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# Gradient Extremum Seeking with Delays ⋆

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**Abstract:** In this paper, we derive the design and analysis for scalar gradient extremum seeking control (ESC) subject to arbitrarily long input-output delays, by employing a predictor with a perturbation-based estimate of the Hessian. Exponential stability and convergence to a small neighborhood of the unknown extremum point can be guaranteed. This result is carried out using backstepping transformation and averaging in infinite dimensions. Some simulation examples are presented to illustrate the performance of the delay-compensated ESC scheme.



Keywords: time delay, adaptive systems, extremum seeking, predictors, backstepping transformation, averaging in infinite dimensions.

## 1. INTRODUCTION

Despite of the large number of publications on extremum seeking control (ESC) [Krstić, 2014, Adetola and Guay, 2007, Tan et al., 2009, Nesić et al., 2010, Ghaffari et al., 2012, Liu and Krstić, 2012, Oliveira et al., 2011, 2012], there is no work which rigorously deals with the problem of ESC in the presence of delays. In the present paper and reference [Oliveira and Krstić, 2015], we give an answer to this question by considering scalar gradient ESC and Newton-based ESC [Krstić, 2014, Ghaffari et al., 2012] under input-output delays, respectively. The proposed solution based on prediction feedback with a perturbation-based estimate of the Hessian as well as the stability analysis are rigorously obtained via backstepping transformation [Krstić, 2009] and averaging in infinite dimensions [Hale and Lunel, 1990, Lehman, 2002]. The perturbation-based (averaging-based) approach rises due to the necessity of estimating the unknown second derivative (Hessian) [Nesić et al., 2010, Ghaffari et al., 2012] of the nonlinear map to be optimized.

We start in Section 2 introducing the predictor design for delay compensation in scalar gradient ESC. Exponential stability with explicit Lyapunov-Krasovskii functionals and the real-time convergence to a small neighborhood of the desired extremum are demonstrated in Section 3. Section 4 presents numerical examples to illustrate the applicability of the proposed extremum seeking with delay compensation. Finally, Section 5 concludes the paper identifying some open problems for future research directions.

**Notation and Norms:** The 2−norm of a finite-dimensional (ODE) state vector $X(t)$ is denoted by single bars, $|X(t)|$. In contrast, norms of functions (of $x$) are denoted by double bars. By default, $\|\cdot\|$ denotes the spatial $L_2[0,D]$ norm, *i.e.*, $\|\cdot\| = \|\cdot\|_{L_2[0,D]}$. Since the PDE state variable $u(x,t)$ is a function of two arguments, we should emphasize that taking a norm in one of the variables makes the norm a function of the other variable. For example, the $L_2[0,D]$ norm of $u(x,t)$ in $x \in [0,D]$ is $\|u(t)\| = \left(\int_0^D u^2(x,t)dx\right)^{1/2}$.

## 2. EXTREMUM SEEKING WITH DELAYS

Scalar ESC considers applications in which the goal is to maximize (or minimize) the output $y \in \mathbb{R}$ of an unknown nonlinear static map $Q(\theta)$ by varying the input $\theta \in \mathbb{R}$. Here, we additionally assume that there is a *constant and known* delay $D \geq 0$ in the actuation path or measurement system such that the measured output is given by

$$y(t) = Q(\theta(t-D)). \tag{1}$$

For notation clarity, we assume that our system is output-delayed in the following presentation and block diagrams. However, the results in this paper can be straightforward extended to the input-delay case since any input delay can be moved to the output of the static map. The case when input delays $D_{\rm in}$ and output delays $D_{\rm out}$ occur simultaneously could also be handled, by assuming that the total delay to be counteract would be $D = D_{\rm in} + D_{\rm out}$, with $D_{\rm in}, D_{\rm out} \geq 0$. Without loss of generality, let us consider the maximum seeking problem such that the maximizing value of $\theta$ is denoted by $\theta^*$. For the sake of simplicity, we also assume that the nonlinear map is quadratic, *i.e.*,

$$Q(\theta) = y^* + \frac{H}{2}(\theta - \theta^*)^2, \tag{2}$$

where besides the constants $\theta^* \in \mathbb{R}$ and $y^* \in \mathbb{R}$ being unknown, the scalar $H < 0$ is the unknown Hessian of the static map. By plugging (2) into (1), we obtain the *quadratic static map with delay* of interest:

