NONCOMMUTATIVE REAL ALGEBRAIC GEOMETRY & ANALYSIS

$\underset{NC}{\overset{\shortparallel}{\text{NONCOMMUTATIVE}}}$ $\underbrace{\text{REAL ALGEBRAIC GEOMETRY}}_{RAG}$

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I. POSITIVE (NC) POLYNOMIALS

II. NC NULLSTELLENSÄTZE

III. NC CONVEXITY

IV. NC FUNCTION THEORY

V. NONLINEAR COMPLETELY POSITIVE MAPS

NC RAG ~ study of positivity in general (not necc. commutative) algebras.

E.g.

- $\mathbb{R}[t_1 \ldots t_d]$ = commutative polynomial rings

  $M_n(\mathbb{R}[t])$

  !!! $\longrightarrow$

- $\mathbb{R}\langle x_1 \ldots x_d \rangle$ = free algebra

  $\mathbb{R}\langle x \rangle \otimes \mathbb{R}\langle y \rangle$ = "bifree" algebra

- $\mathbb{R}[G]$ = group algebra, such as $\mathbb{R}[\overset{\text{free group}}{\mathbb{F}_d}]$ or $\mathbb{R}[\mathbb{F}_d \times \mathbb{F}_d]$

- $A_1(\mathbb{R}) = \dfrac{\mathbb{R}\langle x, y \rangle}{(xy - yx - 1)}$ Weyl algebra

- $\mathbb{R}\langle\!\langle x \rangle\!\rangle$ = free skew field = nc rational functions

- Ring of generic matrices $GM_n(\mathbb{R})$, trace ring $\mathbb{T}_n$.

I. POSITIVE NC POLYS

$d \in \mathbb{N}$, $x = (x_1, \ldots, x_d)$ freely nc variables

$\langle x \rangle$ = words in x = free monoid on x

$\emptyset =: 1$ empty word $\in \langle x \rangle$

$\langle x \rangle_\delta$ = all words of length=degree at most δ

$\langle x \rangle_{=\delta}$ = - - - - - - exactly δ

$*$ involution on $\langle x \rangle$ that reverses words:

$$(x_{i_1} \cdots x_{i_r})^* = x_{i_r} \cdots x_{i_1}$$

In particular, $x_j^* = x_j$ (all j).

<u>Def</u>: The free algebra $\mathbb{R}\langle x \rangle$ is the set of all linear combination of words.

$*$ extends to $\mathbb{R}\langle x \rangle$ by linearity.

Puzzle of the Day

If $A = A^T$, $B = B^T$ have norm ≤ 1, then

$$2 \cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - B^2A \succeq 0$$

$\mathbb{R}\langle x \rangle_\delta$ = span $\langle x \rangle_\delta$ = nc polys of degree $\leq \delta$

<u>Ex</u>: $p = x_1^2 x_2 + x_2 x_1 x_2 - 3 x_2 x_1^2 + 4 \in \mathbb{R}\langle x_1, x_2 \rangle$

Given $X_1, X_2 \in M_n(\mathbb{R})$, we define

$$p(X_1, X_2) = X_1^2 X_2 + X_2 X_1 X_2 - 3 X_2 X_1^2 + 4 \cdot I_n$$

This gives a representation $\mathbb{R}\langle x \rangle \to M_n(\mathbb{R})$

If $X \in M_n(\mathbb{R})_{sa}^d$, then the evaluation $p \mapsto p(X)$ is a $*$-representation $\mathbb{R}\langle x \rangle \longrightarrow M_n(\mathbb{R})$

<u>Remark</u>: $M \succeq 0$ means M is positive semidefinite (PSD)

$M \succ 0$ - - - positive definite (PD)

Observation. Suppose $f = g^* g$ for some $g \in \mathbb{R}\langle x \rangle$.

