5D Action Reduction Pipeline

From Brane-World Gravity to Effective 1D Collective Dynamics
(Companion C to Paper 3: NJSR Edition)

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Repository: github.com/igorgrcman/elastic-diffusive-cosmology
(Public artifacts for this paper are in the edc_papers folder.)

Related Documents:
Neutron Lifetime from 5D Membrane Cosmology (DOI: 10.5281/zenodo.18262721)
Framework v2.0 (DOI: 10.5281/zenodo.18299085)
Companions:
A: *Effective Lagrangian* (DOI) · B: *WKB Prefactor* (DOI)
D: *Selection Rules* (DOI) · E: *Symmetry Ops* (DOI)
F: *Proton Junction* (DOI) · G: *Mass Difference* (DOI)
H: *Weak Interactions* (DOI)

Abstract

This companion note documents the dimensional reduction pipeline from the 5D brane-world action to the effective 1D mechanical system $S_{\text{eff}}[q] = \int dt \left(\frac{1}{2}M(q)\dot{q}^2 - V(q)\right)$. We present: (i) the bulk geometry ansatz (warped AdS_5); (ii) the brane embedding with collective coordinate q; (iii) the induced geometry and extrinsic curvature; (iv) the Israel junction conditions; and (v) the extraction of $M(q)$ and $V(q)$ from the action. All steps are tagged with epistemic status (**[Der]**, **[Dc]**, **[P]**, **[OPEN]**). The full forensic-level worked derivation is in the main paper [1], Appendix J; this note provides the conceptual roadmap and key results.

Contents

1 Overview and Reduction Chain

1.1 The Complete Pipeline

The dimensional reduction proceeds in stages:

$$\boxed{\underbrace{S_{5D}}_{\text{bulk + GHY + brane}} \xrightarrow{\text{embedding}} \underbrace{S[\phi(x;q)]}_{\text{defect profile}} \xrightarrow{\text{collective coord}} \underbrace{S_{\text{eff}}[q]}_{\text{1D mechanics}}} \tag{1}$$

Stage	Content	Status
0	Bulk geometry: AdS_5 from Einstein equations	[Dc]
1	Bulk metric ansatz: $ds_5^2 = e^{-2\|y\|/\ell}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$	[P]
2	Brane embedding: $X^A(\sigma^\mu; q) = (\sigma^\mu, f(r;q))$	[P]
3	Induced metric $h_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$	[Der]
4	Israel junction conditions	[Der]
5	Defect profile: Gaussian $f(r;q) = A(q)e^{-r^2/2w^2}$	[P]
6	$M(q)$ from kinetic term	[Dc]
7	$V(q)$ from static energy	[Dc]

1.2 Epistemic Legend

Epistemic Status Tags

[Der] **Derived** — explicit calculation from stated premises
[Dc] **Decisively constrained** — conditional on approximations
[Dd] **Definitional** — structural/mathematical identity
[P] **Proposed** — ansatz, not derived from $\delta S = 0$
[M] **Mathematics** — pure mathematical theorem
[OPEN] **Open** — not yet derived

2 Stage 0–1: Bulk Geometry

2.1 From 5D Einstein Equations to AdS_5 (Conditional)

Postulate 2.1 (Symmetry Assumptions). **[Dd]**

(A1) 5D spacetime: $\mathcal{M}^5 = \mathcal{M}^4 \times I$

(A2) 4D Poincaré invariance on constant-ξ slices

(A3) Static configuration

(A4) Gaussian normal coordinates

Lemma 2.2 (Symmetry-Restricted Metric). ***[Dd]****Under (A1)–(A4), the 5D metric takes the form:*

$$ds_5^2 = e^{-2A(\xi)}\eta_{\mu\nu}dx^\mu dx^\nu + d\xi^2 \tag{2}$$

Postulate 2.3 (Dynamical Assumptions). **[P]**

(A5) 5D Einstein-Hilbert action: $S_{\text{bulk}} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}(R - 2\Lambda_5)$

(A6) Negative cosmological constant: $\Lambda_5 < 0$

(A7) $\mathbb{Z}_2$ orbifold symmetry

(A8) Thin brane with tension σ at $\xi = 0$

Theorem 2.4 (Warp Factor Solution). ***[Dc]****From the* $(\xi\xi)$*-component of Einstein's equations:*

$$\boxed{A(\xi) = k|\xi|, \quad k^2 = -\frac{\Lambda_5}{6}} \tag{3}$$

Corollary 2.5 (Working Metric). ***[P]****We adopt the warped* AdS_5 *metric:*

$$\boxed{ds_5^2 = e^{-2|y|/\ell}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2} \tag{4}$$

where $\ell = 1/k$ *is the AdS radius.*

Remark. The physical origin of (A5)–(A8) from EDC Plenum dynamics remains **[OPEN]**. We proceed with (4) as an ansatz.

