On the admissibility of associated fibrations

Compositional Systems and Methods group seminar at TalTech

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October 18, 2024

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Formal category theory

The pillars of formal category theory

1) Representability
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Interplay between central notions of formal category theory

- Internalization -*internal* categories

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- Enhancement - *enhanced* categories
- Shape - *double, triple, multi*-categories

The unknown role of associated fibrations in foundations

The associated (co)fibrations (co)lax monads are fundamental to:

- generalized Bénabou's fibrations

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- generalized multi-categories

The formal theory of adjunctions

In development of the theory I use the minimum of

- (lax, local) adjunctions

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- (lax, local) adjunctions
- (lax) Kan extensions
- (lax) (co)limits
- (relative) pseudo-monads

[redacted]

A colax monad on a 2-category $\mathscr{K}$ consists of the following data:

1) a colax functor $T\colon \mathscr{K} \to \mathscr{K}$

such that the following axioms are satisfied:

[redacted]

A colax monad on a 2-category $\mathscr{K}$ consists of the following data:

1) a colax functor $T\colon \mathscr{K} \to \mathscr{K}$
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3) a colax transformation $\mu \colon T^2 \to T$
4) families $\lambda_X \colon \mu_X T(\eta_X) \Rightarrow 1_{TX}$, $\rho_X \colon 1_{TX} \Rightarrow \mu_X \circ \eta_{TX}$ and $\alpha_X \colon T\mu_X \circ \mu_X \Rightarrow \mu_{TX} \circ \mu_X$ of 2-cells in $\mathscr{K}$

such that the following axioms are satisfied:

1) $(\alpha_X \circ \eta_{T^2X)})(\mu_X \circ \eta_{\mu_X})(\rho_X \circ \mu_X) = \mu_X \circ \rho_{TX} \colon \mu_X \Rightarrow \mu_X \circ \mu_{TX} \circ \eta_{T^2X}$

$$\begin{tikzcd}
T^2X \arrow[rr, "\eta_{T^2X}"] \arrow[drrr, equal] \arrow[dd, "\mu_X"'] \arrow[rr, phantom, shift right=3, near end, "\rho_{TX}\Nearrow"'] \arrow[ddrr, phantom, "\eta_{\mu_X}\Nearrow"] & & T^3X \arrow[dr, "\mu_{TX}"] \arrow[dd, dashed, "T(\mu_X)"'] \arrow[ddr, phantom, "\alpha_X\Nearrow"] & \\
& & & T^2X \arrow[dd, "\mu_X"] \\
TX \arrow[rr, dashed, "\eta_{TX}"] \arrow[drrr, equal] \arrow[rr, phantom, shift right=3, near end, "\rho_X\Nearrow"'] \arrow[urr, phantom, near end, "\alpha_X\Nearrow"] & & T^2X \arrow[dr, dashed, "\mu_X"] & \\
& & & TX
\end{tikzcd}$$

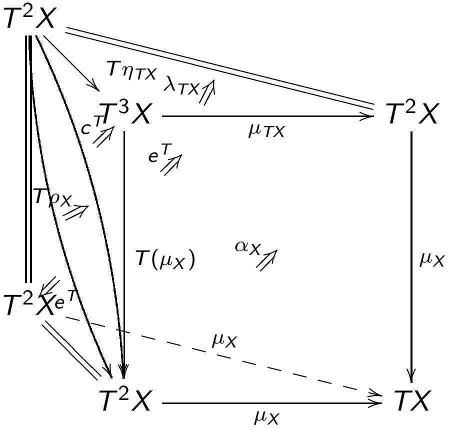
2) $(\mu_f \circ \eta_{TX)}(\mu_{X'} \circ \eta_{Tf})(\rho_{X'} \circ Tf) = Tf \circ \rho_X \colon Tf \Rightarrow Tf \circ \mu_X \circ \eta_{TX'}$,
$\forall f \colon X \to X'$

$$
\begin{tikzcd}
TX \arrow[rr, "\eta_{TX}"] \arrow[drrr, equal] \arrow[dd, "Tf"'] \arrow[rr, phantom, shift right=3, "\rho_X \Uparrow"] & & T^2X \arrow[dr, "\mu_{TX}"] \arrow[dd, dashed, "T^2f"'] & \\
& \arrow[r, phantom, "\eta_{Tf} \Uparrow"] & \arrow[r, phantom, "\mu_f \Uparrow"] & TX \arrow[dd, "Tf"] \\
TX' \arrow[rr, dashed, "\eta_{TX'}"] \arrow[drrr, equal] \arrow[rr, phantom, shift right=3, "\rho_{X'} \Uparrow"] & \arrow[ur, phantom, "\alpha_X \Uparrow"] & T^2X' \arrow[dr, dashed, "\mu_{X'}"] & \\
& & & TX'
\end{tikzcd}
$$

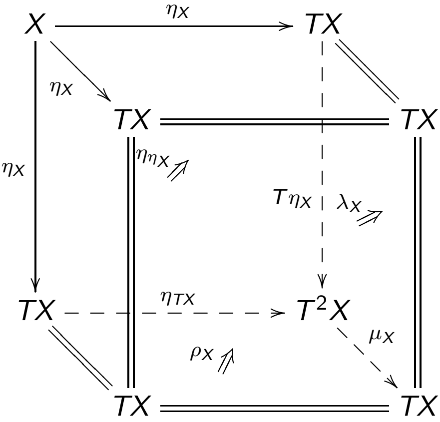
3) $(Tf \circ \lambda_X)(\mu_f \circ T\eta_X)(\mu_{X'} \circ c^T)(\mu_{X'} \circ T\eta_f) = (\lambda_{X'} \circ Tf)(\mu_{X'} \circ c^T) \colon \mu_{X'} \circ T(\eta_{X'} \circ f) \Rightarrow Tf, \forall f \colon X \to X'$

$$
\begin{tikzcd}
TX \arrow[dr, "T\eta_X"] \arrow[drr, equal, "\lambda_X"] \arrow[dd, "Tf"'] \arrow[dddr, bend left=10, "T\eta_f"] \arrow[dddr, bend left=25, "c^T"] & & \\
& T^2X \arrow[r, "\mu_X"'] \arrow[dd, "T^2f"] \arrow[ddr, phantom, "\mu_f"] & TX \arrow[dd, "Tf"] \\
TX' \arrow[d, phantom, "c^T"] \arrow[dr, "T(\eta_{X'})"'] \arrow[drr, dashed, equal, "\lambda_{X'}"] & & \\
& T^2X' \arrow[r, "\mu_{X'}"'] & TX
\end{tikzcd}
$$

