III. NC CONVEXITY

$\mathbb{S}_\delta := M_\delta(\mathbb{R})_{sa}$

Def: Let $A_0, A_1, \ldots, A_d \in \mathbb{S}_\delta$

$L(x) := A_0 + A_1 x_1 + \cdots + A_d x_d$

is a linear pencil.

If $A_0 = I$, then L is monic

Inequality $L(x) \geq 0$ is a linear matrix ineq (LMI).

Its solution set

$$\mathcal{D}_L(1) = \{x \in \mathbb{R}^d \mid L(x) \geq 0\}$$

is a spectrahedron or LMI domain.

Now let $X = (X_1 \ldots X_d) \in \mathbb{S}_m^d$. Then

$$L(X) := A_0 \otimes I_m + \sum_{j=1}^d A_j \otimes X_j$$

Define $\mathcal{D}_L(m) = \{X \in \mathbb{S}_m^d \mid L(X) \geq 0\}$

$\mathcal{D}_L = \bigcup_{m \in \mathbb{N}} \mathcal{D}_L(m)$ ... free spectrahedron

Properties: (1) $\mathcal{D}_L(m)$ is convex

$X, Y \in \mathcal{D}_L(m)$

$$L\left(\frac{X+Y}{2}\right) = \frac{1}{2} L(X) + \frac{1}{2} L(Y) \geq 0$$

(2) $X \in \mathcal{D}_L(n)$, $Y \in \mathcal{D}_L(m)$

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{D}_L(n+m)$$

(3) $\overset{(X_1 \ldots X_d)}{X} \in \mathcal{D}_L(m)$, V $m \times n$ isometry ($V^* V = I_n$)

Then $\underset{\substack{\| \\ (V^* X_1 V, \ldots, V^* X_d V)}}{V^* X V} \in \mathcal{D}_L(n)$

$$L(V^* X V) = A_0 \otimes I + \sum A_j \otimes V^* X_j V$$

$$= (I \otimes V)^* \underbrace{\left(A_0 \otimes I + \sum A_j \otimes X_j\right)}_{L(X)} (I \otimes V) \succeq 0$$

- $\mathcal{D}_L(1)=\emptyset \iff \mathcal{D}_L=\emptyset$
- if $\operatorname{int}\mathcal{D}_L(1)=\emptyset$, then $\mathcal{D}_L(1)\subseteq\{\sum a_i x_i=b\}$

Solve for some x_i's & repeat if needed.

So "wlog" $\operatorname{int}\mathcal{D}_L(1)\neq\emptyset$.

Translate so that $0\in\operatorname{int}\mathcal{D}_L(1)$.

Then $\exists$ monic $\tilde{L}$ st. $\mathcal{D}_L=\mathcal{D}_{\tilde{L}}$.

Proof. $L(0)=A_0\succeq 0$. Since $0\in\operatorname{int}\mathcal{D}_L(1)$, $\exists\,\varepsilon>0$ st. $A_0\succeq\pm\varepsilon A_j$ $\forall j$.

$V:=\operatorname{range}A_0\subseteq\mathbb{R}^\delta$, $\tilde{A}_0:=A_0|_V\ V\to V$ is invertible, so pos. def.

Puzzle of the day #3

Toss biased coin prob (head) $=2/3$, 300 times. Which is more likely:

(a) Getting ≥ 200 heads

(b) Getting <200 heads?

Claim: range $A_j\subseteq V$.

Suppose $x\perp V$, that is, $A_0x=0$

Then $0=x^*A_0x\geq\pm\varepsilon\, x^*A_jx$

So $x^*A_jx=0$.

From $A_0+\varepsilon A_j\succeq 0$ & $x^*(A_0+\varepsilon A_j)x=0$

we deduce $(A_0+\varepsilon A_j)x=0$, whence $A_jx=0$

Let $\tilde{A}_j:=A_j|_V: V\to V$ and then $\tilde{L}(x):=\tilde{A}_0+\sum\tilde{A}_jx_j$ has $\mathcal{D}_L=\mathcal{D}_{\tilde{L}}$.