3 Stage 2: Brane Embedding

3.1 Embedding Map

Postulate 3.1 (Brane Embedding). **[P]**The 3-brane worldvolume Σ^4 is embedded via:

$$\boxed{X^A : \Sigma^4 \to \mathcal{M}^5, \quad X^A(\sigma^\mu; q) = (\sigma^\mu, f(r; q))} \tag{5}$$

where:

- $\sigma^\mu = (t, x^1, x^2, x^3)$ are worldvolume coordinates
- $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ is the radial distance
- $q \in [0, 1]$ is the collective coordinate
- $f(r; q)$ is the defect profile function

3.2 Collective Coordinate Definition

Definition 3.2 (Collective Coordinate). **[P]**The coordinate q parameterizes the junction asymmetry:

- $q = 0$: Symmetric Y-junction $\Rightarrow$ proton (stable)
- $q = 1$: Maximum asymmetry $\Rightarrow$ neutron (metastable)
- $q \in (0, 1)$: Intermediate configurations $\Rightarrow$ transition states

4 Stage 3: Induced Geometry

4.1 Tangent Vectors and Induced Metric

Proposition 4.1 (Tangent Vectors). ***[Der]***

$$e^A_\mu = \frac{\partial X^A}{\partial \sigma^\mu} = (\delta^\nu_\mu, \partial_\mu f) \tag{6}$$

Proposition 4.2 (Induced Metric). ***[Der]***

$$\boxed{h_{\mu\nu} = g_{AB}e^A_\mu e^B_\nu = a^2(f)\eta_{\mu\nu} + \partial_\mu f \partial_\nu f} \tag{7}$$

where $a(f) = e^{-|f|/\ell}$ *is the warp factor evaluated on the brane.*

4.2 Unit Normal

Proposition 4.3 (Unit Normal)**.** ***[Der]***

$$n^A = \frac{1}{\sqrt{1 + a^{-2}|\nabla f|^2}} \left(-a^{-2}\partial^\mu f, 1\right) \tag{8}$$

4.3 Extrinsic Curvature

Definition 4.4 (Extrinsic Curvature)**. [Dd]**

$$K_{\mu\nu} = -\nabla_\mu n_\nu = -e^A_\mu e^B_\nu \nabla_A n_B \tag{9}$$

Proposition 4.5 (Explicit Form)**.** ***[Der]***

$$\boxed{K_{\mu\nu} = \frac{1}{\sqrt{1 + a^{-2}|\nabla f|^2}} \left[\nabla_\mu \nabla_\nu f - k \,\mathrm{sgn}(f)\, a^2 \eta_{\mu\nu}\right]} \tag{10}$$

5 Stage 4: Israel Junction Conditions

Theorem 5.1 (Israel Junction Conditions)**.** ***[Der]****For a hypersurface with $\mathbb{Z}_2$ symmetry:*

$$\boxed{[K_{\mu\nu}] - h_{\mu\nu}[K] = -\kappa_5^2 S_{\mu\nu}} \tag{11}$$

where $S_{\mu\nu} = -\sigma h_{\mu\nu} + \tau^{\text{defect}}_{\mu\nu}$ is the brane stress-energy.

Corollary 5.2 (Fine-Tuning Condition)**.** ***[Dc]****For a flat brane ($f = 0$), the Israel condition yields the RS fine-tuning:*

$$\sigma = \frac{6k}{\kappa_5^2} \tag{12}$$

6 Stage 5: Defect Profile Ansatz

Postulate 6.1 (Gaussian Profile)**. [P]**

$$\boxed{f(r; q) = A(q)\, e^{-r^2/2w^2}} \tag{13}$$

where:

- $A(q)$ is the amplitude function (depends on collective coordinate)
- w is the characteristic width (set by defect scale)

Postulate 6.2 (Amplitude Parameterization)**. [P]**

$$A(q) = A_{\max} \cdot q(1 - q) \tag{14}$$

This satisfies boundary conditions $A(0) = A(1) = 0$.

7 Stage 6: Supermetric $M(q)$

Theorem 7.1 (Supermetric Formula). ***[Dc]*** *From the kinetic term in the brane action:*

$$\boxed{M(q) = \sigma \int d^3\sigma \, a^2(f)\sqrt{1 + a^{-2}|\nabla f|^2} \left(\frac{\partial f}{\partial q}\right)^2} \tag{15}$$

Proposition 7.2 (Explicit Form). ***[Dc]****For the Gaussian profile* (13) *with amplitude* (14)*:*

$$\boxed{M(q) = M_0 \cdot (1 - 2q)^2} \tag{16}$$

where $M_0 = 4\pi\sigma a_0^2 A_{\text{max}}^2 w^3 \cdot \mathcal{J}$.

Remark. The supermetric vanishes at $q = 1/2$ (barrier top), reflecting the geometric singularity in configuration space.

