Z_N Mode Selection Robustness

Under Non-Quadratic Anchor Potential $W(u)$

EDC Project

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Abstract

We prove that the mode selection result—$\cos(N\theta)$ is the leading anisotropic mode under Z_N delta-pinning—is **robust** when the anchor potential $W(u)$ is not purely quadratic. The key insight is that mode *index* selection is determined by the Hessian (second variation) of the energy functional, which depends only on $W''(u_0)$. Nonlinear terms W''', W'''', etc., generate higher harmonics and amplitude corrections but do not change the leading mode index near equilibrium.

Contents

1 Setup: General Smooth Potential

1.1 Energy Functional with General W

Definition 1.1 (General Anchor Potential). **[P]** Let $W : \mathbb{R} \to \mathbb{R}$ be a C^4 function with a stable minimum at u_0:

$$W'(u_0) = 0 \quad \text{(equilibrium)} \tag{1}$$
$$W''(u_0) = \kappa > 0 \quad \text{(stability)} \tag{2}$$

Definition 1.2 (Full Energy Functional). **[P]**

$$E[u] = \frac{T}{2}\int_0^{2\pi} (u')^2\, d\theta + \lambda \sum_{n=0}^{N-1} W(u(\theta_n)) \tag{3}$$

where $\theta_n = 2\pi n/N$ are the Z_N fixed points.

1.2 Taylor Expansion of W

Proposition 1.3 (Potential Expansion). ***[Der]*** *Setting $\eta = u - u_0$ (perturbation around equilibrium):*

$$W(u_0 + \eta) = W_0 + \frac{\kappa}{2}\eta^2 + \frac{g}{6}\eta^3 + \frac{h}{24}\eta^4 + O(\eta^5) \tag{4}$$

where:

$$W_0 = W(u_0) \quad \textit{(constant, irrelevant for dynamics)} \tag{5}$$
$$\kappa = W''(u_0) > 0 \quad \textit{(quadratic stiffness)} \tag{6}$$
$$g = W'''(u_0) \quad \textit{(cubic coupling)} \tag{7}$$
$$h = W''''(u_0) \quad \textit{(quartic coupling)} \tag{8}$$

2 Second Variation and Mode Selection

2.1 First and Second Variations

Theorem 2.1 (Second Variation Controls Mode Selection). ***[Der]*** *Let $u = u_0 + \eta$ with η small. The energy expands as:*

$$E[u_0 + \eta] = E_0 + \delta E[\eta] + \frac{1}{2}\delta^2 E[\eta, \eta] + O(\eta^3) \tag{9}$$

First variation:

$$\delta E[\eta] = \lambda \sum_{n=0}^{N-1} W'(u_0) \cdot \eta(\theta_n) = 0 \tag{10}$$

(vanishes at equilibrium since $W'(u_0) = 0$).

Second variation (Hessian):

$$\boxed{\delta^2 E[\eta, \eta] = T\int_0^{2\pi} (\eta')^2\, d\theta + \lambda\kappa \sum_{n=0}^{N-1} \eta(\theta_n)^2} \tag{11}$$

Proof. From (3) with $u = u_0 + \eta$:

$$E[u_0 + \eta] = \frac{T}{2} \int (\eta')^2 \, d\theta + \lambda \sum_n W(u_0 + \eta_n) \tag{12}$$

$$= \frac{T}{2} \int (\eta')^2 \, d\theta + \lambda \sum_n \left[W_0 + \frac{\kappa}{2}\eta_n^2 + \frac{g}{6}\eta_n^3 + \dots \right] \tag{13}$$

The quadratic part is exactly $\frac{1}{2}\delta^2 E$. Higher powers contribute to $O(\eta^3)$ and beyond. □

2.2 The Central Observation

Key Insight: Mode Index Selection is Linear

The second variation (11) has **exactly the same form** as the quadratic energy functional analyzed in the previous derivation:

$$\delta^2 E[\eta, \eta] = T \int (\eta')^2 \, d\theta + \lambda\kappa \sum_n \eta_n^2 \tag{14}$$

This depends only on $\kappa = W''(u_0)$, not on $g = W'''(u_0)$, $h = W''''(u_0)$, etc. Therefore:

- The eigenmode structure of $\delta^2 E$ is identical to the quadratic case
- The Selection Lemma still holds: only $m = kN$ modes couple to anchors
- The gradient ordering still holds: $m = N$ has lowest gradient energy among coupled modes
- **The leading anisotropic mode remains** $\cos(N\theta)$

Theorem 2.2 (Mode Index Robustness). ***[Der]*** *For any C^2 potential W with stable minimum at u_0 ($W'(u_0) = 0$, $W''(u_0) > 0$), the leading anisotropic eigenmode of the Hessian $\delta^2 E$ is $\cos(N\theta)$.*

The mode index $m = N$ is determined by:

1. *Z_N symmetry of anchor positions (Selection Lemma)*
2. *Gradient energy ordering ($E_{grad} \propto m^2$)*

Neither depends on W''', W'''', or any higher derivatives.

3 Nonlinear Corrections: Amplitude and Harmonics

3.1 Leading-Order Solution

Proposition 3.1 (Linear Solution). ***[Der]*** *At leading order, the perturbation takes the form:*

$$\eta^{(1)}(\theta) = A\cos(N\theta) \tag{15}$$

where A is the amplitude, determined by the source/forcing.