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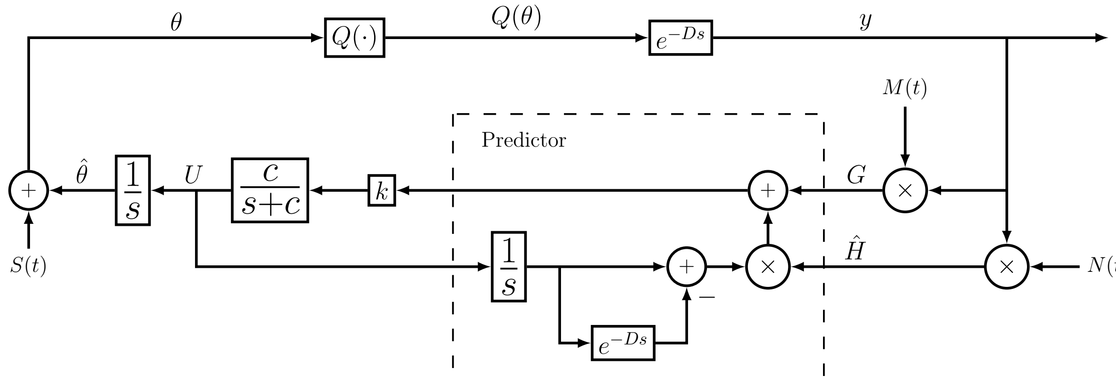


Fig. 1. Block diagram of the basic prediction scheme for output-delay compensation in gradient ESC. The predictor feedback with a perturbation-based estimate of the Hessian obeys equation (18), the dither signals are given by $S(t) = a\sin(\omega(t + D))$ and $M(t) = \frac{2}{a}\sin(\omega t)$ and the demodulating signal $N(t) = -\frac{8}{a^2}\cos(2\omega t)$.

$$y(t) = y^* + \frac{H}{2}(\theta(t-D) - \theta^*)^2 . \tag{3}$$

### 2.1 System Signals

Let $\hat{\theta}$ be the estimate of $\theta^*$ and

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^* \tag{4}$$

be the *estimation error*. From Figure 1, the *error dynamics* can be written as

$$\dot{\tilde{\theta}}(t-D) = U(t-D) . \tag{5}$$

Moreover, one has

$$G(t) = M(t)y(t) , \quad \theta(t) = \hat{\theta}(t) + S(t) , \tag{6}$$

where the dither signals are given by

$$S(t) = a\sin(\omega(t+D)) , \quad M(t) = \frac{2}{a}\sin(\omega t) , \tag{7}$$

with nonzero perturbation amplitude $a$ and frequency $\omega$. The signal

$$\hat{H}(t) = N(t)y(t) , \tag{8}$$

is applied to obtain an estimate of the unknown Hessian $H$, where the demodulating signal $N(t)$ is given by

$$N(t) = -\frac{8}{a^2}\cos(2\omega t) . \tag{9}$$

In [Ghaffari et al., 2012], it was proved that

$$\frac{1}{\Pi}\int_0^{\Pi} N(\sigma)y d\sigma = H , \quad \Pi = 2\pi/\omega , \tag{10}$$

if a quadratic map as in (2) is considered. In other words, the average version $\hat{H}_{\rm av} = (Ny)_{\rm av} = H$.

### 2.2 Predictor Feedback with a Estimate of the Hessian

By using the averaging analysis, we can verify that the average version of the signal $G(t)$ in (6) is given by

$$G_{\rm av}(t) = H\tilde{\theta}_{\rm av}(t-D) . \tag{11}$$

From (5), the following average models can be obtained

$$\dot{\tilde{\theta}}_{\rm av}(t-D) = U_{\rm av}(t-D) , \quad \dot{G}_{\rm av}(t) = HU_{\rm av}(t-D) , \tag{12}$$

where $U_{\rm av} \in \mathbb{R}$ is the resulting average control for $U \in \mathbb{R}$.

In order to motivate the predictor feedback design, the idea here is to compensate for the delay by feeding back the future state $G(t+D)$, or $G_{\rm av}(t+D)$ in the equivalent average system. Given any stabilizing gain $k > 0$ for the undelayed system, our wish is to have a control that achieves

$$U_{\rm av}(t) = kG_{\rm av}(t+D) , \quad \forall t \geq 0 , \tag{13}$$

and it appears to be non implementable since it requires future values of the state. However, by applying the variation of constants formula to (12) we can express the future state as

$$G_{\rm av}(t+D) = G_{\rm av}(t) + H\int_{t-D}^{t} U_{\rm av}(\sigma)d\sigma , \tag{14}$$

which gives the future state $G_{\rm av}(t+D)$ in terms of the average control signal $U_{\rm av}(\sigma)$ from the past window $[t-D, t]$. It yields the following feedback law

$$U_{\rm av}(t) = k\left[G_{\rm av}(t) + H\int_{t-D}^{t} U_{\rm av}(\sigma)d\sigma\right] . \tag{15}$$

Hence, from (14) and (15), the average feedback law (13) can be obtained indeed as desired. Consequently,

$$\dot{\tilde{\theta}}_{\rm av}(t) = kG_{\rm av}(t+D) , \quad \forall t \geq 0 . \tag{16}$$

Therefore, from (11), one has

$$\frac{d\tilde{\theta}_{\rm av}(t)}{dt} = kH\tilde{\theta}_{\rm av}(t) , \quad \forall t \geq D , \tag{17}$$

with an exponentially attractive equilibrium $\tilde{\theta}^e_{\rm av} = 0$, since $k > 0$ in the control design and $H < 0$ by assumption.

