2512942, 3746333, 5498978, 7963527, 11386406, 16058844, 22353988, 30759714, 41839872, 56238352, 74757850, 98340264, 127996579, 164875626, 210268514, 265518647, 332061277, 411414715, 504958674, 614075484, 740272276, 884534812, 1047413235, 1229737728, 1431700846, 1652211880, 1890258403, 2144499030, 2411711632, 2688028475, 2969737924, 3251282972, 3525714941, 3786625399, 4026316754, 4235611752, 4406074739, 4529134791, 4595116230, 4595217203, 4521988761, 4367186737, 4123660173, 3785800928, 3347488055, 2805032872, 2167875057, 1486173810, 866316389, 416461908, 164327289, 54789051, 16347103, 4650644, 1305284, 367812, 107719, 34044, 11050, 3538, 1133, 379, 110, 25, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
4	2	3	0.(3)	1, 3, 3, 1
5	2	10	1	1, 3, 6, 10, 16, 24, 29, 21, 6, 3, 1
6	2	8	0.1	1, 3, 6, 11, 16, 17, 12, 5, 1
7	2	21	1	1, 3, 6, 12, 20, 34, 58, 96, 158, 234, 334, 456, 562, 672, 731, 737, 583, 280, 45, 12, 5, 1
8	2	15	0.0286	1, 3, 6, 12, 21, 36, 61, 96, 146, 199, 231, 201, 100, 27, 10, 2
9	2	38	1	1, 3, 6, 12, 22, 38, 68, 120, 206, 349, 570, 914, 1412, 2123, 3133, 4504, 6327, 8625, 11505, 15021, 19099, 23613, 28154, 32397, 35872, 37701, 37249, 34491, 28822, 18996, 8738, 2270, 381, 85, 31, 12, 6, 3, 1
10	2	24	0.0079	1, 3, 6, 12, 22, 39, 70, 125, 216, 361, 590, 945, 1457, 2109, 2835, 3551, 4159, 4437, 3904, 2556, 1114, 258, 25, 4, 1
11	2	57	1	1, 3, 6, 12, 22, 40, 72, 130, 230, 397, 682, 1154, 1937, 3191, 5111, 8001, 12251, 18465, 27423, 39892, 57045, 80628, 112557, 154908, 210147, 281345, 371408, 483838, 621765, 785674, 976164, 1194403, 1436429, 1696458, 1970193, 2248594, 2519496, 2767436, 2969625, 3104916, 3151450, 3079981, 2867329, 2488212, 1936233, 1270751, 645163, 238422, 61536, 12173, 2461, 679, 244, 85, 22, 6, 3, 1
12	2	36	0.0022	1, 3, 6, 12, 22, 40, 73, 132, 235, 409, 704, 1198, 2019, 3359, 5490, 8754, 13455, 20040, 29013, 40559, 54910, 71309, 88725, 105254, 117594, 122887, 117879, 100035, 70970, 39564, 16233, 4532, 988, 297, 80, 16, 3
13	2	82	1	1, 3, 6, 12, 22, 40, 74, 134, 240, 421, 730, 1266, 2173, 3700, 6235, 10395, 17066, 27435, 43210, 66841, 101927, 153313, 227103, 331927, 480041, 687370, 973409, 1363835, 1892662, 2602661, 3547099, 4789101, 6405188, 8489779, 11156231, 14533053, 18754480, 23958597, 30304552, 37980778, 47167690, 58025966, 70716104, 85383149, 102116959, 120956373, 141908283, 164907428, 189709842, 215886929, 242919508, 270199807, 296975805, 322328991, 345119584, 364018096, 377739711, 385092514, 384801443, 375470221, 355696676, 323955463, 279173963, 221713589, 156066489, 92855837, 44427067, 16520829, 4791012, 1143528, 244323, 53209, 13909, 4662, 1746, 640, 217, 76, 30, 12, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
5	3	10	1	1, 3, 6, 10, 16, 24, 29, 21, 6, 3, 1
6	3	4	0.0(3)	1, 3, 6, 9, 5
7	3	23	1	1, 3, 6, 12, 20, 34, 55, 83, 124, 185, 274, 395, 558, 726, 808, 739, 540, 323, 104, 26, 14, 6, 3, 1
8	3	30	1	1, 3, 6, 12, 21, 36, 60, 95, 150, 237, 366, 545, 795, 1155, 1666, 2327, 3086, 3854, 4530, 4996, 5119, 4583, 3383, 1984, 882, 305, 88, 25, 6, 3, 1
9	3	12	0.0018	1, 3, 6, 12, 22, 38, 63, 95, 122, 129, 105, 49, 3
10	3	51	0.0079	1, 3, 6, 12, 22, 39, 67, 112, 188, 314, 517, 838, 1329, 2045, 3068, 4545, 6708, 9838, 14207, 20110, 27999, 38546, 52466, 70349, 92556, 119062, 149223, 182190, 217730, 254890, 290234, 318231, 334637, 336252, 317313, 273013, 206542, 133786, 75747, 39938, 19829, 8816, 3454, 1259, 475, 178, 67, 27, 12, 6, 3, 1
11	3	63	1	1, 3, 6, 12, 22, 40, 69, 117, 200, 340, 569, 945, 1546, 2472, 3851, 5866, 8767, 12985, 19172, 28062, 40379, 57215, 80280, 111736, 154288, 210919, 284697, 378893, 496405, 639448, 810437, 1012363, 1246697, 1511100, 1797997, 2101015, 2409595, 2708575, 2972426, 3172407, 3276488, 3252060, 3063097, 2680655, 2110960, 1453697, 871460, 467175, 235761, 117422, 57204, 27172, 12461, 5294, 2184, 1004, 448, 199, 85, 33, 14, 6, 3, 1
12	3	23	8.66e-05	1, 3, 6, 12, 22, 40, 70, 119, 205, 350, 583, 949, 1510, 2311, 3426, 4720, 5903, 6586, 6257, 4778, 2641, 864, 112, 4
13	3	92	1	1, 3, 6, 12, 22, 40, 71, 121, 210, 364, 621, 1049, 1762, 2928, 4817, 7848, 12553, 19622, 30058, 45380, 67899, 100730, 147792, 214222, 307279, 437826, 620357, 872504, 1216894, 1684195, 2315857, 3163482, 4287274, 5758596, 7661077, 10094145, 13179609, 17060694, 21895790, 27855846, 35135571, 43952016, 54538109, 67115475, 81877523, 98965494, 118474162, 140458743, 164889502, 191539031, 219968432, 249529785, 279423663, 308717795, 336343816, 360949515, 380867599, 394310610, 399501048, 394694487, 378039911, 347686777, 302978792, 245959041, 183359347, 125334721, 80100780, 49450637, 30201815, 18322790, 10941718, 6378296, 3617024, 1996753, 1081169, 582241, 315314, 171967, 91407, 45728, 22078, 10648, 5203, 2545, 1212, 555, 246, 98, 33, 12, 6, 3, 1
14	3	109	?	1, 3, 6, 12, 22, 40, 71, 122, 212, 369, 633, 1075, 1814, 3036, 5041, 8312, 13607, 21989, 34807, 53928, 82205, 123995, 185510, 274721, 401826, 581387, 835038, 1192816, 1693041, 2384645, 3333355, 4628066, 6387591, 8768269, 11969785, 16235440, 21849617, 29149608, 38560443, 50630164, 66027193, 85510675, 109946940, 140388166, 178108441, 224552945, 281278284, 349968777, 432470242, 530761076, 646910325, 783033788, 941158639, 1122931324, 1329263656, 1560086116, 1814503087, 2091191012, 2388419140, 2703339336, 3031164434, 3365083502, 3696863442, 4017241516, 4315221587, 4577453038, 4788861931, 4934035687, 4998126405, 4966307104, 4822417543, 4548622627, 4129967064, 3566881565, 2893336550, 2185716764, 1540881308, 1029215420, 665793822, 424233089, 267900994, 167718459, 104169416, 64321940, 39444443, 23932599, 14356906, 8511230, 4960840, 2832401, 1589099, 883548, 485912, 257156, 127600, 59967, 28128, 13577, 6369, 2725, 1030, 371, 149, 73, 41, 23, 12, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
6	4	8	0.1	1, 3, 6, 11, 16, 17, 12, 5, 1
7	4	23	1	1, 3, 6, 12, 20, 34, 55, 83, 124, 185, 274, 395, 558, 726, 808, 739, 540, 323, 104, 26, 14, 6, 3, 1
8	4	6	0.0016	1, 3, 6, 12, 19, 17, 6
9	4	39	1	1, 3, 6, 12, 22, 38, 65, 107, 172, 268, 408, 620, 931, 1366, 1978, 2860, 4101, 5793, 8068, 11018, 14787, 19688, 25815, 32767, 39607, 44606, 45330, 40343, 29532, 17722, 8791, 3664, 1417, 597, 252, 92, 23, 6, 3, 1
10	4	24	0.0079	1, 3, 6, 12, 22, 39, 67, 112, 184, 294, 464, 733, 1138, 1704, 2452, 3334, 4258, 4842, 4519, 3051, 1265, 273, 23, 3, 1
11	4	61	1	1, 3, 6, 12, 22, 40, 69, 117, 196, 320, 520, 842, 1354, 2139, 3286, 4933, 7310, 10781, 15867, 23180, 33416, 47461, 66519, 92372, 127568, 175124, 238080, 319959, 425365, 559227, 726454, 931740, 1177770, 1465544, 1793585, 2153821, 2529419, 2893000, 3208036, 3423860, 3508459, 3421308, 3131851, 2631465, 1978755, 1309347, 763280, 393177, 184843, 81767, 33388, 12566, 4495, 1720, 701, 234, 73, 29, 14, 6, 3, 1
12	4	17	1.08e-05	1, 3, 6, 12, 22, 40, 70, 119, 199, 320, 488, 678, 844, 908, 813, 502, 149, 10
13	4	89	1	1, 3, 6, 12, 22, 40, 71, 121, 206, 344, 572, 954, 1574, 2578, 4197, 6749, 10635, 16390, 24760, 36970, 54980, 81494, 119892, 174219, 249764, 354342, 499383, 700865, 979980, 1362835, 1880218, 2569012, 3475394, 4660381, 6208151, 8225431, 10828539, 14138596, 18296413, 23486970, 29952677, 37984475, 47897868, 59998973, 74550360, 91755339, 111745066, 134564029, 160149616, 188367639, 218910399, 251287740, 284670301, 317739035, 348884372, 376440273, 398743635, 413998149, 420085770, 414236962, 393386088, 355531766, 301781832, 237759748, 172705947, 116007949, 72870469, 43363193, 24742132, 13674143, 7352284, 3836528, 1925457, 923936, 425844, 187653, 77819, 30642, 12410, 5319, 2302, 969, 384, 143, 54, 25, 12, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
7	5	21	1	1, 3, 6, 12, 20, 34, 58, 96, 158, 234, 334, 456, 562, 672, 731, 737, 583, 280, 45, 12, 5, 1
8	5	30	1	1, 3, 6, 12, 21, 36, 60, 95, 150, 237, 366, 545, 795, 1155, 1666, 2327, 3086, 3854, 4530, 4996, 5119, 4583, 3383, 1984, 882, 305, 88, 25, 6, 3, 1
9	5	39	1	1, 3, 6, 12, 22, 38, 65, 107, 172, 268, 408, 620, 931, 1366, 1978, 2860, 4101, 5793, 8068, 11018, 14787, 19688, 25815, 32767, 39607, 44606, 45330, 40343, 29532, 17722, 8791, 3664, 1417, 597, 252, 92, 23, 6, 3, 1
10	5	8	4.4091e-05	1, 3, 6, 12, 22, 37, 46, 28, 5
11	5	59	1	1, 3, 6, 12, 22, 40, 69, 117, 196, 320, 515, 813, 1263, 1949, 2977, 4451, 6522, 9486, 13802, 20007, 28733, 40838, 57261, 78870, 107061, 144266, 193601, 259069, 346004, 460027, 605751, 787463, 1011603, 1287822, 1625786, 2029039, 2486582, 2974318, 3447404, 3835812, 4035702, 3941321, 3488449, 2724519, 1829973, 1058798, 541720, 254325, 107982, 40699, 14139, 5375, 2345, 1037, 390, 110, 25, 6, 3, 1
12	5	70	1	1, 3, 6, 12, 22, 40, 70, 119, 201, 332, 541, 869, 1379, 2184, 3444, 5338, 8073, 11968, 17546, 25620, 37389, 54463, 78766, 112421, 157847, 218413, 299481, 408443, 553914, 746444, 1000256, 1333612, 1768417, 2330500, 3049951, 3961613, 5108108, 6543496, 8327072, 10504397, 13093568, 16091345, 19485592, 23246974, 27294044, 31443674, 35382422, 38631941, 40598844, 40677650, 38410724, 33718019, 27123253, 19732023, 12841214, 7457800, 3899160, 1857003, 814782, 332784, 127508, 45517, 15256, 5193, 1798, 554, 144, 33, 6, 3, 1
13	5	86	1	1, 3, 6, 12, 22, 40, 71, 121, 206, 344, 567, 925, 1495, 2411, 3873, 6163, 9626, 14706, 22029, 32574, 47942, 70478, 103323, 150507, 217133, 309754, 436875, 609928, 845017, 1163814, 1592804, 2163383, 2914951, 3898639, 5180440, 6846261, 9005514, 11793471, 15377846, 19966714, 25808146, 33174934, 42341427, 53575349, 67161647, 83435448, 102789299, 125628427, 152263963, 182789174, 217007693, 254394698, 294103101, 334671841, 374116596, 409755080, 438298392, 455882110, 458378041, 442524726, 407210952, 353702049, 286520504, 213728578, 145442593, 89899196, 50567121, 26119653, 12625399, 5827355, 2611900, 1130390, 463034, 178946, 65600, 23662, 8801, 3202, 1112, 386, 150, 72, 35, 17, 8, 3, 1
14	5	101	?	1, 3, 6, 12, 22, 40, 71, 122, 208, 349, 579, 951, 1551, 2527, 4101, 6611, 10581, 16721, 25889, 39219, 58499, 86675, 128189, 189032, 276895, 401777, 577041, 820971, 1159417, 1628933, 2278750, 3170739, 4377895, 5986448, 8105543, 10882174, 14518469, 19283785, 25512528, 33595215, 43994803, 57309006, 74343562, 96149055, 124017979, 159451146, 204083803, 259599658, 327743177, 410504861, 510366789, 630376965, 773934922, 944395742, 1144711338, 1377277761, 1643988873, 1946279650, 2284696892, 2658002871, 3061905991, 3488042472, 3923523680, 4351095592, 4750131167, 5097508707, 5368954008, 5540589468, 5588536542, 5487479959, 5212855233, 4750688847, 4113989138, 3354118308, 2555758702, 1814150199, 1202392351, 749363585, 442713883, 249315448, 134392877, 69706468, 34956057, 17003448, 8055675, 3734904, 1692058, 739356, 310194, 126383, 50649, 20154, 7946, 2973, 989, 280, 72, 23, 12, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
8	6	15	0.0286	1, 3, 6, 12, 21, 36, 61, 96, 146, 199, 231, 201, 100, 27, 10, 2
9	6	12	0.0018	1, 3, 6, 12, 22, 38, 63, 95, 122, 129, 105, 49, 3
10	6	24	0.0079	1, 3, 6, 12, 22, 39, 67, 112, 184, 294, 464, 733, 1138, 1704, 2452, 3334, 4258, 4842, 4519, 3051, 1265, 273, 23, 3, 1
11	6	59	1	1, 3, 6, 12, 22, 40, 69, 117, 196, 320, 515, 813, 1263, 1949, 2977, 4451, 6522, 9486, 13802, 20007, 28733, 40838, 57261, 78870, 107061, 144266, 193601, 259069, 346004, 460027, 605751, 787463, 1011603, 1287822, 1625786, 2029039, 2486582, 2974318, 3447404, 3835812, 4035702, 3941321, 3488449, 2724519, 1829973, 1058798, 541720, 254325, 107982, 40699, 14139, 5375, 2345, 1037, 390, 110, 25, 6, 3, 1
12	6	10	8.02e-07	1, 3, 6, 12, 22, 40, 68, 96, 92, 41, 3
13	6	83	1	1, 3, 6, 12, 22, 40, 71, 121, 206, 344, 567, 925, 1489, 2371, 3734, 5841, 9060, 13806, 20578, 30188, 44078, 64410, 93921, 136102, 195785, 279255, 393431, 545860, 747048, 1012781, 1365174, 1835237, 2465632, 3308669, 4420900, 5860264, 7689939, 9990005, 12872262, 16490911, 21045622, 26779418, 33984492, 43023428, 54340675, 68437191, 85827409, 107018827, 132523670, 162864781, 198532390, 239777665, 286324496, 337039807, 389672673, 440630325, 484728386, 515453094, 524956696, 505759640, 453901386, 372827768, 275681050, 181542407, 106728110, 56833648, 28031099, 13037188, 5770715, 2417544, 965062, 382127, 159909, 70306, 30568, 12634, 4974, 1815, 572, 146, 28, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
9	7	38	1	1, 3, 6, 12, 22, 38, 68, 120, 206, 349, 570, 914, 1412, 2123, 3133, 4504, 6327, 8625, 11505, 15021, 19099, 23613, 28154, 32397, 35872, 37701, 37249, 34491, 28822, 18996, 8738, 2270, 381, 85, 31, 12, 6, 3, 1
10	7	51	1	1, 3, 6, 12, 22, 39, 67, 112, 188, 314, 517, 838, 1329, 2045, 3068, 4545, 6708, 9838, 14207, 20110, 27999, 38546, 52466, 70349, 92556, 119062, 149223, 182190, 217730, 254890, 290234, 318231, 334637, 336252, 317313, 273013, 206542, 133786, 75747, 39938, 19829, 8816, 3454, 1259, 475, 178, 67, 27, 12, 6, 3, 1
11	7	61	1	1, 3, 6, 12, 22, 40, 69, 117, 196, 320, 520, 842, 1354, 2139, 3286, 4933, 7310, 10781, 15867, 23180, 33416, 47461, 66519, 92372, 127568, 175124, 238080, 319959, 425365, 559227, 726454, 931740, 1177770, 1465544, 1793585, 2153821, 2529419, 2893000, 3208036, 3423860, 3508459, 3421308, 3131851, 2631465, 1978755, 1309347, 763280, 393177, 184843, 81767, 33388, 12566, 4495, 1720, 701, 234, 73, 29, 14, 6, 3, 1
12	7	70	1	1, 3, 6, 12, 22, 40, 70, 119, 201, 332, 541, 869, 1379, 2184, 3444, 5338, 8073, 11968, 17546, 25620, 37389, 54463, 78766, 112421, 157847, 218413, 299481, 408443, 553914, 746444, 1000256, 1333612, 1768417, 2330500, 3049951, 3961613, 5108108, 6543496, 8327072, 10504397, 13093568, 16091345, 19485592, 23246974, 27294044, 31443674, 35382422, 38631941, 40598844, 40677650, 38410724, 33718019, 27123253, 19732023, 12841214, 7457800, 3899160, 1857003, 814782, 332784, 127508, 45517, 15256, 5193, 1798, 554, 144, 33, 6, 3, 1
13	7	83	1	1, 3, 6, 12, 22, 40, 71, 121, 206, 344, 567, 925, 1489, 2371, 3734, 5841, 9060, 13806, 20578, 30188, 44078, 64410, 93921, 136102, 195785, 279255, 393431, 545860, 747048, 1012781, 1365174, 1835237, 2465632, 3308669, 4420900, 5860264, 7689939, 9990005, 12872262, 16490911, 21045622, 26779418, 33984492, 43023428, 54340675, 68437191, 85827409, 107018827, 132523670, 162864781, 198532390, 239777665, 286324496, 337039807, 389672673, 440630325, 484728386, 515453094, 524956696, 505759640, 453901386, 372827768, 275681050, 181542407, 106728110, 56833648, 28031099, 13037188, 5770715, 2417544, 965062, 382127, 159909, 70306, 30568, 12634, 4974, 1815, 572, 146, 28, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
10	8	24	0.0079	1, 3, 6, 12, 22, 39, 70, 125, 216, 361, 590, 945, 1457, 2109, 2835, 3551, 4159, 4437, 3904, 2556, 1114, 258, 25, 4, 1
11	8	63	1	1, 3, 6, 12, 22, 40, 69, 117, 200, 340, 569, 945, 1546, 2472, 3851, 5866, 8767, 12985, 19172, 28062, 40379, 57215, 80280, 111736, 154288, 210919, 284697, 378893, 496405, 639448, 810437, 1012363, 1246697, 1511100, 1797997, 2101015, 2409595, 2708575, 2972426, 3172407, 3276488, 3252060, 3063097, 2680655, 2110960, 1453697, 871460, 467175, 235761, 117422, 57204, 27172, 12461, 5294, 2184, 1004, 448, 199, 85, 33, 14, 6, 3, 1
12	8	17	1.08e-05	1, 3, 6, 12, 22, 40, 70, 119, 199, 320, 488, 678, 844, 908, 813, 502, 149, 10
13	8	86	1	1, 3, 6, 12, 22, 40, 71, 121, 206, 344, 567, 925, 1495, 2411, 3873, 6163, 9626, 14706, 22029, 32574, 47942, 70478, 103323, 150507, 217133, 309754, 436875, 609928, 845017, 1163814, 1592804, 2163383, 2914951, 3898639, 5180440, 6846261, 9005514, 11793471, 15377846, 19966714, 25808146, 33174934, 42341427, 53575349, 67161647, 83435448, 102789299, 125628427, 152263963, 182789174, 217007693, 254394698, 294103101, 334671841, 374116596, 409755080, 438298392, 455882110, 458378041, 442524726, 407210952, 353702049, 286520504, 213728578, 145442593, 89899196, 50567121, 26119653, 12625399, 5827355, 2611900, 1130390, 463034, 178946, 65600, 23662, 8801, 3202, 1112, 386, 150, 72, 35, 17, 8, 3, 1

n	k	Diameter	Size $/n!$	Growth
11	9	57	1	1, 3, 6, 12, 22, 40, 72, 130, 230, 397, 682, 1154, 1937, 3191, 5111, 8001, 12251, 18465, 27423, 39892, 57045, 80628, 112557, 154908, 210147, 281345, 371408, 483838, 621765, 785674, 976164, 1194403, 1436429, 1696458, 1970193, 2248594, 2519496, 2767436, 2969625, 3104916, 3151450, 3079981, 2867329, 2488212, 1936233, 1270751, 645163, 238422, 61536, 12173, 2461, 679, 244, 85, 22, 6, 3, 1
12	9	23	8.66e-05	1, 3, 6, 12, 22, 40, 70, 119, 205, 350, 583, 949, 1510, 2311, 3426, 4720, 5903, 6586, 6257, 4778, 2641, 864, 112, 4
13	9	89	1	1, 3, 6, 12, 22, 40, 71, 121, 206, 344, 572, 954, 1574, 2578, 4197, 6749, 10635, 16390, 24760, 36970, 54980, 81494, 119892, 174219, 249764, 354342, 499383, 700865, 979980, 1362835, 1880218, 2569012, 3475394, 4660381, 6208151, 8225431, 10828539, 14138596, 18296413, 23486970, 29952677, 37984475, 47897868, 59998973, 74550360, 91755339, 111745066, 134564029, 160149616, 188367639, 218910399, 251287740, 284670301, 317739035, 348884372, 376440273, 398743635, 413998149, 420085770, 414236962, 393386088, 355531766, 301781832, 237759748, 172705947, 116007949, 72870469, 43363193, 24742132, 13674143, 7352284, 3836528, 1925457, 923936, 425844, 187653, 77819, 30642, 12410, 5319, 2302, 969, 384, 143, 54, 25, 12, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
12	10	36	0.0022	1, 3, 6, 12, 22, 40, 73, 132, 235, 409, 704, 1198, 2019, 3359, 5490, 8754, 13455, 20040, 29013, 40559, 54910, 71309, 88725, 105254, 117594, 122887, 117879, 100035, 70970, 39564, 16233, 4532, 988, 297, 80, 16, 3
13	10	92	1	1, 3, 6, 12, 22, 40, 71, 121, 210, 364, 621, 1049, 1762, 2928, 4817, 7848, 12553, 19622, 30058, 45380, 67899, 100730, 147792, 214222, 307279, 437826, 620357, 872504, 1216894, 1684195, 2315857, 3163482, 4287274, 5758596, 7661077, 10094145, 13179609, 17060694, 21895790, 27855846, 35135571, 43952016, 54538109, 67115475, 81877523, 98965494, 118474162, 140458743, 164889502, 191539031, 219968432, 249529785, 279423663, 308717795, 336343816, 360949515, 380867599, 394310610, 399501048, 394694487, 378039911, 347686777, 302978792, 245959041, 183359347, 125334721, 80100780, 49450637, 30201815, 18322790, 10941718, 6378296, 3617024, 1996753, 1081169, 582241, 315314, 171967, 91407, 45728, 22078, 10648, 5203, 2545, 1212, 555, 246, 98, 33, 12, 6, 3, 1

n	k	Diameter	Size $/n!$	Growth
13	11	82	1	1, 3, 6, 12, 22, 40, 74, 134, 240, 421, 730, 1266, 2173, 3700, 6235, 10395, 17066, 27435, 43210, 66841, 101927, 153313, 227103, 331927, 480041, 687370, 973409, 1363835, 1892662, 2602661, 3547099, 4789101, 6405188, 8489779, 11156231, 14533053, 18754480, 23958597, 30304552, 37980778, 47167690, 58025966, 70716104, 85383149, 102116959, 120956373, 141908283, 164907428, 189709842, 215886929, 242919508, 270199807, 296975805, 322328991, 345119584, 364018096, 377739711, 385092514, 384801443, 375470221, 355696676, 323955463, 279173963, 221713589, 156066489, 92855837, 44427067, 16520829, 4791012, 1143528, 244323, 53209, 13909, 4662, 1746, 640, 217, 76, 30, 12, 6, 3, 1

Growth computations have been performed in the LRX(n,k) Permutations Growth

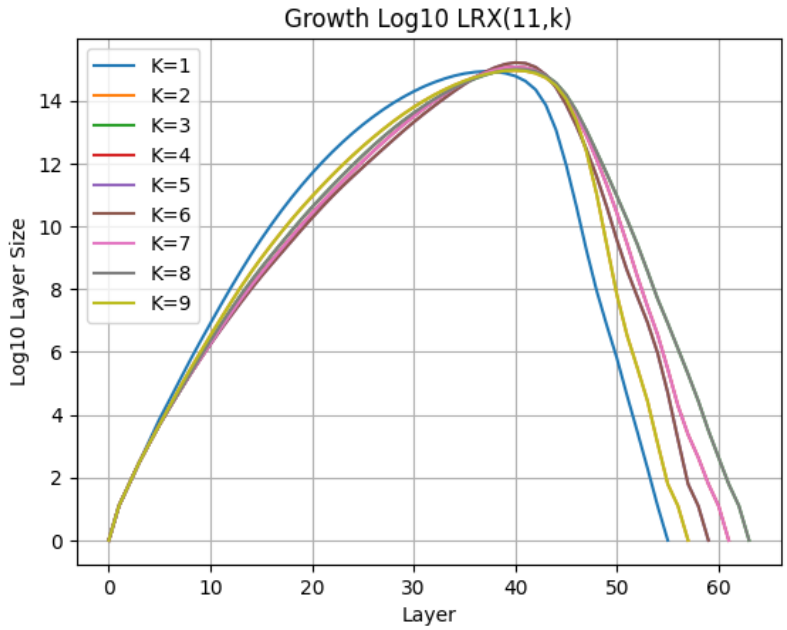


Figure 8: LRX(11,k)

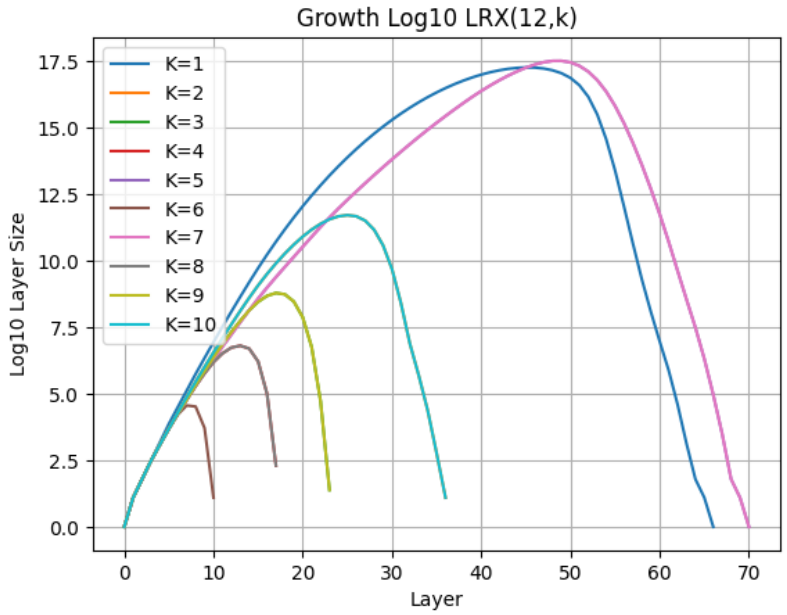


Figure 9: LRX(12,k)

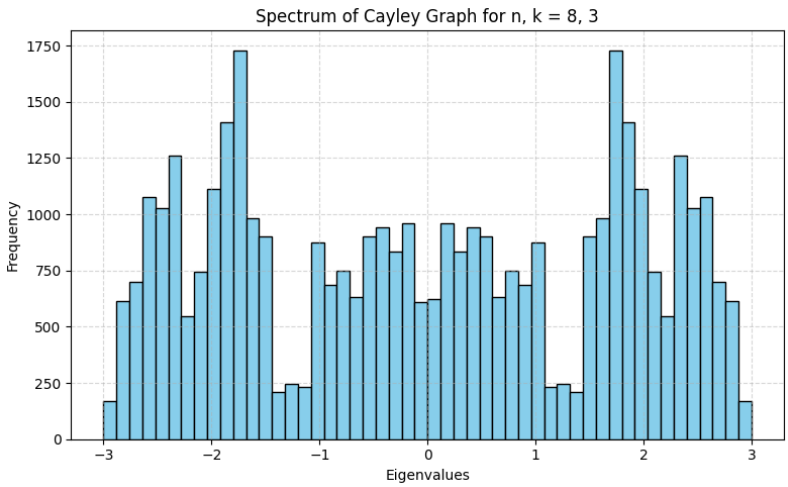


Figure 10: LRX(8,2) Spectrum

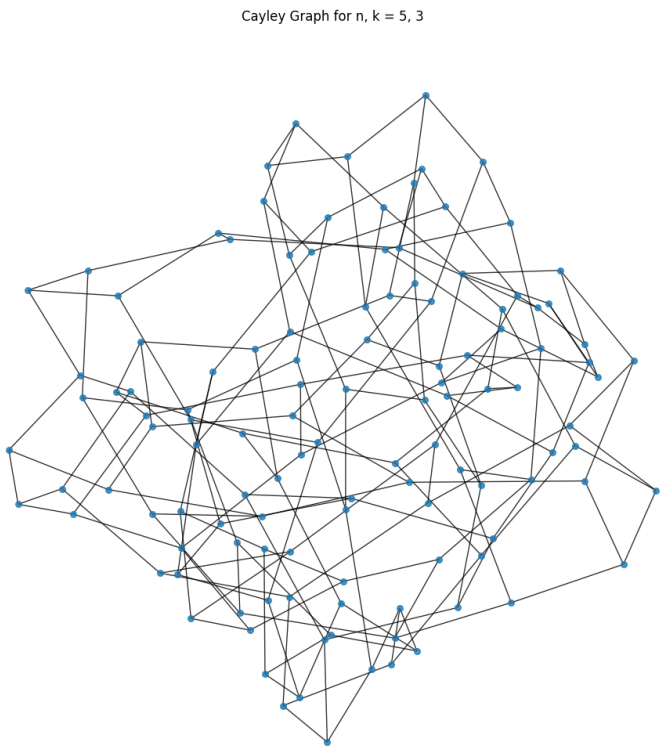


Figure 11: LRX(5,3) Graph

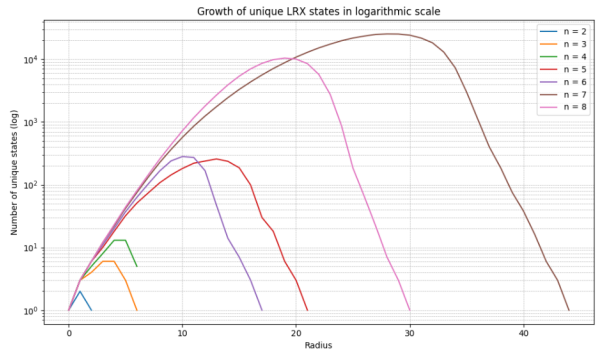


Figure 12: Two minuses LRX states grows

7.6.3 Signed LRX

The diameter of the LRX permutation group where X changes *both* signs can be *estimated* as follows. It is *at least* the diameter of the *un*signed group, which is at least $n(n-1)/2 - n/2 - 1$.

On the other hand, the diameter of the signed permutation group is also estimated from above as follows. Solve the *un*signed permutation. Then we end up with some combination of signs. After some combination of transpositions and rotations every element returns to its original place, this time with the right signs.

If n is even, then the number of transpositions of every element has the same parity as the number of rotations. Therefore, the number of sign changes of every element is the same, and once the unsigned permutation is solved, the signed one is such that all signs are the same. If they are negative, then we apply $(LX)^{n-1}$. Then all elements stay at the same place and change the signs.

Next we consider the case when n is odd. Recall that the algorithm for solving the unsigned permutation by using all cyclic transpositions required us to move every element to its rightful place. If we revise the algorithm by choosing the different path, then the sign of all elements, except for the moved one, changes. Therefore, if the solution to the unsigned problem made some elements negative, then we change the path of precisely these (or of precisely the positive ones). Therefore, the number of operations used in a cycle of m elements is at most $m + \sum_{cycle} 2[dist(i, \sigma(i))$ or $n - dist(i, \sigma(i)))]$. Therefore, the total amount of operations is at most

$$3n + \sum 2[dist(i, \sigma(i)) \text{ or } n - dist(i, \sigma(i)))].$$

Summing the numbers in two different cases, we obtain $6n + 2n^2$, and one of the numbers is at most $n^2 + 3n$.

n	Full	Diameter	Growth
2	Yes	2	1,2,1
3	Yes	6	1,3,4,6,6,3,1
4	Yes	6	1,3,5,8,13,13,5
5	Yes	21	1,3,6,10,18,32,51,74,107,143,181,220,238,257,236,186,99,30,18,6,3,1
6	Yes	17	1,3,6,11,20,37,63,103,165,239,281,272,167,47,14,7,3,1
7	Yes	44	1,3,6,12,22,42,75,132,225,363,567,856,1244,1752,2436,3298,4344,5612,7112,8817,10793,12889,15138,17404,
8	Yes	30	1,3,6,12,23,44,80,144,255,433,722,1167,1797,2682,3894,5368,7012,8603,9832,10395,10031,8510,5727,2749,8

Growth calculations have been performed in the LRXsign Permutations Growth. This version contains two minuses in the X permutation. Here, below, we can see the growth of unique LRX states of this case on log scale:

n	Full	Diameter	Growth
2	Yes	3	1,2,3,2
3	Yes	5	1,3,5,9,5,1
4	Yes	10	1,3,6,13,27,55,91,105,65,16,2
5	Yes	12	1,3,7,15,34,73,141,250,409,486,359,119,23
6	Yes	19	1,3,7,16,36,81,169,340,671,1278,2319,3904,5983,8212,9221,7744,4360,1414,273,48
7	Yes	25	1,3,7,17,38,88,189,396,818,1630,3143,5851,10402,17051,25883,36449,46253,52508,51536,40153,21755,6772,
8	Yes	34	1,3,7,17,39,90,196,416,877,1801,3620,7156,13834,25996,47401,82662,137849,221602,341736,505969,715529

Growth calculations have been performed in the LRXsign V1 Permutations Growth. This version contains one minus in the X permutation. Here, below, we can see the growth of unique LRX states of this case on log scale:

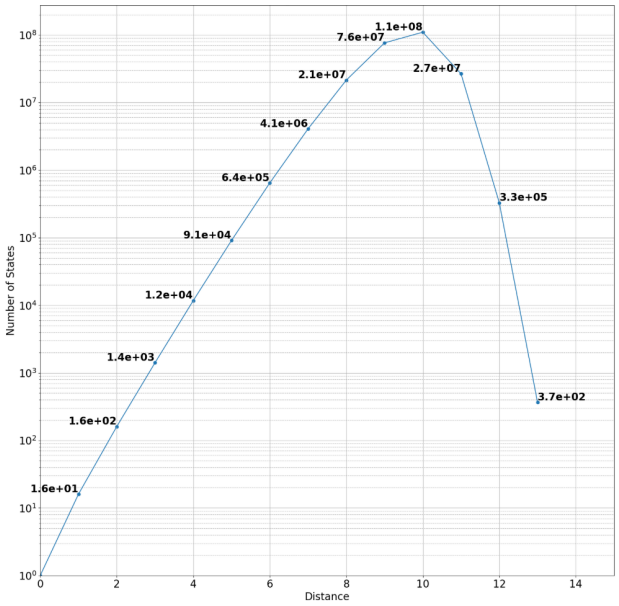


Figure 13: One minus LRX states grows

7.7 Transposition and sub-long cycle

7.8 3-cycles

7.9 Consecutive 3-cycles

7.10 0ij 3-cycles

7.11 Consecutive k-cycles

7.12 Consecutive wrapped k-cycles

7.13 Prefix cycles $(1, 2, ..., i)$

Consider the set $CP = \{(1, \ldots, k) \in \mathrm{Sym}_n, 1 < k \leqslant n\}$ of cyclic permutations of the first k elements [20, 29]. The corresponding Cayley graph is $CP_n = Cay(\mathrm{Sym}_n, CP)$. The diameter of CP_n is equal to $D(n) = n-1$, which is established by V.P. Zyazin (1971) (according to [28]). Given the set of generators M_n, the asymptotical lower bound on the diameter of the Cayley graph associated with M_n is $D(n) \gtrsim \frac{n \log_2 n}{\log_2 |M_n|}$ for $n \to \infty$. One can see that $|M_n| = n-1$, so that $D(n) \gtrsim n$, while $D(n) = n-1$ matches this lower bound and, therefore, is optimal [28]. In addition, it is shown that the cardinalities of the metric balls $B_i = \{v \in V(CP_n), \tilde{d}(I_n, v) \leqslant i\}$, where $\tilde{d}$ denotes the number of generators, without their inverses (unlike d), needed to come to v starting from I_n, are $|B_1| = n-1$, $|B_2| = (n-1)^2$, and $|B_i| = n(n-1)\cdots(n-i+1)$ for $2 < i < n-1$. Note that $B_i \subset B_{i+1}$ for any i. The difference between the definitions of d and $\tilde{d}$ leads to different notions of growth. In figure 16, we compare the classical growth $(|S_i|)_{i=0}^{D(n)}$ of CP_n with the theoretical values $(|B_{i+1} \setminus B_i|)_{i=0}^{D(n)}$ taken from [28].

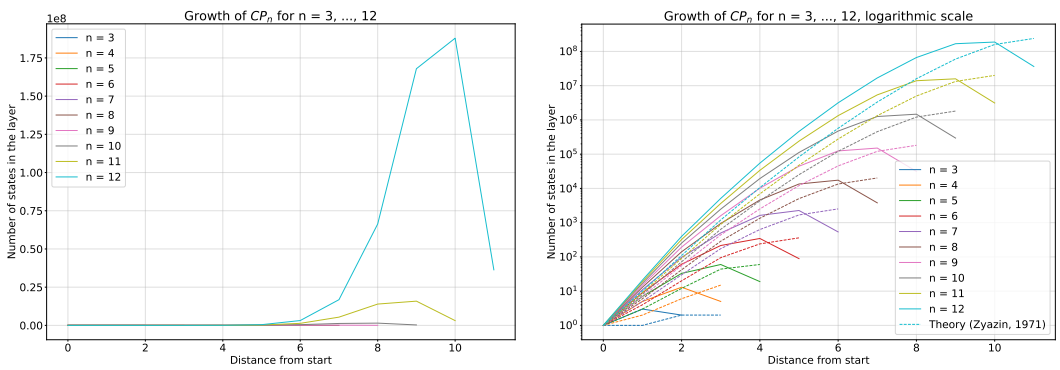


Figure 14: CP_n growth, $n = 3, ..., 12$.

n	$D(n)$	$(\lvert S_i \rvert)_{i=0}^{D(n)}$
3	2	1, 3, 2
4	3	1, 5, 13, 5
5	4	1, 7, 33, 60, 19
6	5	1, 9, 61, 215, 345, 89
7	6	1, 11, 97, 499, 1631, 2265, 536
8	7	1, 13, 141, 947, 4584, 13452, 17397, 3785
9	8	1, 15, 193, 1595, 10122, 44503, 124306, 150714, 31431
10	9	1, 17, 253, 2479, 19384, 112042, 472504, 1256861, 1469983, 295276
11	10	1, 19, 321, 3635, 33714, 240254, 1333473, 5386904, 13945587, 15854699, 3118193
12	11	1, 21, 397, 5099, 54660, 462054, 3160407, 16812578, 66316525, 167939249, 187901775, 36348834

Table 3: CP_n growth, $n = 3, ..., 12$.

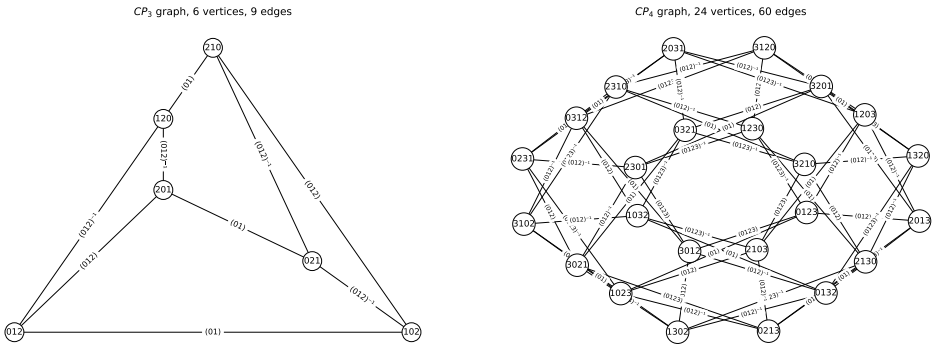


Figure 15: CP_n graph for $n = 3, 4$

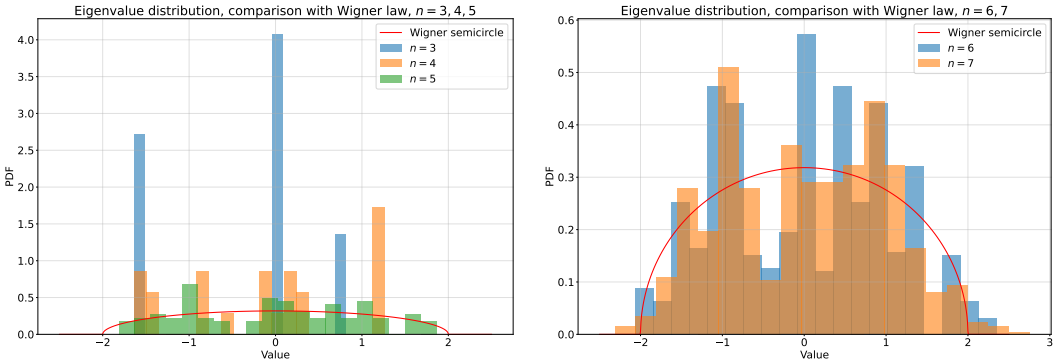


Figure 16: Comparison of spectrum of CP_n with Wigner law for $n = 3, ..., 7$.

n	layer_sizes
3	1, 4, 1
4	1, 9, 13, 1
5	1, 16, 61, 41, 1
6	1, 25, 181, 381, 131, 1
7	1, 36, 421, 1821, 2332, 428, 1
8	1, 49, 841, 6105, 17557, 14337, 1429, 1
9	1, 64, 1513, 16465, 83029, 167449, 89497, 4861, 1
10	1, 81, 2521, 38281, 296326, 1100902, 1604098, 569794, 16795, 1
11	1, 100, 3961, 79861, 874886, 5122877, 14516426, 15555398, 3704504, 58785, 1
12	1, 121, 5941, 153341, 2250887, 18943343, 87116283, 192422979, 153315999, 24584693, 208011, 1

Table 4: CP_n growth, $n = 3, ..., 12$

7.14 Downcycles: $(i, i+1, ..., j), i < j$

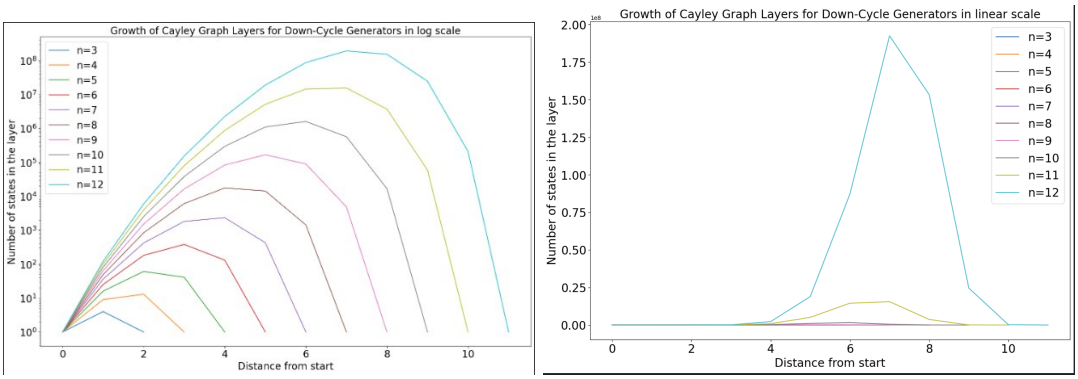


Figure 17: CP_n graph for $n = 3, 4$

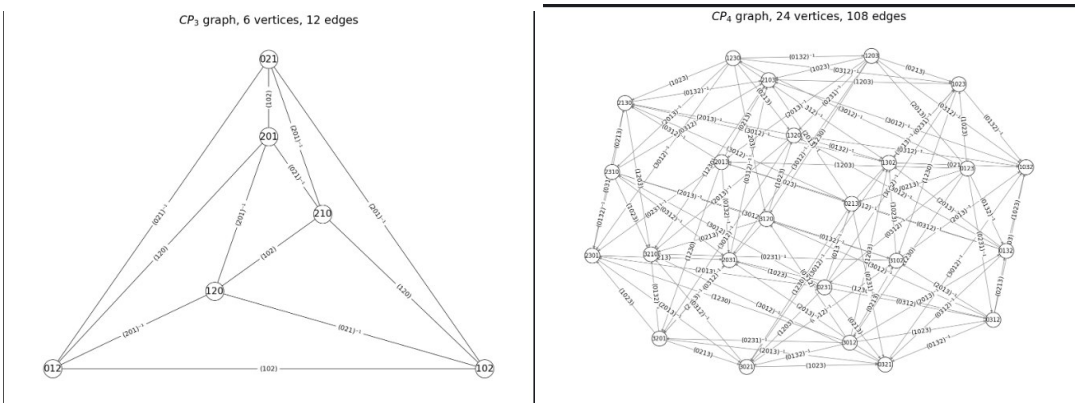


Figure 18: CP_n growth, $n = 3, ..., 12$

7.15 Increasing cycles $(i_1, i_2, ..., i_k), i_1 < i_2 < ... < i_k$.

7.16 Rapaport M1

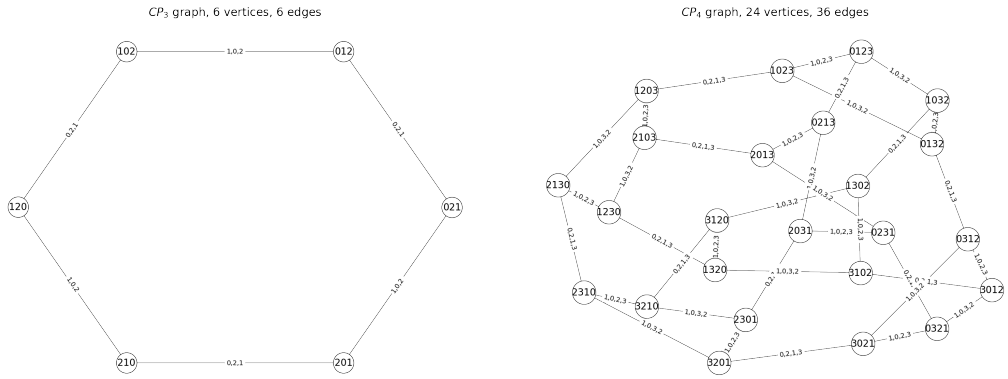


Figure 19: ...

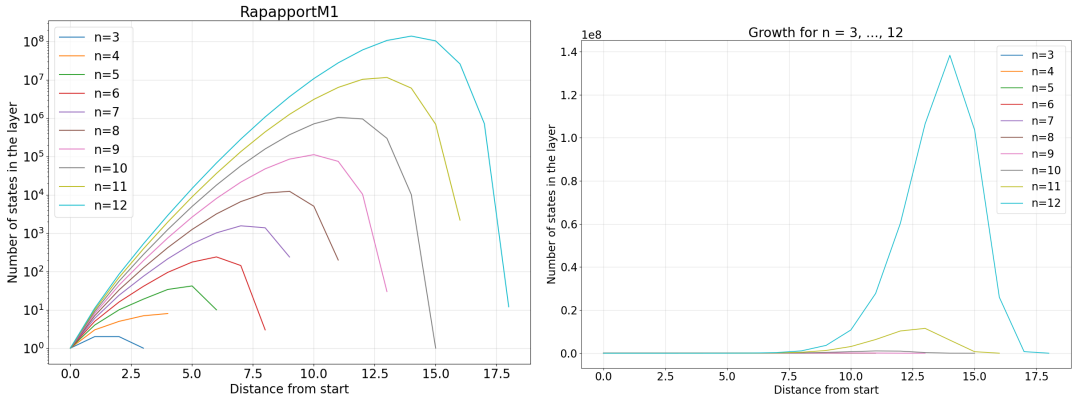


Figure 20: ...

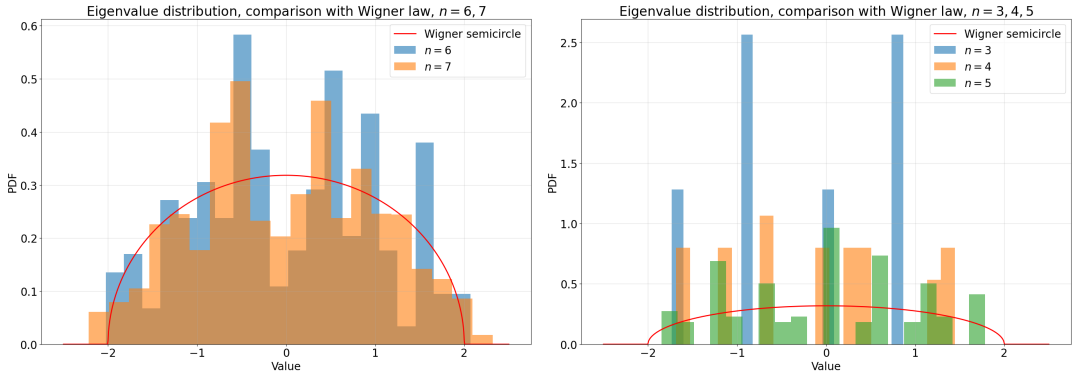


Figure 21: ...

7.17 Rapaport M2

Rapaport M2 generators

In this section, we provide a detailed analysis of the properties of Cayley graphs for the symmetric group S_n generated by the set of generators we denote as M2.

General Information and Generators

The M2 generator set for the symmetric group S_n was notably studied by Elvira Strasser Rapaport. Based on the provided materials [?Glukhov1999], the set is defined as:

$$M_2 = \{(1,2),(1,2)(3,4),\dots\} \cup \{(2,3),(2,3)(4,5),\dots\}$$

More formally, the generators are products of disjoint transpositions of two forms:

1. Products of transpositions of the form $(2k-1, 2k)$.
2. Products of transpositions of the form $(2k, 2k+1)$.

All generators in this set are involutions, i.e., $s^2 = e$ for any $s \in M2$, which implies that the resulting Cayley graph is undirected. This generator set is of particular interest in the study of Hamiltonian paths on Cayley graphs [?Rapaport1959].

Growth Function

The growth function describes the number of elements at a given distance from the identity. The table below shows the layer sizes (number of elements at distance k) for S_n with the M2 generator set, based on our computations.

Graph Diameter

The diameter of the Cayley graph, which corresponds to the maximum shortest distance between any two vertices, is given in Table 6.

Our analysis indicates that the diameter $D(n)$ can be closely approximated by the quadratic function:

$$D(n) \approx 0.583n^2 - 0.724n - 1.422$$

This polynomial growth is a significant characteristic of this generator set.

Algorithmic Complexity and Subgroup Structure

- **Existence of an Algorithm:** An algorithm to find the shortest path from any permutation to the identity is the standard Breadth-First Search (BFS), which was used to compute the growth function.
- **Optimal Algorithm:** For an unweighted graph, BFS is guaranteed to find a shortest path, making it an optimal algorithm in terms of path length.
- **Generated Subgroup:** This set of generators is known to generate the entire symmetric group S_n. If it generated a proper subgroup (e.g., the alternating group A_n), the Cayley graph would have fewer vertices or be disconnected.

Growth and Spectrum Visualization

The growth function and adjacency spectrum provide key insights into the graph's structure.

Further Properties

- **Growth Approximation:** For $n > 5$, the distribution of layer sizes, $|S(k)|$, visually resembles a Gaussian distribution. This behavior is characteristic of graphs with good expansion properties.
- **Analytic Formula for Growth:** Currently, no closed-form analytic formula is known for the growth sequence $\{|S(k)|\}_{k\geq 0}$ with this generator set. Obtaining such formulas is a notoriously difficult problem in algebraic combinatorics.
- **Spectrum Properties:** The spectrum is generally not integer-valued, distinguishing these graphs from simpler structures like cycle graphs. For S_4, the spectrum is symmetric about zero, which suggests the graph may be bipartite. The study of the limiting distribution of eigenvalues as $n \to \infty$ is an interesting open problem.

