A MATRIX POSITIVSTELLENSATZ WITH LIFTING POLYNOMIALS

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Abstract. Given the projections of two semialgebraic sets defined by polynomial matrix inequalities, it is in general difficult to determine whether one is contained in the other. To address this issue we propose a new matrix Positivstellensatz that uses lifting polynomials. Under the classical archimedean condition and some convexity assumptions, we prove that such a containment holds if and only if the proposed matrix Positivstellensatz is satisfied. The corresponding certificate can be searched for by solving a semidefinite program. An important application is to certify when a spectrahedrop (i.e., the projection of a spectrahedron) is contained in another one.

1. Introduction

A basic question of fundamental importance in convex geometry and optimization is to determine whether or not containment holds between two given convex sets. The simplest convex sets are polyhedra, defined by a finite set of scalar linear inequalities. Containment problems for polyhedra have been studied extensively and are well understood [FO85, GK94]. Another class of thoroughly studied convex sets are spectrahedra. They arise as feasible sets of semidefinite programs [deK02, WSV00] and are defined by linear matrix inequalities (LMIs). Denote by $\mathcal{S}^k$ the space of all $k \times k$ real symmetric matrices. A tuple $A := (A_0, A_1, \ldots, A_n) \in (\mathcal{S}^k)^{n+1}$ gives rise to the linear pencil

$$A(\mathtt{x}) := A_0 + \mathtt{x}_1 A_1 + \cdots + \mathtt{x}_n A_n,$$

in the variables $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n)$. It determines the *spectrahedron* (i.e., a set that is defined by a linear matrix inequality)

$$S_A := \{x \in \mathbb{R}^n : A(x) \succeq 0\}.$$

(Here, $C \succeq 0$ means the symmetric matrix C is positive semidefinite. Similarly, we use $C \succ 0$ to express that C is positive definite.)

An important special case of the containment question is the matrix cube problem of Ben-Tal & Nemirovski [B-TN02, Nem06]. It asks for the largest hypercube contained in a given spectrahedron. The problem is known to be NP-hard. Numerous problems of robust control, such as Lyapunov stability analysis for uncertain dynamical systems, are special cases of the matrix cube problem. This is also the case for maximizing a positive definite quadratic form over the unit cube, one of the fundamental problems in combinatorial optimization.

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More generally, given another tuple $B := (B_0, B_1, \dots, B_n) \in (\mathcal{S}^t)^{n+1}$, where t might be different from k, one is interested in a certificate for the containment

$$S_A \subseteq S_B. \tag{1.1}$$

Clearly, if there exist matrices V_i $(i = 0, \dots, \ell)$ such that

$$B(\mathtt{x}) = V_0^T V_0 + \sum_{i=1}^{\ell} V_i^T A(\mathtt{x}) V_i, \tag{1.2}$$

then $S_A \subseteq S_B$. If S_A has nonempty interior (this is the case e.g. if $A_0 = I_d$, the $d \times d$ identity matrix), then (1.2) holds if and only if the matricial relaxation of S_A is contained in the matricial relaxation of S_B [HKM12, HKM13]. When $B(\mathtt{x})$ is the normal form of an ellipsoid or polytope, the certificate (1.2) is necessary and sufficient for $S_A \subseteq S_B$, as shown by Kellner, Theobald and Trabandt [KTT13]. More general spectrahedral containment is also addressed by the same authors in [KTT15].

In general, the certificate (1.2) is sufficient but not necessary for ensuring $S_A \subseteq S_B$. A more general certificate than (1.2) is

$$B(\mathtt{x}) = V_0(\mathtt{x})^T V_0(\mathtt{x}) + \sum_{i=1}^{\ell} V_i(\mathtt{x})^T A(\mathtt{x}) V_i(\mathtt{x}), \tag{1.3}$$

for matrix polynomials $V_0(\mathtt{x}), V_1(\mathtt{x}), \dots, V_\ell(\mathtt{x})$. To guarantee (1.3), we typically need that S_A is bounded and $B(\mathtt{x}) \succ 0$ on S_A. The boundedness of S_A is equivalent to archimedeanness of the quadratic module associated to the linear pencil $A(\mathtt{x})$; see [KS13]. Hence if S_A is bounded and $B(x) \succ 0$ on S_A, then $B(x)$ can be expressed as in (1.3). This is a consequence of the classical matrix Positivstellensatz [HN10, KS10, SH06]. It can be used to check containment of spectrahedra [KTT15].

