

# NONCOMMUTATIVE RATIONAL FUNCTIONS INVARIANT UNDER THE ACTION OF A FINITE SOLVABLE GROUP

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ABSTRACT. This paper describes the structure of invariant skew fields for linear actions of finite solvable groups on free skew fields in  $d$  generators. These invariant skew fields are always finitely generated, which contrasts with the free algebra case. For abelian groups or solvable groups  $G$  with a well-behaved representation theory it is shown that the invariant skew fields are free on  $|G|(d-1)+1$  generators. Finally, positivity certificates for invariant rational functions in terms of sums of squares of invariants are presented.

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## 1. INTRODUCTION

Classical invariant theory studies polynomials that are preserved under linear group actions [Kra84, Stu08, DK15]. By the Chevalley–Shepard–Todd theorem [Stu08, Theorem 2.4.1] for a finite group  $G \subseteq \mathrm{GL}_n(\mathbb{C})$ , the ring of invariants  $\mathbb{C}[x_1, \dots, x_n]^G$  is isomorphic to a polynomial ring (in the same number of variables) if and only if  $G$  is a complex reflection group. Similarly, one considers the rational invariants  $\mathbb{C}(x_1, \dots, x_n)^G$ . **Noether’s problem** asks when this invariant field is rational, that is, isomorphic to a field of rational functions. This is a subtle question which heavily depends on the structure of the group; however, in recent years much progress has been made along the lines of [Sal84, CTS07, Pey08, CHKK10, Mor12, CHHK15, JS]. Examples of invariant fields give the simplest negative answers to the **Lüroth problem**, i.e., examples of unirational varieties which are not rational [AM72, Sal84]. Lüroth’s problem has a positive answer in one variable (every field between  $K$  and  $K(x)$  must be  $K$  or purely transcendental over  $K$ ), and in two variables over  $\mathbb{C}$ .

We study the free noncommutative analogue of the above program over an algebraically closed field  $\mathbb{F}$  of characteristic 0. Let  $x = (x_1, \dots, x_d)$  be a tuple of noncommuting indeterminates. A **noncommutative polynomial** is a formal linear combination of words in  $x$  with coefficients in  $\mathbb{F}$ . For example,

$$17x_1^4 + 13x_1x_2 - 9x_2x_1 + 39.$$

We denote the free associative algebra of noncommutative polynomials on  $d$  generators by  $\mathbb{F}\langle x_1, \dots, x_d \rangle$ . A **noncommutative rational expression** is a syntactically valid combination of noncommutative polynomials, arithmetic operations  $+$ ,  $\cdot$ ,  $^{-1}$ , and parentheses, e.g.

$$(216x_1^3x_2^4x_1^5 - ((x_1x_2 - x_2x_1)^{-1} + 3)^8)^{-1}.$$

These expressions can be naturally evaluated on  $d$ -tuples of matrices. An expression is called nondegenerate if it is valid to evaluate it on at least one such tuple of matrices. Two nondegenerate expressions with the same evaluations wherever they are both defined are equivalent. A **noncommutative rational function** is an equivalence class of a nondegenerate rational expression. They form the **free skew field**  $\mathbb{F}\langle\!\langle x_1, \dots, x_d \rangle\!\rangle$ , which is the universal skew field of fractions of the free algebra  $\mathbb{F}\langle x_1, \dots, x_d \rangle$ . We refer the reader to [BGM05, Coh06, HMV06, BR11, KVV12, Vol18] for more on the free skew field.

We analyze the invariants in a free skew field under the action of a finite solvable group. For example, a symmetric noncommutative rational function  $r$  in two variables satisfies the equation

$$r(x, y) = r(y, x).$$

Naturally, this corresponds to an action of the symmetric group with two elements,  $S_2$ , and we denote the ring of symmetric noncommutative rational functions in two variables by  $\mathbb{F}\langle\!\langle x, y \rangle\!\rangle^{S_2}$ . Similarly,  $\mathbb{F}\langle x, y \rangle^{S_2}$  is the ring of symmetric noncommutative polynomials. By a theorem of Wolf, see e.g. [Coh06, §6.8], we have

$$\mathbb{F}\langle x, y \rangle^{S_2} \cong \mathbb{F}\langle u_1, u_2, \dots \rangle.$$

In fact, the polynomial invariants  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$  are, except in some trivial cases, always isomorphic to a free algebra in countably infinitely many variables [Coh06, §6.8]. As was observed by Agler and Young [AY14] (see also [CPTD18]),

$$\mathbb{F}\langle x, y \rangle^{S_2} \subseteq \mathbb{F}\langle x + y, (x - y)^2, (x - y)(x + y)(x - y) \rangle,$$

where  $\mathbb{F}\langle x + y, (x - y)^2, (x - y)(x + y)(x - y) \rangle$  denotes the subfield of  $\mathbb{F}\langle x, y \rangle$  generated by  $x + y$ ,  $(x - y)^2$ , and  $(x - y)(x + y)(x - y)$ . What is perhaps less clear, and follows from our Theorem 4.1, is that

$$\begin{aligned} \mathbb{F}\langle x, y \rangle^{S_2} &= \mathbb{F}\langle x + y, (x - y)^2, (x - y)(x + y)(x - y) \rangle \\ &\cong \mathbb{F}\langle a, b, c \rangle. \end{aligned}$$

Moreover, the isomorphism  $\varphi : \mathbb{F}\langle a, b, c \rangle \rightarrow \mathbb{F}\langle x, y \rangle^{S_2}$  satisfies  $\varphi(a) = x + y$ ,  $\varphi(b) = (x - y)^2$ , and  $\varphi(c) = (x - y)(x + y)(x - y)$ .

The equality and further isomorphism are remarkable for a few reasons. First, in the noncommutative case, it is nontrivial to show that the set of symmetric noncommutative polynomials generate the free skew field of symmetric rational functions. For example, expressions for relatively simple symmetric rational functions may require complicated expressions in terms of the generators, as is shown by the equalities expressing  $x^{-1} + y^{-1}$  :

$$\begin{aligned} x^{-1} + y^{-1} &= 4(x + y - (x - y)^2((x - y)(x + y)(x - y))^{-1}(x - y)^2)^{-1} \\ &= \varphi((a - bc^{-1}b)^{-1}) \end{aligned}$$

(It is somewhat hard to even find an elementary way of showing the equality, which we manufactured using realization theory which will be a key ingredient of the proof of our main result.) Secondly, it is interesting that  $x + y$ ,  $(x - y)^2$ ,  $(x - y)(x + y)(x - y)$  satisfy no hidden rational relations, which follows from a result of Lewin [Lew74, Theorem 1]. Unlike in the commutative case, it does not suffice to test only polynomial relations; for example,  $x, xy, xy^2$  satisfy no polynomial relations and generate a proper free subalgebra in  $\mathbb{F}\langle x, y \rangle$ , while they satisfy a rational relation and the skew field they generate in  $\mathbb{F}\langle x, y \rangle$  is  $\mathbb{F}\langle x, y \rangle$  itself.

