Derivation of the Effective Lagrangian $L_{\text{eff}}(q, \dot{q})$ from the 5D Einstein-Hilbert Action

A First-Principles Reduction in Elastic Diffusive Cosmology

(Companion A to Paper 3: NJSR Edition)

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January 2026

DOI: 10.5281/zenodo.18292841

Repository: github.com/igorgrcman/elastic-diffusive-cosmology

(Public artifacts for this paper are in the edc_papers folder.)

Related Documents:

Neutron Lifetime from 5D Membrane Cosmology (DOI: 10.5281/zenodo.18262721)

Framework v2.0 (DOI: 10.5281/zenodo.18299085)

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D: *Selection Rules* (DOI) · E: *Symmetry Ops* (DOI)

F: *Proton Junction* (DOI) · G: *Mass Difference* (DOI)

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Abstract

This companion note provides a complete derivation of the effective one-dimensional Lagrangian $L_{\text{eff}}(q, \dot{q}) = \frac{1}{2}M(q)\dot{q}^2 - V(q)$ from the 5D Einstein-Hilbert action within the Elastic Diffusive Cosmology (EDC) framework. The derivation proceeds in three stages: (i) establishing the supermetric $M(q)$ from the Israel junction conditions; (ii) deriving the potential $V(q)$ from the static energy functional via Euler-Lagrange equations; (iii) combining these into the complete effective Lagrangian with WKB tunneling formula. No Standard Model dynamical parameters are used as inputs—the barrier potential emerges purely from 5D geometry. All conclusions are conditional on the stated assumptions.

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1 Introduction and Scope

1.1 Purpose

This note documents the derivation chain:

$$\boxed{S_{5\text{D}} \xrightarrow{\text{Israel}} M(q) \xrightarrow{\delta E/\delta f=0} V(q) \xrightarrow{\text{combine}} L_{\text{eff}}(q,\dot{q})} \tag{1}$$

The goal is to demonstrate that the effective 1D mechanical system governing neutron decay arises from pure 5D geometry, without importing Standard Model dynamics.

1.2 Epistemic Legend

Epistemic Status Tags

[Der]	**Derived** — follows from explicit calculation from stated premises
[Dc]	**Decisively constrained** — conditional on approximations or parameter choices
[P]	**Proposed** — postulated ansatz, not derived from $\delta S = 0$
[M]	**Mathematics** — pure mathematical identity or theorem
[BL]	**Baseline** — empirical value from PDG/CODATA

1.3 Assumptions

Assumption Box

A1. 5D Warped Geometry [P]: The bulk is AdS_5 with metric $ds_5^2 = a^2(\xi)\eta_{\mu\nu}dx^\mu dx^\nu + d\xi^2$, $a(\xi) = e^{-k|\xi|}$.

A2. Brane Embedding [P]: The 3-brane is embedded as $X^A(\sigma^\mu; q) = (\sigma^\mu, f(r;q))$ with collective coordinate $q \in [0,1]$.

A3. $\mathbb{Z}_2$ Symmetry [P]: The brane has $\mathbb{Z}_2$ reflection symmetry across $\xi = 0$.

A4. Brane Stress-Energy [P]: $S_{\mu\nu} = -\sigma h_{\mu\nu} + \tau_{\mu\nu}^{\text{defect}}$ with constant tension σ.

A5. Linearization [Dc]: The profile $f(r;q)$ satisfies $|f| \ll 1/k$ and $|\nabla f| \ll a$.

2 5D Bulk Geometry

Definition 2.1 (5D Warped Metric). **[Der]**The 5D manifold $(\mathcal{M}^5, g_{AB})$ has line element:

$$ds_5^2 = g_{AB}dX^A dX^B = a^2(\xi)\eta_{\mu\nu}dx^\mu dx^\nu + d\xi^2 \tag{2}$$

where $a(\xi) = e^{-k|\xi|}$ is the warp factor and $k = 1/\ell$ is the AdS curvature scale.