4) $(\mu_X \circ \lambda_{TX})(\alpha_X \circ T\eta_{TX})(\mu_X \circ c^T)(\mu_X \circ T\rho_X) =$
$\mu_X \circ e^T : \mu_X \circ T1_{TX} \Rightarrow \mu_X \circ 1_{T^2X}$



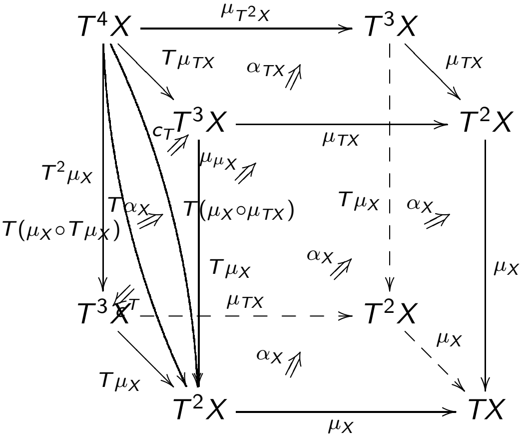
5) $(\lambda_X \circ \eta_X)(\mu_X \circ \eta_{\eta_X})(\rho_X \circ \eta_X) = 1_{\eta_X}$



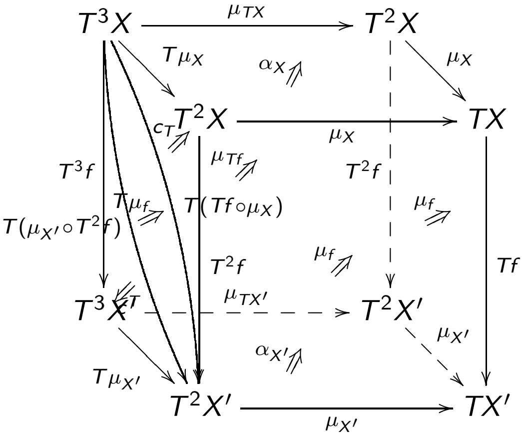
6) $(\lambda_X \circ \mu_X)(\mu_X \circ \mu_{\eta_X})(\alpha_X \circ T^2\eta_X)(\mu_X \circ c^T) =$
$(\mu_X \circ e^T)(\mu_X \circ T\lambda_X) \colon \mu_X \circ T(\mu_X \circ T\eta_X) \Rightarrow \mu_X$

$$\begin{tikzcd}
T^2X \arrow[rr, "\mu_X"] \arrow[dr, "T^2\eta_X"] \arrow[ddr, bend right=40, equal] \arrow[ddr, bend right=30, equal] \arrow[ddr, bend right=10] \arrow[ddr, bend left=10, "T(\mu_X\circ T\eta_X)"] \arrow[dr, phantom, "c_T\Uparrow"'] \arrow[ddr, phantom, "e^T\Leftarrow\ T\lambda_X\Leftarrow"] \arrow[drr, phantom, "\mu_{\eta_X}\Uparrow"] & & TX \arrow[dr, "T\eta_X"] \arrow[ddr, bend right=20, equal] \arrow[ddr, phantom, "\Swarrow\lambda_X"] & \\
& T^3X \arrow[rr, "\mu_{TX}"'] \arrow[d, "T\mu_X"] \arrow[drr, phantom, "\alpha_X\Uparrow"] & & T^2X \arrow[d, "\mu_X"] \\
& T^2X \arrow[rr, "\mu_X"'] & & TX
\end{tikzcd}$$

7) $(\alpha_X \circ \mu_{T^2X})(\mu_X \circ \mu_{\mu_X})(\alpha_X \circ T^2\mu_X)(\mu_X \circ c^T) = (\mu_X \circ \alpha_{TX})(\alpha_X \circ T\mu_{TX})(\mu_X \circ c^T)(\mu_X \circ T\alpha_X) \colon \mu_X \circ T(\mu_X \circ T\mu_X) \Rightarrow \mu_X \circ \mu_{TX} \circ \mu_{T^2X}$



8) $(\mu_f \circ \mu_{TX})(\mu_{X'} \circ \mu_{Tf})(\alpha_X \circ T^3 f)(\mu_{X'} \circ c^T) = (Tf \circ \alpha_X)(\mu_f \circ T\mu_X)(\mu_{X'} \circ c^T)(\mu_{X'} \circ T\mu_f) \colon \mu_{X'} \circ T(\mu_{X'} \circ T^2 f) \Rightarrow Tf \circ \mu_X \circ \mu_{TX}, \forall f : X \to X'$



[redacted]

Let $T\colon \mathscr{K} \to \mathscr{K}$ be an underlying colax functor of a colax monad $(T, \eta, \mu, \lambda, \rho, \alpha)$ on the 2-category $\mathscr{K}$. A lax T-algebra $(X, \xi, \iota_\xi, \kappa_\xi)$ consists of:

1) an object X of $\mathscr{K}$

such that the following axioms are satisfied:

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3) a 2-cell $\iota_\xi\colon 1_X \Rightarrow \xi \circ \eta_X$ of $\mathscr{K}$
4) a 2-cell $\kappa_\xi\colon \xi \circ T\xi \Rightarrow \xi \circ \mu_X$ of $\mathscr{K}$

such that the following axioms are satisfied:

