A RANDOM COPOSITIVE MATRIX IS COMPLETELY POSITIVE WITH POSITIVE PROBABILITY

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Abstract. An $n \times n$ symmetric matrix A is copositive if the quadratic form $x^T Ax$ is nonnegative on the nonnegative orthant $\mathbb{R}^n_{\geq 0}$. The cone of copositive matrices strictly contains the cone of completely positive matrices, i.e., all matrices of the form BB^T for some $n \times r$ matrix B with nonnegative entries. The main result, proved using Blekherman's real algebraic geometry inspired techniques and tools of convex geometry, shows that asymptotically, as n goes to infinity, the ratio of volume radii of the two cones is strictly positive. Consequently, the same holds true for the ratio of volume radii of any two cones sandwiched between them, e.g., the cones of positive semidefinite matrices, matrices with nonnegative entries, their intersection and their Minkowski sum. Further, a free probability inspired construction of exceptional copositive matrices, i.e., copositive matrices that are not sums of a positive semidefinite matrix and a nonnegative one, is given.

1. Introduction

For $n \in \mathbb{N}$ let $M_n(\mathbb{R})$ be the vector space of $n \times n$ real matrices and let $\mathbb{S}_n = \{A \in M_n(\mathbb{R})\colon A^T = A\}$ be its subspace of real symmetric matrices, where ^{T} stands for the usual transposition of matrices. Let $\mathbb{R}[\mathtt{x}]$ be the vector space of real polynomials in the variables $\mathtt{x} = (x_1, \ldots, x_n)$. Let $\mathbb{R}[\mathtt{x}]_k$ be the subspace of **forms of degree** k, i.e., homogeneous polynomials from $\mathbb{R}[\mathtt{x}]$ of degree k. To a matrix $A = [a_{ij}]_{i,j=1}^n \in \mathbb{S}_n$ we associate the quadratic form

$$p_A(\mathtt{x}) := \mathtt{x}^T A\mathtt{x} = \sum_{i,j=1}^n a_{ij}x_ix_j \in \mathbb{R}[\mathtt{x}]_2 \tag{1.1}$$

and the quartic form

$$q_A(\mathtt{x}) := p_A(x_1^2, \ldots, x_n^2) \in \mathbb{R}[\mathtt{x}]_4. \tag{1.2}$$

The first contribution of this paper are the estimates of asymptotic volumes (as n goes to infinity) of the cones of the following classes of matrices.

Definition 1.1. A matrix $A \in \mathbb{S}_n$ is:

(1) **copositive** if p_A is nonnegative on the nonnegative orthant

$$\mathbb{R}^n_{\geq 0} := \{(x_1, \ldots, x_n)\colon x_i \geq 0, i = 1, \ldots, n\},$$

i.e., $p_A(\mathtt{x}) \geq 0$ for every $\mathtt{x} \in \mathbb{R}^n_{\geq 0}$. Equivalently, A is copositive iff q_A is nonnegative on $\mathbb{R}^n$. We write COP_n for the cone of all $n \times n$ copositive matrices.

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(2) **positive semidefinite (PSD)** if all of its eigenvalues are nonnegative. Equivalently, A is PSD iff $p_A(\mathtt{x}) \geq 0$ for all $\mathtt{x} \in \mathbb{R}^n$ iff $A = BB^T$ for some matrix $B \in M_n(\mathbb{R})$. We write $A \succeq 0$ to denote that A is PSD and PSD_n stands for the cone of all $n \times n$ PSD matrices.

(3) **nonnegative (NN)** if all of its entries are nonnegative, i.e., $A = [a_{ij}]_{i,j=1}^n$ with $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$. We write NN_n for the cone of all $n \times n$ NN matrices.

(4) **SPN** (sum of a positive semidefinite matrix and a nonnegative one) if it is of the form $A = P+N$, where $P \in \mathrm{PSD}_n$ and $N \in \mathrm{NN}_n$. We write $\mathrm{SPN}_n := \mathrm{PSD}_n + \mathrm{NN}_n$ for the cone of all $n \times n$ SPN matrices.

(5) **doubly nonnegative (DNN)** if it is PSD and NN. We write $\mathrm{DNN}_n := \mathrm{PSD}_n \cap \mathrm{NN}_n$ for the cone of all $n \times n$ DNN matrices.

(6) **completely positive (CP)**[1] if $A = BB^T$ for some $r \in \mathbb{N}$ and $n \times r$ entrywise nonnegative matrix B. We write CP_n for the cone of all $n \times n$ CP matrices.

Clearly,

$$\mathrm{COP}_n \supseteq \mathrm{SPN}_n \supseteq \mathrm{PSD}_n \cup \mathrm{NN}_n \supseteq \mathrm{DNN}_n \supseteq \mathrm{CP}_n\,. \tag{1.3}$$

The cones from Definition 1.1 are of great importance in optimization, since many combinatorial problems can be formulated as conic linear programs over the largest cone COP_n or the smallest cone CP_n among them [KP02, Bur09, RRW10, DR21]. However, deciding whether a given matrix belongs to COP_n is co-NP-complete [MK87] and NP-hard for CP_n [DG14]. To get a tractable approximation of the cone COP_n based on semidefinite programming, Parrilo [Par00] proposed an increasing hierarchy of inner approximating cones $K_n^{(r)} := \{A \in \mathbb{S}_n \colon (\sum_{i=1}^n x_i^2)^r \cdot p_A(\mathtt{x})$ is a sum of squares of forms$\}$. Clearly,

$$\bigcup_{r\in\mathbb{N}_0} K_n^{(r)} \subseteq \mathrm{COP}_n, \tag{1.4}$$

and by a result of Pólya [Pól28], $\mathrm{int}(\mathrm{COP}_n) \subseteq \bigcup_{r\in\mathbb{N}_0} K_n^{(r)}$. By [Par00, p. 63–64], $K_n^{(0)} = \mathrm{SPN}_n$. Since $\mathrm{SPN}_n = \mathrm{COP}_n$ for $n \leq 4$ [MM62], it follows that $K_n^{(0)} = \mathrm{COP}_n$ and the inclusion in (1.4) is an equality for $n \leq 4$ (see also [Dia62]). For $n \geq 5$, $K_n^{(0)}$ is strictly contained in COP_n; the so-called Horn matrix [HN63]

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix} \tag{1.5}$$

is a standard example of a copositive matrix that is not SPN.

Further, $H \in K_5^{(1)}$ [Par00], but $\mathrm{COP}_5 \neq K_5^{(r)}$ for any $r \in \mathbb{N}$ [DDGH13]. It has been very recently shown [LV22b, SV-] that for $n = 5$ the inclusion in (1.4) is still the equality, while for $n \geq 6$, the inclusion is strict [LV22a]. For a nice exposition on the classes of matrices defined above we refer the reader to [BSM21]. Some open problems regarding COP_n, CP_n are presented in [BDSM15].

[1] Despite the similar name, the CP matrices considered here are not related to the CP maps ubiquitous in operator algebra [Pau02].

1.1. **Main results and reader's guide.** The contribution of this paper is twofold. First, we compare the cones of quartic forms corresponding to the cones from Definition 1.1 by the correspondence (1.2), by estimating volumes of their compact base sections. For this we lean on powerful techniques, developed by Blekherman [Ble04, Ble06] and Barvinok-Blekherman [BB05] for comparing the cones of positive polynomials and sums of squares, fundamental objects of real algebraic geometry [BCR98, Mar08, Lau09, Sce09]. In addition to these we will rely on some classical results of convex analysis [RS57, BM87, MP90].

Let

$$\mathcal{Q} := \Big\{ f \in \mathbb{R}[\mathtt{x}]_4 \colon f(\mathtt{x}) = \sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2 \quad \text{where each } a_{ij} \in \mathbb{R} \text{ and } \forall i,j : a_{ij} = a_{ji} \Big\} \tag{1.6}$$

be a vector subspace in $\mathbb{R}[\mathtt{x}]_4$. The elements of $\mathcal{Q}$ are called **even quartics**. Note that $\mathcal{Q}$ can be identified with the space of all real symmetric matrices $\mathbb{S}_n$ by the correspondence $\mathbb{S}_n \to \mathcal{Q}, A \mapsto q_A$ with q_A as in (1.2). Under this correspondence we have the following bijections:

(1) COP_n corresponds to nonnegative forms from $\mathcal{Q}$:

$$\mathrm{POS}_{\mathcal{Q}} := \{ f \in \mathcal{Q} \colon f(\mathtt{x}) \geq 0 \quad \text{for all } \mathtt{x} \in \mathbb{R}^n \}.$$

(2) PSD_n corresponds to sums of squares of quadratic forms in the variables $x_1^2, \dots, x_n^2$:

$$\mathrm{PSD}_{\mathcal{Q}} := \Big\{ f \in \mathcal{Q} \colon f = \sum_i f_i^2 \quad \text{for some } f_i(\mathtt{x}) = \sum_{j=1}^{n} f_j^{(i)} x_j^2 \in \mathbb{R}[\mathtt{x}]_2 \Big\},$$

(3) NN_n corresponds to forms in $\mathcal{Q}$ with nonnegative coefficients:

$$\mathrm{NN}_{\mathcal{Q}} := \Big\{ f \in \mathcal{Q} \colon f(\mathtt{x}) = \sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2 \quad \text{where } a_{ij} \geq 0 \text{ for } i,j = 1, \dots, n \Big\}.$$

(4) SPN_n corresponds to forms in $\mathcal{Q}$ that are sums of a form from $\mathrm{PSD}_{\mathcal{Q}}$ and a form from $\mathrm{NN}_{\mathcal{Q}}$:

$$\mathrm{SPN}_{\mathcal{Q}} := \mathrm{PSD}_{\mathcal{Q}} + \mathrm{NN}_{\mathcal{Q}}$$

(5) DNN_n corresponds to forms in $\mathcal{Q}$ that belong to the intersection of the cones $\mathrm{PSD}_{\mathcal{Q}}$ and $\mathrm{NN}_{\mathcal{Q}}$:

$$\mathrm{DNN}_{\mathcal{Q}} := \mathrm{PSD}_{\mathcal{Q}} \cap \mathrm{NN}_{\mathcal{Q}}$$

(6) CP_n corresponds to sums of squares of quadratic forms in variables $x_1^2, \dots, x_n^2$ with nonnegative coefficients:

$$\mathrm{CP}_{\mathcal{Q}} := \Big\{ f \in \mathcal{Q} \colon f = \sum_{i=1} f_i^2 \ \text{ for some } f_i(\mathtt{x}) = \sum_{j=1}^{n} f_j^{(i)} x_j^2 \in \mathbb{R}[\mathtt{x}]_2 \text{ with } f_j^{(i)} \geq 0 \text{ for each } i,j \Big\}.$$

There are also two additional distinguished subsets of $\mathcal{Q}$.

(7) The set of sums of squares of all quadratic forms which belong to $\mathcal{Q}$:

$$\mathrm{SOS}_{\mathcal{Q}} := \mathcal{Q} \bigcap \Big\{ f \in \mathbb{R}[\mathtt{x}]_4 \colon f = \sum_i f_i^2 \quad \text{for some } f_i \in \mathbb{R}[\mathtt{x}]_2 \Big\},$$

It turns out that $\mathrm{SOS}_{\mathcal{Q}} = \mathrm{SPN}_{\mathcal{Q}}$ [Par00, p. 63–64].

(8) The set of all projections to $\mathcal{Q}$ of 4-th powers of linear forms:

$$\mathrm{LF}_{\mathcal{Q}} := \Big\{ \mathrm{pr}_{\mathcal{Q}}(f) \in \mathbb{R}[\mathtt{x}]_4 \colon f = \sum_i f_i^4 \quad \text{for some } f_i \in \mathbb{R}[\mathtt{x}]_1 \Big\},$$

where $\mathrm{pr}_{\mathcal{Q}} : \mathbb{R}[\mathtt{x}]_4 \to \mathcal{Q}$ is defined by:

$$\mathrm{pr}_{\mathcal{Q}}\big(\sum_{1\le i\le j\le k\le \ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \big) = \sum_{1\le i\le j\le n} a_{iijj} x_i^2 x_j^2. \tag{1.7}$$

Let $\mathcal{C}$ be the set of cones we are interested in:

$$\mathcal{C} := \{\mathrm{POS}_{\mathcal{Q}}, \mathrm{SOS}_{\mathcal{Q}} = \mathrm{SPN}_{\mathcal{Q}}, \mathrm{NN}_{\mathcal{Q}}, \mathrm{PSD}_{\mathcal{Q}}, \mathrm{DNN}_{\mathcal{Q}}, \mathrm{LF}_{\mathcal{Q}}, \mathrm{CP}_{\mathcal{Q}}\}. \tag{1.8}$$

We shall compare these cones by passing to the corresponding cones of forms from $\mathbb{R}[\mathtt{x}]_4$.