In the next section, we show that the control objectives can still be achieved if a simple modification of the above basic predictor-based controller, which employs a low-pass filter, is applied. In this case, we propose the following infinite-dimensional and averaging-based predictor feedback in order to compensate the delay [Krstić, 2008]

$$U(t) = \frac{c}{s+c}\left\{k\left[G(t) + \hat{H}(t)\int_{t-D}^{t} U(\tau)d\tau\right]\right\} , \tag{18}$$

where $c > 0$ is sufficiently large, i.e., the predictor feedback is of the form of a low-pass filtered of the non average version of (15). This low pass filtering is particularly required in the stability analysis when the averaging theorem in infinite dimensions [Hale and Lunel, 1990, Lehman, 2002] is invoked. The predictor feedback (18) is averaging-base (perturbation-based) because $\hat{H}$ is updated according to the estimate (8) of the unknown Hessian $H$, satisfying the averaging property (10).

## 3. STABILITY ANALYSIS

The main stability/convergence results for the closed-loop system are summarized in the next theorem.

*Theorem 1. Consider the closed-loop system in Figure 1 with delayed output (3). There exists $c^* > 0$ such that, $\forall c \geq c^*$, $\exists\ \omega^*(c) > 0$ such that, $\forall \omega > \omega^*$, the closed-loop delayed system (5) and (18), with $G(t)$ in (6), $\hat{H}(t)$ in (8) and state $\tilde{\theta}(t-D)$, $U(\sigma)$, $\forall \sigma \in [t-D, t]$, has a unique exponentially stable periodic solution in $t$ of period $\Pi = 2\pi/\omega$, denoted by $\tilde{\theta}^{\Pi}(t-D)$, $U^{\Pi}(\sigma)$, $\forall \sigma \in [t-D, t]$, satisfying, $\forall t \geq 0$:*

$$\left(\left|\tilde{\theta}^{\Pi}(t-D)\right|^2 + \left[U^{\Pi}(t)\right]^2 + \int_{t-D}^{t} \left[U^{\Pi}(\tau)\right]^2 d\tau\right)^{1/2} \leq \mathcal{O}(1/\omega) \quad (19)$$

*Furthermore,*

$$\limsup_{t\to+\infty} |\theta(t) - \theta^*| = \mathcal{O}(a + 1/\omega)\,, \quad (20)$$

$$\limsup_{t\to+\infty} |y(t) - y^*| = \mathcal{O}(a^2 + 1/\omega^2)\,. \quad (21)$$

*Proof:* The demonstration follows the steps 1 to 8 below.

***Step 1:*** *Transport PDE for Delay Representation*

According to [Krstić, 2009], the delay in (5) can be represented using a transport PDE as

$$\dot{\tilde{\theta}}(t-D) = u(0,t)\,, \quad (22)$$

$$u_t(x,t) = u_x(x,t)\,, \quad x \in [0, D]\,, \quad (23)$$

$$u(D,t) = U(t)\,, \quad (24)$$

where the solution of (23)–(24) is

$$u(x,t) = U(t + x - D)\,. \quad (25)$$

***Step 2:*** *Equations of the Closed-loop System*

First, substituting $S(t)$ given in (7) into $\theta(t)$ in (6), we obtain

$$\theta(t) = \hat{\theta}(t) + a\sin(\omega(t+D))\,. \quad (26)$$

Now, plug (4) and (26) into (3) so that the output is given in terms of $\tilde{\theta}$:

$$y(t) = y^* + \frac{H}{2}(\tilde{\theta}(t-D) + a\sin(\omega t))^2\,. \quad (27)$$

By plugging $M(t)$ given in (7) into $G(t)$ in (6), (9) into (8) and representing the integrand in (18) using the transport PDE state, one has

$$U(t) = \frac{c}{s+c}\left\{k\left[G(t) + \hat{H}(t)\int_0^D u(\sigma,t)d\sigma\right]\right\}\,, \quad (28)$$

$$G(t) = \frac{2}{a}\sin(\omega t)y(t)\,, \quad (29)$$

$$\hat{H}(t) = -\frac{8}{a^2}\cos(2\omega t)y(t)\,. \quad (30)$$

Plug (27) into (29) and (30), and then the resulting (29) and (30) into (28). By extracting the common factor $y$ in the resulting version of (28), one has

$$U(t) = \frac{c}{s+c}\left\{k\left[y^* + \frac{H}{2}(\tilde{\theta}(t-D) + a\sin(\omega t))^2\right] \times \left[\frac{2}{a}\sin(\omega t) - \frac{8}{a^2}\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma\right]\right\}\,. \quad (31)$$

