Comma categories and 2-(co)monads in foundations and theoretical computer science

Transactions in Category Theory 2025

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Formal category theory

The pillars of formal category theory

1) Representability
2) Coherence
3) Duality

Interplay between central notions of formal category theory

- Internalization -*internal* categories

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- Enhancement - *enhanced* categories
- Shape - *double, triple, multi*-categories

The unknown role of associated fibrations in foundations

The associated (co)fibrations (co)lax monads are fundamental to:

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- generalized multi-categories

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- (lax, local) adjunctions

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- (lax) (co)limits
- (relative) (co)lax-2-(co)monads

Bénabou's theory of generalized fibrations

Bénabou's theory of cartesian functors

Definition

Consider a diagram

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "F"] \arrow[dr, "P"'] & & \mathscr{D} \arrow[dl, "Q"] \\ & \mathscr{B} & \end{tikzcd}$$

where P is a prefoliation and Q an arbitrary functor. We say that F is a cartesian functor if the following conditions are satisfied:

(i) It preserves cartesian maps, i.e. $k \in K(P) \Rightarrow Fk \in K(P')$.

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(ii) $\forall f' : Y' \to F(X)$ in $\mathscr{D}$, $\exists f : Y \to X$ in $\mathscr{E}$ and $v : Y' \to F(Y)$

$$\begin{tikzcd} Y' \arrow[d, "v"'] \arrow[dr, "f'"] & \\ F(Y) \arrow[r, "F(f)"'] & F(X) \end{tikzcd}$$

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(1) *F is faithful iff every F_B is.*

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3. *F is essentially surjective iff every F_B is.*
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(5) *F is flat iff every F_B is.*

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4. *F is final iff every F_B is.*
5. *F is flat iff every F_B is.*
6. *F has a left adjoint iff every F_B has.*

Extension of the definition of the associated split fibration

We consider functors as *generalized fibrations* (following Bénabou)

1) from the 2-category $(\mathscr{C}at, \mathscr{B})$ whose 1-cells are triangles

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "F"] \arrow[dr, "P"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & \mathscr{D} \arrow[dl, "Q"] \\ & \mathscr{B} & \end{tikzcd}$$

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2) to the 2-category $\mathscr{C}at^2_c$ whose 1-cells are colax squares

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\beta \Nearrow"] & \mathscr{D} \arrow[d, "Q"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd}$$

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3) ultimately to the double category $\mathbb{C}at^2$ whose horizontal (vertical) cells are (co)lax squares.

Comma 2-comonad

The basic scenery

$\mathscr{C}at^2_c$ is a 2-category: objects are functors, 1-cells are colax squares

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "S"'] \arrow[dr, phantom, "\phi \Nearrow"] & \mathscr{E} \arrow[d, "G"] \\
\mathscr{B} \arrow[r, "V"'] & \mathscr{X}
\end{tikzcd}$$

2-cells are cylinders

$$\begin{tikzcd}
\mathscr{E} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "U"'] \arrow[dr, phantom, "\phi \Nearrow"] & \mathscr{A} \arrow[d, "G"] \\
\mathscr{B} \arrow[r, bend left, dashed, "V"] \arrow[r, bend right, "V'"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathscr{X}
\end{tikzcd}$$

The basic 2-adjunction

There is a canonical 2-functor

$$I\colon \mathscr{C}at \to \mathscr{C}at^2_c$$

which sends a category $\mathscr{B}$ to the identity functor $I_{\mathscr{B}}\colon \mathscr{B} \to \mathscr{B}$.

Theorem

There exists a strict 2-adjunction

$$I \dashv D$$

where $D\colon \mathscr{C}at^2_c \to \mathscr{C}at$ sends any functor $U\colon \mathscr{A} \to \mathscr{X}$ to its comma category $(\mathscr{X}, U)$.

Comma 2-comonad

Theorem

There is a strict 2-comonad $(\mathscr{D}, E, C)$ on the 2-category $\mathscr{C}at_c^2$ whose underlying 2-functor

$$\mathscr{D} \colon \mathscr{C}at_c^2 \to \mathscr{C}at_c^2$$

is a composition $\mathscr{D} := ID$ of the pair of adjoint 2-functors.

Colax $\mathscr{D}$-coalgebras

Definition

A colax $\mathscr{D}$-coalgebra consists of the following data:

1) a 1-cell $\boldsymbol{F}_G = (\boldsymbol{F}_1, \varphi, \boldsymbol{F_0})\colon G \to \mathscr{D}(G)$

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2) a 2-cell $\zeta \colon \iota_G \Rightarrow \boldsymbol{\delta}_G \boldsymbol{F}_G$

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2) a 2-cell $\zeta\colon \iota_G \Rightarrow \boldsymbol{\delta}_G \boldsymbol{F}_G$
3) a 2-cell $\boldsymbol{\theta}\colon \mathscr{D}(\boldsymbol{F}_G)\boldsymbol{F}_G \Rightarrow \boldsymbol{\xi}_G \boldsymbol{F}_G$

The data for colax $\mathscr{D}$-coalgebras

$$
\begin{tikzcd}
\mathscr{A} \arrow[r, "(H{,}\chi{,}Q)"] \arrow[d, "G"'] \arrow[dr, phantom, "(\omega{,}\varepsilon)\Nearrow"] & (\mathscr{X}, G) \arrow[d, equal] \\
\mathscr{X} \arrow[r, "(C{,}\eta{,}K)"'] & (\mathscr{X}, G)
\end{tikzcd}
\qquad
\begin{tikzcd}
CG(A) \arrow[r, "\omega_A"] \arrow[d, "\eta_{G(A)}"'] & H(A) \arrow[d, "\chi_A"] \\
GKG(A) \arrow[r, "G(\varepsilon_A)"'] & GQ(A)
\end{tikzcd}
$$

1) $H \colon \mathscr{A} \to \mathscr{X}$, $Q \colon \mathscr{A} \to \mathscr{A}$ and $\chi \colon H \Rightarrow GQ$

The data for colax $\mathscr{D}$-coalgebras

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "{(H,\chi,Q)}"] \arrow[d, "G"'] \arrow[dr, phantom, "{(\omega,\varepsilon)\Nearrow}"] & (\mathscr{X}, G) \arrow[d, equal] \\
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\end{tikzcd}$$

1) $H\colon \mathscr{A} \to \mathscr{X}$, $Q\colon \mathscr{A} \to \mathscr{A}$ and $\chi\colon H \Rightarrow GQ$
2) $C\colon \mathscr{X} \to \mathscr{X}$, $K\colon \mathscr{X} \to \mathscr{A}$ and $\omega\colon CG \Rightarrow H$

