A Practical Introduction to Integer Linear Programming

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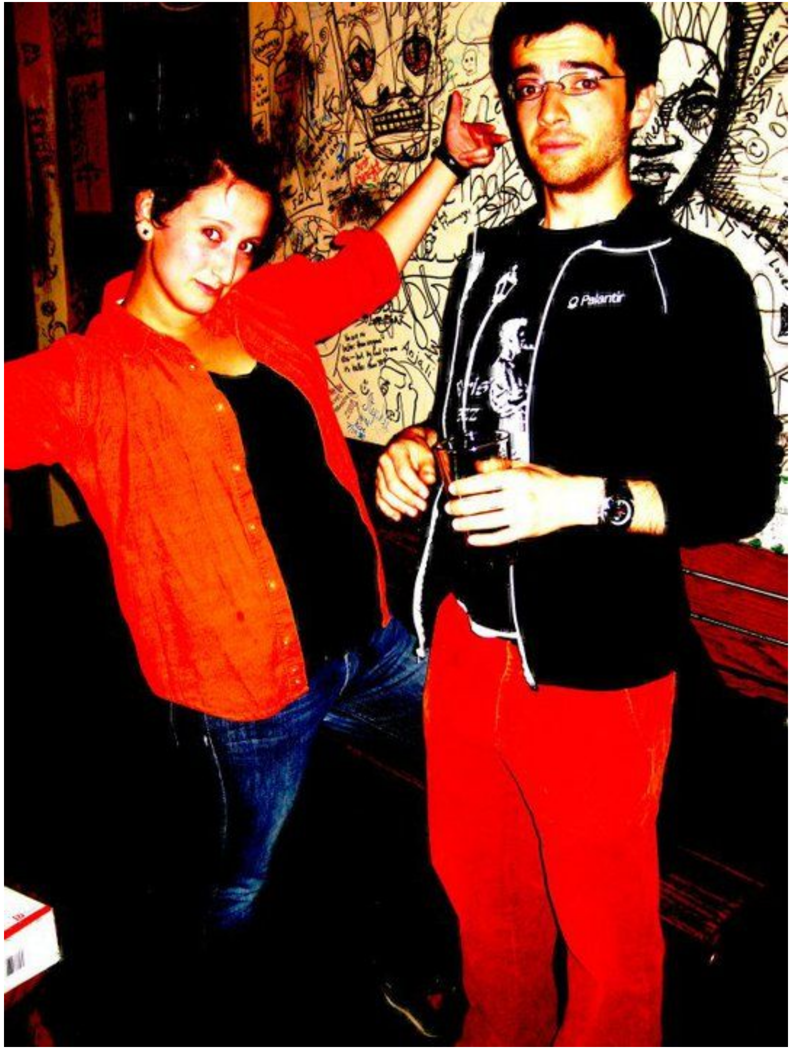
mobikit

Outline

- What is Integer Linear Programming and what is it good for?
- The vehicle fleet assignment problem
- Using Google's OR-Tools library
- ILP in practice

Some History

- Integer Linear Programming is a variant of Linear Programming
- Both part of Operations Research - rigorous quantitative methods for optimal decision-making
- Modern OR developed during WWII
- Computers expanded the scope and scale of applications: manufacturing, planning, routing, scheduling, etc.


Palantir

Examples

- Given a list of work shifts and a group of employees, schedule employees for all shifts while respecting employee availability and preferences.

- Given a list of potential factory sites, choose factory locations in a way that minimizes operating and transportation costs.

- Given a list of routes and a set of vehicles, assign a vehicle to each route in a way that maximizes fleet efficiency and utilization.

ILP is a powerful technique for solving **optimization** problems that require making **assignment** decisions for a set of discrete **resources** while being subject to **constraints**.

You can use this technique if you are able to represent your problem in a particular mathematical form.

Examples

- Given a list of work shifts and a group of employees, **schedule employees** for all shifts while respecting employee availability and preferences.
- Given a list of potential factory sites, **choose factory locations** in a way that minimizes operating and transportation costs.
- Given a list of routes and a set of vehicles, **assign a vehicle to each route** in a way that maximizes fleet efficiency and utilization.