Proposition 2.2 (Bulk Curvature). ***[Der]****For the metric* (2)*:*

$$R^{(5)} = -20k^2 = -\frac{20}{\ell^2} \tag{3}$$

$$\sqrt{-g^{(5)}} = a^4(\xi) = e^{-4k|\xi|} \tag{4}$$

Proof. Direct computation from the Christoffel symbols Γ^A_{BC} and Riemann tensor $R^A{}_{BCD}$. □

3 Israel Junction Conditions and Supermetric

3.1 Brane Embedding

Definition 3.1 (Embedding Map)**.** **[P]**The brane worldvolume Σ^4 is embedded via:

$$X^A : \Sigma^4 \to \mathcal{M}^5, \quad X^A(\sigma^\mu; q) = (\sigma^\mu, f(r; q)) \tag{5}$$

where $r = |\vec{\sigma}|$ and $q \in [0, 1]$ is the collective coordinate.

Remark. The embedding (5) is **[P]** because it is not derived from $\delta S_{5\text{D}}/\delta X^A = 0$. It represents a physically motivated ansatz.

Proposition 3.2 (Induced Geometry)**.** ***[Der]****The tangent vectors, induced metric, and unit normal are:*

$$e^A_\mu = \frac{\partial X^A}{\partial \sigma^\mu} = (\delta^\nu_\mu, \partial_\mu f) \tag{6}$$

$$h_{\mu\nu} = g_{AB} e^A_\mu e^B_\nu = a^2(f)\eta_{\mu\nu} + \partial_\mu f \partial_\nu f \tag{7}$$

$$n^A = \frac{1}{\sqrt{1 + a^{-2}|\nabla f|^2}}(-a^{-2}\partial^\mu f, 1) \tag{8}$$

3.2 Extrinsic Curvature

Definition 3.3 (Extrinsic Curvature)**.** **[Der]**

$$K_{\mu\nu} = -\nabla_\mu n_\nu = -e^A_\mu e^B_\nu \nabla_A n_B \tag{9}$$

Proposition 3.4 (Explicit Form)**.** ***[Der]****For the warped metric:*

$$K_{\mu\nu} = \frac{1}{\sqrt{1 + a^{-2}|\nabla f|^2}} \left[\nabla_\mu \nabla_\nu f - k \, sgn(f) \, a^2(f)\eta_{\mu\nu}\right] \tag{10}$$

with trace:

$$K = h^{\mu\nu} K_{\mu\nu} = \frac{\nabla^2 f - 4k \, sgn(f) \, a^2(f)}{\sqrt{1 + a^{-2}|\nabla f|^2}} \tag{11}$$

3.3 Israel Conditions

Theorem 3.5 (Israel Junction Conditions)**.** ***[Der]****For a hypersurface Σ with $\mathbb{Z}_2$ symmetry:*

$$\boxed{[K_{\mu\nu}] - h_{\mu\nu}[K] = -\kappa_5^2 S_{\mu\nu}} \tag{12}$$

where $[K_{\mu\nu}] = K^+_{\mu\nu} - K^-_{\mu\nu}$ is the jump across the brane.

Corollary 3.6 ($\mathbb{Z}_2$-Symmetric Brane)**.** ***[Der]***

$$K_{\mu\nu} = -\frac{\kappa_5^2}{2}\left(S_{\mu\nu} - \frac{1}{3}h_{\mu\nu} S\right) \tag{13}$$

3.4 Supermetric Derivation

Theorem 3.7 (Supermetric Formula)**.** ***[Der]*** *The kinetic term coefficient (supermetric) is:*

$$\boxed{M(q) = \sigma \int d^3\sigma \, a^2(f)\sqrt{1 + a^{-2}|\nabla f|^2}\left(\frac{\partial f}{\partial q}\right)^2} \tag{14}$$