The data for colax $\mathscr{D}$-coalgebras - counit

$$
\begin{tikzcd}
\mathscr{A} \arrow[rr, equal] \arrow[dr, "{(H,\chi,Q)}"'] \arrow[dd, "G"'] & & \mathscr{A} \arrow[dr, equal] \arrow[dd, equal, dashed] \arrow[dl, phantom, "\zeta^0 \Nearrow"] & \\
& (\mathscr{X},G) \arrow[rr, "\boldsymbol{d}_0"] \arrow[dd, equal] \arrow[dl, phantom, "{(\omega,\varepsilon)} \Nearrow"] & & \mathscr{A} \arrow[dd, "G"] \\
\mathscr{X} \arrow[rr, equal, dashed] \arrow[dr, "{(C,\eta,K)}"'] & & \mathscr{X} \arrow[dr, equal] \arrow[ul, phantom, "\delta_G \Nearrow"] \arrow[dl, phantom, "\zeta^1 \Nearrow"] & \\
& (\mathscr{X},G) \arrow[rr, "\boldsymbol{d}_1"'] & & \mathscr{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
CG(A) \arrow[rr, "\zeta^1_{G(A)}"] \arrow[dr, "\omega_A"] \arrow[dd, "\eta_{G(A)}"'] & & G(A) \arrow[dr, equal] \arrow[dd, equal, dashed] & \\
& H(A) \arrow[rr, "G(\zeta^0_A)\chi_A"'] \arrow[dd, "\chi_A"] & & G(A) \arrow[dd, equal] \\
GKG(A) \arrow[rr, dashed, "G(\zeta^0_A\varepsilon_A)"] \arrow[dr, "G(\varepsilon_A)"'] & & G(A) \arrow[dr, equal] & \\
& GQ(A) \arrow[rr, "G(\zeta^0_A)"'] & & G(A)
\end{tikzcd}
$$

1) Natural transformations $\zeta^0 \colon Q \Rightarrow I_{\mathscr{A}}$ and $\zeta^1 \colon C \Rightarrow I_{\mathscr{X}}$

The data for colax $\mathscr{D}$-coalgebras - coassociativity 1

$$
\begin{tikzcd}
C(X) \arrow[rr, "\theta^{100}_X"] \arrow[ddd, "\eta_X"'] \arrow[ddr, "1_{C(X)}"] & & CC(X) \arrow[dr, "C(\eta_X)"] \arrow[ddd, dashed, "\eta_{C(X)}"' near start] & & \\
 & & & CGK(X) \arrow[dr, "\omega_{K(X)}"] \arrow[ddd, dashed, "\eta_{GK(X)}"' near end] & \\
 & C(X) \arrow[rrr, "\theta^{101}_X"] \arrow[ddd, "\eta_X"'] & & & HK(X) \arrow[ddd, "\chi_{K(X)}"] \\
GK(X) \arrow[rr, dashed, "G(\theta^{110}_X)"] \arrow[ddr, "G(1_{K(X)})"'] & & GKC(X) \arrow[dr, dashed, "GK(\eta_X)"'] & & \\
 & & & GKGK(X) \arrow[dr, "G(\varepsilon_{K(X)})"'] & \\
 & GK(X) \arrow[rrr, "G(\theta^{111}_X)"'] & & & GQK(X)
\end{tikzcd}
$$

The data for colax $\mathscr{D}$-coalgebras - coassociativity 2

$$
\begin{tikzcd}
H(A) \arrow[rrr, "\theta_A^{000}"] \arrow[ddrr, "1_{H(A)}"] \arrow[ddd, "\chi_A"'] & & & CH(A) \arrow[dr, "C(\chi_A)"] \arrow[ddd, dashed, "\eta_{H(A)}"'] & & \\
& & & & CGQ(A) \arrow[dr, "\omega_{Q(A)}"] \arrow[ddd, dashed, "\eta_{GQ(A)}"] & \\
& & H(A) \arrow[rrr, "\theta_A^{001}"] \arrow[ddd, "\chi_A"'] & & & HQ(A) \arrow[ddd, "\chi_{Q(A)}"] \\
GQ(A) \arrow[rrr, dashed, "G(\theta_A^{010})"] \arrow[ddrr, "G(1_{Q(A)})"'] & & & GKH(A) \arrow[dr, dashed, "GK(\chi_A)"'] & & \\
& & & & GKGQ(A) \arrow[dr, dashed, "G(\varepsilon_{Q(A)})"'] & \\
& & GQ(A) \arrow[rrr, "G(\theta_A^{011})"'] & & & GQQ(A)
\end{tikzcd}
$$

Axioms for colax $\mathscr{D}$-coalgebras

$$
\begin{tikzcd}
G \arrow[rrr, equal] \arrow[dr, "\boldsymbol{F}_G"] \arrow[ddd, "\boldsymbol{F}_G"'] & & \arrow[dl, phantom, "\zeta \Uparrow"] & G \arrow[dr, equal] \arrow[ddd, dashed, "\boldsymbol{F}_G"' near start] & \\
& \mathscr{D}(G) \arrow[rrr, "\boldsymbol{\delta}_G"] \arrow[ddd, "\mathscr{D}(\boldsymbol{F}_G)"] \arrow[dl, phantom, "\boldsymbol{\theta} \Uparrow"] & & & G \arrow[ddd, "\boldsymbol{F}_G"] \\
& & & & \\
\mathscr{D}(G) \arrow[rrr, equal, dashed] \arrow[dr, "\boldsymbol{\xi}_G"'] & & & \mathscr{D}(G) \arrow[dr, equal] & \\
& \mathscr{D}^2(G) \arrow[rrr, "\boldsymbol{\delta}_{\mathscr{D}(G)}"'] & & & \mathscr{D}(G)
\end{tikzcd}
$$

$$\boldsymbol{F}_G \boldsymbol{\zeta} \cdot \boldsymbol{\delta}_{\mathscr{D}(G)} \boldsymbol{\theta} = \iota_{\boldsymbol{F}_G}$$

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$$
\begin{tikzcd}
G \arrow[rr, "\boldsymbol{F}_G"] \arrow[dr, equal] \arrow[dd, "\boldsymbol{F}_G"'] & & \mathscr{D}(G) \arrow[dr, equal] \arrow[dd, dashed, "\mathscr{D}(\boldsymbol{F}_G)" near start] & \\
& G \arrow[rr, "\boldsymbol{F}_G" near start] \arrow[dd, "\boldsymbol{F}_G" near start] \arrow[r, phantom, "\theta \nearrow" near start] & & \mathscr{D}(G) \arrow[dd, equal] \arrow[dl, phantom, "\mathscr{D}(\zeta) \nearrow"] \\
\mathscr{D}(G) \arrow[rr, dashed, "\boldsymbol{\xi}_G" near end] \arrow[dr, equal] & & \mathscr{D}^2(G) \arrow[dr, dashed, "\mathscr{D}(\boldsymbol{\xi}_G)"'] & \\
& \mathscr{D}(G) \arrow[rr, equal] & & \mathscr{D}(G)
\end{tikzcd}
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$$\mathscr{D}(\zeta)\boldsymbol{F}_G \cdot \mathscr{D}(\boldsymbol{\delta}_G)\boldsymbol{\theta} = \iota_{\boldsymbol{F}_G}$$

