Comma 2-Comonad and Its Cousins

6th ItaCa Workshop

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Motivation

The role (co)lax 2-(co)monads built by a comma construction:

- The operation which associates to any pair of categories $\mathscr{A}$ and $\mathscr{B}$ a category $(\mathscr{A}, \mathscr{B})$ which came to be known by the name "comma category" was introduced by Lawvere in his thesis for the purpose of a foundational clarification, in particular of the notion of *adjointness*.

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- 'pursuing' formal theory of adjunctions

A 2-category $\mathscr{C}at_c^2$ of functors and colax squares

$\mathscr{C}at_c^2$ is a 2-category; objects are functors, 1-cells are colax squares

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "S"'] \arrow[dr, phantom, "\overset{\phi}{\Nearrow}"] & \mathscr{E} \arrow[d, "G"] \\
\mathscr{B} \arrow[r, "V"'] & \mathscr{X}
\end{tikzcd}$$

with 2-cells

$$\begin{tikzcd}
\mathscr{E} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "U"'] \arrow[dr, phantom, "\overset{\phi'}{\Nearrow}"] & \mathscr{A} \arrow[d, "G"] \\
\mathscr{B} \arrow[r, bend left, dashed, "V"] \arrow[r, bend right, "V'"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathscr{X}
\end{tikzcd}$$

The basic 2-adjunction

There is a canonical 2-functor

$$I\colon \mathscr{C}at \to \mathscr{C}at^2_c$$

which sends a category $\mathscr{B}$ to the identity functor $I_{\mathscr{B}}\colon \mathscr{B} \to \mathscr{B}$.

Theorem

There exists a strict 2-adjunction

$$I \dashv D$$

where $D\colon \mathscr{C}at^2_c \to \mathscr{C}at$ is a comma category construction; it takes any functor $G\colon \mathscr{A} \to \mathscr{X}$ to the comma category $D(G) := (\mathscr{X}, G)$.

The comma 2-comonad induced by the basic 2-adjunction

Theorem

There is a strict 2-comonad on the 2-category $\mathscr{C}at^2_c$ whose underlying 2-functor

$$\mathscr{D} : \mathscr{C}at^2_c \to \mathscr{C}at^2_c$$

is a composition $\mathscr{D} := ID$ of the pair of basic adjoint 2-functors.

A colax $\mathscr{D}$-coalgebra: data

Definition

A colax $\mathscr{D}$-coalgebra (on a functor G) consists of the data:

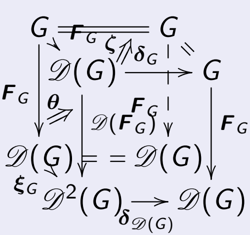
- a 1-cell $\boldsymbol{F}_G = (\boldsymbol{F}_1, \varphi, \boldsymbol{F}_0)\colon G \to \mathscr{D}(G)$ in $\mathscr{C}at_c^2$

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\boldsymbol{F}_0"] \arrow[d, "G"'] \arrow[dr, phantom, "\varphi \nearrow"] & (\mathscr{X}, G) \arrow[d, equal] \\ \mathscr{X} \arrow[r, "\boldsymbol{F}_1"] & (\mathscr{X}, G) \end{tikzcd}$$

- 2-cells $\zeta\colon \iota_G \Rightarrow \boldsymbol{\delta}_G \boldsymbol{F}_G$ and $\boldsymbol{\theta}\colon \mathscr{D}(\boldsymbol{F}_G)\boldsymbol{F}_G \Rightarrow \boldsymbol{\xi}_G \boldsymbol{F}_G$ in $\mathscr{C}at_c^2$

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "\boldsymbol{F}_G"'] \arrow[dr, phantom, "\zeta \nearrow"] & G \arrow[d, equal] \\ \mathscr{D}(G) \arrow[r, "\boldsymbol{\delta}_G"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\boldsymbol{F}_G"] \arrow[d, "\boldsymbol{F}_G"'] \arrow[dr, phantom, "\boldsymbol{\theta} \nearrow"] & \mathscr{D}(G) \arrow[d, "\mathscr{D}(\boldsymbol{F}_G)"] \\ \mathscr{D}(G) \arrow[r, "\boldsymbol{\xi}_G"'] & \mathscr{D}^2(G) \end{tikzcd}$$

Definition


G
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$\mathscr{D}^2(G)$
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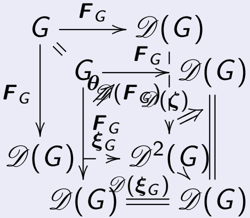
$\delta_{\mathscr{D}(G)}$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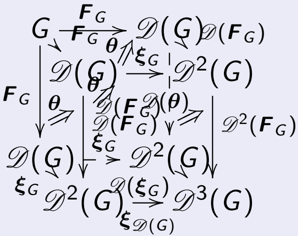
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$\mathscr{D}^2(G)$
$\mathscr{D}(\xi_G)$
$\mathscr{D}^3(G)$
$\xi_{\mathscr{D}(G)}$

Coherence conditions for the components of the counit ζ

$$
\begin{tikzcd}
\mathscr{A} \arrow[rr, equal] \arrow[dr, "{(H,\chi,Q)}"'] \arrow[dd, "G"'] & & \mathscr{A} \arrow[dr, equal] \arrow[dd, dashed, "G"] & \\
& (\mathscr{X}, G) \arrow[rr, "\boldsymbol{d}_0"] \arrow[dd, equal] & & \mathscr{A} \arrow[dd, "G"] \\
\mathscr{X} \arrow[rr, equal, dashed] \arrow[dr, "{(C,\varrho,K)}"'] & & \mathscr{X} \arrow[dr, equal, dashed] & \\
& (\mathscr{X}, G) \arrow[rr, "\boldsymbol{d}_1"'] & & \mathscr{X}
\arrow[from=1-1, to=2-4, phantom, "\zeta^0 \Nearrow"]
\arrow[from=1-1, to=4-2, phantom, "{(\omega,\varepsilon)} \Nearrow"]
\arrow[from=1-3, to=4-4, phantom, "\delta_G \Nearrow"]
\arrow[from=3-1, to=4-4, phantom, "\zeta^1 \Nearrow"]
\end{tikzcd}
\qquad
\begin{tikzcd}
CG(A) \arrow[r, "\omega_A"] \arrow[d, "\varrho_{G(A)}"'] & H(A) \arrow[d, "\chi_A"] \\
GKG(A) \arrow[r, "G(\varepsilon_A)"'] & GQ(A)
\end{tikzcd}
$$

$$G(\zeta^0_A)\chi_A\omega_A = \zeta^1_{G(A)} = G(\zeta^0_A)G(\varepsilon_A)\varrho_{G(A)}$$

The component of the comultiplication θ^1

$$
\begin{tikzcd}
C(X) \arrow[rr, "\theta^{100}_X"] \arrow[ddd, "\varrho_X"'] \arrow[ddr, "1_{C(X)}"] & & CC(X) \arrow[dr, "C(\varrho_X)"] \arrow[ddd, dashed, "\varrho_{C(X)}"'] & & \\
& & & CGK(X) \arrow[dr, "\omega_{K(X)}"] \arrow[ddd, dashed, "\varrho_{GK(X)}"'] & \\
& C(X) \arrow[rrr, "\theta^{101}_X"] \arrow[ddd, "\varrho_X"'] & & & HK(X) \arrow[ddd, "\chi_{K(X)}"] \\
GK(X) \arrow[rr, dashed, "G(\theta^{110}_X)"] \arrow[ddr, "G(1_{K(X)})"'] & & GKC(X) \arrow[dr, dashed, "GK(\varrho_X)"'] & & \\
& & & GKGK(X) \arrow[dr, dashed, "G(\varepsilon_{K(X)})"'] & \\
& GK(X) \arrow[rrr, "G(\theta^{111}_X)"'] & & & GQK(X)
\end{tikzcd}
$$