Remark 1.2. It turns out that the set $\mathrm{LF}_{\mathcal{Q}}$ is crucial to estimate the size of COP_n, since sections of both cones with a suitable hyperplane are dual to each other in the so-called differential metric [Ble06, Section 5], originally called an apolar inner product (see [Rez82, p. 11] for a historical account). To get the needed lower bound on the size of $\mathrm{LF}_{\mathcal{Q}}$ we compare it to $\mathrm{NN}_{\mathcal{Q}}$ for which the lower bound is obtained by a version of the reverse Blaschke-Santaló inequality in the differential metric together with a self-duality of $\mathrm{NN}_{\mathcal{Q}}$.

We will estimate the gap between the cones of forms by estimating the volumes of compact sections obtained by intersecting each with a suitably chosen affine hyperplane $\mathcal{L} \subset \mathbb{R}[\mathtt{x}]_4$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and σ the rotation invariant probability measure on S^{n-1}. The natural L^2 inner product (resp. L^2 norm) on $\mathbb{R}[\mathtt{x}]_4$ is given by

$$\langle f, g\rangle = \int_{S^{n-1}} fg \,\mathrm{d}\sigma \qquad \Big(\text{resp. } \|f\|_2^2 = \int_{S^{n-1}} |f|^2 \,\mathrm{d}\sigma.\Big)$$

Remark 1.3. The projection $\mathrm{pr}_{\mathcal{Q}}$, defined by (1.7), is the orthogonal projection onto $\mathcal{Q}$ w.r.t. the L^2 inner product. This follows by noticing that $\int_{S^{n-1}} x_{k_1}\cdots x_{k_8} \mathrm{d}\sigma$, where $k_1,\ldots,k_8 \in \{1,\ldots,n\}$, is zero if there is $j \in \{1,\ldots,n\}$ which appears an odd number of times among the indices $k_1,\ldots,k_8$ [Bar02, Lemma 8].

Let $\mathcal{L}$ be the affine hyperplane of forms from $\mathbb{R}[\mathtt{x}]_4$ of average 1 on S^{n-1}, i.e.,

$$\mathcal{L} = \left\{ f \in \mathbb{R}[\mathtt{x}]_4 \colon \int_{S^{n-1}} f \,\mathrm{d}\sigma = 1 \right\}.$$

For every $K \in \mathcal{C}$ let K' be its intersection with $\mathcal{L}$, that is,

$$K' = K \cap \mathcal{L}.$$

Note that for all $K \in \mathcal{C}$ the section K' is a convex, compact, full-dimensional set in the finite-dimensional affine hyperplane $\mathcal{L}$. For technical reasons we translate every section by subtracting the polynomial $(x_1^2+\ldots+x_n^2)^2$:

$$\widetilde{K} := K' - (x_1^2+\ldots+x_n^2)^2 = \left\{ f \in \mathcal{Q} \colon f + (x_1^2+\ldots+x_n^2)^2 \in K' \right\}.$$

Let $\mathcal{M}$ be the hyperplane of forms from $\mathcal{Q}$ with average 0 on S^{n-1}, i.e.,

$$\mathcal{M} = \left\{ f \in \mathcal{Q} \colon \int_{S^{n-1}} f \,\mathrm{d}\sigma = 0 \right\}. \tag{1.9}$$

Notice that for every $K \in \mathcal{C}$,

$$\widetilde{K} \subseteq \mathcal{M}.$$

With respect to the L^2 inner product $\mathcal{M}$ is a subspace of $\mathcal{Q}$ of dimension $\dim\mathcal{M} = \frac{n(n+1)}{2} - 1$ and so it is isomorphic to $\mathbb{R}^{\dim\mathcal{M}}$ as a Hilbert space. Let $S_{\mathcal{M}}$, $B_{\mathcal{M}}$ be the unit sphere and the unit ball in $\mathcal{M}$, respectively. Let $\psi : \mathbb{R}^{\dim\mathcal{M}} \to \mathcal{M}$ be a unitary isomorphism and $\psi_*\mu$ the pushforward of the Lebesgue measure μ on $\mathbb{R}^{\dim\mathcal{M}}$ to $\mathcal{M}$, i.e., $\psi_*\mu(E) := \mu(\psi^{-1}(E))$ for every Borel measurable set $E \subseteq \mathcal{M}$.

Lemma 1.4. *The measure of a Borel set $E \subseteq \mathcal{M}$ does not depend on the choice of the unitary isomorphism ψ, i.e., if $\psi_1 : \mathbb{R}^{\dim\mathcal{M}} \to \mathcal{M}$ and $\psi_2 : \mathbb{R}^{\dim\mathcal{M}} \to \mathcal{M}$ are unitary isomorphisms, then $(\psi_1)_*\mu(E) = (\psi_2)_*\mu(E)$.*

Proof. The proof of Lemma 1.4 is the same as the proof of [KMŠZ19, Lemma 1.4]. ∎

Let V be a finite-dimensional Hilbert space equipped with the pushforward measure of the Lebesgue measure on $\mathbb{R}^{\dim V}$. The **volume radius** $\mathrm{vrad}(C)$ of a compact measurable set C is defined by

$$\mathrm{vrad}(C) = \left(\frac{\mathrm{Vol}(C)}{\mathrm{Vol}(B)}\right)^{1/\dim V},$$

where B is the unit ball in V.

We will compare the sizes of the convex cones K from (1.8) by comparing the volumes of their sections $\widetilde{K}$. For us the proper measure of the size of $\widetilde{K}$ is $\mathrm{Vol}(\widetilde{K})^{1/\dim\mathcal{M}}$, since we are concerned with the asymptotic behavior as n goes to infinity and we thus need to eliminate the dimension effect when dilating K by some factor c. Namely, such dilation multiplies the volume by $c^{\dim\mathcal{M}}$, but a more appropriate effect would be multiplication by c.

The first main result of the paper is as follows.

Theorem 1.5. *For all $K \in \{\mathrm{POS}_{\mathcal{Q}}, \mathrm{SOS}_{\mathcal{Q}} = \mathrm{SPN}_{\mathcal{Q}}, \mathrm{NN}_{\mathcal{Q}}, \mathrm{PSD}_{\mathcal{Q}}, \mathrm{DNN}_{\mathcal{Q}}, \mathrm{LF}_{\mathcal{Q}}, \mathrm{CP}_{\mathcal{Q}}\}$ we have that*

$$\mathrm{vrad}(\widetilde{K}) = \Theta(n^{-1}).$$

In Section 2 we state some classical inequalities for the volume of a compact section of a convex cone and establish some preliminary results for even quartic forms. These results are then applied in a novel way in the proof of Theorem 1.5, presented in Section 3.

The second contribution of the paper is the construction of matrices from $\mathrm{DNN}_n \setminus \mathrm{CP}_n$ (resp. $\mathrm{COP}_n \setminus \mathrm{SPN}_n$) for any $n \geq 5$, called **exceptional DNN matrices** (resp. **exceptional copositive matrices**). The construction is inspired by the free probability construction in [CHN17] of positive maps between matrix spaces that are not completely positive and uses semidefinite programming as a technical tool. In the following subsection we present the construction, while the justification and some examples are presented in Subsections 4.1 and 4.2, respectively.

Recently in [JS22] the notions of *completely positive completely positive (CPCP)* and *completely positive doubly nonnegative (CPDNN) maps* have been introduced. By the results of the present paper, it follows that asymptotically the ratio of the volume radii of the compact sections of CPCP (resp. CPDNN) and CP cones is strictly positive as n goes to infinity. Moreover, our construction can be used to generate CPDNN maps that are not CPCP. Namely, to obtain such a $*$-linear map $\Phi : M_n(\mathbb{R}) \to M_m(\mathbb{R})$, one only needs to construct an exceptional DNN matrix of size $nm \times nm$ representing the Choi matrix $C = (\Phi(E_{ij}))_{i,j}$ of Φ; here E_{ij} are the $n \times n$ matrix units.

1.1.1. *Construction of exceptional DNN and exceptional copositive matrices of size $n \times n$ for $n \geq 5$.* Let

$$\mathcal{B} := \{1\} \cup \{\sqrt{2}\cos(2k\pi) \colon k \in \mathbb{N}\} \cup \{\sqrt{2}\sin(2k\pi) \colon k \in \mathbb{N}\} \tag{1.10}$$

be the standard orthonormal basis of $L^2[0,1]$ and for $f \in L^\infty[0,1]$ let

$$M_f : L^2[0,1] \to L^2[0,1], \qquad M_f(g) = fg$$

be the corresponding multiplication operator. Note that M_f can be represented as an infinite matrix w.r.t. the basis $\mathcal{B}$.

For a closed subspace $\mathcal{H} \subseteq L^2[0,1]$ denote by $P_{\mathcal{H}} : L^2[0,1] \to \mathcal{H}$ the orthogonal projection onto $\mathcal{H}$. Our idea is to find an infinite dimensional $\mathcal{H}$ and an $f \in \mathcal{H}$ such that $M_f^{\mathcal{H}} := P_{\mathcal{H}} M_f P_{\mathcal{H}}$ has all finite principal submatrices DNN but not CP.

The setting in which we construct such an example is the following:

$$\begin{aligned} &f \text{ is of the form } 1 + 2\sum_{k=1}^{m} a_k \cos(2k\pi), \quad m \in \mathbb{N},\ a_1 \geq 0, \ldots, a_m \geq 0, \\ &\mathcal{H} \subseteq L^2[0,1] \text{ is spanned by the functions } \cos(2k\pi), k \in \mathbb{N}_0. \end{aligned} \tag{1.11}$$

Observe that for $f \in \mathcal{H}$, the operator $M_f^{\mathcal{H}}$ is in fact a multiplication operator on $\mathcal{H}$, and can be represented by an infinite matrix w.r.t. a basis of $\mathcal{H}$.

For $n \in \mathbb{N}$, let $\mathcal{H}_n$ be the finite-dimensional subspace of $\mathcal{H}$ spanned by the functions

$$1, \sqrt{2}\cos(2\pi), \ldots, \sqrt{2}\cos(2(n-1)\pi)$$

and $P_n : \mathcal{H} \to \mathcal{H}_n$ the orthogonal projection onto $\mathcal{H}_n$. To certify that matrices

$$A^{(n)} := P_n M_f^{\mathcal{H}} P_n,\ n \in \mathbb{N} \tag{1.12}$$

are PSD, we impose the condition f is SOS, i.e.,

$$f = v^T B v, \quad \text{where} \quad B \in \mathrm{PSD}_{m'+1} \tag{1.13}$$

and

$$v^T = \begin{pmatrix} 1 & \cos(2\pi x) & \cdots & \cos(2m'\pi x) \end{pmatrix} \quad \text{for some } m' \leq m.$$

Finally, to achieve that $A^{(n)} \notin \mathrm{CP}_n$ for $n \geq 5$, we demand that

$$\langle A^{(5)}, H \rangle < 0, \tag{1.14}$$

where H is the Horn matrix of (1.5) and $\langle \cdot, \cdot \rangle$ is the usual Frobenius inner product on symmetric matrices, i.e., $\langle A, B \rangle = \mathrm{tr}(AB)$.

Now let $m = 6$. The above construction can be implemented with the help of the following feasibility SDP

$$\begin{aligned} &\mathrm{tr}(A^{(5)}H) = -\epsilon, \\ &f = v^{\mathrm{T}} B v \quad \text{with} \quad B \succeq 0, \\ &a_i \geq 0, \quad i = 1, \ldots, 6, \end{aligned} \tag{1.15}$$

where $\epsilon > 0$ is predetermined (small enough). Solving (1.15) for different values of ϵ and $m' \leq 6$, Mathematica's semidefinite optimization solver gives an exceptional DNN matrix $A^{(5)}$ (see Subsection 4.2 for an explicit example).