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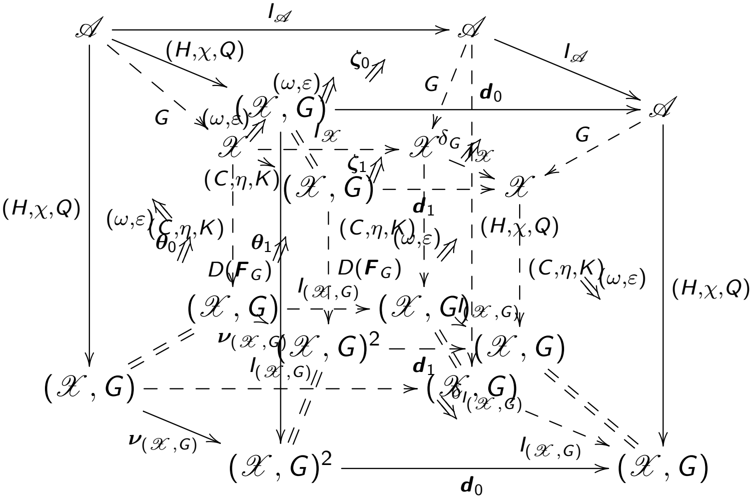
G \arrow[rr, "\boldsymbol{F}_G"] \arrow[dr, "\boldsymbol{F}_G"] \arrow[dd, "\boldsymbol{F}_G"'] & & \mathscr{D}(G) \arrow[dr, "\mathscr{D}(\boldsymbol{F}_G)"] \arrow[dd, dashed, "\mathscr{D}(\boldsymbol{F}_G)" near start] & \\
& \mathscr{D}(G) \arrow[rr, "\boldsymbol{\xi}_G"] \arrow[dd, "\mathscr{D}(\boldsymbol{F}_G)"] \arrow[ur, phantom, "\boldsymbol{\theta}\Uparrow"] \arrow[dr, phantom, "\boldsymbol{\theta}\Uparrow"] & & \mathscr{D}^2(G) \arrow[dd, "\mathscr{D}^2(\boldsymbol{F}_G)"] \arrow[dl, phantom, "\mathscr{D}(\boldsymbol{\theta})\Uparrow"] \\
\mathscr{D}(G) \arrow[rr, dashed, "\boldsymbol{\xi}_G" near end] \arrow[dr, "\boldsymbol{\xi}_G"'] \arrow[ur, phantom, "\boldsymbol{\theta}\Uparrow"] & & \mathscr{D}^2(G) \arrow[dr, dashed, "\mathscr{D}(\boldsymbol{\xi}_G)"] & \\
& \mathscr{D}^2(G) \arrow[rr, "\boldsymbol{\xi}_{\mathscr{D}(G)}"'] & & \mathscr{D}^3(G)
\end{tikzcd}
$$

$$\mathscr{D}^2(\boldsymbol{F}_G)\boldsymbol{\theta} \cdot \boldsymbol{\xi}_{\mathscr{D}(G)}\boldsymbol{\theta} = \mathscr{D}(\boldsymbol{\theta})\boldsymbol{F}_G \cdot \mathscr{D}(\boldsymbol{\xi}_G)\boldsymbol{\theta}$$


$\mathscr{A}$
$I_{\mathscr{A}}$
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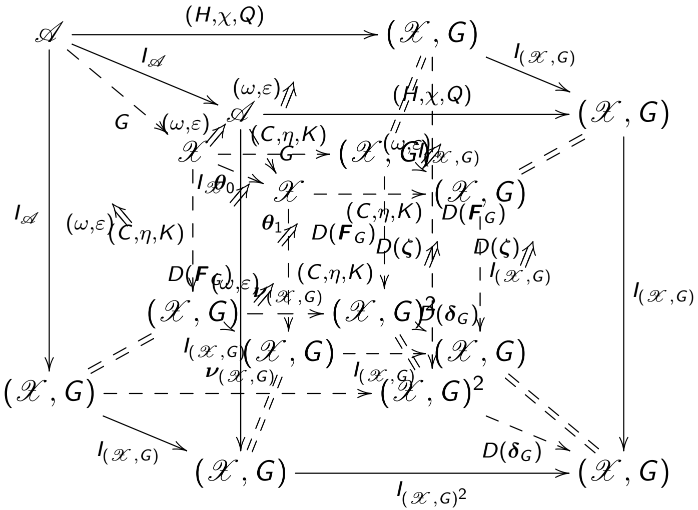
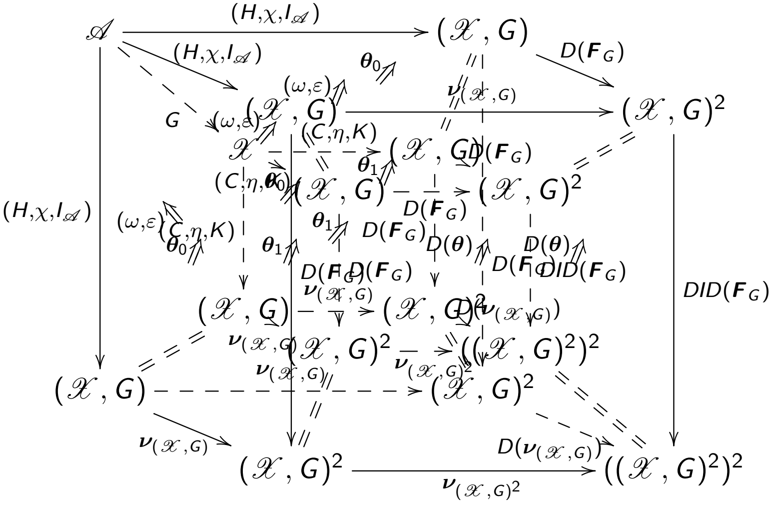
ζ_0
(ω,ε)
G
d_0
$(\mathscr{X},G)$
$I_{\mathscr{X}}$
$\mathscr{X}$
δ_G
ζ_1
(C,η,K)
d_1
θ_0
θ_1
$D(\boldsymbol{F}_G)$
$I_{(\mathscr{X},G)}$
$\nu_{(\mathscr{X},G)}$
$(\mathscr{X},G)^2$