1) $(\kappa_\xi \circ \eta_{TX})(\xi \circ \eta_\xi)(\iota_\xi \circ \xi) = \xi \circ \rho_X : \xi \Rightarrow \xi \circ \mu_X \circ \eta_{TX}$

$$
\begin{tikzcd}
TX \arrow[rr, "\eta_{TX}"] \arrow[drrr, equal] \arrow[dd, "\xi"'] \arrow[rr, phantom, shift right=2ex, "\rho_X \Nearrow"] & & T^2X \arrow[dr, "\mu_X"] \arrow[dd, dashed, "T\xi"'] & \\
& \eta_\xi \Nearrow & & TX \arrow[dd, "\xi"] \arrow[dl, phantom, "\kappa_\xi \Nearrow"] \\
X \arrow[rr, dashed, "\eta_X"] \arrow[drrr, equal] \arrow[rr, phantom, shift right=2ex, "\iota_\xi \Nearrow"] & & TX \arrow[dr, dashed, "\xi"] & \\
& & & X
\end{tikzcd}
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2) $(\xi \circ \lambda_X)(\kappa_\xi \circ T\eta_X)(\xi \circ c^T)(\xi \circ T\iota_\xi) = \xi \circ e^T \colon \xi \circ T1_X \Rightarrow \xi \circ 1_{TX}$

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\begin{tikzcd}
TX \arrow[rr, "T\eta_X"] \arrow[rrr, equal, ""{name=A}] \arrow[dd, equal] \arrow[ddrr, bend right, "T1_X"'] \arrow[ddrr, bend left, "T(\xi\circ\eta_X)"] & & T^2X \arrow[r, "\mu_X"] \arrow[dd, dashed, "T\xi"'] \arrow[dr, phantom, "\Downarrow\lambda_X"] & TX \arrow[dd, "\xi"] \arrow[ddl, phantom, "\kappa_\xi \Rightarrow"] \\
& & & \\
TX \arrow[rr, equal, dashed] \arrow[drrr, "\xi"'] \arrow[urr, phantom, "T\iota_\xi \Uparrow"] \arrow[u, phantom, "T1_X \Downarrow e^T"] & & TX \arrow[dr, dashed, "\xi"] \arrow[uu, phantom, "c^T \Uparrow"] & \\
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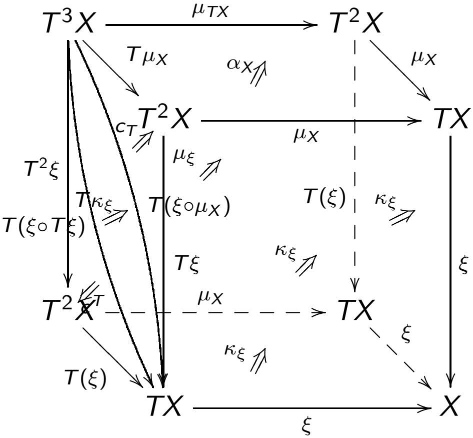
3) $(\kappa_\xi \circ \mu_{TX})(\xi \circ \mu_\xi)(\kappa_\xi \circ T^2\xi)(\xi \circ c^T) =$
$(\xi\circ\alpha_X)(\kappa_\xi\circ T\mu_X)(\xi\circ c^T)(\xi\circ T\kappa_\xi)\colon \xi\circ T(\xi\circ T\xi) \Rightarrow \xi\circ\mu_X\circ\mu_{TX}$



Bénabou's theory of cartesian functors

[redacted]

Consider a diagram

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "F"] \arrow[dr, "P"'] & & \mathscr{D} \arrow[dl, "Q"] \\ & \mathscr{B} & \end{tikzcd}$$

where P is a prefoliation and Q an arbitrary functor. We say that F is a cartesian functor if the following conditions are satisfied:

(i) It preserves cartesian maps, i.e. $k \in K(P) \Rightarrow Fk \in K(P')$.

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(i) It preserves cartesian maps, i.e. $k \in K(P) \Rightarrow Fk \in K(P')$.

(ii) $\forall f' : Y' \to F(X)$ in $\mathscr{D}$, $\exists f \colon Y \to X$ in $\mathscr{E}$ and $v \colon Y' \to F(Y)$

$$\begin{tikzcd} Y' \arrow[d, "v"'] \arrow[dr, "f'"] & \\ F(Y) \arrow[r, "F(f)"'] & F(X) \end{tikzcd}$$

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where P is a prefoliation, Q arbitrary and F a cartesian functor.

(1) *F is faithful iff every F_B is.*

If moreover P is a foliation, then F is conservative iff every F_B is.

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Bénabou's theory of cartesian functors

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(5) *F is flat iff every F_B is.*

(6) *F has a left adjoint iff every F_B has.*

If moreover P is a foliation, then F is conservative iff every F_B is.

Extension of the definition of the associated split fibration

We consider functors as *generalized fibrations* (following Bénabou) in order to extend the definition of associated split fibration

1) from the 2-category $(\mathscr{C}at, \mathscr{B})$ whose 1-cells are triangles

$$\begin{tikzcd}
\mathscr{E} \arrow[rr, "F"] \arrow[dr, "P"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & \mathscr{D} \arrow[dl, "Q"] \\
& \mathscr{B} &
\end{tikzcd}$$

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$$\begin{tikzcd} \mathscr{E} \arrow[rr, "F"] \arrow[dr, "P"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & \mathscr{D} \arrow[dl, "Q"] \\ & \mathscr{B} & \end{tikzcd}$$

2) to the 2-category $\mathscr{C}at^2_c$ whose 1-cells are colax squares

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] & \mathscr{D} \arrow[d, "Q"] \\ \mathscr{B} \arrow[r, "U"'] \arrow[ur, phantom, "\overset{\beta}{\nearrow}"] & \mathscr{C} \end{tikzcd}$$

Extension of the definition of the associated split fibration

We consider functors as *generalized fibrations* (following Bénabou) in order to extend the definition of associated split fibration

1) from the 2-category $(\mathscr{C}at, \mathscr{B})$ whose 1-cells are triangles

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "F"] \arrow[dr, "P"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & \mathscr{D} \arrow[dl, "Q"] \\ & \mathscr{B} & \end{tikzcd}$$

2) to the 2-category $\mathscr{C}at^2_c$ whose 1-cells are colax squares

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\beta \nearrow"] & \mathscr{D} \arrow[d, "Q"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd}$$

3) ultimately to the double category $\mathbb{C}at^2$ whose horizontal (vertical) cells are (co)lax squares.