However, in applications, convex sets are often not spectrahedra. A much more general class of convex sets are projections of spectrahedra, which we call *spectrahedrops*. They are sometimes called semidefinitely representable sets or spectrahedral shadows. The Lasserre type moment relaxations [Las09a, Las15, NPS10] produce a nested hierarchy of spectrahedrops approximating and closing down on the (convex hull of a) semialgebraic set. Many convex semialgebraic sets are spectrahedrops [HN09, HN10, Sce11]; however, not all of them are [Sce18].

Consider the linear pencils ($\mathtt{y} := (\mathtt{y}_1, \dots, \mathtt{y}_r)$, $\mathtt{z} := (\mathtt{z}_1, \dots, \mathtt{z}_s)$)

$$\left\{ \begin{array}{rcl} A(\mathtt{x}, \mathtt{y}) & := & A_0 + \mathtt{x}_1 A_1 + \cdots + \mathtt{x}_n A_n + \mathtt{y}_1 A_{n+1} + \cdots + \mathtt{y}_r A_{n+r}, \\ B(\mathtt{x}, \mathtt{z}) & := & B_0 + \mathtt{x}_1 B_1 + \cdots + \mathtt{x}_n B_n + \mathtt{z}_1 B_{n+1} + \cdots + \mathtt{z}_s B_{n+s}, \end{array} \right. \tag{1.4}$$

where A_i, B_i are all real symmetric matrices. They define the spectrahedrops

$$\left\{ \begin{array}{rcl} P_A & := & \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^r,\ A(x, y) \succeq 0\}, \\ P_B & := & \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^s,\ B(x, z) \succeq 0\}. \end{array} \right. \tag{1.5}$$

A natural question is: how can we check the containment

$$P_A \subseteq P_B? \tag{1.6}$$

If $P_A \subseteq P_B$, then for all $x \in P_A$ there exists $z \in \mathbb{R}^s$ such that $B(x, z) \succeq 0$. When there are no lifting variables $\mathtt{y}, \mathtt{z}$, we have $P_A = S_A$ and $P_B = S_B$, so the containment (1.6) simply reduces to (1.1) and can be certified by (1.2) or (1.3). However, when there are lifting variables y, z, (1.2) and (1.3) do not apply, because the ranges of y, z depend on x. While a Positivstellensatz describing polynomials positive on spectrahedrops is given in [GN11] (see also [HKM17]), to the best of the authors' knowledge, the question of a satisfactory certificate for (1.6) is widely open.

Contributions. In this paper, we study how to check the containment between projections of two semialgebraic sets that are given by polynomial matrix inequalities. By Tarski's transfer principle [BCR98], the projection of a semialgebraic set is again semialgebraic. However, it is generally a challenge to find a concrete description for the projection. For computational efficiency we usually need to work directly with the original semialgebraic descriptions, including the extra variables. For this purpose, we propose a new matrix Positivstellensatz that uses *lifting polynomials*, which we call a *lifted matrix Positivstellensatz*.

Denote by $\mathcal{S}\mathbb{R}[\mathtt{x}, \mathtt{y}]^{k\times k}$ the space of all real $k \times k$ symmetric matrix polynomials in $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n)$ and $\mathtt{y} := (\mathtt{y}_1, \ldots, \mathtt{y}_r)$. The space $\mathcal{S}\mathbb{R}[\mathtt{x}, \mathtt{z}]^{t\times t}$ is similarly defined, with $\mathtt{z} := (\mathtt{z}_1, \ldots, \mathtt{z}_s)$ and an integer $t > 0$. For $G(\mathtt{x}, \mathtt{y}) \in \mathcal{S}\mathbb{R}[\mathtt{x}, \mathtt{y}]^{k\times k}$ and $Q(\mathtt{x}, \mathtt{z}) \in \mathcal{S}\mathbb{R}[\mathtt{x}, \mathtt{z}]^{t\times t}$, consider the projections of semialgebraic sets defined by them,

$$\begin{array}{rcl} P_G & := & \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^r,\, G(x, y) \succeq 0\}, \\ P_Q & := & \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^s,\, Q(x, z) \succeq 0\}. \end{array}$$