(H,χ,Q)
I_𝒜
(ω,ε)
(C,η,K)
D(F_G)
D(ζ)
D(δ_G)
θ_0
θ_1
ν_(𝒳,G)
I_(𝒳,G)
I_(𝒳,G)^2


$\mathscr{A}$
$(H,\chi,I_{\mathscr{A}})$
$(\mathscr{X}, G)$
$D(\boldsymbol{F}_G)$
$(\mathscr{X}, G)^2$
G
(ω,ε)
(C,η,K)
$\boldsymbol{\theta}_0$
$\boldsymbol{\theta}_1$
$\mathscr{X}$
$\boldsymbol{\nu}_{(\mathscr{X},G)}$
$D(\boldsymbol{\theta})$
$DID(\boldsymbol{F}_G)$
$((\mathscr{X}, G)^2)^2$
$\boldsymbol{\nu}_{(\mathscr{X},G)^2}$
$D(\boldsymbol{\nu}_{(\mathscr{X},G)})$

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Interpretation of the data

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6) $\theta_X^{001} \colon (H(A), \theta_A^{000}) \to (HQ(A), \theta_{Q(A)}^{000})$ is a morphism of C-coalgebras
7) $\theta_X^{110} \colon (K(X), \theta_X^{111}) \to (KC(X), \theta_{C(X)}^{111})$ is a morphism of Q-coalgebras
8) $\theta_A^{010} \colon (Q(A), \theta_A^{011}) \to (KH(A), \theta_{H(A)}^{111})$ is a morphism of Q-coalgebras

Liftings to the Kleisli category

There exists a lifing

$$\begin{tikzcd} \mathscr{X}_C \arrow[r, "\widetilde{K}"] \arrow[d, "U_C"'] & \mathscr{A}_Q \arrow[d, "U_Q"] \\ \mathscr{X} \arrow[r, "K"'] & \mathscr{A} \end{tikzcd}$$

where the functor $\widetilde{K}_G \colon \mathscr{X}^C \to \mathscr{A}^Q$ acts as K on objects and takes any Kleisli morphism $f \colon C(X) \to Y$ to

$$QK(X) \xrightarrow{Q(\theta^{110}_X)} QKC(X) \xrightarrow{QK(f)} QK(Y) \xrightarrow{\zeta^0_{K(Y)}} K(Y).$$

Liftings to the Kleisli category

There exists a lifing

$$\begin{tikzcd} \mathscr{A}_Q \arrow[r, "\widetilde{H}"] \arrow[d, "U_Q"'] & \mathscr{X}_C \arrow[d, "U_C"] \\ \mathscr{A} \arrow[r, "H"'] & \mathscr{X} \end{tikzcd}$$

where the functor $\widetilde{K}_G \colon \mathscr{X}^C \to \mathscr{A}^Q$ acts as K on objects and takes any Kleisli morphism $f \colon C(X) \to Y$ to

$$CH(A) \xrightarrow{C(\theta_A^{001})} CHQ(A) \xrightarrow{CH(a)} CH(B) \xrightarrow{\zeta^1_{H(B)}} H(B).$$

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$$\begin{tikzcd} \mathscr{A}_Q \arrow[r, "\widetilde{G}"] \arrow[d, "U_Q"'] & \mathscr{X}_C \arrow[d, "U_C"] \\ \mathscr{A} \arrow[r, "G"'] & \mathscr{X} \end{tikzcd}$$

where the functor $\widetilde{K}_G \colon \mathscr{X}^C \to \mathscr{A}^Q$ acts as K on objects and takes any Kleisli morphism $f \colon C(X) \to Y$ to

$$CG(A) \xrightarrow{\omega_A} H(A) \xrightarrow{\chi_A} GQ(A) \xrightarrow{G(a)} G(A').$$

Theorem

Let $\pi\colon HK \to I_{\mathscr{X}}$ be a natural transformation which satisfies the following conditions

$$
\begin{tikzcd}
HQ(A) \arrow[d, "H(\theta^{010}_A)"'] \arrow[dr, "H(\zeta^0_A)"] & \\
HKH(A) \arrow[r, "\pi_{H(A)}"'] & H(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
CH(A) \arrow[d, "\theta^{101}_{H(A)}"'] \arrow[dr, "\zeta^1_{H(A)}"] & \\
HKH(A) \arrow[r, "\pi_{H(A)}"'] & H(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
QK(X) \arrow[d, "\theta^{010}_{K(X)}"'] \arrow[dr, "\zeta^0_{K(X)}"] & \\
KHK(X) \arrow[r, "K(\pi_X)"'] & K(X)
\end{tikzcd}
$$

Then the diagonal of the naturality square

$$
\begin{tikzcd}
CHK(X) \arrow[r, "\zeta^1_{HK(X)}"] \arrow[d, dash] & HK(X) \arrow[d, dash] \\
{} & {}
\end{tikzcd}
$$

Examples

1) Jacobs comprehension categories

Examples

1) Jacobs comprehension categories
2) Ehrhard D-categories

Examples

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3) Fumex tC-opfibrations

Examples

1) Jacobs comprehension categories
2) Ehrhard D-categories
3) Fumex tC-opfibrations
4) Lawvere categories

Morita's strongly adjoint pairs

Example

By taking $G = H$, and $\chi = \iota_G = \omega$ we obtain what Morita called a *strongly adjoint pair* consisting of an adjoint triple

$$G \dashv K \dashv G$$

where G is simultaneously left and right adjoint of K.

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where G is simultaneously left and right adjoint of K.

Ambidextrous adjunctions

Example

By keeping ω and χ as mutually invertible natural transformations we end up with an ambidextrous adjunction

$$H \dashv K \dashv G$$

which were pivotal in the work of Lauda who showed that every Frobenius object M in a monoidal category $\mathscr{M}$ arises from an ambijunction in some 2-category $\mathscr{D}$ into which M fully and faithfully embeds. This result also shows that every 2D TQFT is obtained from an ambijunction in some 2-category since it is well known that a 2D topological quantum field theory is equivalent to a commutative Frobenius algebra.

Associated split fibration 2-monad

Associated split fibration 2-monad

Consider the following square

$$\begin{tikzcd}
(\mathscr{B}, P) \arrow[rrr, "E_P"] \arrow[dr, "\mathscr{F}(F{,}\beta{,}U)"'] \arrow[dd, "\mathscr{F}(P)"'] & & & \mathscr{E} \arrow[dr, "F"] \arrow[dd, dashed, "P"' near start] & \\
& (\mathscr{C}, Q) \arrow[rrr, "E_Q"] \arrow[dd, "\mathscr{F}(Q)" near start] \arrow[r, phantom, "\varphi_P \nearrow" near start] & & & \mathscr{F} \arrow[dd, "Q"] \arrow[l, phantom, "\beta \nearrow"] \\
\mathscr{B} \arrow[rrr, equal] \arrow[dr, "U"'] & & \arrow[r, phantom, "\varphi_Q \nearrow"] & \mathscr{B} \arrow[dr, "U"] & \\
& \mathscr{C} \arrow[rrr, equal] & & & \mathscr{C}
\end{tikzcd}$$

$\mathscr{F}(P)\colon (\mathscr{B}, P) \to \mathscr{B}$ and $E_P\colon (\mathscr{B}, P) \to \mathscr{E}$ send any object (B, p, E) in $(\mathscr{B}, P)$ (where $p\colon B \to P(E)$) to B and E respectively.

