

NONCOMMUTATIVE NC

REAL ALGEBRAIC GEOMETRY & ANALYSIS RAG

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I. POSITIVE (NC) POLYNOMIALS

II. NC NULLSTELLEN SÄTZE

III. NC CONVEXITY

IV. NC FUNCTION THEORY

V. NONLINEAR COMPLETELY POSITIVE MAPS

NC RAG \sim Study of positivity in general
(not necc. commutative) algebras.

E.g. • $\mathbb{R}[t_1 \dots t_d]$ = commutative polynomial rings

$M_n(\mathbb{R}[t])$

III

• $\mathbb{R}\langle x_1 \dots x_d \rangle$ = free algebra

$\mathbb{R}\langle x \rangle \otimes \mathbb{R}\langle y \rangle$ = "bifree" algebra

• $\mathbb{R}[G] =$ group algebra, such as $\mathbb{R}\{\mathbb{F}_d\}$ or $\mathbb{R}\{\mathbb{F}_d \times \mathbb{F}_d\}$
 \downarrow free group

• $A_1(\mathbb{R}) = \frac{\mathbb{R}\langle x, y \rangle}{(xy - yx - 1)}$ Weyl algebra

• $\mathbb{R}\langle X \rangle$ = free skew field = nc rational functions

• $\mathbb{R}\text{hg. of generic matrices } (M_n(\mathbb{R}), \text{ trace ring } T_n)$

I. POSITIVE NC POLYS

$d \in \mathbb{N}$, $x = (x_1, \dots, x_d)$ freely nc variables

$\langle x \rangle$ = words in x = free monoid on x

$\emptyset = 1$ empty word $\in \langle x \rangle$

$\langle x \rangle_\delta$ = all words of length-degree at most δ

$\langle x \rangle_{\leq \delta}$ = \dots exactly δ

* involution on $\langle x \rangle$ that reverses words:

$$(x_{i_1} \cdots x_{i_r})^* = x_{i_r} \cdots x_{i_1}$$

In particular $\underline{x_j^*} = x_j$ (all j).

Puzzle of the Day

If $A = A^T$, $B = B^T$ have norm ≤ 1 , then

$$2 \cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - B^2A \geq 0$$

Def: The free algebra $\mathbb{R}\langle x \rangle$ is the set of all linear combination of words.

* extends to $\mathbb{R}\langle x \rangle$ by linearity.

$$\mathbb{R}\langle x \rangle_\delta = \text{span } \langle x \rangle_\delta = \text{nc polys of degree} \leq \delta$$

If $X \in M_n(\mathbb{R})_{sa}^d$, then the evaluation $p \mapsto p(X)$
is a *-representation $\mathbb{R}\langle x \rangle \rightarrow M_n(\mathbb{R})$

Remark: $M \succeq 0$ means M is positive semidefinite (PSD)
 $M \succ 0$ — — — positive definite (PD)

$$\text{Ex: } p = x_1^2 x_2 + x_2 x_1 x_2 - 3 x_2 x_1^2 + 4 \in \mathbb{R}\langle x_1, x_2 \rangle$$

Given $X_1, X_2 \in M_n(\mathbb{R})$, we define

$$p(X_1, X_2) = X_1^2 X_2 + X_2 X_1 X_2 - 3 X_2 X_1^2 + 4 \cdot I_n$$

This gives a representation $\mathbb{R}\langle x \rangle \rightarrow M_n(\mathbb{R})$

Observation. Suppose $f = g^*g$ for some $g \in \mathbb{R}\langle x \rangle$.

Then $\forall X \in (\mathbb{M}_n)_{sa}^d$ we have $f(X) \succeq 0$

$$\text{Indeed, } f(X) = (g^*g)(X) = g^*(X) \cdot g(X) = g(X)^* \cdot g(X) \succeq 0$$

Theorem (Helton²⁰⁰², McCullough²⁰⁰¹):
Let $N(\delta) = \dim \mathbb{R}\langle x \rangle_\delta = 1 + d + d^2 + \dots + d^\delta$

Suppose $p \in \mathbb{R}\langle x \rangle_\delta$ and

$$p(X) \succeq 0 \text{ for all } N(\delta) \times N(\delta) \text{ s.a. } X$$

Then $\exists r_1, \dots, r_{N(\delta)} \in \mathbb{R}\langle x \rangle_\delta$ s.t.