Table 5: Summary of Cayley graph properties for various values of n

n	**Layer Sizes**	**Max Layer Size**	**Graph Diameter**	**Total States**	**Farthest States Sample**
3	[1, 2, 2, 1]	2	3	6	[2 1 0]
4	[1, 3, 5, 7, 8]	8	4	24	[3 2 1 0], [2 3 1 0], [1 3 2 0], [2 3 0], [3 2 0 1], [3 0 2 1], [0 3 1], [3 0 1 2]
5	[1, 3, 5, 8, 13, 20, 26, 33, 11]	33	8	120	[2 1 4 3 0], [1 2 4 3 0], [2 1 3 4 0], [0 3 4 2 1], [0 4 3 1 2], [4 1 3 2], [4 0 1 3 2], [4 1 0 2 3], [2 1 0 4], [3 2 0 1 4]
6	[1, 3, 5, 8, 13, 21, 32, 47, 64, 90, 124, 122, 104, 52, 22, 10, 2]	124	16	720	[0 1 4 5 3 2], [0 1 5 4 2 3]
7	[1, 3, 5, 8, 13, 21, 33, 50, 74, 106, 153, 219, 301, 403, 526, 612, 643, 648, 568, 389, 193, 61, 10]	648	22	5040	[5 1 2 4 6 3 0], [6 1 2 3 5 4 0], 3 2 1 5 6 0], [0 4 3 5 6 2 1], [0 2 3 5 4 1], [0 6 3 2 4 5 1], [0 1 4 2 3], [6 1 2 5 3 0 4], [0 6 5 2 4], [6 3 2 1 0 4 5]
8	[1, 3, 5, 8, 13, 21, 33, 51, 77, 114, 171, 250, 364, 519, 724, 1015, 1393, 1888, 2476, 3164, 3761, 4242, 4486, 4469, 3832, 3019, 2143, 1259, 584, 223, 12]	4486	30	40320	[2 1 7 6 3 5 4 0], [7 1 0 4 6 5 3 [1 0 4 5 7 6 3 2], [6 0 1 5 7 4 2 [4 5 1 0 7 6 2 3], [5 4 0 1 7 6 2 [2 5 1 6 7 0 4 3], [1 2 6 7 5 3 0 [5 2 0 7 6 1 3 4], [2 3 6 7 1 0 5
9	[1, 3, 5, 8, 13, 21, 33, 51, 78, 117, 176, 264, 388, 564, 810, 1161, 1664, 2351, 3277, 4503, 6120, 8213, 10805, 13985, 17691, 21873, 25983, 29628, 32734, 34224, 33248, 30263, 25674, 20827, 15452, 10871, 6438, 2612, 658, 85, 8]	34224	40	362880	[2 1 8 7 3 4 6 5 0], [2 1 7 8 4 3 0], [1 2 7 8 4 3 5 6 0], [3 8 2 0 5 4 1], [3 8 2 0 7 6 4 5 1], [8 1 5 7 6 3 2], [8 1 0 5 4 6 7 2 3], [1 5 4 6 7 2 3]
10	[1, 3, 5, 8, 13, 21, 33, 51, 78, 118, 179, 269, 400, 590, 857, 1249, 1818, 2620, 3747, 5309, 7500, 10528, 14641, 20031, 27085, 36120, 47671, 61740, 78961, 99498, 123487, 150609, 180223, 210176, 238556, 263604, 279352, 282394, 275840, 261268, 236528, 204901, 169751, 132167, 92078, 57920, 32235, 13520, 2897, 150]	282394	49	3628800	[2 1 5 9 8 4 6 7 3 0], [1 5 6 2 9 3 7 4 0], [1 6 7 3 2 9 8 4 5 0], [6 3 2 8 9 4 5 0], [1 4 7 3 8 9 2 6 0], [9 3 4 2 1 7 8 6 5 0], [9 4 3 7 8 6 5 0], [9 1 3 2 4 7 8 6 5 0], 1 2 6 9 3 4 8 5 0], [1 9 7 4 2 3 6 5 0]
11	[1, 3, 5, 8, 13, 21, 33, 51, 78, 118, 180, 272, 405, 599, 879, 1289, 1888, 2746, 3968, 5700, 8169, 11679, 16581, 23384, 32756, 45552, 62789, 85632, 115407, 153293, 201304, 262023, 337047, 427781, 536869, 666661, 817922, 991852, 1190281, 1409960, 1642381, 1881544, 2115512, 2324272, 2492006, 2598690, 2634931, 2605882, 2518678, 2373853, 2166559, 1902771, 1591937, 1266483, 956837, 669050, 415602, 219298, 93056, 27705, 4322, 228, 4]	2634931	62	39916800	[2 1 8 7 3 10 9 4 6 5 0], [0 6 7 9 3 8 4 5 10 1], [10 1 0 4 7 9 8 3 2 5], [0 10 4 5 7 8 1 2 6 3 9]
12	[1, 3, 5, 8, 13, 21, 33, 51, 78, 118, 180, 273, 408, 604, 888, 1309, 1930, 2818, 4096, 5929, 8564, 12358, 17730, 25327, 36038, 51162, 72345, 101566, 141268, 195021, 266907, 361965,	2764989	74	479001600	[8 0 5 11 2 3 10 6 7 9 4 1], [1 11 5 10 2 7 8 0 9 6 3], [1 2 7 8 6 9 3 5 10 0 4], [10 0 1 7 11 8 4 2 3 6], [10 2 1 7 9 8 4 0 3 5 11 6], [1 8 6 9 11 3 5 4 0 10]

Table 6: Diameter of the Cayley graph of S_n with M2 generators.

n	**Diameter**	**n**	**Diameter**
3	3	9	40
4	4	10	49
5	8	11	62
6	16	12	74
7	22	13	87
8	30		

Table 7: M2 Cayley graph eigenvalues for different n (format: value^{multiplicity})

n	Eigenvalues (value ^{multiplicity})
3	$-3.0^1,\ -1.0^2,\ 1.0^2,\ 3.0^1$
4	$-4.0^1,\ -2.0^4,\ 0.0^6,\ 2.0^4,\ 4.0^1$
5	$-5.0^1,\ -3.0^5,\ -1.0^9,\ 1.0^9,\ 3.0^5,\ 5.0^1$
6	$-6.0^1,\ -4.0^9,\ -2.0^{25},\ 0.0^{36},\ 2.0^{25},\ 4.0^9,\ 6.0^1$
7	$-6.0^1,\ -5.0^{30},\ -4.0^{19},\ 4.0^{19},\ 5.0^{30},\ 6.0^1$

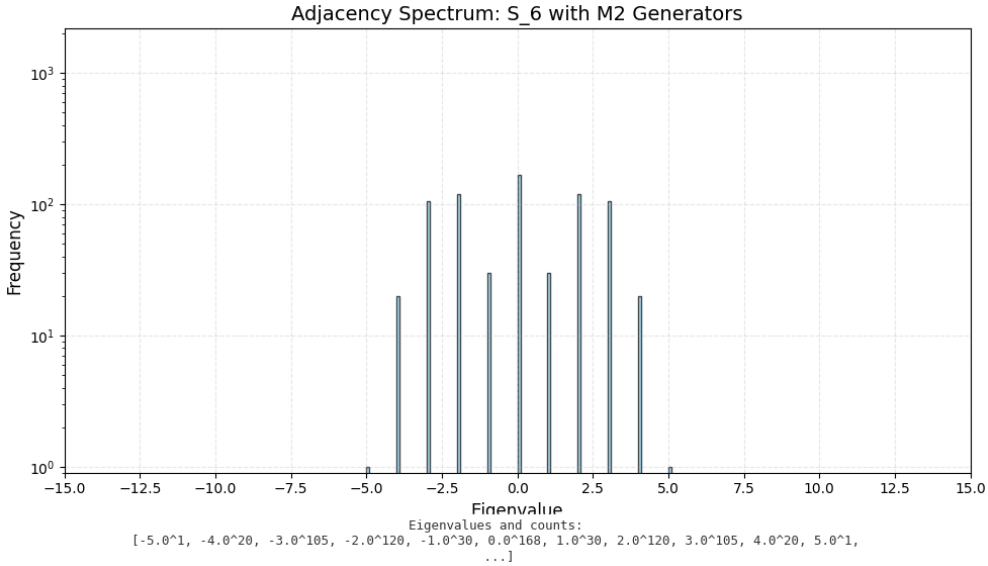


Figure 22: Histogram of the adjacency spectrum for the Cayley graph of S_4 with M2 generators.

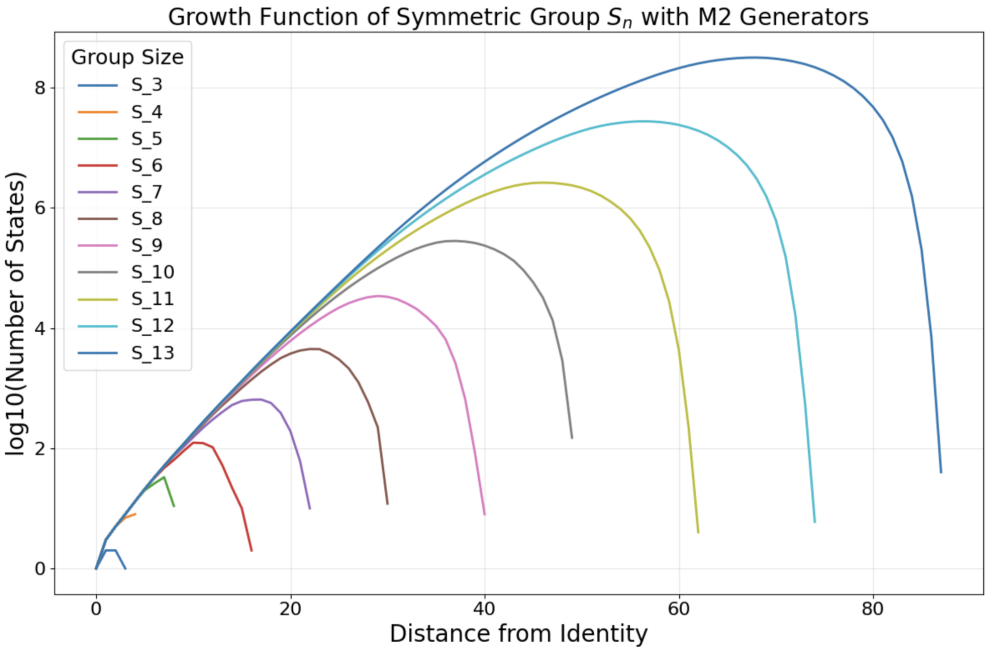


Figure 23: Growth function for S_n with M2 generators (log-scale)

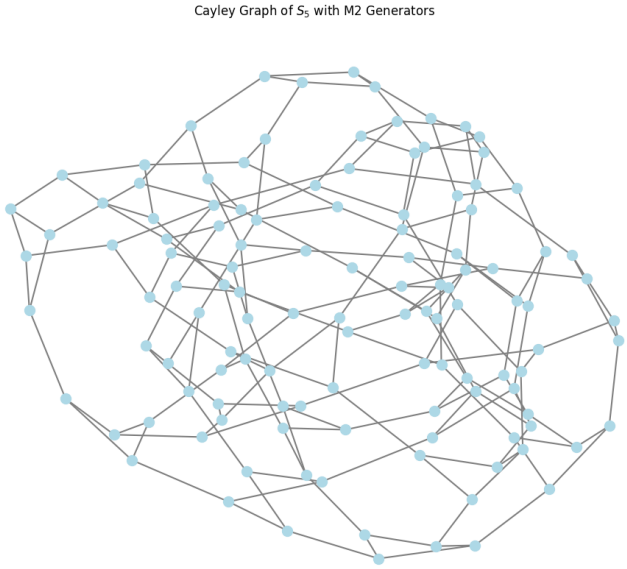


Figure 24: A 2D visualization of the Cayley graph for S_5 with the M2 generator set

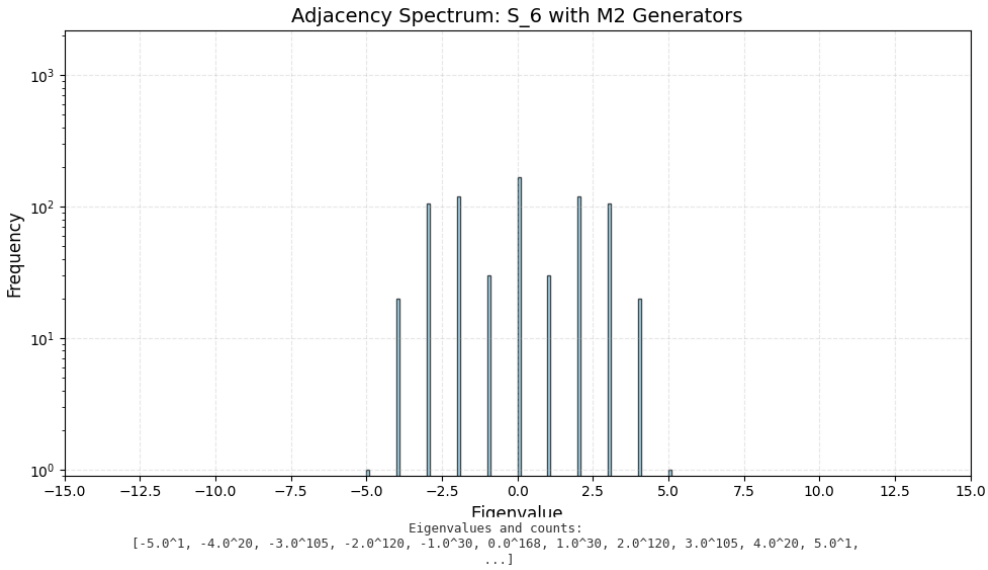


Figure 25: Histogram of the adjacency spectrum for the Cayley graph of S_4 with M2 generators.

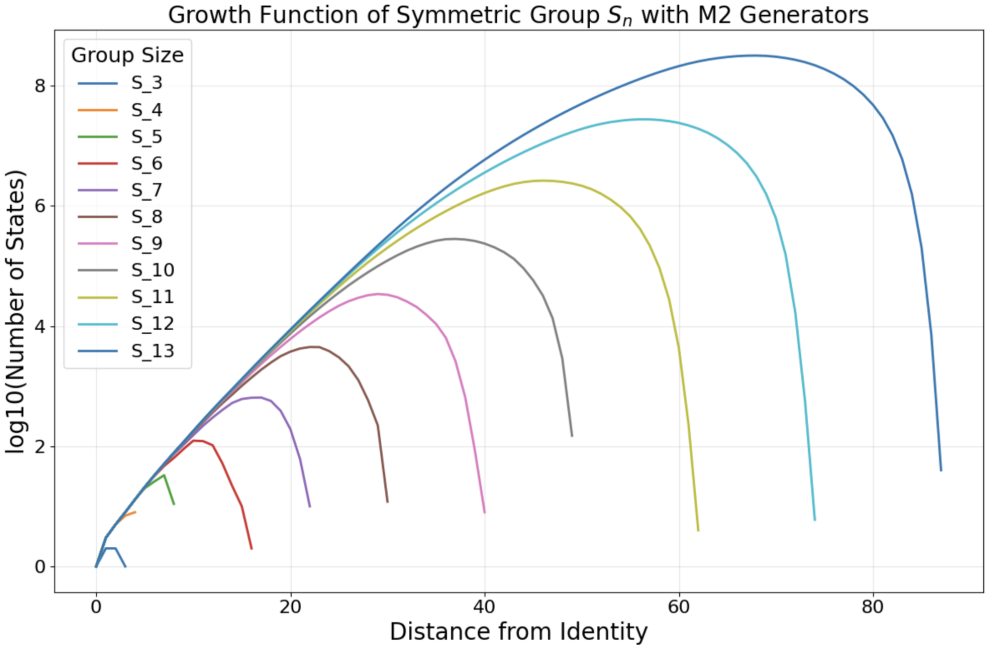


Figure 26: Growth Function of Symmetric Group S_n with M2 Generators

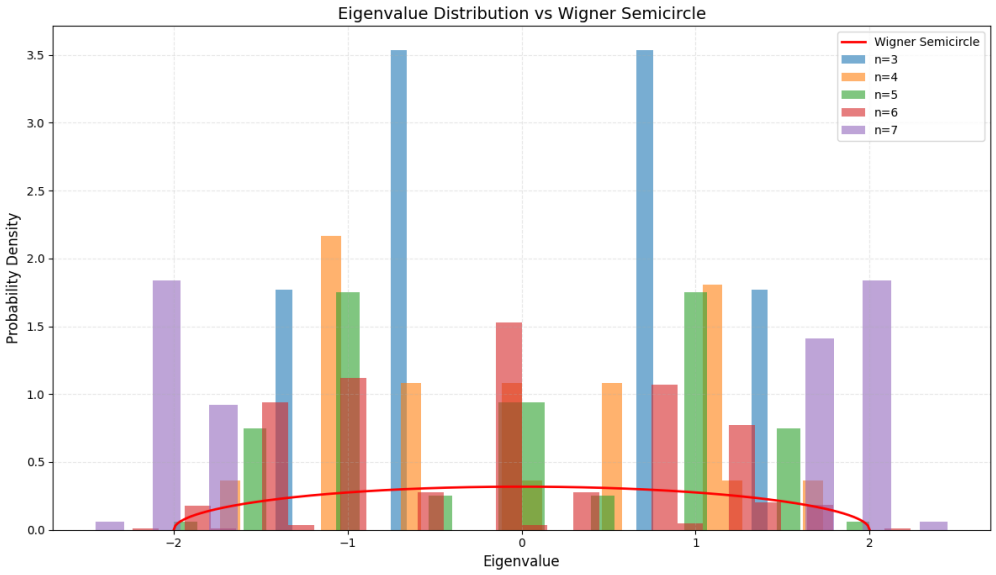


Figure 27: Histogram of eigenvalues with Weigner circle

7.18 Rubik's cubes

For the Rubik's Cube, there are different ways to count moves, which affect both the number of generators and the structure of the corresponding subgroup.

Several well-established move-counting conventions exist, based on the physical structure of the puzzle. These are commonly referred to as move count metrics or turn metrics:

- Half Turn Metric (HTM): In this metric, a single move corresponds to a rotation of any face by either 90 or 180 degrees, and it always counts as one move. Thus, the turn of any inner layer is counted as two moves in HTM.
- Quarter Turn Metric (QTM): One move is defined as a 90-degree clockwise or counterclockwise rotation of any face.
- Slice Turn Metric (STM): Here, one move corresponds to a rotation of any layer – that is, faces and inner layers (slices) – by either 90 or 180 degrees. Every such rotation is counted as a single move.

It is important to note that whole cube rotations do not count as moves in any of the above metrics.

A well-known approach is to consider the Rubik's Cube solution problem from the Cayley graph point of view. In this approach, the vertices represent the states of the puzzle (elements of the underlying permutation group). Any of the metrics described above corresponds to a specific set of generators for this group, and these generators define the edges of the Cayley graph.

Thus, the diameter of this Cayley graph provides the "solution diameter," called the *God's Number*. It is the greatest number of moves required to solve the puzzle from any possible state. For the metrics described above, the known results for the 3×3×3 Rubik's Cube are as follows: HTM — the God's Number is 20; QTM — the God's Number is 26; STM — the God's Number is between 18 and 20.

When defining the God's number, it is assumed that the orientation of the whole cube is fixed —i.e., states obtained by rotations of the entire cube are considered distinct.

[37]

- OEIS sequence for growth function for 2x2x2 cube QTM fixed corners: https://oeis.org/A079762
- OEIS sequence for growth function for 2x2x2 cube QTM fixed corners: https://oeis.org/A080630
- OEIS sequence for growth function for 3x3x3 cube QTM: https://oeis.org/A080602
- OEIS sequence for growth function for 3x3x3 cube HTM: https://oeis.org/A080601

Cube	Generators		Size (number of configurations)	Diameter	Growth
	Type	Number			
2x2x2	QTM fixed corners	6	Yes	14	1, 6, 27, 120, 534, 2256, 8969, 33058, 114149, 360508, 930588, 1350852, 782536, 90280, 276
2x2x2	QTM full	12	Yes	14	1, 12, 114, 924, 6539, 39528, 199926, 806136, 2761740, 8656152, 22334112, 32420448, 18780864, 2166720, 6624
3x3x3	QTM		Yes	26	1, 12, 114, 1068, 10011, 93840, 878880, 8221632, 76843595, 717789576, 6701836858, 62549615248, 583570100997, 5442351625028, 50729620202582, 472495678811004, 4393570406220123, 40648181519827392, 368071526203620348, about 3000000000000000000, about 14000000000000000000, about 7000000000000000000, about 240000000000000000, about 150000
3x3x3	HTM		Yes	20	1, 18, 243, 3240, 43239, 574908, 7618438, 100803036, 1332343288, 17596479795, 232248063316, 3063288809012, 40374425656248, 531653418284628, 6989320578825358, 91365146187124313, about 1100000000000000000, about 12000000000000000000, about 29000000000000000000, about 150000000000000000, about 490000000
3x3x3 Max			No		

7.19 TopSpin (Oval track puzzle)

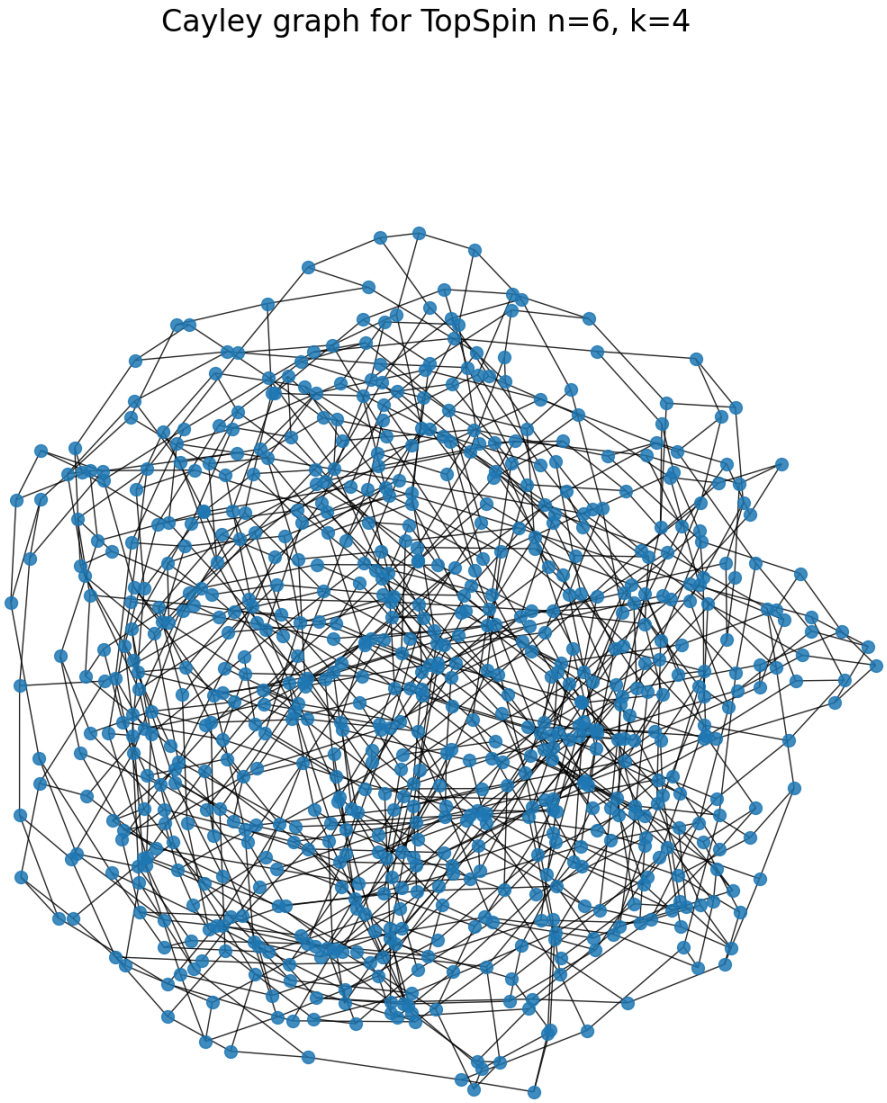


Figure 28: Top Spin Puzzle

The oval track group $OT_{n,k}$ is a group that represents the well-known puzzle called "Top Spin" (also known as the "Oval Track"), where each puzzle arrangement (yellow plates in Figure 49) is viewed as an element of the symmetric group S_n. In the more general setting, one can translate n plates around the track, and the turntable allows flipping k plates at a time. These two basic moves — translation L and flip X — generate a subgroup of S_n, which is called the oval track group $OT_{n,k}$.

In the general case of the Top Spin puzzle game, the problem of a "broken" puzzle is completely solved [51]. A puzzle is considered broken when all the plates in a track are fall out and must be placed back. The question is how to return the plates so that the resulting configuration of a track lies within the subgroup generated by the allowed moves — in other words, how to avoid trivial cases or placements that make the

puzzle unsolvable. Such cases are referred to as *degenerate.*

The degenerate cases [51] include all instances with $1 \leq n \leq 3$, as well as any case where $k = 1$, $k = n$, or $k = n - 1$. In these cases, the oval track group takes the following forms: $OT_{n,1} \cong \mathbb{Z}_n$, $OT_{2,2} \cong S_2$ and for $n \geq 3$ the group is given by $OT_{n,n} \cong OT_{n,n-1} \cong D_n$.

Solvability of the oval track group was also established, and $OT_{n,k}$ is described through various types of subgroups of S_n. Here, more precisely, A_n is the classical subgroup of even permutations. Additionally, a special type of subgroup of S_n appears, which is denoted by PS_n and consists of the following permutations:

1. Type I: Permutations that act independently on the odd-numbered positions and even-numbered positions. The permutations of the odd positions form a group denoted by $S^{\text{odd}}n/2$, and those on the even positions form $S^{\text{even}}n/2$. Thus, all such permutations lie in the group $S^{\text{odd}}n/2 \times S^{\text{even}}n/2$.

2. Type II: Permutations that map every odd-numbered position to an even-numbered one and vice versa.

There is also a subgroup $A^{\text{odd}}n/2 \times A^{\text{even}}n/2$ of S_n, which consists of the compositions of even permutations of Type I.

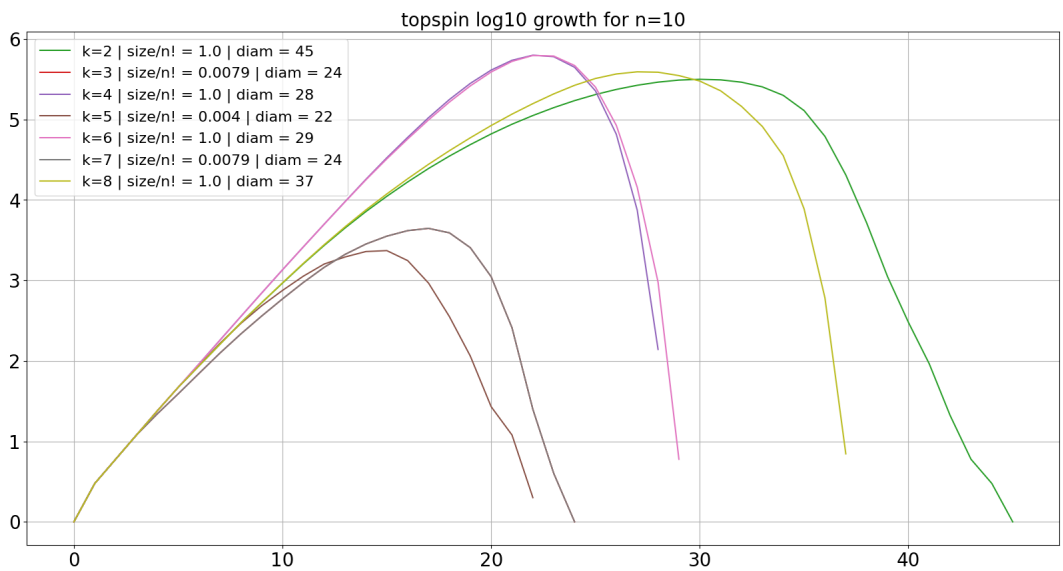


Figure 29: Top Spin growth for n=10

Solvability [51] was shown for parameters $n \geq 4$ and $1 < k < n - 1$, and the oval track group $OT_{n,k}$ is described in the following theorem:

Theorem. *If $n \geq 4$ and $1 < k < n - 1$, then*

1. $OT_{n,k} \cong S_n$ if n is even and k is even, or if n is odd and $k \equiv 2, 3 \pmod 4$.

2. $OT_{n,k} \cong A_n$ if n is odd and $k \equiv 0, 1 \pmod 4$.

3. $OT_{n,k} \cong PS_n$ if n is even and $k \equiv 3 \pmod 4$.

4. $OT_{n,k} = H \cup \tau H$, where $H = S^{\text{odd}}n/2 \times S^{\text{even}}n/2 \cap A_n$, if n is even and $k \equiv 5 \pmod 8$, or if $n \equiv 0 \pmod 4$ and $k \equiv 1 \pmod 8$. In this case, $OT_{n,k}$ consists of permutations of Type I whose overall parity is even, i.e., compositions of the form (even) (even) or (odd) (odd). The order of this group is exactly $(n/2)! \cdot (n/2)!$.

5. $OT_{n,k} = K \cup \tau K$, where $K = A^{\text{odd}}n/2 \times A^{\text{even}}n/2$. So $OT_{n,k}$ in this case consists only of permutations of Type I formed by composing even permutations on both sides, i.e., (even) (even), if $n \equiv 2 \pmod 4$ and $k \equiv 1 \pmod 8$. The order of this group is exactly $\frac{(n/2)! \cdot (n/2)!}{2}$.

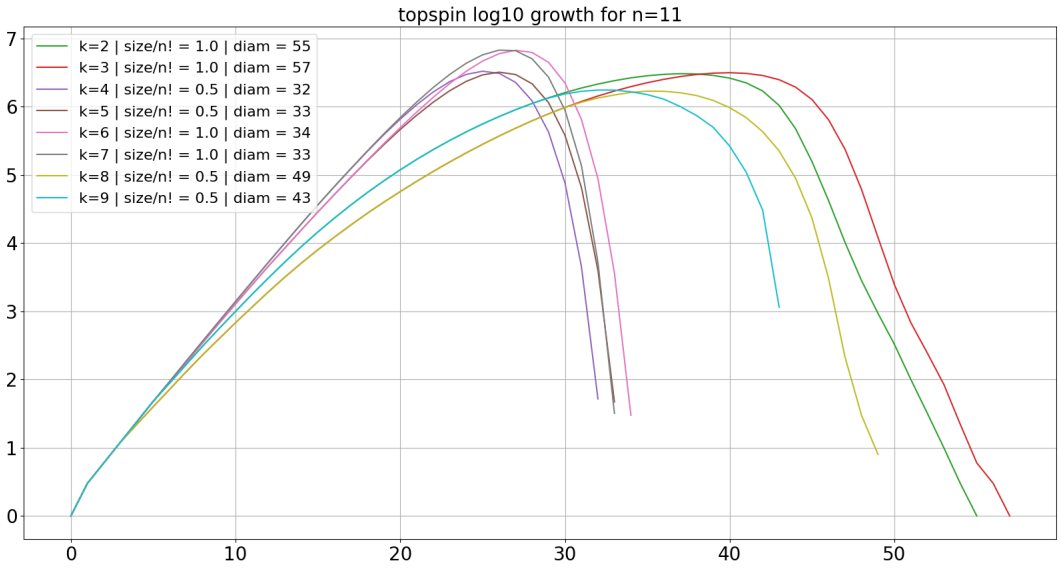


Figure 30: Top Spin growth for n=11

The puzzle is discussed in: [50], [27], [65], [44], [12], [24], [39], [40], [25], [51], [9] and web-pages Jamie Mulholand: "Oval Track puzzle", Jaap Scherphuis, "Topspin", Ivars Peterson ("Math Tourist") , the first two web-pages contain Java-applets to virtually play the puzzles.

Relation of TopSpin with prefix reversal generators has been analysed in [14].

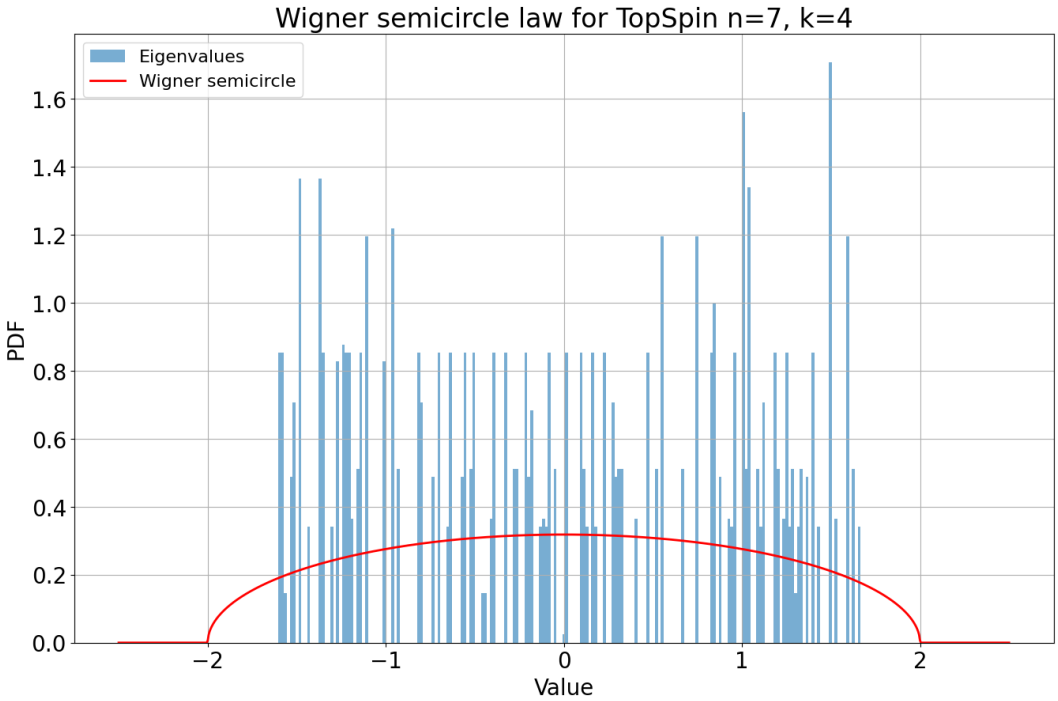


Figure 31: Topspin spectrum

n	k	*Diameter*	*Growth*
4	2	6	1, 3, 5, 6, 5, 3, 1
5	2	10	1, 3, 6, 10, 16, 24, 29, 21, 6, 3, 1
5	3	10	1, 3, 6, 10, 16, 24, 29, 21, 6, 3, 1
6	2	15	1, 3, 6, 11, 20, 35, 55, 81, 109, 128, 126, 95, 40, 6, 3, 1
6	3	8	1, 3, 6, 11, 16, 17, 12, 5, 1
6	4	12	1, 3, 6, 11, 20, 37, 63, 103, 165, 152, 106, 46, 7
7	2	21	1, 3, 6, 12, 22, 42, 73, 124, 203, 303, 425, 559, 678, 713, 746, 611, 355, 122, 28, 10, 3, 1
7	3	21	1, 3, 6, 12, 20, 34, 58, 96, 158, 234, 334, 456, 562, 672, 731, 737, 583, 280, 45, 12, 5, 1
7	4	17	1, 3, 6, 12, 20, 34, 58, 96, 158, 234, 334, 430, 410, 326, 258, 112, 24, 4
7	5	15	1, 3, 6, 12, 22, 42, 73, 124, 203, 303, 425, 475, 402, 265, 144, 20
8	2	28	1, 3, 6, 12, 23, 44, 80, 142, 247, 411, 662, 1019, 1481, 2059, 2745, 3465, 4126, 4633, 4913, 4777, 4163, 3079, 1612, 488, 94, 25, 6, 3, 1
8	3	15	1, 3, 6, 12, 21, 36, 61, 96, 146, 199, 231, 201, 100, 27, 10, 2
8	4	22	1, 3, 6, 12, 23, 44, 85, 148, 244, 389, 621, 996, 1584, 2484, 3764, 5452, 6968, 7138, 6292, 3036, 848, 166, 16
8	5	13	1, 3, 6, 12, 21, 36, 61, 96, 114, 97, 72, 42, 12, 3
8	6	23	1, 3, 6, 12, 23, 44, 80, 144, 255, 433, 722, 1167, 1797, 2682, 3894, 5180, 6018, 6087, 5306, 3636, 1888, 771, 170, 1
9	2	36	1, 3, 6, 12, 24, 46, 85, 156, 283, 492, 845, 1415, 2258, 3476, 5188, 7448, 10303, 13816, 17827, 22311, 26810, 31091, 34664, 36935, 37516, 35651, 31354, 23197, 13382, 4680, 1212, 268, 88, 25, 8, 3, 1
9	3	38	1, 3, 6, 12, 22, 38, 68, 120, 206, 349, 570, 914, 1412, 2123, 3133, 4504, 6327, 8625, 11505, 15021, 19099, 23613, 28154, 32397, 35872, 37701, 37249, 34491, 28822, 18996, 8738, 2270, 381, 85, 31, 12, 6, 3, 1
9	4	23	1, 3, 6, 12, 24, 46, 88, 168, 322, 613, 1137, 2095, 3745, 6385, 10697, 17134, 24461, 30412, 32269, 27633, 17101, 5884, 1112, 92
9	5	23	1, 3, 6, 12, 24, 46, 88, 168, 322, 613, 1137, 2095, 3779, 6482, 10912, 16945, 24168, 30555, 32373, 27657, 16711, 6123, 1138, 82
9	6	38	1, 3, 6, 12, 22, 38, 68, 120, 206, 349, 570, 914, 1412, 2123, 3133, 4504, 6327, 8625, 11505, 15021, 19099, 23613, 28154, 32397, 35872, 37701, 37249, 34491, 28822, 18996, 8738, 2270, 381, 85, 31, 12, 6, 3, 1
9	7	36	1, 3, 6, 12, 24, 46, 85, 156, 283, 492, 845, 1415, 2258, 3476, 5188, 7448, 10303, 13816,

n	k	max excess over n*(n-1) / 2 for fixed k
4	2	0
5	2,3	0
6	2,3,4	0
7	2,3,4,5	0
8	2,3,4,5,6	0
9	3, 6	2
10	2,3,4,5,6,7,8	0
11	3	2
12	2,3,4,5,6,7,8,9,10	0
13	3, 10	4

Table 8: Excess over n(n-1)/2 estimated upper bound for diameter

Dima: (Note: for odd n, $n >= 7$, exactly $n!/2$ permutations are reachable)

n	k	Diameter	Growth (Incomplete)
15	4	33	1, 3, 6, 12, 24, 48, 96, 192, 382, 759, 1501, 2971, 5875, 11574, 22749, 44631, 87012, 168879, 326130, 626788, 1198995, 2285881, 4338282, 8197342, 15422382, 28889568, 53838080, 99789865, 183849006, 336320443, 609856717, 1094267586, 1938747096, 3382275643
16	4	32	1, 3, 6, 12, 24, 48, 96, 192, 383, 761, 1506, 2983, 5905, 11646, 22921, 45025, 87904, 170904, 330779, 637558, 1223349, 2340954, 4462266, 8475406, 16044513, 30275496, 56933890, 106700721, 199314350, 370957082, 687500108, 1268310412, 2328066376
17	4	32	1, 3, 6, 12, 24, 48, 96, 192, 384, 763, 1511, 2995, 5933, 11710, 23069, 45355, 88638, 172533, 334361, 645527, 1241125, 2380402, 4547928, 8660991, 16444730, 31133332, 58760160, 110572517, 207507035, 388241511, 723928986, 1345076107, 2489975239
18	4	32	1, 3, 6, 12, 24, 48, 96, 192, 384, 764, 1513, 3000, 5945, 11738, 23135, 45513, 88998, 173345, 336197, 649644, 1250549, 2402100, 4597757, 8773233, 16694580, 31688822, 59988716, 113267013, 213387763, 401039805, 751740415, 1405396502, 2620712418
19	4	32	1, 3, 6, 12, 24, 48, 96, 192, 384, 765, 1515, 3005, 5957, 11766, 23199, 45663, 89334, 174093, 337861, 653315, 1258643, 2420025, 4637715, 8862610, 16891302, 32117759, 60918097, 115273891, 217693311, 410215033, 771206257, 1446531796, 2707295840
20	4	31	1, 3, 6, 12, 24, 48, 96, 192, 384, 765, 1516, 3007, 5962, 11778, 23227, 45727, 89486, 174439, 338639, 655057, 1262507, 2428594, 4656935, 8906353, 16991500, 32343993, 61422983, 116395625, 220173994, 415660897, 783095519, 1472350772

7.20 Globes (similar to "Masterball" puzzles)

The "Globe" was introduced on Kaggle Santa 2023 challenge. It is similar to "Masterball" puzzle, but not exactly it. It does not have mechanical realization in contrast to "Masterball".

The structure of the generated subgroup was described by Jaap Scherphuis MathStackexchange 2024

A	B	Diameter	Growth
1	1	4	1, 4, 10, 8, 1
2	1	5	1, 5, 14, 18, 9, 1
3	1	7	1, 6, 23, 72, 179, 202, 85, 8
4	1	8	1, 7, 29, 95, 251, 381, 287, 93, 8
5	1	10	1, 8, 40, 170, 596, 1656, 3544, 4502, 2695, 576, 36
6	1	11	1, 9, 48, 210, 766, 2252, 5200, 8046, 7197, 3271, 612, 36
7	1	12	1, 10, 61, 320, 1434, 5208, 15438, 38464, 79797, 107558, 71173, 11936, 376
8	1	13	1, 11, 71, 381, 1754, 6642, 20646, 53902, 118261, 187355, 178731, 83109, 12312, 376
9	1	15	1, 12, 86, 530, 2865, 12822, 47138, 145836, 389675, 905110, 1725096, 2303362, 1751771, 612440, 64680, 1200
1	2	14	1, 8, 48, 220, 575, 1100, 1952, 3520, 5312, 6304, 7168, 7264, 5056, 1568, 224
2	2	16	1, 10, 65, 324, 1063, 2470, 4727, 8524, 14304, 20448, 25088, 27904, 26752, 18944, 8416, 2016, 224

A	B	Diameter	Growth (Incomplete)
?	?	?	??

7.21 Hungarian Rings ("Devil's circles", "wreath")

[37] (chapter 4.2). Group structure has been determined in Jaap Scherphuis on Mathematics Stack Exchange 4842187 .

A	B	Diameter	Growth
?	?	?	??

A	B	Diameter	Growth (Incomplete)
?	?	?	??

7.22 Generalized Olympic Rings

7.23 All cycles (Birkhoff polytope graphs)

The Birkhoff polytope graph, denoted as $G(\Omega_n)$, is formed from the symmetric group of degree n, which consists of all possible permutations of n items. The vertices of this graph are these permutations, and two vertices are connected if one can be transformed into the other by a specific operation involving cycles. The graph is vertex transitive, Hamiltonian, and regular with diameter 2.

A clique in a graph is defined as a subset of vertices such that every two distinct vertices in the subset are adjacent. The clique number of a graph, denoted as $\omega(G)$, is the size of the largest clique within that graph. This number provides insight into the structure and connectivity of the graph. The exact clique number $\omega(G)$ of the Birkhoff polytope graph remains unknown due to the intricate structure of the graph, which is based on the symmetric group Sym(n) with n! vertices. Adjacency in this graph is defined via geometric relations from the Birkhoff polytope, making the analysis significantly more complex than in standard algebraic graphs.

A specific class of cliques, known as k-cycle cliques, consisting of cycles of fixed length k, has been partially explored. In particular, maximal k-cycle cliques have been studied for small values of k, such as 2 and 3, providing insights into the local structure of the graph. However, the general clique number $\omega(G)$ remains an open problem, with only lower and upper bounds known.[33]

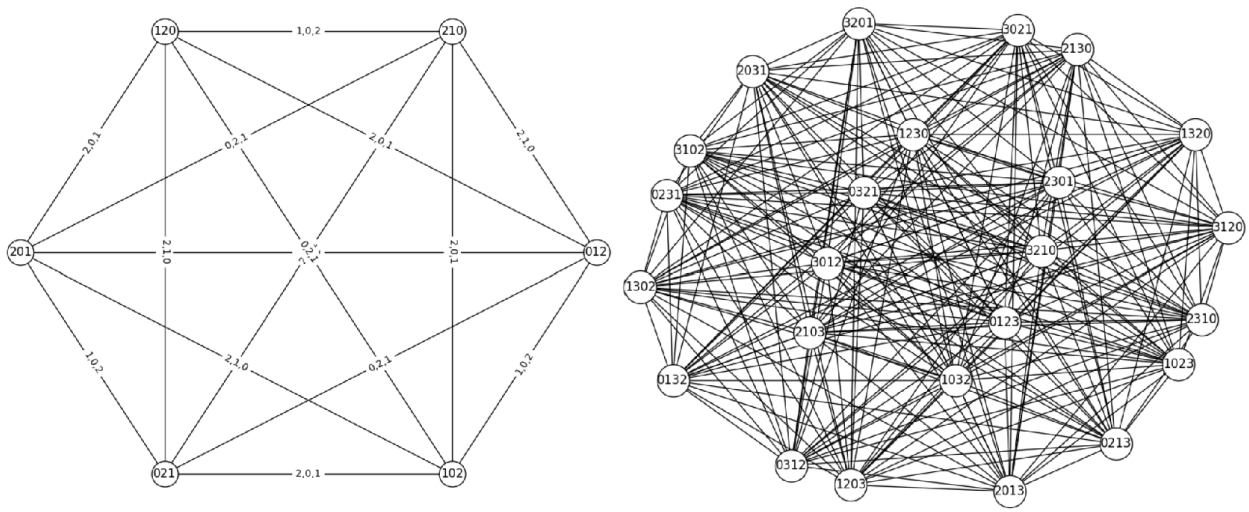


Figure 32: Graphs for n = 3, 4

n	Generators
2	10
3	102, 201, 021, 120, 210
4	1023, 2103, 3120, 0231, 0321, 0132, 1203, 2013, 1320, 3021, 2130, 3102, 0231, 0312, 1230, 1302, 2310, 2031, 3201, 3012
5	10234, 21034, 31204, 41230, 02134, 03124, 04123, 01324, 01432, 01243, 12034, 20134, 13204, 30214, 14230, 40231, 21304, 31024, 21430, 41032, 31240, 41203, 02314, 03124, 02431, 04123, 03241, 04213, 01342, 01423, 12304, 13024, 23104, 20314, 32014, 30124, 12430, 14032, 24130, 20431, 42031, 40132, 13240, 14203, 34210, 30241, 43201, 40213, 21340, 21403, 31420, 31042, 41302, 41023, 02341, 02413, 03421, 03142, 04312, 04132, 12340, 12403, 13420, 13042, 14302, 14023, 23140, 24103, 24310, 20341, 23401, 20413, 32410, 34201, 34012, 30421, 32041, 30142, 24301, 43021, 43120, 30412, 32014, 30124

Table 9: Generators for $n = 2$ to $n = 5$

n	Eigenvalues (value ^{multiplicity})
3	-1.0^{5}, 5.0^{1}
4	-4.0^{5}, 0.0^{18}, 20.0^{1}
5	-8.0^{25}, -6.0^{16}, 0.0^{25}, 4.0^{53}, 84.0^{1}
6	-41.0^{1}, -23.0^{25}, -21.0^{81}, -11.0^{181}, 4.0^{256}, 7.0^{25}, 13.0^{25}, 19.0^{125}, 409.0^{1}
7	-85.0^{36}, -77.0^{196}, -65.0^{196}, -29.0^{400}, -23.0^{1225}, -17.0^{196}, 5.0^{441}, 19.0^{1225}, 23.0^{441}, 25.0^{196}, 41.0^{225}, 83.0^{261}, 223.0^{1}, 2365.0^{1}

Table 10: Eigenvalues for different n (format: value^{multiplicity})

n	Growth
2	1, 1
3	1, 5
4	1, 20, 3
5	1, 84, 35
6	1, 409, 310
7	1, 2365, 2674
8	1, 16064, 24255

Table 11: Growth for different n

n	Number of cycles
1	0
2	1
3	5
4	20
5	84
6	409
7	2365
8	16064
9	125664
10	1112073

Table 12: Number of cycles for different n. For more details, see sequence OEIS A006231.

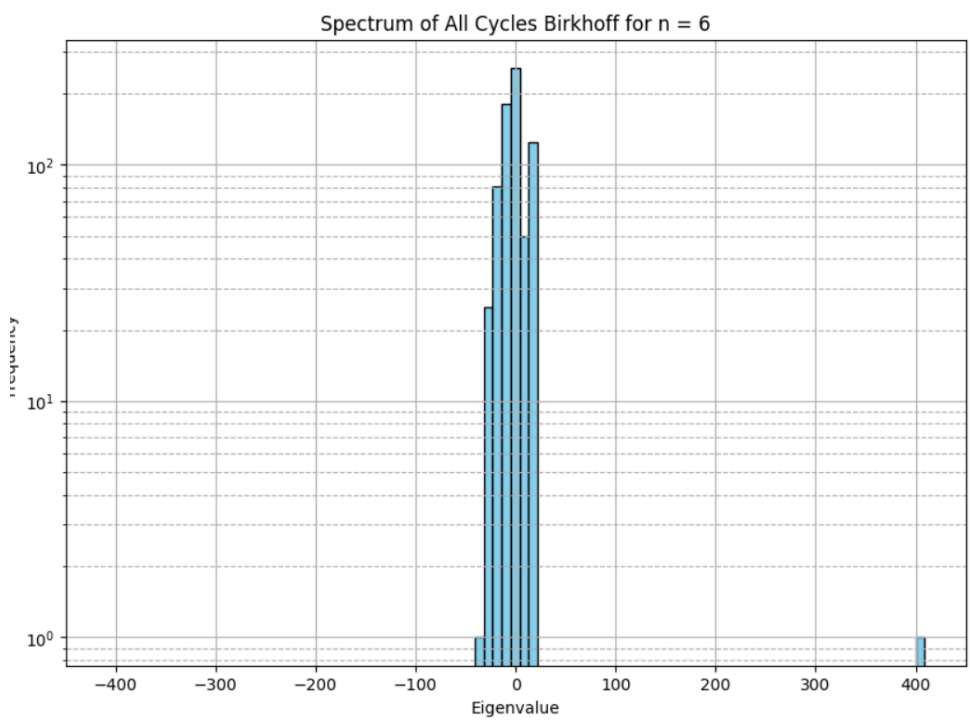


Figure 33: Spectrum for n = 6

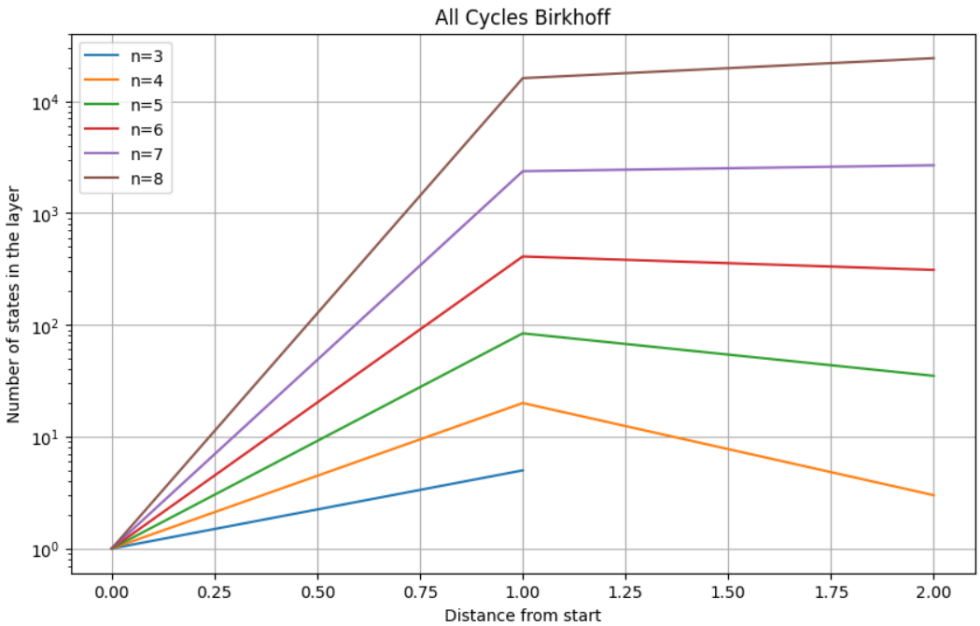


Figure 34: Growth for Birkhoff cycles

7.24 Derangements

A derangement is a permutation of a finite set where no element appears in its original position, representing a fundamental concept in combinatorics with significant theoretical and practical applications. The number of derangements of n elements, denoted as $D(n)$ or $!n$, follows the recursive formula $D(n) = (n-1)[D(n-1)+D(n-2)]$ and can be approximated by $n!/e$ for large n, where the probability that a random permutation is a derangement approaches $1/e \approx 0.368$ as n increases. This concept extends naturally to partial derangement, where exactly k elements remain in their original positions, leading to the inclusion-exclusion principle's elegant application in counting such arrangements.

n	Generators
2	[1, 0]
3	[1, 2, 0], [2, 0, 1]
4	[1, 0, 3, 2], [1, 2, 3, 0], [1, 3, 0, 2], [2, 0, 3, 1], [2, 3, 0, 1], [2, 3, 1, 0], [3, 0, 1, 2], [3, 2, 0, 1], [3, 2, 1, 0]
5	[1, 0, 3, 4, 2], [1, 0, 4, 2, 3], [1, 2, 0, 4, 3], [1, 2, 3, 4, 0], [1, 2, 4, 0, 3], [1, 3, 0, 4, 2], [1, 3, 4, 0, 2], [1, 3, 4, 2, 0], [1, 4, 0, 2, 3], [1, 4, 3, 0, 2], [1, 4, 3, 2, 0], [2, 0, 1, 4, 3], [2, 0, 3, 4, 1], [2, 0, 4, 1, 3], [2, 3, 0, 4, 1], [2, 3, 1, 4, 0], [2, 3, 4, 0, 1], [2, 3, 4, 1, 0], [2, 4, 0, 1, 3], [2, 4, 1, 0, 3], [2, 4, 3, 0, 1], [2, 4, 3, 1, 0], [3, 0, 1, 4, 2], [3, 0, 4, 1, 2], [3, 0, 4, 2, 1], [3, 2, 0, 4, 1], [3, 2, 1, 4, 0], [3, 2, 4, 0, 1], [3, 2, 4, 1, 0], [3, 4, 0, 1, 2], [3, 4, 0, 2, 1], [3, 4, 1, 0, 2], [3, 4, 1, 2, 0], [4, 0, 1, 2, 3], [4, 0, 3, 1, 2], [4, 0, 3, 2, 1], [4, 2, 0, 1, 3], [4, 2, 1, 0, 3], [4, 2, 3, 0, 1], [4, 2, 3, 1, 0], [4, 3, 0, 1, 2], [4, 3, 0, 2, 1], [4, 3, 1, 0, 2], [4, 3, 1, 2, 0]

Table 13: Generators for n=2 to n=5

n	Eigenvalues with multiplicities
2	1.00, -1.00
3	2.00, -1.00^2
4	9.00, 3.00^4, 1.00^9, -3.00^10
5	44.00, 4.00^62, -1.00^16, -4.00^25, -11.00^16
6	265.00, 15.00^81, 13.00^100, 7.00^25, 5.00^81, 1.00^25, -5.00^357, -11.00^25, -53.00^25

Table 14: Eigenvalues for n=2 to n=6

n	Layer sizes
2	1, 1
3	1, 2
4	1, 9, 14
5	1, 44, 75
6	1, 265, 454
7	1, 1854, 3185
8	1, 14833, 25486

Table 15: Layer sizes for n=2 to n=8

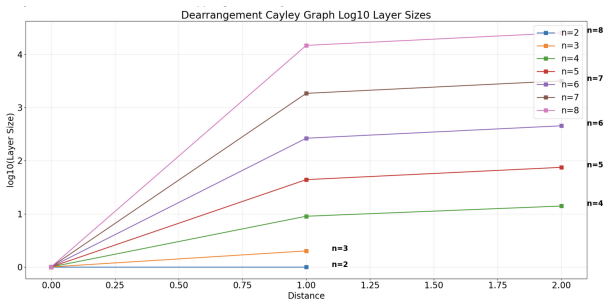


Figure 35: Dearrangement Cayley Graph growth for n=2 to n=8

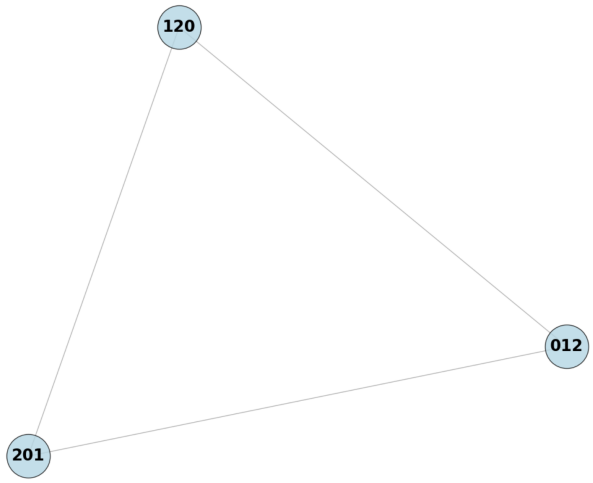


Figure 36: Dearrangement Cayley Graph for n=3

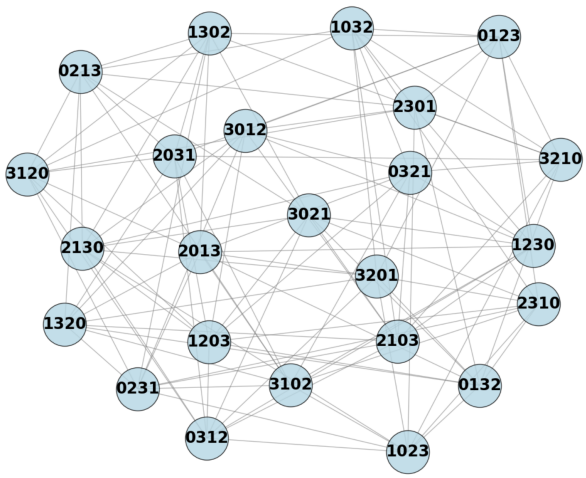


Figure 37: Dearrangement Cayley Graph for n=4

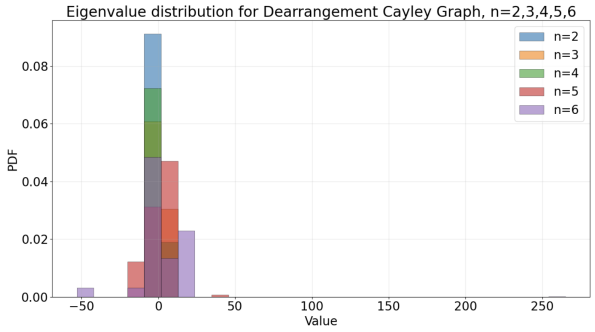


Figure 38: Eigenvalue distribution for DearranpassLeaf1gements Cayley Graph, n=2,3,4,5,6

7.25 Involutive derangements (Breaking Enigma - WW2 encryption system)

Enigma

8 Pancake (prefix reversals) and variations

8.1 Pancake (prefix reversals)

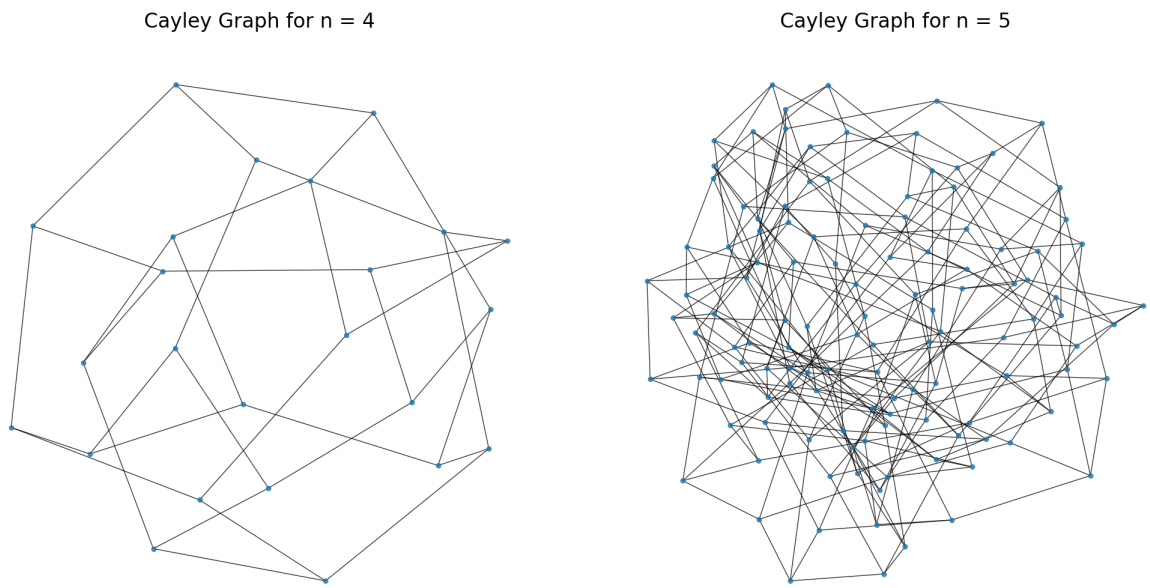


Figure 39: Cayley Graph for $n = 4, 5$

The Pancake graph $P_n = Cay(\mathrm{Sym}_n, PR), n \geqslant 2$, is the Cayley graph over the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2 \ldots \pi_n]$ written as strings in one-line notation, where $\pi_i = \pi(i)$ for any $1 \leqslant i \leqslant n$, with the generating set $PR = \{r_i \in \mathrm{Sym}_n : 2 \leqslant i \leqslant n\}$, $|PR| = n - 1$, of all prefix reversals r_i inversing the order of any substring $[1, i]$, $2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e. $[\underline{\pi_1 \ldots \pi_i}\pi_{i+1} \ldots \pi_n]r_i = [\underline{\pi_i \ldots \pi_1}\pi_{i+1} \ldots \pi_n]$. This graph is well known because of the open combinatorial Pancake problem of finding its diameter [22]. A stack of n pancakes is represented by a permutation on n elements and the problem is to find the least number of flips (prefix reversals) needed to transform a permutation into the identity permutation. This number of flips corresponds to the diameter D of the Pancake graph. The known diameters D for P_n, $4 \leqslant n \leqslant 19$, are given as follows [2, 18]:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
D	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22

At the first, bounds on the diameter D of the Pancake graph were obtained by Gates and Papadimitriou [26] as $17n/16 \leqslant D \leqslant (5n + 5)/3$. Improved upper and lower bounds are known as $15n/14 \leqslant D$ and $D \leqslant 18n/11$. These bounds were obtained by Heydari and Sudborough [31], and by Sudborough in cooperation with a team of researchers at University of Texas at Dallas [15], respectively.