The coherence conditions for the comultiplication $\boldsymbol{\theta}^1$

$$
\begin{gathered}
C(\theta_X^{100})\theta_X^{100} = \theta_{C(X)}^{100}\theta_X^{100} \\
C(\theta_X^{101})\theta_X^{100} = \theta_{K(X)}^{000}\theta_X^{101} \\
H(\theta_X^{110})\theta_X^{101} = \theta_{C(X)}^{101}\theta_X^{100} \\
H(\theta_X^{111})\theta_X^{101} = \theta_{K(X)}^{001}\theta_X^{101} \\
Q(\theta_X^{110})\theta_X^{111} = \theta_{C(X)}^{111}\theta_X^{110} \\
Q(\theta_X^{111})\theta_X^{111} = \theta_{K(X)}^{011}\theta_X^{111} \\
K(\theta_X^{100})\theta_X^{110} = \theta_{C(X)}^{110}\theta_X^{110} \\
K(\theta_X^{101})\theta_X^{110} = \theta_{K(X)}^{010}\theta_X^{111}
\end{gathered}
$$

The component of the comultiplication $\boldsymbol{\theta}^0$

$$
\begin{tikzcd}
H(A) \arrow[rr, "\theta^{000}_A"] \arrow[ddr, "1_{H(A)}"] \arrow[ddd, "\chi_A"'] & & CH(A) \arrow[dr, "C(\chi_A)"] \arrow[ddd, dashed, "\varrho_{H(A)}"'] & & \\
 & & & CGQ(A) \arrow[dr, "\omega_{Q(A)}"] \arrow[ddd, dashed, "\varrho_{GQ(A)}"] & \\
 & H(A) \arrow[rrr, "\theta^{001}_A"] \arrow[ddd, "\chi_A"'] & & & HQ(A) \arrow[ddd, "\chi_{Q(A)}"] \\
GQ(A) \arrow[rr, dashed, "G(\theta^{010}_A)"] \arrow[ddr, "G(1_{Q(A)})"'] & & GKH(A) \arrow[dr, dashed, "GK(\chi_A)"'] & & \\
 & & & GKGQ(A) \arrow[dr, dashed, "G(\varepsilon_{Q(A)})"'] & \\
 & GQ(A) \arrow[rrr, "G(\theta^{011}_A)"'] & & & GQQ(A)
\end{tikzcd}
$$

The coherence conditions for the comultiplication $\boldsymbol{\theta}^0$

$$
\begin{aligned}
C(\theta_A^{000})\theta_A^{000} &= \theta_{H(A)}^{100}\theta_A^{000}\\
C(\theta_A^{001})\theta_A^{000} &= \theta_{Q(A)}^{000}\theta_A^{001}\\
H(\theta_A^{010})\theta_A^{001} &= \theta_{H(A)}^{101}\theta_A^{000}\\
H(\theta_A^{011})\theta_A^{001} &= \theta_{Q(A)}^{001}\theta_A^{001}\\
Q(\theta_A^{010})\theta_A^{011} &= \theta_{H(A)}^{111}\theta_A^{010}\\
Q(\theta_A^{011})\theta_A^{011} &= \theta_{Q(A)}^{011}\theta_A^{011}\\
K(\theta_A^{000})\theta_A^{010} &= \theta_{H(A)}^{110}\theta_A^{010}\\
K(\theta_A^{001})\theta_A^{010} &= \theta_{Q(A)}^{010}\theta_A^{011}
\end{aligned}
$$

The compatibility of φ and $\boldsymbol{\theta}$

$$
\begin{gathered}
\varrho_{C(X)}\theta_X^{100} = \mathscr{G}(\theta_X^{110})\varrho_X \\
\chi_{K(X)}\theta_X^{101} = G(\theta_X^{111})\varrho_X \\
\omega_{K(X)}C(\varrho_X)\theta_X^{100} = \theta_X^{101} \\
\varepsilon_{K(X)}K(\varrho_X)\theta_X^{110} = \theta_X^{111} \\
\varrho_{H(A)}\theta_A^{000} = G(\theta_A^{010})\chi_A \\
\chi_{Q(A)}\theta_A^{001} = G(\theta_A^{011})\chi_A \\
\omega_{Q(A)}C(\chi_A)\theta_A^{000} = \theta_A^{001} \\
\varepsilon_{Q(A)}K(\chi_A)\theta_A^{010} = \theta_A^{011}
\end{gathered}
$$

The compatibility of ζ and $\boldsymbol{\theta}$

$$
\begin{gathered}
C(\zeta_X^1)\theta_X^{100} = 1_{C(X)} \\
K(\zeta_X^1)\theta_X^{110} = 1_{K(X)} \\
H(\zeta_A^0)\theta_A^{001} = 1_{H(A)} \\
Q(\zeta_A^0)\theta_A^{011} = 1_{Q(A)} \\
\zeta_{C(X)}^1\theta_X^{100} = 1_{C(X)} \\
\zeta_{K(X)}^0\theta_X^{111} = 1_{K(X)} \\
\zeta_{H(A)}^1\theta_A^{000} = 1_{H(A)} \\
\zeta_{Q(A)}^0\theta_A^{011} = 1_{Q(A)}
\end{gathered}
$$

A colax $\mathscr{D}$-coalgebra in a nutshell

$$\begin{tikzcd}
\mathscr{A} \arrow[loop above, "Q"] \arrow[d, shift right=6, "H"'] \arrow[d, shift left=6, "G"] \\
\mathscr{X} \arrow[loop below, "C"] \arrow[u, "K"'] \arrow[u, phantom, shift left=3, "\dashv"]
\end{tikzcd}$$

A colax $\mathscr{D}$-coalgebra structure on a functor G consists of:

- a functor K left adjoint G,

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\mathscr{X} \arrow[u, "K"'] \arrow[u, phantom, shift left=3, "\dashv"] \arrow[loop below, "C"]
\end{tikzcd}
$$

A colax $\mathscr{D}$-coalgebra structure on a functor G consists of:

- a functor K left adjoint G,
- a functor H (which may or may not be an adjoint to K),

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\mathscr{X} \arrow[u, "K"] \arrow[loop below, "C"]
\end{tikzcd}
$$

A colax $\mathscr{D}$-coalgebra structure on a functor G consists of:

- a functor K left adjoint G,
- a functor H (which may or may not be an adjoint to K),
- a comonad $(C, \zeta^1, \theta^{100})$ on $\mathscr{X}$

A colax $\mathscr{D}$-coalgebra in a nutshell

$$
\begin{tikzcd}[row sep=huge]
\mathscr{A} \arrow[loop above, "Q"] \arrow[d, shift right=6, "H"'] \arrow[d, shift left=6, "G"] \\
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- a comonad $(Q, \zeta^0, \theta^{011})$ on $\mathscr{A}$

A colax $\mathscr{D}$-coalgebra in a nutshell

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\begin{tikzcd}
\mathscr{A} \arrow[loop above, "Q"] \arrow[d, shift right=6, "H"'] \arrow[d, shift left=6, "G"] \\
\mathscr{X} \arrow[loop below, "C"] \arrow[u, "K"'] \arrow[u, phantom, shift left=3, "\dashv"]
\end{tikzcd}
$$

A colax $\mathscr{D}$-coalgebra structure on a functor G consists of:

- a functor K left adjoint G,
- a functor H (which may or may not be an adjoint to K),
- a comonad $(C, \zeta^1, \theta^{100})$ on $\mathscr{X}$
- a comonad $(Q, \zeta^0, \theta^{011})$ on $\mathscr{A}$
- natural transformations satisfying 34 equations...

A colax $\mathscr{D}$-coalgebra is a comonad in $\mathscr{C}at^2_c$

Theorem

Every colax $\mathscr{D}$-coalgebra on G induces a comonad on G in $\mathscr{C}at^2_c$!

Proof.