**1.1. Main results.** Let  $G \subseteq \mathrm{GL}_d(\mathbb{F})$  be a finite group. The **skew field of rational invariants**, denoted by  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$ , is the skew field of elements of  $\mathbb{F}\langle x_1, \dots, x_d \rangle$  that are invariant under the action of  $G$ , that is,

$$\mathbb{F}\langle x_1, \dots, x_d \rangle^G = \{r \in \mathbb{F}\langle x_1, \dots, x_d \rangle : r(g \cdot x) = r(x) \text{ for all } g \in G\}.$$

Our first main result states that for solvable groups the skew field of rational invariants is always finitely generated:

**Theorem 1.1.** *Let  $G \subseteq \mathrm{GL}_d(\mathbb{F})$  be a finite solvable group. Then  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$  is finitely generated.*

For solvable groups  $G$  with a well-behaved representation theory we can give a finer structure of the invariant skew field  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$ .

**Definition 1.2.** Let  $G$  be a finite group.

- (1) Let  $N$  be a nontrivial normal abelian subgroup. We say that  $G$  is **unramified** over  $N$  if for every irreducible representation  $\pi$  of  $G$ ,  $\pi|_N$  is trivial or  $\pi|_N$  splits into distinct irreducible representations of  $N$ .
- (2) We say a group  $G$  is **totally unramified** if either  $G$  is the trivial group or there exists a nontrivial normal abelian subgroup  $N$  such that  $G$  is unramified over  $N$  and  $G/N$  is totally unramified.

**Theorem 1.3.** *Let  $G$  be a totally unramified group acting on  $\mathbb{F}^G$  via the left regular representation. Then*

$$\mathbb{F}\langle x_1, \dots, x_{|G|} \rangle^G \cong \mathbb{F}\langle u_1, \dots, u_{|G|(|G|-1)+1} \rangle.$$

Examples of totally unramified groups include abelian groups,  $S_3$ ,  $S_4$ , and dihedral groups; furthermore, all groups of order  $< 24$  are totally unramified. For these, noncommutative Noether's problem is tractable in the sense that for left regular representations the answer is affirmative. In fact, we conjecture that invariant skew fields of finite groups are always free.

The smallest non-example of a totally unramified group is the group  $\mathrm{SL}_2(\mathbb{F}_3)$  of order 24. The next non-examples are given by eight groups of order 48: the four groups in the isoclinism class of  $(\mathbb{Z}_4 \times D_4) \rtimes \mathbb{Z}_2$ , the isoclinism class of  $\mathrm{GL}_2(\mathbb{F}_3)$  with two groups, and the isoclinism class of  $\mathbb{Z}_2 \times \mathrm{SL}_2(\mathbb{F}_3)$  again containing two groups. Then we find 32 groups that are not totally unramified among the 267 groups of order 64. The smallest odd order examples are found among groups with 243 elements.

**1.2. Reader's guide.** The paper is organized as follows. After Section 2 with preliminaries we establish Theorem 1.1 in Section 3. Theorem 1.3 for abelian groups  $G$  is proved in Section 4, followed by the proof of the theorem itself, and a strengthening thereof (Theorem 5.1) in Section 5. Finally, Section 6 discusses relationships with real algebraic geometry; positivity certificates for invariant positive rational functions can be chosen to be invariant themselves.

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## 2. PRELIMINARIES ON GROUP REPRESENTATIONS

In this section we give some background and introduce notions which will be necessary for the sequel.

**2.1. Pontryagin duality.** Let  $N$  be a finite abelian group. Define  $N^*$  to be the group of multiplicative homomorphisms  $\chi : N \rightarrow \mathbb{F}^*$ . The group  $N^*$  is non-canonically isomorphic to  $N$ . Every representation  $\pi$  of  $N$  decomposes into a direct sum of elements of  $N^*$ , that is,  $N^*$  consists of all the irreducible representations of  $N$ . For more details see [Ser77, Rud90].

**2.2. Complete representations.** A faithful representation  $\pi$  of a group  $G$  is **complete** if it decomposes as  $\pi = \pi_B \oplus \pi_J$  and there is a nontrivial normal abelian subgroup  $N \subseteq G$  such that:

- (1)  $\pi_B|_N$  contains exactly the nontrivial representations of  $N$  as direct summands with multiplicity 1;

(2) The representation

$$\pi_{N^\tau} \oplus (\pi_B \otimes \pi \oplus \pi \otimes \pi_B)_{N^\tau} \oplus (\pi_B \otimes \pi \otimes \pi_B)_{N^\tau}$$

is a complete representation of  $G/N$ . Here, for a representation  $\varrho$ ,  $\varrho_{N^\tau}$  denotes the summands of  $\varrho$  which are trivial on  $N$  and thus naturally gives rise to a representation of  $G/N$ .

**2.3. Unramified groups.** The interplay between subgroups and representations is the subject of Clifford theory, see e.g. [Isa76]. We now give a reinterpretation of what it means for  $G$  to be unramified over  $N$ . There is a natural action of  $G/N$  on  $N^*$  given by

$$gN : \chi \mapsto (n \mapsto \chi(g^{-1}ng)) = \chi^g.$$

Let  $\pi$  an irreducible representation of  $G$ , such that  $\pi|_N$  decomposes as  $\bigoplus_i \chi_i$ . For any  $gN \in \text{Stab } \chi_i$ , we have that  $\pi(g)e_i \in \text{span } e_i$ , where  $e_i$  form the basis corresponding to the decomposition of  $\pi|_N$  into one-dimensional representations  $\bigoplus_i \chi_i$ . That is,  $G/N$  acts on the characters composing  $\pi$ , so if they are all distinct, as in the case of a totally unramified group, it permutes them.

