A Practical Introduction to Integer Linear Programming

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mobikit

Outline

- What is Integer Linear Programming and what is it good for?
- The vehicle fleet assignment problem
- Using Google’s OR-Tools library
- Why real world data problems require both art and science

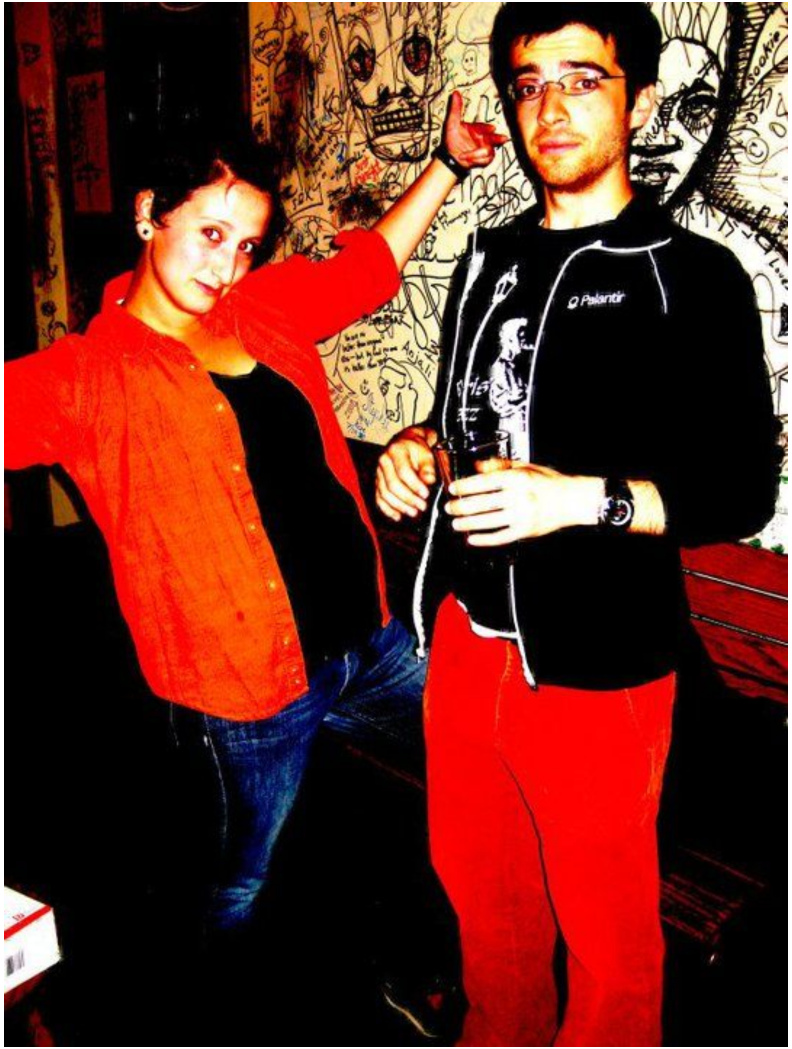
Some History

- Integer Linear Programming is a variant of Linear Programming

- Both part of Operations Research - rigorous quantitative methods for optimal decision-making

- Modern OR developed during WWII

- Applications in manufacturing, planning, routing, scheduling, many many more.


Palantir

Examples

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- Given a list of potential factory sites, choose factory locations in a way that minimizes operating and transportation costs.

- Given a list of routes and a fleet of vehicles, assign a vehicle to each route in a way that maximizes fleet efficiency and utilization.

Optimization problems that require making **assignment** decisions while being subject to **constraints**

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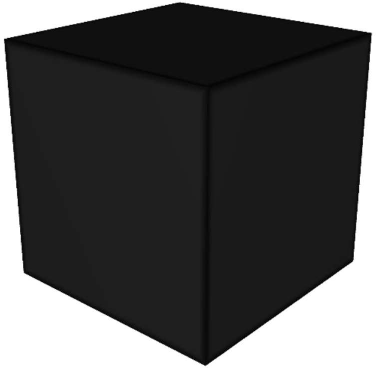
You can use ILP if you can represent your problem in a particular mathematical form.

You can use an ILP solver as a black box!

your assignment problem in “a particular mathematical form”



An optimal assignment

What's this "particular mathematical form"?