What's this "particular mathematical form"?

Say you want to assign resources to categories.

For every pair of resource **r** and category **c**, define a variable

$$x_{rc} = \begin{cases} 1 \text{ if resource } \mathbf{r} \text{ is assigned to category } \mathbf{c} \\ 0 \text{ if it is not} \end{cases}$$

What's this "particular mathematical form"?

You can use ILP if you can

1) Write constraints as linear inequalities over these variables, e.g.

$$x_{A1} + 3x_{B1} + 4.2x_{A2} + 1.5x_{B2} \leq 10$$

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1) Write constraints as linear inequalities over these variables, e.g.

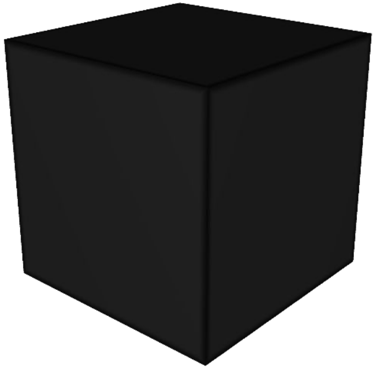
$$x_{A1} + 3x_{B1} + 4.2x_{A2} + 1.5x_{B2} \leq 10$$

2) Write the function to be optimized as a linear sum of these variables, e.g.

$$\textit{minimize} \quad 8x_{A1} + 2x_{B1} + 0.7x_{A2} - x_{B2}$$

You can use an ILP solver as a black box!

constraints in the form of linear inequalities
+
linear function to optimize



A value of 0 or 1 for each x_{rc}

Vehicle fleet assignment

- Assignment: assign vehicles to routes
- Optimization: minimize wasted capacity, i.e. use smallest available vehicles

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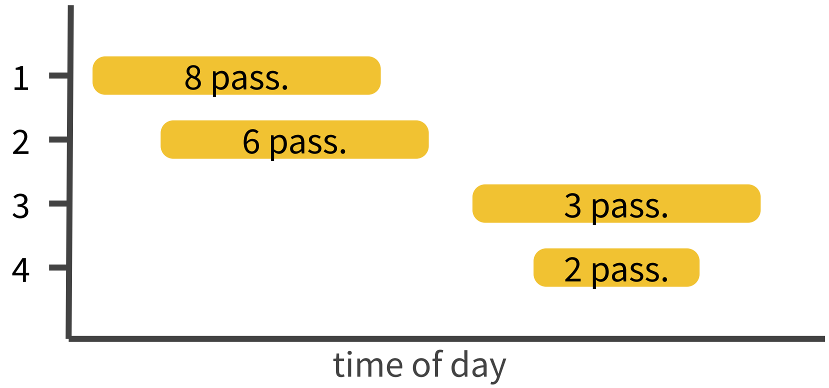
Four Routes

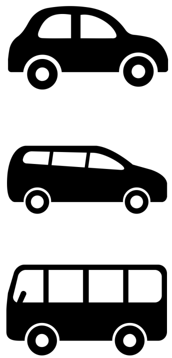


Three Vehicles

A (4 pass.)

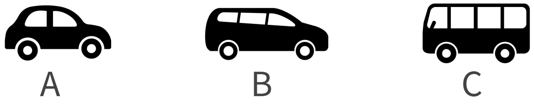
B (6 pass.)

C (10 pass.)