To construct exceptional copositive matrices of any size we then proceed as follows. Let $A^{(n)}$, $n \geq 5$, be a DNN matrix constructed by the procedure described above. To obtain an exceptional copositive matrix C of size $n \times n$ we impose the conditions

$$\begin{aligned} &\langle A^{(n)}, C\rangle < 0, \\ &\Big(\sum_{i=1}^{n} x_i^2\Big)^k q_C \text{ is SOS for some } k \in \mathbb{N} \end{aligned} \tag{1.16}$$

with q_C as in (1.2). Searching for C satisfying (1.16) for fixed k can again be formulated as a feasibility SDP. For an explicit example obtained in this way see Subsection 4.2.

1.1.2. *Difference with the original work of Blekherman.* In [Ble06] Blekherman established estimates on the volume radii of compact sections of the cones of nonnegative forms and sums of squares forms. For a fixed degree bigger than 2, as the number of variables goes to infinity, the ratio between the volume radii goes to 0.

Specializing Blekherman's techniques, in [KMŠZ19] two types of cones of linear maps between matrix spaces were compared. Namely, the larger cone of positive maps and the smaller cone of completely positive maps. Using the correspondence analogous to (1.2) the problem is equivalent to comparing the cone of nonnegative biquadratic biforms with the smaller cone of biforms that are sums of squares of bilinear ones. The conclusion is similar to the case of all forms [Ble06]: as the number of variables goes to infinity, the ratio between the volume radii goes to 0.

However, as we explain next, several adaptations of Blekherman's original approach are needed to give sufficiently tight estimates on the volume radii of the cones in (1.8) we are interested in. To study these cones from Definition 1.1 one could a priori use either of the one-to-one correspondences (1.1) or (1.2) and then compare sizes of the corresponding cones in either quadratic or quartic forms, respectively. However, in quadratic forms establishing volume estimates does not seem to be straightforward with specializing Blekherman's approach; it is not clear how to handle copositivity to derive asymptotically tight estimates. Working with the cones within even quartics using (1.2) is more convenient. Here copositive matrices correspond to nonnegative even quartics, while SPN matrices correspond to sum of squares (SOS) even quartics. In the latter case one can specialize Blekherman's techniques, but the obtained estimates are not tight enough: the upper bound on the volume radius of even SOS quartics turns out to be larger than the lower bound on nonnegative ones. Therefore one has to adapt the methods to obtain tighter estimates. The crucial observation to achieve this is to notice that dilations of the difference bodies of the compact sections of $\mathrm{LF}_{\mathcal{Q}}$, resp. $\mathrm{CP}_{\mathcal{Q}}$, by absolute constants independent of the dimension, contain the compact section of $\mathrm{NN}_{\mathcal{Q}}$ (see Lemma 3.4 below). This fact together with the reverse Blaschke-Santaló inequality suffices to obtain conclusive estimates. In particular, perhaps slightly surprisingly, the difference between the cones of nonnegative even quartics and SOS even ones does not grow arbitrarily large as the number of variables grows to infinity.

In [BR21], the authors compared the cone of nonnegative symmetric quartics and the cone of symmetric quartics that are sums of squares of quadratics. Similarly as in the setting of the present paper and in sharp contrast to [Ble06], asymptotically the ratio on the volume radii of compact sections of these cones is strictly positive as the number of variables goes to infinity. However, the subspaces of quartic forms we study and the ones from [BR21] are essentially different. For instance, the subspace of even symmetric quartics has empty interior in the subspace of even quartics. Thus different methods are required to obtain Theorem 1.5.

2. Preliminaries

In this section we recall some classical inequalities for the volume of a compact set and establish some properties of even quartic forms needed in the proof of our main result on volume estimates in the next section. These properties are obtained by specializing the results from [Ble04, Ble06] to our setting.

2.1. **Blaschke-Santaló inequality and its reverse.** Let K be a bounded convex set in $\mathbb{R}^n$ with origin in its interior and $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{R}^n$. The **polar** K° of K is defined by

$$K^\circ = \{y \in \mathbb{R}^n \colon \langle x, y\rangle \le 1 \quad \forall x \in K\}.$$

For $z \in \mathbb{R}^n$ let

$$K^z = \{y \in \mathbb{R}^n \colon \langle x - z, y - z\rangle \le 1 \quad \forall x \in K\},$$

i.e., z is translated to the origin. We denote by $p(K)$ the **volume product** of K:

$$p(K) = \inf\{\operatorname{Vol}(K)\operatorname{Vol}(K^z) \colon z \text{ is an interior point of } K\}. \tag{2.1}$$

It is clear that $p(K)$ is affine invariant, i.e., for every affine linear invertible transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ it holds that $p(K) = p(TK)$. It turns out that there is a unique point z [MP90, p. 85] , where the infimum in (2.1) is attained. This point is called the **Santaló point** of K and we denote it by $s(K)$. The following upper bound holds for the volume product.

Theorem 2.1 (Blaschke-Santaló inequality, [MP90, p. 90])**.** *For a bounded convex set K in $\mathbb{R}^n$ with a non-empty interior it holds that*

$$\operatorname{Vol}(K)\operatorname{Vol}(K^{s(K)}) \le (\operatorname{Vol}(B))^2,$$

where B is the unit ball w.r.t. the inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$.

There is also a lower bound for the volume product.

Theorem 2.2 (Reverse Blaschke-Santaló inequality)**.** *For a bounded convex set K in $\mathbb{R}^n$ with the origin in its interior, it holds that*

$$4^{-n}\pi n(\operatorname{Vol}(B))^2 < \operatorname{Vol}(K)\operatorname{Vol}(K^\circ), \tag{2.2}$$

where B is the unit ball w.r.t. the inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$.

Bourgain and Milman [BM87, Corollary 6.1] established a weaker version of Theorem 2.2 with $4^{-n}\pi n$ replaced by c^n for an absolute constant $c > 0$ independent of the dimension n and the set K. Later, Kupperberg [Kup08, Corollary 1.8] proved that $\operatorname{Vol}(K)\operatorname{Vol}(K^\circ) \ge 4^n\big(\frac{(n!)^2}{(2n!)}\big)^2(\operatorname{Vol}(B))^2$. Since $4^n\big(\frac{(n!)^2}{(2n!)}\big)^2 = \Big(2^n\big(\frac{(n!)^2}{(2n!)}\big)\Big)^2 \ge 2^{-2n} = 4^{-n}$, one immediately gets $4^n(\operatorname{Vol}(B))^2$ as a lower bound on $\operatorname{Vol}(K)\operatorname{Vol}(K^\circ)$. To obtain the lower bound from Theorem 2.2, we use the following asymptotically sharp estimate:

$$4^n\Big(\frac{(n!)^2}{(2n!)}\Big)^2 = 4^n\Big(\binom{2n}{n}\Big)^{-2} > \big(2^{-n}\sqrt{\pi n}\big)^2 = 4^{-n}\pi n.$$

Indeed, following the idea from [UL-mo] one can estimate the central binomial coefficient as follows:

$$\begin{aligned}\binom{2n}{n} &= \frac{4^n}{\pi}\int_{-\pi/2}^{\pi/2}\cos^{2n}x\,dx \leq \frac{4^n}{\pi}\int_{-\pi/2}^{\pi/2}e^{-nx^2}x\,dx \\ &< \frac{4^n}{\pi}\int_{-\infty}^{\infty}e^{-nx^2}x\,dx = \frac{4^n}{\sqrt{\pi n}}.\end{aligned} \tag{2.3}$$

The first equality can be obtained either by induction and integration by parts or by writing $\cos x = (e^{\mathrm{i}x} + e^{-\mathrm{i}x})/2$, expanding $\cos^{2n} x$ by the binomial theorem and directly computing the integrals involving the exponential function. For the first inequality we estimate $\cos x \leq e^{-x^2/2}$ on $|x| \leq \pi/2$ by noting that

$$\log\cos x + \frac{x^2}{2} \leq 0$$

for $|x| < \pi/2$. Indeed, the function $\log\cos x + \frac{x^2}{2}$ is even and vanishes at $x = 0$, but it is also concave since its second derivative $1 - 1/\cos^2 x$ is negative on $0 < |x| < \pi/2$. Asymptotical sharpness of the upper bound in (2.3) follows by Stirling's formula.

2.2. **Rogers-Shepard inequality.** Let K be a bounded convex set in $\mathbb{R}^n$ with a non-empty interior. The **difference body** $\mathrm{Diff}(K)$ of K [RS57] is defined by

$$\mathrm{Diff}(K) := K - K.$$

The following inequality compares the volumes of K and $\mathrm{Diff}(K)$.

Theorem 2.3 (Rogers-Shepard inequality, [RS57, Theorem 1])**.** *Let K be a bounded convex set in $\mathbb{R}^n$ with a non-empty interior. Then*

$$\mathrm{Vol}(\mathrm{Diff}(K)) \leq \binom{2n}{n}\mathrm{Vol}(K)$$

and hence

$$\mathrm{vrad}(\mathrm{Diff}(K)) \leq 4\,\mathrm{vrad}(K).$$

Remark 2.4. Let K_1, K_2 be any of the cones in (1.8). In this remark we discuss one possible approach, based on applying Theorem 2.3, to establish asymptotic behavior of the ratio of volume radii of the compact bases $\widetilde{K}_1, \widetilde{K}_2$ as n goes to infinity. As we explain next, using this approach gives tight estimates only for cones contained in $\mathrm{NN}_{\mathcal{Q}}$.

Assume that $K_1 \subseteq K_2$ and $\widetilde{K}_2 \subseteq c \cdot \mathrm{Diff}(\widetilde{K}_1)$ for some constant c. Then by Theorem 2.3, the ratio $\frac{\mathrm{vrad}\,K_1}{\mathrm{vrad}\,K_2}$, as n goes to infinity, is bounded below by $\frac{1}{4c}$. To prove that this lower bound is strictly positive as n goes to infinity, one has to argue that c can be chosen independently of n.

To derive a dilation constant c from the previous paragraph, it suffices to consider the extreme points of $\widetilde{K}_2$. Let p be such an extreme point and let c_p be the smallest constant such that $p \in c_p \cdot \mathrm{Diff}(\widetilde{K}_1)$. A good choice for c is then

$$c = \sup\{c_p \colon p \text{ is an extreme point of } \widetilde{K}_2\}.$$

Let p be of the form $c_{ij}x_i^2x_j^2 - (\sum_{i=1}^n x_i^2)^2$ for $c_{ij} \in \mathbb{R}$, where $c_{ij} \in \mathbb{R}$ is such that $p \in \widetilde{K}_2$. It turns out by a simple computation that p belongs to $4\,\mathrm{Diff}(\mathrm{CP}_{\mathcal{Q}})$ (see (3.5) below). Since all extreme points of $\widetilde{\mathrm{NN}}_{\mathcal{Q}}$ are of this form, this implies that $\widetilde{K}_2 \subseteq 4 \cdot \mathrm{Diff}(\widetilde{K}_1)$ for any pair of cones K_1, K_2 sandwiched between $\mathrm{CP}_{\mathcal{Q}}$ and $\mathrm{NN}_{\mathcal{Q}}$.

However, not all extreme points of $\widetilde{\mathrm{POS}}_{\mathcal{Q}}$ and $\widetilde{\mathrm{SOS}}_{\mathcal{Q}}$ are of the simple form from the previous paragraph. For such extreme points p it is not clear whether some dilation constant c_p as above can be chosen independently of n. This is the main limitation preventing us from establishing Theorem 1.5 solely by applying Theorem 2.3.

2.3. **Membership in $\mathcal{M}$, $\mathcal{L}$ and the space of harmonic polynomials.** Let

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

be the Laplace operator. A form $f \in \mathbb{R}[\mathtt{x}]_k$ is called **harmonic** if $\Delta(f) = 0$. We denote by $\mathcal{H}_4$ the space of all harmonic forms of degree 4.

The following lemma characterizes the membership of $f \in \mathcal{Q}$ in $\mathcal{L}$, $\mathcal{M}$, $\mathcal{H}_4$ in terms of its coefficients.