The structure 1-cell

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "Q"] \arrow[d, "G"'] & \mathscr{A} \arrow[d, "G"] \\
\mathscr{X} \arrow[r, "C"'] & \mathscr{X}
\end{tikzcd}$$

□

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\mathscr{A} \arrow[r, "Q"] \arrow[d, "G"'] \arrow[dr, "H"'] & \mathscr{A} \arrow[d, "G"] \\
\mathscr{X} \arrow[r, "C"'] \arrow[ur, phantom, "\omega\Nearrow" near start] & \mathscr{X}
\end{tikzcd}$$

$\square$

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$$\begin{tikzcd}
\mathscr{A} \arrow[r, "Q"] \arrow[d, "G"'] \arrow[dr, "H"] \arrow[d, phantom, shift left=8ex, "\overset{\omega}{\Nearrow}"] & \mathscr{A} \arrow[d, "G"] \arrow[dl, phantom, shift left=2ex, "\overset{\chi}{\Nearrow}"] \\
\mathscr{X} \arrow[r, "C"'] & \mathscr{X}
\end{tikzcd}$$

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$$\begin{tikzcd}
\mathscr{A} \arrow[r, "Q"] \arrow[d, "G"'] \arrow[dr, "K" near end] & \mathscr{A} \arrow[d, "G"] \arrow[dl, phantom, "\Nearrow\chi" near start] \\
\mathscr{X} \arrow[r, "C"'] \arrow[ur, dashed, "H"'] \arrow[ur, phantom, "\Nearrow\omega" near start] & \mathscr{X}
\end{tikzcd}$$

$\square$

A colax $\mathscr{D}$-coalgebra is a comonad in $\mathscr{C}at^2_c$

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Every colax $\mathscr{D}$-coalgebra on G induces a comonad on G in $\mathscr{C}at^2_c$!

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The structure 1-cell

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathscr{A} \arrow[r, "Q"] \arrow[d, "G"'] \arrow[dr, "K"] & \mathscr{A} \arrow[d, "G"] \arrow[dl, phantom, "\overset{\chi}{\Nearrow}"] \\
\mathscr{X} \arrow[r, "C"'] \arrow[ur, dashed, "H"] \arrow[ur, phantom, "\overset{\omega}{\Nearrow}" near start] \arrow[r, phantom, "\varrho\Uparrow" above, shift left=4] & \mathscr{X}
\end{tikzcd}$$

□

A colax $\mathscr{D}$-coalgebra is a comonad in $\mathscr{C}at^2_c$

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\begin{tikzcd}
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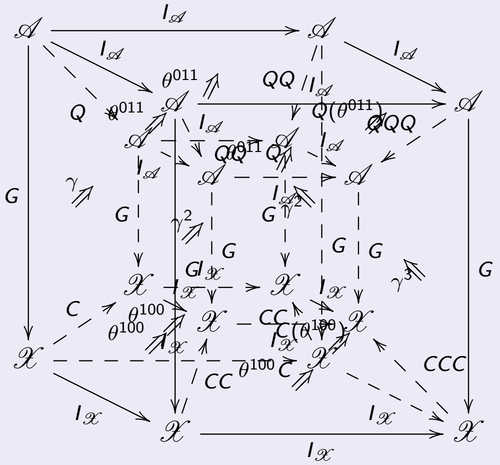
\mathscr{X} \arrow[r, "C"'] \arrow[ur, dashed, "H"] \arrow[r, phantom, shift left=5, "\varrho\Uparrow"] \arrow[ur, phantom, shift left=3, "\omega"] \arrow[ur, phantom, shift right=3, "\chi"] & \mathscr{X}
\end{tikzcd}
$$

□

Proof.



$\mathscr{A}$
$I_{\mathscr{A}}$
$\mathscr{X}$
$I_{\mathscr{X}}$
G
Q
QQ
QQQ
C
CC
CCC
θ^{011}
θ^{100}
$Q(\theta^{011})$
$C(\theta^{100})$
γ
γ^2
γ^3

□

Split colax $\mathscr{D}$-coalgebras

Definition

We say that a colax $\mathscr{D}$-coalgebra $(G, \boldsymbol{F}_G, \zeta, \boldsymbol{\theta})$ is split if satisfies the following conditions

$$K(\theta_X^{100}) = \theta_{C(X)}^{110}$$

$$\zeta^0_{K(X)}\epsilon_{K(X)}K(\varrho_X) = K(\zeta^1_X)$$

Adjunctions

Theorem

Every adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[d, shift left=3, "G"] \arrow[d, phantom, "\dashv"] \\ \mathscr{X} \arrow[u, shift left=3, "K"] \end{tikzcd}$$

induces a colax $\mathscr{D}$-coalgebra structure with $H = G$ and the two comonads defined by $(Q = KG, \epsilon, K\eta G)$ and $C = I_{\mathscr{X}}$.

Adjunctions

Theorem

Every adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[d, shift right=6, "G"'] \arrow[d, shift left=6, "G"] \\ \mathscr{X} \arrow[u, "K"'] \arrow[u, phantom, shift left=3, "\dashv"] \end{tikzcd}$$

induces a colax $\mathscr{D}$-coalgebra structure with $H = G$ and the two comonads defined by $(Q = KG, \epsilon, K\eta G)$ and $C = I_{\mathscr{X}}$.

Adjunctions

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Every adjunction

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\begin{tikzcd}
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\mathscr{X} \arrow[u, "K"']
\end{tikzcd}
$$

induces a colax $\mathscr{D}$-coalgebra structure with $H = G$ and the two comonads defined by $(Q = KG, \epsilon, K\eta G)$ and $C = I_{\mathscr{X}}$.

Theorem

Every adjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[loop above, "KG"] \arrow[d, shift right=6, "G"'] \arrow[d, shift left=6, "G"] \arrow[d, phantom, shift left=3, "\dashv"] \\
\mathscr{X} \arrow[u, "K"] \arrow[loop below, "I_{\mathscr{X}}"]
\end{tikzcd}$$

induces a colax $\mathscr{D}$-coalgebra structure with $H = G$ and the two comonads defined by $(Q = KG, \epsilon, K\eta G)$ and $C = I_{\mathscr{X}}$.

Proof.

$$\begin{tikzcd}
X \arrow[rr, "1_X"] \arrow[ddr, "1_X"] \arrow[ddd, "\eta_X"'] & & X \arrow[dr, "\eta_X"] \arrow[ddd, dashed, "\eta_X"'] & & \\
& & & GK(X) \arrow[dr, "1_{GK(X)}"] \arrow[ddd, dashed, "\eta_{GK(X)}"'] & \\
& X \arrow[rrr, "\eta_X"] \arrow[ddd, "\eta_X"'] & & & GK(X) \arrow[ddd, "\eta_{GK(X)}"] \\
GK(X) \arrow[rr, dashed, "G(1_{K(X)})"] \arrow[ddr, "G(1_{K(X)})"'] & & GK(X) \arrow[dr, "GK(\eta_X)"'] & & \\
& & & GKGK(X) \arrow[dr, "G(1_{KGK(X)})"'] & \\
& GK(X) \arrow[rrr, "GK(\eta_X)"'] & & & GKGK(X)
\end{tikzcd}$$

$$\zeta^0_{K(X)}\theta^{111}_X = 1_{K(X)} = \epsilon_{K(X)}K(\eta_X)$$

□

Proof.

$$\begin{tikzcd}
G(A) \arrow[rr, "1_{G(A)}"] \arrow[ddr, "1_{G(A)}"] \arrow[ddd, "\eta_{G(A)}"'] & & G(A) \arrow[dr, "\eta_{G(A)}"] \arrow[ddd, dashed, "\eta_{G(A)}"] & & \\
& & & GKG(A) \arrow[dr, "1_{GKG(A)}"] \arrow[ddd, dashed, "\eta_{GKG(A)}"] & \\
& G(A) \arrow[rrr, "\eta_{G(A)}"] \arrow[ddd, "\eta_{G(A)}"'] & & & GKG(A) \arrow[ddd, "\eta_{GKG(A)}"] \\
GKG(A) \arrow[rr, dashed, "G(1_{KG(A)})"] \arrow[ddr, "G(1_{KG(A)})"'] & & GKG(A) \arrow[dr, "GK(\eta_{G(A)})"] & & \\
& & & GKGKG(A) \arrow[dr, "G(1_{KGKG(A)})"] & \\
& GKG(A) \arrow[rrr, "GK(\eta_{G(A)})"'] & & & GKGKG(A)
\end{tikzcd}$$

$$G(\zeta^0_A)\chi_A\omega_A = \zeta^1_{G(A)} = 1_{G(A)} = G(\epsilon_A)\eta_{G(A)}$$

□

Definition

An ionad is a set X together with a left exact comonad Int_X on the category $\mathscr{S}et^X$. An ionad is bounded if the comonad is accessible.

Example

Every ionad on a set X induces a surjective geometric morphism

$$\begin{tikzcd} \mathscr{S}et^X \arrow[d, shift left=2, "f_*"] \arrow[d, phantom, "\dashv"] \\ \Omega(X) \arrow[u, shift left=2, "f^*"] \end{tikzcd}$$

where $\Omega(X)$ is a topos of coalgebras for the comonad Int_X.