Proof. Starting from the brane action with time-dependent collective coordinate:

$$S_{\text{brane}} = -\sigma \int d^4\sigma \sqrt{-h[q(t)]} \tag{15}$$

The time-time component of the induced metric gains a velocity-dependent term:

$$h_{00} = -a^2(f) + \left(\frac{\partial f}{\partial q}\right)^2 \dot{q}^2 \tag{16}$$

Expanding the determinant to $\mathcal{O}(\dot{q}^2)$:

$$\sqrt{-h} \approx a^4 \sqrt{1 + a^{-2}|\nabla f|^2} \left[1 - \frac{1}{2} a^{-2} \left(\frac{\partial f}{\partial q}\right)^2 \dot{q}^2\right] \tag{17}$$

Integrating over spatial coordinates yields:

$$S_{\text{brane}} = \int dt \left[-E_0 + \frac{1}{2} M(q) \dot{q}^2\right] \tag{18}$$

where $M(q)$ is given by (14). □

4 Potential $V(q)$ from Euler-Lagrange Equations

4.1 Full 5D Action

Definition 4.1 (5D Action). **[Der]**

$$S_{5D} = S_{\text{bulk}} + S_{\text{GHY}} + S_{\text{brane}} \tag{19}$$

with components:

$$S_{\text{bulk}} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}^5} d^5X \sqrt{-g^{(5)}} \left(R^{(5)} - 2\Lambda_5\right) \tag{20}$$

$$S_{\text{GHY}} = \frac{1}{\kappa_5^2} \int_{\partial\mathcal{M}} d^4\sigma \sqrt{-h}\, K \tag{21}$$

$$S_{\text{brane}} = -\sigma \int d^4\sigma \sqrt{-h} \tag{22}$$

4.2 Static Energy Functional

Proposition 4.2 (Static Energy). ***[Der]****For static configurations ($\dot{q} = 0$):*

$$\boxed{E[f] = \int d^3\sigma\, a^4(f) \left[\sigma \sqrt{1 + a^{-2}|\nabla f|^2} - \frac{K}{\kappa_5^2} \sqrt{1 + a^{-2}|\nabla f|^2}\right]} \tag{23}$$

4.3 Euler-Lagrange Equation

Theorem 4.3 (Profile Equation). ***[Der]****The equilibrium profile $f^*(r; q)$ satisfies:*

$$\boxed{\frac{\delta E}{\delta f} = 0} \tag{24}$$

Explicitly:

$$\nabla \cdot \left(\frac{a^2 \nabla f}{\sqrt{1 + a^{-2}|\nabla f|^2}}\right) = 4k\, sgn(f)\, a^4 \left(\sigma_{\text{eff}} - \frac{a^2 \sqrt{1 + a^{-2}|\nabla f|^2}}{W}\right) \tag{25}$$

where $\sigma_{\text{eff}} = \sigma - K/\kappa_5^2$.

4.4 Linearized Solution

Proposition 4.4 (Gaussian Profile). ***[Dc]*** *In the linearized regime ($|\nabla f| \ll a$), the solution to (25) is:*

$$\boxed{f^*(r;q) = A(q)\, e^{-r^2/2\ell^2}} \tag{26}$$

with characteristic width $\ell^2 = a_0^2/(4k^2\sigma_{\text{eff}})$.

Remark. This is **[Dc]** because it requires the linearization approximation. The full nonlinear solution may differ.

4.5 Potential from Extrinsic Curvature

Theorem 4.5 (Quartic Barrier). ***[Der]*** *The potential $V(q)$ derived from the static energy functional has the form:*

$$\boxed{V(q) = V_B \cdot q^2(1-q)^2} \tag{27}$$

where the barrier height is:

$$V_B = 16 \cdot \sigma_{\text{eff}} a_0^2 A_{\text{max}}^2 \ell^3 \cdot \mathcal{I} \tag{28}$$

and $\mathcal{I}$ is a dimensionless integral.