**Example 2.1.** We show that  $S_4$  is totally unramified. The maximal abelian normal subgroup is given by  $\{e, (12)(34), (13)(24), (14)(23)\}$  which is an isomorphic copy of the Klein four group  $V = \mathbb{Z}_2 \times \mathbb{Z}_2$ . The character table of the representations of  $V$  is given by:

$\pi_1$	1	1	1	1
$\pi_2$	1	-1	1	-1
$\pi_3$	1	1	-1	-1
$\pi_4$	1	-1	-1	1

The character table of  $S_4$  is:

	{}	{2}	{2,2}	{3}	{4}
$\tau_1$	1	1	1	1	1
$\tau_2$	1	-1	1	1	-1
$\tau_3$	3	1	-1	0	-1
$\tau_4$	3	-1	-1	0	1
$\tau_5$	2	0	2	-1	0

The only irreducible characters that are nontrivial on  $V$  are  $\tau_3$  and  $\tau_4$ . From the tables we see  $\tau_3|_V = \tau_4|_V = \pi_2 + \pi_3 + \pi_4$ . Therefore  $S_4$  is unramified over  $V$ .

Now it remains to see that  $S_4/V \cong S_3$  is totally unramified. The group  $S_3$  has three irreducible representations, two of which are one-dimensional. The two-dimensional representation restricted to the  $\mathbb{Z}_3$  subgroup has two distinct characters on the diagonal.

**Example 2.2.** The dihedral groups  $D_n = \langle a, b : a^n = b^2 = abab = e \rangle$  are totally unramified. The irreducible two-dimensional representations  $\pi$  are given by

$$\pi : a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where  $\omega$  is a primitive  $n$ -th root of unity. The restriction of  $\pi$  to  $\langle a \rangle$  splits into two distinct irreducible representations. For  $n \geq 4$  such a representation is clearly not complete.

### 3. SOLVABLE GROUPS AND THEIR INVARIANTS

In this section we prove Theorem 1.1. A main technical ingredient are realizations of noncommutative rational functions, i.e., a canonical-type forms for them.

**3.1. Realizations.** Each rational function can be written in the form

$$r = c^* L^{-1} b \quad (3.1)$$

where  $b, c \in \mathbb{F}^n$ , and

$$L = A_0 + \sum_{i=1}^d A_i x_i$$

for some  $A_i \in M_n(\mathbb{F})$ . This formula is nondegenerate if and only if  $L$  admits an invertible matrix evaluation. For a comprehensive study of noncommutative rational functions we refer to [Coh06, BR11] or [BGM05, KVV12]. We will need the realization formula in (3.1) to prove for abelian groups (and thus for solvable groups via a later inductive argument) that the noncommutative polynomial invariants generate the rational invariants.

**3.2. Proof of Theorem 1.1.** We prove that if  $G/H$  is abelian and  $\mathbb{F}\langle x_1, \dots, x_d \rangle^H$  is finitely generated, then  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$  is finitely generated. Note that this suffices for proving Theorem 1.1 by a simple inductive argument.

**Lemma 3.1.** *Let  $H$  be a normal subgroup of  $G$  such that  $G/H = N$  is abelian. Suppose there are finitely many  $q_i \in \mathbb{F}\langle x_1, \dots, x_d \rangle^H$  such that every  $r \in \mathbb{F}\langle x_1, \dots, x_d \rangle^H$  is of the form*

$$r(x) = c^* \left( A_0 + \sum_i A_i q_i(x) \right)^{-1} b, \quad (3.2)$$

where the formula on the right-hand side is nondegenerate. Then, there exist finitely many  $\tilde{q}_j \in \mathbb{F}\langle x_1, \dots, x_d \rangle^G$  such that for every  $r \in \mathbb{F}\langle x_1, \dots, x_d \rangle^G$  we have

$$r(x) = \tilde{c}^* \left( \tilde{A}_0 + \sum_i \tilde{A}_i \tilde{q}_i(x) \right)^{-1} \tilde{b},$$

where the formula on the right-hand side is nondegenerate.

*Proof.* Let  $V = \text{span} \{n \cdot q_i(x) : n \in N\}$ . By Pontryagin duality there exists a basis  $\{v_i\}_i$  for  $V$  such that  $n \cdot v_i = \chi_i(n) v_i$ . Without loss of generality,  $q_i = v_i$ . For each nontrivial  $\chi$  in the subgroup of  $N^*$  generated by the  $\chi_i$ , pick a monomial  $m_\chi$  in the  $q_i$  such that  $n \cdot m_\chi = \chi(n) m_\chi$ . Without loss of generality, the subgroup generated by the  $\chi_i$  is the whole of  $N^*$ . Given  $r(x)$

as in (3.2), define the new realization

$$\begin{aligned}
r(x) &= \frac{1}{|N|} (c^{\oplus |N|})^* \left( I \otimes A_0 + \oplus_{n \in N} \sum_i A_i q_i(n \cdot x) \right)^{-1} (b^{\oplus |N|}) \\
&= \frac{1}{|N|} (c^{\oplus |N|})^* \left( I \otimes A_0 + \oplus_{n \in N} \sum_i A_i \chi_i(n) q_i(x) \right)^{-1} (b^{\oplus |N|}) \\
&= \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}^* \left( I \otimes A_0 + \sum_i \tilde{\chi}_i \otimes A_i q_i(x) \right)^{-1} \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= \tilde{c}^* \left( I \otimes A_0 + \sum_i \tilde{\chi}_i \otimes A_i q_i(x) \right)^{-1} \tilde{b},
\end{aligned}$$

where  $\tilde{\chi}_i$  is the permutation matrix that maps  $e_\nu$  to  $e_{\chi_i \nu}$ , and is conjugate to  $\oplus_{n \in N} \chi_i(n)$ , the left regular representation of  $N^*$ . Let  $M$  be a diagonal matrix with columns indexed by elements of  $N^*$  such that the diagonal entries are  $m_\chi$  for nontrivial  $\chi$  and 1 otherwise. Similarly, let  $\hat{M}$  be diagonal with diagonal entries  $m_{\chi^{-1}}$  for nontrivial  $\chi$  and 1 otherwise. Without loss of generality we always list the identity representation in  $N^*$  first.