Ionads

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where $\Omega(X)$ is a topos of coalgebras for the comonad Int_X.

Endofunctors generating a cofree comonad

Definition

An endofunctor $F\colon \mathscr{A} \to \mathscr{A}$ generates a cofree comonad if

$$\begin{tikzcd} \mathscr{A} \arrow[d, shift left, "G"] \\ Coalg(F) \arrow[u, shift left, "K"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

where $K\colon Coalg(F) \to \mathscr{A}$ is the forgetful functor generating a comonad $(Q_F = KG, \epsilon, K\eta G)$. This says that $Coalg(F) \simeq \mathscr{A}^{Q_F}$.

Example

When is $Coalg(F)$ a topos?

- F is idempotent, $\mathscr{A}$ is a category with products.
- F is accessible , $\mathscr{A}$ is locally presentable.
- F is polynomial , $\mathscr{A}$ is a slice topos with NNO.

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Bases of algebras as coalgebras

For an arbitrary monad (T, η, μ) on a category $\mathscr{X}$, we cosider the free-forgetful adjunction

$$\begin{tikzcd} Alg(T) \arrow[loop above, "\bar{T}"] \arrow[d, shift left, "U"] \\ \mathscr{X} \arrow[u, shift left, "F"] \end{tikzcd}$$

from the category $Alg(T)$ of Eilenberg-Moore algebras. This adjunction induced a comonad $(\bar{T} = FU, F\eta U, \epsilon)$ on $Alg(T)$.

Definition

A basis of an algebra $(\bar{T}X \xrightarrow{x} X)$ is a $\bar{T}$-coalgebra on this algebra.

A reader comonad on category with finite products

Let us suppose that $(\mathscr{X}, \times, 1_{\mathscr{X}})$ is a category with finite products. For any object I in $\mathscr{X}$ there is a triple of functors

$$\begin{tikzcd} * \arrow[d, shift right=6, "I"'] \arrow[d, shift left=6, "\top"] \\ \mathscr{X} \arrow[u, "!", "\dashv"'] \end{tikzcd}$$

and a comonad on $\mathscr{X}$ whose underlying endofunctor is defined by

$$X \mapsto I \times X, \qquad f \mapsto I \times f$$

for any object X and any morphism $f \colon X \to Y$ in $\mathscr{X}$ respectively. The counit and the comultiplication are given by morphisms

$$\pi_2 \colon I \times X \to X \qquad \langle \pi_1, 1_{I \times X} \rangle \colon I \times X \to I \times (I \times X)$$

where π_1 and π_2 are the first and second projections respectively.

Lambek's categories with indeterminates

Theorem

Every object I in a category $(\mathscr{X}, \times, 1_{\mathscr{X}})$ with finite products induces a colax $\mathscr{D}$-coalgebra

$$\begin{tikzcd} & * \arrow[dl, "I"'] \arrow[d, phantom, ""] \\ & \mathscr{X} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] * \arrow[d, "I"'] \\ \mathscr{X} \end{tikzcd} \quad \begin{tikzcd} * \arrow[r, phantom] & \\ \mathscr{X} \arrow[u, "!"] \arrow[u, phantom, shift left=2ex, "\dashv"] \end{tikzcd} \quad \begin{tikzcd} * \arrow[d, "\top"] \\ \mathscr{X} \end{tikzcd}$$

with comonads $(C = I \times (-), \pi_2, \langle \pi_1, 1_{I \times X} \rangle)$ and $Q = I_{\mathscr{A}}$.

Theorem

Every object I in a category $(\mathscr{X}, \times, 1_{\mathscr{X}})$ with finite products induces a colax $\mathscr{D}$-coalgebra

$$\begin{tikzcd} * \arrow[d, shift right=6, "I"'] \arrow[d, shift left=3, phantom, "\dashv"] \arrow[d, shift left=6, "\top"] \\ \mathscr{X} \arrow[u, "!"] \arrow[loop below, "I\times(-)"] \end{tikzcd}$$

with comonads $(C = I \times (-), \pi_2, \langle \pi_1, 1_{I\times X}\rangle)$ and $Q = I_{\mathscr{A}}$.

Proof.

$$
\begin{tikzcd}
I \times X \arrow[rrr, "\langle \pi_1, 1_{I \times X} \rangle"] \arrow[dr, "1_{I \times X}"] \arrow[ddd, "!"'] & & & I \times (I \times X) \arrow[dr, "I \times !"] \arrow[ddd, dashed, "!"'] & & \\
& & & & I \times \top \arrow[dr, "\pi_1"] \arrow[ddd, dashed, "!"'] & \\
& I \times X \arrow[rrrr, "\pi_1"] \arrow[ddd, "!"'] & & & & I \arrow[ddd, "!"] \\
\top \arrow[rrr, dashed, "1_\top"] \arrow[ddr, "1_\top"'] & & & \top \arrow[dr, dashed, "1_\top"'] & & \\
& & & & \top \arrow[dr, dashed, "1_\top"'] & \\
& \top \arrow[rrrr, "1_\top"'] & & & & \top
\end{tikzcd}
$$

$\square$

Proof.

$$\begin{tikzcd}
I \arrow[rrr, "\Delta"] \arrow[ddrr, "1_I"] \arrow[ddd, "!"'] & & & I \times I \arrow[dr, "I \times !"] \arrow[ddd, dashed, "!"'] & & \\
& & & & I \times \top \arrow[dr, "\pi_1"] \arrow[ddd, dashed, "!"'] & \\
& & I \arrow[rrr, "1_I"] \arrow[ddd, "!"'] & & & I \arrow[ddd, "!"] \\
\top \arrow[rrr, dashed, "1_\top"] \arrow[ddrr, "1_\top"'] & & & \top \arrow[dr, dashed, "1_\top"'] & & \\
& & & & \top \arrow[dr, dashed, "1_\top"'] & \\
& & \top \arrow[rrr, "1_\top"'] & & & \top
\end{tikzcd}$$

Idempotent adjoint triples

Lemma

Let we have an idempotent adjoint triple

$$H \dashv K \dashv G$$

with $\kappa\colon I_{\mathscr{A}} \Rightarrow KH$ *and* $\lambda\colon HK \Rightarrow I_{\mathscr{X}}$ *a unit and a counit of the first adjunction respectively, and* $\eta\colon I_{\mathscr{X}} \Rightarrow GK$ *and* $\epsilon\colon KG \Rightarrow I_{\mathscr{A}}$ *of the second one. Then there exists a morphism in the category* (H, G)

$$\begin{tikzcd} HKG(A) \arrow[r, "H(\epsilon_A)"] \arrow[d, "\lambda_{G(A)}"'] & H(A) \arrow[d, "\eta_{H(A)}"] \\ G(A) \arrow[r, "G(\kappa_A)"'] & GKH(A) \end{tikzcd}$$

Fully faithful adjoint triples

Theorem

Every fully faithful adjoint triple

$$H \dashv K \dashv G$$

induces a colax $\mathscr{D}$*-coalgebra with* $(C = HK, \lambda, H\kappa K)$ *and* $Q = I_{\mathscr{A}}$.

Proof.

Consider the diagonal of the following commutative diagram

$$
\begin{tikzcd}
HKG(A) \arrow[rr, "H(\epsilon_A)"] \arrow[dr, "H(\epsilon^{-1}_{KG(A)})"'] \arrow[dd, "\eta_{HKG(A)}"'] \arrow[ddr, dotted, "\varrho_{G(A)}"] & & H(A) \arrow[dr, "H(\epsilon_A^{-1})"] \arrow[dd, dashed, "\eta_{H(A)}"'] \arrow[ddr, dotted, "\chi_A"] & \\
& HKGKG(A) \arrow[rr, "HKG(\epsilon_A)"'] \arrow[dd, "\lambda_{GKG(A)}"] & & HKG(A) \arrow[dd, "\lambda_{G(A)}"] \\
GKHKG(A) \arrow[rr, dashed, "GKH(\epsilon_A)"] \arrow[dr, "G(\kappa^{-1}_{KG(A)})"'] & & GKH(A) \arrow[dr, "G(\kappa_A^{-1})"] & \\
& GKG(A) \arrow[rr, "G(\epsilon_A)"'] & & G(A)
\end{tikzcd}
$$

$$\omega_A = H(\epsilon_A), \quad \varepsilon_A = \epsilon_A$$

Proof.