By expanding the binome in (31), we obtain

$$\begin{aligned} U(t) = \frac{c}{s+c}\Bigg\{k\Bigg[&y^* + \frac{H}{2}\tilde{\theta}^2(t-D) \\ &+ Ha\sin(\omega t)\tilde{\theta}(t-D) + \frac{a^2 H}{2}\sin^2(\omega t)\Bigg] \\ &\times \left[\frac{2}{a}\sin(\omega t) - \frac{8}{a^2}\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma\right]\Bigg\}\,. \end{aligned} \quad (32)$$

Finally, substituting (32) into (24), we can rewrite (22)–(24) as

$$\dot{\tilde{\theta}}(t-D) = u(0,t)\,, \quad (33)$$

$$\partial_t u(x,t) = \partial_x u(x,t)\,, \quad x \in [0, D]\,, \quad (34)$$

$$\begin{aligned} u(D,t) = &\frac{c}{s+c}\Bigg\{k\Bigg[y^* + \frac{H}{2}\tilde{\theta}^2(t-D) + \\ &Ha\sin(\omega t)\tilde{\theta}(t-D) + \frac{a^2H}{2}\sin^2(\omega t)\Bigg] \\ &\times\left[\frac{2}{a}\sin(\omega t) - \frac{8}{a^2}\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma\right]\Bigg\} \\ = &\frac{c}{s+c}\Bigg\{k\Bigg[y^*\frac{2}{a}\sin(\omega t) - y^*\frac{8}{a^2}\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma \\ &+\frac{H}{a}\tilde{\theta}^2(t-D)\sin(\omega t) \\ &-\frac{4H}{a^2}\tilde{\theta}^2(t-D)\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma \\ &+2H\sin^2(\omega t)\tilde{\theta}(t-D) \\ &-\frac{8H}{a}\sin(\omega t)\tilde{\theta}(t-D)\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma \\ &+aH\sin^3(\omega t) - 4H\sin^2(\omega t)\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma\Bigg]\Bigg\} \\ = &\frac{c}{s+c}\Bigg\{k\Bigg[y^*\frac{2}{a}\sin(\omega t) - y^*\frac{8}{a^2}\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma \\ &+\frac{H}{a}\tilde{\theta}^2(t-D)\sin(\omega t) \\ &-\frac{4H}{a^2}\tilde{\theta}^2(t-D)\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma \\ &+H\tilde{\theta}(t-D) - H\cos(2\omega t)\tilde{\theta}(t-D) \\ &-\frac{4H}{a}[\sin(3\omega t) - \sin(\omega t)]\tilde{\theta}(t-D)\int_0^D u(\sigma,t)d\sigma \\ &+\frac{3aH}{4}\sin(\omega t) - \frac{aH}{4}\sin(3\omega t) - 2H\cos(2\omega t)\int_0^D u(\sigma,t)d\sigma \\ &+[H + H\cos(4\omega t)]\int_0^D u(\sigma,t)d\sigma\Bigg]\Bigg\}\,. \end{aligned} \quad (35)$$

***Step 3:*** *Average Model of the Closed-loop System*

Now denoting

$$\tilde{\vartheta}(t) = \tilde{\theta}(t-D)\,, \tag{36}$$

the average version of system (33)–(35) is:

$$\dot{\tilde{\vartheta}}_{\rm av}(t) = u_{\rm av}(0,t)\,, \tag{37}$$

$$\partial_t u_{\rm av}(x,t) = \partial_x u_{\rm av}(x,t)\,, \quad x\in[0,D]\,, \tag{38}$$

$$\frac{d}{dt}u_{\rm av}(D,t) = -cu_{\rm av}(D,t) + ckH\left[\tilde{\vartheta}_{\rm av}(t) + \int_0^D u_{\rm av}(\sigma,t)d\sigma\right] \tag{39}$$

where in the last line we have simply set all the averages of the sine and cosine functions of $\omega$, $2\omega$, $3\omega$ and $4\omega$ to zero. Moreover, the filter $c/s+c$ is also represented in the state-space form. The solution of the transport PDE (38)–(39) is given by

$$u_{\rm av}(x,t) = U_{\rm av}(t+x-D)\,. \tag{40}$$

***Step 4:*** *Backstepping transformation, its inverse and the target system*

Consider the infinite-dimensional backstepping transformation of the delay state

$$w(x,t) = u_{\rm av}(x,t) - kH\left[\tilde{\vartheta}_{\rm av}(t) + \int_0^x u_{\rm av}(\sigma,t)d\sigma\right]\,, \tag{41}$$