Then,

$$\begin{aligned}
r(x) &= \tilde{c}^* \left( I \otimes A_0 + \sum_i \tilde{\chi}_i \otimes A_i q_i(x) \right)^{-1} \tilde{b} \\
&= \tilde{c}^* (M M^{-1} \otimes I) \left( A_0^{\oplus |N|} + \sum_i \tilde{\chi}_i \otimes A_i q_i(x) \right)^{-1} (\hat{M}^{-1} \hat{M} \otimes I) \tilde{b} \\
&= \tilde{c}^* \left( \hat{M} M \otimes A_0 + \sum_i \hat{M} \tilde{\chi}_i M q_i(x) \otimes A_i \right)^{-1} \tilde{b}.
\end{aligned}$$

Note that  $\hat{M} M$  is invariant. Now we need to show that  $\hat{M} \tilde{\chi}_i q_i(x) M$  is invariant. The nonzero elements of  $\hat{M} \tilde{\chi}_i q_i(x) M$  are  $m_{(\chi_i \nu)^{-1}} q_i(x) m_\nu$  which are clearly invariant by the choice of  $q_i$  and  $m_\chi$ , so we are done since the new form is defined wherever  $M$  was invertible, and thus non-degenerate.  $\square$

#### 4. THE ABELIAN CASE

The next theorem shows that the invariant fields for abelian groups are free and can be explicitly described.

**Theorem 4.1.** *Let  $G \subset \text{GL}_d(\mathbb{F})$  be abelian. Then*

$$\mathbb{F} \langle x_1, \dots, x_d \rangle^G \cong \mathbb{F} \langle u_1, \dots, u_{|G|(d-1)+1} \rangle.$$

*If  $G$  is diagonal, that is  $G = \oplus \chi_i$  where  $\chi_i \in G^*$ , then any minimal set of generators for the subgroup of the free group on  $d$  generators given by the words  $x_{i_1}^{j_1} \dots x_{i_k}^{j_k}$  with  $\chi_{i_1}^{j_1} \dots \chi_{i_k}^{j_k} = 1$ . can serve as the preimage of the  $u_j$ .*

*Proof.* We note that, under an appropriate change of coordinates, the action of  $G$  is diagonal. That is, there are linear polynomials  $w_1, \dots, w_d$  such that

$$g \cdot w_i = \chi_i(g)w_i,$$

where  $\chi_i$  belongs to the character group of  $G$ , denoted  $\hat{G}$ , and  $w_i$  form an orthonormal basis (in the sense that the coefficients are orthogonal) for the space of all linear polynomials in  $\mathbb{F}\langle x_1, \dots, x_d \rangle$ . By [CPTD18, Theorem 7.4] the elements of  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$  are spanned by monomials of the form  $w_{i_1} \dots w_{i_k}$  such that  $\chi_{i_1} \dots \chi_{i_k} = 1$ . By embedding  $\mathbb{F}\langle x_1, \dots, x_d \rangle$  into the group algebra of the free group on  $d$  generators by mapping the  $w_i$  to the said group generators, one obtains that the invariants in the free group algebra (of noncommutative Laurent polynomials) are generated by  $|G|(d-1) + 1$  elements via the Nielsen-Schreier theorem [LS01, Section I.3] as in [CPTD18, Theorem 7.5]. Concretely, we have a surjective homomorphism from the free group with  $d$  generators to  $\hat{G}$  which is itself non-canonically isomorphic to  $G$ . The kernel of this homomorphism is a subgroup of the free group with  $d$  generators, which must be free and have  $|G|(d-1) + 1$  generators via the Nielsen-Schreier formula. The generating elements can satisfy no rational relations by [Lew74] (see also [Lin00]), that is, their rational closure is a free skew field on these generators.

Therefore it suffices to show that  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$  generates the whole of  $\mathbb{F}\langle\!\langle x_1, \dots, x_d \rangle\!\rangle^G$ . This follows a similar line of reasoning as used in the proof of Lemma 3.1. Let  $r \in \mathbb{F}\langle\!\langle x_1, \dots, x_d \rangle\!\rangle^G$ . By the realization theory, one can write any element of the free skew field as  $r = c^*(A_0 + \sum_i A_i w_i)^{-1}b$ . Now  $g \cdot r = r$ , so

$$r = c^* \left( A_0 + \sum_i A_i \chi_i(g) w_i \right)^{-1} b = \frac{1}{|G|} \hat{c}^* \left( \oplus_{g \in G} A_0 + \sum_i \oplus_{g \in G} A_i \chi_i(g) w_i \right)^{-1} \hat{b}.$$

The representation  $\oplus_{g \in G} \chi_i(g)$  is conjugate to the left regular representation of  $\hat{G}$ , the elements of which we will denote by  $\tilde{\chi}$ . After a change of basis we get

$$r = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}^* \left( I \otimes A_0 + \sum_i \tilde{\chi}_i \otimes A_i w_i \right)^{-1} \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Fix polynomials  $v_\chi$  such that  $g \cdot v_\chi = \chi(g)v_\chi$  and  $v_\tau = 1$ , where  $\tau$  is the trivial representation. Let  $V$  be a diagonal matrix whose diagonal entries are the  $v_\chi$ . Similarly, let  $\hat{V}$  be diagonal with diagonal entries  $v_{\chi^{-1}}$ . Now

$$\begin{aligned} r &= \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}^* \left( I \otimes A_0 + \sum_i \tilde{\chi}_i \otimes A_i w_i \right)^{-1} \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}^* (V V^{-1} \otimes I) \left( I \otimes A_0 + \sum_i \tilde{\chi}_i \otimes A_i w_i \right)^{-1} (\hat{V}^{-1} \hat{V} \otimes I) \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}^* \left( \hat{V} V \otimes A_0 + \sum_i (\hat{V} \tilde{\chi}_i w_i V) \otimes A_i \right)^{-1} \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned}$$



As in the proof of Lemma 3.1,  $\hat{V}V$  and  $\hat{V}\tilde{\chi}_i w_i V$  are invariant under the action of  $G$ . We get  $(\tilde{\chi}_i)_{\eta, \nu} = 1$  if  $\eta = \chi_i \nu$  and 0 otherwise. Now,  $(\hat{V}\tilde{\chi}_i w_i V)_{\chi_i \nu, \nu} = v_{\chi_i^{-1} \nu^{-1}} w_i v_\nu$  is clearly invariant, so we are done.  $\square$

**Corollary 4.2.** *Let  $G$  be an abelian group acting on  $\mathbb{F}^d$  via a complete representation  $\pi = \pi_B \oplus \pi_J$ , where  $\pi_B$  acts on  $\mathbb{F}^{G^* \setminus \{\tau\}}$  and  $\tau$  denotes the trivial representation. Let  $b_\chi$  and  $j_i$  diagonalize  $\pi_B$  and  $\pi_J$ , respectively. Then*

$$\mathbb{F}\langle x_1, \dots, x_d \rangle^G \cong \mathbb{F}\langle u_1, \dots, u_{|G|(d-1)+1} \rangle,$$

and preimages of the  $u_j$  are of the form  $b_\chi b_\eta b_{(\chi\eta)^{-1}}, b_\chi j_i b_{(\chi\eta_i)^{-1}}$ , where  $b_\tau = 1$ .