Lemma 2.5. *Let $f(\mathtt{x}) = \sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2 \in \mathcal{Q}$. The following statements hold:*

(1) $f \in \mathcal{L} \quad \Leftrightarrow \quad 3\Big(\sum_{i=1}^{n} a_{ii}\Big) + \sum_{i \neq j} a_{ij} = n(n+2).$

(2) $f \in \mathcal{M} \quad \Leftrightarrow \quad 3\Big(\sum_{i=1}^{n} a_{ii}\Big) + \sum_{i \neq j} a_{ij} = 0.$

(3) $f \in \mathcal{H}_4 \quad \Leftrightarrow \quad a_{ii} = -\frac{1}{6} \sum_{\substack{j=1,\ldots,n,\\ j \neq i}} (a_{ij} + a_{ji})$ *for* $i = 1, \ldots, n.$

(4) $\mathcal{H}_4 \subseteq \mathcal{M}$.

(5) $\dim(\mathcal{H}_4 \cap \mathcal{Q}) = \frac{n(n-1)}{2}$.

Proof. (1) and (2) follow by using [Bar02, Lemma 8] to check that

$$\int_{S^{n-1}} x_i^4 \, \mathrm{d}\sigma = \frac{3}{n(n+2)} \quad \text{for } i = 1, \ldots, n, \tag{2.4}$$

$$\int_{S^{n-1}} x_i^2 x_j^2 \, \mathrm{d}\sigma = \frac{1}{n(n+2)} \quad \text{for } 1 \leq i \neq j \leq n. \tag{2.5}$$

(3) follows from the following computation:

$$\begin{aligned}
\Delta\Big(\sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2\Big) &= \sum_{i=1}^{n} (12 a_{ii} x_i^2) + \sum_{i \neq j} (2 a_{ij}(x_i^2 + x_j^2)) \\
&= \sum_{i=1}^{n} \Big(x_i^2 \big(12 a_{ii} + \sum_{\substack{j=1,\ldots,n\\ j \neq i}} 2(a_{ij} + a_{ji})\big)\Big).
\end{aligned}$$

Hence,

$$\Delta\Big(\sum_{i,j=1}^{n} a_{ij} x_i^2 x_j^2\Big) = 0 \quad \Leftrightarrow \quad 12 a_{ii} + \sum_{\substack{j=1,\ldots,n\\ j \neq i}} 2(a_{ij} + a_{ji}) = 0 \quad \forall i = 1, \ldots, n,$$

which proves (3).

(4) follows easily from (3) and (2), while the following computation using (3),

$$\dim(\mathcal{H}_4 \cap \mathcal{Q}) = \dim \mathcal{Q} - n = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2},$$

proves (5). ■

2.4. **The differential metric.** In this subsection we follow [Ble06, Section 5]. For a form

$$f(\mathtt{x}) = \sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \in \mathbb{R}[\mathtt{x}]_4$$

the **differential operator** $D_f : \mathbb{R}[\mathtt{x}]_4 \to \mathbb{R}$ is defined by

$$D_f(g) = \sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} \frac{\partial^4 g}{\partial x_i \partial x_j \partial x_k \partial x_\ell}.$$

The **differential metric** on $\mathbb{R}[\mathtt{x}]_4$ is given by

$$\langle f, g\rangle_d = D_f(g).$$

For a point $v = (v_1, \ldots, v_n) \in S^{n-1}$, we denote by v^4 the fourth power of a linear form:

$$v^4 := (v_1 x_1 + \ldots + v_n x_n)^4.$$

For us the following operator $T : \mathbb{R}[\mathtt{x}]_4 \to \mathbb{R}[\mathtt{x}]_4$ will be important:

$$(Tf)(\mathtt{x}) = \int_{S^{n-1}} f(v) v^4 \mathrm{d}\sigma(v).$$

Let

$$\mathcal{H}_0 := \Big\{ c\Big(\sum_{i=1}^n x_i^2\Big)^2 : c \in \mathbb{R}\Big\},$$

$$\mathcal{H}_2 := \Big\{ g \in \mathbb{R}[\mathtt{x}]_4 : g = \Big(\sum_{i=1}^n x_i^2\Big) \cdot h \quad \text{for some harmonic form } h \in \mathbb{R}[\mathtt{x}]_2 \Big\}.$$

It is well-known that every $f \in \mathbb{R}[\mathtt{x}]_4$ can be uniquely written as a sum $f = f_0 + f_1 + f_2$, where $f_i \in \mathcal{H}_{2i}$, $i = 0, 1, 2$ [Ble04, Theorem 2.1]. We denote by $\ell_i : \mathbb{R}[\mathtt{x}]_4 \to \mathcal{H}_{2i}$, $i = 0, 1, 2$, the corresponding projection operators, i.e., $\ell_i(f) = f_i$.

We collect some properties of T in the next lemma.

Lemma 2.6. *The operator T satisfies:*

(1) $\langle Tf, g\rangle_d = 4!\langle f, g\rangle$ *for every* $f, g \in \mathbb{R}[\mathtt{x}]_4$.

(2) $\mathcal{Q}$ *is an invariant subspace of* T.

(3) $T\Big(\Big(\sum_{i=1}^n x_i^2\Big)^2\Big) = \dfrac{3}{n(n+2)}\Big(\sum_{i=1}^n x_i^2\Big)^2.$

(4) $\dfrac{n(n+2)}{3} T(f) = \ell_0(f) + \dfrac{4}{n+4}\ell_1(f) + \dfrac{8}{(n+4)(n+6)}\ell_2(f)$ *for every* $f \in \mathbb{R}[\mathtt{x}]_4$.

(5) *The restriction* $T|_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ *of* T *to* $\mathcal{Q}$ *is bijective.*

Proof. (1) is [Ble06, Lemma 5.1].

Next, we establish (2). Let $f(\mathtt{x}) = \sum_{i,j=1}^n a_{ij}x_i^2x_j^2 \in \mathcal{Q}$. Then

$$\begin{aligned}(Tf)(\mathtt{x}) &= \int_{S^{n-1}} f(v)v^4 \mathrm{d}\sigma(v) = \int_{S^{n-1}} \left(\sum_{i,j=1}^n a_{ij}v_i^2v_j^2\right) v^4 \mathrm{d}\sigma(v) \\ &= \sum_{i,j=1}^n \sum_{k_1,k_2,k_3,k_4=1}^n \left(a_{ij}x_{k_1}x_{k_2}x_{k_3}x_{k_4}\int_{S^{n-1}} \left(v_i^2v_j^2v_{k_1}v_{k_2}v_{k_3}v_{k_4}\right) \mathrm{d}\sigma(v)\right)\end{aligned}$$

By [Bar02, Lemma 8], the integral of $v_i^2v_j^2v_{k_1}v_{k_2}v_{k_3}v_{k_4}$ over S^{n-1} is nonzero iff all the exponents at the coordinates of v are even. Hence, either $k_1 = k_2 = k_3 = k_4$ or the indices k_i split into two equal pairs, i.e., $k_{i_1} = k_{i_2}$ and $k_{i_3} = k_{i_4}$, where i_1, i_2, i_3, i_4 is some permutation of the indices $1, 2, 3, 4$. In both cases $x_{k_1}x_{k_2}x_{k_3}x_{k_4} \in \mathcal{Q}$ and hence $Tf \in \mathcal{Q}$.

(3) is a special case (for $k = 2$) of the first paragraph of the proof of [Ble06, Lemma 5.2] showing that

$$T\Big(\big(\sum_{i=1}^n x_i^2\big)^2\Big) = c\big(\sum_{i=1}^n x_i^2\big)^2 \quad \text{for } c = \int_{S^{n-1}} x_1^4 \mathrm{d}\sigma = \frac{3}{n(n+2)}. \tag{2.6}$$

(4) follows by [Ble04, Lemma 7.4], where it is shown that

$$\frac{1}{c}T(f) = c_0\ell_0(f) + c_1\ell_1(f) + c_2\ell_2(f),$$

with c as in (2.6) and $c_0 = 1, c_1 = \frac{4}{n+4}, c_2 = \frac{8}{(n+4)(n+6)}$.

It remains to prove (5). Using the fact that $\mathbb{R}[x]_4$ is a direct sum of the subspaces $\mathcal{H}_0$, $\mathcal{H}_2$ and $\mathcal{H}_4$ [Ble04, Theorem 2.1] and (4), it follows that $T : \mathbb{R}[x]_4 \to \mathbb{R}[x]_4$ is bijective. In particular, $T|_{\mathcal{Q}}$ is injective and since $T|_{\mathcal{Q}}$ maps to the finite-dimensional $\mathcal{Q}$ by (2), (5) follows. ∎

Given a full-dimensional cone $L \subseteq \mathcal{Q}$ such that $\big(\sum_{i=1}^n x_i^2\big)^2$ is in the interior of L and $\int_{S^{n-1}} f \mathrm{d}\sigma > 0$ for every nonzero $f \in L$, we define the sets

$$L' = L \cap \mathcal{L} \qquad \text{and} \qquad \widetilde{L} = \left\{ f \in \mathcal{M} \colon f + \big(\sum_{i=1}^n x_i^2\big)^2 \in L \right\}.$$

Let L^* and L_d^* be the duals of L in the L^2 metric and the differential metric, respectively:

$$\begin{aligned} L^* &= \{f \in \mathcal{Q} \colon \langle f, g\rangle \geq 0 \quad \forall g \in L\}, \\ L_d^* &= \{f \in \mathcal{Q} \colon \langle f, g\rangle_d \geq 0 \quad \forall g \in L\}. \end{aligned}$$

The following lemma is an analog of [Ble06, Lemma 5.2] for $\mathcal{Q}$.

Lemma 2.7. *Let L be a full-dimensional cone in $\mathcal{Q}$ such that $\big(\sum_{i=1}^n x_i^2\big)^2$ is in the interior of L and $\int_{S^{n-1}} f \mathrm{d}\sigma > 0$ for every nonzero $f \in L$. Then:*

$$\frac{8}{(n+4)(n+6)} \leq \left(\frac{\operatorname{Vol} \widetilde{L_d^*}}{\operatorname{Vol} \widetilde{L^*}}\right)^{1/\dim \mathcal{M}} = \frac{\operatorname{vrad} \widetilde{L_d^*}}{\operatorname{vrad} \widetilde{L^*}} \leq \left(\frac{8}{(n+4)(n+6)}\right)^{1-\frac{2n-1}{n^2+n-1}}.$$

Proof. We follow the proof of [Ble06, Lemma 5.2]. By (1) of Lemma 2.6, for every $f, g \in \mathcal{Q}$ we have

$$\langle f, g\rangle \geq 0 \quad \Leftrightarrow \quad \langle Tf, g\rangle_d \geq 0.$$

Since by (5) of Lemma 2.6, T maps $\mathcal{Q}$ bijectively to $\mathcal{Q}$, it follows that

$$T(L^*) = L_d^*. \tag{2.7}$$

Observe that $\mathcal{M} = \widetilde{\mathcal{H}_2} \oplus \widetilde{\mathcal{H}_4}$, where $\widetilde{\mathcal{H}_{2i}} = \mathcal{H}_{2i} \cap \mathcal{M}$, $i = 1, 2$. Since $\mathcal{H}_2$, $\mathcal{H}_4$ are invariant subspaces of $\frac{n(n+2)}{3}T$ by (4) of Lemma 2.6, it follows that $\mathcal{M}$ is also an invariant subspace of $\frac{n(n+2)}{3}T$, which in addition also fixes $\big(\sum_{i=1}^n x_i^2\big)^2$. This, together with (2.7), implies that

$$\Big(\frac{n(n+2)}{3}T\Big)(\widetilde{L^*}) = \widetilde{L_d^*}. \tag{2.8}$$

Since by (4) of Lemma 2.6, the operator $\frac{n(n+2)}{3}T$ acts as a contraction on the subspaces $\widetilde{\mathcal{H}_{2i}}$, $i = 1, 2$, with the smallest contraction coefficient $\frac{8}{(n+4)(n+6)}$, this establishes the lower bound in the statement of the lemma. To get the upper bound observe that the largest contraction occurs in $\widetilde{\mathcal{H}_4}$ where $\dim \widetilde{\mathcal{H}_4} = \frac{n(n-1)}{2}$ by (5) of Lemma 2.5. Since

$$\frac{\dim \widetilde{\mathcal{H}_4}}{\dim \mathcal{M}} = \frac{\frac{n(n-1)}{2}}{\frac{n(n+1)-1}{2}} = \frac{n(n-1)}{n(n+1)-1} = 1 - \frac{2n-1}{n^2+n-1},$$

this establishes the upper bound of the lemma. ∎

The following proposition establishes a connection between the polar of the section of a cone and the section of its dual in the L^2 metric.