Vehicle fleet assignment

- Assignment: assign vehicles to routes
- Optimization: minimize wasted capacity, i.e. use smallest available vehicles
- Constraints:
    - [coverage] every route should have exactly one vehicle assigned to it
    - [capacity] a vehicle assigned to a route must have sufficient capacity
    - [overlap] two routes that overlap in time can't use the same vehicle

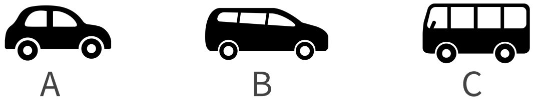


Variables

$$x_{A1}, x_{A2}, x_{A3}, x_{A4}$$

$$x_{B1}, x_{B2}, x_{B3}, x_{B4}$$

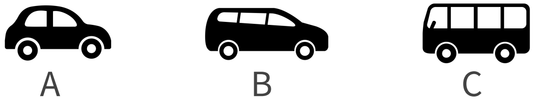
$$x_{C1}, x_{C2}, x_{C3}, x_{C4}$$



Coverage constraints

$$x_{A1} + x_{B1} + x_{C1} = 1 \qquad x_{A3} + x_{B3} + x_{C3} = 1$$

$$x_{A2} + x_{B2} + x_{C2} = 1 \qquad x_{A4} + x_{B4} + x_{C4} = 1$$

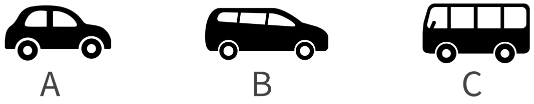


Capacity constraints

$$x_{A1} = 0$$

$$x_{A2} = 0$$

$$x_{B1} = 0$$

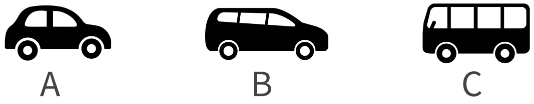


Overlap constraints

$$x_{A1} + x_{A2} \leq 1 \qquad x_{A3} + x_{A4} \leq 1$$

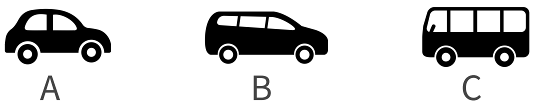
$$x_{B1} + x_{B2} \leq 1 \qquad x_{B3} + x_{B4} \leq 1$$

$$x_{C1} + x_{C2} \leq 1 \qquad x_{C3} + x_{C4} \leq 1$$



Function to minimize

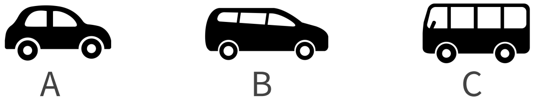
$$\min \left\{ \begin{array}{l} 4x_{A1} + 6x_{B1} + 10x_{C1} + \\ 4x_{A2} + 6x_{B2} + 10x_{C2} + \\ 4x_{A3} + 6x_{B3} + 10x_{C3} + \\ 4x_{A4} + 6x_{B4} + 10x_{C4} \end{array} \right\}$$



Function to minimize

$$
\min \left\{
\begin{array}{l}
4x_{A1} + 6x_{B1} + 10x_{C1} + \\
4x_{A2} + 6x_{B2} + 10x_{C2} + \\
4x_{A3} + 6x_{B3} + 10x_{C3} + \\
4x_{A4} + 6x_{B4} + 10x_{C4}
\end{array}
\right\}
$$