The proof follows from the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\kappa_A"] \arrow[d, "\kappa_A"'] & KH(A) \arrow[r, "KH(\epsilon_A^{-1})"] \arrow[d, "K(\eta_{H(A)})"] & KHKG(A) \arrow[d, "K(\lambda_{G(A)})"] \\
KH(A) \arrow[r, "K(\eta_{H(A)})"'] \arrow[d, "1_{KH(A)}"'] & KGKH(A) \arrow[r, "KG(\kappa_A^{-1})"'] \arrow[d, "\epsilon_{KH(A)}"] & KG(A) \arrow[d, "\epsilon_A"] \\
KH(A) \arrow[r, "1_{KH(A)}"'] & KH(A) \arrow[r, "\kappa_A^{-1}"'] & A
\end{tikzcd}$$

$\square$

Proof.

$$
\begin{tikzcd}
HK(X) \arrow[rr, "H(\kappa_{K(X)})"] \arrow[dddd, "\chi_{K(X)}"'] \arrow[dddr, "H(1_{K(X)})"] & & HKHK(X) \arrow[dr, "HKH(\epsilon^{-1}_{K(X)})"] \arrow[dddd, dashed, "\chi_{KHK(X)}"'] & & & \\
& & & HKHKGK(X) \arrow[dr, "HK(\lambda_{GK(X)})"] \arrow[dddd, dashed, "\chi_{KHKGK(X)}"] & & \\
& & & & HKGK(X) \arrow[dr, "H(\varepsilon_{K(X)})"] \arrow[dddd, dashed, "\chi_{KGK(X)}"] & \\
& HK(X) \arrow[rrrr, "H(1_{K(X)})"] \arrow[dddd, "\chi_{K(X)}"'] & & & & HK(X) \arrow[dddd, "\chi_{K(X)}"] \\
GK(X) \arrow[rr, dashed, "G(\kappa_{K(X)})"] \arrow[dddr, "G(1_{K(X)})"'] & & GKHK(X) \arrow[dr, "GKH(\epsilon^{-1}_{K(X)})"'] & & & \\
& & & GKHKGK(X) \arrow[dr, "GK(\lambda_{GK(X)})"'] & & \\
& & & & GKGK(X) \arrow[dr, "G(\epsilon_{K(X)})"] & \\
& GK(X) \arrow[rrrr, "G(1_{K(X)})"'] & & & & GK(X)
\end{tikzcd}
$$

□

Fully faithful adjoint triples

Proof.

$$
\begin{tikzcd}
H(A) \arrow[rr, "H(\kappa_A)"] \arrow[dddr, "H(1_A)"] \arrow[dddd, "\chi_A"'] & & HKH(A) \arrow[dr, "HKH(\epsilon_A^{-1})"] \arrow[dddd, dashed, "\chi_{KH(A)}"'] & & & \\
& & & HKHKG(A) \arrow[dr, "HK(\lambda_{G(A)})"] \arrow[dddd, dashed, "\chi_{KHKG(A)}"'] & & \\
& & & & HKG(A) \arrow[dr, "H(\varepsilon_A)"] \arrow[dddd, dashed, "\chi_{KG(A)}"] & \\
& H(A) \arrow[rrrr, "H(1_A)"] \arrow[dddd, "\chi_A"'] & & & & H(A) \arrow[dddd, "\chi_A"] \\
G(A) \arrow[rr, dashed, "G(\kappa_A)"] \arrow[dddr, "G(1_A)"'] & & GKH(A) \arrow[dr, "GKH(\epsilon_A^{-1})"'] & & & \\
& & & GKHKG(A) \arrow[dr, "GK(\lambda_{G(A)})"'] & & \\
& & & & GKG(A) \arrow[dr, "G(\epsilon_A)"] & \\
& G(A) \arrow[rrrr, "G(1_A)"'] & & & & G(A)
\end{tikzcd}
$$

$\square$

A comonad inducing a simple fibration

Let us suppose that $(\mathscr{X}, \times, 1_{\mathscr{X}})$ is a category with finite products. There is a triple of functors

$$\begin{tikzcd}[column sep=small] Pt(\mathscr{C}) \arrow[d, shift right=6, "H"'] \arrow[d, phantom, shift right=3, "\dashv"] \arrow[d, phantom, shift left=3, "\dashv"] \arrow[d, shift left=6, "G"] \\ \mathscr{X} \times \mathscr{X} \arrow[u] \end{tikzcd}$$

from the fibration of points $Pt(\mathscr{C})$ over $\mathscr{X}$ defined by

$$H\left(I \underset{p}{\overset{s}{\rightleftarrows}} X\right) = (I, X), K(X, Y) = \left(X \underset{1_X}{\overset{1_X}{\rightleftarrows}} X\right), G\left(I \underset{p}{\overset{s}{\rightleftarrows}} X\right) = (X, I)$$

for any object $\left(I \underset{p}{\overset{s}{\rightleftarrows}} X\right)$ in $Pt(\mathscr{X})$ (such that $ps = 1_I$). There exists a comonad on $\mathscr{X} \times \mathscr{X}$ whose underlying endofunctor C is

$$(I, X) \mapsto (I, I \times X), \qquad (u, f) \mapsto (u, u \times f)$$

Fibration of points

Lemma

Let us suppose that $(\mathscr{X}, \times, 1_{\mathscr{X}})$ is a category with finite products. Then there exists a fibered comonad Q on $Pt(\mathscr{X})$ whose counit and comultiplication are given by the diagrams

$$
\begin{tikzcd}
I \times X \arrow[r, "\pi_2"] \arrow[d, shift right, "1_I \times p"'] & X \arrow[d, shift right, "p"'] \\
I \times I \arrow[r, "\pi_2"'] \arrow[u, shift right, "1_I \times s"'] & I \arrow[u, shift right, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I \times X \arrow[r, "\Delta_I \times \langle p, 1_X \rangle"] \arrow[d, shift right, "1_I \times p"'] & (I \times I) \times (I \times X) \arrow[d, shift right, "(1_I \times 1_I) \times (1_I \times p)"'] \\
I \times I \arrow[r, "\Delta_I \times \Delta_I"'] \arrow[u, shift right, "1_I \times s"'] & (I \times I) \times (I \times I) \arrow[u, shift right, "(1_I \times 1_I) \times (1_I \times s)"']
\end{tikzcd}
$$

Fibration of points and a simple fibration

Theorem

Any category $(\mathscr{X}, \times, 1_{\mathscr{X}})$ *with finite products induces a colax* $\mathscr{D}$*-coalgebra on a triple of functors*

$$\begin{tikzcd}[row sep=huge]
Pt(\mathscr{C}) \arrow[loop above, "Q"] \arrow[d, shift right=5, "H"'] \arrow[d, shift left=5, "G"] \\
\mathscr{X} \times \mathscr{X} \arrow[u] \arrow[loop below, "C"]
\end{tikzcd}$$

$$H \dashv \cdot \dashv G$$

with respects to the two comonads C *and* Q *just defined.*

Definition

Let us suppose that we have a fully faithful adjoint quadruple

$$L \dashv H \dashv K \dashv G$$

such that the units $\chi \colon I_{\mathscr{X}} \Rightarrow HL$ and $\eta \colon I_{\mathscr{X}} \Rightarrow GK$ of the first and the last adjunction respectively, and the counit $\lambda \colon HK \Rightarrow I_{\mathscr{X}}$ of the second one are natural isomorphisms. We say that the adjoint quadruple is *distributive* if for the induced adjoint triple

$$LH \dashv KH \dashv KG$$

of the comonad LH left adjoint to the monad KH left adjoint to the comonad KG there exists a distributive law

$$\lambda \colon KGLH \Rightarrow LHKG$$

Distributive laws between idempotent comonads

Lemma

The natural transformation $\lambda\colon KGLH \Rightarrow LHKG$ is a distributive law of (KG, ϵ) over (LH, μ) if and only if the following diagram

$$\begin{tikzcd}
 & KG(A) & \\
KGLH(A) \arrow[ur, "KG(\mu_A)"] \arrow[rr, "\lambda_A"'] \arrow[dr, "\epsilon_{LH(A)}"'] & & LHKG(A) \arrow[ul, "\mu_{KG(A)}"'] \arrow[dl, "LH(\epsilon_A)"] \\
 & LH(A) &
\end{tikzcd}$$

commutes.