Proposition 2.8. *Let L be a full-dimensional cone in $\mathcal{Q}$ such that $\big(\sum_{i=1}^n x_i^2\big)^2$ is in the interior of L, and assume $\int_{S^{n-1}} f \mathrm{d}\sigma > 0$ for every nonzero $f \in L$. Then*

$$(\widetilde{L})^\circ = -\widetilde{L^*}.$$

Proof. We have that:

$$\begin{aligned}
\widetilde{L^*} &= \Big\{ f \in \mathcal{M} \colon f + (\sum_{i=1}^n x_i^2)^2 \in L^* \Big\} \\
&= \Big\{ f \in \mathcal{M} \colon \langle f + (\sum_{i=1}^n x_i^2)^2, h \rangle \geq 0 \quad \forall h \in L \Big\} \\
&= \Big\{ f \in \mathcal{M} \colon \langle f + (\sum_{i=1}^n x_i^2)^2, h \rangle \geq 0 \quad \forall h \in L' \Big\} \\
&= \{ f \in \mathcal{M} \colon \langle f, h \rangle \geq -1 \quad \forall h \in L' \} \\
&= \{ f \in \mathcal{M} \colon \langle -f, h \rangle \leq 1 \quad \forall h \in L' \} \\
&= \Big\{ f \in \mathcal{M} \colon \langle -f, g + (\sum_{i=1}^n x_i^2)^2 \rangle \leq 1 \quad \forall g \in \widetilde{L} \Big\} \\
&= \Big\{ f \in \mathcal{M} \colon \langle -f, g \rangle \leq 1 \quad \forall g \in \widetilde{L} \Big\} \\
&= -(\widetilde{L})^\circ,
\end{aligned}$$

where in the third equality we used the homogeneity of the inner product, in the fourth the equality $\big\langle (\sum_{i=1}^n x_i^2)^2, h \big\rangle = 1$ for every $h \in L'$ and in the seventh the equality $\big\langle f, (\sum_{i=1}^n x_i^2)^2 \big\rangle = 0$ for every $f \in \mathcal{M}$. This concludes the proof of the proposition. ∎

Corollary 2.9. *Let L be a full-dimensional cone in $\mathcal{Q}$ such that $\left(\sum_{i=1}^n x_i^2\right)^2$ is in the interior of L, and assume $\int_{S^{n-1}} f \mathrm{d}\sigma > 0$ for every nonzero $f \in L$. Then*

$$\frac{2}{(n+4)(n+6)} \leq \operatorname{vrad}(\widetilde{L}) \operatorname{vrad}(\widetilde{L_d^*}). \tag{2.9}$$

Moreover, if $\left(\sum_{i=1}^n x_i^2\right)^2$ is in addition the Santaló point of L, then

$$\operatorname{vrad}(\widetilde{L}) \operatorname{vrad}(\widetilde{L_d^*}) \leq \left(\frac{8}{(n+4)(n+6)}\right)^{1-\frac{2n-1}{n^2+n-1}}, \tag{2.10}$$

Proof. Using Theorem 2.2 and Proposition 2.8 we have that

$$\frac{1}{4} \leq \operatorname{vrad}(\widetilde{L}) \operatorname{vrad}(\widetilde{L^*}). \tag{2.11}$$

Using (2.11) and Lemma 2.7 implies (2.9). Using Theorem 2.1 instead of Theorem 2.2 in the reasoning above we obtain the moreover part. ∎

3. Volume radii estimates of our cones

In this section we prove our main result on the estimates of volume radii of the cones under investigation:

Theorem 3.1. *Let*

$$\mathcal{C} := \{\mathrm{POS}_{\mathcal{Q}}, \mathrm{SOS}_{\mathcal{Q}} = \mathrm{SPN}_{\mathcal{Q}}, \mathrm{NN}_{\mathcal{Q}}, \mathrm{PSD}_{\mathcal{Q}}, \mathrm{DNN}_{\mathcal{Q}}, \mathrm{LF}_{\mathcal{Q}}, \mathrm{CP}_{\mathcal{Q}}\}$$

be the set of cones in the vector space of even quartics $\mathcal{Q}$. We have that

$$(2^4\sqrt{2})^{-1} n^{-1} \leq \operatorname{vrad}(\widetilde{\mathrm{CP}_{\mathcal{Q}}}) \leq \operatorname{vrad}(\widetilde{\mathrm{POS}_{\mathcal{Q}}}) \leq 2^8\sqrt{2} n^{-1}. \tag{3.1}$$

In particular, for every $K \in \mathcal{C}$ it holds that

$$\operatorname{vrad}(\widetilde{K}) = \Theta(n^{-1}).$$

Before proving Theorem 3.1 we need three lemmas. The first lemma compares the section $\widetilde{\mathrm{NN}_{\mathcal{Q}}}$ with the sections $\widetilde{\mathrm{LF}_{\mathcal{Q}}}$ and $\widetilde{\mathrm{CP}_{\mathcal{Q}}}$.

Lemma 3.2. *The following inclusions hold:*

1. $\widetilde{\mathrm{LF}_{\mathcal{Q}}} \subseteq \widetilde{\mathrm{NN}_{\mathcal{Q}}} \subseteq 2\,\mathrm{Diff}(\widetilde{\mathrm{LF}_{\mathcal{Q}}})$.
2. $\widetilde{\mathrm{CP}_{\mathcal{Q}}} \subseteq \widetilde{\mathrm{NN}_{\mathcal{Q}}} \subseteq 4\,\mathrm{Diff}(\widetilde{\mathrm{CP}_{\mathcal{Q}}})$.

Proof. The first inclusions in (1) and (2) are clear. To prove the remaining inclusions of the lemma note that it suffices to prove that every extreme point of $\widetilde{\mathrm{NN}_{\mathcal{Q}}}$ is contained in the corresponding set. Note that the extreme points of $\widetilde{\mathrm{NN}_{\mathcal{Q}}}$ are of two types:

$$\frac{n(n+2)}{3} x_i^4 - \Big(\sum_{i=1}^n x_i^2\Big)^2 \quad \text{for some } i = 1, \ldots, n, \tag{3.2}$$

$$n(n+2) x_i^2 x_j^2 - \Big(\sum_{i=1}^n x_i^2\Big)^2 \quad \text{for some } i, j = 1, \ldots, n, i \neq j. \tag{3.3}$$

The extreme points of the form (3.2) clearly belong to both sections $\widetilde{\mathrm{LF}}_\mathcal{Q}$, $\widetilde{\mathrm{CP}}_\mathcal{Q}$, hence also to the dilations of their difference bodies. So it remains to study the extreme points of the form (3.3). For the case of $\widetilde{\mathrm{LF}}_\mathcal{Q}$ we have the following computation:

$$
\begin{aligned}
&n(n+2)x_i^2x_j^2 - (\sum_{i=1}^n x_i^2)^2 = \\
&= \frac{n(n+2)}{6}\left(\mathrm{pr}_\mathcal{Q}\left((x_i+x_j)^4 - x_i^4 - x_j^4\right)\right) - (\sum_{i=1}^n x_i^2)^2 \\
&= 2\underbrace{\left(\frac{n(n+2)}{12}\mathrm{pr}_\mathcal{Q}((x_i+x_j)^4) - (\sum_{i=1}^n x_i^2)^2\right)}_{p_1} - \frac{1}{2}\underbrace{\left(\frac{n(n+2)}{3}x_i^4 - (\sum_{i=1}^n x_i^2)^2\right)}_{p_2} \\
&\quad - \frac{1}{2}\underbrace{\left(\frac{n(n+2)}{3}x_j^4 - (\sum_{i=1}^n x_i^2)^2\right)}_{p_3} = p_1 + \frac{1}{2}(p_1-p_2) + \frac{1}{2}(p_1-p_3) \\
&\in \widetilde{\mathrm{LF}}_\mathcal{Q} + \frac{1}{2}\mathrm{Diff}(\widetilde{\mathrm{LF}}_\mathcal{Q}) + \frac{1}{2}\mathrm{Diff}(\widetilde{\mathrm{LF}}_\mathcal{Q}) \subseteq 2\,\mathrm{Diff}(\widetilde{\mathrm{LF}}_\mathcal{Q}),
\end{aligned}
\tag{3.4}
$$

where we used (2.4) and (2.5) in the containment of the last line. This concludes the proof of (1). Similarly for the case of $\widetilde{\mathrm{CP}}_\mathcal{Q}$ the following computation holds:

$$
\begin{aligned}
&n(n+2)x_i^2x_j^2 - (\sum_{i=1}^n x_i^2)^2 = \\
&= \frac{n(n+2)}{2}\left((x_i^2+x_j^2)^2 - x_i^4 - x_j^4)\right) - (\sum_{i=1}^n x_i^2)^2 \\
&= 4\underbrace{\left(\frac{n(n+2)}{8}(x_i^2+x_j^2)^2) - (\sum_{i=1}^n x_i^2)^2\right)}_{p_4} - \frac{3}{2}\left(\frac{n(n+2)}{3}x_i^4 - (\sum_{i=1}^n x_i^2)^2\right) \\
&\quad - \frac{3}{2}\left(\frac{n(n+2)}{3}x_j^4 - (\sum_{i=1}^n x_i^2)^2\right) = p_4 + \frac{3}{2}(p_4-p_2) + \frac{3}{2}(p_4-p_3) \\
&\in \widetilde{\mathrm{CP}}_\mathcal{Q} + \frac{3}{2}\mathrm{Diff}(\widetilde{\mathrm{CP}}_\mathcal{Q}) + \frac{3}{2}\mathrm{Diff}(\widetilde{\mathrm{CP}}_\mathcal{Q}) \subseteq 4\,\mathrm{Diff}(\widetilde{\mathrm{CP}}_\mathcal{Q}),
\end{aligned}
\tag{3.5}
$$

where p_2, p_3 are as in (3.4) and we used (2.4) and (2.5) in the containment of the last line. This concludes the proof of (2) and the lemma. ∎

Remark 3.3. By the same reasoning as in the proof of Lemma 3.2 one can show that

$$\widetilde{\mathrm{NN}}_\mathcal{Q} \subseteq 2\,\mathrm{Diff}(\widetilde{\mathrm{PSD}}_\mathcal{Q}).$$

Indeed, for the extreme points of $\widetilde{\mathrm{NN}}_{\mathcal{Q}}$ of the form (3.3) this inclusion follows by the following computation:

$$\begin{aligned}
&n(n+2)x_i^2x_j^2 - \big(\sum_{i=1}^n x_i^2\big)^2 = \\
&= \frac{n(n+2)}{4}\left((x_i^2+x_j^2)^2-(x_i^2-x_j^2)^2\right) - \big(\sum_{i=1}^n x_i^2\big)^2 \\
&= 2\underbrace{\left(\frac{n(n+2)}{8}(x_i^2+x_j^2)^2 - \big(\sum_{i=1}^n x_i^2\big)^2\right)}_{p_1} - \underbrace{\left(\frac{n(n+2)}{4}(x_i^2-x_j^2)^2 - \big(\sum_{i=1}^n x_i^2\big)^2\right)}_{p_2} \\
&= p_1 + (p_1-p_2) \in \widetilde{\mathrm{PSD}}_{\mathcal{Q}} + \mathrm{Diff}(\widetilde{\mathrm{PSD}}_{\mathcal{Q}}) \subseteq 2\,\mathrm{Diff}(\widetilde{\mathrm{PSD}}_{\mathcal{Q}}).
\end{aligned}$$

The second lemma needed in the proof of Theorem 3.1 establishes two dualities in the differential metric between the sections of the cones from Theorem 3.1.

Lemma 3.4. *We have the following dualities in the differential metric:*

(1) $(\widetilde{\mathrm{NN}}_{\mathcal{Q}})_d^* = \widetilde{\mathrm{NN}}_{\mathcal{Q}}$.

(2) $(\widetilde{\mathrm{LF}}_{\mathcal{Q}})_d^* = \widetilde{\mathrm{POS}}_{\mathcal{Q}}$.

