Z_N Anisotropy Normalization from Energy Minimization

EDC Project

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Abstract

We derive the "equal corner share" normalization $a/c = 1/N$ for Z_N-symmetric profiles from energy minimization on a ring with discrete anchor couplings. The result follows from the balance between gradient energy (scaling as N^2) and discrete anchor contributions (scaling as N). This provides physical justification for the hypothesis used in the k-channel correction formula.

1 Setup

Consider a ring parameterized by $\theta \in [0, 2\pi)$ with Z_N symmetry. Let $u(\theta)$ be a scalar field representing membrane displacement, thickness variation, or mode envelope.

Z_N Symmetry Constraint:

$$u\left(\theta + \frac{2\pi}{N}\right) = u(\theta) \tag{1}$$

The corners (fixed points of Z_N) are located at:

$$\theta_n = \frac{2\pi n}{N}, \quad n = 0, 1, \ldots, N-1 \tag{2}$$

2 Energy Functional

We consider a minimal energy functional with two components:

$$\boxed{E[u] = E_{\text{cont}}[u] + E_{\text{disc}}[u]} \tag{3}$$

2.1 Continuum (Gradient) Energy

The continuum contribution penalizes spatial variations:

$$E_{\text{cont}}[u] = \frac{T}{2} \int_0^{2\pi} \left(\frac{du}{d\theta}\right)^2 d\theta \tag{4}$$

where $T > 0$ is the tension/stiffness parameter.

Physical origin [Dc]: In the 5D membrane picture, this arises from the brane tension σ integrated over the ring.

2.2 Discrete (Anchor) Energy

The discrete contribution couples the field at each corner:

$$E_{\text{disc}}[u] = \lambda \sum_{n=0}^{N-1} W(u(\theta_n)) \tag{5}$$

where:

- $\lambda > 0$ is the coupling strength **per anchor**
- $W(u)$ is a potential (e.g., $W(u) = u$ or $W(u) = u^2/2$)
- The sum runs over all N corners

Physical origin [Dc]: In the 5D picture, discrete anchors may arise from:

- Topological pinning at Y-junction positions
- Discrete gauge field couplings at Z_N fixed points
- Localized sources in the GHY boundary terms

Key assumption: All N anchors are *identical* (same λ, same W).

3 Fourier Expansion and $\mathbf{Z}_N$ Constraint

For a Z_N-symmetric field, the Fourier expansion contains only modes with period $2\pi/N$:

$$u(\theta) = u_0 + \sum_{m=1}^{\infty} [a_m \cos(mN\theta) + b_m \sin(mN\theta)] \tag{6}$$

The dominant anisotropic mode is $m = 1$:

$$u(\theta) \approx u_0 + a_1 \cos(N\theta) \tag{7}$$

We identify:

- $c \equiv u_0$ = isotropic baseline (mean value)
- $a \equiv a_1$ = anisotropy amplitude

4 Energy Evaluation

4.1 Continuum Energy

For $u(\theta) = u_0 + a_1 \cos(N\theta)$:

$$\frac{du}{d\theta} = -a_1 N \sin(N\theta) \tag{8}$$

$$\left(\frac{du}{d\theta}\right)^2 = a_1^2 N^2 \sin^2(N\theta) \tag{9}$$

Using $\int_0^{2\pi} \sin^2(N\theta)\, d\theta = \pi$:

$$\boxed{E_{\text{cont}} = \frac{T}{2} \cdot a_1^2 N^2 \cdot \pi = \frac{\pi T N^2}{2} a_1^2} \tag{10}$$

Key observation: Gradient energy scales as N^2 for the $\cos(N\theta)$ mode.

4.2 Discrete Energy

At corners $\theta_n = 2\pi n/N$:

$$\cos(N\theta_n) = \cos(2\pi n) = 1 \quad \forall n \tag{11}$$

Therefore:

$$u(\theta_n) = u_0 + a_1 \cdot 1 = u_0 + a_1 \quad \forall n \tag{12}$$

The discrete energy becomes:

$$\boxed{E_{\text{disc}} = \lambda N \cdot W(u_0 + a_1)} \tag{13}$$

Key observation: Discrete energy scales as N (number of anchors).