Associated split fibration 2-monad

From the universal property of comma squares there exists a unique functor $\mathscr{F}(F, \beta, U)\colon (\mathscr{B}, P) \to (\mathscr{C}, Q)$ which takes any object (B, p, E) in $(\mathscr{B}, P)$ to $(U(B), \beta_E U(p), F(E))$ and any morphism $(u, e)\colon (B, p, E) \to (B', p', E')$ to the morphism $\mathscr{F}(F, \beta, U)(u, e) := (U(u), F(e))$ represented by a diagram

$$\begin{tikzcd}
U(B) \arrow[r, "U(u)"] \arrow[d, "U(p)"'] & U(B') \arrow[d, "U(p')"] \\
UP(E) \arrow[r, "UP(e)"'] \arrow[d, "\beta_E"'] & UP(E') \arrow[d, "\beta_{E'}"] \\
QF(E) \arrow[r, "QF(e)"'] & QF(E')
\end{tikzcd}$$

Theorem

There exists a colax idempotent 2-monad whose underlying 2-functor

$$\mathscr{F} : \mathscr{C}at^2_c \to \mathscr{C}at^2_c$$

is given by the above construction.

$$
\begin{tikzcd}
(\mathscr{B},P) \arrow[rr, "E_P"] \arrow[dr, bend right, "\mathscr{F}(F,\beta,U)"'] \arrow[dr, bend left, "\mathscr{F}(G,\gamma,V)"] \arrow[dr, phantom, "(\theta,\mathscr{F}(\tau))"] \arrow[dd, "\mathscr{F}(P)"'] & & \mathscr{E} \arrow[dr, bend right, "F"'] \arrow[dr, bend left, "G"] \arrow[dr, phantom, "\tau"] \arrow[dd, "P"] & \\
& (\mathscr{C},Q) \arrow[rr, "E_Q"] \arrow[dd, "\mathscr{F}(Q)"] & & \mathscr{D} \arrow[dd, "Q"] \\
\mathscr{B} \arrow[rr, dashed, "I_{\mathscr{B}}"] \arrow[dr, bend right, "U"'] \arrow[dr, "V"] \arrow[dr, phantom, "\theta"] & \arrow[r, phantom, "\beta"] & \mathscr{B} \arrow[dr, dashed, bend right, "U"'] \arrow[dr, "V"] \arrow[dr, phantom, "\theta"] & \arrow[l, phantom, "\gamma"] \\
& \mathscr{C} \arrow[rr, "I_{\mathscr{C}}"'] & & \mathscr{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
U(B) \arrow[rr, "\theta_B"] \arrow[dr, "U(u)"] \arrow[dd, "U(p)"'] & & V(B) \arrow[dr, "V(u)"] \arrow[dd, dashed, "V(p)"] & \\
& U(B') \arrow[rr, "\theta_{B'}"] \arrow[dd, "U(p')"] & & \mathscr{F} \arrow[dd, "V(p')"] \\
UP(E) \arrow[rr, dashed, "\theta_{P(E)}"] \arrow[dr, "UP(e)"] \arrow[dd, "\beta_E"'] & & VP(E) \arrow[dr, dashed, "VP(e)"] \arrow[dd, dashed, "\gamma_E"] & \\
& UP(E') \arrow[rr, "\theta_{P(E')}"] \arrow[dd, "\beta_{E'}"] & & VP(E') \arrow[dd, "\gamma_{E'}"] \\
QF(E) \arrow[rr, dashed, "Q(\tau_E)"] \arrow[dr, "QF(e)"'] & & QG(E) \arrow[dr, dashed, "QG(e)"] & \\
& QF(E') \arrow[rr, "Q(\tau_{E'})"'] & & QG(E')
\end{tikzcd}
$$

Functors $\mathscr{F}(F,\beta,U)$ and $\mathscr{F}(G,\gamma,V)$ take an object (B,p,E) to $\mathscr{F}(F,\beta,U)(B,p,E) := (U(B),\beta_E U(p),F(E))$ and $\mathscr{F}(G,\gamma,V)(B,p,Q) := (V(B),\gamma_E V(p),G(E))$ respectively.

Lax-Gray-monoids

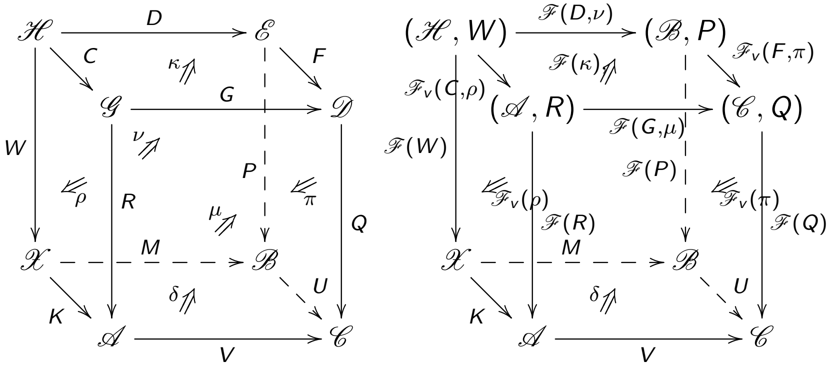
Theorem

An associated split fibration 2-monad $(\mathscr{C}at^2_c, \mathscr{F}, N, \widetilde{\eta}, M)$ is a lax-Gray-monoid in the Gray-category $\mathscr{G}ray_l$ of strict 2-categories, strict 2-functors, lax natural transformations and modifications with respect to a lax-Gray tensor product $\otimes_l$.

Theorem

An associated split cofibration 2-monad $(\mathscr{C}at^2_c, \mathscr{F}^\circ, N^\circ, \widetilde{\eta}^\circ, , M^\circ,)$ is a colax-Gray-monoid in the Gray-category $\mathscr{G}ray_c$ of strict 2-categories, strict 2-functors, colax natural transformations and modifications with respect to a colax-Gray tensor product $\otimes_c$.

The associated split fibration $\mathscr{F}$ double monad



The definition requires the existence of (certain) pullbacks in base categories! Its domain is a double 2-category $(\mathbb{C}at, \mathbb{C}art)$ where $\mathbb{C}art$ is an (enhanced) 2-category of categories with pullbacks.