Associated split fibration 2-monad

Consider the following square

$$
\begin{tikzcd}
(\mathscr{B}, P) \arrow[rr, "E_P"] \arrow[dr, "\mathscr{F}(F{,}\beta{,}U)"'] \arrow[dd, "\mathscr{F}(P)"'] & & \mathscr{E} \arrow[dr, "F"] \arrow[dd, dashed, "P"' near start] & \\
& (\mathscr{C}, Q) \arrow[rr, "E_Q"] \arrow[dd, "\mathscr{F}(Q)"' near start] \arrow[r, phantom, "\varphi_P \nearrow"] & \arrow[r, phantom, "\beta \nearrow"] & \mathscr{F} \arrow[dd, "Q"] \\
\mathscr{B} \arrow[rr, equal] \arrow[dr, "U"'] & \arrow[r, phantom, "\varphi_Q \nearrow"] & \mathscr{B} \arrow[dr, "U"] & \\
& \mathscr{C} \arrow[rr, equal] & & \mathscr{C}
\end{tikzcd}
$$

$\mathscr{F}(P)\colon (\mathscr{B}, P) \to \mathscr{B}$ and $E_P\colon (\mathscr{B}, P) \to \mathscr{E}$ send any object (B, p, E) in $(\mathscr{B}, P)$ (where $p\colon B \to P(E)$) to B and E respectively.

Associated split fibration 2-monad

From the universal property of comma squares there exists a unique functor $\mathscr{F}(F,\beta,U)\colon(\mathscr{B},P)\to(\mathscr{C},Q)$ which takes any object (B,p,E) in $(\mathscr{B},P)$ to $(U(B),\beta_E U(p),F(E))$ and any morphism $(u,e)\colon(B,p,E)\to(B',p',E')$ to the morphism $\mathscr{F}(F,\beta,U)(u,e):=(U(u),F(e))$ represented by a diagram

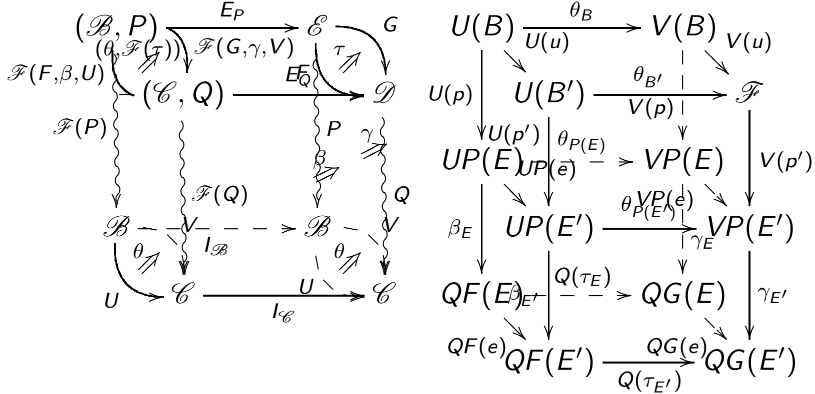
$$\begin{tikzcd}
U(B) \arrow[r, "U(u)"] \arrow[d, "U(p)"'] & U(B') \arrow[d, "U(p')"] \\
UP(E) \arrow[r, "UP(e)"'] \arrow[d, "\beta_E"'] & UP(E') \arrow[d, "\beta_{E'}"] \\
QF(E) \arrow[r, "QF(e)"'] & QF(E')
\end{tikzcd}$$

[redacted]

There exists a colax idempotent 2-monad whose underlying 2-functor

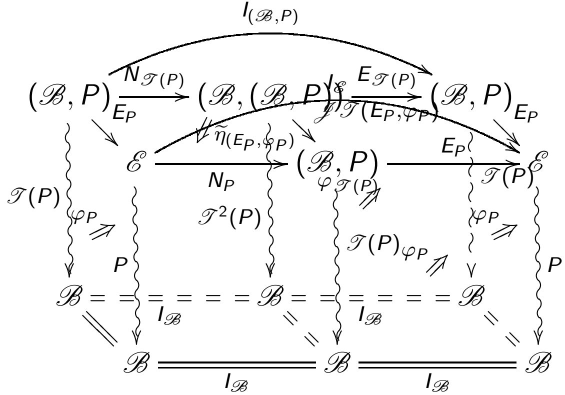
$$\mathscr{F} : \mathscr{C}at^2_c \to \mathscr{C}at^2_c$$

is given by the above construction.



Functors $\mathscr{F}(F, \beta, U)$ and $\mathscr{F}(G, \gamma, V)$ take an object (B, p, E) to $\mathscr{F}(F, \beta, U)(B, p, E) := (U(B), \beta_E U(p), F(E))$ and $\mathscr{F}(G, \gamma, V)(B, p, Q) := (V(B), \gamma_E V(p), G(E))$ respectively.