In molecular biology and computer science, a pancake stack is considered as a data structure due to the NP–hard problem of sorting by prefix reversals also known as pancake flipping/sorting [11].

The Pancake graph P_n is a connected $(n-1)$-regular graph of order $n!$. It is almost pancyclic since it contains cycles of length ℓ, $6 \leqslant \ell \leqslant n!$, but doesn't contain cycles of length $3, 4$ or 5 [38, 58]. Let the metric sphere $S_i = \{v \in V(P_n),\ d(I_n, v) = i\}$ contains all vertices being at distance i, $0 \leqslant i \leqslant D$, from the identity permutation $I_n = [1\ 2 \ldots n]$ in the Pancake graph P_n. Due to the cyclic structure of P_n, it is obvious that $|S_0| = 1$, $|S_1| = n-1$, and $|S_2| = (n-1)(n-2)$ for any $n \geqslant 3$. For any $n \leqslant 16$, the cardinalities $|S_i|$, $0 \leqslant i \leqslant D$ are shown as *Growth* in the table below.

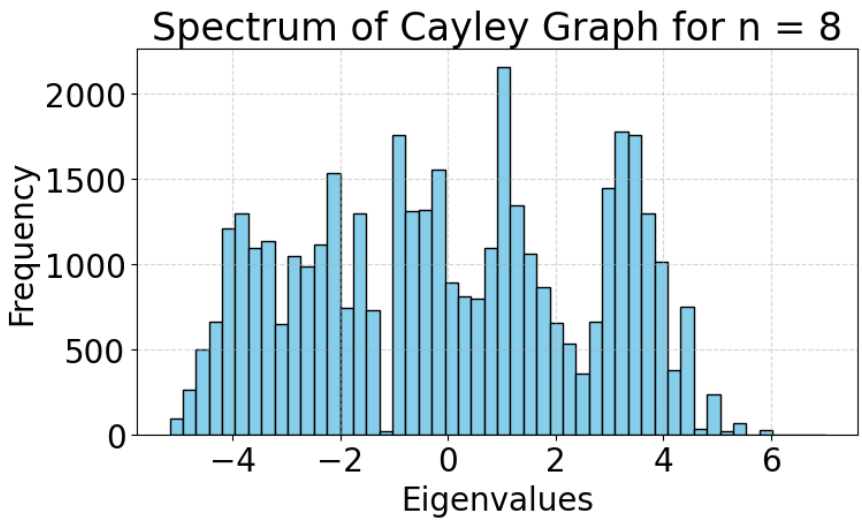


Figure 40: Spectrum Cayley

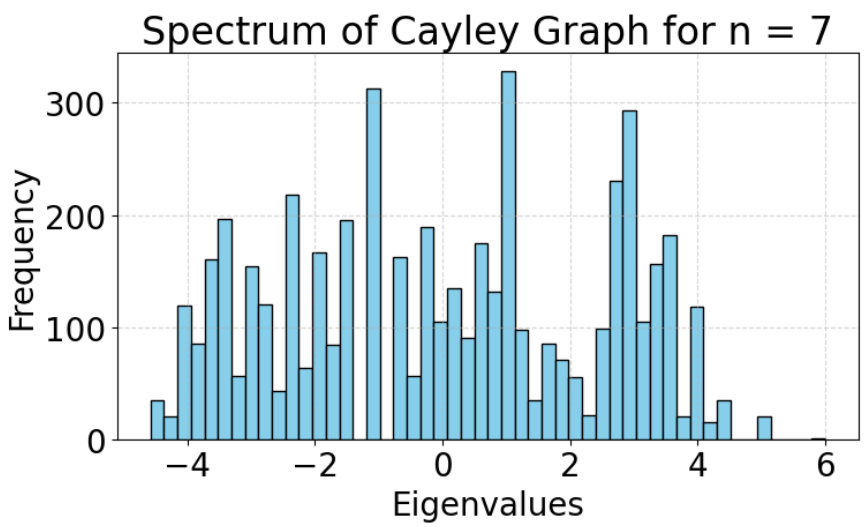


Figure 41: Spectrum Cayley

n	Full	Diameter D	Growth
1	Yes	0	1
2	Yes	1	1,1
3	Yes	3	1, 2, 2, 1
4	Yes	4	1, 3, 6, 11, 3
5	Yes	5	1, 4, 12, 35, 48, 20
6	Yes	7	1, 5, 20, 79, 199, 281, 133, 2
7	Yes	8	1, 6, 30, 149, 543, 1357, 1903, 1016, 35
8	Yes	9	1, 7, 42, 251, 1191, 4281, 10561, 15011, 8520, 455
9	Yes	10	1, 8, 56, 391, 2278, 10666, 38015, 93585, 132697, 79379, 5804
10	Yes	11	1, 9, 72, 575, 3963, 22825, 106461, 377863, 919365, 1309756, 814678, 73232
11	Yes	13	1, 10, 90, 809, 6429, 43891, 252737, 1174766, 4126515, 9981073, 14250471, 9123648, 956354, 6
12	Yes	14	1, 11, 110, 1099, 9883, 77937, 533397, 3064788, 14141929, 49337252, 118420043, 169332213, 111050066, 13032704, 167
13	Yes	15	1, 12, 132, 1451, 14556, 130096, 1030505, 7046318, 40309555, 184992275, 639783475, 1525125357, 2183056566, 1458653648, 186874852, 2001
14	Yes	16	1, 13, 156, 1871, 20703, 206681, 1858149, 14721545, 100464346, 572626637, 2609061935, 8950336881, 21189628403, 30330792508, 20584311501, 2824234896, 24974
15	No	No	1, 14, 182, 2365, 28603, 315305, 3169675, 28528986, 226016576, 1539349289, 8728663805, 39491264561
16	No	No	1, 15, 210, 2939, 38559, 465001, 5165641, 52027677, 468966948, 3713254017
17	No	No	1, 16, 240, 3599, 50898, 666342, 8102491, 90238067, 911274131, 8219206485
18	No	No	1, 17, 272, 4351, 65971, 931561, 12301949, 150044655, 1677036683
19	No	No	1, 18, 306, 5201, 84153, 1274671, 18161133, 240665410, 2947991636
20	No	No	1, 19, 342, 6155, 105843, 1711585, 26163389, 374193014, 4982872288

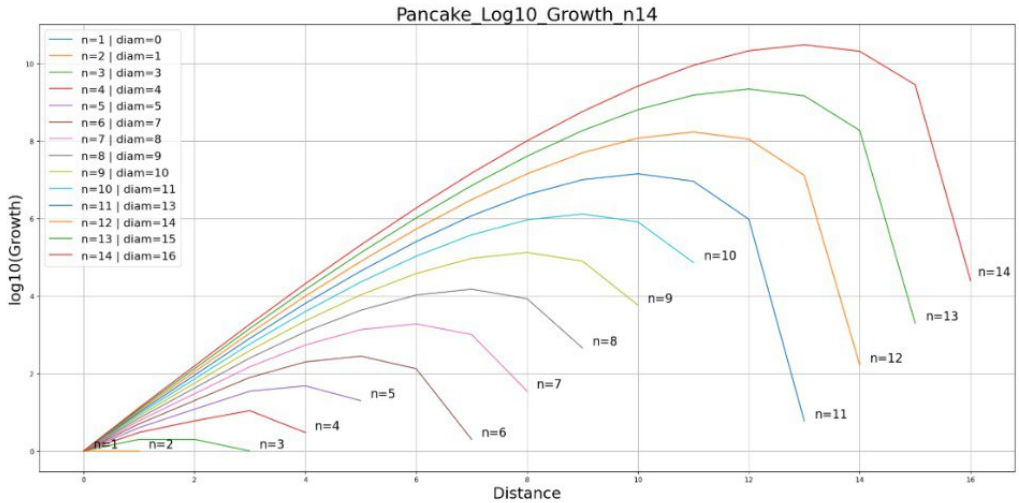
Dima: (OEIS sequence for growth function: https://oeis.org/A092113, computed up to N=12, which means we can extend it. OEIS sequence for size of the last layer: https://oeis.org/A067607, computed up to N=17.)

Growth for N= 16...20 was computed on two processors Intel(R) Xeon(R) Platinum 8480+ and RAM 2TB.

8.1.1 Pancake of zeros and ones

Any sequence of k zeros and l ones can be transformed into 0^k1^l in at most $2l$ or into 1^l0^k in at most $2k$ moves. In the first case we simply deliver every 1 onto the first place, then throw it onto the place of the last zero. The second case is treated similarly.

Figure 42: Pancake Log10 growth for n = 14:



8.2 Burnt Pancake

n	Full	Diameter D	Growth
7	Yes	15	1, 7, 42, 252, 1386, 6230, 24024, 71568, 159326, 222995, 136301, 21951, 1021, 15, 1
8	Yes	16	1, 8, 56, 392, 2548, 14056, 68656, 276136, 901970, 2195663, 3531887, 2743477, 562095, 24627, 347, 1
9	Yes	18	1, 9, 72, 576, 4320, 28224, 166740, 843822, 3636954, 12675375, 33773653, 60758618, 57953163, 15244962, 701298, 6721, 51, 1
10	No	No	1, 10, 90, 810, 6885, 51960, 359928, 2193534, 11738418, 53257425, 198586153, 570362563, 1138201788
11	No	No	1, 11, 110, 1100, 10450, 89430, 710358, 5060220, 32328648, 180577749, 865120003, 3417630744
12	No	No	1, 12, 132, 1452, 15246, 145860, 1306448, 10645866, 79016157, 523887705, 3063740977
13	No	No	1, 13, 156, 1872, 21528, 227656, 2269410, 20812077, 175905015, 1349251761
14	No	No	1, 14, 182, 2366, 29575, 342524, 3760484, 38319281, 363216425, 3162393520
15	No	No	1, 15, 210, 2940, 39690, 499590, 5988892, 67117596, 704903862, 6865359930
16	No	No	1, 16, 240, 3600, 52200, 709520, 9220512, 112694400, 1298683412
17	No	No	1, 17, 272, 4352, 67456, 984640, 13787272, 182483644, 2288941268

8.3 Cubic pancake graphs (only 3 prefix reversals)

The cubic Pancake graph $P_n^3 = Cay(\mathrm{Sym}_n, S)$ is the Cayley graph over the symmetric group whose generating set S contains only three prefix reversals. Any cubic Pancake graph is a subgraph of the Pancake graph P_n. Cubic Pancake graphs were considered by Bass and Sudborough [4] as models for interconnections networks (see also [56, 66] of restricted degree. The authors have found the following necessary conditions for a set of three prefix–reversals to generate the symmetric group:

- one of the prefix-reversals should be of size n; otherwise, there is no way to move an element to/from position n;
- one of the prefix-reversals should be of size i, where $\lfloor \frac{n}{2} \rfloor < i < n$; otherwise, there is no way to move an element from the first half of a permutation to the second half;
- one of the prefix-reversal should be of even size; otherwise, no element can be moved from an odd-numbered position to an even-numbered position at a permutation;
- one of the prefix-reversals should be an odd permutation; otherwise, one can only get the set of all even permutations giving the alternating group A_n;
- the generating set should include at least two prefix-reversals with relatively prime sizes; otherwise, if $k > 1$ is the greatest common divisor of the prefix-reversals sizes then some of the blocks of k contiguous elements can never be separated from each other.

The known list of the generating sets under the necessary conditions is as follows [4, 47, 59]:

$S_{2,-1} = \{r_n, r_{n-1}, r_2\}$ for any $n \geqslant 4$;

$S_{3,-1} = \{r_n, r_{n-1}, r_3\}$ for any odd $n \geqslant 5$;

$S_{-1,-2} = \{r_n, r_{n-1}, r_{n-2}\}$ for any $n \geqslant 4$;

$S_{-1,-3} = \{r_n, r_{n-1}, r_{n-3}\}$ for any odd $n \geqslant 4$;

$S_{2,-2} = \{r_n, r_{n-2}, r_2\}$ for any odd $n \geqslant 5$;

$S_{3,-2} = \{r_n, r_{n-2}, r_3\}$ for any even $n \geqslant 6$;

$S_{-2,-3} = \{r_n, r_{n-2}, r_{n-3}\}$ for any $n \geqslant 5$.

The set $S_{2,-2}$ was found in [59] while others were found in [4]. The set $S_{-1,-2}$ is known as 'big-3' flips, and the corresponding cubic Pancake graph is called as the Big-3 Pancake network. It was conjectured in [56] that this graph has cyclic Gray codes. The diameters of the cubic Pancake graphs generated by the sets above are unknown.

set	n	Diameter	Growth
S_1	4	4	1, 3, 6, 11, 3
S_1	5	7	1, 3, 6, 12, 22, 39, 28, 9
S_1	6	12	1, 3, 6, 12, 22, 40, 70, 114, 150, 177, 94, 30, 1
S_1	7	18	1, 3, 6, 12, 22, 40, 71, 121, 200, 314, 477, 697, 891, 895, 663, 390, 187, 48, 2
S_1	8	24	1, 3, 6, 12, 22, 40, 71, 122, 207, 338, 536, 840, 1300, 1939, 2787, 3845, 5070, 6150, 6205, 5082, 3701, 1699, 307, 34, 3
S_1	9	30	1, 3, 6, 12, 22, 40, 71, 122, 208, 345, 557, 889, 1408, 2195, 3342, 5001, 7375, 10649, 14893, 20199, 26801, 34692, 42804, 48226, 47089, 40720, 30348, 17373, 6267, 1149, 73
S_1	10	39	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 564, 910, 1454, 2296, 3568, 5493, 8401, 12715, 18923, 27682, 39805, 56377, 78575, 107587, 144662, 190511, 245003, 305290, 363016, 404374, 414888, 387986, 327040, 239876, 148601, 70337, 19787, 2164, 72, 12
S_1	11	47	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 565, 917, 1475, 2339, 3663, 5699, 8830, 13589, 20678, 31108, 46454, 68852, 100892, 145627, 207037, 290692, 403518, 552945, 746292, 989772, 1288885, 1650531, 2077980, 2558937, 3051000, 3486024, 3791691, 3895687, 3758592, 3388082, 2812244, 2097947, 1357627, 714248, 273434, 64695, 7235, 186
S_1	12	57	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 565, 918, 1482, 2360, 3706, 5791, 9033, 14001, 21503, 32763, 49702, 75102, 112873, 168257, 248336, 362739, 524683, 751603, 1065445, 1492788, 2065463, 2821039, 3803264, 5062826, 6657292, 8648694, 11089756, 14025509, 17494555, 21493500, 25922016, 30494343, 34800439, 38295048, 40517664, 41127195, 39804330, 36457037, 31359066, 24951971, 17883333, 11105799, 5623803, 2061714, 447723, 42597, 1139, 4
S_1	13	67	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 565, 918, 1483, 2367, 3727, 5834, 9122, 14198, 21899, 33540, 51230, 78078, 118491, 178825, 268344, 400557, 594721, 876945, 1281740, 1855953, 2663397, 3787781, 5335388, 7437259, 10251742, 13973003, 18843557, 25157662, 33252748, 43501324, 56318890, 72170783, 91549435, 114940310, 142793712, 175451728, 213011859, 255053445, 300310863, 346553244, 390250126, 427795461, 455618211, 470875937, 470652463, 452403225, 415719298, 363059017, 298000720, 226566075, 156102138, 94120666, 46598554, 16949519, 3757958, 382030, 11846, 58
S_1	14	78	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 565, 918, 1483, 2368, 3734, 5855, 9165, 14287, 22093, 33933, 51994, 79569, 121375, 184284, 278538, 419623, 630449, 944194, 1407559, 2085533, 3068713, 4483260, 6501274, 9352976, 13340097, 18851959, 26388893, 36589347, 50265028, 68440816, 92383544, 123607699, 163911755, 215486791, 280946794, 363303827, 465963252, 592722579, 747666017, 935193250, 1159921200, 1426368507, 1738672745, 2099810124, 2510408167, 2967175806, 3460926127, 3975099578, 4486205249, 4964902508, 5380529510, 5700056835, 5894355083, 5932993524, 5792641218, 5463201949, 4955477451, 4299553033, 3543991786, 2749798307, 1978010758, 1287198115, 726802165, 330551093, 107018713, 20161781, 1653937, 39524, 186

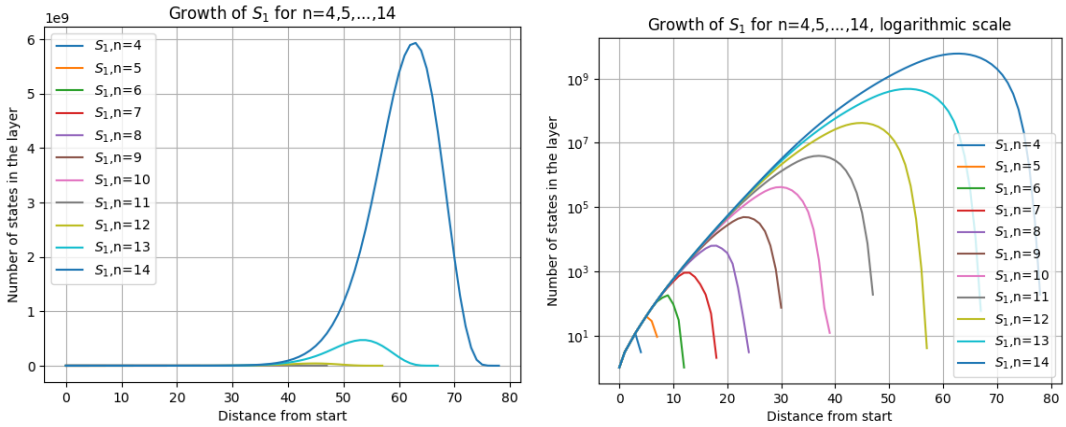


Figure 43: S_1 growth

set	n	Diameter	Growth
S_2	4	4	1, 2, 2, 2, 1
S_2	5	8	1, 3, 6, 12, 21, 21, 24, 21, 11
S_2	6	6	1, 3, 6, 12, 21, 22, 7
S_2	7	16	1, 3, 6, 12, 22, 40, 71, 122, 199, 302, 440, 637, 873, 1020, 862, 384, 46
S_2	8	13	1, 3, 6, 12, 22, 40, 71, 123, 187, 232, 232, 159, 57, 7
S_2	9	30	1, 3, 6, 12, 22, 40, 71, 123, 210, 347, 558, 881, 1366, 2071, 3075, 4488, 6426, 9060, 12694, 17676, 24201, 32187, 41164, 48490, 51557, 47560, 35071, 17891, 5019, 588, 22
S_2	10	21	1, 3, 6, 12, 22, 40, 71, 123, 210, 348, 567, 916, 1449, 2128, 3043, 4109, 4910, 4605, 3580, 1996, 596, 65
S_2	11	49	1, 3, 6, 12, 22, 40, 71, 123, 210, 348, 568, 925, 1493, 2365, 3681, 5671, 8666, 13100, 19558, 28841, 42035, 60631, 86673, 122884, 172643, 239880, 329288, 446810, 600233, 799127, 1053252, 1370335, 1752934, 2191421, 2665003, 3139487, 3552935, 3840981, 3931337, 3751771, 3296190, 2627679, 1852793, 1113660, 542642, 196984, 45894, 5320, 272, 2
S_2	12	33	1, 3, 6, 12, 22, 40, 71, 123, 210, 348, 568, 926, 1502, 2401, 3773, 5855, 9009, 13731, 20702, 30711, 44567, 62692, 84178, 106807, 126654, 139233, 138276, 116081, 76403, 36967, 12160, 2456, 298, 14
S_2	13	70	1, 3, 6, 12, 22, 40, 71, 123, 210, 348, 568, 926, 1503, 2410, 3809, 5953, 9227, 14195, 21707, 33034, 49984, 75134, 112198, 166392, 244880, 357407, 517257, 742817, 1059407, 1500288, 2106810, 2929513, 4032176, 5498419, 7436007, 9975202, 13266676, 17486593, 22848135, 29611961, 38090835, 48653768, 61726278, 77774355, 97278327, 120709272, 148503153, 180903653, 217904653, 258931988, 302393991, 346169186, 386730542, 420680797, 445050273, 457618591, 455623224, 438000957, 404113071, 355400754, 294965200, 226625832, 156890695, 94567448, 47371075, 18295642, 4990465, 859090, 82106, 4000, 155
S_2	14	46	1, 3, 6, 12, 22, 40, 71, 123, 210, 348, 568, 926, 1503, 2411, 3818, 5989, 9319, 14385, 22069, 33714, 51258, 77471, 116285, 173391, 256988, 378314, 551935, 795280, 1126871, 1566414, 2131611, 2825077, 3626280, 4504641, 5307633, 5811373, 5838499, 5319809, 4320466, 3018720, 1738401, 801036, 282990, 73331, 12361, 1169, 58

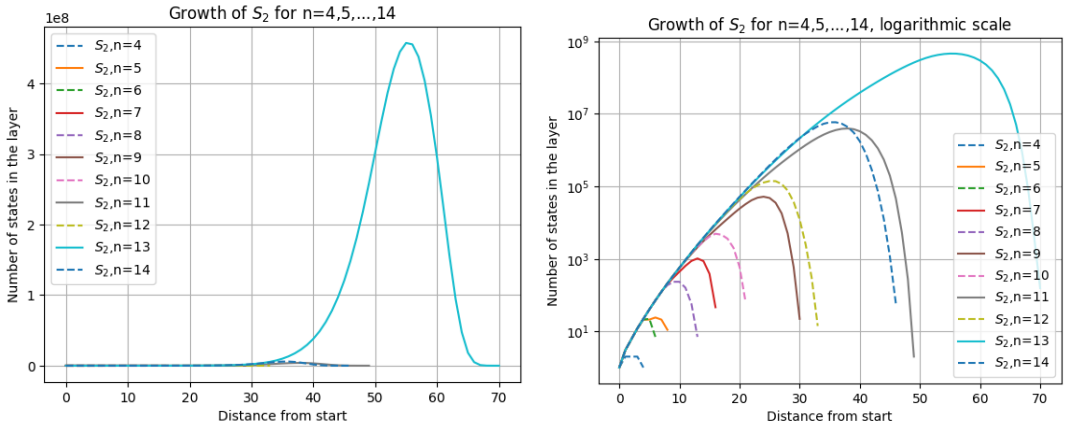


Figure 44: S_2 growth

S_3	4	4	1, 3, 6, 11, 3
S_3	5	8	1, 3, 6, 12, 21, 21, 24, 21, 11
S_3	6	12	1, 3, 6, 12, 22, 39, 66, 108, 172, 163, 102, 24, 2
S_3	7	15	1, 3, 6, 12, 22, 40, 73, 129, 222, 381, 625, 884, 1092, 1011, 475, 64
S_3	8	21	1, 3, 6, 12, 22, 40, 74, 135, 242, 429, 745, 1278, 2096, 3185, 4652, 6379, 7329, 6653, 4593, 2026, 395, 25
S_3	9	27	1, 3, 6, 12, 22, 40, 74, 136, 249, 450, 797, 1415, 2441, 4084, 6707, 10374, 15487, 22541, 31417, 41601, 51495, 56772, 52108, 37609, 19612, 6336, 1031, 60
S_3	10	35	1, 3, 6, 12, 22, 40, 74, 136, 250, 456, 818, 1467, 2578, 4507, 7742, 12825, 20851, 32620, 49697, 74116, 107678, 151731, 207388, 275776, 356789, 436979, 483687, 473262, 401045, 285614, 160262, 63957, 14512, 1703, 180, 16
S_3	11	42	1, 3, 6, 12, 22, 40, 74, 136, 250, 457, 825, 1488, 2634, 4661, 8146, 14070, 24000, 39437, 63646, 100292, 154755, 234816, 348884, 506290, 718977, 998206, 1356768, 1806566, 2351202, 2985117, 3680992, 4296387, 4630254, 4533705, 4008753, 3155148, 2129629, 1157006, 459855, 122474, 19416, 1356, 44
S_3	12	50	1, 3, 6, 12, 22, 40, 74, 136, 250, 457, 826, 1494, 2656, 4717, 8303, 14579, 25306, 43203, 72940, 119306, 191823, 303545, 472825, 724620, 1092170, 1614885, 2344571, 3343215, 4683761, 6455848, 8745809, 11637027, 15227951, 19563833, 24691830, 30579774, 37000755, 43066509, 47302595, 48584920, 46597857, 41433894, 33558879, 23987820, 14678564, 7317814, 2732964, 673768, 92265, 5120, 58
S_3	13	59	1, 3, 6, 12, 22, 40, 74, 136, 250, 457, 826, 1495, 2663, 4738, 8364, 14754, 25785, 44863, 77390, 130749, 218449, 356317, 572368, 908189, 1422297, 2194723, 3338456, 4994983, 7361783, 10684216, 15276415, 21542939, 29949089, 41036859, 55455270, 73877378, 97075461, 125731809, 160622680, 202365713, 251384514, 307780341, 370858525, 436620783, 496248585, 538918539, 557559490, 548957332, 511033276, 443486945, 353725464, 255294711, 162678781, 87283426, 36657668, 10948910, 2075045, 203417, 6994, 32
S_3	14	68	1, 3, 6, 12, 22, 40, 74, 136, 250, 457, 826, 1495, 2664, 4744, 8387, 14814, 25967, 45480, 79029, 136290, 233237, 390728, 647299, 1053604, 1693180, 2690750, 4229184, 6554479, 10031239, 15139079, 22541245, 33131753, 48060103, 68867450, 97482790, 136292249, 188281421, 257001914, 346768700, 462349415, 609499886, 794363446, 1023730501, 1304513488, 1643800194, 2047712704, 2522331093, 3070739778, 3695416671, 4389440438, 5124318858, 5828745054, 6417008773, 6819553904, 6992736415, 6886100586, 6461902981, 5732048203, 4780428852, 3707980300, 2622142529, 1636394874, 863101365, 362985302, 112276447, 22623792, 2515384, 117541, 1325

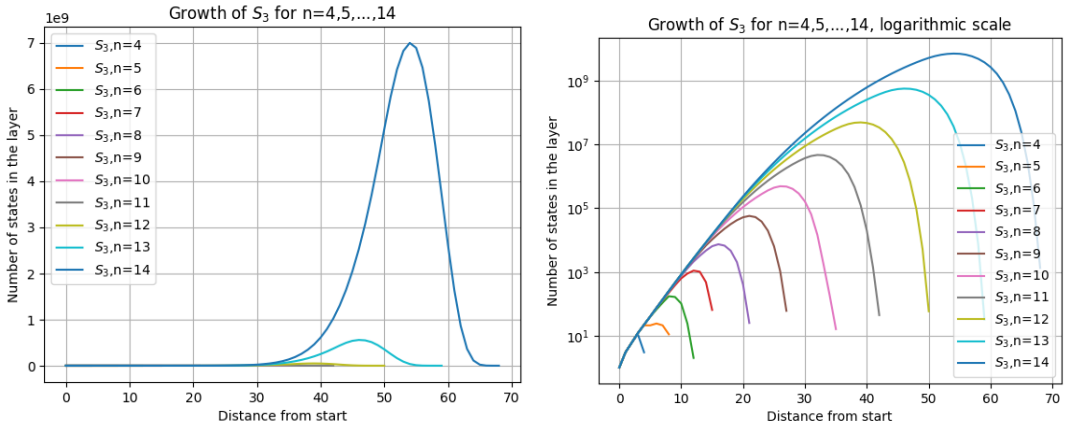


Figure 45: S_3 growth

S_4	4	4	1, 2, 2, 2, 1
S_4	5	7	1, 3, 6, 12, 22, 39, 28, 9
S_4	6	6	1, 3, 6, 12, 21, 22, 7
S_4	7	15	1, 3, 6, 12, 24, 48, 86, 151, 262, 428, 656, 902, 1106, 978, 354, 23
S_4	8	12	1, 3, 6, 12, 24, 48, 86, 132, 200, 272, 262, 96, 10
S_4	9	26	1, 3, 6, 12, 24, 48, 89, 164, 306, 565, 1024, 1824, 3143, 5068, 7897, 12105, 17969, 25790, 35717, 46974, 56102, 57120, 47839, 29531, 11599, 1878, 82
S_4	10	20	1, 3, 6, 12, 24, 48, 90, 168, 310, 566, 1029, 1829, 3032, 4522, 5720, 5703, 3748, 1624, 343, 21, 1
S_4	11	40	1, 3, 6, 12, 24, 48, 90, 168, 318, 598, 1112, 2066, 3815, 6828, 11917, 20410, 34091, 55384, 88017, 137022, 209273, 312538, 456428, 652403, 914881, 1257584, 1693157, 2231497, 2876467, 3607022, 4287708, 4712690, 4725955, 4290445, 3427460, 2262255, 1162181, 397232, 72797, 4848, 49
S_4	12	28	1, 3, 6, 12, 24, 48, 90, 168, 317, 594, 1107, 2063, 3822, 6932, 12302, 21285, 35978, 58823, 91903, 132854, 171453, 185245, 158217, 99446, 42170, 10627, 1264, 44, 2
S_4	13	56	1, 3, 6, 12, 24, 48, 90, 168, 318, 598, 1118, 2091, 3906, 7212, 13162, 23749, 42288, 73748, 126401, 212595, 351309, 570732, 911941, 1432323, 2215180, 3370872, 5046623, 7439001, 10813489, 15502586, 21919248, 30567815, 42058976, 57078537, 76403267, 100870177, 131374047, 168740640, 213728205, 266873852, 328353245, 396978959, 466358119, 525539495, 565417142, 580368920, 563081576, 508887561, 423579867, 320173979, 213241222, 116933119, 47374594, 11619008, 1295420, 42019, 197
S_4	14	37	1, 3, 6, 12, 24, 48, 90, 168, 318, 598, 1123, 2111, 3947, 7303, 13394, 24316, 43638, 77230, 134678, 229989, 385845, 634347, 1018835, 1595169, 2423191, 3534305, 4900905, 6325978, 7391074, 7574085, 6573053, 4577635, 2349952, 797085, 164807, 17283, 653, 1

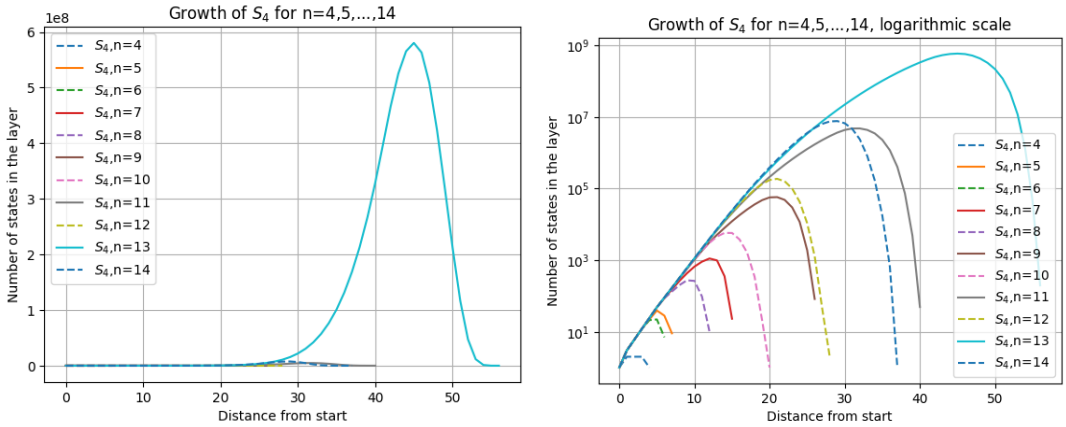


Figure 46: S_4 growth

S_5	4	4	1, 2, 2, 2, 1
S_5	5	7	1, 3, 6, 11, 17, 23, 28, 31
S_5	6	6	1, 3, 6, 12, 18, 6, 2
S_5	7	20	1, 3, 6, 12, 20, 30, 44, 63, 91, 133, 195, 265, 352, 470, 627, 763, 825, 724, 339, 67, 10
S_5	8	9	1, 3, 6, 12, 20, 30, 30, 17, 7, 2
S_5	9	42	1, 3, 6, 12, 20, 30, 44, 63, 89, 123, 170, 239, 337, 473, 642, 866, 1177, 1533, 2015, 2693, 3518, 4647, 6143, 7968, 10240, 12931, 16151, 20042, 24617, 29594, 34036, 38030, 39736, 38082, 32319, 20942, 9378, 2936, 748, 180, 74, 28, 4
S_5	10	12	1, 3, 6, 12, 20, 30, 44, 63, 64, 42, 24, 10, 1
S_5	11	64	1, 3, 6, 12, 20, 30, 44, 63, 89, 123, 170, 239, 334, 456, 617, 843, 1160, 1573, 2121, 2865, 3801, 5084, 6589, 8581, 11378, 14683, 19363, 25102, 32946, 43030, 55523, 71191, 90293, 114463, 145227, 183549, 231983, 289354, 361499, 451551, 561839, 697696, 855931, 1047043, 1267182, 1517754, 1806863, 2135997, 2492414, 2846888, 3159913, 3393338, 3486611, 3374780, 3007255, 2395101, 1702071, 1066342, 568487, 246447, 82733, 21748, 5229, 1069, 110
S_5	12	15	1, 3, 6, 12, 20, 30, 44, 63, 89, 112, 122, 113, 85, 50, 17, 1
S_5	13	93	1, 3, 6, 12, 20, 30, 44, 63, 89, 123, 170, 239, 334, 456, 617, 843, 1156, 1568, 2106, 2834, 3816, 5139, 6916, 9225, 12337, 16305, 21602, 28138, 36508, 48067, 61494, 80706, 103935, 136796, 176297, 229375, 297388, 382759, 489218, 622187, 788688, 998710, 1261616, 1590096, 1990593, 2482962, 3092950, 3863778, 4828384, 6000713, 7461014, 9242173, 11392582, 13964358, 17037572, 20717661, 25176862, 30572805, 37122368, 44915486, 54170623, 65086581, 77894136, 92955347, 110866733, 131976017, 155970268, 182740105, 212580268, 245309689, 280783774, 318986845, 358294521, 395701706, 428283809, 451888692, 461020175, 452232419, 423212612, 367954183, 288764683, 199805875, 119877431, 61143025, 26374870, 9932299, 3615508, 1436340, 596957, 215090, 58004, 10818, 1086, 18
S_5	14	17	1, 3, 6, 12, 20, 30, 44, 63, 89, 123, 170, 218, 264, 286, 207, 144, 85, 27

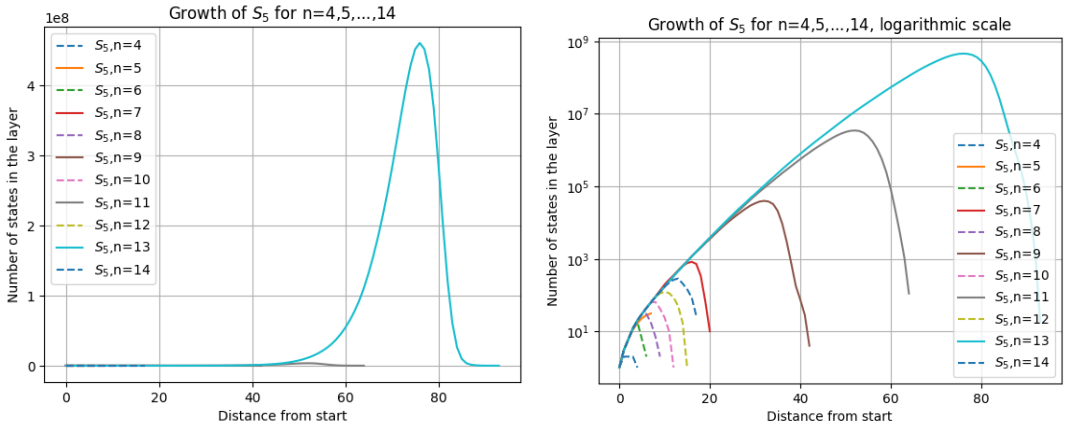


Figure 47: S_5 growth

S_6	4	4	1, 3, 6, 11, 3
S_6	5	6	1, 2, 2, 2, 2, 2, 1
S_6	6	12	1, 3, 6, 12, 22, 40, 70, 114, 150, 177, 94, 30, 1
S_6	7	8	1, 3, 6, 12, 21, 30, 39, 27, 5
S_6	8	24	1, 3, 6, 12, 22, 40, 71, 122, 207, 338, 536, 840, 1300, 1939, 2787, 3845, 5070, 6150, 6205, 5082, 3701, 1699, 307, 34, 3
S_6	9	16	1, 3, 6, 12, 22, 40, 65, 96, 134, 167, 164, 124, 75, 33, 12, 5, 1
S_6	10	39	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 564, 910, 1454, 2296, 3568, 5493, 8401, 12715, 18923, 27682, 39805, 56377, 78575, 107587, 144662, 190511, 245003, 305290, 363016, 404374, 414888, 387986, 327040, 239876, 148601, 70337, 19787, 2164, 72, 12
S_6	11	22	1, 3, 6, 12, 22, 40, 71, 122, 198, 304, 435, 598, 770, 927, 976, 901, 775, 552, 322, 112, 32, 16, 5
S_6	12	57	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 565, 918, 1482, 2360, 3706, 5791, 9033, 14001, 21503, 32763, 49702, 75102, 112873, 168257, 248336, 362739, 524683, 751603, 1065445, 1492788, 2065463, 2821039, 3803264, 5062826, 6657292, 8648694, 11089756, 14025509, 17494555, 21493500, 25922016, 30494343, 34800439, 38295048, 40517664, 41127195, 39804330, 36457037, 31359066, 24951971, 17883333, 11105799, 5623803, 2061714, 447723, 42597, 1139, 4
S_6	13	32	1, 3, 6, 12, 22, 40, 71, 122, 208, 343, 548, 847, 1260, 1799, 2477, 3287, 4188, 5072, 5803, 6299, 6452, 6173, 5302, 4047, 2831, 1771, 917, 377, 139, 43, 14, 5, 1
S_6	14	78	1, 3, 6, 12, 22, 40, 71, 122, 208, 346, 565, 918, 1483, 2368, 3734, 5855, 9165, 14287, 22093, 33933, 51994, 79569, 121375, 184284, 278538, 419623, 630449, 944194, 1407559, 2085533, 3068713, 4483260, 6501274, 9352976, 13340097, 18851959, 26388893, 36589347, 50265028, 68440816, 92383544, 123607699, 163911755, 215486791, 280946794, 363303827, 465963252, 592722579, 747666017, 935193250, 1159921200, 1426368507, 1738672745, 2099810124, 2510408167, 2967175806, 3460926127, 3975099578, 4486205249, 4964902508, 5380529510, 5700056835, 5894355083, 5932993524, 5792641218, 5463201949, 4955477451, 4299553033, 3543991786, 2749798307, 1978010758, 1287198115, 726802165, 330551093, 107018713, 20161781, 1653937, 39524, 186

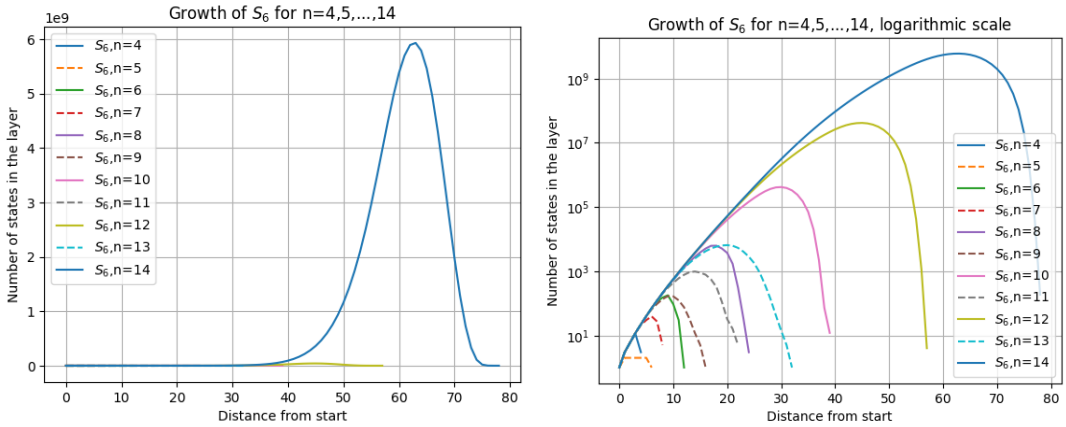


Figure 48: S_6 growth

S_7	4	4	1, 2, 2, 2, 1
S_7	5	7	1, 3, 6, 11, 17, 23, 28, 31
S_7	6	12	1, 3, 6, 12, 22, 40, 70, 114, 150, 177, 94, 30, 1
S_7	7	17	1, 3, 6, 12, 24, 42, 67, 105, 160, 240, 366, 556, 756, 1005, 961, 597, 119, 20
S_7	8	20	1, 3, 6, 12, 24, 48, 91, 168, 310, 567, 984, 1639, 2521, 3839, 5750, 7483, 7711, 5897, 2599, 616, 51
S_7	9	23	1, 3, 6, 12, 24, 48, 95, 187, 365, 710, 1342, 2528, 4651, 8530, 15015, 25557, 40759, 58674, 72594, 70201, 45203, 14870, 1487, 18
S_7	10	27	1, 3, 6, 12, 24, 48, 96, 192, 377, 740, 1458, 2848, 5552, 10737, 20565, 38870, 72628, 132014, 230277, 378215, 561644, 715754, 717458, 498157, 202304, 37017, 1799, 4
S_7	11	32	1, 3, 6, 12, 24, 48, 96, 192, 380, 751, 1473, 2885, 5630, 10830, 20759, 39660, 75229, 141273, 262252, 480170, 862456, 1507771, 2535697, 4015616, 5817238, 7369880, 7615821, 5774702, 2719843, 613166, 42534, 400, 2
S_7	12	36	1, 3, 6, 12, 24, 48, 96, 192, 383, 764, 1514, 3000, 5945, 11674, 22846, 44470, 86325, 166961, 321344, 615003, 1168247, 2198843, 4091706, 7491923, 13395656, 23134297, 37958470, 57674289, 78233168, 90073440, 82128928, 53787348, 21837733, 4294703, 250720, 1517, 1
S_7	13	40	1, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1526, 3031, 6015, 11899, 23468, 45892, 89586, 174484, 338604, 654767, 1259819, 2412803, 4595857, 8696292, 16316406, 30283116, 55429176, 99611416, 174523282, 295041555, 474010446, 707989707, 953440738, 1109880674, 1054799881, 757276217, 367770079, 100970812, 11123134, 238379, 205
S_7	14	45	1, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1531, 3052, 6064, 12042, 23840, 46820, 91857, 179678, 350280, 680713, 1317985, 2543726, 4890103, 9361198, 17834445, 33783525, 63571769, 118702912, 219540864, 401252103, 722318431, 1275091195, 2193508455, 3645060349, 5779780425, 8597147392, 11717463160, 14179792784, 14613741414, 12137210838, 7534036011, 3126507410, 718263102, 63180007, 993578, 608

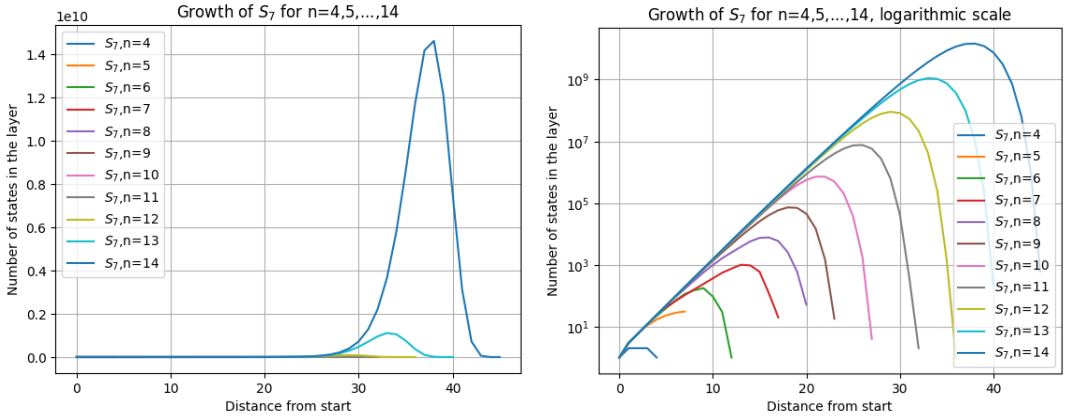


Figure 49: S_7 growth

8.4 New cubic pancakes

1. $S_{3,-3} = \{r_n, r_{n-3}, r_3\}$. Generates the entire group for $n \bmod 6 = 1$ or 5. ($n \equiv 1, 5 \pmod 6$)

n	$\lvert\langle S_{3,-3}\rangle\rvert$	$n!$	*Diam*	*last level*	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	120	120	7	31	10	-3
6	8	720	4	1	15	-11
7	5040	5040	20	10	21	-1
8	1152	40320	15	2	28	-13
9	48	362880	6	2	36	-30
10	28800	3628800	30	16	45	-15
11	39916800	39916800	62	8	55	7
12	128	479001600	9	2	66	-57
13	6227020800	6227020800	93	74	78	15

Table 16: Calculation results for $S_{3,-3} = \{r_3, r_{n-3}, r_n\}$.

Proof for $S_{3,-3}$. $r_3 r_n r_{n-3} = L^3$ or R^3. If n is divisible by 3, then the permutation is split into groups of 3 that change their order and can be flipped. These groups' order is defined by $r_{n/3}$ and $r_{n/3-1}$.

If n is divisible by 2, but not by 3, then r_3 doesn't change the parity of any element, while r_n and r_{n-3} either change the parity of everyone or don't change the parity of anyone.

Finally, if n is coprime with 6, then L^3 or R^3 generates L and R. Then r_3 is the transposition (13). However, $(13)\ldots(n-2\,n)(n-4\,n-2)\ldots(13) = (1n)$, and (13) and adjoint transpositions generate the entire group. □

The experiment has been done in the notebook.

2. $S_{5,-1} = \{r_5, r_{n-1}, r_n\}$

n	$\lvert S_{5,-1}\rvert$	$n!$	*Diam*	*last level*	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	10	120	5	1	10	-5
6	12	720	6	1	15	-9
7	5040	5040	15	64	21	-6
8	1152	40320	12	10	28	-16
9	181440	362880	24	4	36	-12
10	14400	3628800	26	5	45	-19
11	39916800	39916800	38	12	55	-17
12	1036800	479001600	32	3	66	-34
13	3113510400	6227020800	48	2	78	-30

Table 17: Calculation results for $S_{5,-1} = \{r_5, r_{n-1}, r_n\}$.

Based on the above calculation results, it can be hypothesized that at $n = 4k + 3$, this generative will generate the entire group and for $n = 4k + 1$ the generated subgroup is A_n (even permutations).

Proof. For $n = 4k + 1$ the three generators end up even, and for even n we notice that r_{n-1} and r_5 don't change the parity of any element, while r_n changes the parities of everyone. Therefore, for even n any composition either changes the parities of every element or fails to change any parity.

We also notice that $r_n r_{n-1}$ or $r_{n-1} r_n$ is the cycle $(123 \ldots n)$. Denote this cycle as c_n and consider $r_5 c_n r_5 c_n^{-1}$. It is $(135)(246)$. We can also obtain $(135)(357)^{-1} = (157)$ or $(135)^{-1}(357) = (137)$. The two cycles generate the group of even permutations on $\{1, 3, 5, 7\}$. Since n is odd, we can turn the permutations around and generate all even permutations, or just all permutations for $n = 4k + 3$. $\square$

Based on the above calculation results, it can be hypothesized that at $Diam \equiv 7 or 6 \pmod 8$ for odd n, this generative will generate the entire group.

The experiment has been done in the notebook.

3. $S_{5,-3} = \{r_5, r_{n-3}, r_n\}$

n	$\lvert S_{5,-3}\rvert$	$n!$	*Diam*	*last level*	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	8	120	4	1	10	-6
6	72	720	6	7	15	-9
7	5040	5040	17	20	21	-4
8	12	40320	6	1	28	-22
9	362880	362880	25	2	36	-11
10	28800	3628800	20	7	45	-25
11	2304	39916800	18	1	55	-37
12	518400	479001600	29	10	66	-37

Table 18: Calculation results for $S_{5,-3} = \{r_5, r_{n-3}, r_n\}$.

Based on the above calculation results, it can be hypothesized that for $n = 6k + 1$ and $n = 6k + 3$, the full group will be generated.

Proof attempt for $S_{5,-3}$. Consider three cases: $n \vdots 3, n \bmod 3 = 1, n \bmod 3 = 2$. Recall that $r_3 r_n r_{n-3}$ shifts everything three tiles forward.

For $n \vdots 3$ we have $r_3 r_n r_{n-3} = (14 \ldots n{-}2)(25 \ldots n{-}1)(36 \ldots n)$ and $r_n r_{n-3} = (14 \ldots (n{-}2)\, 3 \ldots n)(25 \ldots n{-}1)$. For $n \bmod 3 = 1$ we have $r_n r_{n-3} = (14 \ldots n)(25 \ldots (n-2)36 \ldots (n-1))$ and for $n \bmod 3 = 2$ we notice that $r_n r_{n-3} = (14 \ldots (n-1)25 \ldots n)(36 \ldots (n-2))$ We also notice that none of these cases has r_n or r_{n-3} affect the split.

For $n \bmod 3 = 2$ the permutation $(15)(24)$ fails to affect the numbers divisible by 3.

For $n \bmod 3 = 1$ $r_n r_{n-3}$ consists of two cycles: of $\frac{n-1}{3} + 1$ elements and $2 \cdot \frac{n-1}{3}$ elements.

For $n \vdots 3$ $r_n r_{n-3} = (14 \ldots (n-2)\, 3 \ldots n)(25 \ldots n-1)$. r_n and r_{n-3} don't affect the split while $(15)(24)$ swaps two pairs of points in different cycles. Applying $r_n r_{n-3}$ $n/3$ times, we obtain $(13)(46)\ldots$, and $(15)(24)$ provides us with permutations like $(35)(26)$. Combining $(15)(24)$ with $(35)(26)$, we obtain $(153)(426)$. Moving it 3 tiles forward for $n > 9$ yields $(486)(7\,5\,9)$. Since the even cycles have two common elements, while the odd cycles have one common element, we can split the cycles. Thus we obtain $(153), (246), (357), \ldots$ and all even permutations.

□

The experiment has been done in the notebook.

4. $S_{4,-3} = \{r_4, r_{n-3}, r_n\}$

n	$\lvert S_{4,-3}\rvert$	$n!$	*Diam*	*last level*	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	120	120	7	9	10	-3
6	720	720	12	1	15	-3
7	24	5040	12	1	21	-9
8	20160	40320	19	24	28	-9
9	362880	362880	23	100	36	-13
10	576	3628800	13	6	45	-32
11	39916800	39916800	36	3	55	-19
12	239500800	479001600	38	62	66	-28
13	7680	6227020800	22	1	78	-56

Table 19: Calculation results for $S_{4,-3} = \{r_4, r_{n-3}, r_n\}$.

Based on the above calculation results, it can be assumed that at $n = 4k + 1$ or $n = 5k + 1$, this generator generates the entire group.

Based on the above calculation results, it can be hypothesized that at $Diam \equiv 7 or 6 \pmod 8$ for odd n, this generative will generate the entire group.

The experiment has been done in the notebook.

5. $S_{5,-2} = \{r_5, r_{n-2}, r_n\}$

n	$\lvert S_{5,-2}\rvert$	$n!$	$Diam$	$last\ level$	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	12	120	6	1	10	-4
6	720	720	12	2	15	-3
7	24	5040	12	1	21	-9
8	40320	40320	20	51	28	-8
9	2880	362880	15	12	36	-21
10	3628800	3628800	29	74	45	-16
11	4320	39916800	18	12	55	-37
12	479001600	479001600	41	5	66	-25
13	3628800	6227020800	31	3	78	-47

Table 20: Calculation results for $S_{5,-2} = \{r_5, r_{n-2}, r_n\}$.

Based on the above calculation results, it can be assumed when n is even this generator generates the entire group.

The experiment has been done in the notebook.