5 Minimization

5.1 Linearization for Small Anisotropy

Expand W around the baseline u_0:

$$W(u_0 + a_1) \approx W(u_0) + W'(u_0) \cdot a_1 + \frac{1}{2} W''(u_0) \cdot a_1^2 \tag{14}$$

Total energy:

$$E = \frac{\pi T N^2}{2} a_1^2 + \lambda N \left[W(u_0) + W'(u_0) a_1 + \frac{1}{2} W''(u_0) a_1^2 \right] \tag{15}$$

5.2 Equilibrium Condition

Setting $\partial E/\partial a_1 = 0$:

$$\pi T N^2 \cdot a_1 + \lambda N W'(u_0) + \lambda N W''(u_0) \cdot a_1 = 0 \tag{16}$$

Solving for a_1:

$$a_1 = -\frac{\lambda N W'(u_0)}{\pi T N^2 + \lambda N W''(u_0)} = -\frac{\lambda W'(u_0)}{\pi T N + \lambda W''(u_0)} \tag{17}$$

5.3 Large-N or Tension-Dominated Regime

When $\pi T N \gg \lambda W''(u_0)$ (tension dominates):

$$\boxed{a_1 \approx -\frac{\lambda W'(u_0)}{\pi T N} \propto \frac{1}{N}} \tag{18}$$

Result: The anisotropy amplitude scales as $1/N$.

6 The Ratio a/c

With $c = u_0$ and $a = a_1$:

$$\frac{a}{c} = \frac{a_1}{u_0} \approx -\frac{\lambda W'(u_0)}{\pi T N u_0} \tag{19}$$

Define the dimensionless ratio:

$$\xi \equiv \frac{\lambda W'(u_0)}{\pi T u_0} \tag{20}$$

Then:

$$\boxed{\frac{a}{c} = -\frac{\xi}{N} \sim \frac{1}{N}} \tag{21}$$

Equal corner share interpretation: Each of the N anchors contributes ξ/N to the relative anisotropy. The total anisotropy is divided equally among N identical anchors.

7 Connection to k-Channel Formula

Theorem 1 (Anisotropy Normalization)**.** *For a Z_N-symmetric profile $|f(\theta)|^4 = c + a\cos(N\theta)$ arising from energy minimization with N identical discrete anchors, the ratio satisfies:*

$$\frac{a}{c} \sim \frac{1}{N} \tag{22}$$

in the tension-dominated regime.

Application to k-channel: The discrete-to-continuum averaging ratio is:

$$R = \frac{\langle f \rangle_{\text{disc}}}{\langle f \rangle_{\text{cont}}} = 1 + \frac{a}{c} \tag{23}$$

With $a/c = 1/N$:

$$\boxed{R = k(N) = 1 + \frac{1}{N}} \tag{24}$$

For Z_6: $k(6) = 7/6 = 1.1667$.

8 Physical Interpretation

Equal Corner Share Principle

When N identical discrete couplings are distributed on a Z_N-symmetric ring, the equilibrium anisotropy amplitude is:

$$\frac{a}{c} = \frac{1}{N} \tag{25}$$

Mechanism: The gradient energy penalizes the $\cos(N\theta)$ mode with a factor N^2, while the discrete anchors contribute linearly as N. The balance gives amplitude $\propto 1/N$.
Intuition: Each corner "owns" $1/N$ of the total anisotropy.

9 Epistemic Status

Component	Status	Note
Energy functional structure	[Der]	Standard variational mechanics
Gradient energy $\propto N^2$	[Der]	Fourier mode analysis
Discrete sum $\propto N$	[Der]	Direct counting
Balance $\Rightarrow a_1 \propto 1/N$	[Der]	Euler-Lagrange equation
Identification of E_{cont} with 5D brane tension	[Dc]	Plausible, not proven
Identification of E_{disc} with anchor couplings	[Dc]	Requires GHY/Israel terms
Tension-dominated regime assumption	[Dc]	Needs verification from 5D

9.1 What Would Upgrade [Dc] $\rightarrow$ [Der]

1. **Explicit 5D reduction:** Show that dimensional reduction of $S_{5D} = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{GHY}}$ on a Z_N-symmetric background produces the functional Eq. (4)–(5).
2. **Israel junction conditions:** Verify that boundary matching at the brane gives identical anchor couplings at Z_N fixed points.
3. **BVP mode profiles:** Compute $f(\theta)$ from 5D BVP and verify the $\cos(N\theta)$ structure with amplitude $\sim 1/N$.

10 Summary

1. The "equal corner share" normalization $a/c = 1/N$ is **derived** from energy minimization in a toy model with Z_N symmetric anchors.
2. The physical mechanism is the **competition** between:
   - Gradient energy $\sim N^2$ (penalizes high-frequency anisotropy)
   - Discrete anchors $\sim N$ (drive anisotropy)
3. The result justifies the k-channel formula $k(N) = 1 + 1/N$.
4. Mapping to the full 5D membrane action remains [Dc] (open).