Only one variable will have value 1 per route



Function to minimize

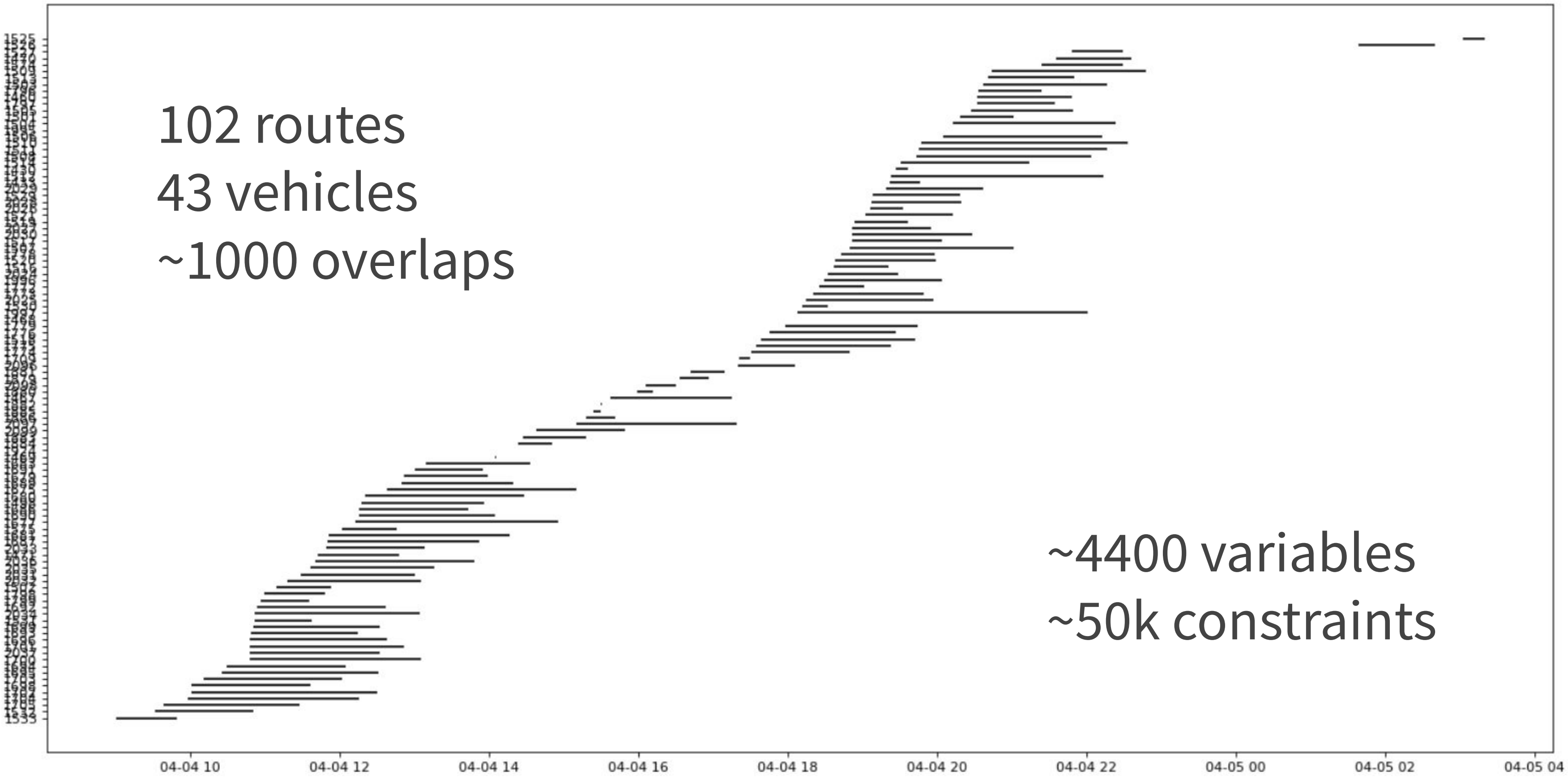
$$\min \left\{ \begin{array}{l} \text{capacity of vehicle assigned to route 1} \\ + \\ \text{capacity of vehicle assigned to route 2} \\ + \\ \text{capacity of vehicle assigned to route 3} \\ + \\ \text{capacity of vehicle assigned to route 4} \end{array} \right\}$$

Google OR-Tools

Life is good!