6. $S_{5,-4} = \{r_5, r_{n-4}, r_n\}$

n	$\lvert S_{5,-4}\rvert$	$n!$	$Diam$	$last\ level$	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	2	120	1	1	10	-9
6	720	720	12	1	15	-3
7	144	5040	8	5	21	-13
8	20160	40320	19	24	28	-9
9	24	362880	12	1	36	-24
10	3628800	3628800	30	2	45	-15
11	86400	39916800	22	5	55	-33
12	239500800	479001600	37	24	66	-29
13	576	6227020800	13	6	78	-65

Table 21: Calculation results for $S_{5,-4} = \{r_5, r_{n-4}, r_n\}$.

Based on the above calculation results, it can be assumed that at $n = 4k + 2$ this generator generates the entire group.

Based on the above calculation results, it can be assumed that at $Diam \equiv 3 \pmod 9$ for even n, this generator generates the entire group.

The experiment has been done in the notebook.

7. $S_{4,-4} = \{r_4, r_{n-4}, r_n\}$

n	$\lvert S_{4,-3}\rvert$	$n!$	$Diam$	$last\ level$	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	10	120	5	1	10	-5
6	48	720	6	2	15	-9
7	5040	5040	20	10	21	-1
8	8	40320	4	1	28	-24
9	181440	362880	34	3	36	-2
10	3840	3628800	25	1	45	-20
11	39916800	39916800	51	16	55	-4
12	48	479001600	6	2	66	-60
13	3113510400	6227020800	75	29	78	-3

Table 22: Calculation results for $S_{4,-4} = \{r_4, r_{n-4}, r_n\}$.

Based on the above calculation results, it can be assumed that at $n = 4k + 3$ this generator generates the entire group.

The experiment has been done in the notebook.

8. $S_{4,-1} = \{r_4, r_{n-1}, r_n\}$

n	$\lvert S_{4,-1}\rvert$	$n!$	$Diam$	$last\ level$	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	10	120	5	1	10	-5
6	720	720	12	2	15	-3
7	5040	5040	15	23	21	-6
8	40320	40320	27	24	28	-1
9	181440	362880	26	12	36	-10
10	3628800	3628800	33	71	45	-12
11	39916800	39916800	38	6	55	-17
12	479001600	479001600	43	17	66	-23
13	3113510400	6227020800	47	30	78	-31

Table 23: Calculation results for $S_{4,-1} = \{r_4, r_{n-1}, r_n\}$.

Based on the above calculation results, it can be assumed that at $n \equiv 1 or 2 or 0 \pmod 4$, this generator generates the entire group.

The experiment has been done in the notebook.

9. $S_{4,-2} = \{r_4, r_{n-2}, r_n\}$

n	$\lvert S_{4,-2}\rvert$	$n!$	$Diam$	$last\ level$	$\frac{n(n-1)}{2}$	$Diam - \frac{n(n-1)}{2}$
5	120	120	8	11	10	-2
6	12	720	6	1	15	-9
7	5040	5040	17	20	21	-4
8	384	40320	11	1	28	-17
9	362880	362880	26	18	36	-10
10	240	3628800	10	3	45	-35
11	39916800	39916800	41	6	55	-14
12	46080	479001600	25	17	66	-41

Table 24: Calculation results for $S_{4,-2} = \{r_4, r_{n-2}, r_n\}$.

Based on the above calculation results, it can be assumed that for odd n, this generator generates the entire group.

The experiment has been done in the notebook.

9 Biologically Relevant Generators

Cayley graph methods have direct applications in computational biology. In comparative genomics, a key task is estimating the evolutionary distance between two genomes. This distance is typically defined as the minimum number of large-scale rearrangement events required to transform one genome into the other.

When genomes are represented as permutations and rearrangements are represented as group generators, this task becomes a distance problem in a Cayley graph.

9.1 Group-theoretic Preliminaries

To model genome rearrangements formally, we represent a genome as a permutation on $\{1, 2, \ldots, n\}$. When gene orientation is relevant, signed permutations are used instead. These form the symmetric group S_n and the hyperoctahedral group B_n, respectively.

Each type of rearrangement defines a specific transformation of the genome and can be treated as a generator of the corresponding group [7]. Let S be a set of allowed rearrangement operations (generators). Then the rearrangement distance $d_S(\pi, \sigma)$ between two genomes π and σ is the smallest number k of rearrangement operations s_i needed to transform σ to π. Formally:

$$d_S(\pi, \sigma) \;=\; \min\{\, k : \pi = s_1\, s_2 \cdots s_k\, \sigma,\; s_i \in S\},$$

This is equivalent to computing the word length of $\pi\sigma^{-1}$ in the Cayley graph of G with respect to S. In the biological applications, S typically includes large-scale rearrangement events such as

$$S \;=\; \{\text{inversion, translocation, fusion, fission}, \ldots\}.$$

To effectively analyze genome rearrangements using Cayley graphs, it's essential to understand how biological events correspond to mathematical operations.

9.2 Core Generators

The most studied genomic rearrangements correspond to the following types of generators:

- **Reversals (Inversions)**: flipping a contiguous segment of a chromosome,

$$\ldots\, a\, \underbrace{b\, c\, \ldots\, d}_{\text{reversed}}\, e\, \ldots \;\longmapsto\; \ldots\, a\, d\, \ldots\, c\, b\, e\, \ldots.$$

  In the signed case, each reversal also toggles the orientation of every gene in the segment [30].

- **Transpositions**: cutting out a segment and reinserting it elsewhere on the same chromosome,

$$\ldots\, u\, \underbrace{v\, \ldots\, w}_{\Delta}\, x\, \ldots\, y\, \ldots \;\longmapsto\; \ldots\, u\, x\, \ldots\, y\, \underbrace{v\, \ldots\, w}_{\Delta}\, \ldots.$$

- **Translocations**: moving a segment from one chromosome to another,

$$\underbrace{\ldots\, A\, \Delta\, B\, \ldots}_{\text{chr. 1}} \;\longleftrightarrow\; \underbrace{\ldots\, C\, \Delta'\, D\, \ldots}_{\text{chr. 2}}.$$

- **Fissions**: splitting a single chromosome into two at one breakpoint,

$$\ldots\, p\, \underbrace{q\, \ldots\, r}_{\Delta}\, s\, \ldots \;\longmapsto\; (\ldots\, p)\, (\underbrace{q\, \ldots\, r}_{\Delta}\, s\, \ldots).$$

- **Fusions**: the inverse of a fission, joining two chromosomes end-to-end.

Different choices of S and G yield distinct distances and computational complexities [10, 13, 52]. In some cases, particularly when the allowed operations do not form a group or when the rearrangement space lacks closure under composition (e.g., unsigned reversals or multi-chromosomal translocations), the problem cannot be modeled via Cayley graphs. Instead, alternative frameworks such as breakpoint graphs, adjacency graphs, or double-cut-and-join graphs are employed [?bafna1996genome, ?yancopoulos2005dcj].

9.3 Key Rearrangement Models and Algorithms

In this work, we specifically focus on genome rearrangement problems that can be formulated and solved on Cayley graphs — that is, problems where the allowed operations form a closed generating set of a group. While other operations such as transpositions can also generate S_n and thus allow a Cayley graph formulation in theory, such representations tend to overlook biologically realistic constraints (e.g., locality or chromosome structure). Depending on the chosen rearrangement model and whether the permutations are signed or unsigned, the computational complexity of finding minimal rearrangement distances varies significantly, as summarized in Table 25.

Table 25: Overview of complexity and approximation in genome rearrangement problems with Cayley-compatible models

Problem	**S Complexity**	**S Approx.**	**UNS Complexity**	**UNS Approx.**
REV	P ($O(n\sqrt{n \log n})$) [61]	1.375 [5]	NP-hard [13]	1.5 [17]
TR	NP-hard [10]	–	NP-hard [23]	1.375 [23]
REV+TR	NP-hard [52]	2 [64]	NP-hard [52]	2.83+ϵ [54]
SSOs	P* [53]	–	P* [53]	–

Approx.: approximation factor (indicates the worst-case ratio between the computed solution and the true minimal rearrangement distance); P: polynomial time; REV: reversals; TR: transpositions; S: signed; SSO: super short operations; UNS: unsigned.
* All known variants of sorting by super short operations are solvable in polynomial time, with time complexity ranging from $O(n)$ to $O(n^4)$, depending on the model (see [53] for references).

9.4 Reversals

The Reversal graph $R_n, n \geqslant 3$, is the Cayley graph over the symmetric group generated by the reversals from the set $R = \{r_{i,j} \in Sym_n, 1 \leqslant i < j \leqslant n\}$, $|R| = \binom{n}{2}$, where any $r_{i,j}$ reverses elements of a permutation $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ within segment $[i, j]$ in a such a way that we have:

$$[\pi_1 \ldots \underline{\pi_i \pi_{i+1} \ldots \pi_{j-1} \pi_j} \ldots \pi_n] r_{i,j} = [\pi_1 \ldots \underline{\pi_j \pi_{j-} \ldots \pi_{i+1} \pi_i} \ldots \pi_n].$$

The distance in this case is defined as the minimal number of reversals transforming one permutation into another. It is known [3, 42] that the diameter of this graph is $(n-1)$, and the only permutations needing this many reversals are the Gollan permutation γ_n and its inverse, where the Gollan permutation, in a cycle notation, is defined as follows:

$$\gamma_n = \begin{cases} (3\,1\,5\,2\,7\,4 \ldots n-3\, n-5\, n-1\, n-4\, n\, n-2), & \text{if } n \text{ is even} \\ (3\,1\,5\,2\,7\,4 \ldots n-6\, n-2\, n-5\, n\, n-3\, n-1), & \text{if } n \text{ is odd.} \end{cases}$$

The analysis of genomes evolving by inversions leads to the combinatorial problem of sorting by reversals. Reversal distance measures the amount of evolution that must have taken place at the chromosome level,

assuming evolution proceeded by inversions. There are two algorithmic subproblems. The first one is to find the reversal distance between two permutations. The second one is to reconstruct a sequence of reversals realizing the distance. The last problem is NP–hard [41]. There are 1.5–approximation [16] and 1.375–approximation [6] algorithms for sorting by reversals.

Thus, the reversal graph $R_n, n \geqslant 3$, is a connected $\binom{n}{2}$-regular graph of order $n!$ and diameter $n-1$. It was shown in [46] that it does not contain neither triangles nor subgraphs isomorphic to $K_{2,4}$. This gives explicit formulas for the cardinality of the metric spheres S_1 and S_2 in this graph as $|S_1| = \binom{n}{2}$ and $|S_2| = \frac{1}{6}(n^4 - 2n^3 - n^2 - 16n + 42)$. For any $n \leqslant 14$, the cardinalities $|S_i|, 0 \leqslant i \leqslant n-1$, are shown as *Growth* in the table below.

Dima: (There are $n(n-1)/2$ generators: for every pair $(i,j) : 0 \leq i < j < n$ we add a permutation that reverses elements $[i..j]$ and doesn't change anything else.)

n	Full	Diameter	Growth
1	Yes	0	1
2	Yes	1	1,1
3	Yes	2	1, 3, 2
4	Yes	3	1, 6, 15, 2
5	Yes	4	1, 10, 52, 55, 2
6	Yes	5	1, 15, 129, 389, 184, 2
7	Yes	6	1, 21, 266, 1563, 2539, 648, 2
8	Yes	7	1, 28, 487, 4642, 16445, 16604, 2111, 2
9	Yes	8	1, 36, 820, 11407, 69863, 169034, 105365, 6352, 2
10	Yes	9	1, 45, 1297, 24600, 228613, 1016341, 1686534, 654030, 17337, 2
11	Yes	10	1, 55, 1954, 48204, 626677, 4398136, 14313789, 16584988, 3900116, 42878, 2
12	Yes	11	1, 66, 2831, 87758, 1510973, 15240603, 81531167, 198802757, 159650162, 22073230, 102050, 2
13	Yes	12	1, 78, 3972, 150707, 3304457, 44997104, 354920337, 1489090761, 2717751441, 1499706234, 116855950, 239756, 2
14	Yes	13	1, 91, 5425, 246787, 6691847, 117687072, 1274360803, 8123911553, 26817217317, 36605339412, 13664302955, 567965207, 562728, 2

Dima: (OEIS sequence for growth function: https://oeis.org/A300003, computed up to N=14, I verified that it matches up to N=12)

9.5 Biological "transpositions" (from "transposons")

10 Puzzle groups

10.1 Skewb

Common information In one row with the famous Rubik's puzzles — the Skewb puzzle was invented by Tony Durham and officially recognized as a competition event by the World Cube Association, but only in 2013.

After some design changes, the puzzle became known for its distinctive mechanism, consisting of 8 corner pieces and 6 fixed centers, where each move is a 120° rotation around a space diagonal.

As the puzzle gained popularity, it also attracted mathematical interest. One of the first known automatic solvers was presented by Huang and Myers (1998), based on first-order logical reasoning.

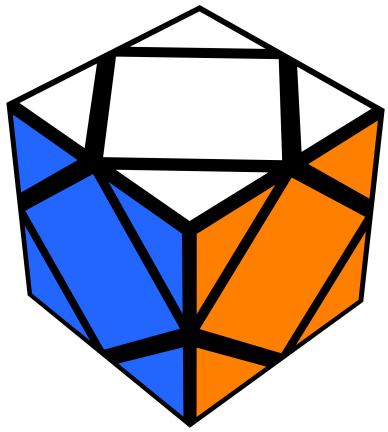


Figure 50: The Skewb puzzle.

The classical results about the God's number (i.e., the minimal number of moves needed to solve any position) were established on: Jaap Scherphuis's website, Tomas Rokicki's Twizzle Explorer, and documented in the OEIS entry A079745.

10.1.1 Generators

Let us consider the basic moves allowed in the puzzle:

- U — up face rotation
- D — down face rotation
- F — front face rotation
- B — back face rotation
- L — left upper face rotation
- R — right upper face rotation

Without fixing a global orientation, the generators of the Cayley graph of the Skewb group are defined by the following permutations (Tomas Rokicki's Twizzle Explorer) of stickers:

$$\begin{aligned}
D &:= (7\ 9\ 8)(13\ 17\ 24)(14\ 18\ 22)(15\ 16\ 23)(25\ 26\ 27),\\
U &:= (1\ 11\ 6)(2\ 12\ 4)(3\ 10\ 5)(19\ 21\ 20)(28\ 30\ 29),\\
L &:= (1\ 8\ 12)(2\ 9\ 10)(3\ 7\ 11)(22\ 24\ 23)(25\ 27\ 30),\\
UR &:= (4\ 6\ 5)(13\ 20\ 18)(14\ 21\ 16)(15\ 19\ 17)(26\ 28\ 29),\\
R &:= (4\ 11\ 9)(5\ 12\ 7)(6\ 10\ 8)(16\ 18\ 17)(26\ 29\ 27),\\
UL &:= (1\ 3\ 2)(13\ 23\ 21)(14\ 24\ 19)(15\ 22\ 20)(25\ 30\ 28),\\
B &:= (10\ 12\ 11)(16\ 20\ 24)(17\ 21\ 22)(18\ 19\ 23)(27\ 29\ 30),\\
F &:= (1\ 5\ 9)(2\ 6\ 7)(3\ 4\ 8)(13\ 15\ 14)(25\ 28\ 26).
\end{aligned}$$

10.1.2 Growth function

It was independently shown by multiple sources that The Skewb puzzle has God's number equal to 11 in the face-turn metric, along with its full growth function. The results computed by `CayleyPy` perfectly match those published on jaapsch.net and Tomas Rokicki's Twizzle Explorer.

Layer	*jaapsch.net* (fixed corner)	*T. Rokicki*
0	1	1
1	8	16
2	48	208
3	288	2280
4	1728	19683
5	10248	125280
6	59304	712032
7	315198	3782376
8	1225483	14705796
9	1455856	17470272
10	81028	972336
11	90	1080

Table 26: Skewb growth.

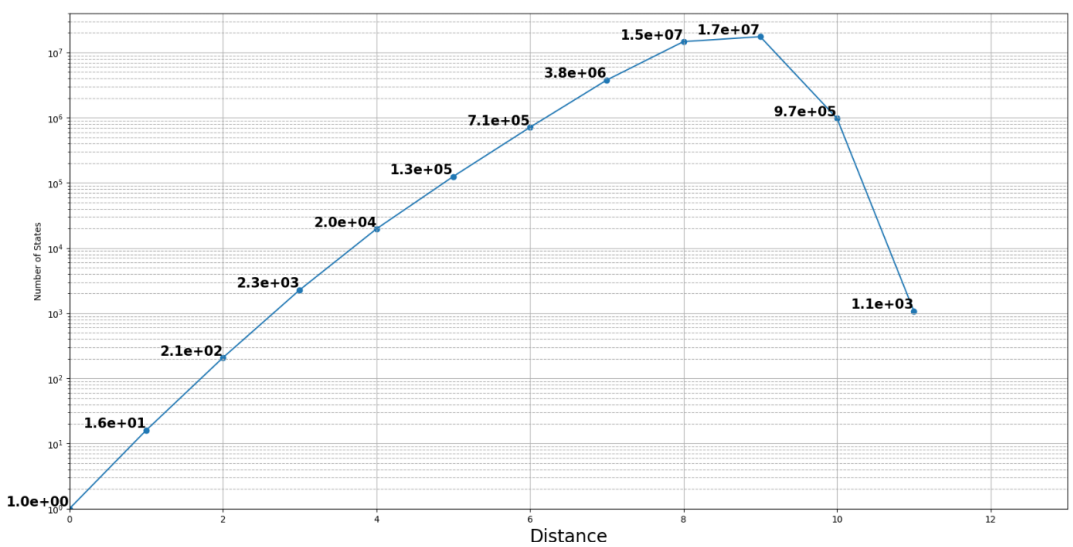


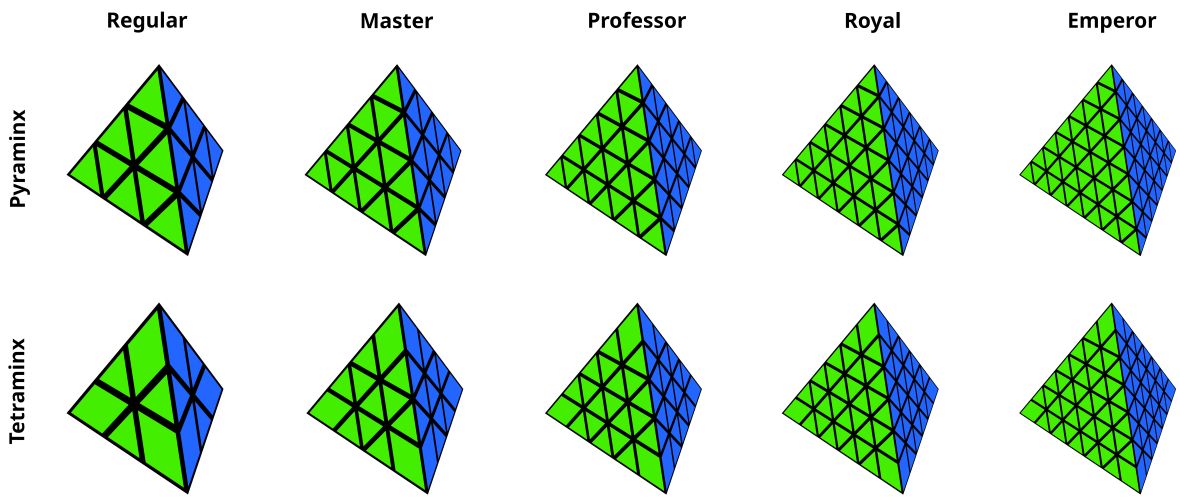
Figure 51: Skewb growth performed by CayleyPy.

10.2 Pyraminx/Tetraminx Family

10.2.1 Common information

Pyraminx and tetraminx are tetrahedron-shaped puzzle with equilateral triangular tiles (see figure (**??**), top) and tetraminx (see figure (**??**), bottom) is a similar puzzle with the tips being "fixed" to the next layer. From the puzzle-solving perspective the difference between pyraminx and tetraminx is minimal (see jaapsch.net) as a solution of a pyraminx always involves aligning tips colors to the respective colors of the mid-tile of

the next layer, effectively reducing it to a tetraminx. After such alignment the algorithms of solving both puzzles are identical.



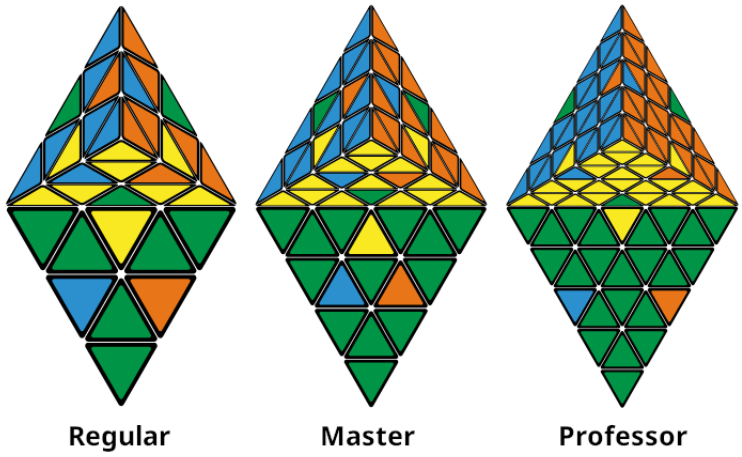
Both pyraminx and tetraminx can have different sizes, which are regular (3 rotating slices along each axis), master (4 slices), professor (5 slices), royal (6 slices) and emperor(7 slices).

The God's number of a regular tetraminx is 11 (see [(wiki). The God's number of pyraminx is 15, since for a given size each solution of the pyraminx can differ from the corresponding tetraminx one no more than 4 moves to align the tips, and there exist confugurations where all 4 moves are required to solve the puzzle.

For a master tetraminx there are confugurations which require at least 20 moves to solve, while the upper bound is 26 (see here.

10.2.2 Superflips

According to randelshofer.ch the superflips (or wedge superflips) of pyraminx are defined as configurations with the edge-central tiles being swapped between the adjacent faces. There are known algorithms for superflip of regular pyraminx, for wedge superflip of master pyraminx and for supeflip of master and for a superflip of professor pyraminx.



10.2.3 Generators

We choose the pyraminx and tetraminx generators as rotations of slices of a puzzle. In order to avoid a full rotation of a puzzle, we don't include rotations of the outer slices (faces). Following cubing.js we denote the generators with a number and text. The text part can be F ("front"), D ("down"), BL ("left, behind"), and BR ("right, behind"). The number represents the slice with 1 being the face slice, 2 – the one next to the face one and so on.

Note, that for a given size pyraminx has 8 more generators than corresponding tetraminx – those generators represent rotations (one clockwise and the other is counterclockwise) of 4 tips which are fixed in tetraminx.

Here we present the generators we used for our computations. For each puzzle only half of the generator's set is present, the other half are inverse generators.

Tetraminx

$$\begin{aligned}
2BL &:=(5,11,9)(6,12,10)(19,20,21),\\
2BR &:=(1,7,6)(2,8,5)(22,23,24),\\
2F &:=(3,12,8)(4,11,7)(16,17,18),\\
2D &:=(1,9,3)(2,10,4)(13,14,15),
\end{aligned}$$

Pyraminx

$$\begin{aligned}
2BL &:=(5,11,9)(6,12,10)(19,20,21),\\
3BL &:=(31,32,33),\\
2BR &:=(1,7,6)(2,8,5)(22,23,24),\\
3BR &:=(34,35,36),\\
2F &:=(3,12,8)(4,11,7)(16,17,18),\\
3F &:=(28,29,30),\\
2D &:=(1,9,3)(2,10,4)(13,14,15),\\
3D &:=(25,26,27),
\end{aligned}$$

Master tetraminx

$2BL := (7,23,13)(8,24,14)(29,35,33)(30,36,34)(49,51,50),$
$3BL := (5,21,17)(6,22,18)(43,44,45),$
$2BR := (1,15,6)(2,16,5)(25,31,30)(26,32,29)(49,52,51),$
$3BR := (7,20,12)(8,19,11)(46,47,48),$
$2F := (9,22,12)(10,21,11)(27,36,32)(28,35,31)(50,51,52),$
$3F := (3,24,16)(4,23,15)(40,41,42),$
$2D := (3,19,17)(4,20,18)(25,33,27)(26,34,28)(49,50,52),$
$3D := (1,13,9)(2,14,10)(37,38,39),$.

Master pyraminx

$2BL := (19,35,25)(20,36,26)(41,47,45)(42,48,46)(61,63,62),$
$3BL := (17,33,29)(18,34,30)(55,56,57),$
$4BL := (7,8,9),$
$2BR := (13,27,18)(14,28,17)(37,43,42)(38,44,41)(61,64,63),$
$3BR := (19,32,24)(20,31,23)(58,59,60),$
$4BR := (1,2,3),$
$2F := (21,34,24)(22,33,23)(39,48,44)(40,47,43)(62,63,64),$
$3F := (15,36,28)(16,35,27)(52,53,54),$
$4F := (10,11,12),$
$2D := (15,31,29)(16,32,30)(37,45,39)(38,46,40)(61,62,64),$
$3D := (13,25,21)(14,26,22)(49,50,51),$
$4D := (4,5,6),$.

mnxminx

$2BL := (5,21,17)(6,22,18)(43,59,49)(44,60,50)(73,78,74)(75,83,81)(85,87,86),$
$3BL := (41,57,53)(42,58,54)(65,71,69)(66,72,70)(76,84,79),$
$4BL := (7,23,13)(8,24,14)(31,32,33),$
$2BR := (7,20,12)(8,19,11)(37,51,42)(38,52,41)(76,82,78)(77,84,81)(85,88,87),$
$3BR := (43,56,48)(44,55,47)(61,67,66)(62,68,65)(73,80,75),$
$4BR := (1,15,6)(2,16,5)(25,26,27),$
$2F := (3,24,16)(4,23,15)(45,58,48)(46,57,47)(74,84,80)(75,82,79)(86,87,88),$
$3F := (39,60,52)(40,59,51)(63,72,68)(64,71,67)(77,83,78),$
$4F := (9,22,12)(10,21,11)(34,35,36),$
$2D := (1,13,9)(2,14,10)(39,55,53)(40,56,54)(73,79,77)(76,83,80)(85,86,88),$
$3D := (37,49,45)(38,50,46)(61,69,63)(62,70,64)(74,82,81),$
$4D := (3,19,17)(4,20,18)(28,29,30),$.

Professor pyraminx

$2BL := (29,45,41)(30,46,42)(55,71,61)(56,72,62)(87,95,93)(88,96,91)(97,99,98),$
$3BL := (53,69,65)(54,70,66)(77,83,81)(78,84,82)(85,90,86),$
$4BL := (19,20,21)(31,47,37)(32,48,38),$
$5BL := (7,8,9),$
$2BR := (31,44,36)(32,43,35)(49,63,54)(50,64,53)(85,92,87)(88,94,90)(97,100,99),$
$3BR := (55,68,60)(56,67,59)(73,79,78)(74,80,77)(89,96,93),$
$4BR := (13,14,15)(25,39,30)(26,40,29),$
$5BR := (1,2,3),$
$2F := (27,48,40)(28,47,39)(57,70,60)(58,69,59)(86,96,92)(89,95,90)(98,99,100),$
$3F := (51,72,64)(52,71,63)(75,84,80)(76,83,79)(87,94,91),$
$4F := (22,23,24)(33,46,36)(34,45,35),$
$5F := (10,11,12),$
$2D := (25,37,33)(26,38,34)(51,67,65)(52,68,66)(85,91,89)(86,94,93)(97,98,100),$
$3D := (49,61,57)(50,62,58)(73,81,75)(74,82,76)(88,95,92),$
$4D := (16,17,18)(27,43,41)(28,44,42),$
$5D := (4,5,6),$.

Royal tetraminx

$2BL := (5,21,17)(6,22,18)(65,81,77)(66,82,78)(99,107,105)(111,119,117)(112,120,115)(123,131,129)$
$(124,132,127),$
$3BL := (41,57,53)(42,58,54)(89,95,93)(90,96,94)(97,102,98)(100,108,103)(121,126,122),$
$4BL := (43,59,49)(44,60,50)(67,83,73)(68,84,74)(109,114,110),$
$5BL := (7,23,13)(8,24,14)(31,32,33),$
$2BR := (7,20,12)(8,19,11)(67,80,72)(68,79,71)(100,106,102)(109,116,111)(112,118,114)(121,128,123)$
$(124,130,126),$
$3BR := (43,56,48)(44,55,47)(85,91,90)(86,92,89)(97,104,99)(101,108,105)(125,132,129),$
$4BR := (37,51,42)(38,52,41)(61,75,66)(62,76,65)(113,120,117),$
$5BR := (1,15,6)(2,16,5)(25,26,27),$
$2F := (3,24,16)(4,23,15)(63,84,76)(64,83,75)(98,108,104)(110,120,116)(113,119,114)(122,132,128)$
$(125,131,126),$
$3F := (39,60,52)(40,59,51)(87,96,92)(88,95,91)(99,106,103)(101,107,102)(123,130,127),$
$4F := (45,58,48)(46,57,47)(69,82,72)(70,81,71)(111,118,115),$
$5F := (9,22,12)(10,21,11)(34,35,36),$
$2D := (1,13,9)(2,14,10)(61,73,69)(62,74,70)(97,103,101)(109,115,113)(110,118,117)(121,127,125)$
$(122,130,129),$
$3D := (37,49,45)(38,50,46)(85,93,87)(86,94,88)(98,106,105)(100,107,104)(124,131,128),$
$4D := (39,55,53)(40,56,54)(63,79,77)(64,80,78)(112,119,116),$
$5D := (3,19,17)(4,20,18)(28,29,30),$.

Royal pyraminx

$2BL := (31,47,37)(32,48,38)(53,69,65)(54,70,66)(112,120,115)(121,126,122)(124,132,127)(133,138,134)$
$(136,144,139)$,
$3BL := (77,93,89)(78,94,90)(101,107,105)(102,108,106)(109,114,110)(111,119,117)(123,131,129)$,
$4BL := (55,71,61)(56,72,62)(79,95,85)(80,96,86)(135,143,141)$,
$5BL := (19,20,21)(29,45,41)(30,46,42)$,
$6BL := (7,8,9)$,
$2BR := (25,39,30)(26,40,29)(55,68,60)(56,67,59)(109,116,111)(121,128,123)(125,132,129)(133,140,135)$
$(137,144,141)$,
$3BR := (79,92,84)(80,91,83)(97,103,102)(98,104,101)(112,118,114)(113,120,117)(124,130,126)$,
$4BR := (49,63,54)(50,64,53)(73,87,78)(74,88,77)(136,142,138)$,
$5BR := (22,23,24)(31,44,36)(32,43,35)$,
$6BR := (10,11,12)$,
$2F := (33,46,36)(34,45,35)(51,72,64)(52,71,63)(113,119,114)(123,130,127)(125,131,126)(135,142,139)$
$(137,143,138)$,
$3F := (75,96,88)(76,95,87)(99,108,104)(100,107,103)(110,120,116)(111,118,115)(122,132,128)$,
$4F := (57,70,60)(58,69,59)(81,94,84)(82,93,83)(134,144,140)$,
$5F := (16,17,18)(27,48,40)(28,47,39)$,
$6F := (4,5,6)$,
$2D := (27,43,41)(28,44,42)(49,61,57)(50,62,58)(110,118,117)(122,130,129)(124,131,128)(134,142,141)$
$(136,143,140)$,
$3D := (73,85,81)(74,86,82)(97,105,99)(98,106,100)(109,115,113)(112,119,116)(121,127,125)$,
$4D := (51,67,65)(52,68,66)(75,91,89)(76,92,90)(133,139,137)$,
$5D := (13,14,15)(25,37,33)(26,38,34)$,
$6D := (1,2,3)$, .

Emperor tetraminx

$2BL := (31,47,37)(32,48,38)(67,83,73)(68,84,74)(124,132,127)(140,148,143)(152,160,155)(161,166,162)$
$(164,172,167)(175,183,181)(176,184,179),$

$3BL := (7,23,13)(8,24,14)(89,105,101)(90,106,102)(121,126,122)(133,135,134)(137,142,138)(139,147,145)$
$(151,159,157),$

$4BL := (91,107,97)(92,108,98)(113,119,117)(114,120,118)(123,131,129)(149,154,150)(163,171,169),$

$5BL := (5,21,17)(6,22,18)(29,45,41)(30,46,42)(173,178,174),$

$6BL := (55,56,57)(65,81,77)(66,82,78),$

$2BR := (25,39,30)(26,40,29)(61,75,66)(62,76,65)(121,128,123)(137,144,139)(149,156,151)(161,168,163)$
$(165,172,169)(173,180,175)(176,182,178),$

$3BR := (1,15,6)(2,16,5)(91,104,96)(92,103,95)(125,132,129)(133,136,135)(140,146,142)(141,148,145)$
$(152,158,154),$

$4BR := (85,99,90)(86,100,89)(109,115,114)(110,116,113)(124,130,126)(153,160,157)(164,170,166),$

$5BR := (7,20,12)(8,19,11)(31,44,36)(32,43,35)(177,184,181),$

$6BR := (58,59,60)(67,80,72)(68,79,71),$

$2F := (33,46,36)(34,45,35)(69,82,72)(70,81,71)(125,131,126)(141,147,142)(153,159,154)(163,170,167)$
$(165,171,166)(174,184,180)(177,183,178),$

$3F := (9,22,12)(10,21,11)(87,108,100)(88,107,99)(123,130,127)(134,135,136)(138,148,144)(139,146,143)$
$(150,160,156),$

$4F := (93,106,96)(94,105,95)(111,120,116)(112,119,115)(122,132,128)(151,158,155)(162,172,168),$

$5F := (3,24,16)(4,23,15)(27,48,40)(28,47,39)(175,182,179),$

$6F := (52,53,54)(63,84,76)(64,83,75),$

$2D := (27,43,41)(28,44,42)(63,79,77)(64,80,78)(122,130,129)(138,146,145)(150,158,157)(162,170,169)$
$(164,171,168)(173,179,177)(174,182,181),$

$3D := (3,19,17)(4,20,18)(85,97,93)(86,98,94)(124,131,128)(133,134,136)(137,143,141)(140,147,144)$
$(149,155,153),$

$4D := (87,103,101)(88,104,102)(109,117,111)(110,118,112)(121,127,125)(152,159,156)(161,167,165),$

$5D := (1,13,9)(2,14,10)(25,37,33)(26,38,34)(176,183,180),$

$6D := (49,50,51)(61,73,69)(62,74,70),$.

Emperor pyraminx

$2BL := (7,23,13)(8,24,14)(77,93,89)(78,94,90)(136,144,139)(147,155,153)(159,167,165)(160,168,163)$
$(171,179,177)(172,180,175)(183,191,189),$

$3BL := (53,69,65)(54,70,66)(103,119,109)(104,120,110)(135,143,141)(148,156,151)(181,186,182)(184,192,187)$
$(193,195,194),$

$4BL := (101,117,113)(102,118,114)(125,131,129)(126,132,130)(133,138,134)(145,150,146)(157,162,158),$

$5BL := (55,71,61)(56,72,62)(79,95,85)(80,96,86)(169,174,170),$

$6BL := (5,21,17)(6,22,18)(31,32,33),$

$7BL := (43,44,45),$

$2BR := (1,15,6)(2,16,5)(79,92,84)(80,91,83)(133,140,135)(148,154,150)(157,164,159)(160,166,162)$
$(169,176,171)(172,178,174)(184,190,186),$

$3BR := (55,68,60)(56,67,59)(97,111,102)(98,112,101)(136,142,138)(145,152,147)(181,188,183)(185,192,189)$
$(193,196,195),$

$4BR := (103,116,108)(104,115,107)(121,127,126)(122,128,125)(137,144,141)(149,156,153)(161,168,165),$

$5BR := (49,63,54)(50,64,53)(73,87,78)(74,88,77)(173,180,177),$

$6BR := (7,20,12)(8,19,11)(34,35,36),$

$7BR := (46,47,48),$

$2F := (9,22,12)(10,21,11)(75,96,88)(76,95,87)(137,143,138)(146,156,152)(158,168,164)(161,167,162)$
$(170,180,176)(173,179,174)(182,192,188),$

$3F := (51,72,64)(52,71,63)(105,118,108)(106,117,107)(134,144,140)(149,155,150)(183,190,187)(185,191,186)$
$(194,195,196),$

$4F := (99,120,112)(100,119,111)(123,132,128)(124,131,127)(135,142,139)(147,154,151)(159,166,163),$

$5F := (57,70,60)(58,69,59)(81,94,84)(82,93,83)(171,178,175),$

$6F := (3,24,16)(4,23,15)(28,29,30),$

$7F := (40,41,42),$

$2D := (3,19,17)(4,20,18)(73,85,81)(74,86,82)(134,142,141)(145,151,149)(157,163,161)(158,166,165)$
$(169,175,173)(170,178,177)(181,187,185),$

$3D := (49,61,57)(50,62,58)(99,115,113)(100,116,114)(133,139,137)(146,154,153)(182,190,189)(184,191,188)$
$(193,194,196),$

$4D := (97,109,105)(98,110,106)(121,129,123)(122,130,124)(136,143,140)(148,155,152)(160,167,164),$

$5D := (51,67,65)(52,68,66)(75,91,89)(76,92,90)(172,179,176),$

$6D := (1,13,9)(2,14,10)(25,26,27),$

$7D := (37,38,39),$.

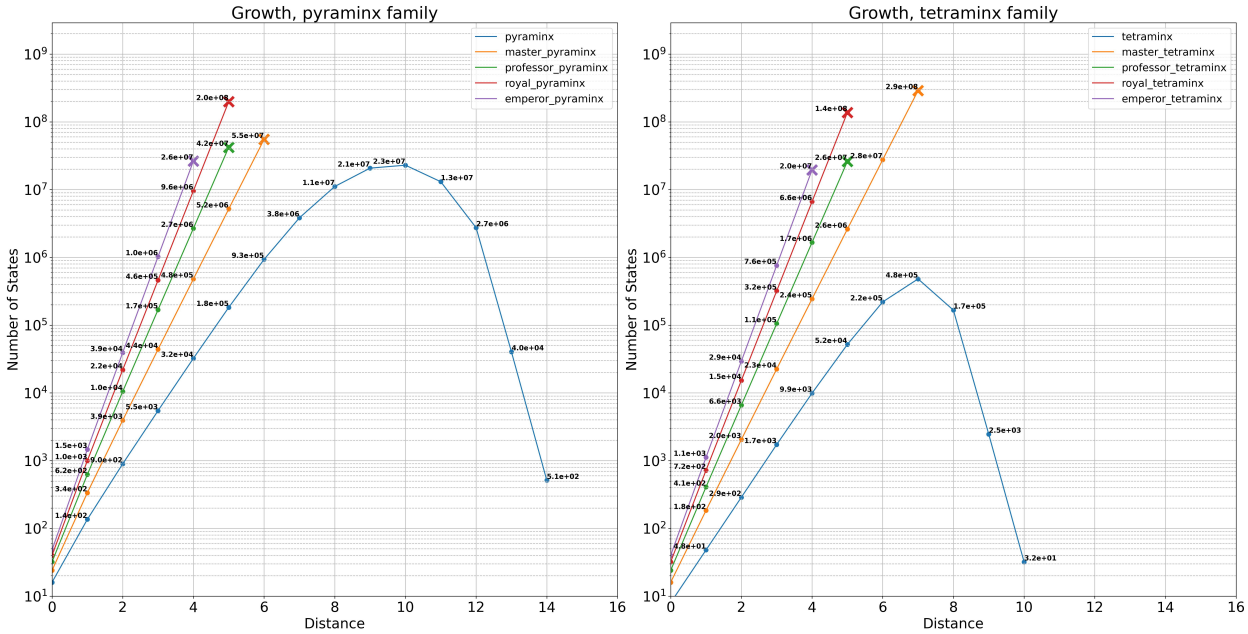
10.2.4 Results

Size	regular	master	professor	royal	emperor
Num layers	15	??	??	??	??
Layers computed	15	7	6	6	5
Layer 1	16	24	32	40	48
Layer 2	136	336	624	1000	1464
Layer 3	896	3936	10464	21824	39328
Layer 4	5456	43840	168448	459920	1023312
Layer 5	32296	479112	2672152	9581072	26357088
Layer 6	182432	5166228	42024236	198363440	
Layer 7	931983	55073497			
Layer 8	3829067				
Layer 9	11108868				
Layer 10	20736353				
Layer 11	22907032				
Layer 12	13067528				
Layer 13	2739808				
Layer 14	40336				
Layer 15	512				

Table 27: Pyraminx Layers Counts

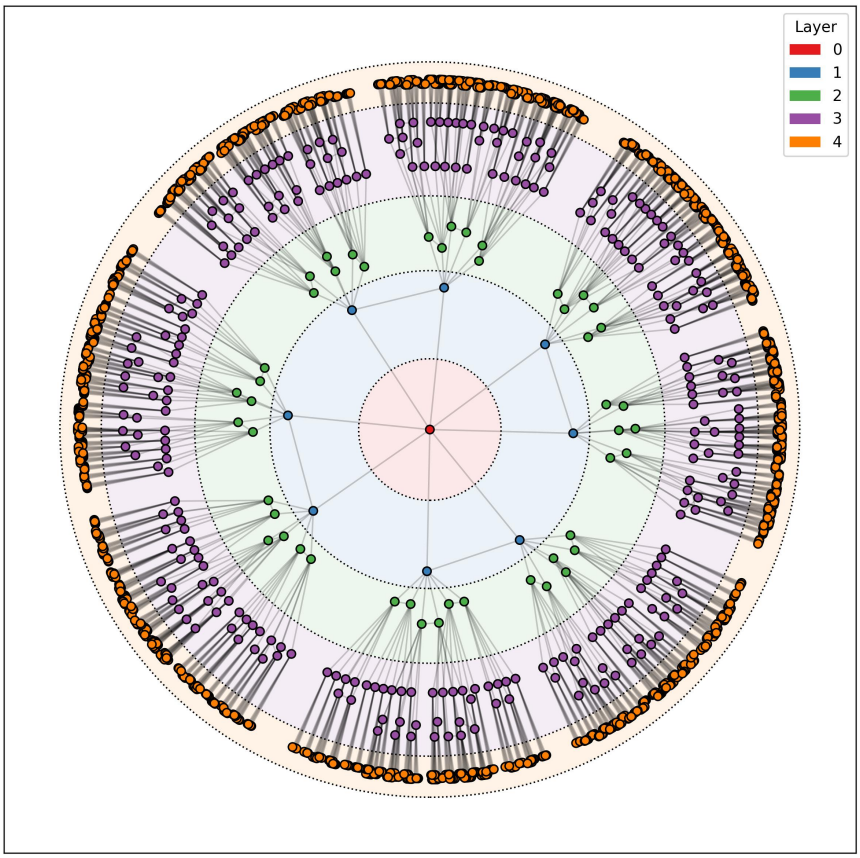
Size	regular	master	professor	royal	emperor
Num layers	11	??	??	??	??
Layers computed	11	7	6	6	5
Layer 1	8	16	24	32	40
Layer 2	48	184	408	720	1120
Layer 3	288	2048	6592	15264	29376
Layer 4	1728	22512	105136	319488	760128
Layer 5	9896	243720	1659416	6635280	19534560
Layer 6	51808	2607700	26008172	137113520	
Layer 7	220111	27609465			
Layer 8	480467	289013734			
Layer 9	166276				
Layer 10	2457				
Layer 11	32				

Table 28: Tetraminx Layers Counts

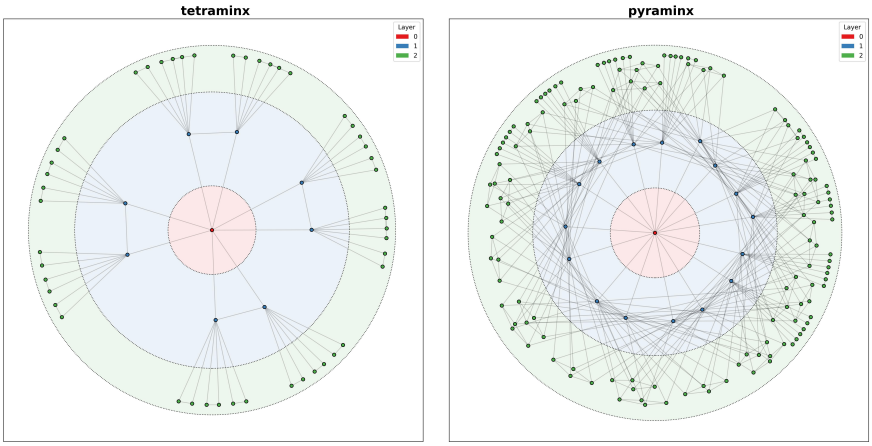


(a) Pyraminx and tetraminx growth

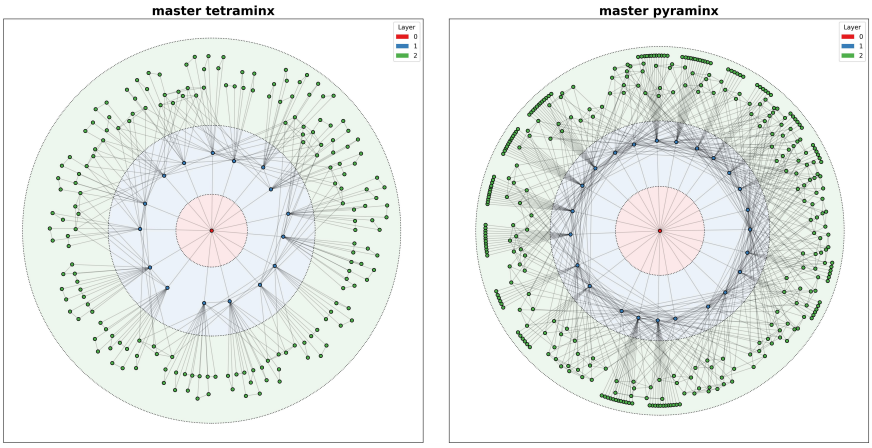
10.2.5 Graphs images



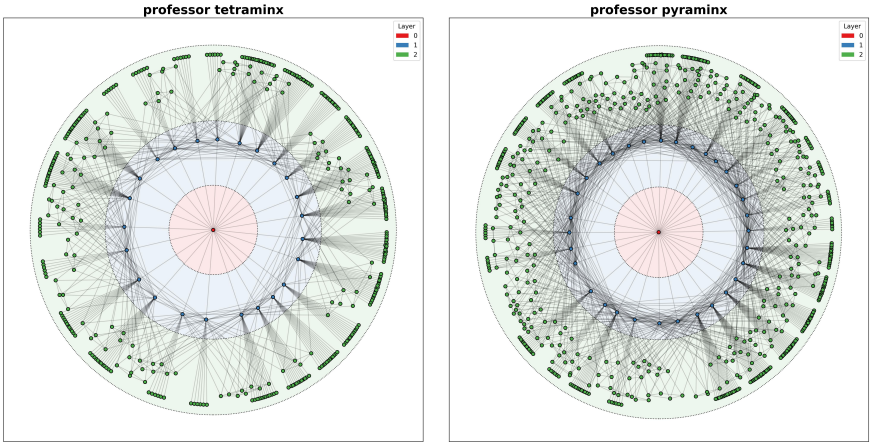
(a) Tetraminx, first 4 layers



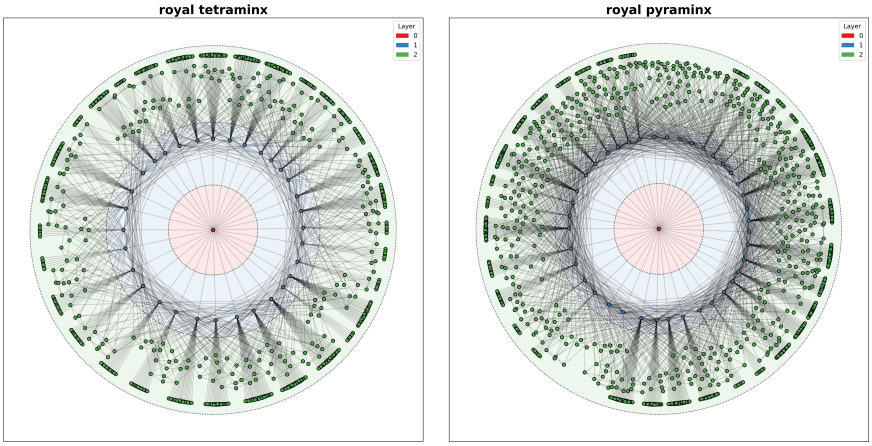
(a) pyraminx and tetraminx Cayley graph, 2 layers



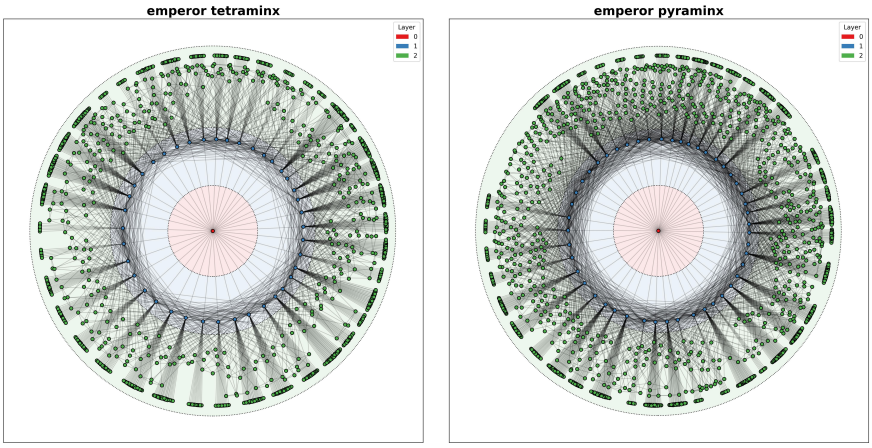
(a) Master pyraminx and tetraminx Cayley graph, 2 layers



(a) Professor pyraminx and tetraminx Cayley graph, 2 layers



(a) Royal pyraminx and tetraminx Cayley graph, 2 layers



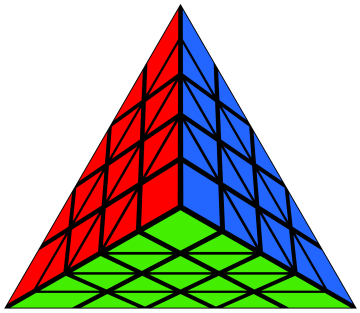
(a) Emperor pyraminx and tetraminx Cayley graph, 2 layers

10.3 Professor Tetraminx

10.3.1 Common information

The Professor Tetraminx is an advanced variant of the classic Tetraminx, sharing its tetrahedral shape but with a more complex internal mechanism. Visually, it appears almost identical to the standard Tetraminx, but its additional layers introduce a higher solving difficulty.

10.3.2 Image



(a) Professor Tetraminx

References:
Image source: Tomas Rokicki's Twizzle site

10.3.3 Generators

Notation:
R – right vertex rotation

L – left vertex rotation
B – back vertex rotation
U – top vertex rotation
BR – right face rotation
BL – left face rotation
F – front face rotation
D – bottom face rotation

Generators:
R:=(7,13,23)(8,14,24)(31,33,32);
2R:=(41,53,57)(42,54,58)(65,69,71)(66,70,72)(76,79,84);
2BL:=(5,21,17)(6,22,18)(43,59,49)(44,60,50)(73,78,74)(75,83,81)(85,87,86);
BL:=(1,12,4)(2,11,3)(9,20,15)(10,19,16)(25,36,28)(26,34,29)(27,35,30)(37,48,40)(38,47,39)(45,56,51)(46,55,52)(61,68,64)(62,67
L:=(1,6,15)(2,5,16)(25,27,26);
2L:=(43,48,56)(44,47,55)(61,66,67)(62,65,68)(73,75,80);
2BR:=(7,20,12)(8,19,11)(37,51,42)(38,52,41)(76,82,78)(77,84,81)(85,88,87);
BR:=(3,21,14)(4,22,13)(9,23,18)(10,24,17)(28,35,31)(29,36,32)(30,34,33)(39,57,50)(40,58,49)(45,59,54)(46,60,53)(63,71,70)(64
B:=(9,12,22)(10,11,21)(34,36,35);
2B:=(39,52,60)(40,51,59)(63,68,72)(64,67,71)(77,78,83);
2F:=(3,24,16)(4,23,15)(45,58,48)(46,57,47)(74,84,80)(75,82,79)(86,87,88);
F:=(1,18,7)(2,17,8)(5,19,14)(6,20,13)(25,30,32)(26,28,33)(27,29,31)(37,54,43)(38,53,44)(41,55,50)(42,56,49)(61,70,65)(62,69,66
U:=(3,17,19)(4,18,20)(28,30,29);
2U:=(37,45,49)(38,46,50)(61,63,69)(62,64,70)(74,81,82);
2D:=(1,13,9)(2,14,10)(39,55,53)(40,56,54)(73,79,77)(76,83,80)(85,86,88);
D:=(5,24,11)(6,23,12)(7,22,15)(8,21,16)(25,31,34)(26,32,35)(27,33,36)(41,60,47)(42,59,48)(43,58,51)(44,57,52)(65,72,67)(66,71

**Note:* The optional number prefix indicates the layer to rotate, counting outward from the vertex or face specified by the letter.

References:
The generators were taken from a special .json file.

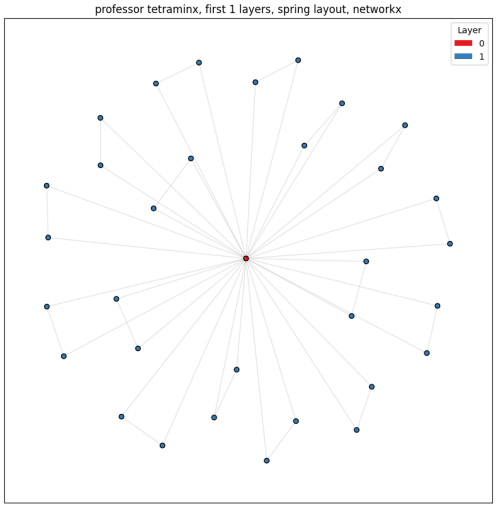
10.3.4 Growth function

Using CayleyPy, we were able to calculate the growth function up to layer 6.

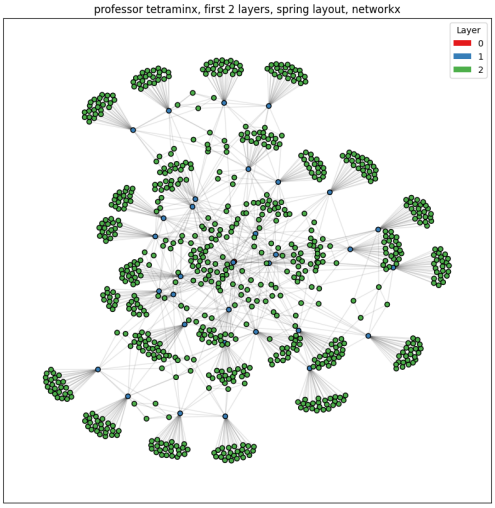
Layer	*Count*
0	1
1	32
2	792
3	18928
4	447568
5	10517672
6	246209428

Table 29: Professor Tetraminx growth.