which maps the system (37)–(39) into the target system:

$$\dot{\tilde{\vartheta}}_{\rm av}(t) = kH\tilde{\vartheta}_{\rm av}(t) + w(0,t)\,, \tag{42}$$

$$w_t(x,t) = w_x(x,t)\,, \quad x\in[0,D]\,, \tag{43}$$

$$w(D,t) = -\frac{1}{c}\partial_t u_{\rm av}(D,t)\,. \tag{44}$$

Using (41) for $x=D$ and the fact that $u_{\rm av}(D,t)=U_{\rm av}(t)$, from (44) we get (39), *i.e.*,

$$U_{\rm av}(t) = \frac{c}{s+c}\left\{kH\left[\tilde{\vartheta}_{\rm av}(t) + \int_0^D u_{\rm av}(\sigma,t)d\sigma\right]\right\}\,. \tag{45}$$

Let us now consider $w(D,t)$. It is easily seen that

$$w_t(D,t) = \partial_t u_{\rm av}(D,t) - kHu_{\rm av}(D,t)\,, \tag{46}$$

where $\partial_t u_{\rm av}(D,t)=\dot{U}_{\rm av}(t)$. The inverse of (41) is given by

$$u_{\rm av}(x,t) = w(x,t) + kH\left[e^{kHx}\tilde{\vartheta}_{\rm av}(t) + \int_0^x e^{kH(x-\sigma)}w(\sigma,t)d\sigma\right]\,. \tag{47}$$

Plugging (44) and (47) into (46), we get

$$w_t(D,t) = -cw(D,t) - kHw(D,t) -(kH)^2\left[e^{kHD}\tilde{\vartheta}_{\rm av}(t) + \int_0^D e^{kH(D-\sigma)}w(\sigma,t)d\sigma\right] \tag{48}$$

***Step 5:*** *Lyapunov-Krasovskii Functional*

Now consider the following Lyapunov functional

$$V(t) = \frac{\tilde{\vartheta}^2_{\rm av}(t)}{2} + \frac{a}{2}\int_0^D (1+x)w^2(x,t)dx + \frac{1}{2}w^2(D,t)\,, \tag{49}$$

where the parameter $a>0$ is to be chosen later. We have

$$\begin{aligned}\dot{V}(t) &= kH\tilde{\vartheta}^2_{\rm av}(t) + \tilde{\vartheta}_{\rm av}(t)w(0,t) \\ &\quad + a\int_0^D (1+x)w(x,t)w_x(x,t)dx + w(D,t)w_t(D,t) \\ &= kH\tilde{\vartheta}^2_{\rm av}(t) + \tilde{\vartheta}_{\rm av}(t)w(0,t) + \frac{a(1+D)}{2}w^2(D,t) \\ &\quad -\frac{a}{2}w^2(0,t) - \frac{a}{2}\int_0^D w^2(x,t)dx + w(D,t)w_t(D,t) \\ &\le kH\tilde{\vartheta}^2_{\rm av}(t) + \frac{\tilde{\vartheta}^2_{\rm av}(t)}{2a} - \frac{a}{2}\int_0^D w^2(x,t)dx \\ &\quad + w(D,t)\left[w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right]\,.\end{aligned}$$

Reminding that $k>0$ and $H<0$, let us choose

$$a = -\frac{1}{kH}\,. \tag{50}$$

Then,

$$\begin{aligned}\dot{V}(t) &\le \frac{kH}{2}\tilde{\vartheta}^2_{\rm av}(t) + \frac{1}{2kH}\int_0^D w^2(x,t)dx \\ &\quad + w(D,t)\left[w_t(D,t) - \frac{(1+D)}{2kH}w(D,t)\right] \\ &= -\frac{1}{2a}\tilde{\vartheta}^2_{\rm av}(t) - \frac{a}{2}\int_0^D w^2(x,t)dx \\ &\quad + w(D,t)\left[w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right]\,.\end{aligned} \tag{51}$$

Now we consider (51) along with (48). With a completion of squares, we obtain

$$\begin{aligned}\dot{V}(t) &\le -\frac{1}{4a}\tilde{\vartheta}^2_{\rm av}(t) - \frac{a}{4}\int_0^D w^2(x,t)dx + a\left|(kH)^2e^{kHD}\right|^2 w^2(D,t) \\ &\quad + \frac{1}{a}\left\|(kH)^2e^{kH(D-\sigma)}\right\|^2 w^2(D,t) \\ &\quad + \left[\frac{a(1+D)}{2} - kH\right]w^2(D,t) - cw^2(D,t)\,.\end{aligned} \tag{52}$$