Then $\forall X \in (M_n)^d_{sa}$ we have $f(X) \succeq 0$

Indeed, $f(X) = (g^* g)(X) = g^*(X) \cdot g(X) = g(X)^* \cdot g(X) \succeq 0$

Theorem (Helton 2002, McCullough 2001):

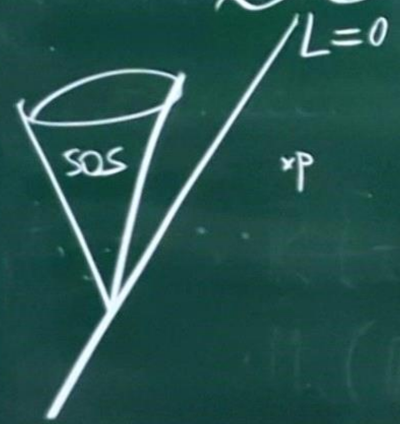
Let $N(\delta) = \dim \mathbb{R}\langle x \rangle_\delta = 1 + d + d^2 + \dots + d^\delta$

Suppose $p \in \mathbb{R}\langle x \rangle_\delta$ and

$p(X) \succeq 0$ for all $N(\delta) \times N(\delta)$ s.a. X

Then $\exists r_1 \dots r_{N(\delta)} \in \mathbb{R}\langle x \rangle_\delta$ s.t.

1st Proof of the theorem



Proposition (Caratheodory convex hull thm)

If $p \in \mathbb{R}\langle x \rangle_\delta$ is a sos (= sum of squares) then $\exists r_1 \dots r_{N(\delta)} \in \mathbb{R}\langle x \rangle_\delta$ st $p = \sum r_j^* r_j$.

Proof: $V = \overrightarrow{\langle x \rangle_\delta}$ (Veronese vector)

Suppose $p = \sum q_j^* q_j$ for $q_j \in \mathbb{R}\langle x \rangle_\delta$

Write $q_j = Q_j^* V$ for a column vector Q_j.

Then $p = \sum q_j^* q_j = $

$$= \sum_j V^* Q_j Q_j^* V$$

$$\left(1 + 2x_1 - x_2^2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}^* \cdot \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \end{pmatrix}\right.$$

$= V^*\underbrace{\left(\sum Q_j Q_j^*\right)}_{=:Q} \cdot V$

$Q \succeq 0$ is a $N(\delta)\times N(\delta)$ matrix

Then $\exists R_1, \dots, R_{N(\delta)}$ s.t. $Q = \sum R_j R_j^*$

Setting $r_j = R_j^* V$ yields $p = \sum r_j^* r_j$. ∎

Lemma (Gelfand-Naimark-Segal construction) GNS

Suppose $\lambda : (\mathbb{R}\langle x\rangle_{2\delta+2})_{sa} \to \mathbb{R}$ is a linear functional that satisfies $\lambda(q^* q) > 0$ $\forall q \in \mathbb{R}\langle x\rangle_{\delta+1} \setminus \{0\}$.

Then $\exists$ Hilbert space $\mathcal{H}$ of dim $N(\delta)$, a $\gamma \in \mathcal{H}$, $X \in B(\mathcal{H})_{sa}^d$

s.t. $\lambda(q^* p) = \langle p(X)\gamma, q(X)\gamma\rangle$ $\forall p, q \in \mathbb{R}\langle X\rangle_\delta$.

Puzzle of the Day

If $A = A^T$, $B = B^T$ have norm ≤ 1, then

$2\cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - B^2A \succeq 0$

Proof: Define inner product

$\langle p, q\rangle := \lambda(q^* p)$ on $\mathbb{R}\langle x\rangle_{\delta+1} =: \mathcal{K}$

$\mathcal{H}$ = subspace $\mathbb{R}\langle x\rangle_\delta$ in $\mathcal{K}$.

Let $P : \mathcal{K} \to \mathcal{H}$ be the orthogonal projection

$S_j : \mathcal{H} \to \mathcal{K}$

$p \mapsto x_j \cdot p$

Extend λ to $\lambda : \mathbb{R}\langle x\rangle_{2\delta+2} \to \mathbb{R}$ by $p \mapsto \frac{1}{2}\lambda(p + p^*)$

Let $X_j = PS_jP : \mathcal{H} \to \mathcal{H}$

X_j is self-adjoint: $p, q \in \mathcal{H}$

$\langle X_j p, q\rangle = \langle PS_jP p, q\rangle = \langle S_j Pp, Pq\rangle = \langle S_j p, q\rangle$

$= \langle x_j p, q\rangle = \lambda(q^* x_j p) = \lambda((x_j q)^* p) = \langle p, x_j q\rangle = \langle p, X_j q\rangle$.