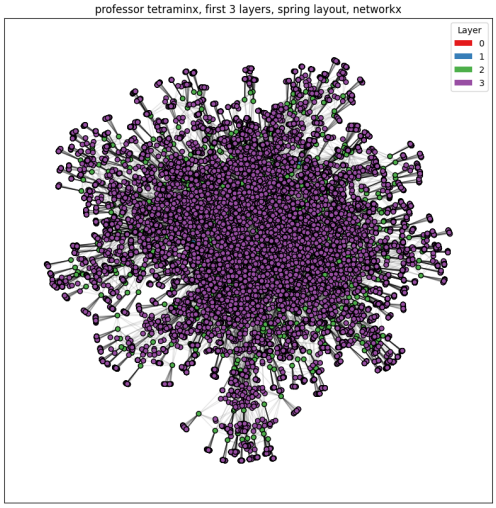
10.3.5 Graph images



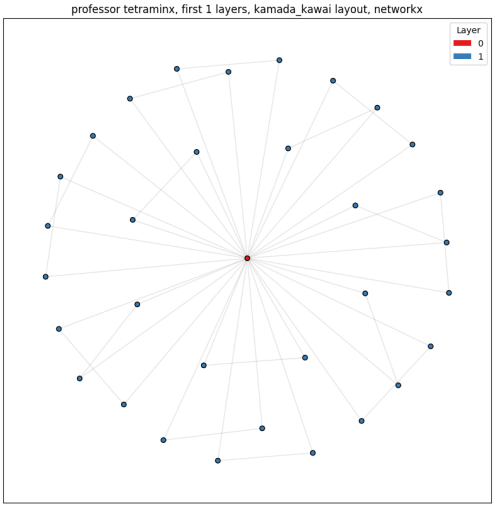
(a)



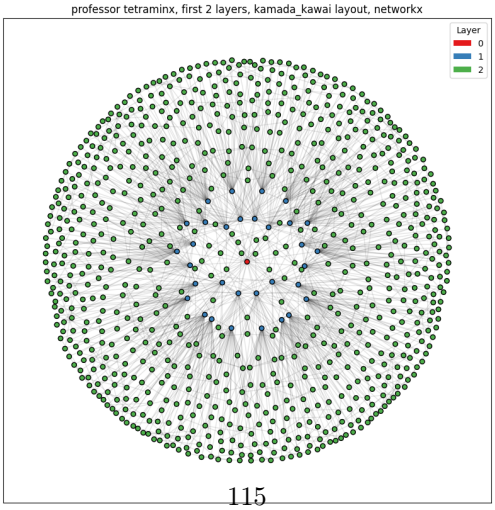
(b)



(c)



(d)



(e)

References:
Our Kaggle notebook in which analysis has been performed

10.4 Megaminx

10.4.1 Common information

This is a variant of the Rubik's cube, in the shape of a dodecahedron. The 3x3x3 version is considered below, there is also a 5x5x5 and similar.

The world record single solve is 22.89 seconds, set by Aidan Grainger of United Kingdom on 18th May, 2025 at Weston-Super-Mare Spring 2025 in Weston-Super-Mare, United Kingdom.

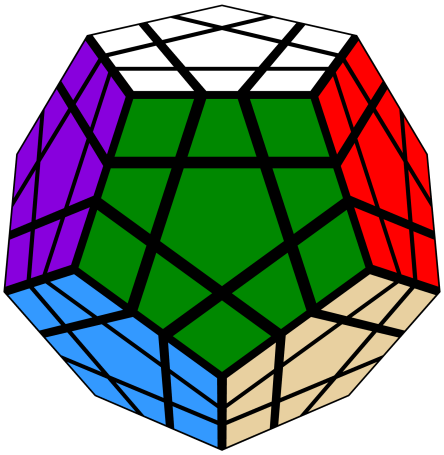
There are $20!/2$ ways to arrange the corners and 319 ways to orient them, since the orientation of the last corner depends on that of the preceding ones. There are $30!/2$ ways to arrange the edges and 229 ways to flip them. $20! \times 3^{19} \times 30! \times 2^{27} \approx 1.01 \times 10^{68}$.

The full number is
100 669 616 553 523 347 122 516 032 313 645 505 168 688 116 411 019 768 627 200 000 000 000
(roughly 101 unvigintillion on the short scale or 101 undecillion on the long scale).

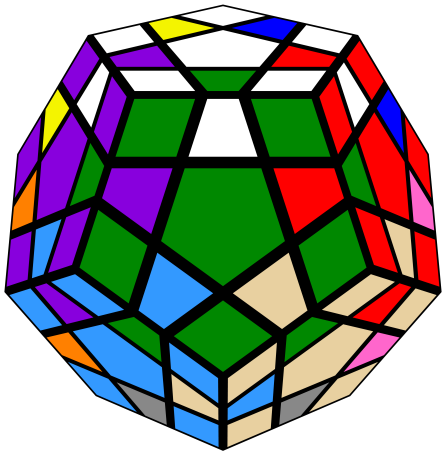
References:
Megaminx on the Wikipedia
Megaminx on Tomas Rokicki's Twizzle site
Our Kaggle notebook in which analysis has been performed

See also:
Megaminx on the Jaapsch
Megaminx on the Speedsolving
Mathematics of the Rubik's cube. W. D. Joyner (4.8 paragraph)

10.4.2 Images



(a) Megaminx



(b) Megaminx - superflip permutation

References:
Picture from Tomas Rokicki's Twizzle site

10.4.3 Generators

Notation:
U - up face rotation
D - down face rotation
F - front face rotation
B - back face rotation
L - left upper face rotation
DR - back right lower face rotation
BL - back left upper face rotation
FR - right lower face rotation
BR - back right upper face rotation
FL - left lower face rotation
R - right upper face rotation
DL - back left lower face rotation

Generators:
U:=(1,4,19,22,7)(2,5,20,23,8)(3,6,21,24,9)(61,63,75,73,77)(62,64,76,74,78);
D:=(37,55,58,52,40)(38,56,59,53,41)(39,57,60,54,42)(93,115,117,111,95)(94,116,118,112,96);
F:=(1,9,31,35,12)(2,7,32,36,10)(3,8,33,34,11)(61,66,87,68,80)(62,65,88,67,79);
B:=(25,41,52,47,30)(26,42,53,48,28)(27,40,54,46,29)(97,112,113,108,99)(98,111,114,107,100);
L:=(1,11,15,16,5)(2,12,13,17,6)(3,10,14,18,4)(63,79,69,81,71)(64,80,70,82,72);
DR:=(43,47,54,59,51)(44,48,52,60,49)(45,46,53,58,50)(101,103,113,117,119)(102,104,114,118,120);
BL:=(4,17,27,28,20)(5,18,25,29,21)(6,16,26,30,19)(71,84,99,86,76)(72,83,100,85,75);
FR:=(31,50,60,56,36)(32,51,58,57,34)(33,49,59,55,35)(87,109,120,115,90)(88,110,119,116,89);
BR:=(19,29,48,43,23)(20,30,46,44,24)(21,28,47,45,22)(73,86,107,103,106)(74,85,108,104,105);
FL:=(10,35,57,38,15)(11,36,55,39,13)(12,34,56,37,14)(67,90,94,91,70)(68,89,93,92,69);
R:=(7,24,43,50,33)(8,22,44,51,31)(9,23,45,49,32)(65,77,105,101,109)(66,78,106,102,110);
DL:=(13,38,40,26,18)(14,39,41,27,16)(15,37,42,25,17)(81,91,95,97,83)(82,92,96,98,84);

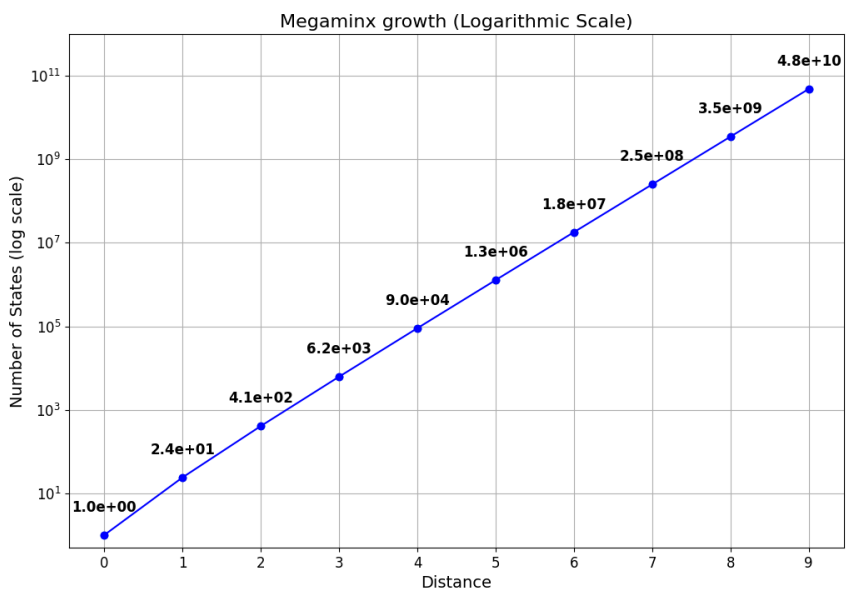
References:
Generators from Tomas Rokicki's Twizzle site

10.4.4 Growth function

Using CayleyPy, we were able to calculate the growth function up to layer 7. All values matched the values calculated by Tomas Rokicki. He was able to calculate values up to layer 9.

Layer	*Count*
0	1
1	24
2	408
3	6208
4	90144
5	1280160
6	17975460
7	250871064
8	3489240564
9	48432813942

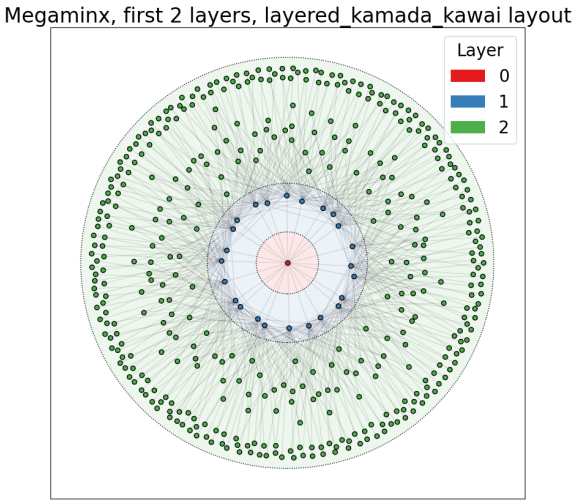
Table 30: Megaminx growth.

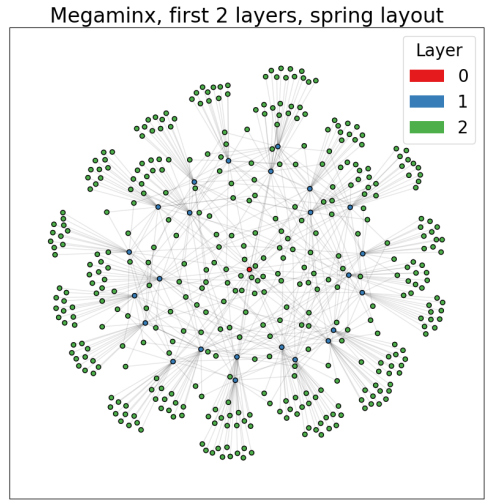


The average growth rate of the linear graph is 14.5.

References:
Our Kaggle notebook in which analysis has been performed
Growth from Tomas Rokicki's GitHub

10.4.5 Graph images





References:
Our Kaggle notebook in which analysis has been performed

10.4.6 Diameter

God's number lower bound for Megaminx (HTM) - 48
God's number lower bound for Megaminx (QTM) - 59

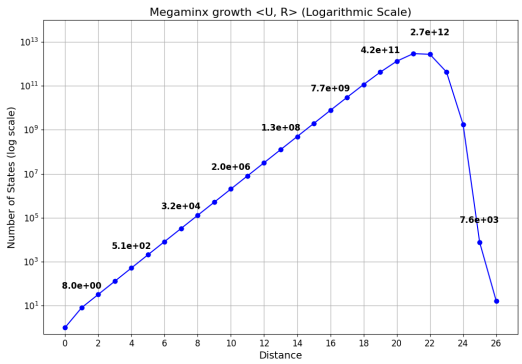
God's number upper bound for Megaminx (HTM) - 194
God's number upper bound for Megaminx (QTM) - unknown

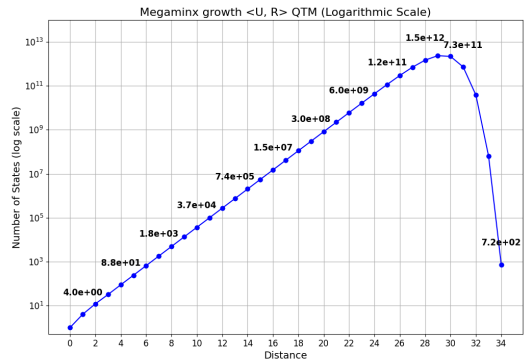
God's number for Megaminx <U,R>: 26
God's number for Megaminx <U,R> QTM: 34

References:
Discussion about the Megaminx diameter on the Speedsolving
Discussion about the Megaminx lower bound on the Speedsolving
Table of God's numbers on the Speedsolving Wiki
Growth values for Megaminx <U,R>

$Layer$	$< U, R >$	$< U, R > QTM$
0	1	1
1	8	4
2	32	12
3	128	32
4	512	88
5	2048	240
6	8176	656
7	32400	1792
8	128608	4896
9	509927	13376
10	2021092	36504
11	8003554	99528
12	31678834	271359
13	125307530	739584
14	495157495	2015884
15	1953478687	5493494
16	7682157029	14965234
17	29984818955	40757056
18	114831820323	110960833
19	417957132495	301946732
20	1325871801018	821049717
21	2934550836931	2230003682
22	2736869164561	6044495440
23	427540149248	16321629020
24	1770867597	43746575372
25	7595	115506765437
26	16	295645071578
27	0	709214579066
28		1487087816072
29		2368788759019
30		2189306113522
31		726260727313
32		38159404443
33		64787090
34		724
35		0

Table 31: Megaminx <U,R> and <U,R> QTM growth.





The average growth rate of the linear graph Megaminx <U, R> is 4 and 2.7 in Megaminx <U, R> QTM.

10.4.7 Superflip

The superflip is a special configuration, in which all the edge and corner pieces are in the correct permutation, and the eight corners are correctly oriented, but all twelve edges are oriented incorrectly ("flipped").

Tomas Rokicki found a sequence that leads to a superflip in 55 steps ((T. Rokicki 2023):
B2' DL' B2 DL' D' DL2 D' DL2' B' DL2 B' DL' B2 DL' D2' B DL2 B D2' B2' DL' D2' B' L F U F R F2 U' L2' U2 L U2 F L F2' L' F2' L U BL DL FL DL D FR BL' D B DR B BL U BR

If we use the generators defined above, we get the following permutation (apply generators from left to right):
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 61, 60, 63, 62, 65, 64, 67, 66, 69, 68, 71, 70, 73, 72, 75, 74, 77, 76, 79, 78, 81, 80, 83, 82, 85, 84, 87, 86, 89, 88, 91, 90, 93, 92, 95, 94, 97, 96, 99, 98, 101, 100, 103, 102, 105, 104, 107, 106, 109, 108, 111, 110, 113, 112, 115, 114, 117, 116, 119, 118.

It is not difficult to verify that the permutation is indeed central, and also squared gives 1.

We believe that the length of the smallest superflip sequence is close to the diameter value.

References:
Another our Kaggle notebook in which analysis about superflip has been performed
Superflip on Wikipedia
Megaminx superflip on Twizzle
Megaminx superflip on Randelshofer
Discussion about the Megaminx superflip

10.4.8 Graph spectrum

TODO

10.5 MiniPypamorphix

10.6 Mastermorphix

10.7 Master Pyramorphix

10.7.1 Common information

The Master Pyramorphix is a more advanced version of the Pyramorphix. It's an edge-turning tetrahedral puzzle that changes shape when twisted, creating many irregular forms.

It has 4 corners, 4 face centers, 6 edges, and 12 non-center face pieces. The edges rotate in place, while the other pieces can move. The face centers and corners (though differently shaped) can swap places, and the non–center face pieces can flip, allowing complex shape-shifting.

10.7.2 Image

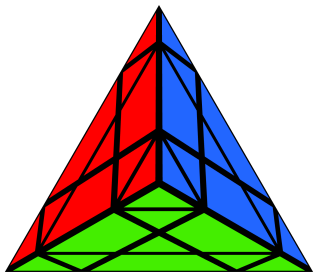


Figure 66: Master Pyramorphix

References:
Picture from Tomas Rokicki's Twizzle site

10.7.3 Generators

DF:=(7,12)(8,10)(9,11)(25,26)(29,30)(33,34)
RL:=(1,4)(2,5)(3,6)(27,28)(31,32)(35,36)
LF:=(1,9)(2,7)(3,8)(13,14)(21,23)(29,31)
RD:=(4,11)(5,12)(6,10)(15,16)(22,24)(30,32)
DL:=(4,9)(5,7)(6,8)(19,20)(22,23)(34,35)
FR:=(1,11)(2,12)(3,10)(17,18)(21,24)(33,36)

10.8 Starminx

10.8.1 Common information

Starminx is a dodecahedral puzzle with faces in the form of inscribed pentagrams
https://rubiks.fandom.com/wiki/Starminx

10.9 Picture

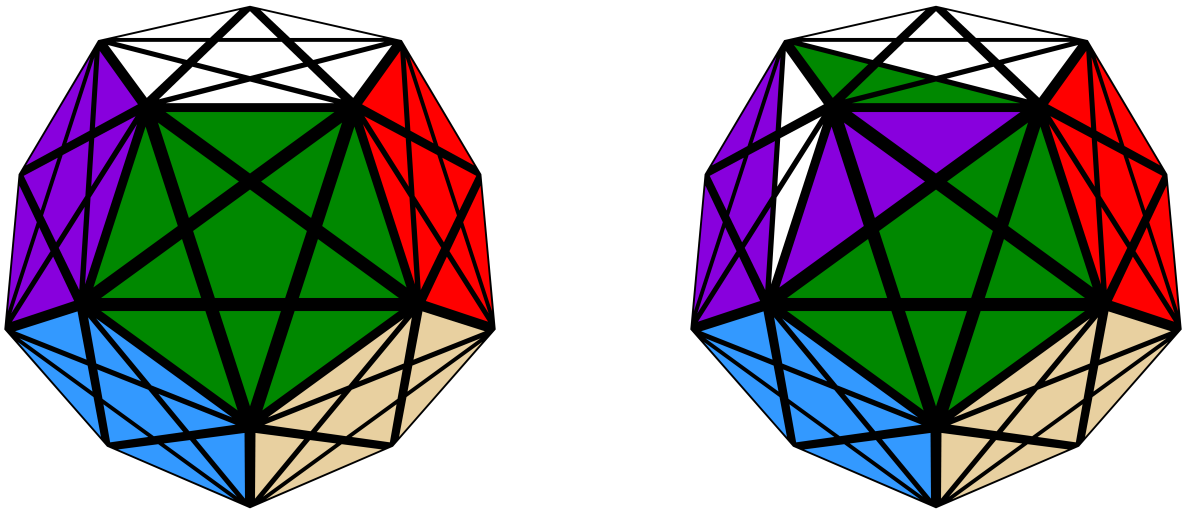


Figure 67: Starminx puzzle

10.9.1 Generators

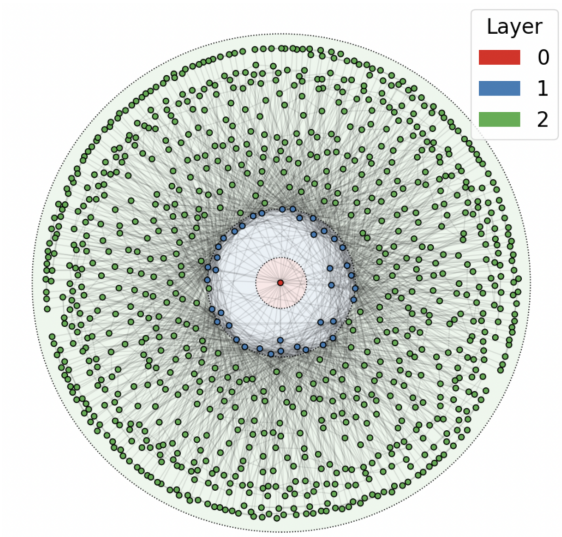
UFR:=(37,45,48)(38,46,47)(103,105,104)
DIBF:=(25,52,40)(26,51,39)(97,99,98)
ULF:=(9,32,45)(10,31,46)(85,87,86)
DBFE:=(19,22,52)(20,21,51)(100,102,101)
AFL:=(9,12,30)(10,11,29)(73,75,74)
BREBF:=(21,58,59)(22,57,60)(82,84,83)
UBLL:=(13,32,35)(14,31,36)(76,78,77)
DEC:=(7,20,41)(8,19,42)(70,72,71)
ILBL:=(15,35,34)(16,36,33)(79,81,80)
RCE:=(3,44,7)(4,43,8)(118,120,119)
UBRBL:=(1,14,55)(2,13,56)(94,96,95)
DCA:=(27,54,42)(28,53,41)(112,114,113)
BFBLBR:=(49,59,55)(50,60,56)(115,117,116)
FAC:=(5,53,11)(6,54,12)(67,69,68)
URBR:=(1,17,48)(2,18,47)(61,63,62)
DAI:=(23,27,26)(24,28,25)(91,93,92)
CRF:=(3,6,38)(4,5,37)(64,66,65)
BLBFI:=(15,40,49)(16,39,50)(106,108,107)
EBRR:=(17,57,44)(18,58,43)(109,111,110)
LIA:=(23,34,29)(24,33,30)(88,90,89)

10.9.2 Growth function

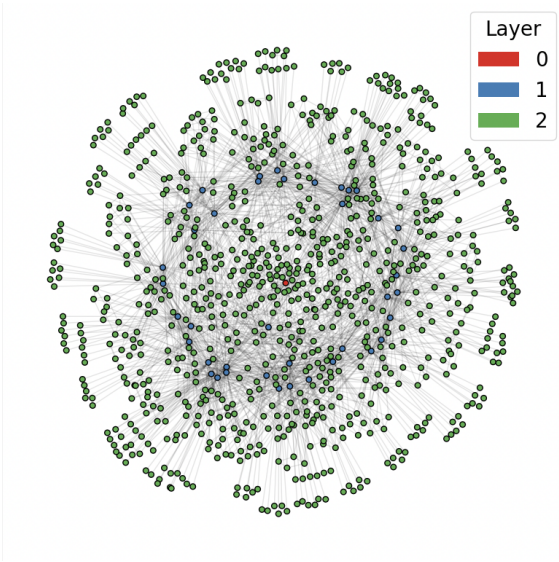
Layer	*Growth*	*Growthratio*
0	1	-
1	40	40
2	880	22
3	6208	7.05
4	14400	2.32
5	198080	13.76
6	2437424	12.31
7	27788320	11.4
8	299849864	10.79
9	3104977640	10.36
10	31146164970	10.03

Table 32: Starminx growth

10.9.3 Graph visualization



(a) Starminx Kamada-Kawai layout for the first 2 layers



(b) Starminx Spring layout for the first two layers

Figure 68: Graph visualizations of Starminx

Layers	Eigenvalues (value ^{multiplicity})
2	-10.0^{1}, -6.68^{20}, -5.88^{3}, -5.84^{4}, -5.52^{4}, -5.35^{5}, -5.14^{3}, 0.0^{841}, 5.68^{20}, 6.14^{3}, 6.35^{5}, 6.52^{4}, 6.84^{4}, 6.88^{3}, 11.0^{1}
3	-12.38^{1}, -10.35^{20}, -9.46^{3}, -9.41^{4}, -8.95^{4}, -8.7^{5}, -8.36^{3}, -7.08^{160}, -6.68^{120}, -6.19^{60}, -6.16^{20}, -6.12^{40}, -5.9^{20}, -5.83^{20}, -5.78^{40}, -5.68^{29}, -5.66^{40}, -5.64^{7}, -5.62^{3}, -5.58^{6}, -5.57^{3}, -5.56^{10}, -5.55^{3}, -5.54^{40}, -5.51^{4}, -5.49^{3}, -5.44^{5}, -5.43^{1}, -5.41^{20}, -5.4^{9}, -5.38^{3}, -5.35^{45}, -5.31^{4}, -5.12^{4}, -5.11^{5}, -5.09^{3}, -5.08^{1}, -5.07^{3}, -5.01^{5}, -5.0^{3}, -4.97^{4}, -4.95^{4}, -4.91^{3}, -4.89^{6}, -4.88^{5}, -4.85^{5}, -4.81^{5}, -4.72^{4}, -4.71^{5}, -4.69^{1}, -4.67^{4}, -4.65^{5}, -4.62^{7}, -4.56^{5}, -4.49^{4}, -4.45^{5}, -4.4^{3}, -4.36^{9}, -4.33^{3}, -4.28^{3}, -4.21^{1}, -4.16^{5}, -4.11^{1}, -4.05^{5}, -0.63^{20}, 0.0^{13520}, 0.64^{19}, 4.85^{1}, 5.08^{160}, 5.46^{40}, 5.54^{20}, 5.66^{40}, 5.68^{120}, 5.73^{40}, 5.8^{20}, 5.83^{20}, 5.99^{5}, 6.0^{40}, 6.08^{1}, 6.14^{5}, 6.16^{20}, 6.18^{20}, 6.19^{40}, 6.2^{3}, 6.27^{3}, 6.28^{4}, 6.3^{3}, 6.31^{5}, 6.35^{4}, 6.37^{9}, 6.41^{4}, 6.43^{11}, 6.44^{5}, 6.47^{8}, 6.48^{20}, 6.53^{10}, 6.55^{3}, 6.6^{45}, 6.64^{12}, 6.71^{5}, 6.73^{3}, 6.77^{4}, 6.8^{9}, 6.86^{5}, 6.87^{3}, 6.89^{1}, 6.9^{5}, 6.91^{4}, 6.94^{3}, 6.95^{3}, 7.0^{8}, 7.03^{7}, 7.04^{4}, 7.07^{5}, 7.08^{1}, 7.14^{3}, 7.19^{3}, 7.25^{5}, 7.3^{4}, 7.37^{4}, 7.43^{5}, 7.44^{1}, 7.45^{5}, 10.16^{20}, 10.48^{3}, 10.83^{5}, 11.1^{4}, 11.61^{4}, 11.67^{3}, 14.61^{1},

Table 33: Eigenvalues for different numbers of layers (format: value^{multiplicity})

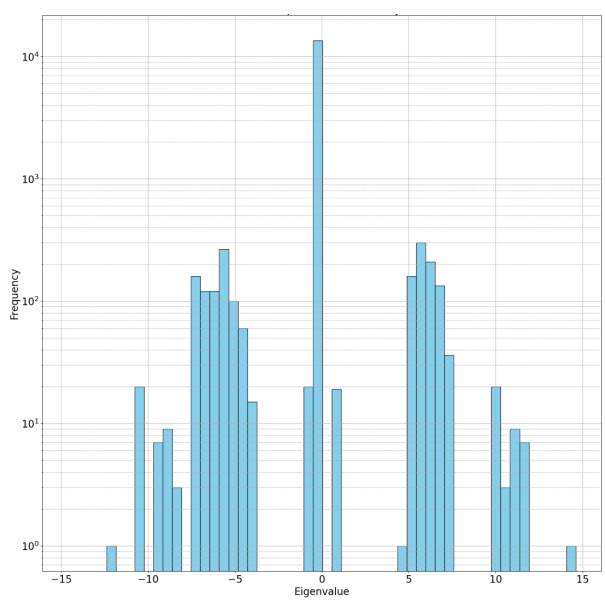


Figure 69: Starminx spectrum first 3 layers

10.10 Starminx 2

10.10.1 Generators

U:=(1,2,8,24,50)(3,10,27,54,59)(4,11,28,55,60)(5,12,29,51,57)(7,9,26,53,58)(61,63,75,65,79)(62,64,76,66,80)(67,82,74,77,108)(6
D:=(14,22,47,39,31)(17,38,46,33,18)(19,42,49,34,20)(21,45,35,30,23)(36,44,41,40,37)(93,115,110,103,96)(94,116,109,104,95)(97
F:=(1,3,14,35,56)(4,10,26,52,57)(8,25,49,17,11)(15,31,53,50,60)(16,27,24,51,38)(61,66,87,115,92)(62,65,88,116,91)(67,80,107,9
B:=(6,19,44,22,7)(12,18,41,23,13)(20,33,43,47,21)(28,32,40,34,29)(36,48,59,58,45)(71,98,113,102,74)(72,97,114,101,73)(83,95,1
L:=(1,4,16,39,58)(2,10,31,45,6)(5,11,27,53,56)(17,32,54,50,57)(18,28,24,52,42)(61,67,91,69,81)(62,68,92,70,82)(63,80,112,93,72
DR:=(3,7,21,37,15)(9,13,20,36,14)(22,23,34,46,48)(25,29,33,44,30)(41,49,51,55,43)(77,85,100,117,88)(78,86,99,118,87)(101,103
BL:=(1,5,18,43,59)(2,11,32,47,7)(6,12,28,54,58)(8,27,39,21,13)(19,33,55,50,56)(61,70,95,85,76)(62,69,96,86,75)(63,82,71,84,73)
FR:=(9,23,40,16,10)(14,30,49,38,15)(25,34,37,31,26)(35,52,51,46,36)(42,57,60,48,44)(87,110,117,116,89)(88,109,118,115,90)(10
BR:=(1,6,20,46,60)(2,12,33,48,3)(7,13,29,55,59)(8,28,43,22,9)(23,25,24,54,47)(63,83,104,105,66)(64,84,103,106,65)(73,86,102,7
FL:=(4,15,36,19,5)(16,31,35,42,17)(26,30,41,32,27)(37,45,56,57,38)(39,53,52,49,40)(67,89,118,97,70)(68,90,117,98,69)(91,112,1
R:=(2,13,34,38,4)(3,9,25,51,60)(8,29,46,15,10)(14,26,24,55,48)(22,30,52,50,59)(65,77,105,87,107)(66,78,106,88,108)(75,102,109
DL:=(5,17,40,21,6)(11,16,37,20,12)(18,32,39,45,19)(35,44,43,54,53)(41,47,58,56,42)(69,93,97,95,71)(70,94,98,96,72)(81,92,119,

10.10.2 Growth function

Layer	*CayleyPy*	*Tomas Rokicki*
0	1	1
1	24	24
2	528	528
3	11568	11568
4	253464	253464
5	5553040	5553040
6	121654680	121654680
7		2665148160
8		58386467340

Table 34: Starminx 2 growth.

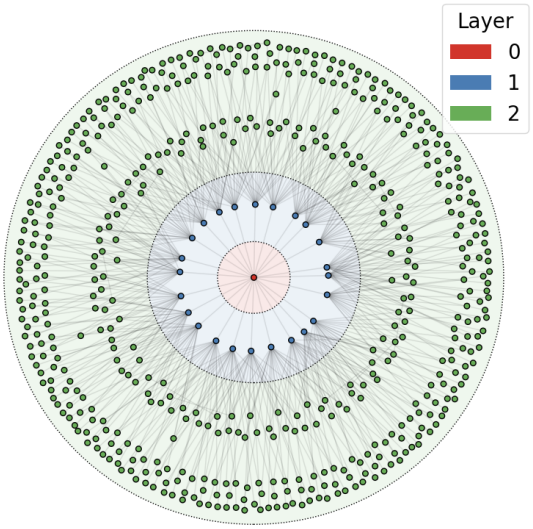
10.10.3 Graph visualization

10.10.4 Eigenvalues and counts (3 layers)

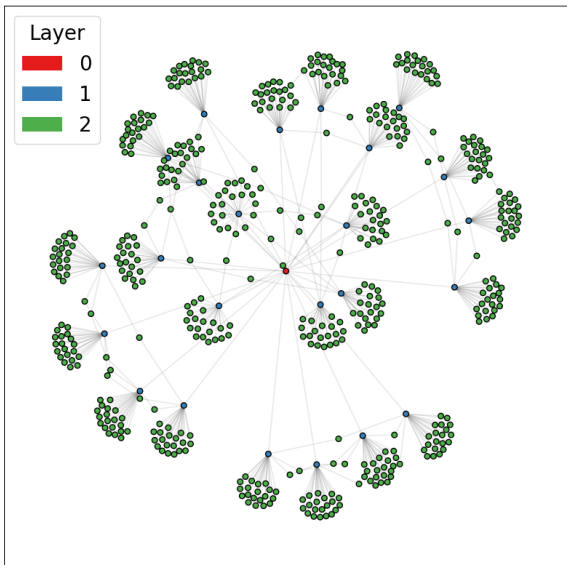
[]

Layers	Eigenvalues (value ^{multiplicity})
2	-7.0^{1}, -5.0^{5}, -4.8^{12}, -4.58^{6}, -0.0^{505}, 4.58^{6}, 4.8^{12}, 5.0^{5}, 7.0^{1}
3	-8.02^{1}, -7.04^{5}, -6.94^{12}, -6.76^{6}, -5.23^{12}, -5.0^{96}, -4.81^{6}, -4.8^{240}, -4.69^{6}, -4.67^{12}, -4.58^{120}, -4.24^{6}, -4.13^{6}, -3.02^{1}, 0.0^{11063}, 3.03^{1}, 4.18^{12}, 4.58^{120}, 4.65^{6}, 4.69^{6}, 4.76^{12}, 4.8^{240}, 5.0^{96}, 5.2^{6}, 5.25^{6}, 6.8^{6}, 6.9^{12}, 7.08^{5}, 8.05^{1}

Table 35: Eigenvalues for different numbers of layers (format: value^{multiplicity})



(a) Starminx 2 Kamada-Kawai layout for the first two layers



(b) Starminx 2 Spring layout for the first two layers

Figure 70: Graph visualizations

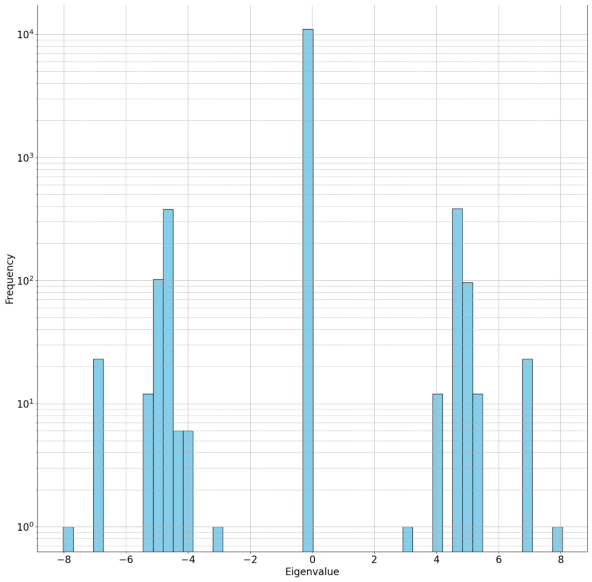


Figure 71: Starminx 2 spectrum first 3 layers

10.11 Dino

The Dino Cube is a combinatorial twisty puzzle consisting of 12 movable edge pieces and no fixed centers. Its full configuration space contains 239500800 distinct states, as computed using the `twsearch` tool developed by Tomas Rokicki [55]. This value represents the total number of reachable states without accounting for rotational symmetries; after factoring out the 12-fold rotational symmetry group, the number of essentially distinct configurations reduces to 19958400 [57]. The legal moves generate a subgroup of the symmetric group on 12 elements, constrained by the puzzle's structural rules, such as edge parity and local move restrictions. The diameter of the state graph has been shown to be 13, meaning any state can be solved from any other in 13 moves or fewer, establishing the *God's number* for this puzzle. This result was confirmed through exhaustive search using `twsearch`, which explores the configuration space using optimized breadth-first enumeration. The basic move set consists of 3-edge rotations around the puzzle's corners (e.g., DFR, ULB), which serve as the generators for the underlying permutation group.

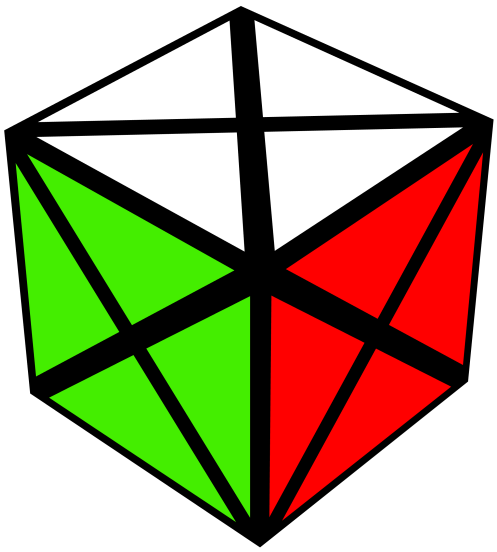


Figure 72: Dino from Twizzle Explorer

Move	Cycle Notation
DRF	(5 18 10)(6 17 9)
UBL	(7 20 14)(8 19 13)
DFL	(11 22 18)(12 21 17)
URB	(3 24 13)(4 23 14)
DBR	(3 15 6)(4 16 5)
ULF	(1 21 8)(2 22 7)
DLB	(11 16 20)(12 15 19)
UFR	(1 24 9)(2 23 10)

Table 36: Generators of the Dino Cube group in cycle notation

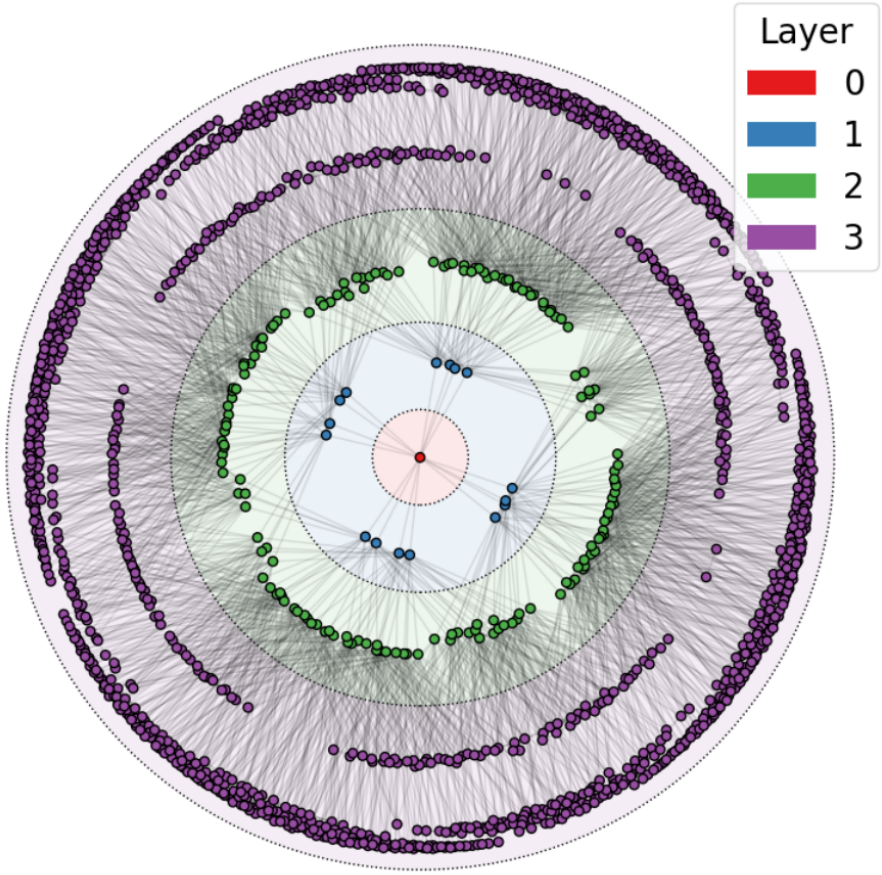


Figure 73: Graph for the first 3 layers (Layered Kamada-Kawai layout)

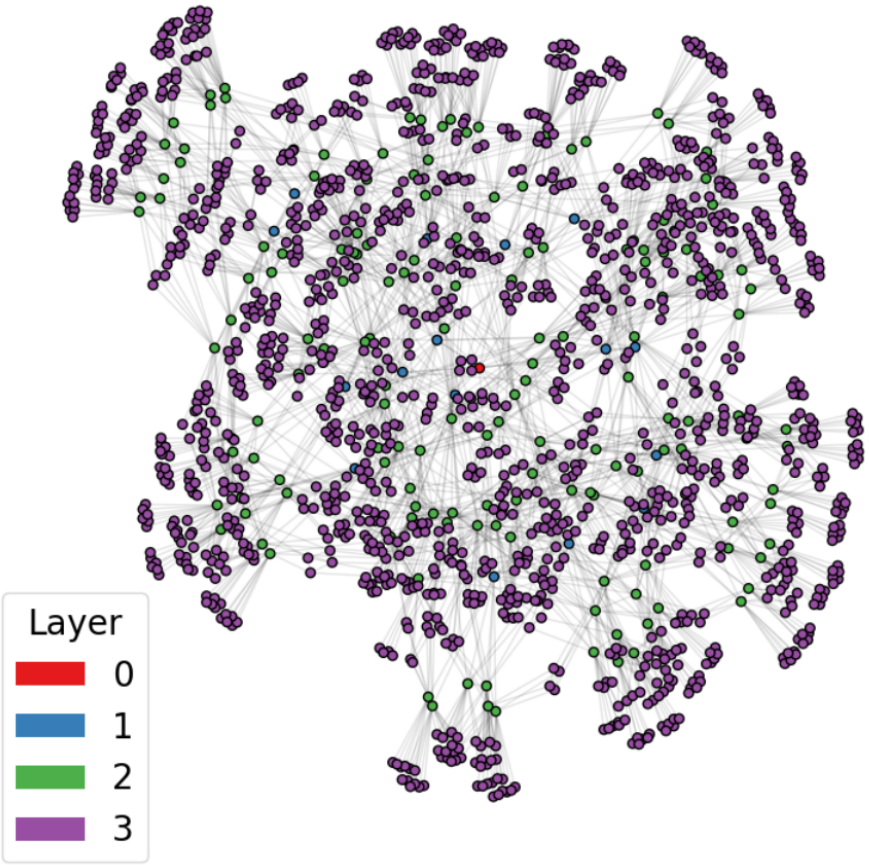


Figure 74: Graph for the first 3 layers (Spring layout)

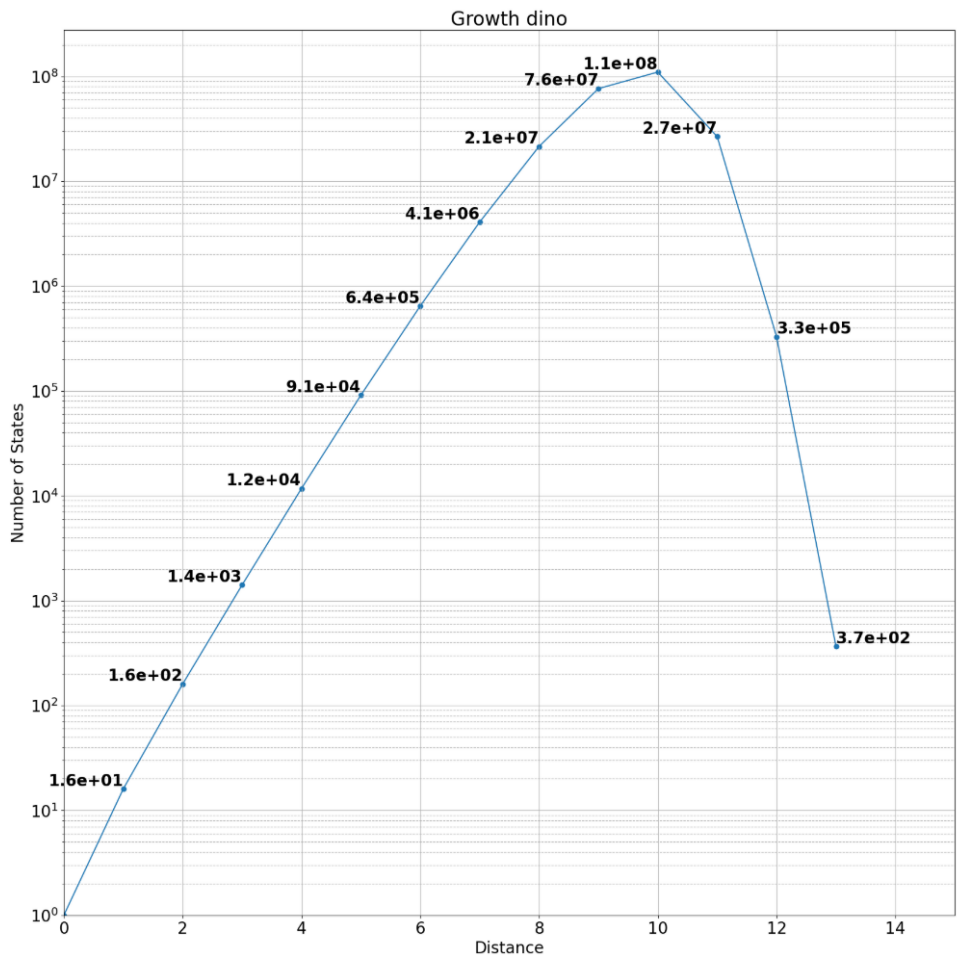


Figure 75: Growth of Dino

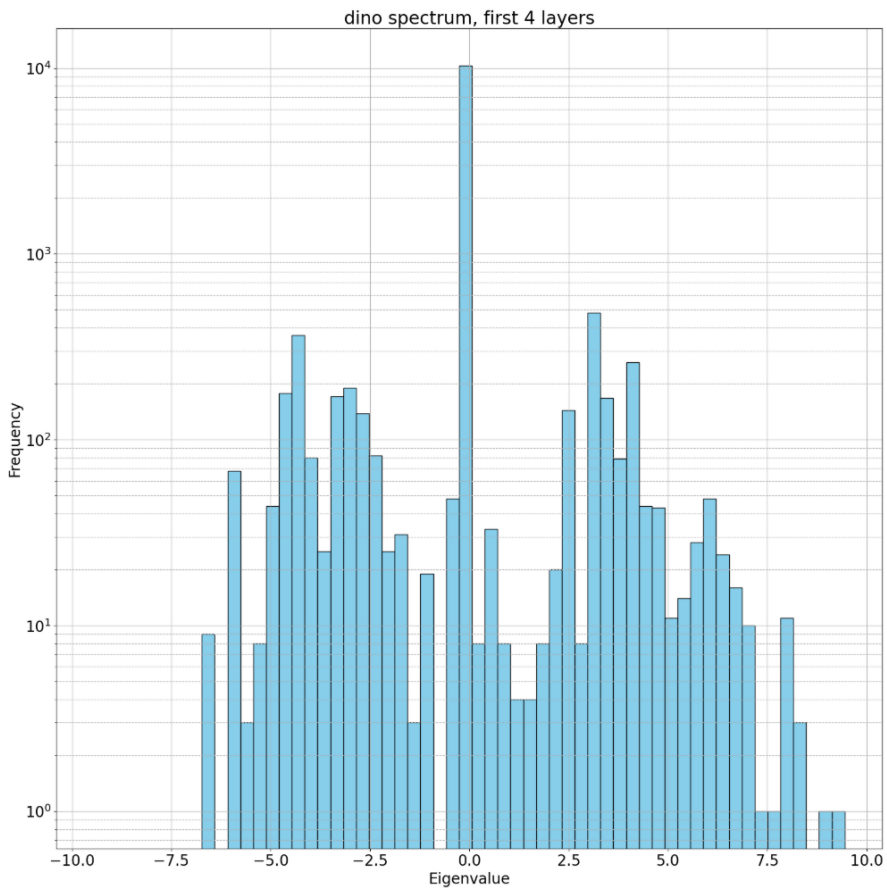


Figure 76: Spectrum for the first 4 layers

Layers	Eigenvalues (value ^{multiplicity})
2	-5.68^{1}, -4.27^{8}, -3.77^{1}, -3.27^{3}, -2.7^{3}, 0.0^{145}, 3.27^{8}, 3.7^{3}, 4.27^{3}, 4.77^{1}, 6.68^{1}
3	-6.39^{1}, -5.84^{8}, -5.35^{1}, -4.99^{3}, -4.61^{16}, -4.54^{3}, -4.27^{48}, -3.85^{8}, -3.52^{1}, -3.38^{16}, -3.37^{1}, -3.18^{3}, -3.17^{3}, -3.14^{4}, -3.08^{8}, -3.0^{6}, -2.75^{16}, -2.55^{1}, -2.51^{3}, -2.5^{3}, -2.32^{3}, -2.05^{1}, -1.8^{4}, -0.57^{8}, 0.0^{1248}, 0.56^{1}, 0.57^{3}, 0.63^{3}, 2.61^{16}, 2.93^{1}, 3.05^{16}, 3.27^{48}, 3.32^{8}, 3.53^{4}, 3.85^{3}, 3.86^{3}, 3.98^{1}, 4.0^{9}, 4.07^{8}, 4.08^{16}, 4.41^{4}, 4.57^{3}, 4.7^{1}, 4.77^{3}, 5.2^{1}, 5.96^{8}, 6.15^{3}, 6.81^{3}, 7.46^{1}, 8.4^{1}
4	-6.73^{1}, -6.47^{8}, -6.03^{1}, -6.01^{12}, -6.0^{4}, -5.93^{3}, -5.85^{20}, -5.84^{16}, -5.83^{12}, -5.64^{3}, -5.38^{8}, -5.07^{16}, -5.02^{1}, -5.0^{8}, -4.95^{2}, -4.93^{2}, -4.92^{3}, -4.91^{3}, -4.87^{1}, -4.86^{8}, -4.78^{3}, -4.76^{3}, -4.61^{144}, -4.6^{16}, -4.49^{12}, -4.42^{3}, -4.41^{33}, -4.36^{12}, -4.34^{3}, -4.32^{3}, -4.3^{15}, -4.29^{6}, -4.28^{3}, -4.27^{240}, -4.24^{1}, -4.2^{12}, -4.16^{5}, -4.15^{27}, -4.14^{3}, -4.12^{5}, -4.06^{12}, -4.02^{3}, -3.93^{12}, -3.86^{12}, -3.85^{28}, -3.84^{8}, -3.74^{4}, -3.67^{1}, -3.54^{12}, -3.53^{8}, -3.46^{8}, -3.44^{3}, -3.42^{8}, -3.41^{12}, -3.4^{1}, -3.39^{4}, -3.38^{48}, -3.37^{19}, -3.36^{1}, -3.35^{12}, -3.28^{1}, -3.27^{16}, -3.26^{2}, -3.23^{3}, -3.22^{6}, -3.21^{19}, -3.2^{5}, -3.18^{3}, -3.17^{4}, -3.16^{18}, -3.15^{6}, -3.14^{2}, -3.13^{9}, -3.11^{6}, -3.1^{21}, -3.09^{5}, -3.08^{21}, -3.07^{17}, -3.06^{3}, -3.05^{6}, -3.04^{3}, -3.02^{12}, -3.01^{8}, -3.0^{32}, -2.99^{3}, -2.97^{3}, -2.96^{2}, -2.95^{3}, -2.94^{3}, -2.91^{3}, -2.84^{1}, -2.83^{8}, -2.81^{12}, -2.8^{12}, -2.78^{3}, -2.76^{4}, -2.75^{37}, -2.74^{10}, -2.73^{21}, -2.72^{8}, -2.68^{2}, -2.67^{12}, -2.66^{3}, -2.65^{1}, -2.62^{3}, -2.55^{1}, -2.52^{8}, -2.51^{16}, -2.5^{3}, -2.49^{5}, -2.47^{2}, -2.46^{5}, -2.42^{3}, -2.41^{1}, -2.32^{16}, -2.29^{1}, -2.28^{3}, -2.25^{16}, -2.21^{3}, -2.2^{8}, -2.18^{9}, -1.97^{2}, -1.95^{6}, -1.87^{3}, -1.8^{16}, -1.79^{3}, -1.68^{1}, -1.67^{6}, -1.64^{2}, -1.42^{3}, -1.23^{3}, -1.16^{4}, -1.15^{12}, -0.58^{12}, -0.57^{36}, 0.0^{10328}, 0.32^{8}, 0.4^{1}, 0.59^{8}, 0.6^{22}, 0.61^{2}, 0.75^{1}, 0.85^{1}, 0.87^{3}, 0.93^{3}, 1.16^{1}, 1.17^{3}, 1.41^{4}, 2.0^{8}, 2.27^{8}, 2.29^{12}, 2.61^{144}, 2.83^{8}, 3.01^{3}, 3.02^{28}, 3.03^{21}, 3.04^{22}, 3.05^{41}, 3.07^{12}, 3.11^{12}, 3.12^{8}, 3.13^{28}, 3.16^{12}, 3.18^{12}, 3.2^{3}, 3.21^{5}, 3.24^{14}, 3.25^{18}, 3.27^{240}, 3.3^{2}, 3.31^{18}, 3.32^{20}, 3.33^{27}, 3.35^{8}, 3.36^{14}, 3.38^{10}, 3.39^{21}, 3.4^{3}, 3.45^{2}, 3.46^{8}, 3.48^{12}, 3.5^{15}, 3.51^{4}, 3.6^{3}, 3.61^{3}, 3.63^{10}, 3.64^{6}, 3.69^{2}, 3.73^{3}, 3.74^{4}, 3.77^{3}, 3.79^{5}, 3.8^{1}, 3.81^{6}, 3.83^{3}, 3.84^{3}, 3.85^{3}, 3.88^{3}, 3.9^{8}, 3.91^{6}, 3.92^{5}, 3.93^{5}, 3.94^{3}, 3.95^{1}, 3.96^{2}, 3.98^{4}, 3.99^{3}, 4.0^{38}, 4.01^{25}, 4.02^{5}, 4.03^{9}, 4.05^{12}, 4.07^{35}, 4.08^{37}, 4.1^{16}, 4.11^{5}, 4.12^{8}, 4.13^{16}, 4.15^{3}, 4.17^{18}, 4.19^{12}, 4.21^{8}, 4.25^{3}, 4.3^{6}, 4.31^{2}, 4.32^{3}, 4.37^{5}, 4.38^{11}, 4.43^{3}, 4.44^{3}, 4.49^{1}, 4.5^{2}, 4.52^{3}, 4.55^{2}, 4.59^{3}, 4.62^{3}, 4.63^{5}, 4.64^{14}, 4.67^{1}, 4.68^{3}, 4.76^{3}, 4.78^{3}, 4.79^{3}, 4.88^{8}, 4.94^{3}, 5.02^{1}, 5.04^{3}, 5.16^{4}, 5.31^{2}, 5.49^{12}, 5.7^{16}, 5.8^{2}, 5.83^{8}, 5.85^{2}, 5.94^{12}, 5.97^{24}, 5.99^{12}, 6.23^{3}, 6.26^{3}, 6.34^{1}, 6.36^{3}, 6.46^{3}, 6.51^{3}, 6.53^{8}, 6.65^{16}, 6.89^{2}, 6.91^{2}, 7.02^{3}, 7.13^{3}, 7.34^{1}, 7.79^{1}, 7.88^{3}, 7.93^{8}, 8.42^{3}, 8.96^{1}, 9.45^{1}

Layer	Number of States	Ratio
1	16	—
2	160	10.00
3	1,408	8.80
4	11,712	8.32
5	90,912	7.76
6	644,756	7.09
7	4,070,826	6.31
8	21,433,009	5.26
9	76,410,122	3.56
10	109,897,795	1.44
11	26,611,502	0.24
12	328,215	0.01
13	366	0.00

Table 38: Number of states per layer and growth ratio for the Dino Cube

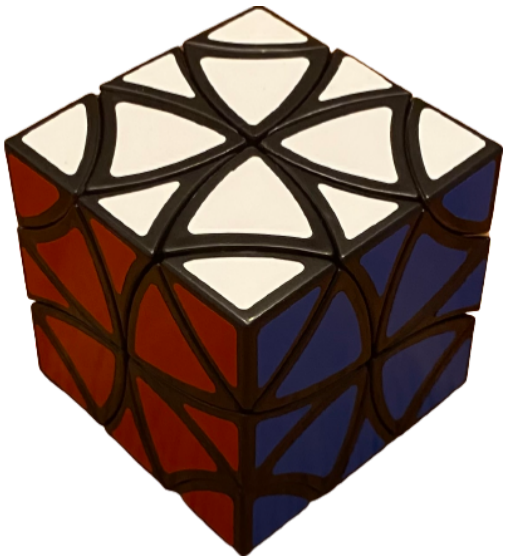
10.12 Little chop

10.13 Curvy copter

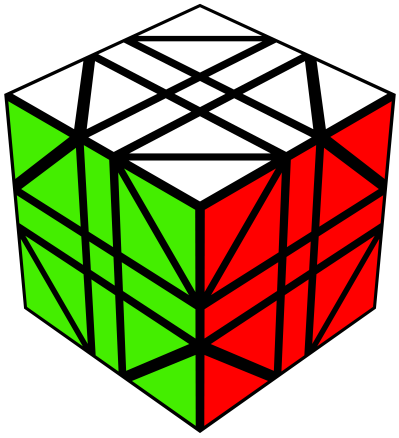
The Curvy Copter is a variant on the Helicopter Cube in which the cutting planes are curved. The effect of this is that each edge has an extra central piece. These extra pieces do not change position but their orientations do change, making this a more difficult puzzle. It was designed by Tom van der Zanden (TomZ). CITE: Jaapsch Helicopter Cube / Curvy Copter

10.13.1 Images

When the jumbling is forbidden, one can represent the Curvy Copter as a cube with "straight forms".