Say you want to assign resources to categories.

For every pair of resource r and category c, define a variable

$$X_{r,c} = \begin{cases} 1 \text{ if resource } r \text{ is assigned to category } c \\ 0 \text{ if it is not} \end{cases}$$

What’s this “particular mathematical form”?

Say you want to assign resources to categories.

For every pair of resource r and category c, define a variable

$X_{r,c} =$ 1 if resource r is assigned to category c
0 if it is not

r = Igor
c = first shift on Tuesday
$X_{r,c} =$ 1 if Igor works that shift, 0 if not

What's this "particular mathematical form"?

You can use ILP if you can

1) Write constraints as linear inequalities over these variables, e.g.

$$X_{\text{Igor,last shift Monday}} + X_{\text{Igor,first shift Tuesday}} \leq 1$$

What’s this “particular mathematical form”?

You can use ILP if you can

1) Write constraints as linear inequalities over these variables, e.g.

$$X_{\text{Igor,last shift Monday}} + X_{\text{Igor,first shift Tuesday}} \leq 1$$

2) Write the function to be optimized as a linear sum of these variables, e.g.

minimize $\displaystyle\sum_{\text{shifts}} 250X_{\text{Igor,shift}} + \sum_{\text{shifts}} 300X_{\text{Rachel,shift}} + \ldots$

Vehicle fleet assignment

- Assignment: assign vehicles to routes

- Optimization: minimize wasted capacity, i.e. use smallest available vehicles

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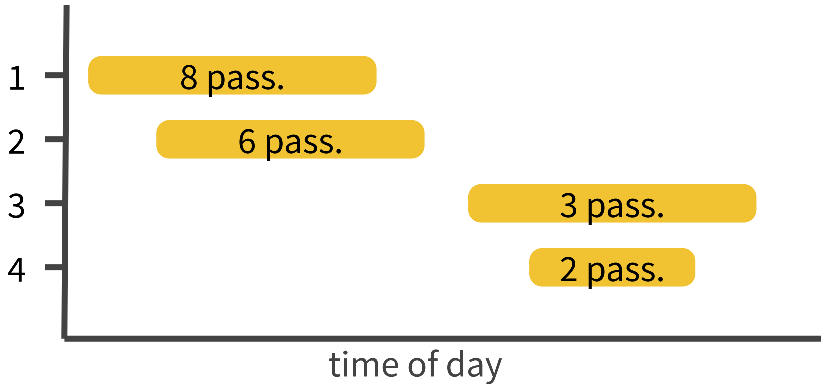
<u>Four Routes</u>

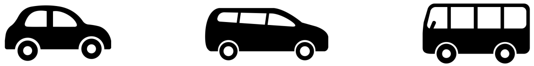


<u>Three Vehicles</u>

sedan (4 pass.)

van (6 pass.)

bus (10 pass.)

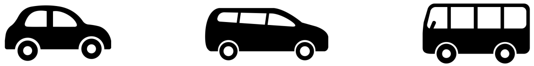


Variables

$$X_{sedan,1} \quad X_{sedan,2} \quad X_{sedan,3} \quad X_{sedan,4}$$

$$X_{van,1} \quad X_{van,2} \quad X_{van,3} \quad X_{van,4}$$

$$X_{bus,1} \quad X_{bus,2} \quad X_{bus,3} \quad X_{bus,4}$$



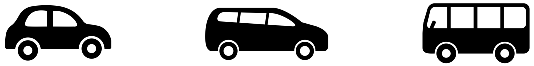
Coverage constraints: every route should have exactly one vehicle assigned to it

$$X_{sedan,1} + X_{van,1} + X_{bus,1} = 1$$

$$X_{sedan,2} + X_{van,2} + X_{bus,2} = 1$$

$$X_{sedan,3} + X_{van,3} + X_{bus,3} = 1$$

$$X_{sedan,4} + X_{van,4} + X_{bus,4} = 1$$

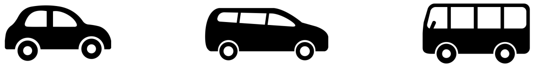


Capacity constraints: a vehicle assigned to a route must have sufficient capacity