We are interested in a certificate for the containment

$$P_G \subseteq P_Q. \tag{1.7}$$

This task is typically very hard. For a given x, checking the existence of z satisfying $Q(x, z) \succeq 0$ is already very difficult, as it amounts to verifying whether a polynomial matrix inequality has a real solution or not. However, we can easily see that $P_G \subseteq P_Q$ if there exist polynomials $p_1(\mathtt{x}), \ldots, p_s(\mathtt{x}) \in \mathbb{R}[\mathtt{x}]$ such that

$$\left\{ \begin{array}{c} Q(\mathtt{x}, \underbrace{(p_1(\mathtt{x}), \ldots, p_s(\mathtt{x}))}_{\mathtt{z}}) = \\ V_0(\mathtt{x}, \mathtt{y})^T V_0(\mathtt{x}, \mathtt{y}) + \sum_{i=1}^{\ell} V_i(\mathtt{x}, \mathtt{y})^T G(\mathtt{x}, \mathtt{y}) V_i(\mathtt{x}, \mathtt{y}), \end{array} \right. \tag{1.8}$$

for certain matrix polynomials $V_0(\mathtt{x}, \mathtt{y}), \ldots, V_\ell(\mathtt{x}, \mathtt{y})$. This is because for every x, if there exists y such that $G(x, y) \succeq 0$ (i.e., $x \in P_G$), then $Q(x, z) \succeq 0$ for $z = (p_1(x), \ldots, p_s(x))$ (i.e., $x \in P_Q$). The representation (1.8) gives a certificate for $P_G \subseteq P_Q$. When $Q(\mathtt{x}, \mathtt{z})$ does not depend on $\mathtt{z}$, (1.8) is reduced to the classical matrix Positivstellensatz [SH06, KS10]. We call each p_i a *lifting polynomial* and call (1.8) a *lifted matrix Positivstellensatz certificate.*

When do there exist polynomials $p_1, \dots, p_s \in \mathbb{R}[\mathtt{x}]$ satisfying (1.8)? Is (1.8) also necessary for $P_G \subseteq P_Q$? If they do exist, how can one compute $p_i(\mathtt{x})$ and $V_i(\mathtt{x}, \mathtt{y})$ satisfying (1.8)? In this paper, we assume that the quadratic module generated by $G(\mathtt{x}, \mathtt{y})$ is archimedean, which is almost equivalent to the compactness of the semialgebraic set $S_G := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^r \colon G(x, y) \succeq 0\}$ and implies the compactness of the projection P_G. The archimedeanness is typically required in a Positivstellensatz [Mar08]. Our major results are:

(I) When $Q(\mathtt{x}, \mathtt{z})$ is linear in $\mathtt{z}$, we show that (1.8) is also a necessary certificate for $P_G \subseteq P_Q$, under the following natural condition: for each $x \in P_G$ there exists z such that $Q(x, z) \succ 0$. The condition essentially means that $P_G \subseteq \operatorname{int}(P_Q)$, the interior of P_Q. Such a condition is generally required. For instance, when $Q(\mathtt{x}, \mathtt{z})$ does not depend on $\mathtt{z}$, strict positivity of $Q(\mathtt{x})$ on P_G is required in the classical matrix Positivstellensatz. The certificate (1.8) can be searched for by solving a semidefinite program, once the degrees for p_i, V_j are fixed. This result is given in Theorem 3.1 in Subsection 3.1.

(II) When $Q(\mathtt{x}, \mathtt{z})$ is nonlinear in $\mathtt{z}$, checking $P_G \subseteq P_Q$ becomes more difficult. In this case, (1.8) gives nonlinear equations for the coefficients of the unknown polynomials $p_i(\mathtt{x})$, i.e., (1.8) is not a convex condition on the $(p_1, \dots, p_m)$. Hence, (1.8) cannot be checked by solving a semidefinite program. This is unsurprising, because for a given x, checking the existence of a z satisfying $Q(x, z) \succeq 0$ is already a difficult problem.