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swArrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \arrow[r, phantom, ""] & {} \arrow[r, dotted, "\mathscr{F}"] & {} & (\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C} & & & \mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

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$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] & & (\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C} \arrow[r, phantom, ""] & {} \arrow[ur, phantom, ""] & \mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$\xrightarrow{\mathscr{F}}$$

$$
\begin{tikzcd}
B \arrow[d, "p"] \\
P(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
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\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
\overset{\mathscr{F}}{\dashrightarrow}
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
U(B) \arrow[d, "U(p)"] \\
UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swArrow_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
\quad \overset{\mathscr{F}}{\dashrightarrow} \quad
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swArrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

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\begin{tikzcd}
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\end{tikzcd}
\overset{\mathscr{F}}{\cdots\cdots\cdots\rightarrow}
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swArrow_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow\pi"] & \mathscr{D} \arrow[d, "Q"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd} \quad \overset{\mathscr{F}}{\dashrightarrow} \quad \begin{tikzcd} (\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F{,}\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow\mathscr{F}_v(\pi)"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd}$$

$$\begin{tikzcd} QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\ QF(E) \arrow[r, "\pi_E"'] & UP(E) \end{tikzcd}$$

$\mathscr{F}_v(B, p, E)$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd} \mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow\pi"] & \mathscr{D} \arrow[d, "Q"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd} \overset{\mathscr{F}}{\dashrightarrow} \begin{tikzcd} (\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F,\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow\mathscr{F}_v(\pi)"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\ \mathscr{B} \arrow[r, "U"'] & \mathscr{C} \end{tikzcd}$$

$$\begin{tikzcd} QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\ QF(E) \arrow[r, "\pi_E"'] & UP(E) \end{tikzcd}$$

$$\mathscr{F}_v(B, p, E) \quad :=$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\swarrow\!\!\!\!/_{\pi}"] & \mathscr{D} \arrow[d, "Q"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
\quad \overset{\mathscr{F}}{\cdots\cdots\cdots\rightarrow} \quad
\begin{tikzcd}
(\mathscr{B}, P) \arrow[r, "\mathscr{F}_v(F{,}\pi)"] \arrow[d, "\mathscr{F}(P)"'] \arrow[dr, phantom, "\swarrow\!\!\!\!/_{\mathscr{F}_v(\pi)}"] & (\mathscr{C}, Q) \arrow[d, "\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[r, "U"'] & \mathscr{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}
$$

$$\mathscr{F}_v(B, p, E) \quad := \quad (QF(E) \times_{UP(E)} U(B), \pi_E^* U(p), F(E))$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd} B \arrow[d, "p"] \\ P(E) \end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
U(B) \arrow[d, "U(p)"] \\
UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd} & U(B) \arrow[d, "U(p)"] \\ QF(E) \arrow[r, "\pi_E"'] & UP(E) \end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$\begin{tikzcd}
QF(E) \times_{UP(E)} U(B) \arrow[r, "\mathscr{F}_V(\pi)_{(B,p,E)}"] \arrow[d, "\pi_E^* U(p)"'] & U(B) \arrow[d, "U(p)"] \\
QF(E) \arrow[r, "\pi_E"'] & UP(E)
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
 & B' \arrow[dd, dashed, "p'"] & \\
QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[ddd, "\pi_E^* U(p)"'] & & U(B) \arrow[ddd, "U(p)"] \\
 & P(E') & \\
 & & \\
QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
 & U(B') \arrow[dd, dashed, "U(p')"] & \\
QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi_E^* U(p)"'] & & U(B) \arrow[dd, "U(p)"] \\
 & UP(E') & \\
QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
 & & U(B') \arrow[dd, dashed, "U(p')"] & \\
 & QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi_E^* U(p)"'] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] & & UP(E') & \\
 & QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dd, "\pi^*_{E'} U(p')"'] & & U(B') \arrow[dd, dashed, "U(p')"] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi^*_E U(p)"] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] & & UP(E') & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dd, "\pi^*_{E'} U(p')"'] & & U(B') \arrow[dr, "U(b)"] \arrow[dd, dashed, "U(p')"] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi^*_E U(p)"] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dr, "QF(e)\times U(b)"] \arrow[dd, "\pi^*_{E'}U(p')"'] & & U(B') \arrow[dr, "U(b)"] \arrow[dd, dashed, "U(p')" near start] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}" near start] \arrow[dd, "\pi^*_E U(p)" near start] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"' near start] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E') \times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dr, "QF(e)\times U(b)"] \arrow[dd, "\pi^*_{E'}U(p')"'] & & U(B') \arrow[dr, "U(b)"] \arrow[dd, dashed, "U(p')" near start] & \\
& QF(E) \times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}" near start] \arrow[dd, "\pi^*_E U(p)" near start] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}" near start] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

$\mathscr{F}_v(E)(b, e) :=$

Definition of the associated split fibration $\mathscr{F}$ on lax squares

$$
\begin{tikzcd}
QF(E')\times_{UP(E')} U(B') \arrow[rr, "\mathscr{F}_v(\pi')_{(B',p',E')}"] \arrow[dr, "QF(e)\times U(b)"] \arrow[dd, "\pi^*_{E'}U(p')"'] & & U(B') \arrow[dr, "U(b)"] \arrow[dd, dashed, "U(p')"] & \\
& QF(E)\times_{UP(E)} U(B) \arrow[rr, "\mathscr{F}_v(\pi)_{(B,p,E)}"] \arrow[dd, "\pi^*_E U(p)"] & & U(B) \arrow[dd, "U(p)"] \\
QF(E') \arrow[rr, dashed, "\pi_{E'}"'] \arrow[dr, "QF(e)"'] & & UP(E') \arrow[dr, dashed, "UP(e)"] & \\
& QF(E) \arrow[rr, "\pi_E"'] & & UP(E)
\end{tikzcd}
$$