~~Life is good!~~

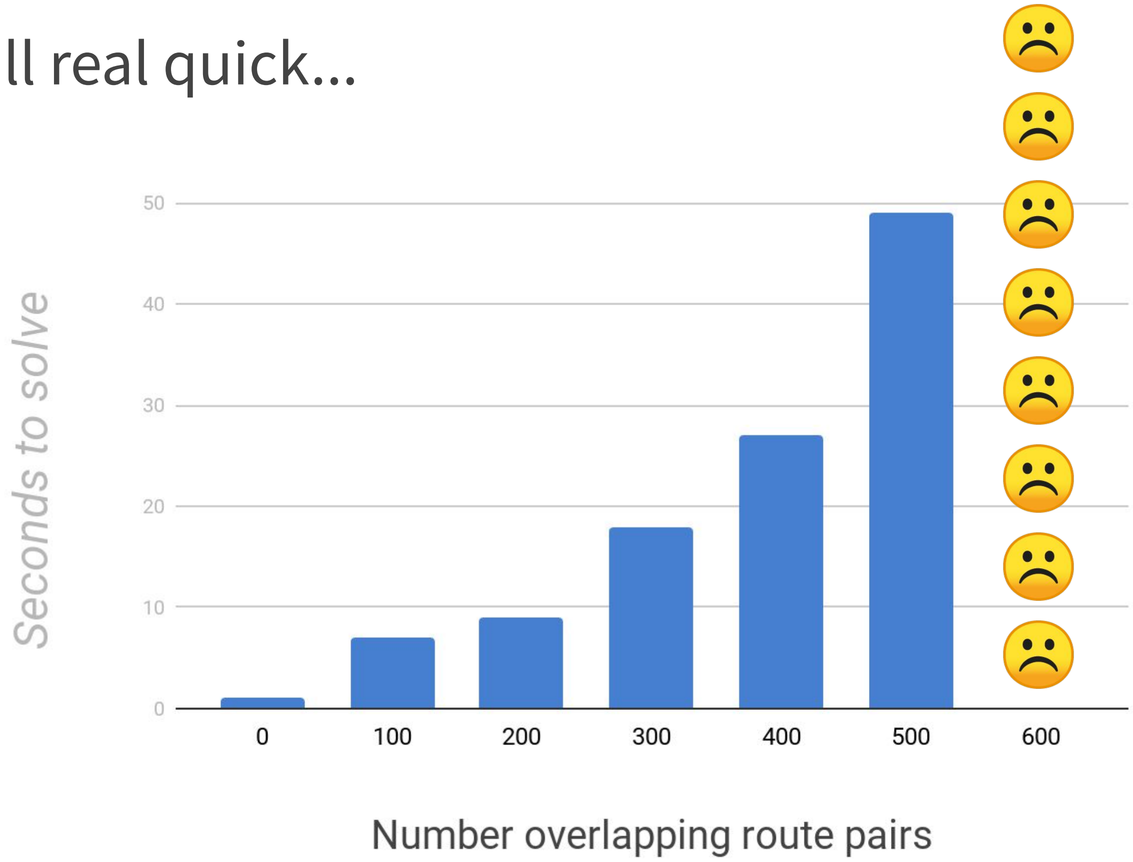
Sorry, ILP is NP-complete...