*Proof.* For  $v_\chi$  from the proof of Theorem 4.1 we take  $b_\chi$ , while for  $w_i$  we take  $b_\chi$  and  $j_i$  for  $i = 1, \dots, d - |G| + 1$ , where  $d - |G| + 1$  is the dimension of  $\pi_J$ . Clearly

$$\{b_\eta b_{\eta^{-1}}, b_\chi b_\eta b_{(\chi\eta)^{-1}}, b_\chi j_i b_{(\chi\eta_i)^{-1}} : \chi \in G^*, \eta \in G^* \setminus \{\tau\}, 1 \leq i \leq t\} \setminus \{1\}$$

generate  $\mathbb{F}\langle x_1, \dots, x_d \rangle^G$ . Since  $b_\eta b_{\eta^{-1}} = b_\tau b_\eta b_{\eta^{-1}} = b_\eta b_{\eta^{-1}} b_\tau$ , there are

$$(|G| - 1) + (|G| - 1)(|G| - 2) + |G|(d - |G| + 1) = |G|(d - 1) + 1$$

generators. By [Coh95, Corollary 5.8.14] they are free generators of the free skew field of invariants.  $\square$

**Example 4.3.** Let  $\omega$  be a third root of unity and  $c$  a generator of  $\mathbb{Z}_3$ . Define a representation of  $\mathbb{Z}_3$  on  $\mathbb{F}^2$  by  $cx = \omega x$  and  $cy = \omega^2 y$ . Then we have

$$\mathbb{F}\langle x, y \rangle^{\mathbb{Z}_3} = \mathbb{F}\langle x^3, xy, yx, y^3 \rangle.$$

## 5. UNRAMIFIED GROUPS AND THEIR INVARIANTS

The following is our main structure theorem for invariant fields of solvable groups.

**Theorem 5.1.** *Let  $G \subset \mathrm{GL}_d(\mathbb{F})$  be a finite group acting on  $\mathbb{F}^d$  via a complete representation. Then*

$$\mathbb{F}\langle x_1, \dots, x_d \rangle^G \cong \mathbb{F}\langle u_1, \dots, u_{|G|(d-1)+1} \rangle.$$

A key step in the proof of Theorem 5.1 will be the following lemma.

**Lemma 5.2.** *Let  $\pi = \pi_B \oplus \pi_J$  be a complete representation of  $G$  and let  $N$  be a normal abelian subgroup as in the definition of complete representation in Section 2.2. Then  $G/N$  acts linearly on the free generators of  $\mathbb{F}\langle x_1, \dots, x_d \rangle^N$  constructed in Corollary 4.2.*

*Proof.* Let  $b_\chi$  and  $j_k$  diagonalize  $\pi_B|_N$  and  $\pi_J|_N$ , respectively. Here  $b_\chi$  are indexed by  $N^* \setminus \{\tau\}$ . Then  $ng \cdot b_\chi = \chi(g^{-1}ng)(g \cdot b_\chi) = \chi^g(n)(g \cdot b_\chi)$ , so  $g \cdot b_\chi$  is a scalar multiple of  $b_{\chi^g}$ .

Denote  $V_\chi = \mathrm{span} \{j_i : n \cdot j_i = \chi(n)j_i\}$ . Since  $ng \cdot j_i = \chi(g^{-1}ng)(g \cdot j_i) = \chi^g(n)(g \cdot j_i)$ ,  $v \in V_\chi$  implies  $g \cdot v \in V_{\chi^g}$ .  $\square$

*Proof of Theorem 5.1.* Let  $\pi$  be a complete representation of  $G$  and  $N$  a nontrivial abelian normal subgroup corresponding to it. Then  $G/N$  acts linearly on

$$\mathbb{F}\langle x_1, \dots, x_d \rangle^N \cong \mathbb{F}\langle u_1, \dots, u_{|N|(d-1)+1} \rangle$$

by Lemma 5.2; furthermore, by the description of the generators  $u_j$  in Corollary 4.2, this action is precisely the representation

$$\pi_{N^\tau} \oplus (\pi_B \otimes \pi \oplus \pi \otimes \pi_B)_{N^\tau} \oplus (\pi_B \otimes \pi \otimes \pi_B)_{N^\tau}.$$

Since it is a complete representation of  $G/N$  by assumption, induction implies

$$\mathbb{F}\langle x_1, \dots, x_d \rangle^G \cong \mathbb{F}\langle u_1, \dots, u_{|N|(d-1)+1} \rangle^{G/N} \cong \mathbb{F}\langle \tilde{u}_1, \dots, \tilde{u}_{|G|(d-1)+1} \rangle. \quad \square$$

*Proof of Theorem 1.3.* We prove that the left regular representation of a totally unramified group is complete. Let  $G$  be unramified over  $N$ . By Clifford's theorem [Isa76, Theorem 6.2], irreducible representations of  $N$  partition into orbits and each orbit is represented by an irreducible representation of  $G$ . Take the nontrivial representatives and define  $\pi_B$  as their sum. The left regular representation of  $G/N$  is then contained in  $\pi_{N^\tau}$  and we are done by recursion.  $\square$

**Example 5.3.** Define a representation of  $S_3$  on  $\mathbb{F}^3$  via  $\sigma x_i = x_{\sigma(i)}$ . The representation of the normal subgroup  $N$  generated by  $(1\ 2\ 3)$  is diagonalized in the basis  $v_1 = x_1 + x_2 + x_3$ ,  $v_2 = x_1 + \omega x_2 + \omega^2 x_3$ ,  $v_3 = x_1 + \omega^2 x_2 + \omega x_3$ , where  $\omega$  is the third root of unity. By Corollary 4.2, we obtain the invariant skew field  $\mathbb{F}\langle x_1, x_2, x_3 \rangle^N = \mathbb{F}\langle v_1, v_2 v_3, v_3 v_2, v_2 v_1 v_3, v_3 v_1 v_2, v_2^3, v_3^3 \rangle = \mathbb{F}\langle z_1, \dots, z_7 \rangle$ .