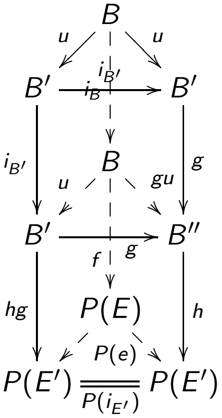
Components of the unit N_P and multiplication M_P of $\mathscr{F}$

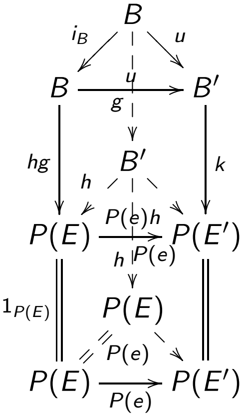


$$N_P(E) = (P(E), 1_{P(E)}, E)$$

$$M_P = \mathscr{F}(E_P,\varphi_P), \qquad M_P(B,f,B',g,E) = (B,gf,E)$$

Universal properties of local units and counits of $\mathscr{F}$





are counit and unit of the fully faithful adjoint triple

$$N_{\mathscr{T}(P)} \dashv \mathscr{T}(E_P) = M_P \dashv \mathscr{T}(N_P)$$

(Co)-lax-Gray-monoids

[redacted]

An associated split fibration 2-monad $(\mathscr{C}at^2_c, \mathscr{F}, N, \widetilde{\eta}, M)$ *is a lax-Gray-monoid in the Gray-category* $\mathscr{G}ray_l$ *of strict 2-categories, strict 2-functors, lax natural transformations and modifications with respect to a lax-Gray tensor product* $\otimes_l$.

[redacted]

An associated split cofibration 2-monad $(\mathscr{C}at^2_c, \mathscr{F}^\circ, N^\circ, \widetilde{\eta}^\circ, , M^\circ,)$ *is a colax-Gray-monoid in the Gray-category* $\mathscr{G}ray_c$ *of strict 2-categories, strict 2-functors, colax natural transformations and modifications with respect to a colax-Gray tensor product* $\otimes_c$.

The associated split fibration $\mathscr{F}$ double monad

$$\begin{tikzcd}
\mathscr{H} \arrow[rrr, "D"] \arrow[dr, "C"] \arrow[dd, "W"'] \arrow[drr, phantom, "\kappa\Uparrow"] \arrow[ddr, phantom, "\Leftarrow\rho"] & & & \mathscr{E} \arrow[dr, "F"] \arrow[dd, dashed, "P"'] \arrow[ddr, phantom, "\Leftarrow\pi"] & \\
& \mathscr{G} \arrow[rrr, "G"] \arrow[dd, "R"] \arrow[drr, phantom, "\nu\Uparrow"] & & & \mathscr{D} \arrow[dd, "Q"] \\
\mathscr{X} \arrow[rrr, dashed, "M"] \arrow[dr, "K"'] \arrow[urr, phantom, "\mu\Uparrow"] \arrow[drr, phantom, "\delta\Uparrow"] & & & \mathscr{B} \arrow[dr, dashed, "U"] & \\
& \mathscr{A} \arrow[rrr, "V"'] & & & \mathscr{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathscr{H}, W) \arrow[rrr, "\mathscr{F}(D{,}\nu)"] \arrow[dr, "\mathscr{F}_v(C{,}\rho)"'] \arrow[dd, "\mathscr{F}(W)"'] \arrow[drr, phantom, "\mathscr{F}(\kappa)\Uparrow"] \arrow[ddr, phantom, "\Leftarrow\mathscr{F}_v(\rho)"] & & & (\mathscr{B}, P) \arrow[dr, "\mathscr{F}_v(F{,}\pi)"] \arrow[dd, dashed, "\mathscr{F}(P)"'] \arrow[ddr, phantom, "\Leftarrow\mathscr{F}_v(\pi)"] & \\
& (\mathscr{A}, R) \arrow[rrr, "\mathscr{F}(G{,}\mu)"'] \arrow[dd, "\mathscr{F}(R)"] & & & (\mathscr{C}, Q) \arrow[dd, "\mathscr{F}(Q)"] \\
\mathscr{X} \arrow[rrr, dashed, "M"] \arrow[dr, "K"'] \arrow[drr, phantom, "\delta\Uparrow"] & & & \mathscr{B} \arrow[dr, dashed, "U"] & \\
& \mathscr{A} \arrow[rrr, "V"'] & & & \mathscr{C}
\end{tikzcd}$$

The definition requires the existence of (certain) pullbacks in base categories! Its domain is a double 2-category $(\mathbb{C}at, \mathbb{C}art)$ where $\mathbb{C}art$ is an (enhanced) 2-category of categories with pullbacks.

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swArrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
\quad \overset{\mathscr{F}}{\dashrightarrow} \quad
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
\overset{\mathscr{F}}{\dashrightarrow}
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[d, "p"] \\
P(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}$$

$$\overset{\mathscr{F}}{\dashrightarrow}$$

$$\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
U(B) \arrow[d, "U(p)"] \\
UP(E)
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
\quad \overset{\mathscr{F}}{\dashrightarrow} \quad
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F{,}\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swArrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \arrow[r, dotted, "\mathscr{F}"] & (\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swArrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C} & \mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \arrow[r, dotted, "\mathscr{F}"] & (\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C} & \mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}
$$

$\mathscr{F}_v(B, p, E)$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd} \quad \overset{\mathscr{F}}{\dashrightarrow} \quad \begin{tikzcd} (\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd}$$

$$\begin{tikzcd} QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi^*_E U(p)"'] & U(B) \arrow[d, "U(p)"] \\ QF(E) \arrow[r, "\pi_E"'] & UP(E) \end{tikzcd}$$

$$\mathscr{F}_v(B, p, E) \qquad :=$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow\pi"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
\quad \overset{\mathscr{F}}{\dashrightarrow} \quad
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow\mathscr{F}_v(\pi)"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}$$