In computation, one often prefers a Positivstellensatz certificate that can be checked by solving a semidefinite program. We show that this is possible when for each fixed $x \in P_G$, the matrix polynomial $Q(x, \mathtt{z})$ is sos-concave in $\mathtt{z}$. Indeed, under the sos-concavity condition, we prove a new lifted matrix Positivstellensatz in Theorem 3.3 in Subsection 3.2: (1.8) is equivalent to a different Positivstellensatz certificate using lifting polynomials, which can again be searched for by solving a semidefinite program.

A key step in the proofs of the above theorems is the existence of a continuous lifting map $P_G \to S_Q$, where some type of convexity assumption is essential, see Example 3.4. When $Q(\mathtt{x}, \mathtt{z})$ is not convex in $\mathtt{z}$, the lifting polynomials might not exist. Hence Theorems 3.1 and 3.3 do not extend to the non-convex case.

(III) The above lifted matrix Positivstellensätze can be applied to check containment between two spectrahedrops. Let P_A, P_B be two spectrahedrops as in (1.5). A certificate for the containment $P_A \subseteq P_B$ is the representation

$$
\left\{
\begin{array}{c}
B(\mathtt{x}, \underbrace{(p_1(\mathtt{x}), \dots, p_s(\mathtt{x}))}_{\mathtt{z}}) = \\
V_0(\mathtt{x}, \mathtt{y})^T V_0(\mathtt{x}, \mathtt{y}) + \sum_{i=1}^{\ell} V_i(\mathtt{x}, \mathtt{y})^T A(\mathtt{x}, \mathtt{y}) V_i(\mathtt{x}, \mathtt{y})
\end{array}
\right.
\tag{1.9}
$$

where $p_1, \dots, p_s$ are scalar polynomials in $\mathtt{x}$ and $V_0, \dots, V_\ell$ are matrix polynomials in $(\mathtt{x}, \mathtt{y})$. In Section 4, we show in Theorem 4.1 that (1.9) is also a necessary certificate for $P_A \subseteq P_B$, under weaker assumptions than in (I). Indeed, the archimedeanness of the quadratic module of $A(\mathtt{x}, \mathtt{y})$ can be weakened to the archimedeanness of its intersection with the ring $\mathbb{R}[\mathtt{x}]^{t\times t}$.

The paper is organized as follows. Section 2 gives preliminaries about matrix polynomials and their quadratic modules. Section 3 presents two lifted matrix Positivstellensätze, gives their proofs and several examples. Section 4 shows how to apply the lifted matrix Positivstellensatz to check containment of spectrahedrops. In Section 5 we apply our results to solve the matrix cube problem and to find maximum inscribing ellipsoids for spectrahedrops. With the help of Lasserre relaxations this leads to an approximation scheme for each of the two problems for general convex semialgebraic sets. Finally, Section 6 gives conclusions and discusses some open questions.

2. Preliminaries

This section reviews some preliminary results about matrix polynomials and the classical matrix Positivstellensatz.

2.1. **Notation.** *Matrix polynomials* are elements of the ring $\mathbb{R}[\mathtt{x}]^{k\times k}$ where $\mathbb{R}[\mathtt{x}]$ is the ring of polynomials in $\mathtt{x} := (\mathtt{x}_1, \dots, \mathtt{x}_n)$ with coefficients from $\mathbb{R}$. The space of all $k \times k$ real symmetric matrix polynomials is denoted as $\mathcal{S}\mathbb{R}[\mathtt{x}]^{k\times k}$. Let I_k denote the $k \times k$ identity matrix. A subset $M \subseteq \mathcal{S}\mathbb{R}[\mathtt{x}]^{k\times k}$ is called a *quadratic module* if

$$I_k \in M, \quad M + M \subseteq M \quad \text{and} \quad a^T M a \subseteq M \ \text{ for all } \ a \in \mathbb{R}[\mathtt{x}]^{k\times k}.$$

Here, the superscript ^{T} denotes the transpose of a matrix. For a finite set $\Gamma \subseteq \mathcal{S}\mathbb{R}[\mathtt{x}]^{k\times k}$, define the semialgebraic set

$$S_\Gamma := \{x \in \mathbb{R}^n \colon \forall g \in \Gamma,\, g(x) \succeq 0\}.$$