To obtain (52), we have used

$$\begin{aligned}&-w(D,t)\left\langle (kH)^2e^{kH(D-\sigma)}, w(\sigma,t)\right\rangle \\ &\le |w(D,t)|\left\|(kH)^2e^{kH(D-\sigma)}\right\|\|w(t)\| \\ &\le \frac{a}{4}\|w(t)\|^2 + \frac{1}{a}\left\|(kH)^2e^{kH(D-\sigma)}\right\|^2 w^2(D,t)\,,\end{aligned} \tag{53}$$

where the first inequality is the Cauchy-Schwartz and the second is Young's, the notation $\langle\cdot,\cdot\rangle$ denotes the inner product in the spatial variable $\sigma\in[0,D]$, on which both $e^{kH(D-\sigma)}$ and $w(\sigma,t)$ depend, and $\|\cdot\|$ denotes the $L_2$ norm in $\sigma$. Then, from (52), we arrive at

$$\dot{V}(t) \le -\frac{1}{4a}\tilde{\vartheta}^2_{\rm av}(t) - \frac{a}{4(1+D)}\int_0^D (1+x)w^2(x,t)dx - (c-c^*)w^2(D,t)\,, \tag{54}$$

where

$$c^* = \frac{a(1+D)}{2} - kH + a\left|(kH)^2e^{kHD}\right|^2 + \frac{1}{a}\left\|(kH)^2e^{kH(D-\sigma)}\right\|^2\,. \tag{55}$$

From (55), it is clear that an upper bound for $c^*$ can be obtained from known lower and upper bounds of the unknown Hessian $H$. Hence, from (54), if $c$ is chosen such that $c>c^*$, we obtain

$$\dot{V}(t) \le -\mu V(t)\,, \tag{56}$$

for some $\mu>0$. Thus, the closed-loop system is exponentially stable in the sense of the full state norm

$$\left(|\tilde{\vartheta}_{\rm av}(t)|^2+\int_0^D w^2(x,t)dx+w^2(D,t)\right)^{1/2}, \qquad (57)$$

*i.e.*, in the transformed variable $(\tilde{\vartheta}_{\rm av}, w)$.

***Step 6:*** *Exponential Stability Estimate (in $L_2$ norm) for the Average System (37)–(39)*

To obtain exponential stability in the sense of the norm $\left(|\tilde{\vartheta}_{\rm av}(t)|^2+\int_0^D u_{\rm av}^2(x,t)dx+u_{\rm av}^2(D,t)\right)^{1/2}$, we need to show there exist positive numbers $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1\Psi(t)\leq V(t)\leq \alpha_2\Psi(t), \qquad (58)$$

where $\Psi(t)\triangleq|\tilde{\vartheta}_{\rm av}(t)|^2+\int_0^D u_{\rm av}^2(x,t)dx+u_{\rm av}^2(D,t)$, or equivalently,

$$\Psi(t)\triangleq|\tilde{\theta}_{\rm av}(t-D)|^2+\int_{t-D}^t U_{\rm av}^2(\tau)d\tau+U_{\rm av}^2(t), \qquad (59)$$

using (36) and (40).

This is straightforward to establish by using (41), (47), (49) and employing the Cauchy-Schwartz inequality and other calculations, as in the proof of Theorem 2.1 in [Krstić, 2009]. Hence, with (56), we get

$$\Psi(t)\leq\frac{\alpha_2}{\alpha_1}e^{-\mu t}\Psi(0), \qquad (60)$$

which completes the proof of exponential stability.

***Step 7:*** *Invoking Averaging Theorem*

First, note that the closed-loop system (5) and (18) can be rewritten as:

$$\dot{\tilde{\theta}}(t-D)=U(t-D), \qquad (61)$$

$$\dot{U}(t)=-cU(t)+c\left\{k\left[G(t)+\hat{H}(t)\int_{t-D}^t U(\tau)d\tau\right]\right\}, \qquad (62)$$

where $z(t)=[\tilde{\theta}(t-D),U(t)]^T$ is the state vector. Moreover, from $G(t)$ in (6) and $\hat{H}(t)$ in (8), one has

$$\dot{z}(t)=f(\omega t,z_t), \qquad (63)$$

where $z_t(\Theta)=z(t+\Theta)$ for $-D\leq\Theta\leq 0$ and $f$ is an appropriate continuous functional, such that the averaging theorem by [Hale and Lunel, 1990] and [Lehman, 2002] can be directly applied considering $\omega=1/\epsilon$.

From (60), the origin of the average closed-loop system (37)–(39) with transport PDE for delay representation is exponentially stable.

Then, according to the averaging theorem [Hale and Lunel, 1990, Lehman, 2002], for $\omega$ sufficiently large, (33)–(35) has a unique exponentially stable periodic solution around its equilibrium (origin) satisfying (19).