$$\mathscr{F}_v(B, p, E) \quad := \quad (QF(E) \times_{UP(E)} U(B), \pi_E^* U(p), F(E))$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd} B \arrow[d, "p"] \\ P(E) \end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd} U(B) \arrow[d, "U(p)"] \\ UP(E) \end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
 & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
 & B' \arrow[dd, dashed, "p'"] & \\
QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi_E^* U(p)"'] & & U(B) \arrow[dd, "U(p)"] \\
 & P(E') & \\
QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
 & U(B') \arrow[dd, dashed, "U(p')"] & \\
QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi_E^* U(p)"'] & & U(B) \arrow[dd, "U(p)"] \\
 & UP(E') & \\
QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
 & & U(B') \arrow[dd, dashed, "U(p')"] & \\
 & QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi_E^* U(p)"'] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] & & UP(E') & \\
 & QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dd, "\pi^*_{E'} U(p')"'] & & U(B') \arrow[dd, dashed, "U(p')"] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi^*_E U(p)"' near start] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] & & UP(E') & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dd, "\pi^*_{E'}U(p')"'] & & U(B') \arrow[dr, "U(b)"] \arrow[dd, dashed, "U(p')"] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi^*_E U(p)"] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dr, "QF(e)\times U(b)"] \arrow[dd, "\pi^*_{E'}U(p')"'] & & U(B') \arrow[dr, "U(b)"] \arrow[dd, dashed, "U(p')"] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi^*_E U(p)"] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dr, "QF(e)\times U(b)"] \arrow[dd, "\pi^*_{E'}U(p')"'] & & U(B') \arrow[dr, "U(b)"] \arrow[dd, dashed, "U(p')"' near start] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}" near start] \arrow[dd, "\pi^*_E U(p)" near start] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"' near start] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

$$\mathscr{F}_v(F)(b,e) :=$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dd, "\pi^*_{E'}U(p')"'] \arrow[dr, "QF(e)\times U(b)"] & & U(B') \arrow[dd, dashed, "U(p')"] \arrow[dr, "U(b)"] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi^*_E U(p)"] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

$$\mathscr{F}_v(F)(b, e) := (QF(e) \times U(b), F(e))$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
X \arrow[d, dashed, "l"] \\
W(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
K(X) \arrow[d, dashed, "K(I)"] \\
KW(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$\begin{tikzcd} & K(X) \arrow[d, dashed, "K(I)"] \\ RC(H) \arrow[r, dashed, "\rho_H"] & KW(H) \end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
RC(H) \times_{KW(H)} K(X) \arrow[r, "\mathscr{F}_v(\rho)_{(X,I,H)}"] \arrow[d, "\rho_H^* K(I)"'] & K(X) \arrow[d, dashed, "K(I)"] \\
RC(H) \arrow[r, dashed, "\rho_H"] & KW(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[r, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[d, "V(\rho_H^* K(I))"'] & VK(X) \arrow[d, dashed, "VK(I)"] \\
VRC(H) \arrow[r, dashed, "V(\rho_H)"] & VKW(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[r, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[d, "V(\rho_H^* K(I))"'] & VK(X) \arrow[d, dashed, "VK(I)"] \\
VRC(H) \arrow[r, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & VKW(H) \\
& QGC(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[dd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dd, dashed, "VK(I)"] \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) \\
& QGC(H) &
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\begin{tikzcd}
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& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & X \arrow[dd, dashed, "I"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & \\
& QGC(H) & & W(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[dd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dd, dashed, "VK(I)"] & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & M(X) \arrow[dd, dashed, "M(I)"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & \\
& QGC(H) & & MW(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

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\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[dd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & M(X) \arrow[dd, dashed, "M(I)"] & \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & & \\
& QGC(H) & & MW(H) \arrow[dr, dashed, "\nu_H"'] & \\
& & & & PD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

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\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & M(X) \arrow[dr, equal] \arrow[ddd, dashed, "M(I)"] & \\
& & & & M(X) \arrow[ddd, "\nu_H M(I)"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & & \\
& QGC(H) & & MW(H) \arrow[dr, dashed, "\nu_H"'] & \\
& & & & PD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

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\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho^*_H K(I))"'] & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho^*_H K(I))"'] & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & & \\
& QGC(H) & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & & & UPD(H)
\end{tikzcd}
$$

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\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rrr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rrr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & & VKW(H) & & \\
& QGC(H) & & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & QFD(H) \arrow[rrr, "\pi_{D(H)}"'] & & & UPD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

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\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rrr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & QFD(H) \times_{UPD(H)} UM(X) \arrow[r, phantom, "\mathscr{F}_v(\pi)"] \arrow[rrr, "\mathscr{F}(D,\nu)_{(X,I,H)}"] \arrow[ddd, "\pi_{D(H)}^* U(\nu_H M(I))"'] & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rrr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & & VKW(H) & & \\
& QGC(H) & & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & QFD(H) \arrow[rrr, "\pi_{D(H)}"'] & & & UPD(H)
\end{tikzcd}
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\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dr, "\delta_X"] \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & QFD(H) \times_{UPD(H)} UM(X) \arrow[rr, "\mathscr{F}_v(\pi)_{\mathscr{F}(D,\nu)(X,I,H)}"] \arrow[ddd, "\pi^*_{D(H)} U(\nu_H M(I))"'] & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) \arrow[dr, dashed, "\delta_{W(H)}"'] & & \\
& QGC(H) & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & QFD(H) \arrow[rr, "\pi_{D(H)}"'] & & UPD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rrr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & & VK(X) \arrow[dr, "\delta_X"] \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & QFD(H) \times_{UPD(H)} UM(X) \arrow[rrr, "\mathscr{F}_v(\pi)_{\mathscr{F}(D,\nu)(X,I,H)}"] \arrow[ddd, "\pi_{D(H)}^* U(\nu_H M(I))"'] & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rrr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & & VKW(H) \arrow[dr, dashed, "\delta_{W(H)}"'] & & \\
& QGC(H) \arrow[dr, "Q(\kappa_H)"'] & & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & QFD(H) \arrow[rrr, "\pi_{D(H)}"'] & & & UPD(H)
\end{tikzcd}
$$

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$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rrr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & & VK(X) \arrow[dr, "\delta_X"] \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dr, "\mathscr{F}(\kappa)_{(X,I,H)}"] \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & QFD(H) \times_{UPD(H)} UM(X) \arrow[rrr, "\mathscr{F}_v(\pi)_{\mathscr{F}(D,\nu)(X,I,H)}"] \arrow[ddd, "\pi_{D(H)}^* U(\nu_H M(I))"'] & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rrr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & & VKW(H) \arrow[dr, dashed, "\delta_{W(H)}"'] & & \\
& QGC(H) \arrow[dr, "Q(\kappa_H)"'] & & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & QFD(H) \arrow[rrr, "\pi_{D(H)}"'] & & & UPD(H)
\end{tikzcd}
$$