Lemma

For any adjoint quadruple the natural transformation $\pi \colon GL \Rightarrow I_{\mathscr{X}}$ whose component indexed by X in $\mathscr{X}$ is defined by a diagram

$$\begin{tikzcd}
GL(X) \arrow[rr, "\lambda^{-1}_{GL(X)}"] \arrow[dr, "GL(\lambda^{-1}_X)"] \arrow[dd, "G(\kappa_{L(X)})"'] & & HKGL(X) \arrow[dd, "H(\epsilon_{L(X)})"] & \\
& GLHK(X) \arrow[dd, "G(\mu_{K(X)})"'] & & \\
GKHL(X) \arrow[rr, dashed, "\eta^{-1}_{HL(X)}"] \arrow[dr, "GK(\chi^{-1}_K)"'] & & HL(X) \arrow[dr] & \\
& GK(X) \arrow[rr, "\chi^{-1}_X", "\eta^{-1}_X"'] & & X
\end{tikzcd}$$

is well-augmented in the sense that $\pi GL = GL\pi$.

A comonad induced by a distributive adjoint quadruple

Theorem

The natural transformation $\pi\colon GL \Rightarrow I_{\mathscr{X}}$ underlies an idempotent comonad (GL, π) if and only if

$$L \dashv H \dashv K \dashv G$$

is distributive.

Proof.

The component of the comultiplication is given by a diagram

$$
\begin{tikzcd}
GL(X) \arrow[rrr, "\eta_{GL(X)}"] \arrow[dr, "GL(\chi_X)"] \arrow[ddd, "\eta_{GL(X)}"'] & & & GKGL(X) \arrow[dr, "GKGL(\chi_X)"] \arrow[ddd, dashed, "\eta_{GLHL(X)}"' near end] & & \\
& GLHL(X) \arrow[rrr, "\eta_{GLHL(X)}"] \arrow[dr, "G(\epsilon_{L(X)})"] \arrow[ddd, "\eta_{GLHL(X)}"'] & & & GKGLHL(X) \arrow[dr, "G(\alpha_{L(X)})"] \arrow[ddd, "G(\alpha_{L(X)})"] & \\
& & GKGLHL(X) \arrow[rrr, "G(\alpha_{L(X)})"] \arrow[ddd, "G(\alpha_{L(X)})"] & & & GLHKGL(X) \arrow[ddd, "GL(\lambda_{GL(X)})"] \\
GKGL(X) \arrow[rrr, dashed, "GLH(\epsilon_{L(X)})"] \arrow[dr, "GKGL(\chi_X)"'] & & & GKGLHL(X) \arrow[dr] & & \\
& GKGLHL(X) \arrow[rrr, "G(\alpha_{L(X)})"] \arrow[dr, "G(\alpha_{L(X)})"'] & & & GLHKGL(X) \arrow[dr, "GL(\lambda_{GL(X)})"] & \\
& & GLHKGL(X) \arrow[rrr, "GL(\lambda_{GL(X)})"'] & & & GLGL(X)
\end{tikzcd}
$$

$\square$

A splitting of a distributive adjoint quadruple

Theorem

For a distributive adjoint quadruple the following is equivalent:

(1) *a composition* $LH(A) \xrightarrow{\mu_A} A \xrightarrow{\kappa_A} KH(A)$ *is an isomorphism.*

A splitting of a distributive adjoint quadruple

Theorem

For a distributive adjoint quadruple the following is equivalent:

(1) *a composition* $LH(A) \xrightarrow{\mu_A} A \xrightarrow{\kappa_A} KH(A)$ *is an isomorphism.*

(2) *the natural transformation* $\sigma H \colon LH \Rightarrow KH$ *is an isomorphism.*

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(2) *the natural transformation* $\sigma H \colon LH \Rightarrow KH$ *is an isomorphism.*

(3) *for any object* A *in* $\mathscr{A}$, $H(A)$ *is a* GL*-coalgebra.*

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(2) *the natural transformation* $\sigma H \colon LH \Rightarrow KH$ *is an isomorphism.*

(3) *for any object* A *in* $\mathscr{A}$, $H(A)$ *is a* GL*-coalgebra.*

(4) *the natural transformation* $\beta \colon H \Rightarrow G$ *whose component indexed by an object* A *in* $\mathscr{A}$ *is defined by*

$$\beta_A := G(\mu_A)\pi^{-1}_{H(A)}$$

splits the natural transformation $\tau \colon G \Rightarrow H$.

A splitting of a distributive adjoint quadruple

Theorem

For a distributive adjoint quadruple the following is equivalent:

1. *a composition* $LH(A) \xrightarrow{\mu_A} A \xrightarrow{\kappa_A} KH(A)$ *is an isomorphism.*
2. *the natural transformation* $\sigma H \colon LH \Rightarrow KH$ *is an isomorphism.*
3. *for any object* A *in* $\mathscr{A}$, $H(A)$ *is a* GL*-coalgebra.*
4. *the natural transformation* $\beta \colon H \Rightarrow G$ *whose component indexed by an object* A *in* $\mathscr{A}$ *is defined by*

$$\beta_A := G(\mu_A)\pi^{-1}_{H(A)}$$

   splits the natural transformation $\tau \colon G \Rightarrow H$.

5. *there exists a distributive law* $\rho \colon KGKH \Rightarrow KHKG$ *of the comonad* KG *over a monad* KH.

Split distributive adjoint quadruples

Definition

Let us suppose that we have a fully faithful adjoint quadruple. We say that the distributive adjoint quadruple

$$L \dashv H \dashv K \dashv G$$

is *split* if it satisfies equivalent conditions of the previous Theorem.

Theorem

Let us suppose that we have a split distributive adjoint quadruple

$$L \dashv H \dashv K \dashv G$$

Then H, K and G together with comonads $(C = GL, \pi)$ and $(Q = KG, \epsilon)$ determine a colax $\mathscr{D}$-coalgebra.

$$
\begin{tikzcd}
\mathscr{A} \arrow[rr, equal] \arrow[dr, "{(H,\chi,I_{\mathscr{A}})}"'] \arrow[dd, "G"'] & & \mathscr{A} \arrow[dr, equal] \arrow[dd, dashed, "G"] & \\
& (\mathscr{X}, G) \arrow[rr, "\boldsymbol{d}_0"] \arrow[dd, equal] & & \mathscr{A} \arrow[dd, "G"] \\
\mathscr{X} \arrow[rr, equal, dashed] \arrow[dr, "{(I_{\mathscr{X}},\varrho,K)}"'] \arrow[ur, phantom, "{(\omega,\varepsilon)\Nearrow}"] & & \mathscr{X} \arrow[dr, equal] & \\
& (\mathscr{X}, G) \arrow[rr, "\boldsymbol{d}_1"'] \arrow[ur, phantom, "\delta_G\Nearrow"] & & \mathscr{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
G(A) \arrow[r, "\omega_A"] \arrow[d, "\eta_{G(A)}"'] & H(A) \arrow[d, "\chi_A"] \\
GKG(A) \arrow[r, "G(\epsilon_A)"'] & GQ(A)
\end{tikzcd}
$$

$$\chi_A^{-1} = \omega_A$$

Coherence conditions for normal colax $\mathscr{D}$-coalgebras

$$
\begin{tikzcd}
\mathscr{A} \arrow[rr, equal] \arrow[dr, "{(H,\chi,I_{\mathscr{A}})}"'] \arrow[dd, "G"'] \arrow[ddr, phantom, "{(\omega,\varepsilon)}\Nearrow"] & & \mathscr{A} \arrow[dr, equal] \arrow[dd, dashed, "G"] & \\
& (\mathscr{X}, G) \arrow[rr, "\boldsymbol{d}_0"] \arrow[dd, equal] & & \mathscr{A} \arrow[dd, "G"] \\
\mathscr{X} \arrow[rr, dashed, equal] \arrow[dr, "{(I_{\mathscr{X}},\varrho,K)}"'] & & \mathscr{X} \arrow[dr, equal] \arrow[ul, phantom, "\delta_G \Nearrow"] & \\
& (\mathscr{X}, G) \arrow[rr, "\boldsymbol{d}_1"'] & & \mathscr{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
G(A) \arrow[r, "\omega_A", "\simeq"'] \arrow[d, "\eta_{G(A)}"'] \arrow[dr, "1_{G(A)}"] & H(A) \arrow[d, "\chi_A", "\simeq"'] \\
GKG(A) \arrow[r, "G(\epsilon_A)"'] & GQ(A)
\end{tikzcd}
$$