Proof. First we prove (1). It is equivalent to establish $(\mathrm{NN}_{\mathcal{Q}})_d^* = \mathrm{NN}_{\mathcal{Q}}$. We have:

$$\begin{aligned}
(\mathrm{NN}_{\mathcal{Q}})_d^* &= \left\{\sum_{i,j=1}^n a_{ij}x_i^2x_j^2 \in \mathcal{Q}\colon \big\langle \sum_{i,j=1}^n a_{ij}x_i^2x_j^2, g\big\rangle_d \geq 0 \quad \forall g \in \mathrm{NN}_{\mathcal{Q}}\right\} \\
&= \left\{\sum_{i,j=1}^n a_{ij}x_i^2x_j^2 \in \mathcal{Q}\colon \big\langle \sum_{i,j=1}^n a_{ij}x_i^2x_j^2, x_k^2x_\ell^2\big\rangle_d \geq 0 \quad \forall k,\ell = 1,\ldots,n\right\} \\
&= \left\{\sum_{i,j=1}^n a_{ij}x_i^2x_j^2 \in \mathcal{Q}\colon a_{k\ell} \geq 0 \quad \forall k,\ell = 1,\ldots,n\right\} \\
&= \mathrm{NN}_{\mathcal{Q}},
\end{aligned}$$

where in the second equality we used that the extreme points of $\mathrm{NN}_{\mathcal{Q}}$ are of the form $c^2x_k^2x_\ell^2$ for some $c \in \mathbb{R}$ and $k,\ell = 1,\ldots,n$, while in the third equality we used that $\big\langle \sum_{i,j=1}^n a_{ij}x_i^2x_j^2, x_k^2x_\ell^2\big\rangle_d = 24a_{k\ell}$ if $k=\ell$ and $4(a_{k\ell}+a_{\ell k}) = 8a_{k\ell} = 8a_{\ell k}$ otherwise.

It remains to prove (2). This easily follows by observing that for $f \in \mathcal{Q}$ we have

$$\Big\langle f, \mathrm{pr}_{\mathcal{Q}}\Big(\big(\sum_{i=1}^n v_ix_i\big)^4\Big)\Big\rangle_d = 24f(v),$$

for any $v = (v_1,\ldots,v_n) \in \mathbb{R}^n$. ■

The third lemma needed in the proof of Theorem 3.1 identifies the Santaló point of $\widetilde{\mathrm{LF}}_{\mathcal{Q}}$.

Lemma 3.5. *The Santaló point of* $\widetilde{\mathrm{LF}}_{\mathcal{Q}}$ *is the origin.*

Proof. Every element O of the orthogonal group $O(n)$ defines a linear map

$$L_O : \mathcal{Q} \to \mathcal{Q}, \qquad (L_Of)(\mathtt{x}) := \mathrm{pr}_{\mathcal{Q}}\big(f(O\mathtt{x})\big), \tag{3.6}$$

where $\mathrm{pr}_{\mathcal{Q}}$ is defined as in (1.7).

We will prove that $\widetilde{\mathrm{LF}}_{\mathcal{Q}}$ is invariant under every map L_O, $O \in O(n)$ and the origin is the only fixed point of $\widetilde{\mathrm{LF}}_{\mathcal{Q}}$. Since the Santaló point of a convex body is unique, the statement of the lemma will follow from these two facts.

First we prove

$$r(\mathtt{x}) := \Big(\sum_{i=1}^{n} x_i^2\Big)^2$$

is a fixed point of every map L_O defined by (3.6). Since $r(O\mathtt{x}) \equiv 1$ on S^{n-1} for every $O \in O(n)$ and $r(\mathtt{x})$ is the only form from $\mathbb{R}[\mathtt{x}]_4$ such that $r(\mathtt{x}) \equiv 1$ on S^{n-1}, it follows that $r(O\mathtt{x}) = r(\mathtt{x})$ for every $O \in O(n)$. Hence,

$$(L_O r)(\mathtt{x}) = \mathrm{pr}_{\mathcal{Q}}\big(r(O\mathtt{x})\big) = \mathrm{pr}_{\mathcal{Q}}\big(r(\mathtt{x})\big) = r(\mathtt{x}) \tag{3.7}$$

and $r(\mathtt{x})$ is indeed a fixed point of every map L_O defined by (3.6).

Next we prove that $\widetilde{\mathrm{LF}}_{\mathcal{Q}}$ is invariant for every map L_O defined by (3.6). Let us choose an arbitrary $g \in \widetilde{\mathrm{LF}}_{\mathcal{Q}}$. Then g is of the form

$$g(\mathtt{x}) = \mathrm{pr}_{\mathcal{Q}}\Big(\sum_i f_i^4\Big) - r(\mathtt{x}),$$

where $f_i \in \mathbb{R}[\mathtt{x}]_1$. For $O \in O(n)$ we have that

$$\begin{aligned}(L_O g)(\mathtt{x}) &= \mathrm{pr}_{\mathcal{Q}}\Big(\mathrm{pr}_{\mathcal{Q}}\Big(\sum_i (f_i(O\mathtt{x}))^4\Big) - r(O\mathtt{x})\Big)\\ &= \mathrm{pr}_{\mathcal{Q}}\Big(\sum_i (f_i(O\mathtt{x}))^4\Big) - r(\mathtt{x}) \in \widetilde{\mathrm{LF}}_{\mathcal{Q}},\end{aligned} \tag{3.8}$$

where we used (3.7) in the second equality, while for the containment we used the fact that $f_i(O\mathtt{x}) \in \mathbb{R}[\mathtt{x}]_1$ for every $O \in O(n)$. Since g was arbitrary, (3.8) proves that $\widetilde{\mathrm{LF}}_{\mathcal{Q}}$ is indeed invariant for every map L_O defined by (3.6).

It remains to prove that the origin is the only fixed point of $\widetilde{\mathrm{LF}}_{\mathcal{Q}}$ for every map L_O, $O \in O(n)$. It suffices to prove that fixed points of $\mathcal{Q}$ are of the form $cr(\mathtt{x})$ for $c \in \mathbb{R}$, since the only c such that $cr(\mathtt{x}) - r(\mathtt{x}) \in \mathcal{M}$ is equal to 1. So let

$$f(\mathtt{x}) = \sum_{1\le i\le j\le n} a_{ij} x_i^2 x_j^2 \in \mathcal{Q}$$

be a fixed point (here we used that the coefficients a_{ij} and a_{ji} in (1.6) are the same for every form in $\mathcal{Q}$). Let $\pi_{k\ell} = (k\ \ell)$, $1 \le k < \ell \le n$, be a transposition. We have

$$\begin{aligned}(L_{\pi_{k\ell}} f)(\mathtt{x}) = {} & a_{kk}x_\ell^4 + a_{\ell\ell}x_k^4 + \sum_{i\notin\{k,\ell\}} a_{ii}x_i^4 + a_{k\ell}x_k^2x_\ell^2 + \sum_{i<k} a_{ik}x_i^2x_\ell^2 + \sum_{\substack{j>k,\\ j\neq\ell}} a_{kj}x_\ell^2x_j^2\\ & + \sum_{\substack{i<\ell,\\ i\neq k}} a_{i\ell}x_i^2x_k^2 + \sum_{j>\ell} a_{\ell j}x_k^2x_j^2 + \sum_{i,j\notin\{k,\ell\}} a_{ij}x_i^2x_j^2.\end{aligned} \tag{3.9}$$

Since $L_{\pi_{k\ell}} f = f$, it follows in particular from (3.9) by considering the coefficient at x_k^4 that $a_{\ell\ell} = a_{kk}$, while the comparison of the coefficients at $x_k^2x_m^2$ for $m \notin \{k,\ell\}$ implies that $a_{\ell m} = a_{km}$ if $k < m$ and $a_{m\ell} = a_{mk}$ if $k > m$. Note that k, ℓ and $m \notin \{k,\ell\}$ were arbitrary. So we conclude that there are constants $c, d \in \mathbb{R}$ such that

$$a_{ii} = c \quad \text{for all } i = 1,\ldots,n \qquad \text{and} \qquad a_{ij} = d \quad \text{for all } 1 \le i < j \le n.$$

Hence,

$$f(\mathtt{x}) = c\Big(\sum_{i=1}^{n} x_i^4\Big) + d\Big(\sum_{1\le i<j\le n} x_i^2x_j^2\Big).$$

Let $O_{k\ell}$, $1 \le k < \ell \le n$, be an orthogonal transformation defined on the standard basis vectors e_i, $i = 1,\dots,n$, having the only nonzero entry in the i-th coordinate which is 1, by $e_k \mapsto \frac{1}{\sqrt{2}}(e_k - e_\ell)$, $e_\ell \mapsto \frac{1}{\sqrt{2}}(e_k + e_\ell)$ and $e_i \mapsto e_i$ if $i \notin \{k,\ell\}$. We have that

$$\begin{aligned}(L_{O_{k\ell}}f)(\mathtt{x}) = \mathrm{pr}_{\mathcal{Q}}\Bigg(& c\Big(\frac{1}{4}(x_k - x_\ell)^4 + \frac{1}{4}(x_k + x_\ell)^4 + \sum_{i\notin\{k,\ell\}} x_i^4\Big)\\ &+ d\Big(\frac{1}{4}(x_k^2 - x_\ell^2)^2 + \sum_{i<k}\frac{1}{2}x_i^2(x_k - x_\ell)^2 + \sum_{\substack{j>k,\\ j\neq\ell}}\frac{1}{2}(x_k - x_\ell)^2x_j^2\\ &+ \sum_{\substack{i<\ell,\\ i\neq k}}\frac{1}{2}x_i^2(x_k + x_\ell)^2 + \sum_{j>\ell}\frac{1}{2}(x_k + x_\ell)^2x_j^2 + \sum_{i,j\notin\{k,\ell\}} x_i^2x_j^2\Big)\Bigg).\end{aligned} \tag{3.10}$$

Since $L_{O_{k\ell}}f = f$, it follows in particular from (3.10) by considering the coefficient at x_k^4 that $c = \frac{1}{2}c + \frac{1}{4}d$, or equivalently, $d = 2c$. Hence,

$$f(\mathtt{x}) = c\Big(\sum_{i=1}^{n} x_i^4 + 2\sum_{1\le i<j\le n} x_i^2x_j^2\Big) = cr(\mathtt{x}).$$

This concludes the proof of the lemma. ■

Finally we can prove Theorem 3.1.

Proof of Theorem 3.1. Using (2.9) of Corollary 2.9 for $L = \mathrm{NN}_{\mathcal{Q}}$ and (1) of Lemma 3.4 implies that

$$\frac{2}{(n+4)(n+6)} \le \big(\mathrm{vrad}(\widetilde{\mathrm{NN}_{\mathcal{Q}}})\big)^2. \tag{3.11}$$

Claim 1. For $n \ge 5$ we have that

$$\frac{1}{2n^2} \le \frac{2}{(n+4)(n+6)}. \tag{3.12}$$

Proof of Claim 1. Multiplying (3.12) by $2n^2(n+4)(n+6)$ and rearranging terms, it follows that (3.12) is equivalent to

$$3n^2 - 10n - 24 \ge 0. \tag{3.13}$$

Since the local minimum x_0 of the quadratic function $f(x) = 3x^2 - 10x - 24$ is equal to $x_0 = \frac{5}{3}$ and $f(5) = 1$, this in particular implies that (3.13) holds true. ■

Now (3.11) and Claim 1 imply that

$$(\sqrt{2}n)^{-1} \le \mathrm{vrad}(\widetilde{\mathrm{NN}_{\mathcal{Q}}}) \tag{3.14}$$

Using Theorem 2.3 for $K = \widetilde{\mathrm{LF}_{\mathcal{Q}}}$ (resp. $K = \widetilde{\mathrm{CP}_{\mathcal{Q}}}$) together with (1) (resp. (2)) of Lemma 3.2 gives

$$(8\sqrt{2}n)^{-1} \le \frac{1}{8}\,\mathrm{vrad}(\widetilde{\mathrm{NN}_{\mathcal{Q}}}) \le \mathrm{vrad}(\widetilde{\mathrm{LF}_{\mathcal{Q}}}), \tag{3.15}$$

$$(16\sqrt{2}n)^{-1} \leq \frac{1}{16}\operatorname{vrad}(\widetilde{\mathrm{NN}_{\mathcal{Q}}}) \leq \operatorname{vrad}(\widetilde{\mathrm{CP}_{\mathcal{Q}}}). \tag{3.16}$$

Since by Lemma 3.5 the origin is the Santaló point of $\widetilde{\mathrm{LF}_{\mathcal{Q}}}$, using (2.10) of Corollary 2.9 for $L = \mathrm{LF}_{\mathcal{Q}}$ and (2) of Lemma 3.4 implies that

$$\operatorname{vrad}(\widetilde{\mathrm{POS}_{\mathcal{Q}}}) \leq \left(\frac{8}{(n+4)(n+6)}\right)^{1-\frac{2n-1}{n^2+n-1}} (\operatorname{vrad}(\widetilde{\mathrm{LF}_{\mathcal{Q}}}))^{-1}. \tag{3.17}$$

Claim 2. For $n \geq 4$ we have that

$$\left(\frac{8}{(n+4)(n+6)}\right)^{1-\frac{2n-1}{n^2+n-1}} \leq \frac{32}{n^2}. \tag{3.18}$$

Before we prove Claim 2 we establish two preliminary results.