The associated split cofibration $\mathscr{F}^\circ$ 2-monad

1. The associated split cofibration $\mathscr{F}^\circ$ is defined as dual to $\mathscr{F}$:

$$\mathscr{F}^\circ(P) := (\mathscr{F}(P^{op}))^{op}$$

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2. It requires no conditions on lax squares
3. It requires the existence of pushouts in base categories for colax squares

Pseudo-distributive law between fibrations and cofibrations

$$\begin{tikzcd}
(\mathscr{B},(P,\mathscr{B})) \arrow[rr, "\lambda_P"] \arrow[dr, "\mathscr{F}\mathscr{F}^{\circ}(U{,}\pi{,}F)"'] \arrow[dd, "\mathscr{F}\mathscr{F}^{\circ}(P)"'] & & ((\mathscr{B},P),\mathscr{B}) \arrow[dr, "\mathscr{F}^{\circ}\mathscr{F}(U{,}\pi{,}F)"] \arrow[dd, dashed, "\mathscr{F}^{\circ}\mathscr{F}(P)"' near start] \arrow[dl, phantom, "\Swarrow\lambda_{(U,\pi,F)}"] & \\
& (\mathscr{C},(Q,\mathscr{C})) \arrow[rr, "\lambda_Q"'] \arrow[dd, "\mathscr{F}\mathscr{F}^{\circ}(Q)" near end] & & ((\mathscr{C},Q),\mathscr{C}) \arrow[dd, "\mathscr{F}^{\circ}\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[rr, dashed, equal] \arrow[dr, "U"'] & & \mathscr{B} \arrow[dr, dashed, "U"] & \\
& \mathscr{C} \arrow[rr, equal] & & \mathscr{C}
\end{tikzcd}$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
 & P(E) \arrow[d, dashed, "q"] \\
B \arrow[r, "b"'] & B_q
\end{tikzcd}
$$

The component of a lax-natural transformation λ

$$\begin{tikzcd} & UP(E) \arrow[dr, dashed, "U(q)"] & \\ U(B) \arrow[rr, "U(b)"'] & & U(B_q) \end{tikzcd}$$

The component of a lax-natural transformation λ

$$\begin{tikzcd}
 & QF(E) \arrow[dd, dashed, "\pi_E"'] \arrow[dr, equal] & \\
 & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
 & UP(E) \arrow[dr, dashed, "U(q)"] & \\
U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}$$

The component of a lax-natural transformation λ

$$\begin{tikzcd}
 & QF(E) \arrow[dd, dashed, "\pi_E"'] \arrow[dr, equal] & \\
U(B)\times_{U(B_q)} QF(E) \arrow[rr, "pr_2"] \arrow[dd, "pr_1"'] & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
 & UP(E) \arrow[dr, dashed, "U(q)"] & \\
U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
 & & QF(E) \arrow[dd, dashed, "\pi_E"'] \arrow[dr, equal] & \\
 & U(B) \times_{U(B_q)} QF(E) \arrow[rr, "pr_2"] \arrow[dd, "pr_1"'] & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
U(B \times_{B_q} P(E)) \arrow[rr, dashed, "U(pr_2)"] \arrow[dr, "U(pr_1)"'] & & UP(E) \arrow[dr, dashed, "U(q)"] & \\
 & U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}
$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
U(B \times_{B_q} P(E)) \times_{UP(E)} QF(E) \arrow[rr, "pr_2 = \mathscr{F}(\pi)"] \arrow[dr, "\lambda_{(U,\pi,F)}"] \arrow[dd, "pr_1"'] & & QF(E) \arrow[dd, dashed, "\pi_E"'] \arrow[dr, equal] & \\
& U(B) \times_{U(B_q)} QF(E) \arrow[rr, "pr_2"] \arrow[dd, "pr_1"' near start] & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
U(B \times_{B_q} P(E)) \arrow[rr, dashed, "U(pr_2)" near start] \arrow[dr, "U(pr_1)"'] & & UP(E) \arrow[dr, dashed, "U(q)"] & \\
& U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}
$$

The associated Beck-Chevalley fibration

- The associated Beck-Chevalley fibrations are pseudoalgebras for the pseudo-distributive law

$$\lambda \colon \mathscr{F}\mathscr{F}^{\circ} \Rightarrow \mathscr{F}^{\circ}\mathscr{F}$$

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- A natural candidate for the domain of its underlying 2-functor

$$\mathscr{F}\mathscr{F}^{\circ}\colon (\mathscr{C}at, \mathcal{QT}op) \to (\mathscr{C}at, \mathcal{QT}op)$$

  is a double comma 2-category $(\mathscr{C}at, \mathcal{QT}op)$ where $\mathcal{QT}op$ is a 2-category of quasitoposes and geometric morphisms

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- A quasitopos is a finitely complete, finitely cocomplete, locally cartesian closed category $\mathscr{C}$ in which there exists an object Ω that classifies strong monomorphisms.

Admissibility

Admissible 1-cells

[redacted]

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a lax idempotent 2-monad on the 2-category $\mathscr{K}$. We say that the 1-cell $f\colon C \to D$ in $\mathscr{K}$ is admissible if its image $T(f)$ has a right adjoint μ_f. In the dual case of a colax idempotent 2-monad we say that the 1-cell $f\colon C \to D$ in $\mathscr{K}$ is admissible if $T(f)$ has a left adjoint ν_f.

Admissible objects

[redacted]

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a (co)lax idempotent 2-monad on the 2-category $\mathscr{K}$ with a terminal object $\top$. We say that an object E of $\mathscr{K}$ is admissible if the unique 1-cell $!_E\colon E \to \top$ is admissible.