(a) Curvy Copter



(b) Curvy Copter uncurved

References:
Picture from Tomas Rokicki's Twizzle site

10.13.2 Generators

DF:=(7,24)(8,22)(9,23)(40,43)(42,45)(67,68)
UB:=(13,18)(14,16)(15,17)(31,32)(39,48)(59,60)
FL:=(1,7)(2,8)(3,9)(25,27)(34,43)(49,50)
BR:=(4,16)(5,17)(6,18)(29,37)(30,39)(57,58)
DB:=(4,21)(5,19)(6,20)(26,33)(29,38)(55,56)
UF:=(1,11)(2,12)(3,10)(25,28)(35,47)(65,66)
DL:=(7,20)(8,21)(9,19)(27,38)(41,45)(69,70)
UR:=(10,17)(11,18)(12,16)(28,37)(32,44)(53,54)
BL:=(13,19)(14,20)(15,21)(31,41)(33,46)(61,62)
FR:=(10,22)(11,23)(12,24)(36,47)(42,44)(71,72)
UL:=(1,15)(2,13)(3,14)(34,48)(35,46)(63,64)
DR:=(4,23)(5,24)(6,22)(26,36)(30,40)(51,52)

10.13.3 Growth function

Layer	*CayleyPy*	*Tomas Rokicki*
0	1	1
1	12	12
2	90	90
3	572	572
4	3409	3409
5	19704	19704
6	112172	112172
7	631128	631128
8	3517264	3517264
9	19446514	19446520
10	106845100	106845240
11	584054747	584056780
12		3178549176
13		17228953120
14		93036718208
15		500561912280

Table 39: Curvy Copter growth.

10.14 Compy cube

10.15 Helicopter

10.16 Pentultimate

10.17 Master pentultimate

10.18 Elite pentultimate

10.19 Starminx combo

10.20 Megaminx + Chopasaurus

10.20.1 Common information

The Megaminx + Chopasaurus is an extended version of the classical Megaminx dodecahedral puzzle, but with added slice-turn operations ("chop" moves) alongside the standard face turns. Added chop moves rotate the central layer between pairs of opposite pentagonal faces, resulting in an expanded set of generators

10.20.2 Picture

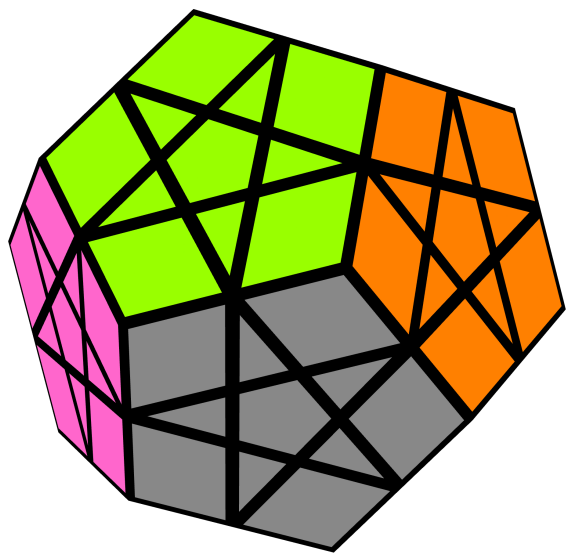


Figure 78: Megaminx + Chopasaurus

References:
Picture from Tomas Rokicki's Twizzle site

10.20.3 Generators

U:=(1,4,13,31,16)(2,5,14,32,17)(3,6,15,33,18)(61,62,78,106,120)(65,73,86,117,110)
D:=(25,58,52,49,40)(26,59,53,50,41)(27,60,54,51,42)(76,100,109,89,81)(90,97,103,94,101)
F:=(4,11,24,28,14)(5,12,22,29,15)(6,10,23,30,13)(64,69,80,104,110)(71,78,79,107,72)
B:=(37,47,57,52,60)(38,48,55,53,58)(39,46,56,54,59)(77,88,96,112,100)(90,102,119,116,98)
L:=(13,29,45,34,32)(14,30,43,35,33)(15,28,44,36,31)(65,71,82,108,113)(84,106,104,95,85)
DR:=(7,20,39,58,27)(8,21,37,59,25)(9,19,38,60,26)(63,67,77,101,70)(76,87,99,115,102)
BL:=(16,33,34,56,48)(17,31,35,57,46)(18,32,36,55,47)(66,73,84,111,116)(74,96,114,120,113)
FR:=(7,26,40,23,12)(8,27,41,24,10)(9,25,42,22,11)(68,87,94,83,69)(70,81,92,107,91)
BR:=(1,18,46,38,21)(2,16,47,39,19)(3,17,48,37,20)(61,66,88,115,118)(67,75,86,114,119)
FL:=(22,41,49,44,30)(23,42,50,45,28)(24,40,51,43,29)(72,83,89,105,95)(80,92,103,93,82)
R:=(1,20,9,10,5)(2,21,7,11,6)(3,19,8,12,4)(62,75,99,91,64)(63,68,79,117,118)
DL:=(34,44,51,53,57)(35,45,49,54,55)(36,43,50,52,56)(74,85,93,109,98)(97,112,111,108,105)
U_F_R:=(1,11,15)(2,12,13)(3,10,14)(4,6,5)(7,29,16)(8,30,17)(9,28,18)(19,23,32)(20,24,33) (21,22,31)
(61,68,104)(62,79,110)(63,80,120)(64,78,117)(65,75,107)(67,92,113)(69,106,118)(70,82,114)(71,86,91)(72,73,99)
(121,122,123)(124,126,127)

D_DL_B:=(25,46,44)(26,47,45)(27,48,43)(34,40,39)(35,41,37)(36,42,38)(49,59,57)(50,60,55)(51,58,56)(52,54,53)
(66,95,87)(74,103,77)(76,116,105)(81,119,108)(83,115,84)(85,94,88)(89,102,111)(90,112,109)(93,101,96)
(97,100,98)(125,128,130)(129,132,131)
U_L_F:=(1,22,35)(2,23,36)(3,24,34)(4,29,33)(5,30,31)(6,28,32)(10,44,17)(11,45,18)(12,43,16)(13,15,14)
(61,69,108)(62,80,113)(64,82,120)(65,78,104)(66,68,105)(71,106,110)(72,84,117)(73,79,95)(83,111,118)(85,86,107)
(121,123,124)(122,128,125)
D_B_DR:=(7,47,50)(8,48,51)(9,46,49)(19,56,40)(20,57,41)(21,55,42)(25,38,54)(26,39,52)(27,37,53)(58,60,59)
(63,116,89)(67,98,81)(70,119,109)(74,92,75)(76,77,90)(87,88,97)(91,114,93)(94,115,112)(96,103,99)(100,101,102)
(126,129,127)(130,131,132)
FL_F_L:=(4,41,34)(5,42,35)(6,40,36)(10,51,31)(11,49,32)(12,50,33)(13,24,44)(14,22,45)(15,23,43)(28,30,29)
(62,81,74)(64,89,113)(65,69,105)(71,80,95)(72,82,104)(78,92,85)(79,103,84)(83,108,110)(91,109,120)(93,106,107)
(121,127,129)(123,128,124)
BR_DR_B:=(1,57,27)(2,55,25)(3,56,26)(7,18,52)(8,16,53)(9,17,54)(19,48,59)(20,46,60)(21,47,58)(37,39,38)
(61,98,70)(63,66,90)(67,88,102)(68,73,97)(75,96,76)(77,115,119)(86,112,87)(94,117,111)(99,114,100)(101,118,116)
(122,125,132)(126,131,130)
U_BL_L:=(1,30,55)(2,28,56)(3,29,57)(4,43,47)(5,44,48)(6,45,46)(13,35,18)(14,36,16)(15,34,17)(31,33,32)
(61,71,111)(62,82,116)(64,93,119)(65,84,120)(66,78,108)(73,106,113)(74,86,104)(75,80,98)(85,114,110)(95,96,117)
(121,124,125)(123,129,126)
D_DR_FR:=(7,59,42)(8,60,40)(9,58,41)(10,38,49)(11,39,50)(12,37,51)(19,52,24)(20,53,22)(21,54,23)(25,27,26)
(63,100,83)(67,109,69)(68,77,89)(70,76,94)(72,118,112)(79,88,105)(81,87,101)(90,92,99)(91,102,103)(97,107,115)
(122,131,128)(127,130,132)
DL_L_BL:=(13,50,46)(14,51,47)(15,49,48)(16,28,54)(17,29,52)(18,30,53)(31,45,56)(32,43,57)(33,44,55)
(34,36,35)(61,72,100)(65,93,116)(66,71,109)(73,82,98)(74,84,108)(78,89,88)(85,111,113)(95,112,120)(96,106,105)
(97,114,104)(121,128,131)(124,129,125)
R_FR_DR:=(1,59,23)(2,60,24)(3,58,22)(4,39,42)(5,37,40)(6,38,41)(7,9,8)(10,21,26)(11,19,27)(12,20,25)
(62,77,83)(63,87,91)(64,67,94)(68,99,70)(69,75,101)(76,107,118)(79,115,81)(80,86,90)(92,117,102)(103,110,119)
(122,130,127)(123,126,132)
U_BR_BL:=(1,32,48)(2,33,46)(3,31,47)(4,36,37)(5,34,38)(6,35,39)(13,55,20)(14,56,21)(15,57,19)(16,18,17)
(61,73,114)(62,84,119)(63,71,112)(65,96,118)(66,86,120)(67,78,111)(74,115,110)(75,106,116)(77,79,108)(88,117,113)
(121,125,126)(122,124,131)
D_FR_FL:=(7,54,30)(8,52,28)(9,53,29)(10,58,45)(11,59,43)(12,60,44)(22,26,51)(23,27,49)(24,25,50)(40,42,41)
(64,102,85)(68,100,82)(69,76,93)(70,97,72)(80,87,109)(81,103,83)(89,92,94)(90,95,91)(98,104,99)(101,105,107)
(123,130,129)(127,132,128)
B_BL_BR:=(1,34,60)(2,35,58)(3,36,59)(16,57,38)(17,55,39)(18,56,37)(19,31,53)(20,32,54)(21,33,52)(46,48,47)
(61,74,102)(62,85,76)(66,96,119)(67,73,112)(75,84,100)(77,86,111)(88,114,116)(90,118,113)(98,115,120)(99,106,109)
(121,129,130)(125,131,126)
F_FL_FR:=(4,26,44)(5,27,45)(6,25,43)(7,49,14)(8,50,15)(9,51,13)(10,42,29)(11,40,30)(12,41,28)(22,24,23)
(63,97,65)(64,81,95)(68,103,71)(69,92,72)(70,105,110)(78,87,93)(79,94,82)(80,107,83)(89,104,91)(101,108,117)
(122,132,124)(123,127,128)
U_R_BR:=(1,3,2)(4,17,21)(5,18,19)(6,16,20)(7,13,48)(8,14,46)(9,15,47)(10,33,39)(11,31,37)(12,32,38)
(61,75,117)(62,86,118)(63,78,114)(64,73,115)(65,88,91)(66,99,110)(67,79,120)(68,106,119)(69,84,102)(87,104,116)
(121,126,122)(123,125,130)
D_FL_DL:=(22,60,36)(23,58,34)(24,59,35)(25,57,28)(26,55,29)(27,56,30)(40,53,45)(41,54,43)(42,52,44)
(49,51,50)(70,96,71)(72,101,74)(76,111,80)(77,113,107)(81,112,82)(83,90,85)(89,97,93)(92,100,108)(94,98,95)
(103,109,105)(124,127,131)(128,132,129)
FR_R_F:=(1,26,28)(2,27,29)(3,25,30)(4,9,23)(5,7,24)(6,8,22)(10,12,11)(13,19,42)(14,20,40)(15,21,41)
(61,76,95)(62,87,72)(63,92,110)(64,68,107)(69,79,91)(70,80,117)(71,75,94)(78,99,83)(81,104,118)(89,106,115)
(121,130,128)(122,127,123)
BL_B_DL:=(16,45,60)(17,43,58)(18,44,59)(31,50,38)(32,51,39)(33,49,37)(34,54,47)(35,52,48)(36,53,46)
(55,57,56)(65,103,67)(66,85,100)(73,93,77)(74,112,116)(82,101,86)(84,109,88)(90,114,108)(96,111,98)(97,119,113)
(102,120,105)(124,132,126)(125,129,131)

DR_BR_R:=(1,39,8)(2,37,9)(3,38,7)(4,46,27)(5,47,25)(6,48,26)(10,16,59)(11,17,60)(12,18,58)(19,21,20)(61,77,91)(62,88,70)(63,75,115)(64,66,101)(67,99,118)(68,86,102)(76,79,114)(78,96,81)(87,117,119)(100,107,120)(121,131,127)(122,126,130)
L_DL_FL:=(13,41,56)(14,42,57)(15,40,55)(22,52,32)(23,53,33)(24,54,31)(28,51,35)(29,49,36)(30,50,34)(43,45,44)(65,83,98)(69,90,73)(71,89,74)(72,109,113)(80,97,84)(82,105,85)(92,112,106)(93,108,95)(94,116,110)(103,111,104)(123,132,125)(124,128,129)

10.20.4 Growth function

Using CayleyPy, we were able to calculate the growth function up to layer 5.

Layer	*Count*
0	1
1	64
2	3328
3	164488
4	7911503
5	370002156

Table 40: Megaminx + Chopasaurus growth.

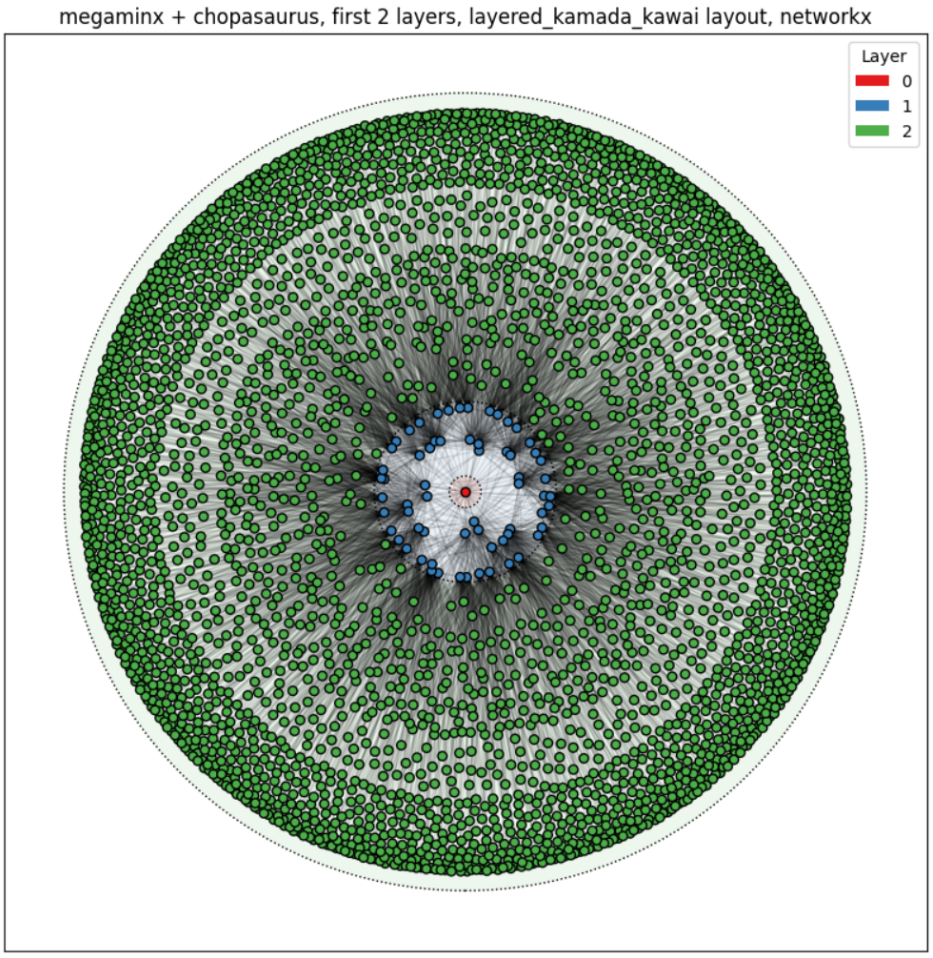


Figure 79: Graph for the first 2 layers (Layered Kamada-Kawai layout)

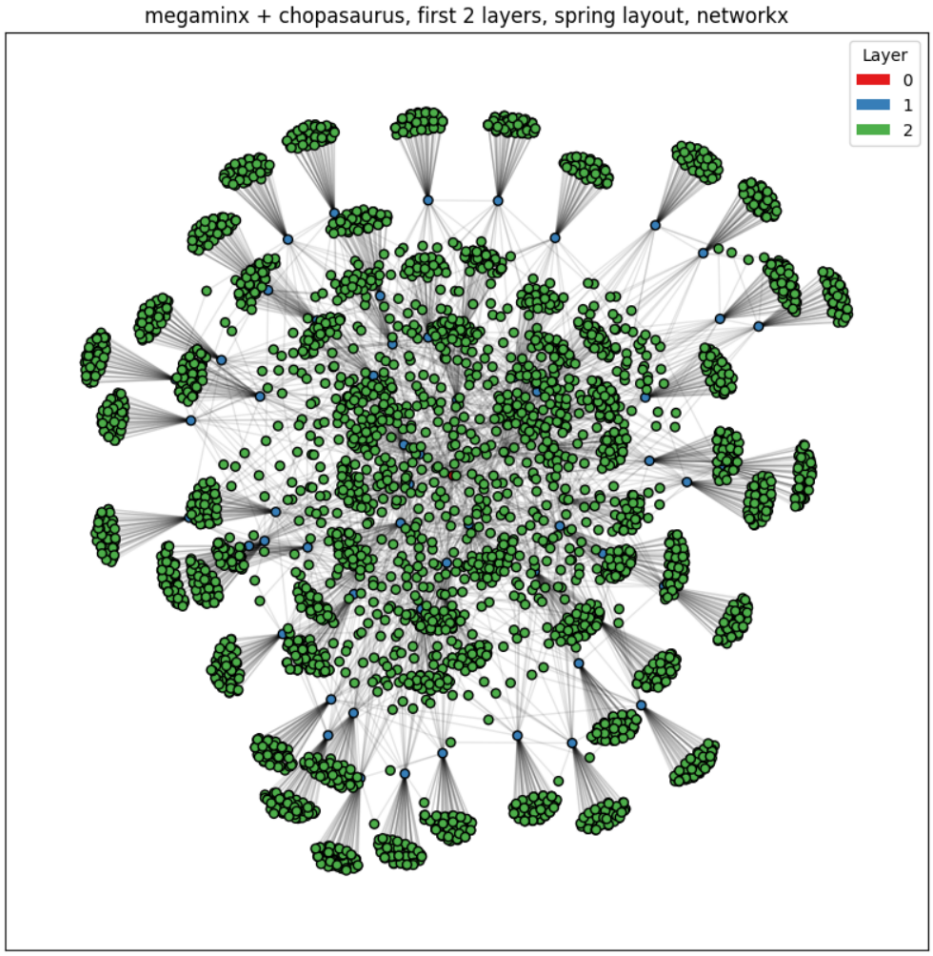


Figure 80: Graph for the first 2 layers (Spring layout)

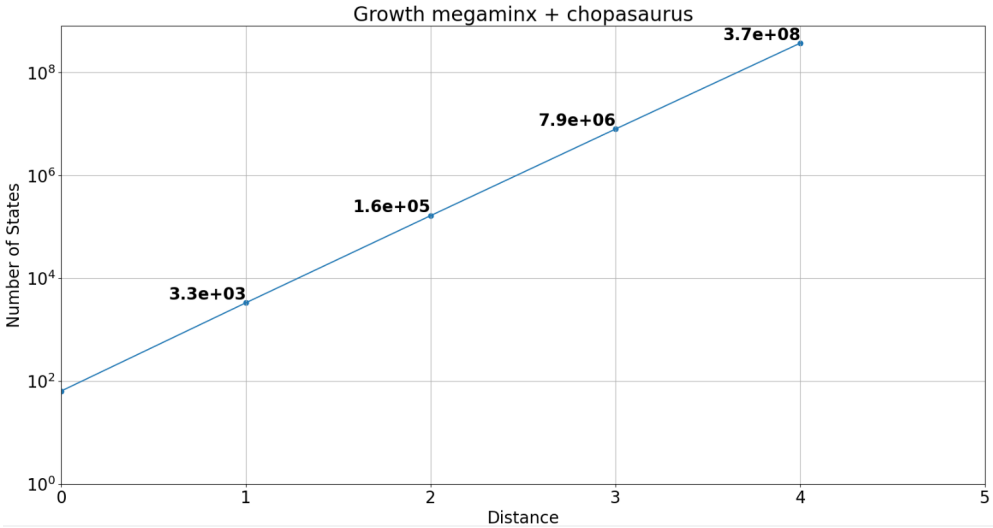


Figure 81: Growth of Megaminx + Chopasaurus

10.21 2x2x2 + dino + little chop

10.22 Dino + little chop

The Dino + Little Chop puzzle is a fascinating extension of the classic Dino Cube. While the traditional Dino Cube involves face turns around its vertices, the "Little Chop" modification introduces additional, often more complex, slice-turn operations. These new moves significantly expand the puzzle's generator set, leading to a much larger state space and different permutation characteristics compared to the standard Dino Cube. This variation provides a unique challenge for group theory analysis and growth function computations.

10.22.1 Picture

Interactive visualization can be found at: Tomas Rokicki's Twizzle site:

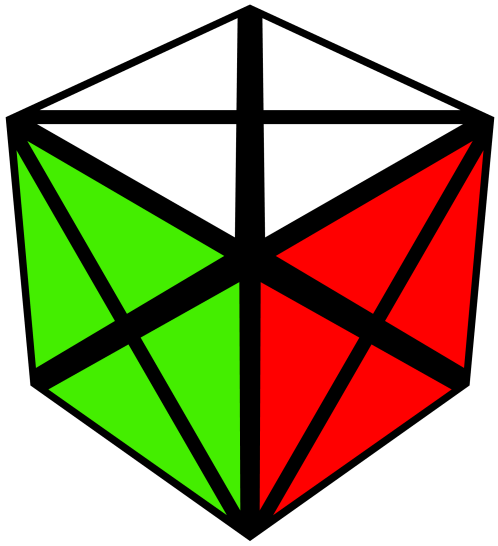


Figure 82: Dino + Little Chop (from Twizzle site)

10.22.2 Generators

DRF:=(3,18,5)(6,16,24);
UBL:=(11,20,15)(14,23,21);
DFL:=(1,3,19)(7,16,17);
URB:=(2,23,13)(8,11,10);
DBR:=(5,13,12)(6,10,9);
ULF:=(1,21,4)(7,20,22);
DLB:=(9,15,17)(12,14,19);
UFR:=(2,18,4)(8,24,22);
DF:=(1,6)(3,16)(5,7)(12,22)(17,18)(19,24);
UB:=(2,15)(4,9)(8,14)(10,21)(11,23)(13,20);
FL:=(1,7)(3,20)(4,17)(15,18)(16,21)(19,22);
BR:=(2,9)(5,11)(6,23)(8,12)(10,13)(14,24);
DB:=(3,11)(5,15)(6,14)(9,12)(10,19)(13,17);
UF:=(1,8)(2,7)(4,22)(16,23)(18,20)(21,24);
DL:=(1,9)(3,15)(6,21)(7,12)(14,16)(17,19);

UR:=(2,8)(4,10)(5,20)(11,18)(13,22)(23,24);
BL:=(7,13)(9,21)(11,19)(12,20)(14,15)(17,23);
FR:=(1,10)(2,16)(3,8)(4,6)(5,22)(18,24);
UL:=(1,11)(4,15)(7,23)(8,19)(14,22)(20,21);
DR:=(2,17)(3,10)(5,6)(9,18)(12,24)(13,16);

10.22.3 Growth function

Layer	Number of States	Power of Growth
0	1	
1	28	28
2	574	20,50
3	10792	18,80
4	194699	18,04
5	3404024	17,48
6	57861970	16,99
7	958196930	16.56

Table 41: Number of growth 2x2x2 Dino Cube+ little chop states at distance n from the solved state (layer n)

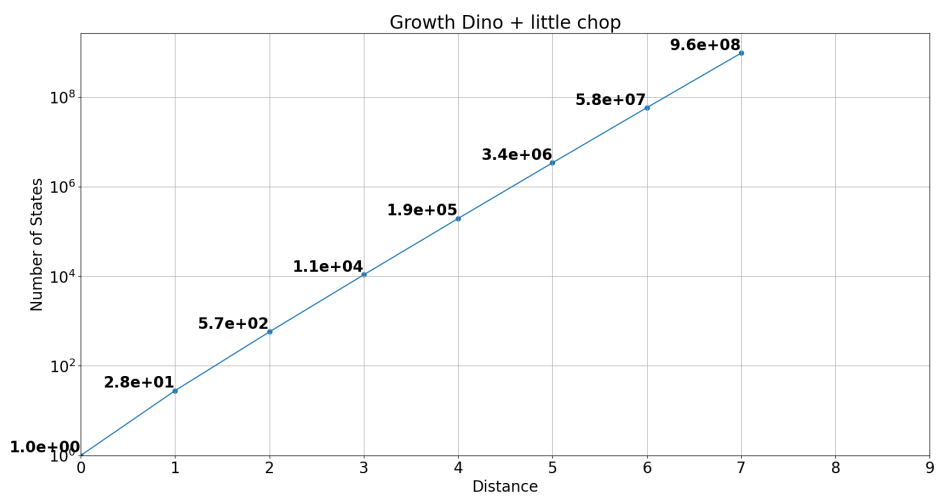


Figure 83: Growth

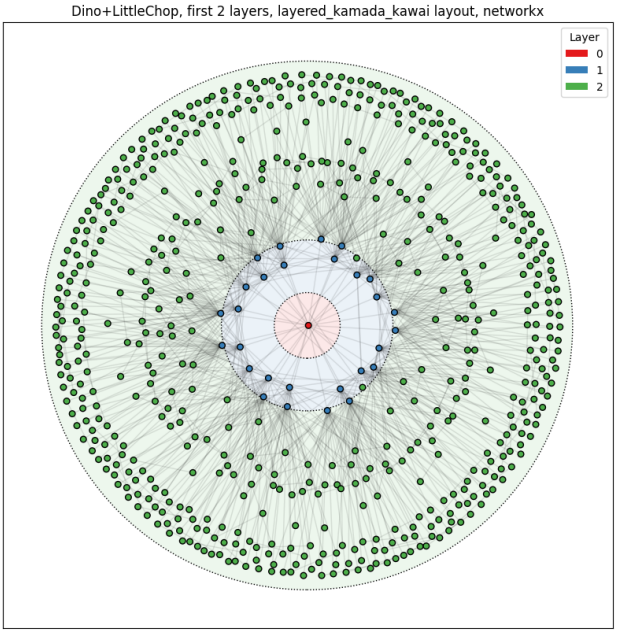


Figure 84: Graph for the first 2 layers (Layered Kamada-Kawai layout)

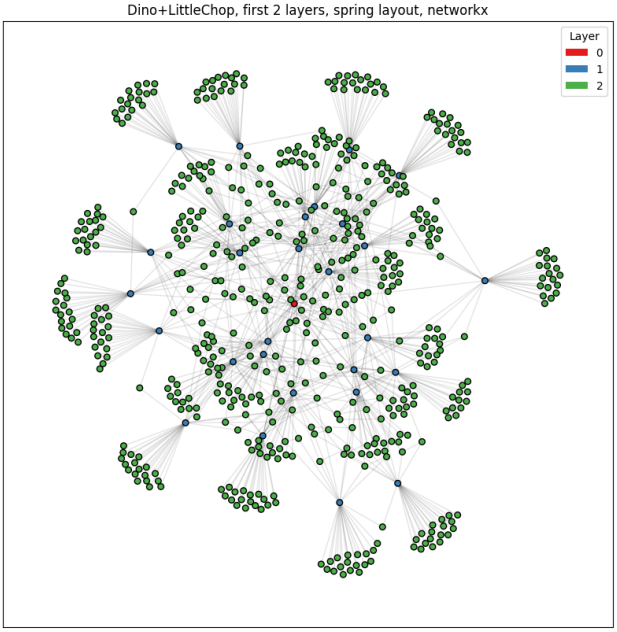


Figure 85: Graph for the first 2 layers (Spring layout)

10.23 2x2x2 + little chop

This puzzle is a hybrid modification of the standard 2×2×2 Pocket Cube, incorporating shell moves from the Little Chop variant. As a result, the standard set of rotations is augmented by additional shell moves, yielding a vastly expanded permutation space. The total number of states is approximately 4.39×10^{29} (github/cs0x7f), while the "super" count—treating identical pieces as distinguishable and accounting for all internal orientations—reaches about 1.92×10^{47} . This hybrid puzzle presents a compelling subject for investigation in combinatorial and Cayley graph theory. Analysis with CayleyPy has been performed in: notebook.

10.23.1 Picture

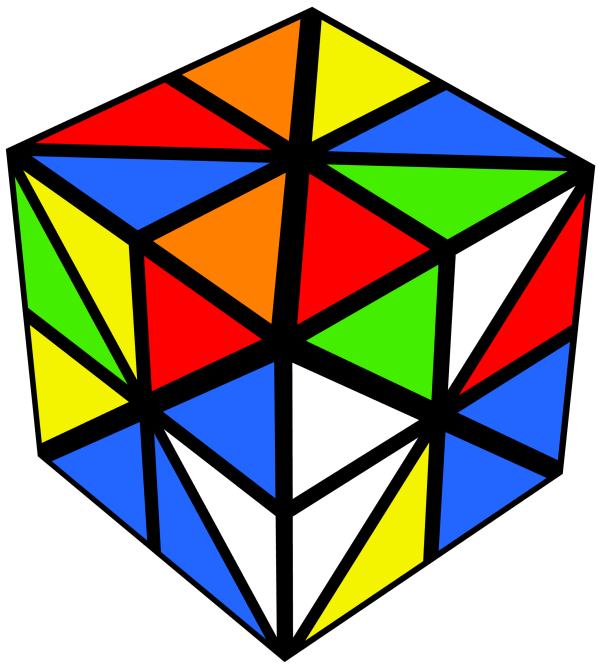


Figure 86: Model 2x2x2 Little Chop

10.23.2 Generators

$$F = (1,2,11,22)(3,13,6,16)(4,20,7,18)(25,26,35,46)(27,37,30,40)(28,44,31,42)$$
$$B = (5,12,24,15)(8,17,10,19)(9,21,23,14)(29,36,48,39)(32,41,34,43)(33,45,47,38)$$
$$D = (4,13,5,14)(6,11,9,19)(8,15,22,18)(28,37,29,38)(30,35,33,43)(32,39,46,42)$$
$$U = (1,23,17,3)(2,20,10,12)(7,16,24,21)(25,47,41,27)(26,44,34,36)(31,40,48,45)$$
$$L = (1,4,19,24)(6,15,23,20)(10,16,22,14)(25,28,43,48)(30,39,47,44)(34,40,46,38)$$
$$R = (2,21,8,13)(3,12,9,18)(5,11,7,17)(26,45,32,37)(27,36,33,42)(29,35,31,41)$$
$$DF = (1,5)(2,14)(3,15)(4,11)(6,18)(13,22)(25,29)(26,38)(28,35)(30,42)(33,44)(37,46)$$
$$UB = (7,19)(8,16)(9,20)(10,21)(12,23)(17,24)(27,39)(31,43)(32,40)(34,45)(36,47)(41,48)$$
$$FL = (1,6)(2,15)(4,16)(11,23)(13,24)(20,22)(25,30)(26,39)(28,40)(31,38)(35,47)(44,46)$$
$$BR = (3,19)(5,21)(7,14)(8,12)(9,17)(10,18)(27,43)(29,45)(32,36)(33,41)(34,42)(37,48)$$
$$DB = (4,17)(5,19)(8,14)(9,15)(10,13)(18,23)(28,41)(29,43)(30,36)(32,38)(33,39)(34,37)$$
$$UF = (1,7)(2,16)(3,20)(6,12)(11,24)(21,22)(25,31)(26,40)(27,44)(35,48)(42,47)(45,46)$$
$$DL = (1,8)(4,15)(5,20)(6,14)(13,23)(19,22)(28,39)(29,44)(30,38)(34,35)(37,47)(43,46)$$
$$UR = (2,17)(3,21)(7,12)(9,16)(10,11)(18,24)(25,32)(26,41)(27,45)(31,36)(33,40)(42,48)$$
$$BL = (4,21)(6,17)(8,20)(10,15)(14,24)(19,23)(29,40)(30,41)(32,44)(34,39)(38,48)(43,47)$$
$$FR = (1,9)(2,18)(3,11)(5,16)(7,13)(12,22)(25,33)(26,42)(27,35)(28,45)(31,37)(36,46)$$
$$UL = (1,10)(2,19)(6,21)(7,15)(16,23)(20,24)(25,34)(30,45)(31,39)(40,47)(41,46)(44,48)$$
$$DR = (3,14)(4,12)(5,18)(8,11)(9,13)(17,22)(26,43)(27,38)(28,36)(29,42)(32,35)(33,37)$$

10.23.3 Growth function

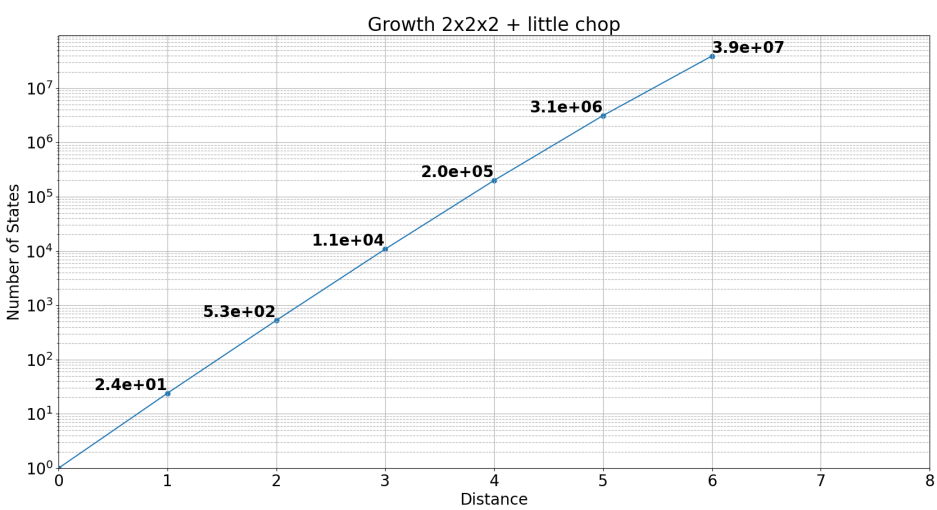


Figure 87: 2x2x2 Little Chop growth

Table 42: Number of 2x2x2 + Little Chop states at distance n from the solved state (layer n)

Layer	Number of States	Ratio
0	1	
1	24	24
2	528	22
3	10,836	20.52
4	199,583	18.41
5	3,117,804	15.62
6	38,976,912	12.50
7	428,348,496	10.99

10.23.4 Graph images

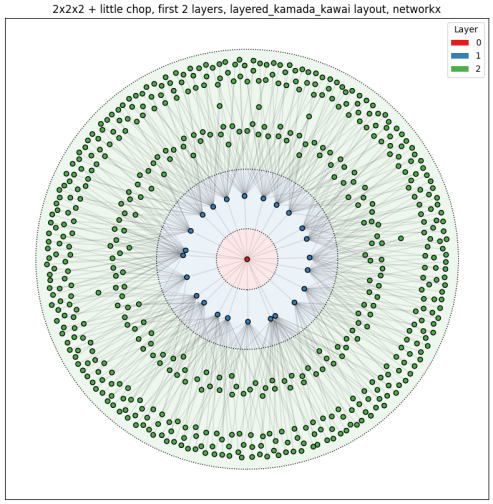


Figure 88: Model 2x2x2 + little chop kamada kawai 2 layers

Figure 89: Caption

10.23.5 Spectrum

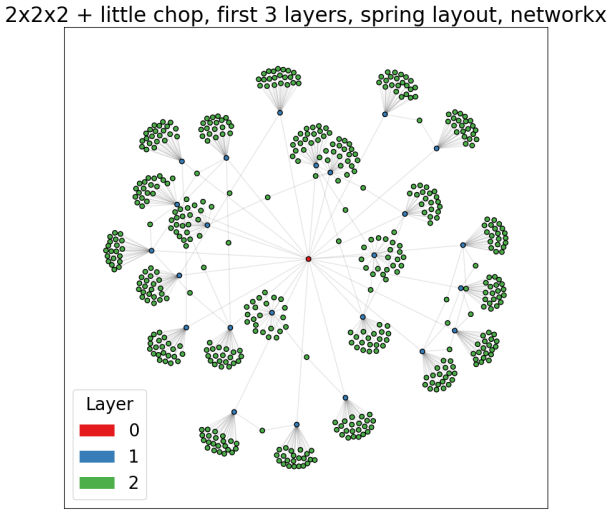


Figure 90: Model 2x2x2 + little chop spring layout for 3 layers

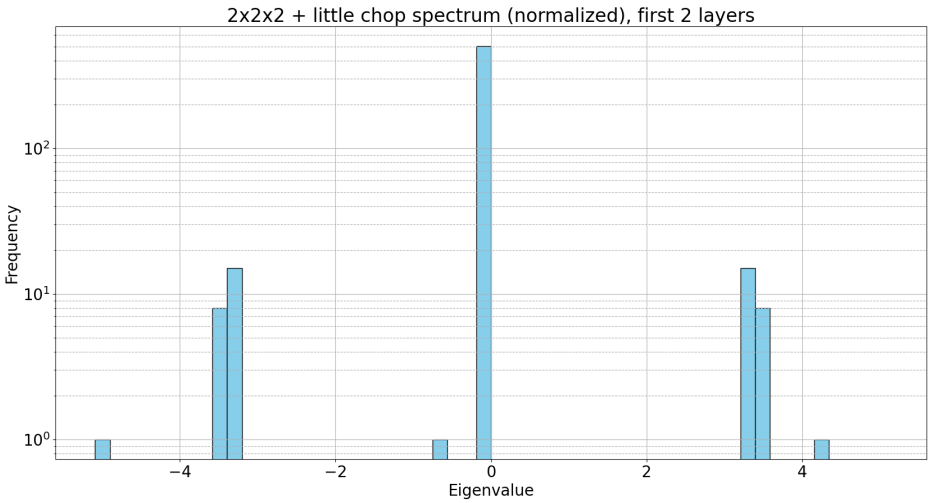


Figure 91: Model 2x2x2 spectrum normalized 2 layers

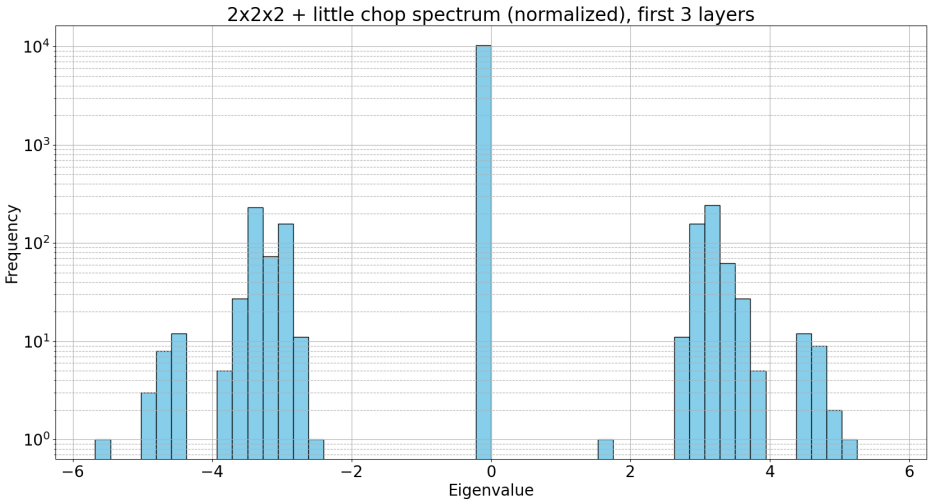


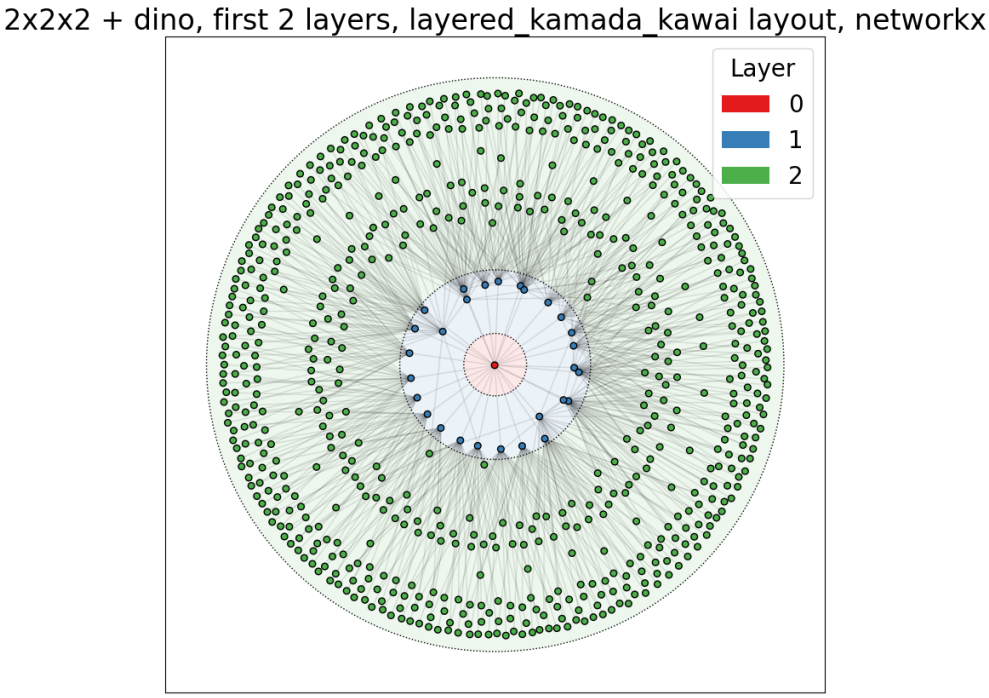
Figure 92: Model 2x2x2 spectrum normalized 3 layers

Table 43: Eigenvalues for 3 layers in 2x2x2 + Little Chop

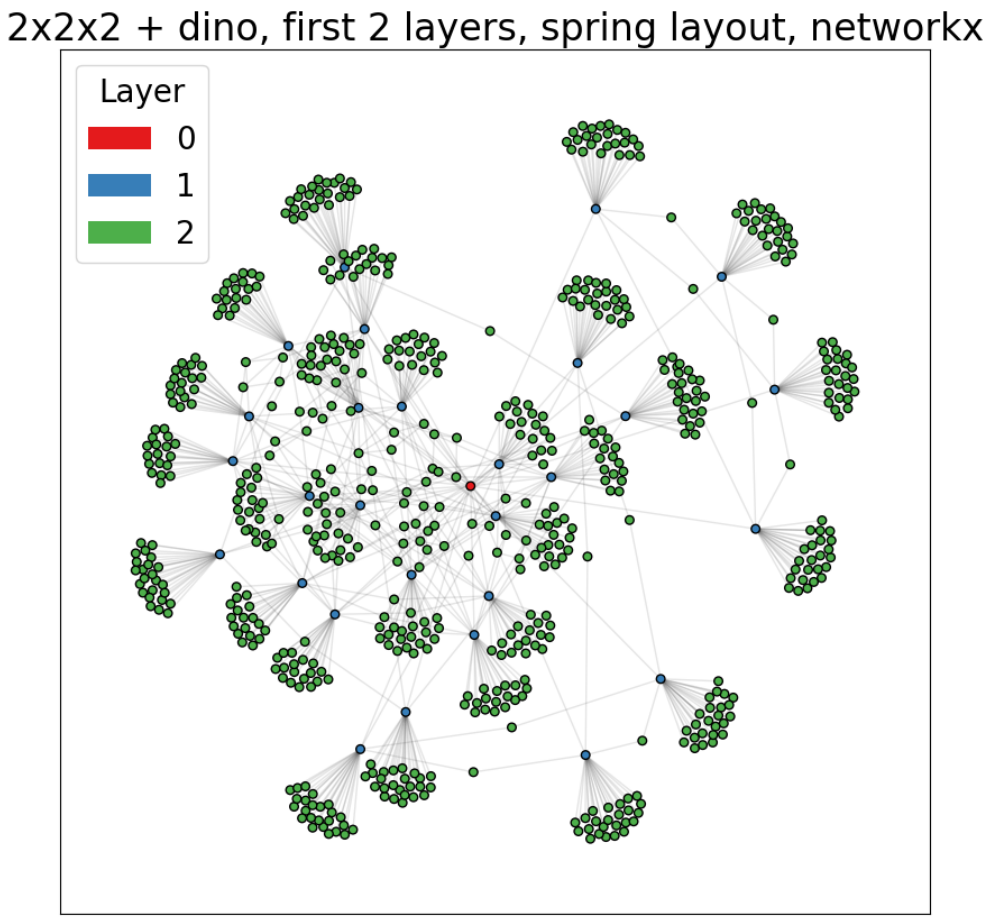
n	Eigenvalues (value^{multiplicity})
1	[-4.18^1, 0.0^{23}, 2.74^1]
2	[-5.09^1, -3.54^2, -3.47^1, -3.4^5, -3.26^{15}, -0.69^1, 0.0^{504}, 3.26^{15}, 3.4^5, 3.47^1, 3.54^2, 4.34^1]
3	[-5.68^1, -4.84^2, -4.81^1, -4.78^2, -4.75^3, -4.66^3, -4.5^6, -4.49^3, -4.48^3, -3.78^1, -3.74^2, -3.72^2, -3.65^3, -3.63^1, -3.58^2, -3.55^3, -3.54^16, -3.51^2, -3.49^3, -3.48^1, -3.47^3, -3.45^2, -3.43^2, -3.42^43, -3.35^1, -3.3^7, -3.28^169, -3.23^3, -3.22^3, -3.2^6, -3.17^9, -3.14^52, -3.0^156, -2.93^1, -2.81^1, -2.73^4, -2.72^3, -2.63^3, -2.5^1, -0.09^1, 0.0^10330, 1.59^1, 2.63^3, 2.72^3, 2.73^4, 2.8^1, 2.91^1, 3.0^156, 3.14^52, 3.17^9, 3.2^6, 3.22^3, 3.23^3, 3.28^169, 3.3^7, 3.32^1, 3.42^43, 3.43^2, 3.45^2, 3.47^3, 3.48^1, 3.49^3, 3.51^2, 3.54^16, 3.55^3, 3.58^2, 3.63^1, 3.65^3, 3.72^2, 3.74^2, 3.78^1, 4.48^3, 4.49^3, 4.5^6, 4.66^3, 4.75^3, 4.78^2, 4.81^1, 4.84^2, 5.25^1]

10.24 2x2x2 + dino

Analysis has been performed in: notebook1 , notebook2 , notebook3 , notebook4



(a) 2x2x2 + Dino



(b) 2x2x2 + Dino spring layout

10.24.1 Growth

Layer	Number of States	Ratio
0	1	-
1	28	28
2	658	23.50
3	14,908	22.65
4	327,755	21.98
5	7,026,680	21.44
6	147,377,530	20.97

Table 44: Growth 2x2x2 + Dino: numbers of states at distance n from the solved state

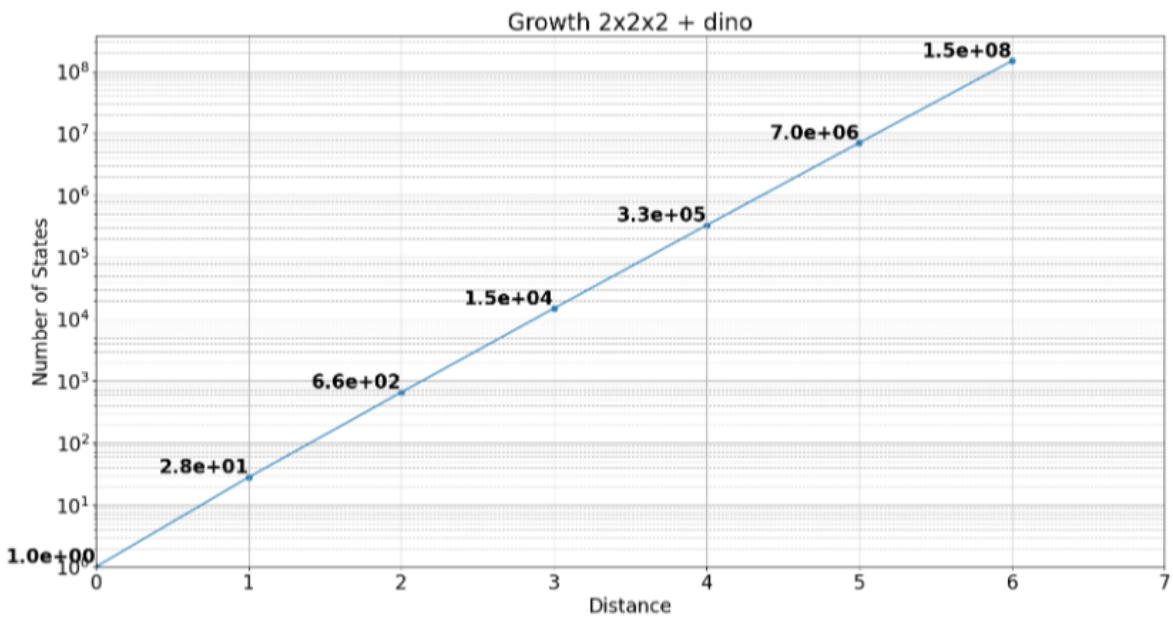


Figure 94: Growth 2x2x2 + Dino: numbers of states at distance n from the solved state

Table 45: Eigenvalues for 2 layers in 2x2x2 + Dino

n	Eigenvalues (value^{multiplicity})
2	[-7.49^1, -5.62^8, -5.48^2, -5.42^1, -5.1^9, -5.0^1, -4.62^3, -4.22^3, -0.0^{6}31, 4.62^8, 5.1^9, 5.22^3, 5.48^2, 5.62^3, 5.78^1, 6.0^1, 8.13^1]

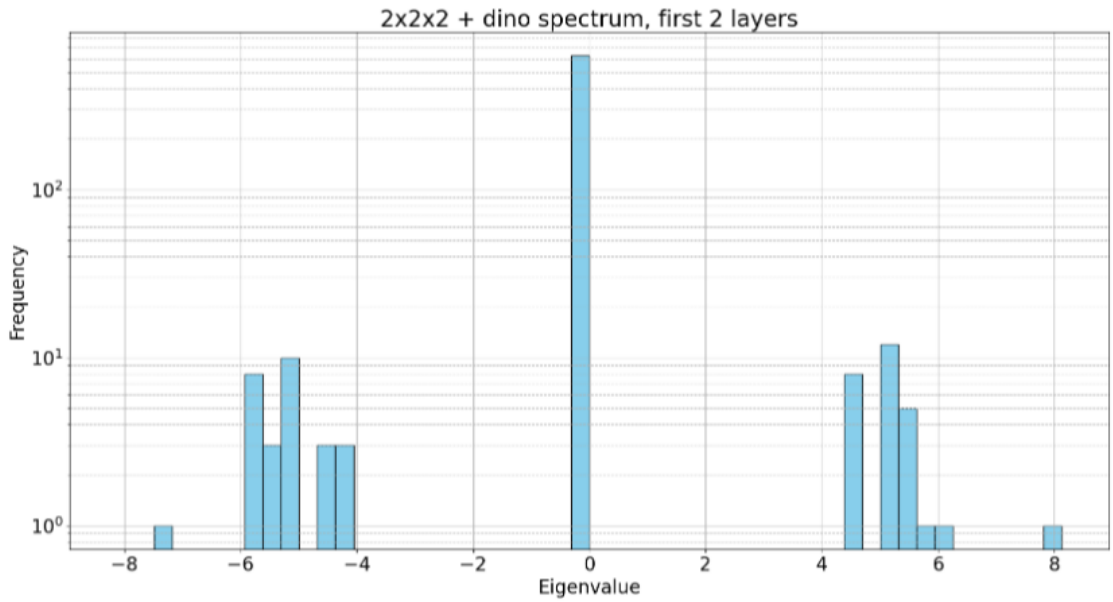


Figure 95: Model 2x2x2 + Dino spectrum first 2 layers

Table 46: Eigenvalues for 3 layers in 2x2x2 + Dino

n	Eigenvalues (value^{multiplicity})
3	[-8.68^1, -7.7^2, -7.69^8, -7.56^1, -7.45^3, -7.23^1, -7.22^3, -7.21^3, -7.0^3, -6.78^3, -6.0^{1}6, -5.96^2, -5.82^{4}8, -5.77^1, -5.72^3, -5.66^6, -5.62^{4}8, -5.61^3, -5.6^2, -5.52^1, -5.48^{3}0, -5.45^8, -5.42^{4}8, -5.38^2, -5.34^1, -5.31^3, -5.3^3, -5.29^{3}7, -5.26^3, -5.25^6, -5.18^6, -5.17^3, -5.16^8, -5.13^3, -5.11^3, -5.1^{1}41, -4.94^3, -4.9^{3}6, -4.82^{2}4, -4.8^{1}6, -4.75^1, -4.63^1, -4.58^8, -4.56^1, -4.52^6, -4.51^4, -4.5^2, -4.42^{4}2, -4.38^3, -4.31^{1}6, -4.0^{1}8, -3.96^3, -3.95^3, -3.85^1, -3.82^3, -3.48^4, -3.19^1, -0.54^8, 0.0^{1}4261, 0.22^1, 0.52^1, 0.54^3, 0.57^3, 3.53^1, 4.0^{1}6, 4.39^3, 4.42^{4}8, 4.5^2, 4.62^{4}8, 4.63^{1}6, 4.68^1, 4.81^8, 4.82^{4}8, 4.9^{3}6, 4.94^3, 5.0^{1}8, 5.1^{1}41, 5.13^3, 5.15^3, 5.19^4, 5.25^3, 5.29^{4}5, 5.32^2, 5.33^6, 5.39^1, 5.41^2, 5.42^{4}2, 5.48^{4}7, 5.5^6, 5.54^3, 5.66^6, 5.77^3, 5.79^{1}2, 5.8^4, 5.82^{2}4, 5.83^3, 5.87^1, 5.89^3, 5.91^2, 5.92^3, 5.98^2, 6.06^1, 6.12^3, 6.18^6, 6.22^1, 6.43^1, 7.53^8, 7.81^3, 7.9^6, 8.15^3, 8.3^3, 8.32^2, 8.55^1, 8.8^1, 9.84^1]

```
using numpy for getting eigenvalues
```

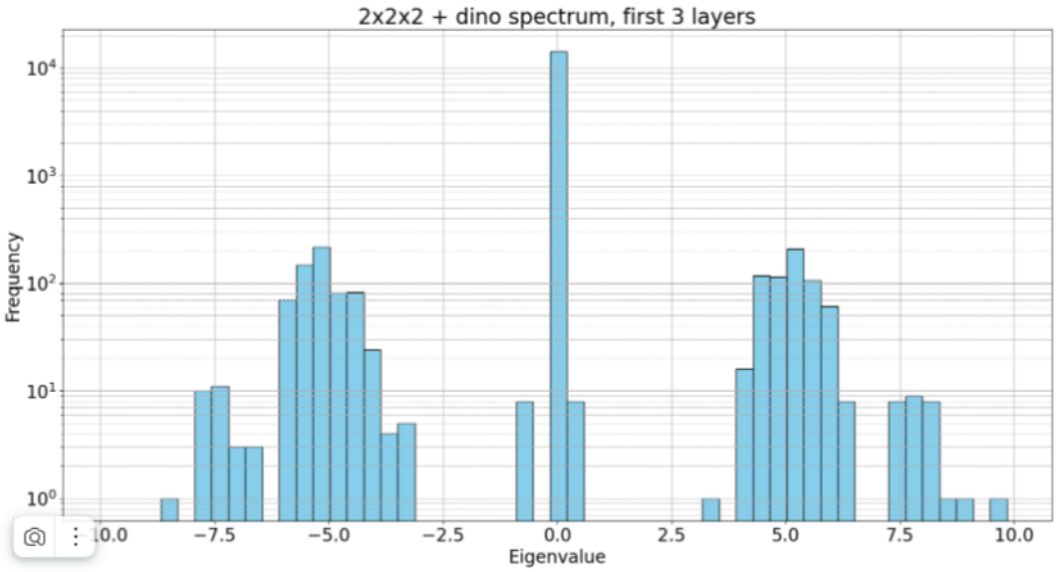


Figure 96: Model 2x2x2 + Dino spectrum first 3 layers

10.25 Eitan's star

10.26 Icosaminx

10.27 Redicosahedron with centers

The Redicosahedron with centers puzzle...