Claim 2.1. For $n \geq 1$ we have that

$$\frac{2n-1}{n^2+n-1} \leq \frac{2}{n}. \tag{3.19}$$

Proof of Claim 2.1. Multiplying (3.19) by $n(n^2+n-1)$ and rearranging terms, it follows that (3.19) is equivalent to $3n-2 \geq 0$, which clearly implies Claim 2.1. □

Claim 2.2. For $n \geq 4$ we have that

$$n^{1/n} \leq 4^{1/4}. \tag{3.20}$$

Proof of Claim 2.2. Let $g(x) = x^{1/x}$. Since $g'(x) = -x^{-2+1/x}(-1+\log x)$, it follows that $g'(x) \leq 0$ for $x \geq e$ and hence g is decreasing on $[e,\infty)$. This proves Claim 2.2. □

Now we are ready to prove Claim 2.

Proof of Claim 2. We have that

$$\begin{aligned}
\left(\frac{8}{(n+4)(n+6)}\right)^{1-\frac{2n-1}{n^2+n-1}} &\leq \left(\frac{8}{n^2}\right)^{1-\frac{2n-1}{n^2+n-1}} = \left(\frac{8}{n^2}\right)\left(\frac{n^2}{8}\right)^{\frac{2n-1}{n^2+n-1}} \\
&= \left(\frac{8}{n^2}\right)(n^2)^{\frac{2n-1}{n^2+n-1}}\left(\frac{1}{8}\right)^{\frac{2n-1}{n^2+n-1}} \\
&< \left(\frac{8}{n^2}\right)(n^2)^{\frac{2n-1}{n^2+n-1}} \\
&\leq \left(\frac{8}{n^2}\right)(n^2)^{\frac{2}{n}} = \left(\frac{8}{n^2}\right)\left((n)^{\frac{1}{n}}\right)^4 \\
&\leq \left(\frac{8}{n^2}\right)\left(4^{\frac{1}{4}}\right)^4 = \frac{32}{n^2},
\end{aligned}$$

where we used that $1 - \frac{2n-1}{n^2+n-1} > 0$ for $n \geq 3$ (Claim 2.1) in the first inequality, $\frac{1}{8} < 1$ in the second, Claim 2.1 in the third and Claim 2.2 in the fourth. □

Using (3.15) and Claim 2 in (3.17), it follows that

$$\operatorname{vrad}(\widetilde{\mathrm{POS}_{\mathcal{Q}}}) \leq 2^8\sqrt{2}n^{-1}. \tag{3.21}$$

Now (3.16) and (3.21) imply (3.1) holds, which proves the theorem. ■

4. Construction of exceptional doubly nonnegative and exceptional copositive matrices

In this section we describe the details of the bootstrap method outlined in Subsection 1.1.1 to find exceptional doubly nonnegative (e-DNN) and exceptional copositive (e-COP) matrices. We first find a seed e-DNN matrix of size 5×5, which then gives rise to a family of e-DNN matrices of arbitrary size ≥ 5. Using the constructed e-DNN matrices we produce a corresponding family of exceptional copositive matrices.

4.1. **Justification of the construction of a family of e-DNN matrices from a seed e-DNN matrix of size 5×5 from Subsection 1.1.1.** First we mention that the technical reasons why we restrict to the subspace $\mathcal{H}$ spanned by the $\cos(2k\pi x)$, $k \in \mathbb{N}_0$, instead of considering the entire $L^2[0,1]$ are discussed in Remark 4.1. The restriction to matrices of size $n \geq 5$ is clear from the introduction since $\mathrm{DNN}_n = \mathrm{CP}_n$ for $n \leq 4$.

To find the general form of the $n \times n$ matrix $A^{(n)} = (A^{(n)}_{jk})_{j,k}$ as in (1.12) note that

$$A^{(n)}_{jk} = \int_0^1 f(x) \cos(2(j-1)\pi x) \cos(2(k-1)\pi x)\, dx \quad \text{for} \quad j, k = 1, \ldots, n,$$

where the integration is with respect to the Lebesgue measure on $[0,1]$. Using the well-known trigonometry formula involving the cosine product identity, the products of different cosine functions can be replaced with linear combinations of cosine functions with higher and lower frequency, i.e.,

$$\cos(2j\pi x)\cos(2k\pi x) = \frac{1}{2}\Big(\cos\big(2(j-k)\pi x\big) + \cos\big(2(j+k)\pi x\big)\Big). \tag{4.1}$$

From (4.1) it follows that

$$\int_0^1 \cos(2j\pi x)\cos(2k\pi x)\cos(2\ell\pi x)dx = \begin{cases} \frac{1}{2}, & \text{if } j = \ell, k = 0, \\ \frac{1}{4}, & \text{if } k \neq 0 \text{ and } j \in \{\ell + k, \ell - k\}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Using (4.2) it is now easy to compute that for $A^{(5)}$ to be the 5×5 compression of a multiplication operator $M_f^{\mathcal{H}}$ for f as in (1.11) with $m = 6$, it must be of the form

$$A^{(5)} = \begin{pmatrix} 1 & \sqrt{2}a_1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \sqrt{2}a_4 \\ \sqrt{2}a_1 & a_2 + 1 & a_1 + a_3 & a_2 + a_4 & a_3 + a_5 \\ \sqrt{2}a_2 & a_1 + a_3 & a_4 + 1 & a_1 + a_5 & a_2 + a_6 \\ \sqrt{2}a_3 & a_2 + a_4 & a_1 + a_5 & 1 + a_6 & a_1 \\ \sqrt{2}a_4 & a_3 + a_5 & a_2 + a_6 & a_1 & 1 \end{pmatrix}. \tag{4.3}$$

Thus demanding that $a_i \geq 0$ for $i = 1, \ldots, 6$ certifies that $A^{(5)}$ is NN. By the same reasoning $A^{(n)}$ is NN for every $n \geq 5$.

Further on, f being of the form (1.13) is equivalent to f being a sum of squares of trigonometric polynomials [Mar08, Lemma 4.1.3]. This implies that all matrices $A^{(n)} = P_n M_f^{\mathcal{H}} P_n$ as in (1.12) are PSD. Indeed, suppose

$$f = \sum_{i=0}^{k} \Big(\underbrace{\sum_{j=0}^{m'} h_{ij} \cos(2j\pi x)}_{h_i} \Big)^2$$

for some k and $h_{ij} \in \mathbb{R}$. Since f and the h_i are in $\mathcal{H}$, clearly $M_f^{\mathcal{H}}$ and the $M_{h_i}^{\mathcal{H}}$ are multiplication operators on $\mathcal{H}$ and

$$M_f^{\mathcal{H}} = \sum_{i=1}^{k} \left(M_{h_i}^{\mathcal{H}}\right)^2.$$

Here each $M_{h_i}^{\mathcal{H}}$ is self-adjoint, from which the claim follows.

Finally, we justify why (1.14) implies that $P_n M_f^{\mathcal{H}} P_n$ is not CP for any $n \geq 5$. Since CP matrices are dual to copositive matrices in the usual Frobenius inner product, (1.14) certifies that $A^{(5)}$ is not CP. Now the equality

$$A^{(5)} = P_5(P_n M_f^{\mathcal{H}} P_n)P_5 = P_5 A^{(n)} P_5 \tag{4.4}$$

for any $n \geq 5$, implies that $A^{(n)}$ is not CP for any $n \geq 5$. Indeed, suppose that $A^{(n)} = BB^{\mathrm{T}}$ for some $n \geq 5$ and (not necessarily square) matrix B with nonnegative entries. By (4.4), $A^{(5)} = P_5 B(P_5 B)^{\mathrm{T}}$ and since P only has $0, 1$ entries, this contradicts $A^{(5)}$ not being CP.

It remains to explain our procedure for constructing exceptional copositive matrices. Since SPN matrices are dual to DNN matrices in the Frobenius inner product, the first condition in (1.16) implies that C is not SPN. The second condition in (1.16) is a relaxation of copositivity of C and it clearly implies that q_C is nonnegative on $\mathbb{R}^n$.

Remark 4.1. We explain the reason for restricting M_f to the closed subspace $\mathcal{H}$ of $L^2[0,1]$ generated by the cosine functions. As in Subsection 1.1.1, with respect to the standard orthonormal basis for $L^2[0,1]$ given by

$$1, \ \sqrt{2}\cos\left(2k\pi x\right), \ \sqrt{2}\sin\left(2k\pi x\right) \tag{4.5}$$

for $k \in \mathbb{N}$, each multiplication operator M_f for $f \in L^\infty[0,1]$ can be represented by an infinite matrix.

It seems natural to start by considering the entire space $L^2[0,1]$ and compressions $\widetilde{P}_n M_f \widetilde{P}_n$ of M_f for some trigonometric polynomial f and $n \geq 2$, onto the $(2n+1)$-dimensional span $\widetilde{\mathcal{H}}_n$ of the functions in (4.5) for $k = 1, \ldots, n$. Here $\widetilde{P}_n : L^2[0,1] \to \widetilde{\mathcal{H}}_n$ are orthogonal projections.

Suppose that A is the 5×5 compression of M_f and is given with respect to the ordered (orthonormal) basis consisting of the functions

$$1, \sqrt{2}\cos(2\pi x), \sqrt{2}\cos(4\pi x), \sqrt{2}\sin(2\pi x), \sqrt{2}\sin(4\pi x)$$

and assume that the corresponding function f has finite Fourier series

$$f(x) = 1 + 2\sum_{k=1}^{m} a_k \cos(2k\pi x) + 2\sum_{k=1}^{m} b_k \sin(2k\pi x) \tag{4.6}$$

for some $m \in \mathbb{N}$ and real numbers a_k, b_k with $k = 1, \ldots, m$. Again, using well-known trigonometry formulas involving product identities, i.e.,

$$\begin{aligned} \sin(2j\pi x)\sin(2k\pi x) &= \frac{1}{2}\Big(\cos\left(2(j-k)\pi x\right) - \cos\left(2(j+k)\pi x\right)\Big) \\ \cos(2j\pi x)\sin(2k\pi x) &= \frac{1}{2}\Big(\sin\left(2(k-j)\pi x\right) + \sin\left(2(j+k)\pi x\right)\Big) \end{aligned} \tag{4.7}$$

in addition to (4.1), it is easy to compute that for A to be the 5×5 compression of a multiplication operator M_f for f as in (4.6) with $m\geq 4$, it must be of the form

$$A=\begin{pmatrix} 1 & \sqrt{2}a_1 & \sqrt{2}a_2 & \sqrt{2}b_1 & \sqrt{2}b_2 \\ \sqrt{2}a_1 & a_2+1 & a_1+a_3 & b_2 & b_1+b_3 \\ \sqrt{2}a_2 & a_1+a_3 & a_4+1 & b_3-b_1 & b_4 \\ \sqrt{2}b_1 & b_2 & b_3-b_1 & 1-a_2 & a_1-a_3 \\ \sqrt{2}b_2 & b_1+b_3 & b_4 & a_1-a_3 & 1-a_4 \end{pmatrix}.$$

Note that since we want all the finite-dimensional compressions of M_f to be NN, f needs to have an infinite Fourier series. Indeed, suppose f has finite Fourier series as in (4.6) for some $m\in\mathbb{N}$. Then for all j,k with $k<j\leq m$ and $m<j+k$, the $(j+1,k+m+1)$-entry of the compression $\widetilde{P}_m M_f \widetilde{P}_m$,

$$\int_0^1 f(x)\,\sqrt{2}\cos(2j\pi x)\,\sqrt{2}\sin(2k\pi x)\,dx =$$
$$\int_0^1 f(x)\sin\big(2(k-j)\pi x)\big)\,dx \;+\; \int_0^1 f(x)\sin\big(2(j+k)\pi x)\big)\,dx,$$

equals $-a_{j-k}$. Furthermore, we see from (4.7) that the Fourier sine coefficients of f must satisfy

$$b_{j+k}\geq b_{j-k}$$

for all k,j. But the containment $f\in L^2[0,1]$ implies that $b_k=0$ for all k. Hence, f has a Fourier cosine series. To avoid technical difficulties, we thus restrict our attention to $\mathcal{H}$.