Note $p(X)\cdot \underset{\gamma}{\underset{\shortparallel}{\mathbb{1}}} = p$ for $p \in \mathcal{H} = \mathbb{R}\langle x\rangle_\delta$

Hence for $p, q \in \mathcal{H}$,

$$\langle p(x)\gamma, q(x)\gamma\rangle = \lambda(q^* p).$$

<u>Corollary</u>: $\exists X \in B\left(\mathbb{R}^{N(\delta)}\right)^d_{sa}$ s.t. for $p \in \mathbb{R}\langle x\rangle_\delta$,

$p(X) = 0 \implies p = 0.$

<u>Theorem</u> (Helton[2002], McCullough[2001]):

Let $N(\delta) = \dim \mathbb{R}\langle x\rangle_\delta = 1 + d + d^2 + \cdots + d^\delta$

Suppose $p \in \mathbb{R}\langle x\rangle_{2\delta}$ and

$p(X) \succeq 0$ for all $N(\delta) \times N(\delta)$ s.a. X

Then $\exists r_1, \ldots, r_{N(\delta)} \in \mathbb{R}\langle x\rangle_\delta$ s.t.

$p = \sum r_j^* r_j$

1st Proof of the theorem

<u>Lemma</u>: $\exists$ linear functional $\mu : \mathbb{R}\langle x\rangle_{2\delta} \to \mathbb{R}$

s.t. $\mu(p^* p) > 0 \quad \forall\, 0 \neq p \in \mathbb{R}\langle x\rangle_\delta$.

Proof: Induct on δ. Suppose $\mu = \mu_{2\delta} : \mathbb{R}\langle x\rangle_{2\delta} \to \mathbb{R}$ is strictly positive. Define an extension $\mu_{2\delta+2}$ of $\mu_{2\delta}$ to $\mathbb{R}\langle x\rangle_{2\delta+2} \to \mathbb{R}$, as follows.

If $\deg v = 2\delta+1$, set $\mu_{2\delta+2}(v) = 0$

If $\deg v = 2\delta+2$ and $v \neq u^* u$, also $\mu_{2\delta+2}(v) = 0$

If $v = u^* u$, set $\mu_{2\delta+2}(v) = C$.

Since $\langle p, q\rangle = \mu_{2\delta}(q^* p)$ is strictly positive,

$\exists\, C >> 1$ s.t. $\mu_{2\delta+2}$ is also strictly positive. $\blacksquare$

$$\left(\begin{array}{c|c} \succ 0 & * \\ \hline * & \begin{matrix} C & 0 \\ 0 & C \end{matrix} \end{array}\right)$$

Proof (Cor): Apply GNS to $\mu_{2\delta+2}$ from last lemma. $\square$

Proposition: The cone of sos of degree at most δ, Σ^2_δ, is closed in $\mathbb{R}\langle x\rangle_{2\delta}$.

Proof: Take X from Corollary (for 2δ),

$\|p\| := \underbrace{\|p(X)\|}_{\text{Operator}}$ is a norm on $\mathbb{R}\langle x\rangle_{2\delta}$

Suppose $(p_n)_n$ is a sequence in Σ^2_d that converges to some $p \in \mathbb{R}\langle x\rangle_{2\delta}$.

Thus $p_n(X) \longrightarrow p(X)$.

So $(p_n(X))_n$ is bounded.

Puzzle of the Day

If $A = A^T$, $B = B^T$ have norm ≤ 1, then

$2 \cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - B^2A \geq 0$

$\exists\, r_{j,n}$ $(j \leq \overset{\text{Caratheodory}}{N(\delta)})$ s.t. $p_n = \sum_j r_{j,n}^* r_{j,n}$

Then $p_n(X) = \sum_j r_{j,n}(X)^* r_{j,n}(X)$, so

$(r_{j,n}(X))_n$ is bounded.

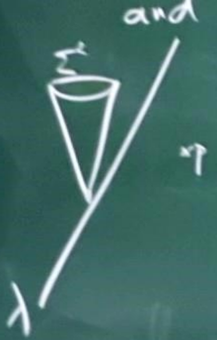
Passing to a subsequence we may assume

$r_{j,n} \longrightarrow r_j \in \mathbb{R}\langle x\rangle_\delta$

Then $p_n(X) = \sum_j r_{j,n}(X)^* r_{j,n}(X) \longrightarrow \sum r_j(X)^* r_j(X)$,

so $p = \sum r_j^* r_j \in \Sigma^2_\delta$.