10.27.1 Images

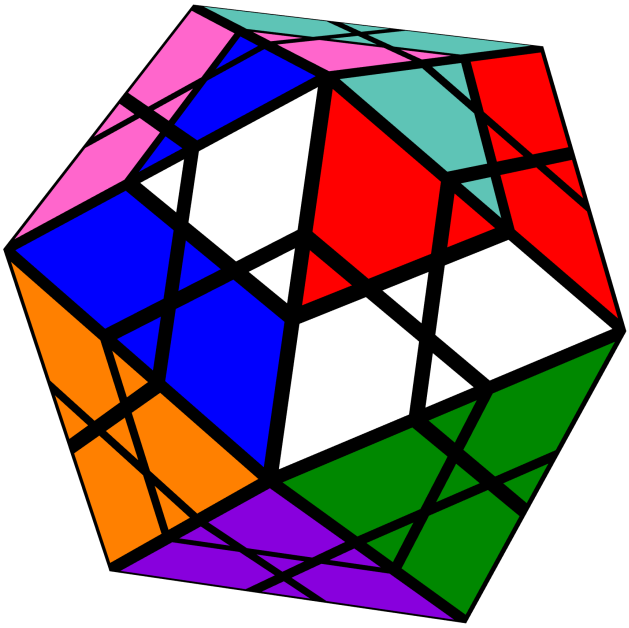
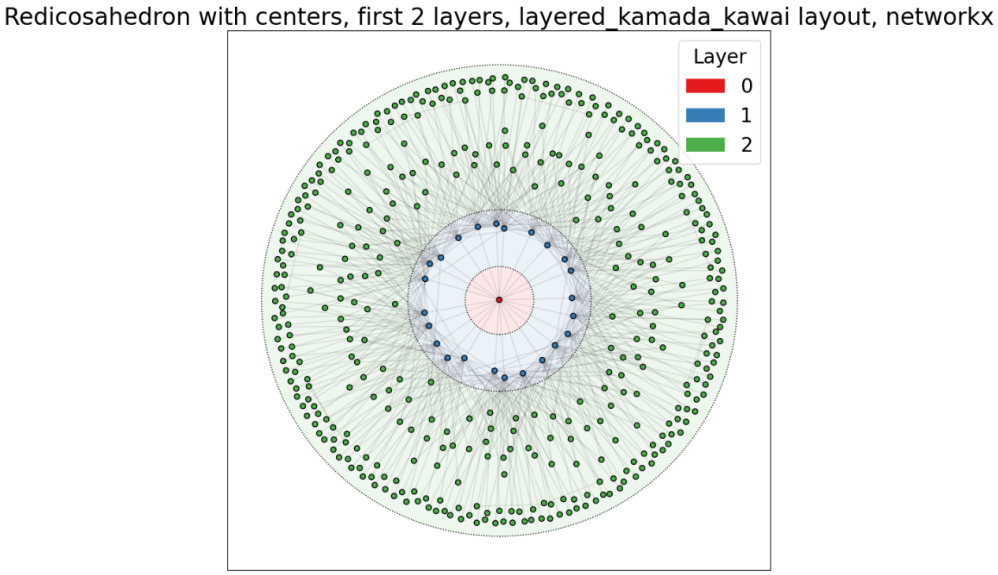
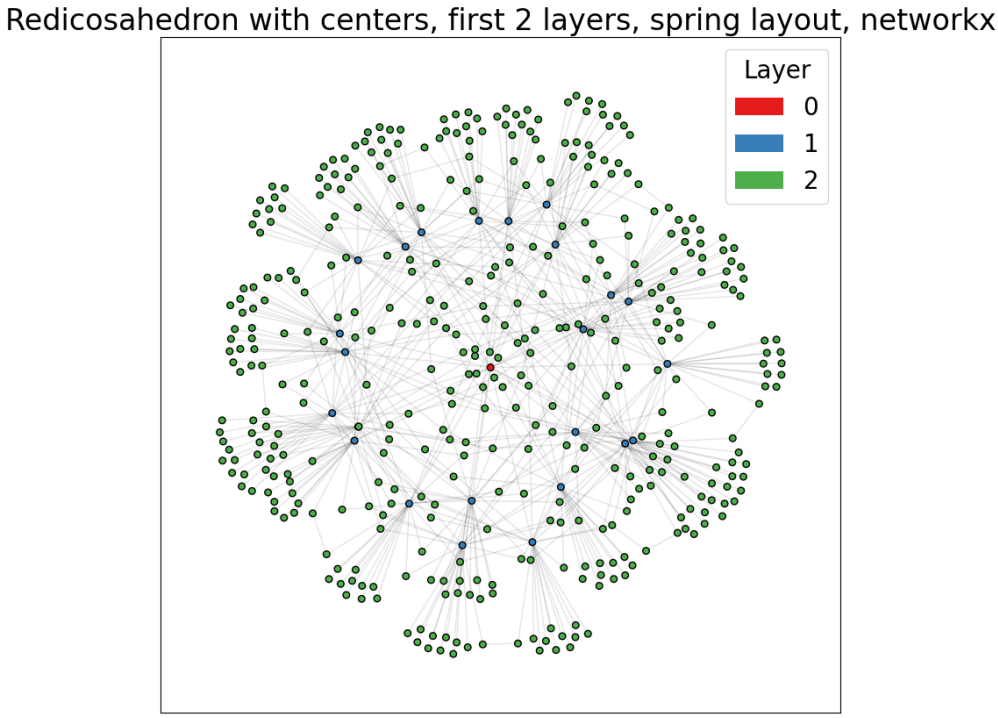


Figure 97: Model Redicosahedron with centers



(a) Redicosahedron with centers



(b) Redicosahedron with centers spring layout

10.27.2 Generators

FREGU:=(5,20,43,17,50)(6,19,44,18,49)(86,90,89,88,87)
OKBDN:=(31,37,36,39,34)(32,38,35,40,33)(96,100,99,98,97)
HIERC:=(1,6,25,30,27)(2,5,26,29,28)(61,65,64,63,62)
MKOPQ:=(23,40,55,51,42)(24,39,56,52,41)(111,115,114,113,112)
FLACR:=(1,8,14,3,20)(2,7,13,4,19)(66,70,69,68,67)

BKMJS:=(35,60,53,47,56)(36,59,54,48,55)(101,105,104,103,102)
NALPO:=(11,34,23,22,13)(12,33,24,21,14)(91,95,94,93,92)
JGEIS:=(25,50,45,53,58)(26,49,46,54,57)(116,120,119,118,117)
FUQPL:=(3,22,41,16,44)(4,21,42,15,43)(81,85,84,83,82)
IHDBS:=(9,30,57,60,37)(10,29,58,59,38)(76,80,79,78,77)
UGJMQ:=(15,52,48,45,18)(16,51,47,46,17)(106,110,109,108,107)
DHCAN:=(7,27,10,31,12)(8,28,9,32,11)(71,75,74,73,72)

10.27.3 Growth

Layer	Number of States
1	24
2	408
3	6,208
4	89,664
5	1,257,040
6	17,302,020
7	235,305,168

Table 47: Number of growth Redicosahedron with centers states at distance n from the solved state (layer n)

10.28 Redicosahedron

10.29 Regular Astrominx + Big Chop

10.29.1 Picture

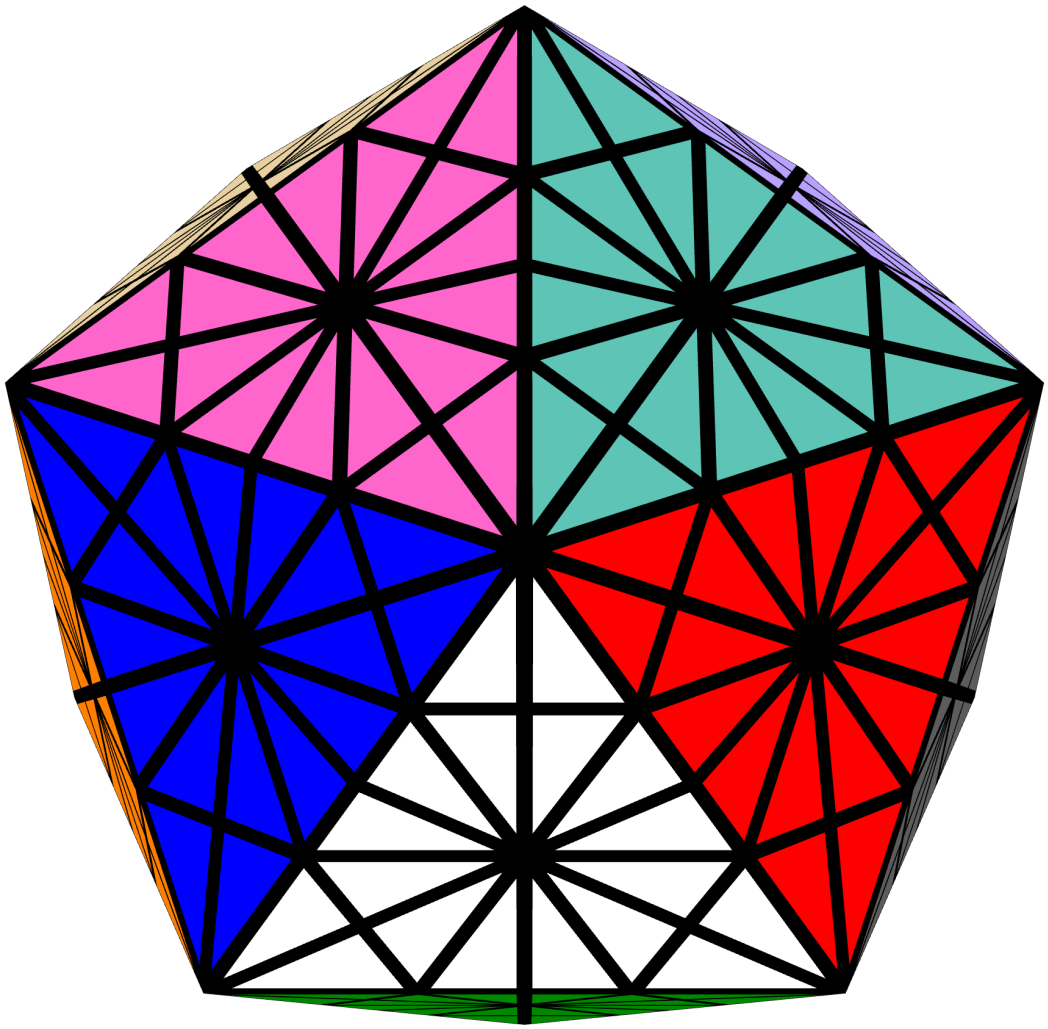


Figure 99: Regular Astrominx + big chop

10.29.2 Generators

```
M_FREGU:=(1,2,23,44,59)(3,25,11,36,58)(4,26,19,31,40)(5,6,27,13,35)(7,55,18,28,53)
    (8,30,60,57,24)(61,62,83,104,119)(63,85,71,96,118)(64,86,79,91,100)
    (67,115,78,88,113)(68,90,120,117,84)(121,122,143,164,179)(123,145,131,156,178)
    (124,146,139,151,160)(127,175,138,148,173)(128,150,180,177,144)
    (181,182,203,224,239)(183,205,191,216,238)(184,206,199,211,220)
    (185,186,207,193,215)(187,235,198,208,233)(188,210,240,237,204)
    (241,242,263,284,299)(243,265,251,276,298)(247,295,258,268,293)
    (248,270,300,297,264)(301,302,323,344,359)(303,325,311,336,358)
    (307,355,318,328,353)(308,330,360,357,324);
M_OKBDN:=(9,38,54,52,33)(10,21,20,17,12)(14,42,49,48,34)(15,47,39,56,32)
    (16,46,45,43,37)(22,51,50,41,29)(70,81,80,77,72)(74,102,109,108,94)
    (75,107,99,116,92)(76,106,105,103,97)(82,111,110,101,89)(130,141,140,137,132)
    (134,162,169,168,154)(135,167,159,176,152)(136,166,165,163,157)
    (142,171,170,161,149)(189,218,234,232,213)(190,201,200,197,192)
    (194,222,229,228,214)(195,227,219,236,212)(196,226,225,223,217)
    (202,231,230,221,209)(250,261,260,257,252)(254,282,289,288,274)
    (255,287,279,296,272)(256,286,285,283,277)(310,321,320,317,312)
    (314,342,349,348,334)(315,347,339,356,332)(316,346,345,343,337);
M_HIERC:=(1,3,38,29,53)(2,24,22,43,6)(4,25,8,28,5)(9,30,59,40,39)(11,44,35,48,47)
    (18,37,49,26,23)(61,63,98,89,113)(62,84,82,103,66)(64,85,68,88,65)
    (69,90,119,100,99)(71,104,95,108,107)(121,123,158,149,173)(122,144,142,163,126)
    (124,145,128,148,125)(129,150,179,160,159)(131,164,155,168,167)
    (181,183,218,209,233)(182,204,202,223,186)(184,205,188,208,185)
    (189,210,239,220,219)(191,224,215,228,227)(198,217,229,206,203)
    (241,243,278,269,293)(242,264,262,283,246)(244,265,248,268,245)
    (249,270,299,280,279)(301,303,338,329,353)(302,324,322,343,306)
    (304,325,308,328,305)(309,330,359,340,339);
M_MKOPQ:=(7,45,17,54,57)(10,16,13,19,15)(12,51,55,27,42)(14,32,52,50,46)
    (20,56,58,60,41)(21,34,31,36,33)(67,105,77,114,117)(70,76,73,79,75)
    (72,111,115,87,102)(74,92,112,110,106)(81,94,91,96,93)(127,165,137,174,177)
    (130,136,133,139,135)(132,171,175,147,162)(134,152,172,170,166)
    (141,154,151,156,153)(187,225,197,234,237)(190,196,193,199,195)
    (192,231,235,207,222)(194,212,232,230,226)(200,236,238,240,221)
    (201,214,211,216,213)(250,256,253,259,255)(252,291,295,267,282)
    (254,272,292,290,286)(261,274,271,276,273)(310,316,313,319,315)
    (312,351,355,327,342)(314,332,352,350,346)(321,334,331,336,333);
M_FLACR:=(1,4,47,33,60)(2,25,9,41,7)(5,49,15,36,59)(6,26,11,30,53)(8,29,45,27,23)
    (19,44,28,43,42)(61,64,107,93,120)(62,85,69,101,67)(65,109,75,96,119)
    (66,86,71,90,113)(68,89,105,87,83)(121,124,167,153,180)(122,145,129,161,127)
    (125,169,135,156,179)(126,146,131,150,173)(128,149,165,147,143)
    (181,184,227,213,240)(182,205,189,221,187)(185,229,195,216,239)
    (186,206,191,210,233)(188,209,225,207,203)(199,224,208,223,222)
    (241,244,287,273,300)(242,265,249,281,247)(246,266,251,270,293)
    (248,269,285,267,263)(301,304,347,333,360)(302,325,309,341,307)
    (306,326,311,330,353)(308,329,345,327,323);
M_BKMJS:=(3,57,50,21,39)(10,48,40,58,52)(12,37,35,31,32)(13,14,17,22,18)
    (16,34,56,54,51)(20,38,24,55,46)(70,108,100,118,112)(72,97,95,91,92)
    (73,74,77,82,78)(76,94,116,114,111)(80,98,84,115,106)(130,168,160,178,172)
    (132,157,155,151,152)(133,134,137,142,138)(136,154,176,174,171)
    (140,158,144,175,166)(183,237,230,201,219)(190,228,220,238,232)
    (192,217,215,211,212)(193,194,197,202,198)(196,214,236,234,231)
    (200,218,204,235,226)(250,288,280,298,292)(252,277,275,271,272)
    (253,254,257,262,258)(256,274,296,294,291)(310,348,340,358,352)
    (312,337,335,331,332)(313,314,317,322,318)(316,334,356,354,351);
```

```
M_NALPO:=(6,49,10,50,7)(9,20,14,19,11)(12,52,60,53,43)(15,33,41,45,42)
    (17,32,36,30,29)(21,46,27,26,47)(66,109,70,110,67)(69,80,74,79,71)
    (72,112,120,113,103)(75,93,101,105,102)(81,106,87,86,107)(126,169,130,170,127)
    (129,140,134,139,131)(132,172,180,173,163)(135,153,161,165,162)
    (141,166,147,146,167)(186,229,190,230,187)(189,200,194,199,191)
    (192,232,240,233,223)(195,213,221,225,222)(197,212,216,210,209)
    (201,226,207,206,227)(246,289,250,290,247)(249,260,254,259,251)
    (255,273,281,285,282)(261,286,267,266,287)(306,349,310,350,307)
    (309,320,314,319,311)(315,333,341,345,342)(321,346,327,326,347);
M_JGEIS:=(1,57,51,37,5)(2,55,16,48,4)(3,24,18,35,40)(8,44,31,56,38)
    (13,34,39,25,23)(22,28,59,58,54)(61,117,111,97,65)(63,84,78,95,100)
    (68,104,91,116,98)(73,94,99,85,83)(82,88,119,118,114)(121,177,171,157,125)
    (123,144,138,155,160)(128,164,151,176,158)(133,154,159,145,143)
    (142,148,179,178,174)(181,237,231,217,185)(182,235,196,228,184)
    (183,204,198,215,220)(188,224,211,236,218)(193,214,219,205,203)
    (202,208,239,238,234)(241,297,291,277,245)(243,264,258,275,280)
    (248,284,271,296,278)(262,268,299,298,294)(301,357,351,337,305)
    (303,324,318,335,340)(308,344,331,356,338)(322,328,359,358,354);
M_FUQPL:=(1,6,42,32,58)(2,26,15,52,57)(7,27,19,36,60)(11,33,50,55,23)
    (13,44,30,41,46)(14,31,59,53,45)(61,66,102,92,118)(62,86,75,112,117)
    (67,87,79,96,120)(71,93,110,115,83)(73,104,90,101,106)(121,126,162,152,178)
    (122,146,135,172,177)(127,147,139,156,180)(131,153,170,175,143)
    (133,164,150,161,166)(181,186,222,212,238)(182,206,195,232,237)
    (187,207,199,216,240)(191,213,230,235,203)(193,224,210,221,226)
    (194,211,239,233,225)(242,266,255,292,297)(247,267,259,276,300)
    (251,273,290,295,263)(253,284,270,281,286)(302,326,315,352,357)
    (307,327,319,336,360)(311,333,350,355,323)(313,344,330,341,346);
M_IHDBS:=(3,54,12,49,4)(5,40,56,17,43)(8,18,16,21,9)(10,47,25,24,51)
    (20,29,28,35,34)(22,37,48,39,38)(63,114,72,109,64)(65,100,116,77,103)
    (68,78,76,81,69)(80,89,88,95,94)(82,97,108,99,98)(123,174,132,169,124)
    (125,160,176,137,163)(128,138,136,141,129)(140,149,148,155,154)
    (142,157,168,159,158)(183,234,192,229,184)(185,220,236,197,223)
    (188,198,196,201,189)(190,227,205,204,231)(200,209,208,215,214)
    (202,217,228,219,218)(243,294,252,289,244)(245,280,296,257,283)
    (260,269,268,275,274)(262,277,288,279,278)(303,354,312,349,304)
    (305,340,356,317,343)(320,329,328,335,334)(322,337,348,339,338);
M_UGJMQ:=(1,7,46,34,40)(2,27,14,56,3)(13,31,58,57,55)(16,35,59,60,50)
    (18,44,36,52,51)(19,32,54,24,23)(62,87,74,116,63)(73,91,118,117,115)
    (76,95,119,120,110)(78,104,96,112,111)(79,92,114,84,83)(122,147,134,176,123)
    (133,151,178,177,175)(136,155,179,180,170)(138,164,156,172,171)
    (139,152,174,144,143)(181,187,226,214,220)(182,207,194,236,183)
    (193,211,238,237,235)(196,215,239,240,230)(198,224,216,232,231)
    (199,212,234,204,203)(253,271,298,297,295)(256,275,299,300,290)
    (258,284,276,292,291)(259,272,294,264,263)(313,331,358,357,355)
    (316,335,359,360,350)(318,344,336,352,351)(319,332,354,324,323);
M_DHCAN:=(4,39,20,45,6)(5,48,21,41,53)(8,22,12,15,11)(9,29,43,49,47)
    (10,33,30,28,37)(17,42,26,25,38)(64,99,80,105,66)(65,108,81,101,113)
    (69,89,103,109,107)(70,93,90,88,97)(77,102,86,85,98)(124,159,140,165,126)
    (125,168,141,161,173)(129,149,163,169,167)(130,153,150,148,157)
    (137,162,146,145,158)(184,219,200,225,186)(185,228,201,221,233)
    (188,202,192,195,191)(189,209,223,229,227)(190,213,210,208,217)
    (197,222,206,205,218)(244,279,260,285,246)(245,288,261,281,293)
    (249,269,283,289,287)(257,282,266,265,278)(304,339,320,345,306)
    (305,348,321,341,353)(309,329,343,349,347)(317,342,326,325,338);
M_ER:=(1,8)(2,28)(3,30)(4,44)(5,23)(6,18)(9,58)(11,40)(24,53)(25,59)(26,35)(29,57)
    (31,47)(36,39)(38,60)(61,68)(62,88)(63,90)(64,104)(65,83)(66,78)(69,118)
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    (71,100)(84,113)(85,119)(86,95)(89,117)(91,107)(96,99)(98,120)(121,128)
    (122,148)(123,150)(124,164)(125,143)(126,138)(127,142)(129,178)(131,160)
    (144,173)(145,179)(146,155)(149,177)(156,159)(158,180)(181,188)(182,208)
    (183,210)(184,224)(185,203)(186,198)(187,202)(189,238)(191,220)(204,233)
    (205,239)(206,215)(209,237)(216,219)(218,240)(241,248)(242,268)(243,270)
    (244,284)(245,263)(246,258)(247,262)(249,298)(251,280)(264,293)(265,299)
    (269,297)(276,279)(278,300)(281,294)(301,308)(302,328)(303,330)(304,344)
    (305,323)(306,318)(307,322)(309,358)(311,340)(324,353)(325,359)(326,335)
    (329,357)(336,339)(338,360);
M_KO:=(7,22)(10,14)(12,46)(13,49)(15,34)(16,42)(17,50)(19,48)(20,52)(21,32)(27,37)
    (33,56)(41,54)(43,55)(45,51)(67,82)(70,74)(72,106)(73,109)(75,94)(76,102)
    (77,110)(79,108)(80,112)(81,92)(87,97)(93,116)(101,114)(103,115)(105,111)
    (130,134)(132,166)(133,169)(135,154)(136,162)(137,170)(139,168)(140,172)
    (141,152)(147,157)(151,167)(153,176)(161,174)(163,175)(165,171)(190,194)
    (192,226)(193,229)(195,214)(196,222)(197,230)(199,228)(200,232)(201,212)
    (207,217)(211,227)(213,236)(221,234)(223,235)(225,231)(250,254)(252,286)
    (253,289)(255,274)(256,282)(257,290)(259,288)(260,292)(261,272)(266,275)
    (267,277)(271,287)(273,296)(283,295)(285,291)(310,314)(312,346)(313,349)
    (315,334)(316,342)(317,350)(319,348)(320,352)(321,332)(327,337)(331,347)
    (333,356)(341,354)(343,355)(345,351);
M_CR:=(1,9)(2,29)(3,41)(4,30)(5,11)(6,8)(7,38)(23,43)(25,53)(26,28)(33,40)(36,48)
    (39,60)(44,49)(47,59)(61,69)(62,89)(63,101)(64,90)(65,71)(66,68)(67,98)(83,103)
    (85,113)(86,88)(93,100)(96,108)(99,120)(104,109)(107,119)(121,129)(122,149)
    (123,161)(124,150)(125,131)(126,128)(127,158)(143,163)(144,165)(145,173)
    (146,148)(153,160)(159,180)(164,169)(167,179)(181,189)(182,209)(183,221)
    (184,210)(185,191)(186,188)(187,218)(203,223)(204,225)(205,233)(206,208)
    (213,220)(219,240)(224,229)(227,239)(241,249)(242,269)(243,281)(244,270)
    (245,251)(246,248)(247,278)(260,297)(263,283)(264,285)(265,293)(266,268)
    (273,280)(279,300)(287,299)(301,309)(302,329)(303,341)(304,330)(305,311)
    (306,308)(307,338)(323,343)(324,345)(325,353)(326,328)(333,340)(339,360)
    (344,349)(347,359);
M_KM:=(10,31)(12,13)(14,51)(15,35)(16,32)(17,55)(18,42)(19,37)(20,57)(21,58)
    (22,27)(24,45)(34,52)(46,54)(50,56)(70,91)(72,73)(74,111)(75,95)(76,92)(77,115)
    (78,102)(79,97)(80,117)(81,118)(82,87)(84,105)(94,112)(106,114)(110,116)
    (130,151)(132,133)(134,171)(135,155)(136,152)(137,175)(138,162)(139,157)
    (140,177)(141,178)(142,147)(154,172)(156,168)(166,174)(170,176)(190,211)
    (192,193)(194,231)(195,215)(196,212)(197,235)(198,222)(199,217)(200,237)
    (201,238)(202,207)(214,232)(216,228)(226,234)(230,236)(250,271)(252,253)
    (254,291)(255,275)(256,272)(257,295)(258,282)(259,277)(261,298)(262,267)
    (274,292)(276,288)(284,289)(286,294)(290,296)(310,331)(312,313)(314,351)
    (315,335)(316,332)(317,355)(318,342)(319,337)(320,357)(321,358)(322,327)
    (334,352)(336,348)(346,354)(350,356);
M_AL:=(1,10)(2,21)(4,50)(5,52)(6,33)(7,47)(9,27)(11,45)(15,53)(19,29)(25,46)
    (26,41)(30,42)(36,43)(49,60)(61,70)(62,81)(64,110)(65,112)(66,93)(67,107)
    (69,87)(71,105)(75,113)(79,89)(85,106)(86,101)(90,102)(96,103)(109,120)
    (121,130)(122,141)(124,170)(126,153)(127,167)(129,147)(131,165)(135,173)
    (139,149)(140,143)(145,166)(146,161)(150,162)(156,163)(169,180)(181,190)
    (182,201)(184,230)(186,213)(187,227)(189,207)(191,225)(195,233)(199,209)
    (200,203)(205,226)(206,221)(210,222)(216,223)(229,240)(241,250)(242,261)
    (244,290)(246,273)(247,287)(249,267)(251,285)(255,293)(259,269)(260,263)
    (265,286)(266,281)(270,282)(279,295)(289,300)(301,310)(302,321)(304,350)
    (306,333)(307,347)(309,327)(311,345)(315,353)(319,329)(320,323)(325,346)
    (326,341)(330,342)(336,343)(349,360);
M_SJ:=(3,16)(8,14)(12,59)(13,38)(17,44)(18,56)(20,23)(22,31)(24,34)(28,32)(35,54)
    (37,58)(39,55)(40,51)(48,57)(63,76)(68,74)(72,119)(73,98)(77,104)(78,116)
    (80,83)(82,91)(84,94)(88,92)(95,114)(97,118)(99,115)(100,111)(108,117)(123,136)
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    (125,172)(128,134)(132,179)(133,158)(137,164)(138,176)(142,151)(144,154)
    (148,152)(155,174)(157,178)(159,175)(160,171)(168,177)(183,196)(185,232)
    (188,194)(192,239)(193,218)(197,224)(198,236)(202,211)(204,214)(208,212)
    (215,234)(217,238)(219,235)(220,231)(228,237)(243,256)(245,292)(248,254)
    (252,299)(253,278)(257,284)(258,296)(262,271)(264,274)(268,272)(275,294)
    (276,283)(277,298)(280,291)(288,297)(303,316)(305,352)(308,314)(312,359)
    (313,338)(317,344)(318,356)(322,331)(324,334)(328,332)(335,354)(337,358)
    (339,355)(340,351)(348,357);
M_FR:=(1,11)(2,30)(3,33)(4,36)(5,19)(6,44)(7,8)(9,57)(15,40)(23,53)(24,41)(25,60)
    (26,59)(27,28)(47,58)(61,71)(62,90)(63,93)(64,96)(65,79)(66,104)(67,68)(69,117)
    (75,100)(83,113)(84,101)(85,120)(86,119)(87,88)(107,118)(121,131)(122,150)
    (123,153)(124,156)(125,139)(126,164)(127,128)(129,177)(143,173)(144,161)
    (145,180)(146,179)(147,148)(149,175)(167,178)(181,191)(182,210)(183,213)
    (184,216)(185,199)(186,224)(187,188)(189,237)(203,233)(204,221)(205,240)
    (206,239)(207,208)(209,235)(227,238)(241,251)(242,270)(243,273)(244,276)
    (246,284)(247,248)(249,297)(263,293)(264,281)(265,300)(266,299)(267,268)
    (269,295)(278,290)(287,298)(301,311)(302,330)(303,333)(304,336)(305,319)
    (306,344)(307,308)(309,357)(323,353)(324,341)(325,360)(326,359)(327,328)
    (329,355)(347,358);
M_KB:=(10,56)(12,34)(13,43)(14,37)(16,17)(18,45)(20,51)(21,54)(22,46)(29,55)
    (31,49)(32,48)(35,42)(38,50)(39,52)(70,116)(72,94)(73,103)(74,97)(76,77)
    (78,105)(80,111)(81,114)(82,106)(89,115)(91,109)(92,108)(95,102)(98,110)
    (99,112)(130,176)(132,154)(133,163)(134,157)(135,160)(136,137)(138,165)
    (140,171)(141,174)(142,166)(151,169)(152,168)(155,162)(158,170)(159,172)
    (190,236)(192,214)(193,223)(194,217)(195,220)(196,197)(198,225)(200,231)
    (201,234)(202,226)(211,229)(212,228)(215,222)(218,230)(219,232)(245,259)
    (250,296)(252,274)(253,283)(254,277)(255,280)(256,257)(258,285)(260,291)
    (261,294)(262,286)(271,289)(272,288)(275,282)(279,292)(310,356)(312,334)
    (313,343)(314,337)(315,340)(316,317)(318,345)(320,351)(321,354)(322,346)
    (331,349)(332,348)(335,342)(338,350)(339,352);
M_PL:=(1,12)(2,10)(6,52)(7,15)(11,46)(14,30)(19,41)(21,23)(26,50)(27,33)(32,53)
    (36,45)(42,60)(47,55)(49,57)(61,72)(62,70)(66,112)(67,75)(71,106)(74,90)
    (79,101)(81,83)(86,110)(87,93)(92,113)(96,105)(102,120)(107,115)(109,117)
    (122,130)(126,172)(127,135)(129,133)(131,166)(134,150)(139,161)(141,143)
    (146,170)(147,153)(152,173)(156,165)(162,180)(167,175)(169,177)(182,190)
    (186,232)(187,195)(189,193)(191,226)(194,210)(199,221)(201,203)(206,230)
    (207,213)(212,233)(216,225)(222,240)(227,235)(229,237)(242,250)(246,292)
    (247,255)(249,253)(251,286)(254,270)(256,265)(259,281)(261,263)(266,290)
    (267,273)(276,285)(282,300)(287,295)(289,297)(302,310)(306,352)(307,315)
    (309,313)(311,346)(314,330)(319,341)(321,323)(326,350)(327,333)(332,353)
    (336,345)(342,360)(347,355)(349,357);
M_SI:=(3,37)(4,51)(5,54)(8,34)(9,13)(16,25)(17,59)(18,39)(20,44)(22,40)(24,48)
    (28,56)(29,31)(35,38)(43,58)(63,97)(64,111)(65,114)(68,94)(69,73)(76,85)
    (77,119)(78,99)(80,104)(82,100)(84,108)(88,116)(89,91)(95,98)(103,118)(121,132)
    (123,157)(124,171)(125,174)(128,154)(136,145)(137,179)(138,159)(140,164)
    (142,160)(144,168)(148,176)(149,151)(155,158)(163,178)(181,192)(183,217)
    (184,231)(185,234)(188,214)(196,205)(197,239)(198,219)(200,224)(202,220)
    (204,228)(208,236)(209,211)(215,218)(223,238)(241,252)(243,277)(244,291)
    (245,294)(248,274)(257,299)(258,279)(260,284)(262,280)(264,288)(268,296)
    (269,271)(272,293)(275,278)(283,298)(301,312)(303,337)(304,351)(305,354)
    (308,334)(316,325)(317,359)(318,339)(320,344)(322,340)(324,348)(328,356)
    (329,331)(335,338)(343,358);
M_UG:=(1,13)(2,31)(3,19)(7,35)(8,52)(11,54)(18,60)(23,58)(24,36)(25,32)(26,56)
    (27,40)(30,51)(44,57)(55,59)(61,73)(62,91)(63,79)(67,95)(68,112)(71,114)
    (78,120)(83,118)(84,96)(85,92)(86,116)(87,100)(90,111)(104,117)(115,119)
    (121,133)(122,151)(123,139)(127,155)(128,172)(131,174)(138,180)(143,178)
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    (144,156)(145,152)(147,160)(148,170)(150,171)(164,177)(175,179)(181,193)
    (182,211)(183,199)(187,215)(188,232)(191,234)(198,240)(203,238)(204,216)
    (205,212)(207,220)(208,230)(210,231)(224,237)(235,239)(241,253)(242,271)
    (243,259)(247,275)(248,292)(251,294)(258,300)(262,273)(263,298)(264,276)
    (265,272)(268,290)(270,291)(284,297)(295,299)(301,313)(302,331)(303,319)
    (307,335)(308,352)(311,354)(318,360)(323,358)(324,336)(325,332)(327,340)
    (328,350)(330,351)(344,357)(355,359);
M_ND:=(4,14)(5,46)(6,34)(9,12)(10,29)(15,38)(16,53)(17,47)(20,49)(21,43)(22,33)
    (28,50)(37,41)(39,42)(45,48)(64,74)(65,106)(66,94)(69,72)(70,89)(75,98)(76,113)
    (77,107)(80,109)(81,103)(82,93)(88,110)(97,101)(99,102)(105,108)(124,134)
    (125,166)(126,154)(129,132)(130,149)(135,158)(136,173)(137,167)(140,169)
    (141,163)(142,153)(146,176)(157,161)(159,162)(165,168)(184,194)(185,226)
    (186,214)(189,192)(190,209)(195,218)(196,233)(197,227)(200,229)(201,223)
    (202,213)(206,236)(217,221)(219,222)(225,228)(244,254)(245,286)(246,274)
    (249,252)(250,269)(255,278)(256,293)(257,287)(260,289)(261,283)(266,296)
    (267,280)(277,281)(279,282)(285,288)(304,314)(305,346)(306,334)(309,312)
    (310,329)(315,338)(316,353)(317,347)(320,349)(321,343)(322,333)(326,356)
    (337,341)(339,342)(345,348);
M_UQ:=(1,14)(2,32)(3,42)(7,31)(11,51)(13,60)(15,24)(18,33)(19,57)(23,52)(26,54)
    (27,58)(36,55)(44,50)(46,59)(61,74)(62,92)(63,102)(67,91)(71,111)(73,120)
    (75,84)(78,93)(79,117)(83,112)(86,114)(87,118)(96,115)(104,110)(106,119)
    (121,134)(122,152)(127,151)(131,171)(133,180)(135,144)(136,150)(138,153)
    (139,177)(143,172)(146,174)(147,178)(156,175)(164,170)(166,179)(181,194)
    (182,212)(187,211)(191,231)(193,240)(195,204)(196,210)(198,213)(199,237)
    (203,232)(206,234)(207,238)(216,235)(224,230)(226,239)(242,272)(247,271)
    (248,250)(251,291)(253,300)(255,264)(256,270)(258,273)(259,297)(263,292)
    (266,294)(267,298)(276,295)(284,290)(286,299)(301,314)(302,332)(307,331)
    (311,351)(313,360)(315,324)(316,330)(318,333)(319,357)(323,352)(326,354)
    (327,358)(336,355)(344,350)(346,359);
M_HD:=(4,17)(5,20)(6,56)(8,10)(9,37)(12,25)(16,30)(21,28)(22,47)(29,48)(34,53)
    (35,41)(38,49)(39,43)(40,45)(64,77)(65,80)(66,116)(68,70)(69,97)(72,85)(76,90)
    (81,88)(82,107)(89,108)(94,113)(95,101)(98,109)(99,103)(100,105)(123,162)
    (124,137)(125,140)(126,176)(128,130)(129,157)(132,145)(141,148)(142,167)
    (149,168)(154,173)(155,161)(158,169)(159,163)(160,165)(183,222)(184,197)
    (185,200)(186,236)(188,190)(189,217)(192,205)(201,208)(202,227)(209,228)
    (214,233)(215,221)(218,229)(219,223)(220,225)(241,254)(243,282)(244,257)
    (245,260)(246,296)(249,277)(252,265)(261,268)(262,287)(269,288)(274,293)
    (275,281)(278,289)(279,283)(280,285)(303,342)(304,317)(305,320)(306,356)
    (308,310)(309,337)(312,325)(321,328)(322,347)(329,348)(334,353)(335,341)
    (338,349)(339,343)(340,345);
M_FL:=(1,15)(2,33)(4,52)(6,36)(7,11)(9,55)(19,53)(23,41)(25,50)(26,60)(27,30)
    (42,59)(44,45)(47,57)(49,58)(61,75)(62,93)(64,112)(66,96)(67,71)(69,115)
    (79,113)(83,101)(85,110)(86,120)(87,90)(102,119)(104,105)(107,117)(109,118)
    (121,135)(122,153)(124,172)(126,156)(127,131)(128,166)(129,175)(139,173)
    (143,161)(145,170)(146,180)(147,150)(162,179)(164,165)(167,177)(181,195)
    (182,213)(184,232)(186,216)(187,191)(188,226)(189,235)(199,233)(203,221)
    (205,230)(206,240)(207,210)(222,239)(224,225)(227,237)(241,255)(242,273)
    (244,292)(246,276)(247,251)(248,286)(249,295)(259,293)(261,264)(263,281)
    (265,290)(266,300)(267,270)(284,285)(287,297)(301,315)(302,333)(304,352)
    (306,336)(307,311)(308,346)(309,355)(319,353)(323,341)(325,350)(326,360)
    (327,330)(342,359)(344,345)(347,357);
M_SB:=(3,10)(5,32)(8,46)(12,40)(13,29)(14,28)(16,38)(17,35)(18,20)(21,24)(22,34)
    (31,43)(37,56)(39,51)(48,54)(63,70)(65,92)(68,106)(72,100)(73,89)(74,88)(76,98)
    (77,95)(78,80)(81,84)(82,94)(91,103)(97,116)(99,111)(108,114)(123,130)(125,152)
    (132,160)(133,149)(134,148)(136,158)(137,155)(138,140)(141,144)(142,154)
    (151,163)(157,176)(159,171)(168,174)(169,178)(183,190)(185,212)(192,220)
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    (193,209)(194,208)(196,218)(197,215)(198,200)(201,204)(202,214)(211,223)
    (217,236)(219,231)(228,234)(229,238)(243,250)(245,272)(252,280)(253,269)
    (254,268)(256,278)(257,275)(258,260)(262,274)(271,283)(277,296)(279,291)
    (282,299)(288,294)(289,298)(303,310)(305,332)(312,340)(313,329)(314,328)
    (316,338)(317,335)(318,320)(321,324)(322,334)(331,343)(337,356)(339,351)
    (348,354)(349,358);
M_JG:=(1,16)(2,34)(3,13)(8,32)(14,25)(18,58)(19,38)(22,36)(23,56)(24,31)(28,52)
    (35,57)(40,55)(44,54)(51,59)(61,76)(62,94)(63,73)(68,92)(74,85)(78,118)(79,98)
    (82,96)(83,116)(84,91)(88,112)(95,117)(100,115)(104,114)(111,119)(121,136)
    (122,154)(123,133)(128,152)(138,178)(139,158)(142,156)(143,176)(144,151)
    (148,172)(155,177)(157,180)(160,175)(164,174)(171,179)(181,196)(182,214)
    (183,193)(188,212)(198,238)(199,218)(202,216)(203,236)(204,211)(208,232)
    (215,237)(217,240)(220,235)(224,234)(231,239)(241,256)(243,253)(248,272)
    (252,270)(258,298)(259,278)(262,276)(263,296)(264,271)(268,292)(275,297)
    (277,300)(280,295)(284,294)(291,299)(301,316)(302,334)(303,313)(308,332)
    (318,358)(319,338)(322,336)(323,356)(324,331)(328,352)(335,357)(337,360)
    (340,355)(344,354)(351,359);
M_NA:=(4,46)(5,50)(6,21)(7,48)(9,42)(10,53)(11,17)(12,30)(15,29)(20,26)(27,39)
    (33,43)(37,60)(41,49)(45,47)(64,106)(65,110)(66,81)(67,108)(69,102)(70,113)
    (71,77)(72,90)(75,89)(80,86)(87,99)(93,103)(97,120)(101,109)(105,107)(124,166)
    (125,170)(126,141)(127,168)(129,162)(130,173)(131,137)(132,150)(134,145)
    (135,149)(140,146)(147,159)(153,163)(161,169)(165,167)(184,226)(185,230)
    (186,201)(187,228)(189,222)(190,233)(191,197)(192,210)(194,205)(195,209)
    (200,206)(207,219)(213,223)(221,229)(225,227)(242,274)(244,286)(245,290)
    (246,261)(247,288)(249,282)(250,293)(251,257)(254,265)(255,269)(260,266)
    (267,279)(273,283)(281,289)(285,287)(304,346)(305,350)(306,321)(307,348)
    (309,342)(310,353)(311,317)(312,330)(314,325)(315,329)(320,326)(327,339)
    (333,343)(341,349)(345,347);
M_HI:=(1,17)(3,43)(4,22)(5,38)(8,48)(9,35)(18,47)(20,59)(21,44)(24,49)(25,37)
    (28,39)(29,40)(30,34)(53,56)(61,77)(63,103)(64,82)(65,98)(68,108)(69,95)
    (78,107)(80,119)(81,104)(84,109)(85,97)(88,99)(89,100)(90,94)(113,116)(121,137)
    (123,163)(124,142)(125,158)(126,174)(128,168)(129,155)(138,167)(140,179)
    (144,169)(145,157)(148,159)(149,160)(150,154)(173,176)(181,197)(183,223)
    (184,202)(185,218)(186,234)(188,228)(189,215)(198,227)(200,239)(204,229)
    (205,217)(208,219)(209,220)(210,214)(233,236)(241,257)(243,283)(244,262)
    (245,278)(246,294)(248,288)(249,275)(260,299)(264,289)(265,277)(268,279)
    (269,280)(270,274)(285,298)(293,296)(301,317)(303,343)(304,322)(305,338)
    (306,354)(308,348)(309,335)(318,347)(320,359)(324,349)(325,337)(328,339)
    (329,340)(330,334)(353,356);
M_PQ:=(2,12)(6,54)(7,32)(10,23)(11,16)(13,33)(14,60)(15,55)(19,50)(26,51)(27,52)
    (31,41)(36,46)(42,57)(45,58)(62,72)(66,114)(67,92)(70,83)(71,76)(73,93)(74,120)
    (75,115)(79,110)(86,111)(87,112)(91,101)(96,106)(102,117)(105,118)(122,132)
    (127,152)(130,143)(131,136)(133,153)(134,180)(135,175)(139,170)(141,164)
    (146,171)(147,172)(151,161)(156,166)(162,177)(165,178)(182,192)(187,212)
    (190,203)(191,196)(193,213)(194,240)(195,235)(199,230)(201,224)(206,231)
    (207,232)(211,221)(216,226)(222,237)(225,238)(242,252)(247,272)(250,263)
    (251,256)(253,273)(254,300)(255,295)(258,287)(259,290)(261,284)(266,291)
    (267,292)(271,281)(276,286)(282,297)(302,312)(307,332)(310,323)(311,316)
    (313,333)(314,360)(315,355)(319,350)(321,344)(326,351)(327,352)(331,341)
    (336,346)(342,357)(345,358);
M_EG:=(1,18)(2,35)(3,44)(4,13)(5,55)(8,58)(11,56)(19,39)(22,60)(23,40)(24,59)
    (25,31)(28,57)(30,54)(36,38)(61,78)(62,95)(63,104)(64,73)(65,115)(68,118)
    (71,116)(79,99)(82,120)(83,100)(84,119)(85,91)(88,117)(90,114)(96,98)(121,138)
    (122,155)(123,164)(124,133)(125,175)(128,178)(131,176)(142,180)(143,160)
    (144,179)(145,151)(148,177)(150,174)(156,158)(171,173)(181,198)(182,215)
    (183,224)(184,193)(185,235)(188,238)(191,236)(202,240)(203,220)(204,239)
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    (205,211)(208,237)(210,234)(216,218)(231,233)(241,258)(242,275)(243,284)
    (245,295)(248,298)(251,296)(262,300)(263,280)(264,299)(265,271)(268,297)
    (269,292)(270,294)(276,278)(291,293)(301,318)(302,335)(303,344)(304,313)
    (305,355)(308,358)(311,356)(322,360)(323,340)(324,359)(325,331)(328,357)
    (330,354)(336,338)(351,353);
M_NO:=(6,16)(7,37)(9,32)(10,45)(12,41)(14,47)(15,20)(17,33)(21,42)(26,34)(27,48)
    (29,52)(43,50)(46,49)(51,53)(66,76)(67,97)(69,92)(70,105)(72,101)(74,107)
    (75,80)(77,93)(81,102)(86,94)(87,108)(89,112)(103,110)(106,109)(111,113)
    (126,136)(127,157)(129,152)(130,165)(132,161)(134,167)(135,140)(137,153)
    (139,159)(141,162)(146,154)(147,168)(149,172)(163,170)(166,169)(186,196)
    (187,217)(189,212)(190,225)(192,221)(194,227)(195,200)(197,213)(199,219)
    (201,222)(206,214)(207,228)(209,232)(223,230)(226,229)(244,253)(246,256)
    (247,277)(249,272)(250,285)(252,281)(254,287)(255,260)(257,273)(259,279)
    (261,282)(266,274)(267,288)(283,290)(286,289)(306,316)(307,337)(309,332)
    (310,345)(312,341)(314,347)(315,320)(317,333)(319,339)(321,342)(326,334)
    (327,348)(329,352)(343,350)(346,349);
M_FU:=(1,19)(2,36)(3,15)(4,32)(6,31)(7,44)(8,50)(11,57)(13,53)(23,60)(24,33)
    (25,52)(26,58)(27,59)(30,55)(61,79)(62,96)(63,75)(64,92)(66,91)(67,104)(68,110)
    (71,117)(73,113)(83,120)(84,93)(85,112)(86,118)(87,119)(90,115)(121,139)
    (122,156)(123,135)(126,151)(127,164)(128,170)(131,177)(133,173)(138,161)
    (143,180)(144,153)(145,172)(146,178)(147,179)(150,175)(181,199)(182,216)
    (183,195)(186,211)(187,224)(188,230)(191,237)(193,233)(198,221)(203,240)
    (204,213)(205,232)(206,238)(207,239)(210,235)(241,259)(242,276)(243,255)
    (247,284)(248,290)(249,291)(251,297)(253,293)(258,281)(263,300)(264,273)
    (265,292)(266,298)(267,299)(270,295)(301,319)(302,336)(303,315)(306,331)
    (307,344)(308,350)(311,357)(313,353)(318,341)(323,360)(324,333)(325,352)
    (326,358)(327,359)(330,355);
M_DB:=(5,14)(9,51)(10,38)(12,39)(16,29)(17,48)(18,41)(20,37)(21,22)(28,46)(34,43)
    (35,45)(40,42)(47,54)(49,56)(65,74)(69,111)(70,98)(72,99)(76,89)(77,108)
    (78,101)(80,97)(81,82)(88,106)(94,103)(95,105)(100,102)(107,114)(109,116)
    (124,152)(125,134)(129,171)(130,158)(132,159)(136,149)(137,168)(140,157)
    (141,142)(148,166)(154,163)(155,165)(160,162)(167,174)(169,176)(184,212)
    (185,194)(189,231)(190,218)(192,219)(196,209)(197,228)(200,217)(201,202)
    (208,226)(214,223)(215,225)(220,222)(227,234)(229,236)(244,272)(245,254)
    (246,271)(250,278)(252,279)(256,269)(257,288)(260,277)(261,262)(268,286)
    (274,283)(275,285)(280,282)(287,294)(289,296)(304,332)(305,314)(309,351)
    (310,338)(312,339)(316,329)(317,348)(320,337)(321,322)(328,346)(334,343)
    (335,345)(340,342)(347,354)(349,356);
M_HC:=(1,20)(3,45)(4,29)(5,9)(6,38)(8,49)(11,37)(21,59)(22,26)(25,43)(28,47)
    (30,48)(33,35)(39,53)(40,41)(61,80)(63,105)(64,89)(65,69)(66,98)(68,109)(71,97)
    (81,119)(82,86)(85,103)(88,107)(90,108)(93,95)(99,113)(100,101)(121,140)
    (122,137)(123,165)(124,149)(125,129)(126,158)(128,169)(131,157)(141,179)
    (142,146)(145,163)(148,167)(150,168)(159,173)(160,161)(181,200)(182,197)
    (183,225)(184,209)(185,189)(186,218)(188,229)(191,217)(201,239)(202,206)
    (205,223)(208,227)(210,228)(219,233)(220,221)(241,260)(242,257)(243,285)
    (244,269)(245,249)(246,278)(247,296)(248,289)(261,299)(262,266)(265,283)
    (268,287)(270,288)(279,293)(280,281)(301,320)(302,317)(303,345)(304,329)
    (305,309)(306,338)(308,349)(311,337)(321,359)(322,326)(325,343)(328,347)
    (330,348)(339,353)(340,341);
M_MQ:=(2,17)(7,56)(10,44)(12,23)(13,52)(14,57)(15,18)(16,36)(19,51)(24,42)(27,54)
    (31,50)(32,55)(34,60)(46,58)(62,77)(67,116)(70,104)(72,83)(73,112)(74,117)
    (75,78)(76,96)(79,111)(84,102)(87,114)(91,110)(92,115)(94,120)(106,118)
    (127,176)(130,164)(132,143)(133,172)(134,177)(135,138)(136,156)(139,171)
    (144,162)(147,174)(151,170)(152,175)(153,155)(154,180)(166,178)(187,236)
    (190,224)(192,203)(193,232)(194,237)(195,198)(196,216)(199,231)(204,222)
    (207,234)(211,230)(212,235)(213,215)(214,240)(226,238)(250,284)(251,277)
```

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    (252,263)(253,292)(254,297)(255,258)(256,276)(259,291)(264,282)(267,294)
    (271,290)(272,295)(273,275)(274,300)(286,298)(307,356)(310,344)(312,323)
    (313,352)(314,357)(315,318)(316,336)(319,351)(324,342)(327,354)(331,350)
    (332,355)(333,335)(334,360)(346,358);
M_CA:=(1,21)(2,20)(4,41)(5,33)(6,9)(7,39)(8,42)(10,59)(11,43)(15,28)(25,45)(26,29)
    (30,49)(47,53)(48,60)(61,81)(62,80)(64,101)(65,93)(66,69)(67,99)(68,102)
    (70,119)(71,103)(75,88)(85,105)(86,89)(90,109)(107,113)(108,120)(121,141)
    (122,140)(124,161)(125,153)(126,129)(127,159)(128,162)(131,163)(135,148)
    (145,165)(146,149)(147,158)(150,169)(167,173)(168,180)(181,201)(182,200)
    (184,221)(185,213)(186,189)(187,219)(188,222)(191,223)(195,208)(205,225)
    (206,209)(207,218)(210,229)(227,233)(228,240)(241,261)(242,260)(243,286)
    (244,281)(245,273)(246,249)(247,279)(248,282)(251,283)(265,285)(266,269)
    (267,278)(270,289)(287,293)(288,300)(301,321)(302,320)(304,341)(305,333)
    (306,309)(307,339)(308,342)(311,343)(315,328)(325,345)(326,329)(327,338)
    (330,349)(347,353)(348,360);
M_JM:=(3,46)(12,44)(13,54)(14,24)(16,58)(17,23)(18,32)(19,22)(27,38)(31,51)(34,57)
    (35,52)(36,37)(40,50)(55,56)(63,106)(72,104)(73,114)(74,84)(76,118)(77,83)
    (78,92)(79,82)(87,98)(91,111)(94,117)(95,112)(96,97)(100,110)(115,116)(123,166)
    (130,179)(132,164)(133,174)(134,144)(136,178)(137,143)(138,152)(139,142)
    (151,171)(154,177)(155,172)(156,157)(160,170)(175,176)(183,226)(190,239)
    (192,224)(193,234)(194,204)(196,238)(197,203)(198,212)(199,202)(211,231)
    (214,237)(215,232)(216,217)(220,230)(235,236)(250,299)(252,284)(253,294)
    (254,264)(255,268)(256,298)(257,263)(258,272)(259,262)(271,291)(274,297)
    (275,292)(276,277)(280,290)(295,296)(303,346)(310,359)(312,344)(313,354)
    (314,324)(316,358)(317,323)(318,332)(319,322)(331,351)(334,357)(335,352)
    (336,337)(340,350)(355,356);
M_EI:=(1,22)(2,37)(3,28)(4,18)(5,24)(8,40)(9,31)(11,34)(23,48)(25,35)(29,58)
    (30,56)(38,59)(39,44)(53,54)(61,82)(62,97)(63,88)(64,78)(65,84)(68,100)(69,91)
    (71,94)(83,108)(85,95)(89,118)(90,116)(98,119)(99,104)(113,114)(121,142)
    (122,157)(123,148)(124,138)(125,144)(128,160)(129,151)(143,168)(145,155)
    (149,178)(150,176)(158,179)(159,164)(163,177)(173,174)(181,202)(182,217)
    (183,208)(184,198)(185,204)(188,220)(189,211)(203,228)(205,215)(209,238)
    (210,236)(218,239)(219,224)(223,237)(233,234)(241,262)(242,277)(243,268)
    (244,258)(245,264)(248,280)(249,271)(257,300)(265,275)(269,298)(270,296)
    (278,299)(279,284)(283,297)(293,294)(301,322)(302,337)(303,328)(304,318)
    (305,324)(308,340)(309,331)(323,348)(325,335)(329,358)(330,356)(338,359)
    (339,344)(343,357)(353,354);
M_PO:=(6,51)(7,12)(10,27)(13,47)(14,33)(15,46)(16,26)(17,60)(19,21)(20,36)(32,41)
    (42,50)(43,57)(45,52)(49,55)(66,111)(67,72)(70,87)(73,107)(74,93)(75,106)
    (76,86)(77,120)(79,81)(80,96)(92,101)(102,110)(103,117)(105,112)(109,115)
    (126,171)(127,132)(130,147)(131,154)(133,167)(134,153)(135,166)(136,146)
    (137,180)(139,141)(140,156)(152,161)(162,170)(165,172)(169,175)(186,231)
    (187,192)(190,207)(191,214)(193,227)(194,213)(195,226)(196,206)(197,240)
    (199,201)(200,216)(212,221)(222,230)(225,232)(229,235)(246,291)(247,252)
    (250,267)(251,274)(253,287)(254,273)(255,286)(256,266)(259,261)(260,276)
    (263,288)(272,281)(282,290)(285,292)(289,295)(306,351)(307,312)(310,327)
    (311,334)(313,347)(314,333)(315,346)(316,326)(317,360)(319,321)(320,336)
    (332,341)(342,350)(345,352)(349,355);
Gen:=[
M_FREGU,M_OKBDN,M_HIERC,M_MKOPQ,M_FLACR,M_BKMJS,M_NALPO,M_JGEIS,M_FUQPL,M_IHDBS,
    M_UGJMQ,M_DHCAN,M_ER,M_KO,M_CR,M_KM,M_AL,M_SJ,M_FR,M_KB,M_PL,M_SI,M_UG,M_ND,
    M_UQ,M_HD,M_FL,M_SB,M_JG,M_NA,M_HI,M_PQ,M_EG,M_NO,M_FU,M_DB,M_HC,M_MQ,M_CA,M_JM
    ,M_EI,M_PO
];
ip:=[
     [1,25,30],[2,11,60],[3,8,59],[4,9,53],[5,29,39],
```