3.2 Cubic Correction

Proposition 3.2 (First Nonlinear Correction). ***[Dc]*** *The cubic term in W generates corrections at order A^2:*

$$W_{cubic} = \frac{g}{6}\eta^3 = \frac{g}{6}A^3\cos^3(N\theta) \tag{16}$$

Using $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$:

$$\cos^3(N\theta) = \frac{3}{4}\cos(N\theta) + \frac{1}{4}\cos(3N\theta) \tag{17}$$

Result: *The cubic term:*

- *Modifies the coefficient of $\cos(N\theta)$ (amplitude shift)*
- *Generates a new harmonic $\cos(3N\theta)$ at order A^3*

3.3 Quartic Correction

Proposition 3.3 (Second Nonlinear Correction). ***[Dc]*** *The quartic term generates corrections at order A^3:*

$$W_{quartic} = \frac{h}{24}\eta^4 = \frac{h}{24}A^4\cos^4(N\theta) \tag{18}$$

Using $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$:

$$\cos^4(N\theta) = \frac{3}{8} + \frac{1}{2}\cos(2N\theta) + \frac{1}{8}\cos(4N\theta) \tag{19}$$

Result: *The quartic term:*

- *Shifts the mean value (constant term)*
- *Generates harmonics $\cos(2N\theta)$ and $\cos(4N\theta)$ at order A^4*

3.4 Harmonic Content Table

Harmonic Content vs Amplitude Order

Order	Source	Harmonics Generated	Amplitude
$O(A)$	Linear (κ)	$\cos(N\theta)$	A
$O(A^2)$	—	(none at this order)	—
$O(A^3)$	Cubic (g)	$\cos(N\theta)$, $\cos(3N\theta)$	$\sim gA^3/\kappa$
$O(A^4)$	Quartic (h)	const, $\cos(2N\theta)$, $\cos(4N\theta)$	$\sim hA^4/\kappa$

Key observation: All generated harmonics are multiples of N: $(N, 2N, 3N, 4N, \ldots)$.
No harmonics with $m < N$ are generated by nonlinear terms!

4 Regime of Validity

4.1 Smallness Conditions

Definition 4.1 (Nonlinearity Parameters)**.** **[Der]** Define dimensionless nonlinearity measures:

$$\varepsilon_3 = \frac{|g|A}{\kappa} \quad \text{(cubic strength)} \tag{20}$$

$$\varepsilon_4 = \frac{|h|A^2}{\kappa} \quad \text{(quartic strength)} \tag{21}$$

Proposition 4.2 (Perturbative Regime)**.** ***[Der]*** *The linear mode selection result holds when:*

$$\boxed{\varepsilon_3 \ll 1 \quad \textit{and} \quad \varepsilon_4 \ll 1} \tag{22}$$

Equivalently, the amplitude must satisfy:

$$|A| \ll \min\left(\frac{\kappa}{|g|}, \sqrt{\frac{\kappa}{|h|}}\right) \tag{23}$$

4.2 Physical Interpretation

Proposition 4.3 (Scale of Validity)**.** ***[Dc]*** *The perturbative regime corresponds to:*

$$|u - u_0| \ll L_W \tag{24}$$

where L_W is the scale over which $W(u)$ deviates significantly from quadratic:

$$L_W \sim \min\left(\frac{\kappa}{|g|}, \sqrt{\frac{\kappa}{|h|}}\right) \tag{25}$$

For a "typical" smooth potential, L_W is comparable to the width of the potential well.

5 The Robustness Theorem

Robustness Theorem: Mode Index Selection Under General W

Theorem 5.1 (Mode Selection Robustness)**.** *[Der] Let $W : \mathbb{R} \to \mathbb{R}$ be a C^2 function with:*

1. *A stable equilibrium at u_0: $W'(u_0) = 0$, $W''(u_0) = \kappa > 0$*
2. *Identical anchors at Z_N fixed points $\theta_n = 2\pi n/N$*

Then for sufficiently small amplitude $|A| \ll L_W$:

$$\boxed{\textit{The leading anisotropic mode is } \cos(N\theta)} \tag{26}$$

Nonlinear effects:

- *Modify the amplitude relationship (energy vs A)*
- *Generate higher harmonics $(2N\theta, 3N\theta, \dots)$ at higher orders*
- *Do NOT change the mode index $m = N$*
- *Do NOT introduce harmonics with $m < N$*

Failure modes (when theorem does not apply):

1. **Non-smooth W:** If W is not C^2, Hessian may not exist
2. **Metastability:** If $W''(u_0) \leq 0$, equilibrium is unstable
3. **Large amplitude:** If $|A| \gtrsim L_W$, perturbation theory fails
4. **Symmetry breaking:** If anchors are not identical or not at Z_N positions
5. **Multiple minima:** If system jumps between different equilibria

6 Summary: What Changes vs What Doesn't

Property	**Quadratic W**	**General W**
Mode index ($m = N$)	Fixed	**Unchanged**
Selection Lemma (coupling)	Exact	**Unchanged**
Gradient ordering	Exact	**Unchanged**
Amplitude relation	Linear in source	Nonlinear corrections
Harmonic content	Pure $\cos(N\theta)$	$\cos(N\theta)$ + higher $(2N, 3N, \dots)$
Shape near equilibrium	Pure cosine	Slightly distorted

Bottom line:

*Mode index selection is a **linear** property, determined by the Hessian. Nonlinearities modify amplitude and shape but do not change the leading harmonic near equilibrium.*

7 Epistemic Status

Result	**Status**	**Comment**
Second variation formula	**[Der]**	Standard calculus of variations
Hessian depends only on W''	**[Der]**	Direct calculation
Mode index robustness theorem	**[Der]**	Follows from Hessian structure
Harmonic generation table	**[Dc]**	Perturbation expansion; exact coefficients not computed
Regime of validity bounds	**[Der]**	Dimensional analysis

The central result—mode index $m = N$ is robust under non-quadratic W—is **fully derived** [Der].