The set Γ generates the following quadratic module in $\mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$,

$$\mathrm{QM}_t(\Gamma) := \left\{ \sum_{i=1}^{L} p_i^T g_i p_i \;\middle|\; \begin{array}{l} g_i \in \{I_k\} \cup \Gamma, \\ L \in \mathbb{N},\, p_i \in \mathbb{R}[\mathtt{x}]^{k\times t} \end{array} \right\}. \tag{2.1}$$

In particular, when Γ is empty, $\mathrm{QM}_t(\emptyset)$ is the set of all sums of hermitian squares in $\mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$, i.e., the *sos matrix polynomials*. Given a matrix polynomial $f \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ and $S \subseteq \mathbb{R}^n$, we write $f \succeq 0$ on S if for all $x \in S$, $f(x) \succeq 0$ (i.e., $f(x)$ is positive semidefinite). Similarly, by writing $f \succ 0$ on S we mean that $f(x) \succ 0$, i.e., $f(x)$ is positive definite for all $x \in S$. Clearly, if $f \in \mathrm{QM}_t(\Gamma)$ then $f \succeq 0$ on S_Γ. Note that the finite set Γ can be replaced by a block-diagonal matrix polynomial. Thus there is no harm in assuming that $\Gamma = \{G\}$. In this case we shall write simply S_G and $\mathrm{QM}_t(G)$ for the semialgebraic set and quadratic module generated by S, respectively.

In a Positivstellensatz, we usually deal with the case that S_G is compact. In fact, we often need a slightly stronger assumption that the quadratic module $\mathrm{QM}_t(G)$ is archimedean. Here, a quadratic module M of $\mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ is said to be *archimedean* if there exists $f \in M$ such that the set S_f is compact. When S_G is bounded, the archimedeanness can be enforced by possibly enlarging G without changing S_G.

2.2. **Matrix Positivstellensatz.** For a matrix polynomial $G \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{k\times k}$, if $f \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ and $f \succeq 0$ on S_G, we might not have $f \in \mathrm{QM}_t(G)$. To guarantee $f \in \mathrm{QM}_t(G)$, we typically need that $\mathrm{QM}_t(G)$ is archimedean (and thus S_G compact) and $f \succ 0$ on S_G. This is the matrix version of Putinar's Positivstellensatz [Put93], which is given by Scherer & Hol [SH06].

Theorem 2.1 ([SH06])**.** *Let $G \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{k\times k}$ be such that $\mathrm{QM}_t(G)$ is archimedean. For $f \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$, if $f \succ 0$ on S_G, then $f \in \mathrm{QM}_t(G)$.*

We refer readers to [HN10, KS10] for further refinements of this result, and to [Cim12, HL06, Scm09] for additional recent results on Positivstellensätze for matrix polynomials.

A matrix polynomial $Q \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ is sos if and only if the scalar polynomial $\mathtt{y}^T Q(\mathtt{x})\mathtt{y}$ is sos in $(\mathtt{x}, \mathtt{y})$, where $\mathtt{y}$ is a new t-tuple of variables. This means that whether Q is an sos matrix polynomial can be checked by solving a semidefinite program. A more direct procedure (see [SH06, Lemma 1]) is as follows. When Q has degree $2d$, Q is sos if and only if there exists a positive semidefinite matrix Z such that

$$Q = (u(\mathtt{x}) \otimes I_t)^T Z (u(\mathtt{x}) \otimes I_t), \tag{2.2}$$

where $\otimes$ is the classical Kronecker product and $u(\mathtt{x})$ is the vector of all monomials in $\mathtt{x}$ of degrees $\le d$. As (2.2) is just a set of linear equations in the entries of a positive semidefinite matrix Z, one can search for a feasible Z by solving a semidefinite program. More generally, for a given finite set $\Gamma \subseteq \mathcal{S}\mathbb{R}[\mathtt{x}]^{k\times k}$, one can check whether or not Q belongs to the *truncated* quadratic module

$$\mathrm{QM}_t(\Gamma)\big|_{2d} := \left\{ \sum_{i=1}^{L} p_i^T g_i p_i \;\middle|\; \begin{array}{l} g_i \in \{I_k\} \cup \Gamma,\ p_i \in \mathbb{R}[\mathtt{x}]^{k\times t}, \\ L \in \mathbb{N},\ \deg(p_i^T g_i p_i) \le 2d \end{array} \right\}. \tag{2.3}$$

This can be done similarly by solving a semidefinite program [SH06, Section 5]. For further recent developments in the area, we refer to positive polynomials [HG05, RT08, Sce09], moment problems [Las09c, Lau09, PV99], convex algebraic geometry [BPR13, FNT17, GPT13, GT13], polynomial optimization [deKL11, HL06, Las01, Las15, Lau14, PS03, Scw05], and semidefinite programs [deK02, HNS16, WSV00].