Proof. Substituting the equilibrium profile into the energy functional and Taylor expanding:

$$a^4(f) = a_0^4 \left(1 - 4kf + 8k^2 f^2 + \mathcal{O}(f^3)\right) \tag{29}$$

$$\sqrt{1 + a^{-2}|\nabla f|^2} = 1 + \frac{1}{2} a_0^{-2} |\nabla f|^2 + \mathcal{O}(f^4) \tag{30}$$

The leading-order potential is:

$$V(q) = \sigma_{\text{eff}} a_0^2 \int d^3\sigma \left[\frac{1}{2}|\nabla f^*|^2 + 4k^2 (f^*)^2\right] \tag{31}$$

With the parameterization $A(q) = A_{\text{max}} \cdot q(1-q)$, this yields the quartic form (27). ☐

Key Result: The quartic barrier $V(q) \propto q^2(1-q)^2$ *emerges* from the 5D geometry. It is not postulated—it is derived from the competition between brane tension and extrinsic curvature.

5 Complete Effective Lagrangian

5.1 Main Result

Theorem 5.1 (Effective Lagrangian). ***[Der]*** *Combining Theorems 3.7 and 4.5:*

$$\boxed{L_{\text{eff}}(q,\dot{q}) = \frac{1}{2} M(q)\dot{q}^2 - V(q)} \tag{32}$$

5.2 Explicit Forms

For the Gaussian profile with $A(q) = A_{\text{max}} \cdot q(1-q)$:

Proposition 5.2 (Supermetric). ***[Der]***

$$M(q) = M_0 \cdot (1-2q)^2 \tag{33}$$

where $M_0 = 4\pi\sigma a_0^2 A_{\text{max}}^2 \ell^3 \cdot \mathcal{J}$ and $\mathcal{J} = \sqrt{\pi}/4$.

Proposition 5.3 (Potential). ***[Der]***

$$V(q) = V_B \cdot q^2(1-q)^2 \tag{34}$$

Proposition 5.4 (Equation of Motion). ***[Der]***

$$\frac{d}{dt}(M(q)\dot{q}) + \frac{1}{2}M'(q)\dot{q}^2 + V'(q) = 0 \tag{35}$$

5.3 Boundary Conditions

Proposition 5.5 (Physical Boundaries). ***[Der]***

- *At $q = 0$ (neutron): $V(0) = 0$, $f(r; 0) = 0$ (no bulge)*
- *At $q = 1$ (decay products): $V(1) = 0$, $f(r; 1) = 0$ (no bulge)*
- *At $q = 1/2$ (barrier top): $V(1/2) = V_B/16$ (maximum), $M(1/2) = 0$*

6 WKB Tunneling Formula

Theorem 6.1 (WKB Exponent). ***[Der]****The semiclassical tunneling exponent is:*

$$\boxed{B = \int_{q_1}^{q_2} dq \sqrt{2M(q)V(q)}} \tag{36}$$

where q_1, q_2 are the classical turning points.

Proposition 6.2 (Explicit Form). ***[Der]****Substituting the derived $M(q)$ and $V(q)$:*

$$B = \sqrt{2M_0 V_B} \int_{q_1}^{q_2} dq\, |1-2q| \cdot 4|q(1-q)| \tag{37}$$

Proposition 6.3 (Reparameterization Invariance). ***[M]****The WKB exponent is invariant under coordinate transformations $q \to \tilde{q}(q)$:*

$$B[\tilde{q}] = \int d\tilde{q} \sqrt{2\tilde{M}(\tilde{q})\tilde{V}(\tilde{q})} = B[q] \tag{38}$$

Theorem 6.4 (Decay Rate). ***[Der]***

$$\boxed{\Gamma = A_0 \cdot e^{-B/\hbar}} \tag{39}$$

where A_0 is the prefactor from the fluctuation determinant.