$\mathscr{F}_v(E)(b,e) := (QF(e)\times U(b), F(e))$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$\begin{tikzcd} X \arrow[d, dashed, "I"] \\ W(H) \end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
K(X) \arrow[d, dashed, "K(I)"] \\
KW(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
& K(X) \arrow[d, dashed, "K(I)"] \\
RC(H) \arrow[r, dashed, "\rho_H"] & KW(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$\begin{tikzcd}
RC(H) \times_{KW(H)} K(X) \arrow[r, "\mathscr{F}_V(\rho)_{(X,I,H)}"] \arrow[d, "\rho_H^* K(I)"'] & K(X) \arrow[d, dashed, "K(I)"] \\
RC(H) \arrow[r, dashed, "\rho_H"] & KW(H)
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[r, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[d, "V(\rho_H^* K(I))"'] & VK(X) \arrow[d, dashed, "VK(I)"] \\
VRC(H) \arrow[r, dashed, "V(\rho_H)"] & VKW(H)
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[r, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[d, "V(\rho_H^* K(I))"'] & VK(X) \arrow[d, dashed, "VK(I)"] \\
VRC(H) \arrow[r, dashed, "V(\rho_H)"] \arrow[d, "\mu_{C(H)}"'] & VKW(H) \\
QGC(H) &
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_V(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[dd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dd, dashed, "VK(I)"] \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) \\
& QGC(H) &
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[dd, "V(\rho^*_H K(I))"'] & & VK(X) \arrow[dd, dashed, "VK(I)"] & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho^*_H K(I))"' near start] & & X \arrow[dd, dashed, "I"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)" near end] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & \\
& QGC(H) & & W(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[dd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dd, dashed, "VK(I)"] & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho_H^* K(I))"' near start] & & M(X) \arrow[dd, dashed, "M(I)"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & \\
& QGC(H) & & MW(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[dd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & M(X) \arrow[dd, dashed, "M(I)"] & \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & & \\
& QGC(H) & & MW(H) \arrow[dr, dashed, "\nu_H"'] & \\
& & & & PD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[dddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & M(X) \arrow[dr, equal] \arrow[ddd, dashed, "M(I)"] & \\
& & & & M(X) \arrow[dddd, "\nu_H M(I)"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & & \\
& QGC(H) & & MW(H) \arrow[dr, dashed, "\nu_H"'] & \\
& & & & PD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho^*_H K(I))"'] & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho^*_H K(I))"'] & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & & \\
& QGC(H) & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & & & UPD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"' near start] & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & & & UM(X) \arrow[dddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) & & \\
& QGC(H) & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & QFD(H) \arrow[rr] & & UPD(H)
\end{tikzcd}$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rrr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & & VK(X) \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"'] & & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & QFD(H) \times_{UPD(H)} UM(X) \arrow[rrr, "\mathscr{F}_v(\pi)_{\mathscr{F}(D,\nu)(X,I,H)}"] \arrow[ddd, "\pi^*_{D(H)} U(\nu_H M(I))"'] & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rrr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & & VKW(H) & & \\
& QGC(H) & & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"] & \\
& & QFD(H) \arrow[rrr] & & & UPD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rrr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & & VK(X) \arrow[dr, "\delta_X"] \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(I))"] & & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & QFD(H) \times_{UPD(H)} UM(X) \arrow[rrr, "\mathscr{F}_v(\pi)_{\mathscr{F}(D,\nu)(X,I,H)}"] \arrow[ddd, "\pi_{D(H)}^* U(\nu_H M(I))"] & & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rrr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & & VKW(H) \arrow[dr, dashed, "\delta_{W(H)}"] & & \\
& QGC(H) & & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"] & \\
& & QFD(H) \arrow[rrr] & & & UPD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H) \times_{KW(H)} K(X)) \arrow[rrr, "V(\mathscr{F}_v(\rho)_{(X,l,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(l))"'] & & & VK(X) \arrow[dr, "\delta_X"] \arrow[ddd, dashed, "VK(l)"] & & \\
& V(RC(H) \times_{KW(H)} K(X)) \arrow[ddd, "\mu_{C(H)} V(\rho_H^* K(l))"'] & & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(l)"] & \\
& & QFD(H) \times_{UPD(H)} UM(X) \arrow[rrr, "\mathscr{F}_v(\pi)_{\mathscr{F}(D,\nu)(X,l,H)}"] \arrow[ddd, "\pi^*_{D(H)} U(\nu_H M(l))"'] & & & UM(X) \arrow[ddd, "U(\nu_H M(l))"] \\
VRC(H) \arrow[rrr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & & VKW(H) \arrow[dr, dashed, "\delta_{W(H)}"'] & & \\
& QGC(H) \arrow[dr, "Q(\kappa_H)"'] & & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"'] & \\
& & QFD(H) \arrow[rrr] & & & UPD(H)
\end{tikzcd}
$$

Definition of the associated split fibration $\mathscr{F}$ on cubes

$$
\begin{tikzcd}
V(RC(H)\times_{KW(H)}K(X)) \arrow[rr, "V(\mathscr{F}_v(\rho)_{(X,I,H)})"] \arrow[dr, equal] \arrow[ddd, "V(\rho_H^* K(I))"'] & & VK(X) \arrow[dr, "\delta_X"] \arrow[ddd, dashed, "VK(I)"] & & \\
& V(RC(H)\times_{KW(H)}K(X)) \arrow[dr, "\mathscr{F}(\kappa)_{(X,I,H)}"] \arrow[ddd, "\mu_{C(H)}V(\rho_H^* K(I))"'] & & UM(X) \arrow[dr, equal] \arrow[ddd, dashed, "UM(I)"] & \\
& & QFD(H)\times_{UPD(H)}UM(X) \arrow[rr, "\mathscr{F}_v(\pi)_{\mathscr{F}(D,\nu)(X,I,H)}"] \arrow[ddd, "\pi^*_{D(H)}U(\nu_H M(I))"'] & & UM(X) \arrow[ddd, "U(\nu_H M(I))"] \\
VRC(H) \arrow[rr, dashed, "V(\rho_H)"] \arrow[dr, "\mu_{C(H)}"'] & & VKW(H) \arrow[dr, dashed, "\delta_{W(H)}"] & & \\
& QGC(H) \arrow[dr, "Q(\kappa_H)"'] & & UMW(H) \arrow[dr, dashed, "U(\nu_H)"] & \\
& & QFD(H) \arrow[rr] & & UPD(H)
\end{tikzcd}
$$

The associated split cofibration $\mathscr{F}^\circ$ 2-monad

1. The associated split cofibration $\mathscr{F}^\circ$ is defined as dual to $\mathscr{F}$:

$$\mathscr{F}^\circ(P) := (\mathscr{F}(P^{op}))^{op}$$

The associated split cofibration $\mathscr{F}^\circ$ 2-monad

1. The associated split cofibration $\mathscr{F}^\circ$ is defined as dual to $\mathscr{F}$:

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The associated split cofibration $\mathscr{F}^\circ$ 2-monad

1. The associated split cofibration $\mathscr{F}^\circ$ is defined as dual to $\mathscr{F}$:

$$\mathscr{F}^\circ(P) := (\mathscr{F}(P^{op}))^{op}$$

2. It requires no conditions on lax squares
3. It requires the existence of pushouts in base categories for colax squares