1st Proof of the theorem



Proposition (Carathéodory convex hull thm)

If $p \in \mathbb{R}\langle x \rangle_\delta$ is a SOS (=sum of squares)

then $\exists r_1, \dots, r_{N(\delta)} \in \mathbb{R}\langle x \rangle_\delta$ s.t. $p = \sum r_j^* r_j$

Proof: $V = \overrightarrow{\mathbb{R}\langle x \rangle_\delta}$ (Veronese vector)

Suppose $p = \sum q_j^* q_j$ for $q_j \in \mathbb{R}\langle x \rangle_\delta$

Write $q_j = Q_j^* V$ for a column vector Q_j

$$\text{Then } p = \sum q_j^* q_j = \left(\begin{matrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1 x_2 \\ x_2 & x_1 x_2 & x_2^2 \end{matrix} \right)^* \left(\begin{matrix} 1 \\ x_1 \\ x_2 \end{matrix} \right) \cdot \left(\begin{matrix} 1 \\ x_1 \\ x_2 \end{matrix} \right)$$

$$= \sum_j V^* Q_j Q_j^* V$$

$$= V^* \left(\underbrace{\sum Q_j Q_j^*}_{=: Q} \right) V$$

Lemma (Gelfand-Naimark-Segal construction)

Suppose $\lambda : (\mathbb{R}\langle x \rangle_{2\delta+2})_{sa} \rightarrow \mathbb{R}$ is a linear

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If $A = A^T, B = B^T$ have
norm ≤ 1 , then
$$2 \cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - B^2A \geq 0$$

$Q \succeq 0$ is a $N(\delta) \times N(\delta)$ matrix

$$\text{Then } \exists R_1, \dots, R_{N(\delta)} \text{ s.t. } Q = \sum R_j R_j^*$$

$$\text{Setting } r_j = R_j^* \text{ yields } p = \sum r_j^* r_j.$$

functional that satisfies $\lambda(q^* q) > 0 \quad \forall q \in \mathbb{R}\langle x \rangle_{\delta+1} \setminus \{0\}$.

Then \exists Hilbert space \mathcal{H} of $\dim N(\delta)$, a $\gamma \in \mathcal{H}$, $X \in B(\mathcal{H})_{sa}$
s.t. $\lambda(q^* p) = \langle p(x)\gamma, q(x)\gamma \rangle \quad \forall p, q \in \mathbb{R}\langle x \rangle_\delta$.

Proof: Define inner product

$$\langle p, q \rangle := \lambda(q^* p) \text{ on } \mathbb{R}\langle x \rangle_{\delta+1} =: \mathcal{K}$$

\mathcal{K} = subspace $\mathbb{R}\langle x \rangle_\delta$ in \mathcal{H} .

Let $P : \mathcal{K} \rightarrow \mathcal{K}$ be the orthogonal projection

$$S_j : \mathcal{H} \rightarrow \mathcal{K} \quad P \mapsto x_j P$$

Extend λ to $\lambda : \mathbb{R}\langle x \rangle_{2\delta+2} \rightarrow \mathbb{R}$ by $p \mapsto \frac{1}{2} \lambda(p + p^*)$

$$\text{Let } X_j := PS_jP : \mathcal{H} \rightarrow \mathcal{H}$$

X_j is self-adjoint: $p, q \in \mathcal{K}$

$$\begin{aligned} \langle X_j p, q \rangle &= \langle PS_j P p, q \rangle = \langle S_j P p, P q \rangle = \langle S_j P, q \rangle \\ &= \langle x_j p, q \rangle = \lambda(q^* x_j p) = \lambda((x_j q)^* p) = \langle p, x_j q \rangle = \langle p, X_j q \rangle. \end{aligned}$$