$$\chi_A^{-1} = \omega_A$$

Theorem

The $\boldsymbol{\theta}$-components of a normal colax $\mathscr{D}$-coalgebra $(G, \boldsymbol{F}_G, \boldsymbol{\theta})$ are

$$
\begin{tikzcd}
X \arrow[rrr, "1_X"] \arrow[dddd, "\varrho_X"'] \arrow[dddrr, "1_X"] & & & X \arrow[dr, "\varrho_X"] \arrow[dddd, dashed, "\varrho_X"'] & & & \\
& & & & GK(X) \arrow[dr, "\theta^{101}_{GK(X)}"] \arrow[dddd, dashed, "\varrho_{GK(X)}"] & & \\
& & & & & HKGK(X) \arrow[dr, "H(\varepsilon_{K(X)})"] \arrow[dddd, dashed, "\chi_{KGK(X)}"] & \\
& & X \arrow[rrrr, "\theta^{101}_X"] \arrow[dddd, "\varrho_X"'] & & & & HK(X) \arrow[dddd, "\chi_{K(X)}"] \\
GK(X) \arrow[rrr, dashed, "G(1_{K(X)})"] \arrow[dddrr, "G(1_{K(X)})"'] & & & GK(X) \arrow[dr, "GK(\varrho_X)"] & & & \\
& & & & GKGK(X) \arrow[dr, "G(1_{KGK(X)})"] & & \\
& & & & & GKGK(X) \arrow[dr, "G(\varepsilon_{K(X)})"] & \\
& & GK(X) \arrow[rrrr, "G(1_{K(X)})"'] & & & & GK(X)
\end{tikzcd}
$$

Theorem

and

$$
\begin{tikzcd}
H(A) \arrow[rrrr, "1_{H(A)}"] \arrow[dddd, "\chi_A"'] \arrow[dddrrr, "1_{H(A)}"] & & & & H(A) \arrow[dr, "\chi_A"] \arrow[dddd, dashed, "\varrho_{H(A)}"'] & & & \\
& & & & & G(A) \arrow[dr, "\theta^{101}_{G(A)}"] \arrow[dddd, dashed, "\varrho_{G(A)}"] & & \\
& & & & & & HKG(A) \arrow[dr, "H(\varepsilon_A)"] \arrow[dddd, dashed, "\chi_{KG(A)}"] & \\
& & & H(A) \arrow[rrrr, "1_{H(A)}"] \arrow[dddd, "\chi_A"'] & & & & H(A) \arrow[dddd, "\chi_A"] \\
G(A) \arrow[rrrr, dashed, "G(\theta^{010}_A)"] \arrow[dddrrr, "G(1_A)"'] & & & & GKH(A) \arrow[dr, "GK(\chi_A)"'] & & & \\
& & & & & GKG(A) \arrow[dr, "G(1_{KG(A)})"'] & & \\
& & & & & & GKG(A) \arrow[dr, "G(\varepsilon_A)"'] & \\
& & & G(A) \arrow[rrrr, "G(1_A)"'] & & & & G(A)
\end{tikzcd}
$$

Example

By taking $G = H$, and $\chi = \iota_G = \omega$ we obtain what Morita called a strongly adjoint pair consisting of an adjoint triple

$$G \dashv K \dashv G$$

where G is simultaneously left and right adjoint of K.

Example

By keeping ω and χ as mutually invertible natural transformations we end up with an ambidextrous adjunction

$$H \dashv K \dashv G$$

(or sometimes *ambiadjunction* for short) which were pivotal in the work of Lauda who showed that every Frobenius object M in a monoidal category $\mathscr{M}$ arises from an ambiadjunction in some 2-category $\mathscr{D}$ into which M fully and faithfully embeds. This result shows that every 2D TQFT is obtained from an ambiadjunction in some 2-category since every 2D topological quantum field theory is equivalent to a commutative Frobenius algebra.

Lawvere's quality types

Definition

A fully faithful functor $q^* \colon \mathscr{X} \to \mathscr{A}$ between extensive categories which is both reflective and coreflective by a single functor $q_! = q_*$

$$q_! \dashv q^* \dashv q_*$$

makes $\mathscr{A}$ a quality type over $\mathscr{X}$.

Example

Every quality type is a normal colax $\mathscr{D}$-coalgebra.

Normal pseudo $\mathscr{D}$-coalgebras

Theorem

Every normal pseudo $\mathscr{D}$-coalgebra is an adjoint equivalence.

Lemma

Every geometric morphism $f : \mathscr{E} \to \mathscr{S}et$ which is either

- *localic*
- *groupoidal*
- *petit étale*
- *étendue*
- *locally separable*

then it is not a quality type unless $f : \mathscr{E} \to \mathscr{S}et$ is an equivalence, in which case is an example of a normal pseudo $\mathscr{D}$-coalgebra.

Definition

A cartesian closed extensive category $\mathscr{A}$ is a category of cohesion relative to another such category $\mathscr{X}$ if it is equipped with an adjoint string of four functors

$$p_! \dashv p^* \dashv p_* \dashv p^!$$

having the further properties:

(a) $p_!$ preserves finite products and $p^!$ is fully faithful.

(b) $p_!$ preserves $\mathscr{X}$-parameterized powers in the sense that

$$p_!(A^{p^*(X)}) = p_!(A)^X$$

is a natural isomorphism for all X in $\mathscr{X}$ and A in $\mathscr{A}$.

(c) The canonical natural transformation $\tau \colon p_* \Rightarrow p_!$ is epi.

Theorem

There exists a canonical distributive law $\lambda\colon \mathscr{D}\mathscr{F} \Rightarrow \mathscr{F}\mathscr{D}$

$$\begin{tikzcd}
(\mathscr{X}, \mathscr{F}(G)) \arrow[r, "\lambda_G"] \arrow[d, equal, "\mathscr{D}\mathscr{F}(G)"'] & (\mathscr{X}, G)^2 \arrow[d, "\mathscr{F}\mathscr{D}(G)"] \\
(\mathscr{X}, \mathscr{F}(G)) \arrow[r, "\mathscr{F}\mathscr{D}(G)\lambda_G"'] & (\mathscr{X}, G)
\end{tikzcd}$$

where $\lambda_G\colon (\mathscr{X}, (\mathscr{X}, G)) \to (\mathscr{X}, G)^2$ *sends* (X, x, Y, f, A) *to*

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "fx"'] & Y \arrow[d, "f"] \\
G(A) \arrow[r, "G(1_A)"'] & G(A)
\end{tikzcd}$$

Theorem

$$\Omega_1 \colon \lambda \odot \mathscr{D}(N) \Rrightarrow N\mathscr{D}$$

$$\begin{tikzcd}
(\mathscr{X}, G) \arrow[rr, equal] \arrow[dr, "D(N_G)"] \arrow[dd, equal, "\mathscr{D}(G)"'] & & (\mathscr{X}, G) \arrow[dr, "N_{\mathscr{D}(G)}"] \arrow[dd, equal, dashed] \arrow[dl, phantom, "\Downarrow\Omega_1"] & \\
& (\mathscr{X}, \mathscr{F}(G)) \arrow[rr, "L^0_G"'] \arrow[dd, equal, "\mathscr{D}\mathscr{F}(G)"] & & (\mathscr{X}, G)^2 \arrow[dd, "\mathscr{F}\mathscr{D}(G)=\boldsymbol{d}_1", "\mathscr{D}(G)"'] \\
(\mathscr{X}, G) \arrow[rr, equal, dashed] \arrow[dr, "D(N_G)"'] & & (X, G) \arrow[dr, equal] & \\
& (\mathscr{X}, \mathscr{F}(G)) \arrow[rr, "\mu_G"'] & & (\mathscr{X}, G)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "1_X"] \arrow[dr, "1_X"] \arrow[dd, "f"'] & & X \arrow[dr, "f"] \arrow[dd, dashed, "f"'] & \\
& X \arrow[rr, "f"] \arrow[dd, "f"] & & G(A) \arrow[dd, "1_{G(A)}"] \\
G(A) \arrow[rr, dashed, "G(1_A)"] \arrow[dr, "G(1_A)"'] & & G(A) \arrow[dr, "G(1_A)"] & \\
& G(A) \arrow[rr, "G(1_A)"'] & & G(A)
\end{tikzcd}$$