The action of  $G/N \cong \mathbb{Z}_2$  on  $\mathbb{F}\langle z_1, \dots, z_7 \rangle$  is given by the action of  $(2\ 3)$  (or any other transposition) on the initial variables. We get a representation given by

$$z_1 \mapsto z_1, z_2 \mapsto z_3, z_3 \mapsto z_2, z_4 \mapsto z_5, z_5 \mapsto z_4, z_6 \mapsto z_7, z_7 \mapsto z_6,$$

which is diagonalized by

$$w_1 = z_1, w_2 = z_2 + z_3, w_3 = z_2 - z_3, w_4 = z_4 + z_5, w_5 = z_4 - z_5, w_6 = z_6 + z_7, w_7 = z_6 - z_7.$$

Finally, applying Corollary 4.2 again, the obtained free generators of  $\mathbb{F}\langle x_1, x_2, x_3 \rangle^{S_3}$  are

$$w_1, w_2, w_4, w_6, w_3^2, w_3 w_5, w_3 w_7, w_5 w_3, w_7 w_3, w_3 w_1 w_3, w_3 w_2 w_3, w_3 w_4 w_3, w_3 w_6 w_3.$$

**Example 5.4.** Even though the standard two-dimensional representation of  $D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$  given by  $a \cdot x = ix$ ,  $a \cdot y = -iy$ ,  $b \cdot x = y$  and  $b \cdot y = x$  is not complete, we can still compute its invariants. The invariants of  $N = \langle a \rangle \cong \mathbb{Z}_4$  are freely generated by  $z_1 = xy$ ,  $z_2 = yx$ ,  $z_3 = x^2 y^2$ ,  $z_4 = y^2 x^2$  and  $z_5' = x^4$ . Then we replace  $z_5'$  by  $z_5 = z_5' z_4^{-1} = x^2 y^{-2}$ . The action of  $D_4/N \cong \mathbb{Z}_2$  on these generators is

$$z_1 \mapsto z_2, z_2 \mapsto z_1, z_3 \mapsto z_4, z_4 \mapsto z_3, z_5 \mapsto z_5^{-1}.$$

Observe that this action is linearized and diagonalized with respect to

$$w_1 = z_1 + z_2, w_2 = z_1 - z_2, w_3 = z_3 + z_4, w_4 = z_3 - z_4, w_5 = (1 + z_5)(1 - z_5)^{-1}.$$

Finally we get nine free generators of the rational invariants of  $D_4$ :

$$w_1, w_2^2, w_2 w_1 w_2, w_2 w_3 w_2, w_2 w_4 w_2, w_2 w_5, w_3, w_4 w_2, w_5 w_2.$$

**Example 5.5.** The smallest example of a not totally unramified group is  $\mathrm{SL}_2(\mathbb{F}_3)$ . It has only one nontrivial normal abelian subgroup  $N \cong \mathbb{Z}_2$ ; it is generated by  $\mathrm{diag}(2, 2)$ . Every irreducible representation restricted to  $N$  is trivial or contains two copies of the sign representation.

Let us describe problems arising in the computation of a generating set for the skew field of invariants. We start with a two-dimensional irreducible representation of  $\mathrm{SL}_2(\mathbb{F}_3)$  on  $\mathbb{F}^2$ , for instance defined by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \omega^2 \\ -\omega & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\omega \\ \omega^2 & 0 \end{pmatrix}.$$

The generator of  $N$  is mapped to  $\mathrm{diag}(-1, -1)$ . Hence the free generators of  $N$ -invariants are  $x^2$ ,  $xy$  and  $yx$ .

The group  $G/N$  has one abelian normal subgroup  $\tilde{N}/N$ ; it is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Representatives of its generators are mapped to

$$\begin{pmatrix} 0 & \omega \\ -\omega^2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\omega & -1 \\ \omega & \omega \end{pmatrix}.$$

The action of these two on  $N$ -invariants is given by

$$x^2 \mapsto \omega y^2 = \omega(yx)(x^2)^{-1}(xy), \quad xy \mapsto -yx, \quad yx \mapsto -xy, \quad (5.1)$$

and

$$x^2 \mapsto \omega(x+y)^2, \quad xy \mapsto -\omega(x+y)(\omega y - x), \quad yx \mapsto -\omega(\omega y - x)(x+y). \quad (5.2)$$

Now the problem is to find a set of free generators of  $N$ -invariants that simultaneously linearizes and diagonalizes both mappings as we have done in Example 5.4. It is straightforward to linearize (5.1) by using a linear fractional transformation in  $xy^{-1} = x^2(yx)^{-1}$  (cf. Example 5.4), but then the action (5.2) becomes unwieldy. We have been unable to determine if  $\mathbb{F}\langle x, y \rangle^{\mathrm{SL}_2(\mathbb{F}_3)}$  is free (on 25 generators).

## 6. POSITIVITY OF INVARIANT RATIONAL FUNCTIONS

In this section we investigate positive invariant noncommutative rational functions and prove an invariant rational Positivstellensatz in Theorem 6.2 for solvable groups  $G$ . A finer structure of constraint positivity is proved in Subsection 6.1. Positivity certificates for invariants are ubiquitous in real algebraic geometry literature, see e.g. [PS85, CKS09, Rie16, Bro98]. Throughout this section let  $\mathbb{F} = \mathbb{C}$  be the field of complex numbers. We endow  $\mathbb{C}\langle x_1, \dots, x_d \rangle$  with the natural involution fixing the  $x_j$  and extending the complex conjugation on  $\mathbb{C}$ .

**Lemma 6.1.** *Let  $G \subset \mathrm{U}_d(\mathbb{C})$  be a finite solvable group,  $H$  its normal subgroup, and assume that  $N = G/H$  is abelian. There exists an invertible matrix  $R_N \in \mathrm{M}_{|N|}(\mathbb{C}\langle x_1, \dots, x_d \rangle^H)$  such that for every  $Q_H \in \mathrm{M}_n(\mathbb{C}\langle x_1, \dots, x_d \rangle^H)$ ,*

$$Q_G = (R_N \otimes I)^* \left( \bigoplus_{n \in N} n \cdot Q_H \right) (R_N \otimes I) \in \mathrm{M}_{|N|n}(\mathbb{C}\langle x_1, \dots, x_d \rangle^G)$$

and every  $r \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$  of the form

$$r(x) = c^* \left( \left( A_0 + \sum_i A_i q_i(x) \right)^{-1} \right)^* Q_H \left( A_0 + \sum_i A_i q_i(x) \right)^{-1} c, \quad (6.1)$$

where the  $q_i \in \mathbb{C}\langle x_1, \dots, x_d \rangle^H$ , can be rewritten as

$$r(x) = \tilde{c}^* \left( \left( \tilde{A}_0 + \sum_i \tilde{A}_i \tilde{q}_i(x) \right)^{-1} \right)^* Q_G \left( \tilde{A}_0 + \sum_i \tilde{A}_i \tilde{q}_i(x) \right)^{-1} \tilde{c}$$

with  $\tilde{q}_i \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$ .