$$X_{sedan,1} = 0$$

$$X_{sedan,2} = 0$$

$$X_{van,1} = 0$$

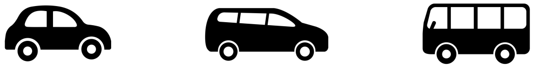


<u>Overlap constraints:</u> two routes that overlap in time can't use the same vehicle

$$X_{sedan,1} + X_{sedan,2} \leq 1 \qquad X_{sedan,3} + X_{sedan,4} \leq 1$$

$$X_{van,1} + X_{van,2} \leq 1 \qquad X_{van,3} + X_{van,4} \leq 1$$

$$X_{bus,1} + X_{bus,2} \leq 1 \qquad X_{bus,3} + X_{bus,4} \leq 1$$



<u>Function to minimize</u>

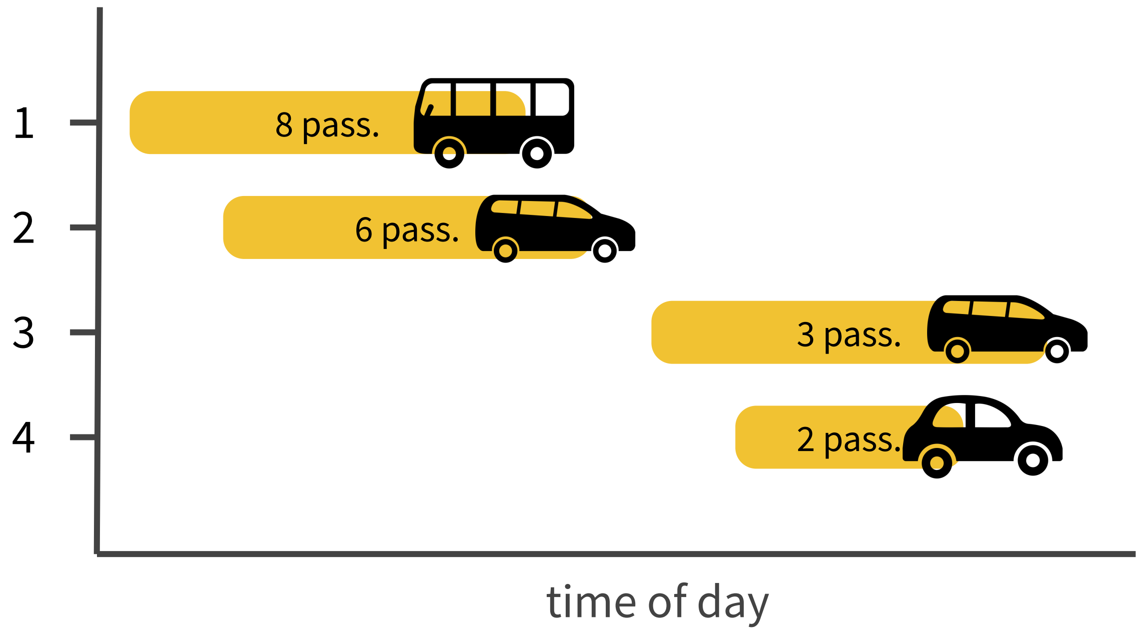
$$\text{minimize} \quad \left\{ \begin{array}{l} 4X_{sedan,1} + 6X_{van,1} + 10X_{bus,1} + \\ 4X_{sedan,2} + 6X_{van,2} + 10X_{bus,2} + \\ 4X_{sedan,3} + 6X_{van,3} + 10X_{bus,3} + \\ 4X_{sedan,4} + 6X_{van,4} + 10X_{bus,4} \end{array} \right\}$$

Function to minimize

$$\text{minimize} \left\{ \begin{aligned} &4X_{sedan,1} + 6X_{van,1} + 10X_{bus,1} + \\ &4X_{sedan,2} + 6X_{van,2} + 10X_{bus,2} + \\ &4X_{sedan,3} + 6X_{van,3} + 10X_{bus,3} + \\ &4X_{sedan,4} + 6X_{van,4} + 10X_{bus,4} \end{aligned} \right\}$$