```
    [6,41,47],[7,26,33],[10,42,46],[12,14,16],[13,32,51],
    [15,27,50],[17,34,37],[18,31,54],[19,52,55],[20,43,48],
    [21,45,49],[22,35,56],[23,36,57],[24,44,58],[28,38,40],
    [61,85,90],[62,71,120],[63,68,119],[64,69,113],[65,89,99],
    [66,101,107],[67,86,93],[70,102,106],[72,74,76],[73,92,111],
    [75,87,110],[77,94,97],[78,91,114],[79,112,115],[80,103,108],
    [81,105,109],[82,95,116],[83,96,117],[84,104,118],[88,98,100],
    [121,145,150],[122,131,180],[123,128,179],[124,129,173],[125,149,159],
    [126,161,167],[127,146,153],[130,162,166],[132,134,136],[133,152,171],
    [135,147,170],[137,154,157],[138,151,174],[139,172,175],[140,163,168],
    [141,165,169],[142,155,176],[143,156,177],[144,164,178],[148,158,160],
    [181,205,210],[182,191,240],[183,188,239],[184,189,233],[185,209,219],
    [186,221,227],[187,206,213],[190,222,226],[192,194,196],[193,212,231],
    [195,207,230],[197,214,217],[198,211,234],[199,232,235],[200,223,228],
    [201,225,229],[202,215,236],[203,216,237],[204,224,238],[208,218,220],
    [241,265,270],[242,251,300],[243,248,299],[244,249,293],[245,269,279],
    [246,281,287],[247,266,273],[250,282,286],[252,254,256],[253,272,291],
    [255,267,290],[257,274,277],[258,271,294],[259,292,295],[260,283,288],
    [261,285,289],[262,275,296],[263,276,297],[264,284,298],[268,278,280],
    [301,325,330],[302,311,360],[303,308,359],[304,309,353],[305,329,339],
    [306,341,347],[307,326,333],[310,342,346],[312,314,316],[313,332,351],
    [315,327,350],[317,334,337],[318,331,354],[319,352,355],[320,343,348],
    [321,345,349],[322,335,356],[323,336,357],[324,344,358],[328,338,340]
];
```

10.29.3 Growth

Layer	Number of States	Ratio
0	1	-
1	54	54
2	2,823	52,28
3	144,978	51,36
4	7,351,128	50,71
5	367,647,256	50,01

Table 48: Growth Regular Astrominx + big chop: numbers of states at distance n from the solved state

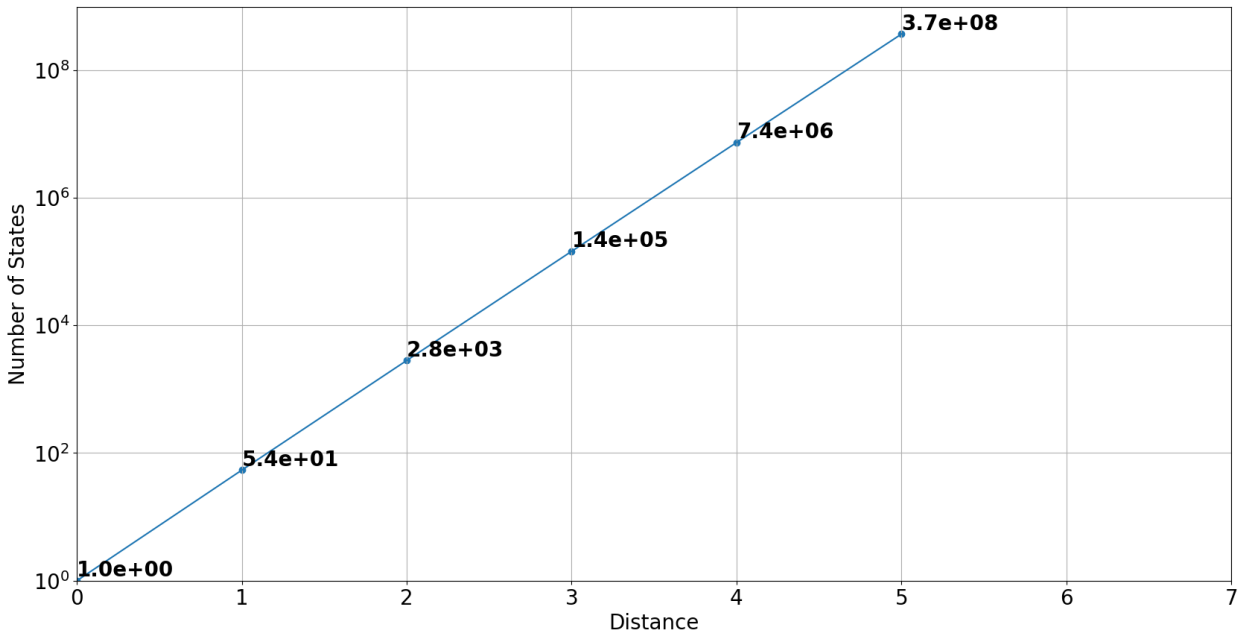

Figure 100: Growth Regular Astrominx + big chop: numbers of states at distance n from the solved state

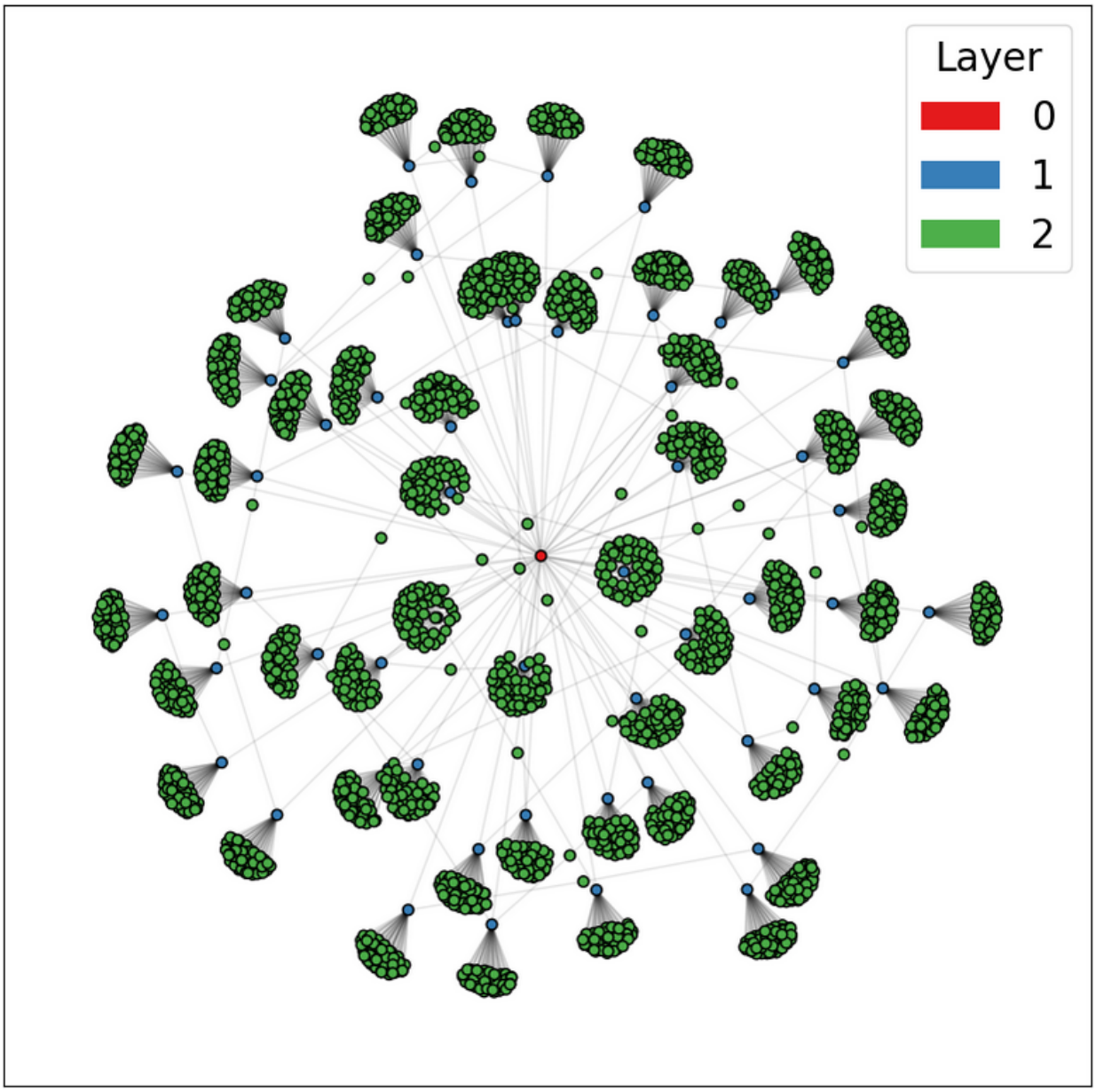

Figure 101: Regular Astrominx + big chop spring layout

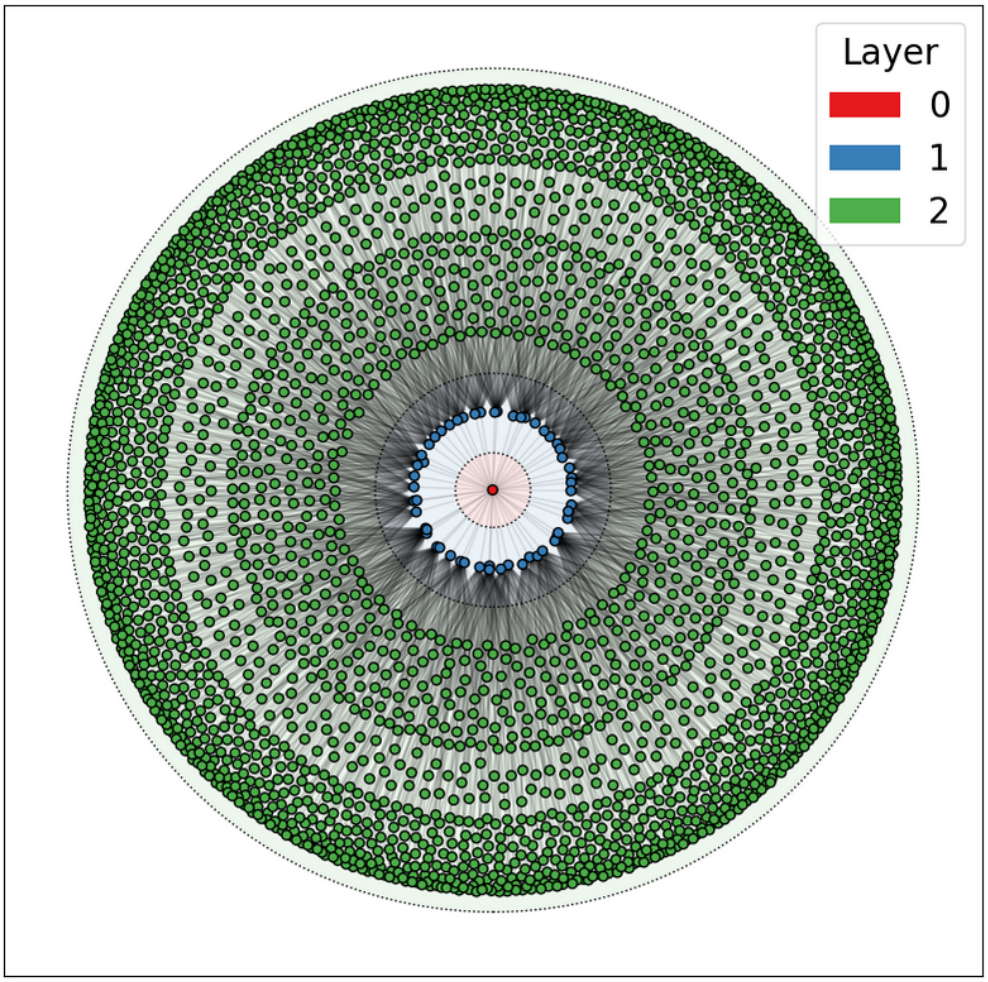

Figure 102: Model Regular Astrominx + big chop, first 2 layers, layered Kamada-Kawai layout

10.30 Regular Astrominx

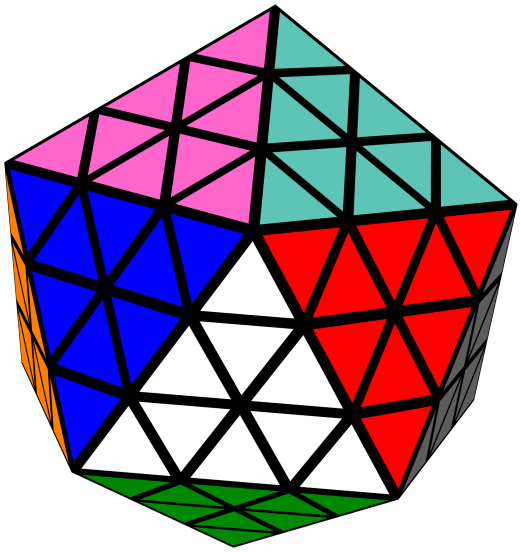

Figure 103: Regular Astrominx

References:
Picture from Tomas Rokicki's Twizzle site

10.31 Icosamate

10.32 Radio chop

The Radio Chop puzzle is a complex vertex-turning and slice-based twisty puzzle. Building on mechanisms similar to puzzles like the Dino Cube, it introduces deeper and more intricate slice moves, resulting in a highly non-trivial permutation structure. These extended moves greatly increase the size and complexity of the generated group, offering rich ground for algebraic exploration, especially in studying growth functions and Cayley graph structures.
More Details

10.32.1 Picture

Interactive visualization can be found at: Tomas Rokicki's Twizzle site:

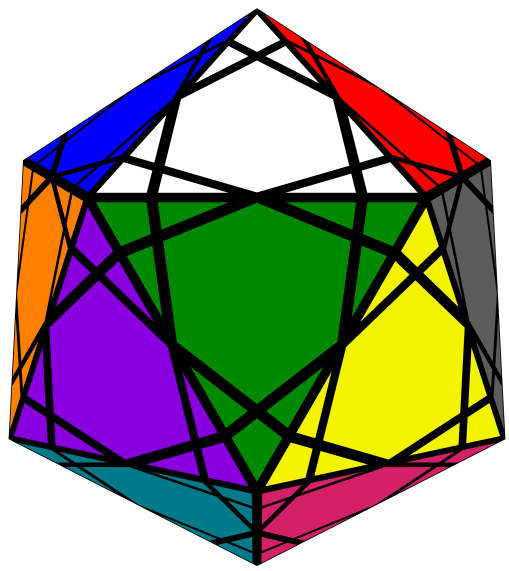

Figure 104: Radio Chop (from Twizzle site)

10.32.2 Generators

M_R := (1,3,23) (2,4,24) (5,28,57) (6,27,58) (13,40,112) (14,39,111) (17,64,118) (18,63,117) (19,59,104) (20,60,103) (33,108,76) (34,107,75) (37,113,83) (38,114,84) (49,115,79) (50,116,80) (51,65,119) (52,66,120) (67,100,95) (68,99,96) (121,128,159) (122,129,160) (123,130,156) (124,126,157) (125,127,158) (131,144,171) (132,145,172) (133,141,173) (134,142,174) (135,143,175) (186,189,195) (188,197,199) (192,198,200);

M_K := (7,46,32) (8,45,31) (9,94,36) (10,93,35) (11,44,25) (12,43,26) (15,77,41) (16,78,42) (21,72,30) (22,71,29) (47,101,81) (48,102,82) (53,91,97) (54,92,98) (55,62,70) (56,61,69) (73,88,106) (74,87,105) (85,89,109) (86,90,110) (136,167,151) (137,168,152) (138,169,153) (139,170,154) (140,166,155) (146,176,161) (147,177,162) (148,178,163) (149,179,164) (150,180,165) (182,185,183) (184,190,187) (191,194,196); M_L := (1,6,43) (2,5,44) (9,49,88) (10,50,87) (19,39,107) (20,40,108) (23,46,27) (24,45,28) (29,95,63) (30,96,64) (51,114,77) (52,113,78) (55,115,94) (56,116,93) (61,74,80) (62,73,79) (81,118,99) (82,117,100) (83,120,89) (84,119,90) (121,134,179) (122,135,180) (123,131,176) (124,132,177) (125,133,178) (141,158,169) (142,159,170) (143,160,166) (144,156,167) (145,157,168) (181,182,183) (184,192,197) (196,199,200);

M_S := (3,7,26) (4,8,25) (11,31,57) (12,32,58) (13,76,53) (14,75,54) (15,110,38) (16,109,37) (17,72,47) (18,71,48) (21,68,101) (22,67,102) (33,60,91) (34,59,92) (35,69,105) (36,70,106) (41,65,86) (42,66,85) (97,103,112)

(98,104,111) (126,137,162) (127,138,163) (128,139,164) (129,140,165) (130,136,161) (146,151,171) (147,152,172) (148,153,173) (149,154,174) (150,155,175) (186,195,193) (187,191,188) (190,198,194);

M_G := (1,7,58) (2,8,57) (3,27,12) (4,28,11) (9,70,79) (10,69,80) (15,65,84) (16,66,83) (17,96,101) (18,95,102) (35,116,74) (36,115,73) (37,52,85) (38,51,86) (49,55,106) (50,56,105) (59,111,75) (60,112,76) (67,117,71) (68,118,72) (121,139,174) (122,140,175) (123,136,171) (124,137,172) (125,138,173) (126,145,147) (127,141,148) (128,142,149) (129,143,150) (130,144,146) (181,183,185) (187,198,199) (188,192,194);

M_N := (5,25,45) (6,26,46) (13,91,20) (14,92,19) (21,81,64) (22,82,63) (23,32,43) (24,31,44) (29,100,48) (30,99,47) (33,97,40) (34,98,39) (41,90,114) (42,89,113) (53,108,103) (54,107,104) (61,87,93) (62,88,94) (77,119,110) (78,120,109) (131,161,167) (132,162,168) (133,163,169) (134,164,170) (135,165,166) (151,176,156) (152,177,157) (153,178,158) (154,179,159) (155,180,160) (184,200,191) (186,193,189) (190,196,197);

M_U := (1,10,72) (2,9,71) (3,30,16) (4,29,15) (7,59,96) (8,60,95) (19,56,68) (20,55,67) (23,78,35) (24,77,36) (27,111,52) (28,112,51) (37,64,92) (38,63,91) (43,75,50) (44,76,49) (73,117,84) (74,118,83) (85,116,107) (86,115,108) (121,143,137) (122,144,138) (123,145,139) (124,141,140) (125,142,136) (156,180,172) (157,176,173) (158,177,174) (159,178,175) (160,179,171) (181,184,187) (183,195,192) (189,194,198);

M_D := (5,13,48) (6,14,47) (11,90,79) (12,89,80) (17,54,87) (18,53,88) (21,42,93) (22,41,94) (25,33,110) (26,34,109) (31,62,100) (32,61,99) (39,66,101) (40,65,102) (45,103,70) (46,104,69) (57,82,119) (58,81,120) (97,114,106) (98,113,105) (126,146,166) (127,147,167) (128,148,168) (129,149,169) (130,150,170) (131,153,164) (132,154,165) (133,155,161) (134,151,162) (135,152,163) (182,186,191) (185,196,200) (188,193,197);

M_H := (3,32,6) (4,31,5) (13,33,103) (14,34,104) (17,21,99) (18,22,100) (23,58,26) (24,57,25) (35,61,50) (36,62,49) (37,42,120) (38,41,119) (47,64,68) (48,63,67) (65,110,114) (66,109,113) (69,87,116) (70,88,115) (79,106,94) (80,105,93) (126,152,132) (127,153,133) (128,154,134) (129,155,135) (130,151,131) (156,171,161) (157,172,162) (158,173,163) (159,174,164) (160,175,165) (181,185,182) (188,190,200) (191,197,198);

M_Q := (1,46,12) (2,45,11) (7,27,43) (8,28,44) (9,55,73) (10,56,74) (15,51,90) (16,52,89) (19,98,75) (20,97,76) (29,102,117) (30,101,118) (39,54,59) (40,53,60) (71,95,82) (72,96,81) (77,86,84) (78,85,83) (91,112,108) (92,111,107) (121,170,149) (122,166,150) (123,167,146) (124,168,147) (125,169,148) (136,144,176) (137,145,177) (138,141,178) (139,142,179) (140,143,180) (184,194,199) (187,192,196) (189,193,195);

M_C := (1,14,87) (2,13,88) (3,34,64) (4,33,63) (5,49,18) (6,50,17) (19,66,93) (20,65,94) (21,39,116) (22,40,115) (23,104,113) (24,103,114) (25,95,38) (26,96,37) (29,60,110) (30,59,109) (45,51,106) (46,52,105) (79,100,119) (80,99,120) (121,151,166) (122,152,167) (123,153,168) (124,154,169) (125,155,170) (126,134,160) (127,135,156) (128,131,157) (129,132,158) (130,133,159) (181,186,197) (182,192,188) (189,198,190);

M_M := (7,92,16) (8,91,15) (9,48,76) (10,47,75) (11,73,82) (12,74,81) (27,98,35) (28,97,36) (31,117,90) (32,118,89) (41,67,108) (42,68,107) (43,54,72) (44,53,71) (55,102,86) (56,101,85) (57,84,62) (58,83,61) (69,111,78) (70,112,77) (136,178,150) (137,179,146) (138,180,147) (139,176,148) (140,177,149) (141,165,174) (142,161,175) (143,162,171) (144,163,172) (145,164,173) (183,193,187) (184,191,195) (185,199,196);

M_I := (1,16,93) (2,15,94) (3,35,21) (4,36,22) (11,33,115) (12,34,116) (13,65,57) (14,66,58) (17,37,105) (18,38,106) (23,72,42) (24,71,41) (31,103,67) (32,104,68) (53,114,117) (54,113,118) (59,69,99) (60,70,100) (75,109,80) (76,110,79) (121,148,135) (122,149,131) (123,150,132) (124,146,133) (125,147,134) (126,173,155) (127,174,151) (128,175,152) (129,171,153) (130,172,154) (181,187,190) (185,186,198) (191,200,195);

M_P := (5,102,51) (6,101,52) (7,19,61) (8,20,62) (9,29,77) (10,30,78) (25,86,49) (26,85,50) (27,39,81) (28,40,82) (43,107,89) (44,108,90) (45,55,95) (46,56,96) (47,111,120) (48,112,119) (63,97,73) (64,98,74) (83,87,92) (84,88,91) (136,158,162) (137,159,163) (138,160,164) (139,156,165) (140,157,161) (141,166,176) (142,167,177) (143,168,178) (144,169,179) (145,170,180) (182,194,192) (183,189,196) (193,199,197);

M_O := (5,53,9) (6,54,10) (7,64,109) (8,63,110) (15,95,33) (16,96,34) (25,91,29) (26,92,30) (27,113,69) (28,114,70) (39,42,56) (40,41,55) (43,87,47) (44,88,48) (45,97,77) (46,98,78) (61,81,89) (62,82,90) (93,101,107) (94,102,108) (131,148,145) (132,149,141) (133,150,142) (134,146,143) (135,147,144) (161,178,170) (162,179,166) (163,180,167) (164,176,168) (165,177,169) (182,193,184) (183,197,191) (189,190,194);

M_E := (1,75,17) (2,76,18) (3,59,37) (4,60,38) (11,100,84) (12,99,83) (13,49,71) (14,50,72) (19,85,21) (20,86,22) (23,111,105) (24,112,106) (31,119,73) (32,120,74) (35,104,52) (36,103,51) (57,79,117) (58,80,118) (65,115,67) (66,116,68) (121,171,126) (122,172,127) (123,173,128) (124,174,129) (125,175,130) (136,155,159) (137,151,160) (138,152,156) (139,153,157) (140,154,158) (181,195,188) (185,200,199) (186,192,187);

M_F := (1,19,83) (2,20,84) (3,39,74) (4,40,73) (7,66,87) (8,65,88) (9,60,119) (10,59,120) (23,107,118)

(24,108,117) (27,116,64) (28,115,63) (29,112,79) (30,111,80) (43,68,113) (44,67,114) (49,95,51) (50,96,52) (71,103,90) (72,104,89) (121,156,141) (122,157,142) (123,158,143) (124,159,144) (125,160,145) (126,170,138) (127,166,139) (128,167,140) (129,168,136) (130,169,137) (181,189,199) (183,198,197) (184,195,200);

M_B := (5,38,55) (6,37,56) (11,97,22) (12,98,21) (13,15,62) (14,16,61) (17,85,46) (18,86,45) (25,106,102) (26,105,101) (31,53,41) (32,54,42) (33,36,82) (34,35,81) (47,109,69) (48,110,70) (57,91,94) (58,92,93) (75,78,99) (76,77,100) (131,175,177) (132,171,178) (133,172,179) (134,173,180) (135,174,176) (146,163,154) (147,164,155) (148,165,151) (149,161,152) (150,162,153) (182,188,194) (185,193,190) (186,187,196);

M_A := (1,21,78) (2,22,77) (3,42,10) (4,41,9) (5,40,119) (6,39,120) (17,98,83) (18,97,84) (19,99,46) (20,100,45) (23,93,30) (24,94,29) (25,108,79) (26,107,80) (33,90,49) (34,89,50) (61,96,104) (62,95,103) (63,114,88) (64,113,87) (126,161,141) (127,162,142) (128,163,143) (129,164,144) (130,165,145) (131,169,160) (132,170,156) (133,166,157) (134,167,158) (135,168,159) (181,190,184) (182,189,200) (186,196,192);

M_J := (7,69,68) (8,70,67) (11,76,86) (12,75,85) (13,51,82) (14,52,81) (15,36,71) (16,35,72) (27,47,66) (28,48,65) (31,60,55) (32,59,56) (37,74,54) (38,73,53) (43,109,116) (44,110,115) (57,112,102) (58,111,101) (91,106,117) (92,105,118) (121,178,152) (122,179,153) (123,180,154) (124,176,155) (125,177,151) (136,147,174) (137,148,175) (138,149,171) (139,150,172) (140,146,173) (183,191,198) (185,195,194) (188,199,193);

10.32.3 Growth function

Layer	*CayleyPy*	*TomasRokicki*
0	1	1
1	40	40
2	1480	1480
3	51960	51960
4	1728399	1728399
5	53193780	53193780
6		1462318260
7		34592510160
8		
9		

Table 49: Curvy Copter growth.

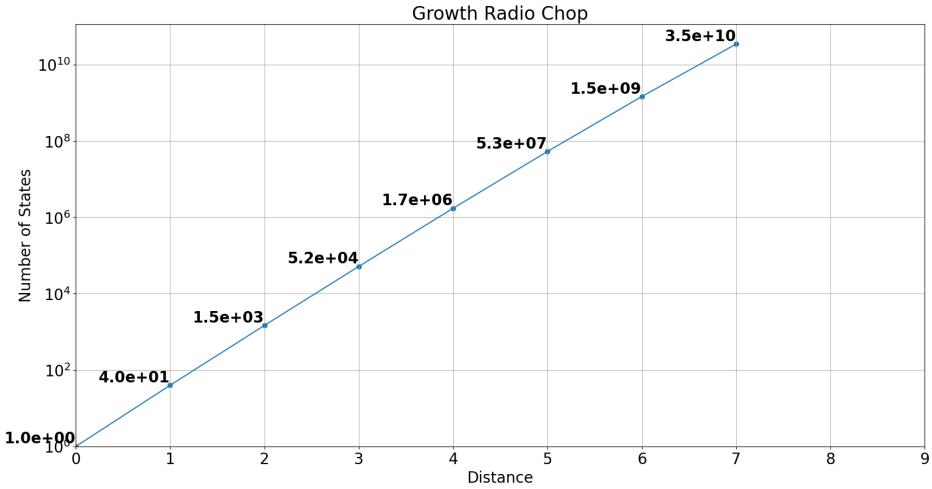

Figure 105: Growth

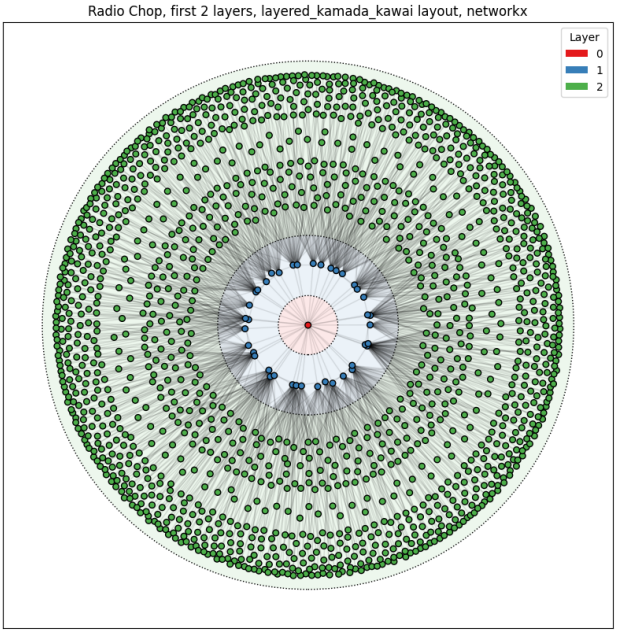

Figure 106: Graph for the first 2 layers (Layered Kamada-Kawai layout)

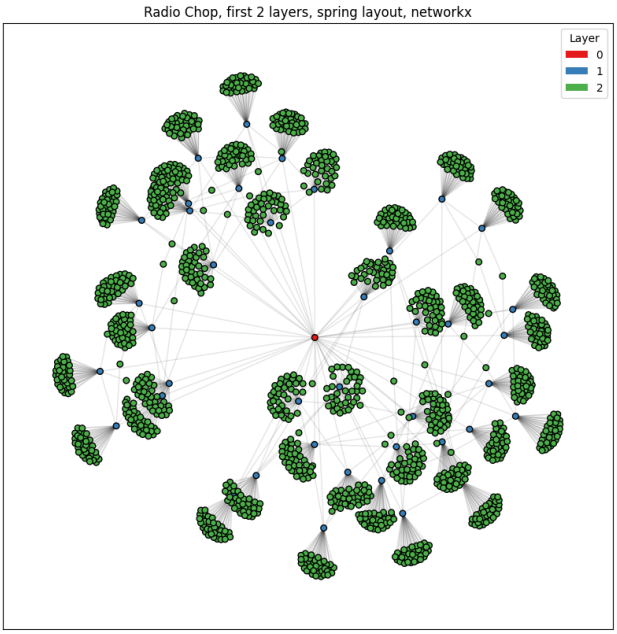

Figure 107: Graph for the first 2 layers (Spring layout)

10.33 Trajber's octahedron

10.34 Octastar

10.35 Christopher's jewel

Similar to the previously considered puzzles, Christopher's Jewel depicted below exhibits a highly non-trivial structure in its permutation group and the corresponding Cayley graph.

More Details

10.35.1 Picture

Interactive visualization can be found at: Tomas Rokicki's Twizzle site:

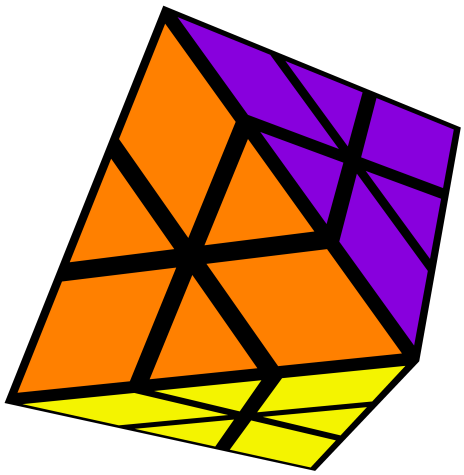

Figure 108: Christopher's jewel (from Twizzle site)

10.35.2 Generators

M_R := (1,3,23) (2,4,24) (5,28,57) (6,27,58) (13,40,112) (14,39,111) (17,64,118) (18,63,117) (19,59,104) (20,60,103) (33,108,76) (34,107,75) (37,113,83) (38,114,84) (49,115,79) (50,116,80) (51,65,119) (52,66,120) (67,100,95) (68,99,96) (121,128,159) (122,129,160) (123,130,156) (124,126,157) (125,127,158) (131,144,171) (132,145,172) (133,141,173) (134,142,174) (135,143,175) (186,189,195) (188,197,199) (192,198,200);

M_K := (7,46,32) (8,45,31) (9,94,36) (10,93,35) (11,44,25) (12,43,26) (15,77,41) (16,78,42) (21,72,30) (22,71,29) (47,101,81) (48,102,82) (53,91,97) (54,92,98) (55,62,70) (56,61,69) (73,88,106) (74,87,105) (85,89,109) (86,90,110) (136,167,151) (137,168,152) (138,169,153) (139,170,154) (140,166,155) (146,176,161) (147,177,162) (148,178,163) (149,179,164) (150,180,165) (182,185,183) (184,190,187) (191,194,196); M_L := (1,6,43) (2,5,44) (9,49,88) (10,50,87) (19,39,107) (20,40,108) (23,46,27) (24,45,28) (29,95,63) (30,96,64) (51,114,77) (52,113,78) (55,115,94) (56,116,93) (61,74,80) (62,73,79) (81,118,99) (82,117,100) (83,120,89) (84,119,90) (121,134,179) (122,135,180) (123,131,176) (124,132,177) (125,133,178) (141,158,169) (142,159,170) (143,160,166) (144,156,167) (145,157,168) (181,182,183) (184,192,197) (196,199,200);

M_S := (3,7,26) (4,8,25) (11,31,57) (12,32,58) (13,76,53) (14,75,54) (15,110,38) (16,109,37) (17,72,47) (18,71,48) (21,68,101) (22,67,102) (33,60,91) (34,59,92) (35,69,105) (36,70,106) (41,65,86) (42,66,85) (97,103,112) (98,104,111) (126,137,162) (127,138,163) (128,139,164) (129,140,165) (130,136,161) (146,151,171) (147,152,172) (148,153,173) (149,154,174) (150,155,175) (186,195,193) (187,191,188) (190,198,194);

M_G := (1,7,58) (2,8,57) (3,27,12) (4,28,11) (9,70,79) (10,69,80) (15,65,84) (16,66,83) (17,96,101) (18,95,102) (35,116,74) (36,115,73) (37,52,85) (38,51,86) (49,55,106) (50,56,105) (59,111,75) (60,112,76) (67,117,71) (68,118,72) (121,139,174) (122,140,175) (123,136,171) (124,137,172) (125,138,173) (126,145,147) (127,141,148) (128,142,149) (129,143,150) (130,144,146) (181,183,185) (187,198,199) (188,192,194);

M_N := (5,25,45) (6,26,46) (13,91,20) (14,92,19) (21,81,64) (22,82,63) (23,32,43) (24,31,44) (29,100,48) (30,99,47) (33,97,40) (34,98,39) (41,90,114) (42,89,113) (53,108,103) (54,107,104) (61,87,93) (62,88,94) (77,119,110) (78,120,109) (131,161,167) (132,162,168) (133,163,169) (134,164,170) (135,165,166) (151,176,156) (152,177,157) (153,178,158) (154,179,159) (155,180,160) (184,200,191) (186,193,189) (190,196,197);

M_U := (1,10,72) (2,9,71) (3,30,16) (4,29,15) (7,59,96) (8,60,95) (19,56,68) (20,55,67) (23,78,35) (24,77,36) (27,111,52) (28,112,51) (37,64,92) (38,63,91) (43,75,50) (44,76,49) (73,117,84) (74,118,83) (85,116,107) (86,115,108) (121,143,137) (122,144,138) (123,145,139) (124,141,140) (125,142,136) (156,180,172) (157,176,173) (158,177,174) (159,178,175) (160,179,171) (181,184,187) (183,195,192) (189,194,198);

M_D := (5,13,48) (6,14,47) (11,90,79) (12,89,80) (17,54,87) (18,53,88) (21,42,93) (22,41,94) (25,33,110) (26,34,109) (31,62,100) (32,61,99) (39,66,101) (40,65,102) (45,103,70) (46,104,69) (57,82,119) (58,81,120) (97,114,106) (98,113,105) (126,146,166) (127,147,167) (128,148,168) (129,149,169) (130,150,170) (131,153,164) (132,154,165) (133,155,161) (134,151,162) (135,152,163) (182,186,191) (185,196,200) (188,193,197);

M_H := (3,32,6) (4,31,5) (13,33,103) (14,34,104) (17,21,99) (18,22,100) (23,58,26) (24,57,25) (35,61,50) (36,62,49) (37,42,120) (38,41,119) (47,64,68) (48,63,67) (65,110,114) (66,109,113) (69,87,116) (70,88,115) (79,106,94) (80,105,93) (126,152,132) (127,153,133) (128,154,134) (129,155,135) (130,151,131) (156,171,161) (157,172,162) (158,173,163) (159,174,164) (160,175,165) (181,185,182) (188,190,200) (191,197,198);

M_Q := (1,46,12) (2,45,11) (7,27,43) (8,28,44) (9,55,73) (10,56,74) (15,51,90) (16,52,89) (19,98,75) (20,97,76) (29,102,117) (30,101,118) (39,54,59) (40,53,60) (71,95,82) (72,96,81) (77,86,84) (78,85,83) (91,112,108) (92,111,107) (121,170,149) (122,166,150) (123,167,146) (124,168,147) (125,169,148) (136,144,176) (137,145,177) (138,141,178) (139,142,179) (140,143,180) (184,194,199) (187,192,196) (189,193,195);

M_C := (1,14,87) (2,13,88) (3,34,64) (4,33,63) (5,49,18) (6,50,17) (19,66,93) (20,65,94) (21,39,116) (22,40,115) (23,104,113) (24,103,114) (25,95,38) (26,96,37) (29,60,110) (30,59,109) (45,51,106) (46,52,105) (79,100,119) (80,99,120) (121,151,166) (122,152,167) (123,153,168) (124,154,169) (125,155,170) (126,134,160) (127,135,156) (128,131,157) (129,132,158) (130,133,159) (181,186,197) (182,192,188) (189,198,190);

M_M := (7,92,16) (8,91,15) (9,48,76) (10,47,75) (11,73,82) (12,74,81) (27,98,35) (28,97,36) (31,117,90) (32,118,89) (41,67,108) (42,68,107) (43,54,72) (44,53,71) (55,102,86) (56,101,85) (57,84,62) (58,83,61) (69,111,78) (70,112,77) (136,178,150) (137,179,146) (138,180,147) (139,176,148) (140,177,149) (141,165,174) (142,161,175) (143,162,171) (144,163,172) (145,164,173) (183,193,187) (184,191,195) (185,199,196);

M_I := (1,16,93) (2,15,94) (3,35,21) (4,36,22) (11,33,115) (12,34,116) (13,65,57) (14,66,58) (17,37,105) (18,38,106) (23,72,42) (24,71,41) (31,103,67) (32,104,68) (53,114,117) (54,113,118) (59,69,99) (60,70,100) (75,109,80) (76,110,79) (121,148,135) (122,149,131) (123,150,132) (124,146,133) (125,147,134) (126,173,155) (127,174,151) (128,175,152) (129,171,153) (130,172,154) (181,187,190) (185,186,198) (191,200,195);

M_P := (5,102,51) (6,101,52) (7,19,61) (8,20,62) (9,29,77) (10,30,78) (25,86,49) (26,85,50) (27,39,81) (28,40,82) (43,107,89) (44,108,90) (45,55,95) (46,56,96) (47,111,120) (48,112,119) (63,97,73) (64,98,74) (83,87,92) (84,88,91) (136,158,162) (137,159,163) (138,160,164) (139,156,165) (140,157,161) (141,166,176) (142,167,177) (143,168,178) (144,169,179) (145,170,180) (182,194,192) (183,189,196) (193,199,197);

M_O := (5,53,9) (6,54,10) (7,64,109) (8,63,110) (15,95,33) (16,96,34) (25,91,29) (26,92,30) (27,113,69) (28,114,70) (39,42,56) (40,41,55) (43,87,47) (44,88,48) (45,97,77) (46,98,78) (61,81,89) (62,82,90) (93,101,107) (94,102,108) (131,148,145) (132,149,141) (133,150,142) (134,146,143) (135,147,144) (161,178,170) (162,179,166) (163,180,167) (164,176,168) (165,177,169) (182,193,184) (183,197,191) (189,190,194);

M_E := (1,75,17) (2,76,18) (3,59,37) (4,60,38) (11,100,84) (12,99,83) (13,49,71) (14,50,72) (19,85,21) (20,86,22) (23,111,105) (24,112,106) (31,119,73) (32,120,74) (35,104,52) (36,103,51) (57,79,117) (58,80,118) (65,115,67) (66,116,68) (121,171,126) (122,172,127) (123,173,128) (124,174,129) (125,175,130) (136,155,159) (137,151,160) (138,152,156) (139,153,157) (140,154,158) (181,195,188) (185,200,199) (186,192,187);

M_F := (1,19,83) (2,20,84) (3,39,74) (4,40,73) (7,66,87) (8,65,88) (9,60,119) (10,59,120) (23,107,118) (24,108,117) (27,116,64) (28,115,63) (29,112,79) (30,111,80) (43,68,113) (44,67,114) (49,95,51) (50,96,52) (71,103,90) (72,104,89) (121,156,141) (122,157,142) (123,158,143) (124,159,144) (125,160,145) (126,170,138)

(127,166,139) (128,167,140) (129,168,136) (130,169,137) (181,189,199) (183,198,197) (184,195,200);

M_B := (5,38,55) (6,37,56) (11,97,22) (12,98,21) (13,15,62) (14,16,61) (17,85,46) (18,86,45) (25,106,102) (26,105,101) (31,53,41) (32,54,42) (33,36,82) (34,35,81) (47,109,69) (48,110,70) (57,91,94) (58,92,93) (75,78,99) (76,77,100) (131,175,177) (132,171,178) (133,172,179) (134,173,180) (135,174,176) (146,163,154) (147,164,155) (148,165,151) (149,161,152) (150,162,153) (182,188,194) (185,193,190) (186,187,196);

M_A := (1,21,78) (2,22,77) (3,42,10) (4,41,9) (5,40,119) (6,39,120) (17,98,83) (18,97,84) (19,99,46) (20,100,45) (23,93,30) (24,94,29) (25,108,79) (26,107,80) (33,90,49) (34,89,50) (61,96,104) (62,95,103) (63,114,88) (64,113,87) (126,161,141) (127,162,142) (128,163,143) (129,164,144) (130,165,145) (131,169,160) (132,170,156) (133,166,157) (134,167,158) (135,168,159) (181,190,184) (182,189,200) (186,196,192);

M_J := (7,69,68) (8,70,67) (11,76,86) (12,75,85) (13,51,82) (14,52,81) (15,36,71) (16,35,72) (27,47,66) (28,48,65) (31,60,55) (32,59,56) (37,74,54) (38,73,53) (43,109,116) (44,110,115) (57,112,102) (58,111,101) (91,106,117) (92,105,118) (121,178,152) (122,179,153) (123,180,154) (124,176,155) (125,177,151) (136,147,174) (137,148,175) (138,149,171) (139,150,172) (140,146,173) (183,191,198) (185,195,194) (188,199,193);

10.35.3 Growth function

Layer	*CayleyPy*	*TomasRokicki*
0	1	1
1	40	40
2	1480	1480
3	51960	51960
4	1728399	1728399
5	53193780	53193780
6		1462318260
7		34592510160
8		
9		

Table 50: Curvy Copter growth.

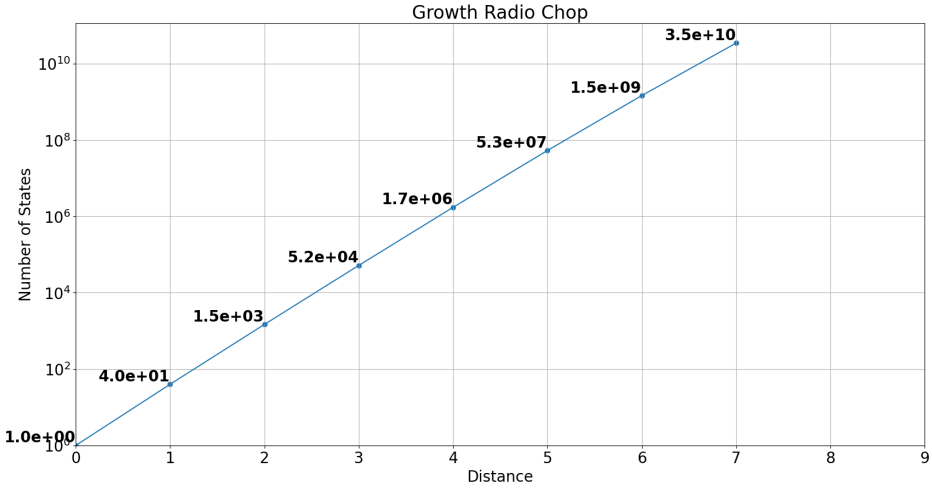

Figure 109: Growth

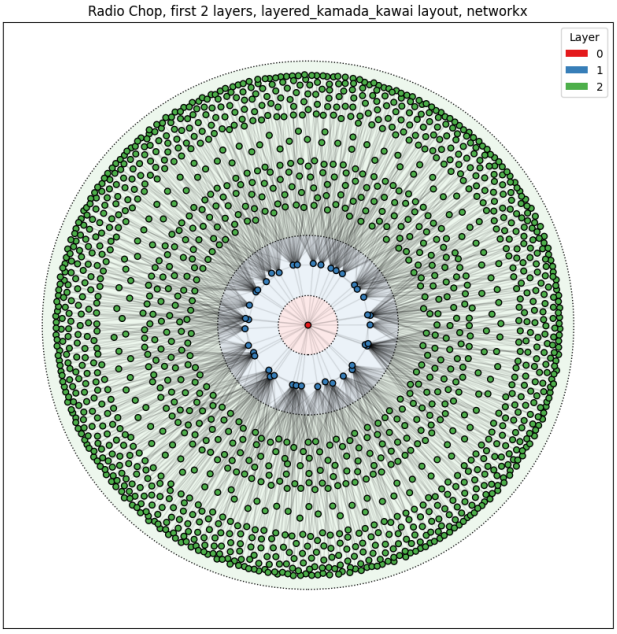

Figure 110: Graph for the first 2 layers (Layered Kamada-Kawai layout)

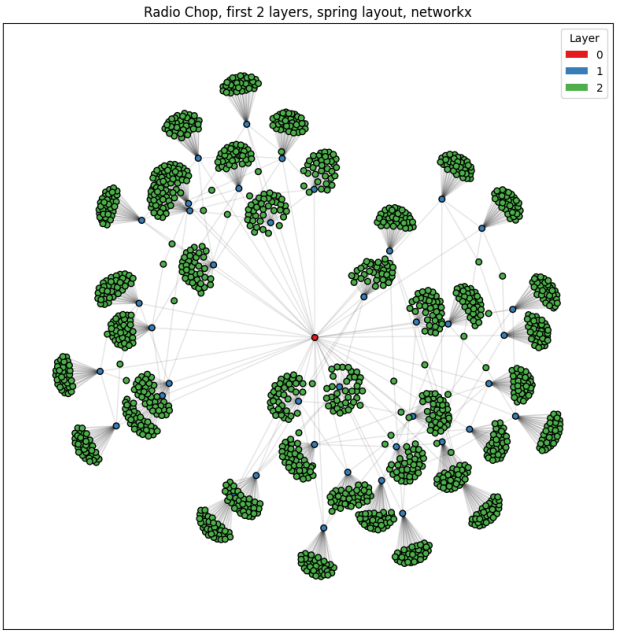

Figure 111: Graph for the first 2 layers (Spring layout)

10.36 Master FTO

10.36.1 Common information

10.36.2 Images

Interactive visualization can be found at: Tomas Rokicki's Twizzle site:

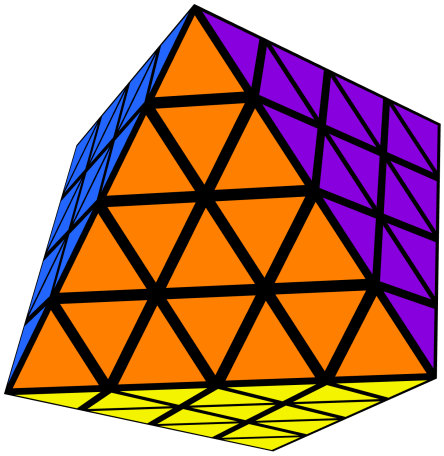

Figure 112: master FTO (from Twizzle site)

10.36.3 Generators

M_F:=(1,6,16)(2,7,13)(3,8,14)(4,5,15)(25,27,35)(26,28,36)(49,51,59)(50,52,60)(73,74,78)(85,86,90)(101,102,106)(107,109,112)(113,114,118);
M_2F:=(31,43,33)(32,44,34)(55,67,57)(56,68,58)(76,82,77)(115,120,123)(119,121,124)(125,126,128);
M_2B:=(29,45,39)(30,46,40)(61,71,65)(62,72,66)(87,95,92)(88,89,94)(98,100,99)(103,111,108);
M_B:=(9,23,20)(10,24,17)(11,21,18)(12,22,19)(37,47,41)(38,48,42)(53,69,63)(54,70,64)(75,83,80)(79,84,81)(91,96,93)(104,105,110)(116,117,122);
M_D:=(5,17,13)(6,18,14)(7,19,15)(8,20,16)(33,39,35)(34,40,36)(51,61,55)(52,62,56)(74,79,76)(77,80,78)(89,92,90)(101,103,109)(113,115,121);
M_2D:=(27,37,31)(28,38,32)(57,63,59)(58,64,60)(85,87,93)(86,91,88)(97,98,99)(102,107,104);
M_2U:=(25,41,29)(26,42,30)(49,65,53)(50,66,54)(73,81,75)(114,116,119)(117,118,120)(126,127,128);
M_U:=(1,21,9)(2,22,10)(3,23,11)(4,24,12)(43,45,47)(44,46,48)(67,69,71)(68,70,72)(82,83,84)(94,95,96)(105,106,108)(110,111,112)(122,123,124);
M_L:=(1,14,23)(2,15,24)(3,16,21)(4,13,22)(25,31,45)(26,32,46)(59,65,67)(60,66,68)(73,76,83)(78,81,82)(85,88,95)(102,108,112)(114,120,124);
M_2L:=(35,41,43)(36,42,44)(49,55,69)(50,56,70)(86,92,96)(90,93,94)(97,99,100)(106,109,110);
M_2BR:=(27,47,39)(28,48,40)(51,71,63)(52,72,64)(74,84,80)(113,123,116)(118,122,121)(125,128,127);
M_BR:=(5,10,19)(6,11,20)(7,12,17)(8,9,18)(29,37,33)(30,38,34)(53,61,57)(54,62,58)(75,79,77)(87,91,89)(101,111,104)(103,107,105)(115,119,117);
M_BL:=(13,19,24)(14,20,21)(15,17,22)(16,18,23)(31,39,41)(32,40,42)(55,63,65)(56,64,66)(76,80,81)(88,92,93)(102,103,110)(104,108,109)(116,120,121);

M_2BL:=(35,37,45)(36,38,46)(59,61,69)(60,62,70)(78,79,83)(113,117,124)(114,115,122)(125,127,126);
M_2R:=(25,47,33)(26,48,34)(51,67,53)(52,68,54)(85,96,89)(90,95,91)(97,100,98)(101,112,105);
M_R:=(1,11,8)(2,12,5)(3,9,6)(4,10,7)(27,43,29)(28,44,30)(49,71,57)(50,72,58)(73,84,77)(74,82,75)
(86,94,87)(106,111,107)(118,123,119);

10.36.4 Growth function

Layer	*CayleyPy*	*TomasRokicki*
0	1	?
1	32	?
2	864	?
3	23168	?
4	620608	?
5	16606288	?
6	443998400	?
7		
8		
9		

Table 51: master FTO growth.

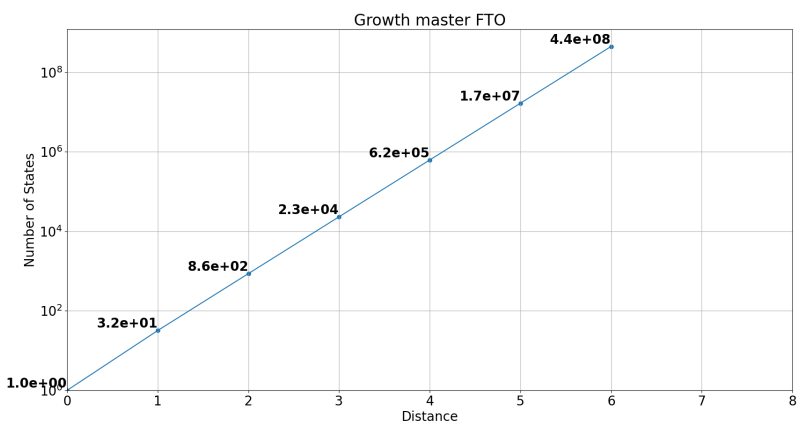

Figure 113: Growth

10.36.5 Graph images

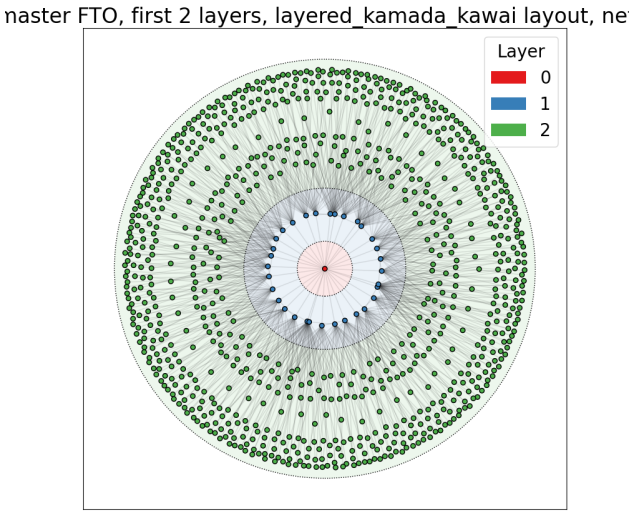

Figure 114: Graph for the first 2 layers (Layered Kamada-Kawai layout)

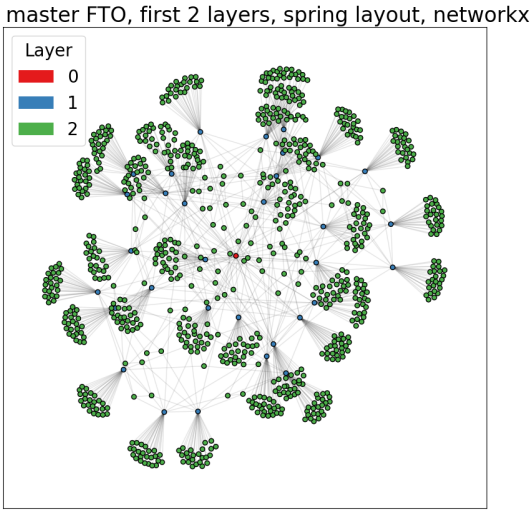

Figure 115: Graph for the first 2 layers (Spring layout)

10.37 FTO

10.38 Skewb diamond

10.39 Big chop

10.39.1 Common information

10.39.2 Images

Interactive visualization can be found at: Tomas Rokicki's Twizzle site:

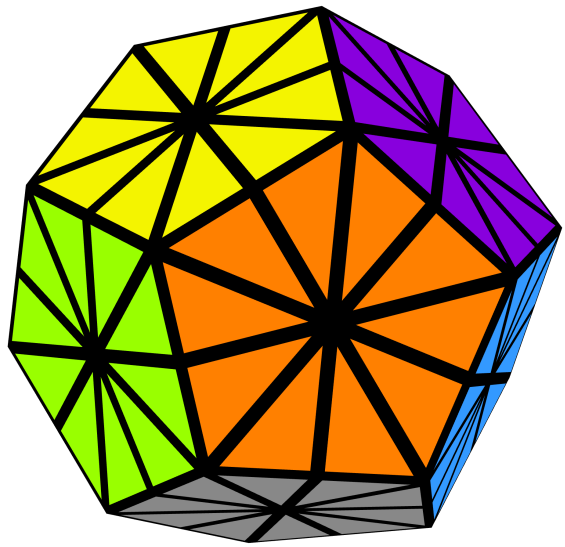

Figure 116: Big chop (from Twizzle site)

10.39.3 Generators

10.39.4 Growth function

Layer	*CayleyPy*	*TomasRokicki*
0	1	1
1		
2		
3		
4		
5		
6		
7		
8		
9		

Table 52: Big chop growth.

10.39.5 Graph images

10.40 Chopasaurus

10.41 Pyraminx crystal

10.42 Yottaminx

10.43 Zetaminx

10.44 Examinx

10.45 Petaminx

10.46 Teraminx

10.47 Gigaminx

10.48 ???

10.49 ???

10.50 ???

11 Matrix groups

11.1 Heisenberg group

11.2 Clifford

12 Machine learning results

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