*Proof.* Consider  $r(x)$  as in (6.1). Tracing through the proof of Theorem 1.1 we see that  $r(x)$  admits a realization

$$\tilde{c}^* \left( \left( \tilde{A}_0 + \sum_i \tilde{A}_i \tilde{q}_i(x) \right)^{-1} \right)^* ((\hat{M}\Gamma^*) \otimes I) \left( \bigoplus_{n \in N} n \cdot Q_H \right) ((\Gamma M) \otimes I) \left( \tilde{A}_0 + \sum_i \tilde{A}_i \tilde{q}_i(x) \right)^{-1} \tilde{c},$$

where  $\Gamma$  is a unitary change of basis matrix (more precisely, columns of  $\Gamma^*$  are eigenvectors for the left regular representation of  $N^*$ ). Note that we can take  $\hat{M} = M^*$ . Hence  $R_N = \Gamma M$  is the desired matrix.  $\square$

**Theorem 6.2.** *Let  $G \subset U_d(\mathbb{C})$  be a finite solvable group. There exists an invertible matrix  $R_G \in M_{|G|}(\mathbb{C}\langle x_1, \dots, x_d \rangle)$  such that for every  $Q \in M_n(\mathbb{C}\langle x_1, \dots, x_d \rangle)$ ,*

$$Q_G = (R_G \otimes I)^* \left( \bigoplus_{g \in G} g \cdot Q \right) (R_G \otimes I) \in M_{|G|n}(\mathbb{C}\langle x_1, \dots, x_d \rangle^G)$$

and every  $r \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$  of the form

$$r(x) = c^* \left( \left( A_0 + \sum_i A_i q_i(x) \right)^{-1} \right)^* Q \left( A_0 + \sum_i A_i q_i(x) \right)^{-1} c,$$

where the  $q_i \in \mathbb{C}\langle x_1, \dots, x_d \rangle$ , can be rewritten as

$$r(x) = \tilde{c}^* \left( \left( \tilde{A}_0 + \sum_i \tilde{A}_i \tilde{q}_i(x) \right)^{-1} \right)^* Q_G \left( \tilde{A}_0 + \sum_i \tilde{A}_i \tilde{q}_i(x) \right)^{-1} \tilde{c}$$

with  $\tilde{q}_i \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$ .

*Proof.* Apply Lemma 6.1 and induction on the derived series of  $G$ .  $\square$

**Corollary 6.3.** *Let  $G \subset U_d(\mathbb{C})$  be a finite solvable group. Then there exist  $q_1, \dots, q_N \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$  such that for every  $r \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$ , if*

$$r = \sum_i s_i^* s_i,$$

then

$$r = \sum_j \tilde{s}_j^* q_{n_j} \tilde{s}_j,$$

where  $\tilde{s}_j \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$ .

*Proof.* Take  $Q = I$  in Theorem 6.2. □

A rational function  $r$  is **positive** if for every  $n \in \mathbb{N}$  and  $X = X^* \in M_n(\mathbb{C})^d$ ,  $r$  is defined at  $X$  and  $r(X)$  is positive semidefinite.

**Corollary 6.4.** *Let  $G \subset U_d(\mathbb{C})$  be a finite solvable group. Then there exist  $q_1, \dots, q_N \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$  such that every positive rational function  $r \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$  is of the form*

$$r = \sum_j \tilde{s}_j^* q_{n_j} \tilde{s}_j,$$

where  $\tilde{s}_j \in \mathbb{C}\langle x_1, \dots, x_d \rangle^G$ .

*Proof.* Since  $r$  is positive semidefinite, it is a sum of hermitian squares by [KPV17, Theorem 4.5]. The conclusion now follows from Corollary 6.3. □

**6.1. Quadratic modules and free semialgebraic sets.** Let  $Q \in M_n(\mathbb{C}\langle x_1, \dots, x_d \rangle)$  be a symmetric matrix. Its **quadratic module** is

$$\text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle}(Q) = \left\{ \sum_i u_i^* u_i + \sum_j v_j^* Q v_j : v_j \in \mathbb{C}\langle x_1, \dots, x_d \rangle^n, u_i \in \mathbb{C}\langle x_1, \dots, x_d \rangle \right\}.$$

It is called **Archimedean** if there is  $N \in \mathbb{N}$  so that  $N - \sum_j x_j^2 \in \text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle}(Q)$ . The associated **free semialgebraic set** is

$$\mathcal{D}_Q = \{X = X^* \in B(\mathcal{H})^d : Q(X) \succeq 0\},$$

where  $\mathcal{H}$  is a separable Hilbert space.

**Corollary 6.5.** *Let  $G \subset U_d(\mathbb{C})$  be a finite solvable group. Then*

$$\text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle}(Q) \cap \mathbb{C}\langle x_1, \dots, x_d \rangle^G = \text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle^G}(Q_G).$$

*Proof.* Use Theorem 6.2. □

Let  $\mathfrak{r}$  be a formal rational expression. We say that  $\mathfrak{r}$  is **(strictly) positive** on  $\mathcal{D}_Q$  if for every  $X \in \mathcal{D}_Q$ ,  $\mathfrak{r}$  is defined at  $X$  and  $\mathfrak{r}(X)$  is a positive semidefinite (definite) operator. In this case we write  $\mathfrak{r} \succeq 0$  ( $\mathfrak{r} \succ 0$ ) on  $\mathcal{D}_Q$ .

The reason for using formal rational expressions is that rational functions (as elements of the free skew field) do not admit unambiguous evaluations on  $B(\mathcal{H})^d$ . For example, the expression  $\mathfrak{r} = x_1(x_2x_1)^{-1}x_2 - 1$  represents the zero element of the free skew field, but admits nonzero evaluations on operators, namely  $\mathfrak{r}(S, S^*) \neq 0$  where  $S$  is the unilateral shift on  $\ell^2(\mathbb{N})$ .