***Step 8:*** *Asymptotic Convergence to a Neighborhood of the Extremum $(\theta^*,y^*)$*

By using the change of variables (36) and then integrating both sides of (22) within the interval $[t,\sigma+D]$, we have:

$$\tilde{\vartheta}(\sigma+D)=\tilde{\vartheta}(t)+\int_t^{\sigma+D}u(0,s)ds. \qquad (64)$$

From (25), we can rewrite (64) in terms of $U$, namely

$$\tilde{\vartheta}(\sigma+D)=\tilde{\vartheta}(t)+\int_{t-D}^{\sigma}U(\tau)d\tau. \qquad (65)$$

Now, note that

$$\tilde{\theta}(\sigma)=\tilde{\vartheta}(\sigma+D), \quad \forall\sigma\in[t-D,t]. \qquad (66)$$

Hence,

$$\tilde{\theta}(\sigma)=\tilde{\theta}(t-D)+\int_{t-D}^{\sigma}U(\tau)d\tau, \quad \forall\sigma\in[t-D,t]. \qquad (67)$$

By applying the supremum norm in both sides of (67), we have

$$\begin{aligned}\sup_{t-D\leq\sigma\leq t}\left|\tilde{\theta}(\sigma)\right| &= \sup_{t-D\leq\sigma\leq t}\left|\tilde{\theta}(t-D)\right|+\sup_{t-D\leq\sigma\leq t}\left|\int_{t-D}^{\sigma}U(\tau)d\tau\right| \\ &\leq \sup_{t-D\leq\sigma\leq t}\left|\tilde{\theta}(t-D)\right|+\sup_{t-D\leq\sigma\leq t}\int_{t-D}^{t}|U(\tau)|\,d\tau \\ &\leq \left|\tilde{\theta}(t-D)\right|+\int_{t-D}^{t}|U(\tau)|\,d\tau \quad \text{(Cauchy-Schwarz)} \\ &\leq \left|\tilde{\theta}(t-D)\right|+\left(\int_{t-D}^{t}d\tau\right)^{1/2}\times\left(\int_{t-D}^{t}|U(\tau)|^2\,d\tau\right)^{1/2} \\ &\leq \left|\tilde{\theta}(t-D)\right|+\sqrt{D}\left(\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2}. \end{aligned} \qquad (68)$$

Now, it is easy to check

$$\left|\tilde{\theta}(t-D)\right|\leq\left(\left|\tilde{\theta}(t-D)\right|^2+\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2}, \qquad (69)$$

$$\left(\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2}\leq\left(\left|\tilde{\theta}(t-D)\right|^2+\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2}. \qquad (70)$$

By using (69) and (70), one has

$$\begin{aligned}\left|\tilde{\theta}(t-D)\right|+\sqrt{D}\left(\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2} &\leq (1+\sqrt{D})\left(\left|\tilde{\theta}(t-D)\right|^2\right. \\ &\left.+\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2}. \end{aligned} \qquad (71)$$

From (68), it is straightforward to conclude that

$$\sup_{t-D\leq\sigma\leq t}\left|\tilde{\theta}(\sigma)\right|\leq(1+\sqrt{D})\left(\left|\tilde{\theta}(t-D)\right|^2+\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2} \qquad (72)$$

and, consequently,

$$\left|\tilde{\theta}(t)\right|\leq(1+\sqrt{D})\left(\left|\tilde{\theta}(t-D)\right|^2+\int_{t-D}^{t}U^2(\tau)d\tau\right)^{1/2}. \qquad (73)$$

Inequality (73) can be given in terms of the periodic solution $\tilde{\theta}^{\Pi}(t-D)$, $U^{\Pi}(\sigma)$, $\forall\sigma\in[t-D,t]$ as follows

$$\begin{aligned}\left|\tilde{\theta}(t)\right| &\leq (1+\sqrt{D})\left(\left|\tilde{\theta}(t-D)-\tilde{\theta}^{\Pi}(t-D)+\tilde{\theta}^{\Pi}(t-D)\right|^2\right. \\ &\left.+\int_{t-D}^{t}\left[U(\tau)-U^{\Pi}(\tau)+U^{\Pi}(\tau)\right]^2d\tau\right)^{1/2}. \end{aligned} \qquad (74)$$

By applying Young's inequality and some algebra, the right-hand side of (74) and $\left|\tilde{\theta}(t)\right|$ can be majorized by

$$\left|\tilde{\theta}(t)\right| \leq \sqrt{2}\,(1+\sqrt{D})\left(\left|\tilde{\theta}(t-D)-\tilde{\theta}^{\Pi}(t-D)\right|^2+\left|\tilde{\theta}^{\Pi}(t-D)\right|^2 + \int_{t-D}^{t}\left[U(\tau)-U^{\Pi}(\tau)\right]^2 d\tau + \int_{t-D}^{t}\left[U^{\Pi}(\tau)\right]^2 d\tau\right)^{1/2} \quad (75)$$