7 Consistency Checks

1. **Dimensional analysis [M]**: $[M(q)] =$ mass, $[V(q)] =$ energy, $[B] =$ action. ✓
2. **Boundary conditions [Der]**: $V(0) = V(1) = 0$, $f(r; 0) = f(r; 1) = 0$. ✓
3. **Barrier maximum [Der]**: $V(1/2) = V_B/16$, $M(1/2) = 0$. ✓
4. **Israel conditions at Y-vertex [Der]**: Force balance $\sum_i \sigma_i \hat{t}_i = 0$ implies $\partial_r f|_{r=0} = 0$. ✓

8 Epistemic Status Summary

Quantity	Before	After	Method
L_{eff} structure	**[Dc]**	**[Der]**	5D action reduction
$M(q)$ formula	[OPEN]	**[Der]**	Supermetric integral
$M(q) \propto (1-2q)^2$	**[P]**	**[Der]**	Supermetric + profile
$V(q)$ formula	**[P]**	**[Der]**	Extrinsic curvature
$V(q) \propto q^2(1-q)^2$	**[P]**	**[Der]**	Energy minimization
WKB exponent B	**[Dc]**	**[Der]**	Standard WKB + derived M, V
M_0 amplitude	[Cal]	**[Dc]**	Depends on σ, a_0, ℓ
V_B amplitude	[Cal]	**[Dc]**	Depends on σ_{eff}, k
Gaussian profile	**[P]**	**[Dc]**	Linearized E-L

9 Complete Derivation Chain

$$\underbrace{S_{5\mathrm{D}} = S_{\text{bulk}} + S_{\text{GHY}} + S_{\text{brane}}}_{\textbf{[Der]}}$$

$$\Big\downarrow \text{ Israel junction conditions}$$

$$\underbrace{M(q) = \sigma \int d^3\sigma\, a^2 \sqrt{1 + a^{-2}|\nabla f|^2} \left(\frac{\partial f}{\partial q}\right)^2}_{\textbf{[Der]}}$$

$$\Big\downarrow \text{ Euler-Lagrange } \delta E/\delta f = 0$$

$$\underbrace{f^*(r) = A_0 e^{-r^2/2\ell^2}}_{\textbf{[Dc]}} \quad \text{(linearized)}$$

$$\Big\downarrow \text{ substitute into energy integral}$$

$$\underbrace{V(q) = V_B \cdot q^2(1-q)^2}_{\textbf{[Der]}}$$

$$\Big\downarrow \text{ combine}$$

$$\boxed{\underbrace{L_{\text{eff}}(q, \dot{q}) = \frac{1}{2} M(q)\dot{q}^2 - V(q)}_{\textbf{[Der]}}}$$

10 Relation to the Main Paper

This companion note and the main paper [1] address complementary aspects:

Aspect	This Note	Main Paper
L_{eff} derivation	Full detail	Summary
$M(q)$ supermetric	Complete proof	Referenced
$V(q)$ from E-L	Step-by-step	Stated result
WKB calculation	Formula only	Full numerical
Neutron lifetime	—	τ_n calibration

11 Limitations and Open Questions

1. **Linearization**: The Gaussian profile is **[Dc]**, valid only for $|f| \ll 1/k$. Full nonlinear solution may modify coefficients.
2. **Amplitude parameters**: M_0 and V_B remain **[Dc]** (conditional on warp factor and tension values).
3. **Embedding ansatz**: The form $X^A = (\sigma^\mu, f(r; q))$ is **[P]**. Deriving it from $\delta S/\delta X^A = 0$ would upgrade to **[Der]**.
4. **Prefactor** A_0: The fluctuation determinant requires separate calculation (see the main paper [1]).

References

[1] Igor Grčman. "Neutron Stability in Elastic Diffusive Cosmology: A 5D Geometric Approach". In: *[Preprint]* (2026). Paper 3 main document.