Theorem (Helton[2002], McCullough[2001]):
Let $N(\delta) = \dim \mathbb{R}\langle x\rangle_\delta = 1 + d + d^2 + \cdots + d^\delta$.
Suppose $p \in \mathbb{R}\langle x\rangle_\delta$ and
$p(X) \geq 0$ for all $N(\delta)\times N(\delta)$ s.a. X.
Then $\exists\, r_1, \dots, r_{N(\delta)} \in \mathbb{R}\langle x\rangle_\delta$ s.t.
$p = \sum r_i^* r_i$.



Proof of the theorem: Suppose $p \geq 0$ on $N(\delta)\times N(\delta)$ s.a. tuples, and $p \notin \Sigma^2_{\delta+1}$. By Hahn–Banach separation,
$\exists\, \lambda : (\mathbb{R}\langle x\rangle_{2\delta+2})_{sa} \to \mathbb{R}$ $\exists\, c \in \mathbb{R}$ s.t.

$$\lambda(p) < c \leq \lambda(q) \quad \forall q \in \Sigma^2_{\delta+1}$$

Since $\Sigma^2_{\delta+1}$ is a cone, w.l.o.g. $c = 0$.

Extend λ to $\lambda : \mathbb{R}\langle x\rangle_{2\delta+2} \to \mathbb{R}$ by $p \mapsto \frac{1}{2}\lambda(p + p^*)$

Pick a strictly positive $\mu : \mathbb{R}\langle x\rangle_{2\delta+2} \to \mathbb{R}$ (lemma)

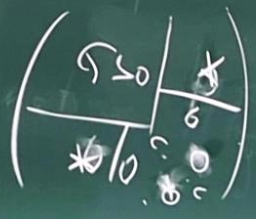
For $\varepsilon > 0$ small enough, $(\lambda + \varepsilon\cdot\mu)(p) < 0$.

Also, $\underline{\lambda + \varepsilon\mu}$ is strictly positive on $\Sigma^2_{\delta+1}$

Apply GNS, get $\underset{\gamma}{X}$ of size $N(\delta)$ s.t. $0 > (\lambda + \varepsilon\mu)(p) = \langle p(X)\gamma, \gamma\rangle \geq 0$.

Contradiction!

If $\deg v = 2\delta + 1$, set $\mu_{2\delta+2}(v) = 0$
If $\deg v = 2\delta + 2$ and $v \neq u^* u$, also $\mu_{2\delta+2}(v) = 0$.
If $v = u^* u$, set $\mu_{2\delta+2}(v) = C$.
Since $\langle p, q\rangle = \mu_{2\delta}(q^* p)$ is strictly positive,
$\exists\, C >> 1$ s.t. $\mu_{2\delta+2}$ is also strictly positive. ∎



Proof (Cor): Apply GNS to $\mu_{2\delta+2}$ from last lemma. □

$\underline{Q}$: Given p, q, does $q \succeq 0 \Rightarrow p \succeq 0$?

For instance, if $p = \sum r_j^* r_j + \sum s_k^* \cdot q \cdot s_k$

then $q \succeq 0 \Rightarrow p \succeq 0$.

Say $q(X) \succeq 0$. Then $p(X) = \sum r_j(X)^* r_j(X) + \underbrace{\sum s_k(X)^* \underbrace{q(X)}_{\succeq 0} s_k(X)}_{\succeq 0}$

$S \subseteq \mathbb{R}\langle x \rangle_{sa}$

The semialgebraic set of S is

$$\mathcal{D}_S = \bigcup_{n \in \mathbb{N}} \left\{ X \in (M_n^d)_{sa} \;\middle|\; \begin{array}{l} s(X) \succeq 0 \\ \forall s \in S \end{array} \right\}$$

$$\mathcal{D}_S^\infty = \left\{ X \in \underbrace{B(\mathcal{H})}_{\text{separable Hilbert space}}{}^d_{sa} \;\middle|\; \begin{array}{l} s(X) \succeq 0 \\ \forall s \in S \end{array} \right\}$$