Note $p(X) \geq 0$ for $p \in \mathcal{H} = \mathbb{R}\langle x \rangle_{\leq d}$

Hence for $p, q \in \mathcal{H}$, we have $\langle p(x), q(x) \rangle \geq 0$

$$\langle p(x)q(x), q(x) \rangle = \lambda(q^*p).$$

Corollary: $\exists X \in \mathbb{B}(\mathbb{R}^{N(d)})^d$ s.t. for $p \in \mathbb{R}\langle x \rangle_{\leq d}$, $p(X) = 0 \Rightarrow p = 0$.

1st Proof of the theorem

Lemma: \exists linear functional $\mu : \mathbb{R}\langle x \rangle_{\leq d} \rightarrow \mathbb{R}$ s.t. $\mu(p^*p) > 0 \quad \forall 0 \neq p \in \mathbb{R}\langle x \rangle_{\leq d}$. (= sum of squares)

Proof: Induct on d . Suppose $\mu = \mu_{\leq d} : \mathbb{R}\langle x \rangle_{\leq d} \rightarrow \mathbb{R}$ is strictly

positive. Define an extension $\mu_{\leq d+2}$ of $\mu_{\leq d}$ to $\mathbb{R}\langle x \rangle_{\leq d+2} \rightarrow \mathbb{R}$, as follows.

Theorem (Helton²⁰⁰², McCullough²⁰⁰¹):
Let $N(d) = \dim \mathbb{R}\langle x \rangle_d = 1 + d + d^2 + \dots + d^d$
Suppose $p \in \mathbb{R}\langle x \rangle_d$ and $p(x) \geq 0$ for all $x \in N(d)$ s.a. X
Then $\exists r_1, \dots, r_{N(d)} \in \mathbb{R}\langle x \rangle_d$ s.t. $p = \sum r_i^* r_i$.

If $\deg v = 2d+1$, set $\mu_{2d+2}(v) = 0$
If $\deg v = 2d+2$ and $v \neq u^* u$, also $\mu_{2d+2}(v) = 0$.
If suppose $p = \sum r_i^* r_i$, $v = u^* u$, set $\mu_{2d+2}(v) = c$.

Since $\langle p, q \rangle = \mu_{2d}(q^* p)$ is strictly positive,

$\exists c > 1$ s.t. μ_{2d+2} is also strictly positive.

$$\begin{pmatrix} r_{2d} & * \\ * & I_0 \end{pmatrix}$$

Proof(Cor): Apply GNS to μ_{2d+2} from last lemma. \square

Proposition: The cone of SOS of degree at most $\sum \delta_j^2$ is closed in $\mathbb{R}(X)_{\leq \sum \delta_j^2}$.

Proof: Take X from Corollary (for $\sum \delta_j^2$),

$\|p\| := \|p(X)\|$ is a norm on $\mathbb{R}(X)_{\leq \sum \delta_j^2}$

Operator

$\exists r_{j,n} \quad (j \leq N(\sum \delta_j^2))$ s.t. $p_n = \sum_j r_{j,n}^* r_{j,n}$ Then $p_n(X) = \sum_j r_{j,n}(X)^* r_{j,n}(X) \rightarrow \sum_j r_j(X)^* r_j(X)$,

Then $p_n(X) = \sum_j r_{j,n}(X)^* r_{j,n}(X)$, so $p = \sum_j r_j^* r_j \in \sum \delta_j^2$.
 $(r_{j,n}(X))_n$ is bounded.

Passing to a subsequence we may assume

$$r_{j,n} \rightarrow r_j \in \mathbb{R}(X)_{\sum \delta_j^2}$$

Suppose $(p_n)_n$ is a sequence in $\sum \delta_j^2$ that converges to some $p \in \mathbb{R}(X)_{\leq \sum \delta_j^2}$.

Thus $p_n(X) \rightarrow p(X)$.

So $(p_n(X))_n$ is bounded.