**Proposition 6.6.** *Let  $G \subset U_d(\mathbb{C})$  be a finite solvable group. Let  $Q \in M_n(\mathbb{C}\langle x_1, \dots, x_d \rangle)$  be such that  $\text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle}(Q)$  is Archimedean. If  $\mathfrak{r}$  is a formal rational expression such that  $\mathfrak{r} \succ 0$  on  $\mathcal{D}_Q$  and  $\mathfrak{r}$  induces a  $G$ -invariant rational function  $r$ , then  $r \in \text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle^G}(Q_G)$ .*

*Proof.* By [Pas18, Theorem 2.1],  $r \in \text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle}(Q)$ . Now Corollary 6.5 finishes the proof.  $\square$

**Proposition 6.7.** *Let  $G \subset \text{U}_d(\mathbb{C})$  be a finite solvable group. Assume  $Q \in \text{M}_n(\mathbb{C}\langle x_1, \dots, x_d \rangle)$  is linear with  $Q(0) = I$ . If  $\mathfrak{r}$  is a formal rational expression such that  $\mathfrak{r} \succeq 0$  on  $\mathcal{D}_Q$  and  $\mathfrak{r}$  induces a  $G$ -invariant rational function  $r$ , then  $r \in \text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle}^G(Q_G)$ .*

*Proof.* By [Pas18, Theorem 3.1],  $r \in \text{QM}_{\mathbb{C}\langle x_1, \dots, x_d \rangle}(Q)$ . Now apply Corollary 6.5.  $\square$

**Example 6.8.** Let  $G = S_2$  act on  $\mathbb{C}\langle x, y \rangle$ . Then

$$a = x + y, \quad b = (x - y)^2, \quad c = (x - y)(x + y)(x - y)$$

are free generators of  $\mathbb{C}\langle x, y \rangle^G$ . The matrix  $R_G$  from Theorem 6.2 equals

$$R_G = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{diag}(1, x - y) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & x - y \\ -1 & x - y \end{pmatrix}.$$

By computing the invariant middle matrix  $Q_G$  we obtain the following Positivstellensätze.

- (1) (Entire space) If  $Q = 1$  then  $Q_G = \text{diag}(1, b)$ . By Corollary 6.4, every positive  $G$ -invariant rational function  $r$  is of the form

$$r = \sum_j u_j^* u_j + \sum_j v_j b v_j^*, \quad u_j, v_j \in \mathbb{C}\langle x, y \rangle^G.$$

- (2) (Disk) If  $Q = 1 - x^2 - y^2$  then

$$Q_G = \text{diag} \left( 1 - \frac{1}{2}(a^2 + b), b - \frac{1}{2}(cb^{-1}c + b^2) \right).$$

Since  $Q$  clearly generates an Archimedean quadratic module, every  $G$ -invariant rational expression strictly positive on the disk

$$\{(X, Y) : X^2 + Y^2 \preceq I\}$$

induces a rational function of the form

$$\sum_j u_j^* u_j + \sum_j v_j (1 - \frac{1}{2}(a^2 + b)) v_j^* + \sum_j w_j (b - \frac{1}{2}(cb^{-1}c + b^2)) w_j^*$$

for  $u_j, v_j, w_j \in \mathbb{C}\langle x, y \rangle^G$  by Proposition 6.6. On the other hand, the disk also admits a linear matrix representation given by

$$Q' = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ x & y & 1 \end{pmatrix},$$

which yields

$$Q'_G = \frac{1}{2} \begin{pmatrix} 2 & 0 & a & 0 & 0 & b \\ 0 & 2 & a & 0 & 0 & -b \\ a & a & 2 & b & -b & 0 \\ 0 & 0 & b & 2b & 0 & c \\ 0 & 0 & -b & 0 & 2b & c \\ b & -b & 0 & c & c & 2b \end{pmatrix}.$$

Note that the free semialgebraic set associated with  $Q'_G$  as a matrix in variables  $a, b, c$  is also convex. By Proposition 6.7 we can use  $Q'_G$  to describe  $G$ -invariant positivity on the disk.

(3) (Bidisk) If  $Q = \text{diag}(1 - x^2, 1 - y^2)$  then

$$Q_G = \frac{1}{2} \begin{pmatrix} 2 - \frac{1}{2}(a^2 + b) & 0 & -\frac{1}{2}(c + ab) & 0 \\ 0 & 2 - \frac{1}{2}(a^2 + b) & 0 & \frac{1}{2}(c + ab) \\ -\frac{1}{2}(c + ba) & 0 & 2b - \frac{1}{2}(cb^{-1}c + b^2) & 0 \\ 0 & \frac{1}{2}(c + ba) & 0 & 2b - \frac{1}{2}(cb^{-1}c + b^2) \end{pmatrix}.$$

Note that  $2Q_G$  is unitarily similar to a direct sum of two copies of

$$S = \begin{pmatrix} 2 - \frac{1}{2}(a^2 + b) & \frac{1}{2}(c + ab) \\ \frac{1}{2}(c + ba) & 2b - \frac{1}{2}(cb^{-1}c + b^2) \end{pmatrix}.$$

Every  $G$ -invariant rational expression strictly positive on the bidisk

$$\{(X, Y) : X^2 \preceq I \text{ \& } Y^2 \preceq I\}$$

induces a rational function in  $\text{QM}_{\mathbb{C}\langle x, y \rangle^G}(S)$  by Proposition 6.6. As in the case of the disk, bidisk can also be represented by a monic linear matrix inequality, which by Proposition 6.7 then gives a description of invariant expressions positive on the bidisk.

(4) (Positive orthant) If  $Q = \text{diag}(x, y)$  then  $2Q_G$  is unitarily similar to a direct sum of two copies of

$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The positive orthant

$$\{(X, Y) : X \succeq 0 \text{ \& } Y \succeq 0\}$$

is a convex semialgebraic set, and after a scalar shift it admits a monic linear matrix representation. Hence rational expressions positive on the orthant induce rational functions in  $\text{QM}_{\mathbb{C}\langle x, y \rangle}(Q)$  by [Pas18, Theorem 3.1]. The  $G$ -invariant rational functions among them then lie to  $\text{QM}_{\mathbb{C}\langle x, y \rangle^G}(S)$  by Corollary 6.5.

**Example 6.9.** Let  $G = \mathbb{Z}_3$  act on  $\mathbb{C}\langle x, y, z \rangle$ . Let  $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$  and

$$q_1 = \omega x + \omega^2 y + z, \quad q_2 = \omega^2 x + \omega y + z.$$

Then

$$R_G = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix} \text{diag}(1, q_2, q_1).$$

For  $Q = 1$  we get  $Q_G = \text{diag}(1, q_1 q_2, q_2 q_1)$ . Therefore all positive semidefinite  $G$ -invariant rational functions in  $\mathbb{C}\langle x, y, z \rangle$  are of the form

$$\sum_j u_j^* u_j + \sum_j v_j q_1 q_2 v_j^* + \sum_j w_j q_2 q_1 w_j^*, \quad u_j, v_j, w_j \in \mathbb{C}\langle x, y, z \rangle^G.$$

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