Each row is the capacity of the vehicle assigned to that route

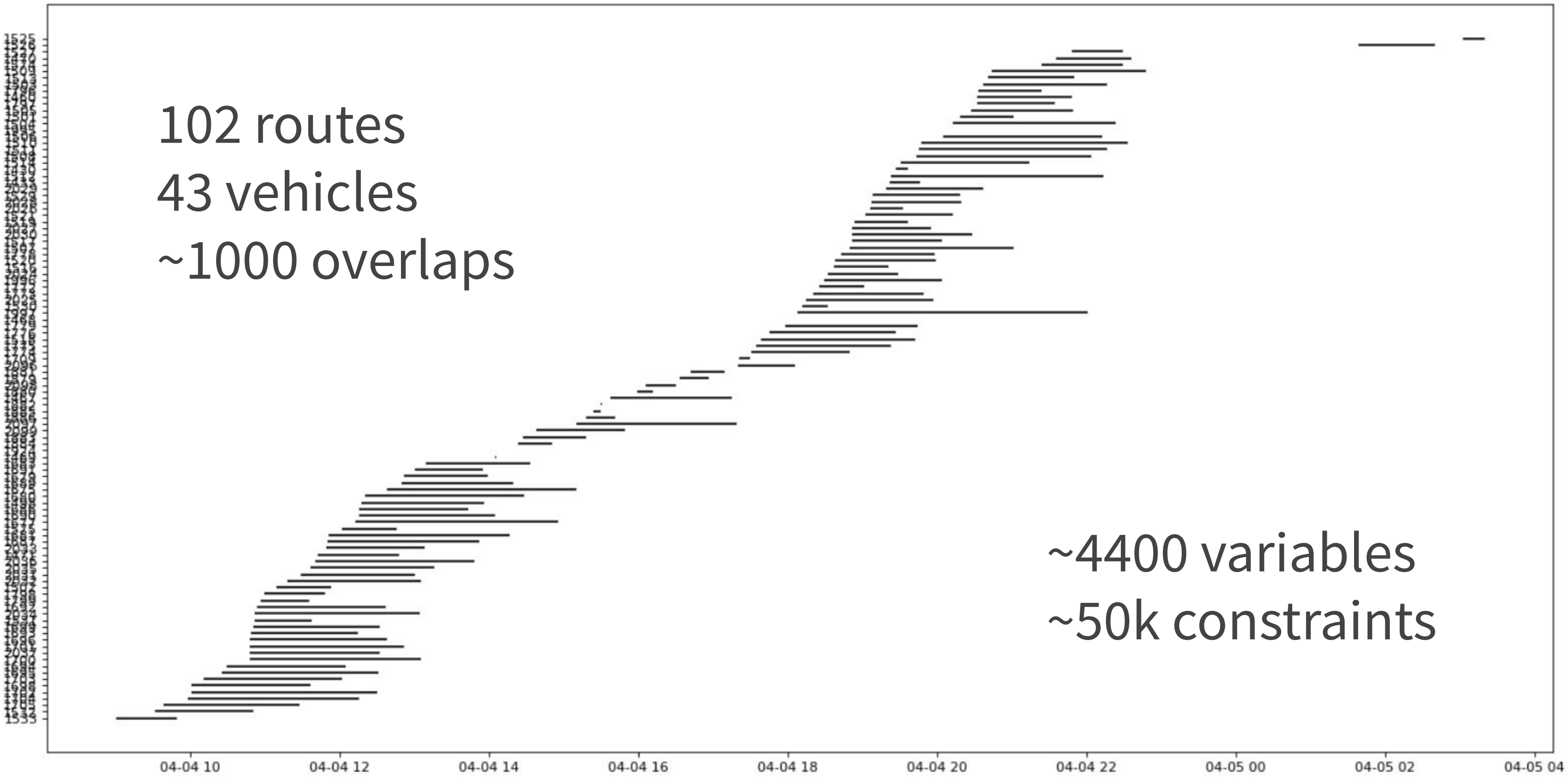
Google OR-Tools


1
2
3
4
8 pass.
6 pass.
3 pass.
2 pass.
time of day

Life is good!