[redacted]

Let $(T, \eta, \mu) \colon \mathscr{K} \to \mathscr{K}$ be a lax idempotent 2-monad on the 2-category $\mathscr{K}$. We say that (T, η, μ) is admissible if the following bicomma object condition holds:

1) the 2-category $\mathscr{K}$ has bicomma objects $f \downarrow g$ of diagrams

$$\begin{tikzcd} & B \arrow[d, rightsquigarrow, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

where 1-cells p and q are admissible.

[redacted]

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1) the 2-category $\mathscr{K}$ has bicomma objects $f \downarrow g$ of diagrams

$$\begin{tikzcd} f \downarrow g \arrow[r, "p"] \arrow[d, squiggly, "q"'] \arrow[dr, phantom, "\nearrow"] & B \arrow[d, squiggly, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

where 1-cells p and q are admissible.

2) the canonical 2-cell $T(q)\mu_p \Rightarrow \mu_f T(g)$ is a 2-isomorphism.

The admissibility of associated split (co)fibrations

[redacted]

The associated split fibration 2-monad is admissible.

The two triangle identities

$$\begin{tikzcd}[column sep=small]
 & \mathscr{F}(F,\beta,U) \arrow[dl, Rightarrow, "\widetilde{\eta}\mathscr{F}(F,\beta,U)"'] \arrow[dr, equal] & \\
\mathscr{F}(F,\beta,U)\mathscr{F}(F^*,\lambda,L)\mathscr{F}(F,\beta,U) \arrow[rr, Rightarrow, "\mathscr{F}(F,\beta,U)\widetilde{\epsilon}"'] & & \mathscr{F}(F,\beta,U)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \mathscr{F}(F^*,\lambda,L) \arrow[dl, Rightarrow, "\mathscr{F}(F^*,\lambda,L)\widetilde{\eta}"'] \arrow[dr, equal] & \\
\mathscr{F}(F^*,\lambda,L)\mathscr{F}(F,\beta,U)\mathscr{F}(F^*,\lambda,L) \arrow[rr, Rightarrow, "\widetilde{\epsilon}\mathscr{F}(F^*,\lambda,L)"'] & & \mathscr{F}(F^*,\lambda,L)
\end{tikzcd}$$

are represented by the following diagrams

$$
\begin{tikzcd}
 & U(B) \arrow[dl, "\eta_{U(B)}"'] \arrow[dr, equal] \arrow[dd, dashed, "U(p)"] & \\
ULU(B) \arrow[rr, "U(\epsilon_B)"'] \arrow[dd, "ULU(p)"'] & & U(B) \arrow[dd, "U(p)"] \\
 & UP(E) \arrow[dl, dashed, "\eta_{UP(E)}"'] \arrow[dr, equal] \arrow[dd, dashed, "\beta_E"] & \\
ULUP(E) \arrow[rr, "U(\epsilon_{P(E)})"'] \arrow[dd, "UL(\beta_E)"'] & & UP(E) \arrow[dd, "\beta_E"] \\
 & QF(E) \arrow[dl, dashed, "\eta_{QF(E)}"'] \arrow[dr, equal] \arrow[dd, dashed, "\beta^*_E"] & \\
ULQF(E) \arrow[dd, "U(\lambda_{F(E)})"'] & & QF(E) \arrow[ll, "\eta_{QF(E)}"] \arrow[dd, "\beta^*_E"] \\
 & UP(E) \arrow[dr, equal] \arrow[dd, dashed, "\beta_E"] & \\
UPF^*F(E) \arrow[ur, dashed, "UP(\widetilde{\epsilon}_E)"] \arrow[rr, "UP(\widetilde{\epsilon}_E)"'] \arrow[d, "\beta_{F^*F(E)}"'] & & UP(E) \arrow[d, "\beta_E"] \\
 {} & QF(E) \arrow[l, phantom, "Q(\widetilde{\eta}_{F(E)})"] & {}
\end{tikzcd}
$$

$$\begin{tikzcd}
 & U(B) \arrow[dl, "\eta_{U(B)}"'] \arrow[dr, equal] \arrow[dd, dashed, "U(p)"] & \\
ULU(B) \arrow[rr, "U(\epsilon_B)"'] \arrow[dd, "ULU(p)"'] & & U(B) \arrow[dd, "U(p)"] \\
 & UP(E) \arrow[dl, dashed, "\eta_{UP(E)}"'] \arrow[dr, dashed, equal] \arrow[dddd, dashed, "\beta_E"] & \\
ULUP(E) \arrow[rr, "U(\epsilon_{P(E)})"'] \arrow[d, "UL(\beta_E)"'] & & UP(E) \arrow[dd, equal] \\
ULQF(E) \arrow[d, "U(\lambda_{F(E)})"'] & & \\
UPF^*F(E) \arrow[rr, "UP(\widetilde{\epsilon}_E)"'] \arrow[dd, "\beta_{F^*F(E)}"'] & & UP(E) \arrow[dd, "\beta_E"] \\
 & QF(E) \arrow[dl, dashed, "Q(\widetilde{\eta}_{F(E)})"'] \arrow[dr, dashed, equal] & \\
QFF^*F(E) \arrow[rr, "QF(\widetilde{\epsilon}_E)"'] & & QF(E)
\end{tikzcd}$$

[redacted]

A functor $U\colon \mathscr{A} \to \mathscr{B}$ is a local right adjoint if the restriction

$$U_A\colon (\mathscr{A}, A) \to (\mathscr{B}, U(A))$$

of U to the slice $(\mathscr{A}, A)$ category for each object A of $\mathscr{A}$ has a left adjoint

$$L_A\colon (\mathscr{B}, U(A)) \to (\mathscr{A}, A).$$

Equivalently, each fiber $U_A\colon (\mathscr{A}, A) \to (\mathscr{B}, U(A))$ of the diagram

$$\begin{tikzcd} \mathscr{A}^2 \arrow[r, "U^2"] \arrow[d, "cod"'] & \mathscr{B}^2 \arrow[d, "cod"] \\ \mathscr{A} \arrow[r, "U"'] & \mathscr{B} \end{tikzcd}$$

has a left adjoint.

[redacted]

Right multiadjoints are admissible objects for the associated split fibration 2-monad.

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