The pencil $\hat{L}(x):=\tilde{A}_0^{-1/2}\,\tilde{L}(x)\,\tilde{A}_0^{-1/2}$ is the desired monic pencil. $\square$

III. NC CONVEXITY

$$\mathbb{S}_n^d \\ \cup\!\!\mid$$

Def: $\mathcal{D} = (\mathcal{D}(n))_{n\in\mathbb{N}}$ is <u>matrix convex</u> if

(a) closed under $\oplus$: $X \in \mathcal{D}(n),\ Y \in \mathcal{D}(m) \Rightarrow X \oplus Y \in \mathcal{D}(n+m)$

(b) closed under isometric conjugation:

$X \in \mathcal{D}(m)$, $V_{m\times n}$ isometry $\Rightarrow V^* X V \in \mathcal{D}(n)$

Rmk: (1) Γ is matrix cvx $\Rightarrow \forall n\ \ \Gamma(n)$ is convex

Given $s,t \in \mathbb{R}$ w/ $s^2+t^2=1$, $X, Y \in \Gamma(n)$,

let $V = \begin{pmatrix} sI_n \\ tI_n \end{pmatrix}$, then

$$V^* \underbrace{\begin{pmatrix} X & \\ & Y \end{pmatrix}}_{\in \Gamma(2n)} V = s^2 X + t^2 Y \in \Gamma(n)$$

(1)' If $A^1, \dots, A^\ell \in \Gamma$ and $V_1, \dots, V_\ell$ are s.t. $\sum V_i^* V_i = I$, then

$V = \begin{pmatrix} V_1 \\ \vdots \\ V_\ell \end{pmatrix}$ is an isometry & $V^*(\oplus A^j)V = \sum V_j^* A^j V_j \in \Gamma$

mtx cvx combination

(2) $0 \in \Gamma$ & Γ is matrix convex, then Γ is closed under contractive conjug

if V is a contraction ($V^*V \leq I$) & $X \in \Gamma$, then $V^*XV \in \Gamma$.

$V \rightsquigarrow W = \begin{bmatrix} V \\ (1-V^*V)^{1/2} \end{bmatrix}$ is an isometry

$$\Gamma \ni W^* (\underbrace{X \oplus 0}_{\in \Gamma}) W = V^* X V$$

For $B \in \Gamma(k)$, $V \in \mathbb{R}^{k\times\delta}$ define $f_{B,V}: \mathbb{S}_\delta \to \mathbb{R}$

$$T \mapsto \operatorname{tr}(VTV^*) - \varphi(V^*BV).$$

Then $\hat{\mathcal{F}} := \{f_{B,V} \mid B\in\Gamma(k),\ V\in\mathbb{R}^{k\times\delta} \text{ contraction}\}$

is convex & $\forall \underset{\substack{\shortparallel \\ f_{B,V}}}{f} \in\hat{\mathcal{F}}$ $\exists T\in\mathcal{T}_\delta$: $f(T)\succeq 0$

Proof: Let w be unit vec w/ $\|Vw\| = \|V\|$,

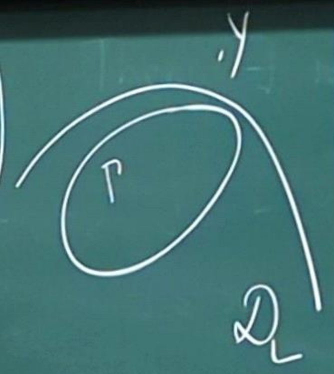
$$T = ww^* \in \mathcal{T}_\delta. \quad f_{B,V}(T) = \operatorname{tr}(Vww^*V^*) - \varphi(V^*BV)$$

$$\overset{\operatorname{tr}(Vw\cdot(Vw)^*)}{=} \|V\|^2 - \varphi(V^*BV) = \|V\|^2\left(1 - \varphi\Big(\underbrace{\frac{V^*}{\|V\|}\cdot B\cdot\frac{V}{\|V\|}}_{\in\Gamma}\Big)\right) \succeq 0$$

$B^j\in\Gamma$, $\lambda_j\in[0,1]$ w/ $\sum\lambda_j = 1$, V_j contractions

$B := \oplus B^j$ $\quad V = \begin{pmatrix}\sqrt{\lambda_1}V_1\\ \vdots\end{pmatrix}$ $\quad$ Then $f_{B,V} = \sum \lambda_j f_{B^j,V_j}$ □

Theorem (nc Hahn-Banach separation theorem) (Effros-Winkler 97)



Suppose Γ is matrix convex, $0\in\Gamma$, and assume Γ is closed. [each $\Gamma(n)$ closed]

Suppose $Y\notin\Gamma$. Then $\exists$ monic linear pencil L of the same size as Y, so that $L|_\Gamma \succeq 0$ & $L(Y)\not\succeq 0$.

Notation: $\mathcal{T}_\delta := \{T\in\mathbb{S}_\delta \mid T\succeq 0,\ \operatorname{tr}T = 1\}$

Lemma: Suppose Y is $\delta\times\delta$, and $\exists\ \varphi: \mathbb{S}_\delta^d \to \mathbb{R}$ s.t. $\varphi(Y) > 1$ & $\varphi(B)\le 1$ $\forall B\in\Gamma(\delta)$.

Puzzle of the day #3

Toss biased coin prob (head) $= 2/3$, 300 times. Which is more likely:

(a) Getting ≥ 200 heads

(b) Getting < 200 heads?