~~Life is good!~~

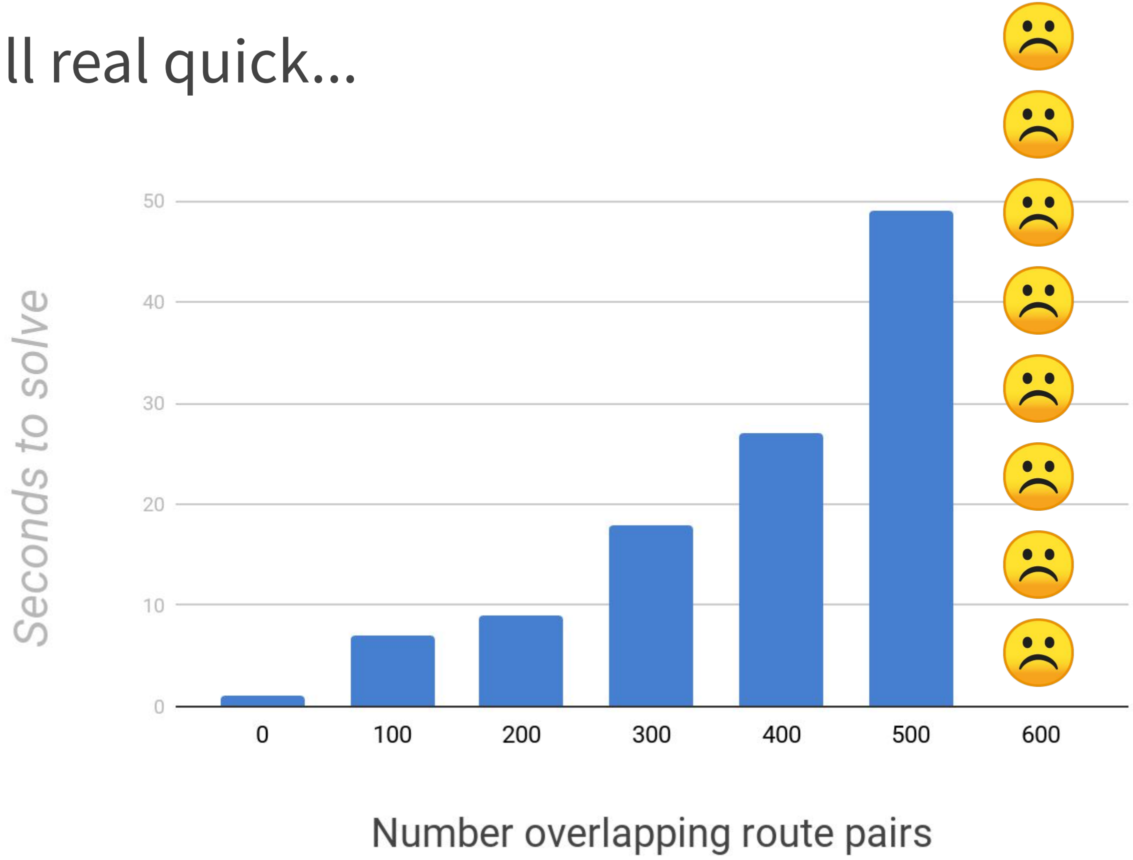
Sorry, ILP is
NP-complete...