Puzzle of the Day

If $A = A^T$, $B = B^T$ have norm ≤ 1, then

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$QM(S)$ = quadratic module generated by S is

$$QM(S) = \left\{ \underbrace{\sum_{i=1}^{r} g_i^* s_i g_i}_{\text{weighted s.o.s.}} \;\middle|\; r \in \mathbb{N},\ g_i \in \mathbb{R}\langle x \rangle,\ s_i \in S \cup \{1\} \right\}$$

- $QM(S) + QM(S) \subseteq QM(S)$ $\quad \forall p \in \mathbb{R}\langle x \rangle \quad p^* \cdot QM(S) \cdot p \subseteq QM(S)$

  $S \cup \{1\} \subseteq QM(S)$

Observation: If $p \in QM(S)$, then $p|_{\mathcal{D}_S^\infty} \succeq 0$ & $p|_{\mathcal{D}_S} \succeq 0$.

if $\exists N \in \mathbb{N}$ s.t. $N - \sum x_j^2 \in QM(S)$

then $\mathcal{D}_S$ and $\mathcal{D}_S^\infty$ are bounded (in norm).

Extend λ to $\lambda : \mathbb{R}\langle X\rangle_{2\delta+2} \to \mathbb{R}$ by $p \mapsto \frac{1}{2}\lambda(p+p^*)$

Pick a strictly positive $\mu : \mathbb{R}\langle X\rangle_{2\delta+2} \to \mathbb{R}$ (lemma)

For $\varepsilon > 0$ small enough, $(\lambda + \varepsilon \cdot \mu)(p) < 0$.

Also, $\underbrace{\lambda + \varepsilon\mu}$ is strictly positive on $\Sigma^2_{\delta+1}$

Apply GNS, get X of size $N(\delta)$, γ s.t. $0 > (\lambda+\varepsilon\mu)(p) = \langle p(X)\gamma, \gamma\rangle \geq 0$.

Contradiction!

If $\deg v = 2\delta+1$, set $\mu_{2\delta+2}(v) = 0$

If $\deg v = 2\delta+2$ and $v \neq u^* u$, also $\mu_{2\delta+2}(v) = 0$.

If $v = u^* u$, set $\mu_{2\delta+2}(v) = C$.

Since $\langle p, q\rangle = \mu_{2\delta}(q^* p)$ is strictly positive,

$\exists\, C \gg 1$ s.t. $\mu_{2\delta+2}$ is also strictly positive. ∎

$$\left(\begin{array}{c|c} \succ 0 & * \\ \hline * \mid 0 & \begin{matrix} c & \\ & c \end{matrix} \end{array}\right)$$

Proof (Cor): Apply GNS to $\mu_{2\delta+2}$ from last lemma. □

Def: We call $QM(S)$ archimedean if $\exists N \in \mathbb{N}$ s.t. $N - \sum x_j^2 \in QM(S)$.

Rmk: If $\mathcal{D}_S$ or $\mathcal{D}_S^\infty$ are bounded, you can add $N - \sum x_j^2$ (for $N \gg 1$) to S w/o changing $\mathcal{D}_S$ or $\mathcal{D}_S^\infty$.

Ex: $S = \{x_1 - 1, x_2 - 1, 8 - x_1 x_2 - x_2 x_1\}$

Then $\mathcal{D}_S^\infty$ is bounded, but $QM(S)$ is not archim.

Theorem (Helton–McCullough 2004)

Suppose $QM(S)$ is archimedean.

If $f|_{\mathcal{D}_S^\infty} \succ 0$, then $f \in QM(S)$.

In particular,

$f|_{\mathcal{D}_S^\infty} \succeq 0 \iff \forall \varepsilon > 0 \;\; f + \varepsilon \in QM(S)$

Letting $P_1 = -2 - x + x^2 - y + y^2$ and

$P_2 = -x - x^2 + y + y^2$, we have

$$2 + 2x + 2y - x^2y - yx^2 - xy^2 - y^2x = \frac{1}{2}\left(P_1^* P_1 + P_2^* P_2\right) + x\left(1 - x^2\right)x + y\left(1 - y^2\right)y.$$

<u>Puzzle of the Day</u>

If $A = A^T$, $B = B^T$ have norm ≤ 1, then

$$2 \cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - B^2A \geq 0$$