8 Stage 7: Potential $V(q)$

Theorem 8.1 (Static Energy Functional). ***[Dc]****The potential arises from the static energy:*

$$V(q) = E_{\text{static}}[f(\cdot; q)] - E_0 \tag{17}$$

where E_{static} *is evaluated on the equilibrium profile.*

Proposition 8.2 (Quartic Barrier). ***[Dc]****For the Gaussian profile:*

$$\boxed{V(q) = V_B \cdot q^2(1 - q)^2} \tag{18}$$

with barrier height V_B *depending on tension and geometry parameters.*

Proposition 8.3 (Combined Potential with Q-Value). ***[Dc]****Including the endpoint energy difference:*

$$\boxed{V(q) = 16V_B \, q^2(1 - q)^2 + Q \cdot q} \tag{19}$$

where $Q = 0.782$ *MeV* ***[BL]*** *is the Q-value of neutron decay.*

9 Final Result: Effective 1D Action

Theorem 9.1 (Effective Action). ***[Dc]****The dimensional reduction yields:*

$$\boxed{S_{\text{eff}}[q] = \int dt \left(\frac{1}{2}M(q)\dot{q}^2 - V(q)\right)} \tag{20}$$

with $M(q)$ *from* (16) *and* $V(q)$ *from* (19).

Key Result: The 5D brane-world action reduces to a 1D mechanical system with q-dependent mass and a quartic barrier potential. The neutron decay is described as tunneling through this barrier.

10 Complete Derivation Chain

$$\begin{array}{rcl}
S_{\text{5D}} = S_{\text{bulk}} + S_{\text{GHY}} + S_{\text{brane}} & \textbf{[P]} & \text{(full action)} \\
\downarrow & & \\
\text{Warped AdS}_5 \text{ metric} & \textbf{[P]} & \text{(geometric ansatz)} \\
\downarrow & & \\
\text{Brane embedding } X^A(\sigma; q) & \textbf{[P]} & \text{(defect parameterization)} \\
\downarrow & & \\
\text{Induced metric } h_{\mu\nu}\text{, normal } n^A & \textbf{[Der]} & \text{(differential geometry)} \\
\downarrow & & \\
\text{Extrinsic curvature } K_{\mu\nu} & \textbf{[Der]} & \text{(second fundamental form)} \\
\downarrow & & \\
\text{Israel conditions} & \textbf{[Der]} & \text{(junction matching)} \\
\downarrow & & \\
\text{Gaussian profile } f(r; q) & \textbf{[P]} & \text{(shape ansatz)} \\
\downarrow & & \\
M(q) \propto (1 - 2q)^2 & \textbf{[Dc]} & \text{(supermetric integral)} \\
\downarrow & & \\
V(q) \propto q^2(1 - q)^2 & \textbf{[Dc]} & \text{(static energy)} \\
\downarrow & & \\
S_{\text{eff}}[q] = \int (\frac{1}{2} M \dot{q}^2 - V) dt & \textbf{[Dc]} & \text{(final result)}
\end{array}$$

11 Epistemic Status Summary

Element	Status	Notes
Warped AdS_5 metric	**[P]**	Consistent with 5D GR + $\Lambda_5 < 0$
Brane embedding ansatz	**[P]**	Y-junction model
Gaussian profile	**[P]**	Tractable; other shapes possible
Induced metric calculation	**[Der]**	Standard differential geometry
Israel junction conditions	**[Der]**	Standard result
Supermetric $M(q)$ formula	**[Dc]**	Conditional on profile
$M(q) \propto (1 - 2q)^2$	**[Dc]**	From Gaussian + amplitude form
Potential $V(q)$ formula	**[Dc]**	Conditional on profile
$V(q) \propto q^2(1 - q)^2$	**[Dc]**	From static energy evaluation
Barrier height V_B	**[OPEN]**	Cannot be derived from action

12 Relation to Other Companions

Aspect	This Note	Other
Reduction pipeline	Full roadmap	—
L_{eff} derivation	Summary	Companion A (detail)
WKB tunneling	Referenced	Companion B
Selection rules	Referenced	Companion D

13 Open Problems

1. **[OPEN] Derive bulk geometry from EDC Plenum:** Why $\Lambda_5 < 0$?
2. **[OPEN] Derive $\mathbb{Z}_2$ symmetry:** Physical origin in EDC?
3. **[OPEN] Profile optimization:** Derive optimal $f(r; q)$ from variational principle.
4. **[OPEN] Barrier height V_B:** Requires weak interaction physics (W boson).
5. **[OPEN] Higher-dimensional corrections:** Beyond leading-order warping.

References

[1] Igor Grčman. "Neutron Lifetime from 5D Membrane Cosmology: WKB Tunneling, Brane-Soliton Structure, and the NJSR Framework". In: *Zenodo* (2026). Paper 3, NJSR Edition.