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 If $A = A^T$, $B = B^T$ have norm ≤ 1 , then
 $2 \cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - BA^2 \geq 0$

Proof of the theorem: Suppose $p \geq 0$ or $N(\delta) \times N(\delta)$ s.a. tuples,

and $p \notin \sum_{i=\delta+1}^{\infty}$. By Hahn-Banach separation,

$$\exists \lambda: (\mathbb{R}(x)_{\leq \delta+2})^* \rightarrow \mathbb{R} \quad \forall c \in \mathbb{R} \text{ s.t.}$$

$$\lambda(p) < c \leq \lambda(q) \quad \forall q \in \sum_{i=\delta+1}^{\infty}$$

Since $\sum_{i=\delta+1}^{\infty}$ is a cone, $k_i \geq 0$, $c=0$.

Extend λ to $\lambda: \mathbb{R}(x)_{\leq \delta+2} \rightarrow \mathbb{R}$ by $p \mapsto \frac{1}{2}\lambda(p+p^*)$

Pick a strictly positive $\mu: \mathbb{R}(x)_{\leq \delta+2} \rightarrow \mathbb{R}$ (lemma)

For $\varepsilon > 0$ small enough, $(\lambda + \varepsilon \cdot \mu)(p) < 0$.

Also, $\lambda + \varepsilon \cdot \mu$ is strictly positive on $\sum_{i=\delta+1}^{\infty}$

Apply GNS: get X of size $N(\delta)$ s.t.

$$\langle (\lambda + \varepsilon \cdot \mu)(p), X \rangle = \langle p(X)X, X \rangle \geq 0$$

Contradiction!

Theorem (Helton²⁰⁰², McCullough²⁰⁰¹):

$$\text{Let } N(\delta) = \dim \mathbb{R}(x)_{\leq \delta} = 1 + d + d^2 + \dots + d^{\delta}$$

Suppose $p \in \mathbb{R}(x)_{\leq \delta}$ and

$$p(x) \geq 0 \text{ for all } N(\delta) \times N(\delta) \text{ s.a. } X$$

Then $\exists r_1, \dots, r_{N(\delta)} \in \mathbb{R}(x)_{\leq \delta}$ s.t.

$$p = \sum r_i^* r_i$$

If $\deg v = 2\delta+1$, set $\mu_{2\delta+2}(v) = 0$

If $\deg v = 2\delta+2$ and $v \neq u^* u$, also $\mu_{2\delta+2}(v) = 0$.

If $\deg v = 2\delta+2$ and $v = u^* u$, set $\mu_{2\delta+2}(v) = C$.

Since $\langle p, q \rangle = \mu_{2\delta}(q^* p)$ is strictly positive,

$\exists C > 1$ s.t. $\mu_{2\delta+2}$ is also strictly positive.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proof (Cor): Apply GNS to $\mu_{2\delta+2}$ from last lemma. \square

Q: Given p, q , does $q \succeq 0 \Rightarrow p \succeq 0$?

For instance, if $p = \sum r_j^* r_j + \sum s_k^* q_* s_k$
then $q_* \succeq 0 \Rightarrow p \succeq 0$.

$$\text{say } q(x) \succeq 0 \text{ Then } p(x) = \sum r_i(x)^* r_i(x) + \sum s_k(x)^* \underbrace{q(x)}_{\geq 0} s_k(x) \geq 0$$

$S \subseteq \mathbb{R}\langle x \rangle_{sa}$

The semialgebraic set of S is

$$\mathcal{D}_S = \bigcup_{n \in \mathbb{N}} \left\{ X \in \left(M_n \right)_{sa} \mid \begin{array}{l} s(X) \succeq 0 \\ \forall x \in S \end{array} \right\}$$

$$\mathcal{D}_S^\infty = \left\{ X \in \mathcal{B}(H)^d \mid \begin{array}{l} s(X) \succeq 0 \\ \forall x \in S \end{array} \right\}$$

separable Hilbert space

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If $A = A^T, B = B^T$ have norm ≤ 1 , then
 $2 \cdot I + 2A + 2B - A^2B - BA^2 - AB^2 - BA \succeq 0$