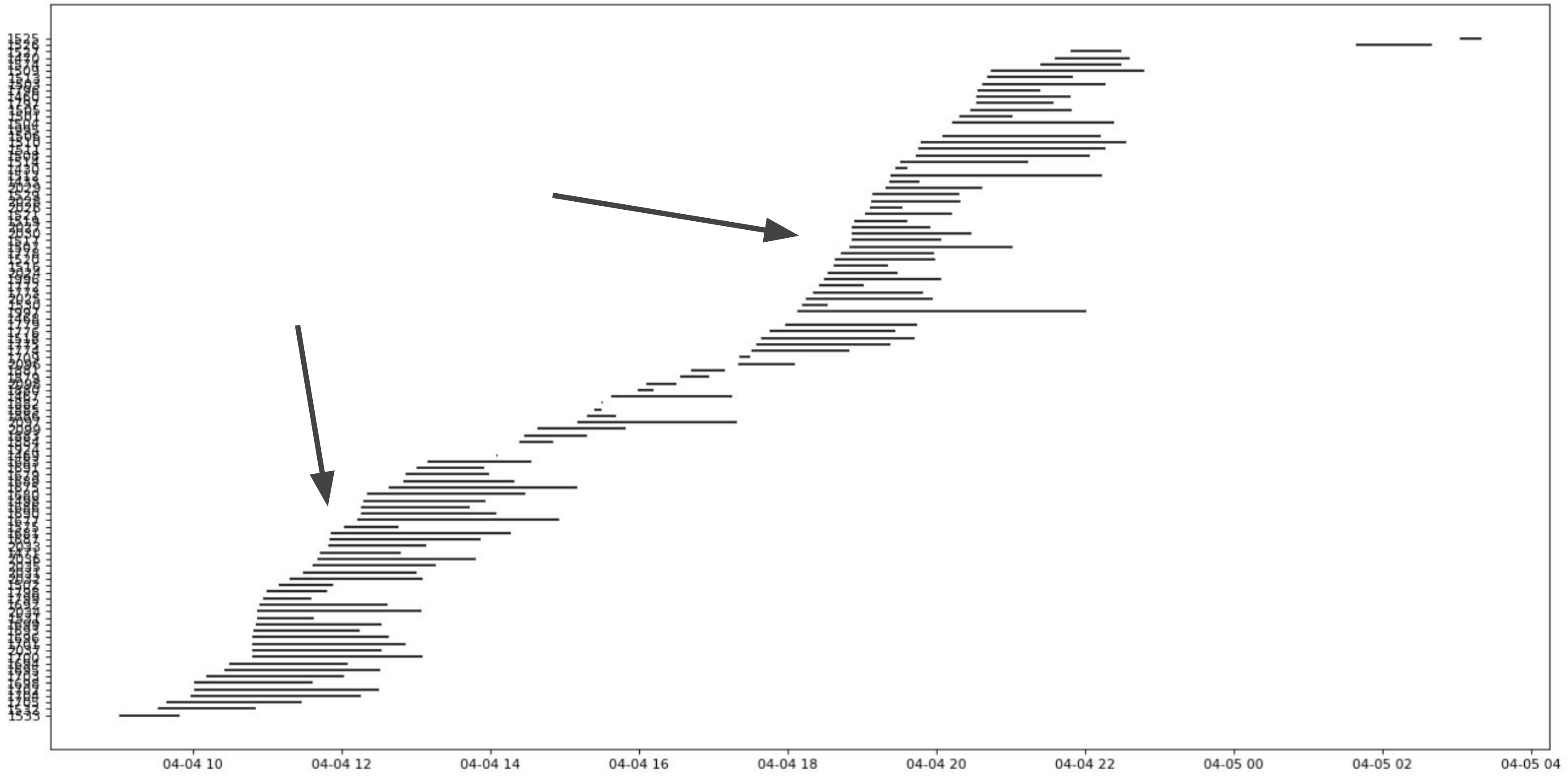


102 routes
43 vehicles
~1000 overlaps

~4400 variables
~50k constraints

We hit a wall real quick...




04-04 10
04-04 12
04-04 14
04-04 16
04-04 18
04-04 20
04-04 22
04-05 00
04-05 02
04-05 04

One of my favorites!

The fleet assignment problem: solving a large-scale integer program [1]

Christopher A. Hane, Cynthia Barnhart, Ellis L. Johnson, Roy E. Marsten, George L. Nemhauser *, Gabriele Sigismondi

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https://www.academia.edu/15152537/The_fleet_assignment_problem_Solving_a_large-scale_integer_program

Among the factors considered in assigning a fleet to a flight leg are passenger demand (both point-to-point and continuing service), revenue, seating capacity, fuel costs, crew size, availability of maintenance at arrival and departure stations, gate availability, and aircraft noise. Many of these factors are captured in the objective coefficient of the decision variable, others are captured by constraints. For example, the potential revenue generated by a flight is determined by forecasting the demand for seats on that flight and multiplying the minimum of it and the seat capacity by the average fare.

The main contribution of this paper is a case study in the solution of a very large mixed-integer program. Using standard default options of a mathematical programming system, we could not come close to solving problems of the size that are required. The solution methodology developed in this paper solves a 150-city, 2500-flight, eleven-fleet daily fleet assignment problem routinely in less than one hour.

Thanks for coming!

github.com/igorferst/pyohio2019

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