Pseudo-distributive law between fibrations and cofibrations

$$
\begin{tikzcd}
(\mathscr{B},(P,\mathscr{B})) \arrow[rr, "\lambda_P"] \arrow[dr, "\mathscr{F}\mathscr{F}^{\circ}(U{,}\pi{,}F)"'] \arrow[dd, "\mathscr{F}\mathscr{F}^{\circ}(P)"'] & & ((\mathscr{B},P),\mathscr{B}) \arrow[dr, "\mathscr{F}^{\circ}\mathscr{F}(U{,}\pi{,}F)"] \arrow[dd, dashed, "\mathscr{F}^{\circ}\mathscr{F}(P)"'] \arrow[dl, phantom, "\Swarrow\lambda_{(U,\pi,F)}"] & \\
& (\mathscr{C},(Q,\mathscr{C})) \arrow[rr, "\lambda_Q"'] \arrow[dd, "\mathscr{F}\mathscr{F}^{\circ}(Q)"] & & ((\mathscr{C},Q),\mathscr{C}) \arrow[dd, "\mathscr{F}^{\circ}\mathscr{F}(Q)"] \\
\mathscr{B} \arrow[rr, dashed, equal] \arrow[dr, "U"'] & & \mathscr{B} \arrow[dr, dashed, "U"] & \\
& \mathscr{C} \arrow[rr, equal] & & \mathscr{C}
\end{tikzcd}
$$

The component of a lax-natural transformation λ

$$\begin{tikzcd} & P(E) \arrow[dr, dashed, "q"] & \\ B \arrow[rr, "b"'] & & B_q \end{tikzcd}$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
 & UP(E) \arrow[dr, dashed, "U(q)"] & \\
U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}
$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
 & QF(E) \arrow[dd, dashed, "\pi_E"'] \arrow[dr, equal] & \\
 & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
 & UP(E) \arrow[dr, dashed, "U(q)"] & \\
U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}
$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
 & QF(E) \arrow[dd, dashed, "\pi_E"'] \arrow[dr, equal] & \\
U(B) \times_{U(B_q)} QF(E) \arrow[rr, "pr_2"] \arrow[dd, "pr_1"'] & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
 & UP(E) \arrow[dr, dashed, "U(q)"] & \\
U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}
$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
 & & QF(E) \arrow[dr, equal] \arrow[dd, dashed, "\pi_E"'] & \\
 & U(B) \times_{U(B_q)} QF(E) \arrow[rr, "pr_2"] \arrow[dd, "pr_1"'] & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
U(B \times_{B_q} P(E)) \arrow[rr, dashed, "U(pr_2)"] \arrow[dr, "U(pr_1)"'] & & UP(E) \arrow[dr, dashed, "U(q)"] & \\
 & U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}
$$

The component of a lax-natural transformation λ

$$
\begin{tikzcd}
U(B \times_{B_q} P(E)) \times_{UP(E)} QF(E) \arrow[rr, "pr_2 = \mathscr{F}(\pi)"] \arrow[dr, "\lambda_{(U,\pi,F)}"] \arrow[dd, "pr_1"'] & & QF(E) \arrow[dr, equal] \arrow[dd, dashed, "\pi_E"'] & \\
& U(B) \times_{U(B_q)} QF(E) \arrow[rr, "pr_2"] \arrow[dd, "pr_1"] & & QF(E) \arrow[dd, "U(q)\pi_E"] \\
U(B \times_{B_q} P(E)) \arrow[rr, dashed, "U(pr_2)"] \arrow[dr, "U(pr_1)"'] & & UP(E) \arrow[dr, dashed, "U(q)"] & \\
& U(B) \arrow[rr, "U(b)"'] & & U(B_q)
\end{tikzcd}
$$

The associated Beck-Chevalley fibration

- The associated Beck-Chevalley fibrations are pseudoalgebras for the pseudo-distributive law

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- A natural candidate for the domain of its underlying 2-functor

$$\mathscr{F}\mathscr{F}^{\circ}\colon (\mathscr{C}at, \mathscr{QT}op) \to (\mathscr{C}at, \mathscr{QT}op)$$

is a double comma 2-category $(\mathscr{C}at, \mathscr{QT}op)$ where $\mathscr{QT}op$ is a 2-category of quasitoposes and geometric morphisms

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is a double comma 2-category $(\mathscr{C}\mathit{at}, \mathscr{QT}\mathit{op})$ where $\mathscr{QT}\mathit{op}$ is a 2-category of quasitoposes and geometric morphisms

- A quasitopos is a finitely complete, finitely cocomplete, locally cartesian closed category $\mathscr{C}$ in which there exists an object Ω that classifies strong monomorphisms.

Admissibility

Admissible 1-cells

Definition

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a lax idempotent 2-monad on the 2-category $\mathscr{K}$. We say that the 1-cell $f\colon C \to D$ in $\mathscr{K}$ is admissible if its image $T(f)$ has a right adjoint μ_f. In the dual case of a colax idempotent 2-monad we say that the 1-cell $f\colon C \to D$ in $\mathscr{K}$ is admissible if $T(f)$ has a left adjoint ν_f.

Admissible objects

Definition

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a (co)lax idempotent 2-monad on the 2-category $\mathscr{K}$ with a terminal object $\top$. We say that an object E of $\mathscr{K}$ is admissible if the unique 1-cell $!_E\colon E \to \top$ is admissible.

Definition

Let $(T, \eta, \mu)\colon \mathscr{K} \to \mathscr{K}$ be a lax idempotent 2-monad on the 2-category $\mathscr{K}$. We say that (T, η, μ) is admissible if the following bicomma object condition holds:

1) the 2-category $\mathscr{K}$ has bicomma objects $f \downarrow g$ of diagrams

$$\begin{tikzcd} & B \arrow[d, squiggly, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

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where 1-cells p and q are admissible.

2) the canonical 2-cell $T(q)\mu_p \Rightarrow \mu_f T(g)$ is a 2-isomorphism.

The admissibility of associated split (co)fibrations

Theorem

The associated split fibration 2-monad is admissible.

Definition

A functor $U\colon \mathscr{A} \to \mathscr{B}$ is a local right adjoint if the restriction

$$U_A\colon (\mathscr{A}, A) \to (\mathscr{B}, U(A))$$

of U to the slice $(\mathscr{A}, A)$ for each object A of $\mathscr{A}$ has a left adjoint

$$L_A\colon (\mathscr{B}, U(A)) \to (\mathscr{A}, A).$$

Equivalently, each fiber $U_A\colon (\mathscr{A}, A) \to (\mathscr{B}, U(A))$ has a left adjoint.

$$\begin{tikzcd} \mathscr{A}^2 \arrow[r, "U^2"] \arrow[d, "cod"'] & \mathscr{B}^2 \arrow[d, "cod"] \\ \mathscr{A} \arrow[r, "U"'] & \mathscr{B} \end{tikzcd}$$

Theorem

Right multiadjoints are admissible objects for the associated split fibration 2-monad.

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