$\bar{QM}(S)$ = quadratic module generated by S is

$$QM(S) = \left\{ \underbrace{\sum_{i=1}^r g_i^* s_i g_i}_{\text{weighted s.o.s.}} \mid r \in \mathbb{N}, g_i \in \mathbb{R}\langle x \rangle, s_i \in S \cup \{I\} \right\}$$

$$QM(S) + QM(S) \subseteq QM(S)$$

$$S \cup \{I\} \subseteq QM(S)$$

Observation: If $p \in QM(S)$, then
 $p|_{\mathcal{D}_S^\infty} \succeq 0$ & $p|_{\mathcal{D}_S} \succeq 0$.

if $\exists N \in \mathbb{N}$ s.t. $N - \sum x_i^2 \in QM(S)$
then \mathcal{D}_S and \mathcal{D}_S^∞ are bounded (in norm)

Extend λ to $\lambda: \mathbb{R}(X)_{2\delta+2} \rightarrow \mathbb{R}$ by $p \mapsto \frac{1}{2}\lambda(p+p^*)$

Pick a strictly positive $\mu: \mathbb{R}(X)_{2\delta+2} \rightarrow \mathbb{R}$ (lemma)

For $\varepsilon > 0$ small enough, $(\lambda + \varepsilon \cdot \mu)(p) < 0$.

Also, $\underline{\lambda + \varepsilon \mu}$ is strictly positive on \sum_{d+1}

Apply GNS: set X of size $N(\delta)$ s.t. $0 > (\lambda + \varepsilon \mu)(p) = \langle p(X)Y, Y \rangle \geq 0$

Contradiction!

Def: We call $QM(S)$ archimedean if $\exists N \in \mathbb{N}$ s.t.,
 $N - \sum x_j^2 \in QM(S)$.

Rmk: If D_S or D_S^∞ are bounded, you can

add $N - \sum x_j^2$ (for $N > 1$)

to S w/o changing D_S or D_S^∞ .

If $\deg v = 2\delta+1$, set $\mu_{2\delta+2}(v) = 0$

If $\deg v = 2\delta+2$ and $v \neq u^* u$, also $\mu_{2\delta+2}(v) = 0$.

If $v = u^* u$, set $\mu_{2\delta+2}(v) = C$.

Since $\langle p, q \rangle = \mu_{2\delta}(q^* p)$ is strictly positive,

$\exists C > 1$ s.t. $\mu_{2\delta+2}$ is also strictly positive. \blacksquare

$$\begin{pmatrix} \mathcal{D}_S & | & * \\ * & \mathcal{D}_S^\infty & * \\ * & * & * \end{pmatrix}$$

Proof(Cor): Apply GNS to $\mu_{2\delta+2}$ from last lemma. \square

Theorem (Helton-McCullough²⁰⁰⁴)

Suppose $QM(S)$ is archimedean.

If $f|_{D_S^\infty} > 0$, then $f \in QM(S)$.

$$\text{Ex: } S = \{x_1 - 1, x_2 - 1, \\ 8 - x_1 x_2 - x_1 x_2\}$$

Then D_S^∞ is bounded, but $QM(S)$ is not archim.

In particular,

$$f|_{D_S^\infty} \geq 0 \Leftrightarrow \forall \varepsilon > 0 \quad f + \varepsilon \in QM(S)$$

Letting $P_1 = -2 - x + x^2 - y + y^2$ and

$P_2 = -x - x^2 + y + y^2$, we have

$$2 + 2x + 2y - x^2 - y^2 - xy^2 - y^2 x = \frac{1}{2} (P_1^* P_1 + P_2^* P_2) + x(1-x^2) + y(1-y^2)y.$$

Puzzle of the Day
If $A = A^T$, $B = B^T$ have
norm ≤ 1 , then
 $2 \cdot I + 2A + 2B - AB - BA - A^2 - B^2 \geq 1$