4.2. **Examples.**

4.2.1. *A seed e-DNN* 5×5 *matrix.* Let $\epsilon=1/20$. Solving the SDP (1.15) with this parameter and rationalizing the solution [PP08, CKP15] yields the 5×5 compression

$$A^{(5)}=\begin{pmatrix} 1 & \frac{16\sqrt{2}}{27} & \frac{\sqrt{2}}{123} & \frac{1}{147\sqrt{2}} & \frac{5\sqrt{2}}{21} \\ \frac{16\sqrt{2}}{27} & \frac{124}{123} & \frac{1577}{2646} & \frac{212}{861} & \frac{1205}{8526} \\ \frac{\sqrt{2}}{123} & \frac{1577}{2646} & \frac{26}{21} & \frac{572}{783} & \frac{1777340\sqrt{2}-2413803}{3254580} \\ \frac{1}{147\sqrt{2}} & \frac{212}{861} & \frac{572}{783} & \frac{1777340\sqrt{2}+814317}{3254580} & \frac{16}{27} \\ \frac{5\sqrt{2}}{21} & \frac{1205}{8526} & \frac{1777340\sqrt{2}-2413803}{3254580} & \frac{16}{27} & 1 \end{pmatrix}. \tag{4.8}$$

By comparing the above $A^{(5)}$ with the general form (4.3) we read off the Fourier coefficients of the corresponding function f as in (1.13), i.e.,

$$\begin{aligned} f(x)=1&+\frac{32}{27}\cos(2\pi x)+\frac{2}{123}\cos(4\pi x)+\frac{1}{147}\cos(6\pi x)\\ &+\frac{10}{21}\cos(8\pi x)+\frac{8}{29}\cos(10\pi x)\\ &+\frac{-2440263+1777340\sqrt{2}}{1627290}\cos(12\pi x). \end{aligned}$$

This function is indeed SOS, since we have $f = v^{\mathrm{T}} B v$ for v as in (1.1.1) with $m' = 3$ and

$$B = \begin{pmatrix} \frac{9}{22} & \frac{7}{37} & -\frac{3}{22} & -\frac{206923}{5678316} \\ \frac{7}{37} & \frac{336929}{243540} - \frac{88867\sqrt{2}}{162729} & \frac{2210}{28971} & \frac{88867}{162729\sqrt{2}} - \frac{200129}{487080} \\ -\frac{3}{22} & \frac{2210}{28971} & \frac{46466763-19550740\sqrt{2}}{35800380} & \frac{4}{29} \\ -\frac{206923}{5678316} & \frac{88867}{162729\sqrt{2}} - \frac{200129}{487080} & \frac{4}{29} & \frac{1777340\sqrt{2}-2440263}{1627290} \end{pmatrix} \succeq 0.$$

4.2.2. *Exceptional copositive matrix from DNN matrix.* Now let $\epsilon' = 1/10$ and $k = 1$. From the matrix $A^{(5)}$ in (4.8) we construct an exceptional copositive matrix C as described in Subsection 1.1.1 by solving the feasibility SDP

$$\begin{aligned} &\operatorname{tr}(CA^{(5)}) = -\epsilon', \\ &\Big(\sum_{i=1}^{n} x_i^2\Big)\, q_C = w^{\mathrm{T}} B w \quad \text{with} \quad B \succeq 0, \end{aligned}$$

where w is the vector with all the degree at most 3 words in the variables $x_1, \ldots, x_n$. Again, after a suitable rationalization, we get an exceptional copositive matrix

$$C = \begin{pmatrix} 17 & -\frac{91}{5} & \frac{33}{2} & \frac{38}{3} & -\frac{36}{5} \\ -\frac{91}{5} & \frac{59}{3} & -\frac{53}{4} & 8 & \frac{33}{4} \\ \frac{33}{2} & -\frac{53}{4} & \frac{39}{4} & -\frac{13}{2} & 8 \\ \frac{38}{3} & 8 & -\frac{13}{2} & \frac{16}{3} & -\frac{13}{3} \\ -\frac{36}{5} & \frac{33}{4} & 8 & -\frac{13}{3} & \frac{1373628701}{353935575} \end{pmatrix}.$$

References

[Bar02] A. Barvinok: *Estimating L^∞ Norms by L^{2k} Norms for Functions on Orbits*, Found. Comput. Math. 2 (2002) 393–412.

[BB05] A. Barvinok, G. Blekherman: *Convex geometry of orbits*, Combinatorial and Computational Geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge University Press, Cambridge, 2005, 51–77.

[BDSM15] A. Berman, M. Dür, N. Shaked-Monderer: *Open problems in the theory of completely positive and copositive matrices*, Electron. J. Linear Algebra 29 (2015) 46–58.

[BSM21] A. Berman, N. Shaked-Monderer: *Copositive and Completely Positive Matrices*, World Scientific Publishing Co., 2021.

[Ble04] G. Blekherman: *Convexity properties of the cone of nonnegative polynomials*, Discrete Comput. Geom. 32 (2004) 345–371.

[Ble06] G. Blekherman: *There are significantly more nonnegative polynomials than sums of squares*, Israel J. Math. 153 (2006) 355–380.

[BR21] G. Blekherman, C. Riener: *Symmetric nonnegative formas and sums of squares*, Discrete Comput. Geom. 65 (2021) 764–799.

[BCR98] J. Bochnack, M. Coste, M.-F. Roy: *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Springer, 1998.

[BM87] J. Bourgain, V.D. Milman: *New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$*, Invent. Math. 88(2) (1987) 319–340.

[Bur09] S. Burer: *On the copositive representation of binary and continuous nonconvex quadratic programs*, Math. Program., Ser. A 120 (2009) 479–495.

[CHN17] B. Collins, P. Hayden, I. Nechita: *Random and Free Positive Maps with Applications to Entanglement Detection*, Int. Math. Res. Not. 3 (2017) 869–894.

[CKP15] K. Cafuta, I. Klep, J. Povh: *Rational sums of hermitian squares of free noncommutative polynomials*, Ars Math. Contemp. 9 (2015) 253–269.

[Dia62] P.H. Diananda: *On non-negative forms in real variables some or all of which are non-negative*, Proc. Camb. Phil. Soc. 58 (1962) 17–25.

[DDGH13] P.J.C. Dickinson, M. Dür, L. Gijben and R. Hildebrand: *Scaling relationship between the copositive cone and Parrilo's first level approximation*, Optim. Lett. 7(8) (2013) 1669–1679.

[DG14] P.J.C. Dickinson, L. Gijben: *On the computational complexity of membership problems for the completely positive cone and its dual*, Comput. Optim. Appl. 57(2) (2014) 403–415.

[DR21] M. Dür, F. Rendl: *Conic optimization: a survey with special focus on copositive optimization and binary quadratic problems*, Eur. J. Comput. Optim. 9 (2021) 100021.

[HN63] Jr.M. Hall, M. Newman: *Copositive and completely positive quadratic forms*, Proc. Camb. Phil. Soc. 59 (1963) 329–333.

[JS22] N. Johnston, J. Sikora: *Completely positive completely positive maps (and a resource theory for non-negativity of quantum amplitudes)*, Linear Algebra Appl. 653 (2022) 395–429.

[KMŠZ19] I. Klep, S. McCullough, K. Šivic, A. Zalar: *There Are Many More Positive Maps Than Completely Positive Maps*, Int. Math. Res. Not. IMRN 11 (2019) 3313–3375.

[KP02] E. de Klerk, D. Pasechnik: *Approximation of the stability number of a graph via copositive programming*, SIAM J. Optim. 12 (2002) 875–892.

[Kup08] G. Kuperberg: *From the Mahler conjecture to Gauss linking integrals*, GAFA Geom. funct. anal. 18 (2008) 870–892.

[Lau09] M. Laurent: *Sums of squares, moment matrices and optimization over polynomials*, Emerging applications of algebraic geometry, 157–270, IMA Vol. Math. Appl., 149, Springer, New York, 2009.

[LV22a] M. Laurent, L.F. Vargas: *Exactness of Parrilo's conic approximations for copositive matrices and associated low order bounds for the stability number of a graph*, Math. Oper. Res. (2022) 1–27. doi: 10.1287/moor.2022.1290

[LV22b] M. Laurent, L.F. Vargas: *On the exactness of sum-of-squares approximations for the cone of* 5×5 *copositive matrices*, Linear Algebra Appl. 651 (2022) 26–50.

[Mar08] M. Marshall: *Positive Polynomials and Sums of Squares*, Mathematical Surveys and Monographs, vol. 146, Amer. Math. Soc., 2008.

[MP90] M. Meyer, A. Pajor: *On the Blaschke-Santalo inequality*, Arch. Math. (Basel) 55 (1990) 82–93.

[MM62] J.E. Maxfield, H. Minc: *On the matrix equation* $X'X = A$, Proc. Edinb. Math. Soc. 13 (1962/63) 125–129.

[MK87] K.G. Murty, S.N. Kabadi: *Some NP-complete probles in quadratic and nonlinear programming*, Math. Program., 39(2) (1987) 117–129.

[Par00] P.A. Parrilo: *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, PhD thesis, California Institute of Technology, 2000.

[Pau02] V.I. Paulsen: *Completely bounded maps and operator algebras*, Cambridge Stud. Adv. Math. 78, Cambridge University Press, Cambridge, 2002.

[PP08] H. Peyrl, P.A. Parrilo: *Computing sum of squares decompositions with rational coefficients*, Theoret. Comput. Sci. 409(2) (2008) 269–281.

[Pól28] G. Pólya: *Über positive Darstellung von Polynomen*, Vierteljschr. Naturforsch. Ges. Ziürich 73 (1928) 141–145.

[Pow11] V. Powers: *Rational certificates of positivity for compact semialgebraic sets*, Pac. J. Math. 251 (2011) 385–391.

[Put93] M. Putinar: *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J. 42 (1993) 969–984.

[RRW10] F. Rendl, G. Rinaldi, A. Wiegele: *Solving max-cut to optimality by intersecting semidefinite and polyhedral relaxations*, Math. Program. 121 (2010) 307–335.

[Rez82] B. Reznick: *Sums of even powers of linear forms*, Memoirs Amer. Math. Soc. Providences, 1982.

[Rez95] B. Reznick: *Uniform denominators in Hilbert's Seventeenth Problem*, Math. Z. 220 (1995) 75–98.

[RS57] C.A. Rogers, G.C. Shepard: *The difference body of a convex body*, Arch. Math. 8 (1957) 220–233.

[Sce09] C. Scheiderer: *Positivity and sums of squares: a guide to recent results*, Emerging applications of algebraic geometry, 271–324, IMA Vol. Math. Appl., 149, Springer, New York, 2009.

[Scm91] K. Schmüdgen: *The* K*–moment problem for compact semi-algebraic sets*, Math. Ann. 289(2) (1991) 203–206.

[Scw03] M. Schweighofer: *Iterated rings of bounded elements and generalizations of Schmüdgen's Positivstellensatz*, J. reine angew. Math. 554 (2003) 19–45.

[SV-] M. Schweighofer, F.L. Vargas: *Polynomial optimization over stable sets in graphs*, in preparation.

[UL-mo] *Upper limit on the central binomial coefficient*, https://mathoverflow.net/questions/133732/upper-limit-on-the-central-binomial-coefficient, Accessed: 20 May 2023.

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