Lemma: $\exists T\in \mathcal{T}_\delta \ \forall f\in \mathcal{F} \quad f(T)\geq 0.$

Proof: $\mathcal{T}_\delta$ is compact, so for $f\in\mathcal{F}$

$\{T\in \mathcal{T}_\delta \mid f(T)\geq 0\}$ is also cpt.

It is enough to show that $\bigcap_{j=1}^{m} \{T\in \mathcal{T}_\delta \mid f_j(T)\geq 0\}\neq\emptyset$

for $f_1,\dots,f_m\in\mathcal{F}$.

Define $F:\mathcal{T}_\delta \longrightarrow \mathbb{R}^m$, $\quad T\longmapsto (f_1(T),\dots,f_m(T))$

Then $F(\mathcal{T}_\delta)$ is cpt, cvx. Claim is $F(\mathcal{T}_\delta)\cap[0,\infty)^m\neq\emptyset$.

Assume this intersection is $\emptyset$. Use H-B to separate.

$\exists$ affine linear $h=\sum_1^m h_j x_j + h_0$ s.t.

$h([0,\infty)^m)\subseteq[0,\infty)$ & $h(F(\mathcal{T}_\delta))\subseteq(-\infty,t]$ for some $t<0$

$\forall \lambda>0 \ \forall k\in\{1 \dots m\}$:

$h(\lambda e_k) = h_0+\lambda h_k \geq 0$

$\Rightarrow h_k\geq 0$, wlog $\sum_1^m h_j = 1$.

$h_0 = h(0)\geq 0$

Set $f := \sum h_j f_j \in \mathcal{F}$.

For every $T\in\mathcal{T}_\delta$ we have

$$f(T) = \sum h_j f_j(T) = h(F(T)) - h_0 \leq h(F(T)) \leq t < 0$$

This contradicts previous lemma.

Proof: Let T be as in prev. lemma

By Riesz repr. thm $\exists\, H_1 \ldots H_d \in \mathbb{S}_\delta$

s.t. $\varphi(C_1 \ldots C_d) = \sum \operatorname{tr}(H_j C_j)$

Let $B \in \mathbb{S}_k^d$, $v = \sum_{\alpha=1}^{\delta} e_\alpha \otimes v_\alpha$

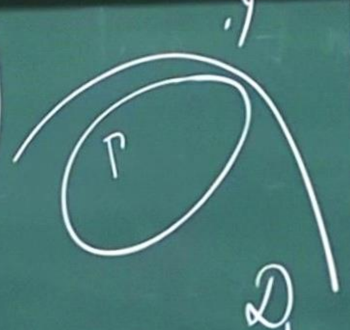
Let $V = (v_1 \ldots v_\delta) \in \mathbb{R}^{k \times \delta}$

Then
$$v^* L(B) v = v^*(T \otimes I)v - \sum_j v^*(H_j \otimes B_j) v$$
$$= \sum_{\alpha,\beta} \langle e_\alpha, T e_\beta\rangle \cdot \langle v_\alpha, I v_\beta\rangle - \sum_{j,\alpha,\beta} \langle e_\alpha, H_j e_\beta\rangle \cdot \langle v_\alpha, B_j v_\beta\rangle$$
$$= \sum_{\alpha,\beta} T_{\alpha,\beta} (V^*V)_{\alpha,\beta} - \sum_j \sum_{\alpha,\beta} (H_j)_{\alpha,\beta} \cdot (V^* B_j V)_{\alpha,\beta}$$
$$= \operatorname{tr}(T^* V^* V) - \sum_j \underbrace{\operatorname{tr}(H_j^* V^* B_j V)}_{\varphi(V^*BV)} = f_{B,V}(T)$$
$$= \operatorname{tr}(V T V^*) - \varphi(V^* B V)$$

<u>Theorem</u> (nc Hahn-Banach separation theorem) (Effros-Winkler 97)

Suppose Γ is matrix convex, $0 \in \Gamma$, and assume Γ is closed. [each $\Gamma(n)$ closed]

Suppose $Y \notin \Gamma$. Then $\exists$ monic linear pencil L of the same size as Y, so that $L|_\Gamma \succeq 0$ & $L(Y) \not\succeq 0$.



<u>Corollary</u> (Keep notation from last 2 lemmas):

$\exists\, T \in \mathcal{T}_\delta$ & $H_j \in \mathbb{S}_\delta$ s.t.

$L := T - \sum H_j x_j$ satisfies

$L|_\Gamma \succeq 0$ & $L(Y) \not\succeq 0$.

<u>Puzzle of the day</u> #3

Toss biased coin prob (head) $= 2/3$, 300 times. Which is more likely:

(a) Getting ≥ 200 heads

(b) Getting < 200 heads