From the averaging theorem [Hale and Lunel, 1990, Lehman, 2002], we have $\tilde{\theta}(t-D)-\tilde{\theta}^{\Pi}(t-D)\to 0$ and $\int_{t-D}^{t}\left[U(\tau)-U^{\Pi}(\tau)\right]^2 d\tau \to 0$, exponentially. Hence,

$$\limsup_{t\to+\infty}|\tilde{\theta}(t)| = \sqrt{2}\,(1+\sqrt{D}) \times\left(\left|\tilde{\theta}^{\Pi}(t-D)\right|^2+\int_{t-D}^{t}[U^{\Pi}(\tau)]^2 d\tau\right)^{1/2}. \quad (76)$$

From (19) and (76), we can write $\limsup_{t\to+\infty}|\tilde{\theta}(t)| = \mathcal{O}(1/\omega)$. From (4) and reminding that $\theta(t)=\hat{\theta}(t)+S(t)$ with $S(t)=a\sin(\omega(t+D))$, one has that

$$\theta(t)-\theta^* = \tilde{\theta}(t)+S(t)\,. \quad (77)$$

Since the first term in the right-hand side of (77) is ultimately of order $\mathcal{O}(1/\omega)$ and the second term is of order $\mathcal{O}(a)$, then

$$\limsup_{t\to+\infty}|\theta(t)-\theta^*| = \mathcal{O}(a+1/\omega)\,. \quad (78)$$

Finally, from (3) and (78), we get (21). □

## 4. SIMULATION EXAMPLE

In order to evaluate the proposed extremum seeking with delay compensation, the following static quadratic map is considered

$$Q(\theta) = 5-(\theta-2)^2\,, \quad (79)$$

subject to an output delay of $D=5$ s. According to (79), the extremum point is $(\theta^*,\ y^*)=(2,\ 5)$ and the Hessian of the map is $H=-2$. In what follows, we present numerical simulations of the predictor (18), where $\hat{H}$ is given by (8) and $c=20$. We perform our tests with the following parameters: $a=0.2$, $\omega=10$, $k=0.2$, and $\theta(0)=0$. Figure 2 shows the system output $y(t)$ in 3 situations: (a) free of output delays, (b) in the presence of output delay but without any delay compensation and (c) with output-delay and predictor based compensation.

## 5. CONCLUSIONS

A new predictor feedback strategy with perturbation-based estimate of the Hessian is introduced to cope with input-output delays in the control-loop of gradient extremum seeking controllers. The resulting approach preserves exponential stability and convergence of the system output to a small neighborhood of the extremum point, despite of the presence of actuator and sensor delays. A rigorous demonstration is given by exploring backstepping transformation and averaging theory in infinite dimensions. While the convergence rate of the gradient method is dictated by the unknown Hessian, the Newton-based scheme addressed in a companion paper [Oliveira and Krstić, 2015] is independent of it and thus the delay compensation can be achieved with an arbitrarily assigned convergence rate, improving the controller performance. The results presented here are given for scalar plants, but the extension to the multivariable case can be found in [Oliveira et al., 2015].

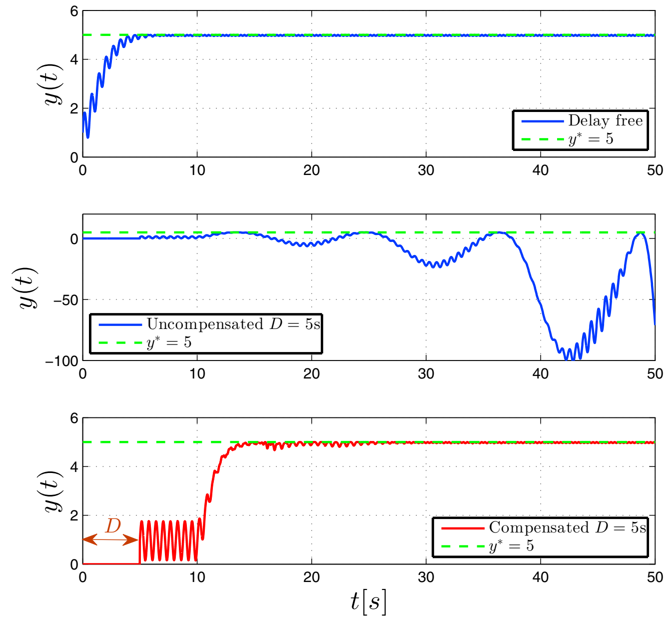


Fig. 2. **Gradient based ESC plus output delay (time response of $y(t)$):** (a) basic ESC works well without delays; (c) ESC goes unstable in the presence of delays; (3) predictor fixes this.

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