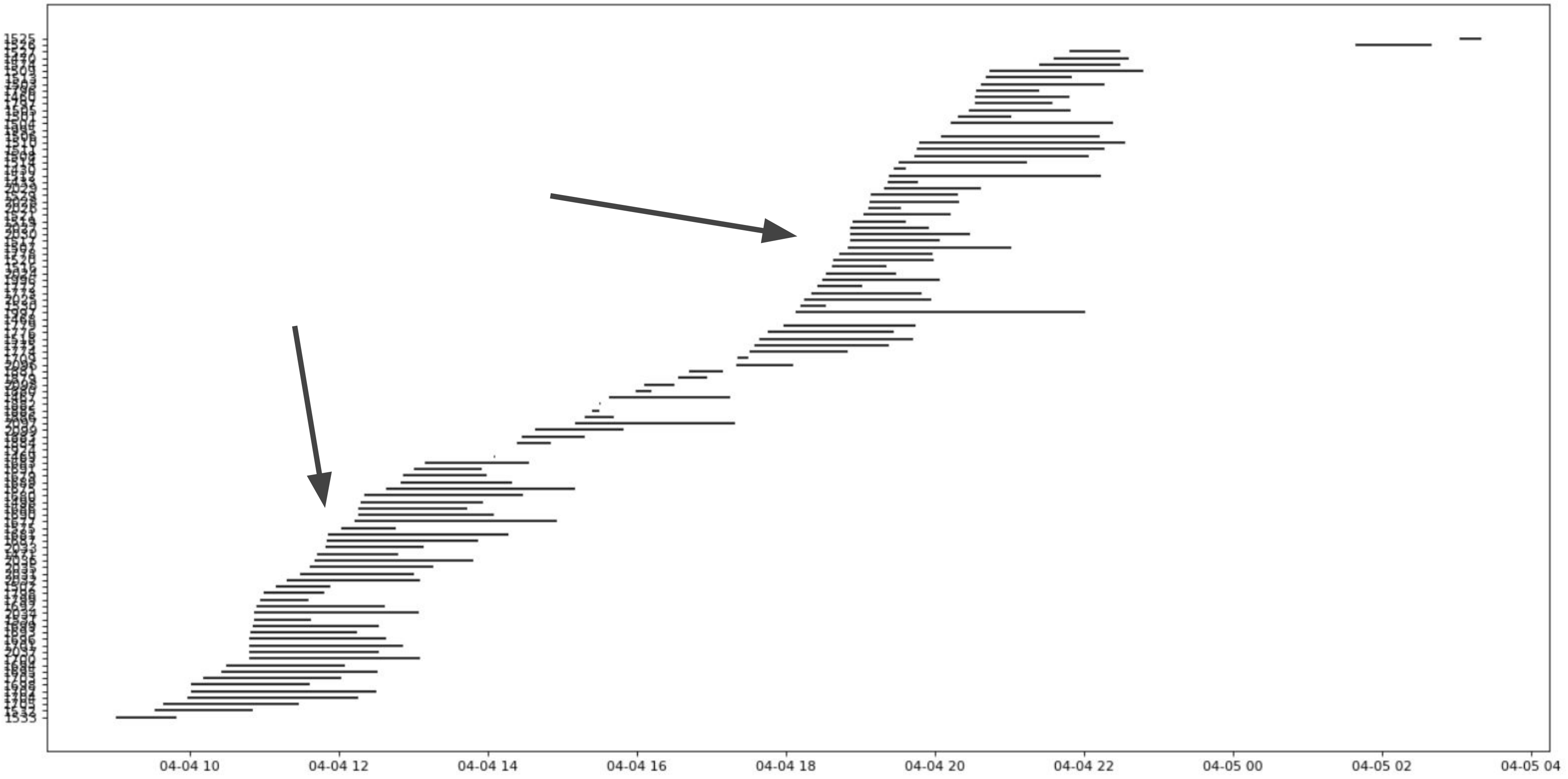


102 routes
43 vehicles
~1000 overlaps

~4400 variables
~50k constraints

We hit a wall real quick...




04-04 10
04-04 12
04-04 14
04-04 16
04-04 18
04-04 20
04-04 22
04-05 00
04-05 02
04-05 04

One of my favorites!

The fleet assignment problem: solving a large-scale integer program [1]

Christopher A. Hane, Cynthia Barnhart, Ellis L. Johnson, Roy E. Marsten, George L. Nemhauser *, Gabriele Sigismondi

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https://www.academia.edu/15152537/The_fleet_assignment_problem_Solving_a_large-scale_integer_program

Among the factors considered in assigning a fleet to a flight leg are passenger demand (both point-to-point and continuing service), revenue, seating capacity, fuel costs, crew size, availability of maintenance at arrival and departure stations, gate availability, and aircraft noise. Many of these factors are captured in the objective coefficient of the decision variable, others are captured by constraints. For example, the potential revenue generated by a flight is determined by forecasting the demand for seats on that flight and multiplying the minimum of it and the seat capacity by the average fare.

The main contribution of this paper is a case study in the solution of a very large mixed-integer program. Using standard default options of a mathematical programming system, we could not come close to solving problems of the size that are required. The solution methodology developed in this paper solves a 150-city, 2500-flight, eleven-fleet daily fleet assignment problem routinely in less than one hour.

Thanks for coming!

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