A canonical distributive law of $\mathscr{D}$ over $\mathscr{F}$

Theorem

$$\Omega_2 \colon \lambda \odot \mathscr{D}(N) \Rrightarrow N\mathscr{D}$$

$$\begin{tikzcd}
(\mathscr{X}, \mathscr{F}(G)) \arrow[rr, "\lambda^0_G"] \arrow[dr, "\boldsymbol{d}_0"] \arrow[dd, equal, "\mathscr{D}\mathscr{F}(G)"'] \arrow[ddr, phantom, "\delta_{\mathscr{F}(G)} \Rrightarrow"] & & (\mathscr{X}, G)^2 \arrow[dr, "\mathscr{F}(\boldsymbol{\delta}_G)"] \arrow[dd, dashed, "\mathscr{F}\mathscr{D}(G)=\boldsymbol{d}_1"] \arrow[dl, phantom, "\Downarrow\Omega_2"] & \\
& (\mathscr{X}, G) \arrow[rr, equal] \arrow[dd, "\mathscr{F}(G)=\boldsymbol{d}_1"] & & (\mathscr{X}, G) \arrow[dd, "\mathscr{F}(G)=\boldsymbol{d}_1"] \\
(\mathscr{X}, \mathscr{F}(G)) \arrow[rr, dashed, "\lambda^1_G"] \arrow[dr, "\boldsymbol{d}_1\boldsymbol{d}_1"'] & & (X, G) \arrow[dr, "\boldsymbol{d}_1"'] & \\
& \mathscr{X} \arrow[rr, equal] & & \mathscr{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "gx"'] & Y \arrow[d, "g"] \\
G(A) \arrow[r, "G(1_A)"'] & G(A)
\end{tikzcd}$$

A canonical distributive law of $\mathscr{D}$ over $\mathscr{F}$

Theorem

$$\Omega_3 \colon \mu\mathscr{D} \odot \mathscr{F}\lambda \odot \lambda\mathscr{F} \Rrightarrow \lambda \odot \mathscr{D}\mu$$

$$\begin{tikzcd}
(\mathscr{X},(\mathscr{X},\mathscr{F}(G))) \arrow[rr, "\lambda^1_{\mathscr{F}(G)}"] \arrow[dr, "D(\mu_G)"] \arrow[dd, equal, "\mathscr{D}\mathscr{F}\mathscr{F}(G)"'] & & (\mathscr{X},(\mathscr{X},G))^2 \arrow[r, "\mathscr{F}(\lambda^1_G)"] \arrow[dd, dashed, "\mathscr{F}\mathscr{D}\mathscr{F}(G)"'] \arrow[r, phantom, shift right=6, "\Omega_3 \Nearrow"] & ((\mathscr{X},G)^2)^2 \arrow[dr, "\mu_{\mathscr{D}(G)}"] \arrow[dd, dashed, "\mathscr{F}\mathscr{F}\mathscr{D}(G)"'] & \\
& (\mathscr{X},(\mathscr{X},G)) \arrow[rrr, "\lambda^1_G"'] \arrow[dd, "\mathscr{D}\mathscr{F}(G)"] & & & (\mathscr{X},G)^2 \arrow[dd, "\mathscr{F}\mathscr{D}(G)"] \\
(\mathscr{X},(\mathscr{X},\mathscr{F}(G))) \arrow[rr, dashed, "\mu_{\mathscr{F}(G)}"] \arrow[dr, "D(\mu_G)"'] & & (\mathscr{X},(\mathscr{X},G)) \arrow[r, dashed, "D(\delta_G)"] & (X,G) \arrow[dr, equal] & \\
& (\mathscr{X},\mathscr{F}(G)) \arrow[rrr, "\mu_G"'] & & & (\mathscr{X},G)
\end{tikzcd}$$

A canonical distributive law of $\mathscr{D}$ over $\mathscr{F}$

Theorem

$$\Omega_3 \colon \mu\mathscr{D} \odot \mathscr{F}\lambda \odot \lambda\mathscr{F} \Rrightarrow \lambda \odot \mathscr{D}\mu$$

$$\begin{tikzcd}
(\mathscr{X},(\mathscr{X},\mathscr{F}(G))) \arrow[rr, "\lambda^1_{\mathscr{F}(G)}"] \arrow[dr, "D(\mu_G)"] \arrow[dd, equals, "\mathscr{D}\mathscr{F}\mathscr{F}(G)" swap] & & (\mathscr{X},(\mathscr{X},G))^2 \arrow[r, "\mathscr{F}(\lambda^1_G)"] \arrow[dd, dashed, "\mathscr{F}\mathscr{D}\mathscr{F}(G)" swap] \arrow[dr, phantom, "\Omega_3 \Nearrow"] & ((\mathscr{X},G)^2)^2 \arrow[dr, "\mu_{\mathscr{D}(G)}"] \arrow[dd, dashed, "\mathscr{F}\mathscr{F}\mathscr{D}(G)" swap] & \\
& (\mathscr{X},(\mathscr{X},G)) \arrow[rrr, "\lambda^1_G" swap] \arrow[dd, "\mathscr{D}\mathscr{F}(G)"] & & & (\mathscr{X},G)^2 \arrow[dd, "\mathscr{F}\mathscr{D}(G)"] \\
(\mathscr{X},(\mathscr{X},\mathscr{F}(G))) \arrow[rr, dashed, "\mu_{\mathscr{F}(G)}"] \arrow[dr, "D(\mu_G)" swap] & & (\mathscr{X},(\mathscr{X},G)) \arrow[r, dashed, "D(\delta_G)"] & (X,G) \arrow[dr, equals] & \\
& (\mathscr{X},\mathscr{F}(G)) \arrow[rrr, "\mu_G" swap] & & & (\mathscr{X},G)
\end{tikzcd}$$

A canonical distributive law of $\mathscr{D}$ over $\mathscr{F}$

Theorem

$$
\begin{tikzcd}
(\mathscr{X},\mathscr{F}(G)) \arrow[rrr, "\lambda_G"] \arrow[dr, "\boldsymbol{\nu}_{(\mathscr{X},\mathscr{F}(G))}"] \arrow[dd, equal, "\mathscr{D}\mathscr{F}(G)"'] & & & (\mathscr{X},G)^2 \arrow[dr, "D(\boldsymbol{\nu}_{\mathscr{D}(G)})"] \arrow[dd, dashed, "\mathscr{F}\mathscr{D}(G)"'] & \\
& (\mathscr{X},\mathscr{F}(G))^2 \arrow[r, "D(\lambda_G,\mu_G)"'] \arrow[dd, equal, "\mathscr{D}\mathscr{D}\mathscr{F}(G)"] & ((\mathscr{X},G),(\mathscr{X},G)^2) \arrow[rr, "\lambda_{\mathscr{D}(G)}"'] \arrow[dd, equal, "\mathscr{D}\mathscr{F}\mathscr{D}(G)"'] & & ((\mathscr{X},G)^2)^2 \arrow[dd, "\mathscr{F}\mathscr{D}\mathscr{D}(G)"] \\
(\mathscr{X},\mathscr{F}(G)) \arrow[rrr, dashed, "\mu_G"'] \arrow[dr, "\boldsymbol{\nu}_{(\mathscr{X},\mathscr{F}(G))}"'] & & & (\mathscr{X},G) \arrow[dr, "\boldsymbol{\nu}_{\mathscr{D}(G)}"'] & \\
& (\mathscr{X},\mathscr{F}(G))^2 \arrow[r, "D(\lambda_G,\mu_G)"'] & ((\mathscr{X},G),(\mathscr{X},G)^2) \arrow[rr, "\mu_{\mathscr{D}(G)}"'] & & (\mathscr{X},G)^2
\end{tikzcd}
$$

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