3. A Lifted Matrix Positivstellensatz

In this section, we prove a lifted matrix Positivstellensatz certifying containment of projections of semialgebraic sets given by polynomial matrix inequalities. For $G \in \mathcal{S}\mathbb{R}[\mathtt{x}, \mathtt{y}]^{k\times k}$ and $Q \in \mathcal{S}\mathbb{R}[x, y]^{t\times t}$, consider the projected semialgebraic sets

$$P_G := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^r,\ G(x, y) \succeq 0\}, \tag{3.1}$$

$$P_Q := \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^s,\ Q(x, z) \succeq 0\}. \tag{3.2}$$

We are going to establish a certificate for the containment $P_G \subseteq P_Q$. Our discussion is divided into two cases. We first analyze the case when $Q(\mathtt{x}, \mathtt{z})$ is linear in $\mathtt{z}$, and then treat the nonlinear case.

3.1. **The case $Q(\mathtt{x}, \mathtt{z})$ is linear in $\mathtt{z}$.** Suppose $Q(\mathtt{x}, \mathtt{z})$ is linear in $\mathtt{z} := (\mathtt{z}_1, \ldots, \mathtt{z}_s)$,

$$Q(\mathtt{x}, \mathtt{z}) := Q_0(\mathtt{x}) + \mathtt{z}_1 Q_1(\mathtt{x}) + \cdots + \mathtt{z}_s Q_s(\mathtt{x}), \tag{3.3}$$

where $Q_0(\mathtt{x}), \ldots, Q_s(\mathtt{x}) \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ are symmetric matrix polynomials. A certificate for the inclusion $P_G \subseteq P_Q$ is the existence of polynomials $p_1(\mathtt{x}), \ldots, p_s(\mathtt{x}) \in \mathbb{R}[\mathtt{x}]$ and matrix polynomials $V_i(\mathtt{x}, \mathtt{y})$ such that

$$\left\{ \begin{array}{c} Q_0(\mathtt{x}) + p_1(\mathtt{x})Q_1(\mathtt{x}) + \cdots + p_s(\mathtt{x})Q_s(\mathtt{x}) = \\ V_0(\mathtt{x}, \mathtt{y})^T V_0(\mathtt{x}, \mathtt{y}) + \sum_{i=1}^{\ell} V_i(\mathtt{x}, \mathtt{y})^T G(\mathtt{x}, \mathtt{y}) V_i(\mathtt{x}, \mathtt{y}). \end{array} \right. \tag{3.4}$$

Indeed, if $x \in P_G$, then there exists $y \in \mathbb{R}^r$ with $G(x, y) \succeq 0$, thus $Q(x, z) \succeq 0$ for $z = (p_1(x), \ldots, p_s(x))$ by (3.4). This certifies that $P_G \subseteq P_Q$.

In the following, we show that (3.4) is almost necessary for ensuring $P_G \subseteq P_Q$. Our main conclusion is that (3.4) must hold if P_G is contained in the interior of P_Q (i.e., $P_G \subseteq \operatorname{int}(P_Q)$), under an archimedean condition. Since G is a matrix polynomial in $(\mathtt{x}, \mathtt{y})$, its quadratic module $\mathrm{QM}_t(G)$ is a subset of $\mathcal{S}\mathbb{R}[\mathtt{x}, \mathtt{y}]^{t\times t}$. The archimedeanness of $\mathrm{QM}_t(G)$ requires the existence of $f \in \mathrm{QM}_t(G)$ such that the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^r : f(x, y) \succeq 0\}$ is compact.

Theorem 3.1. *Let $G(\mathtt{x}, \mathtt{y}) \in \mathcal{S}\mathbb{R}[\mathtt{x}, \mathtt{y}]^{k\times k}$ and let $Q(\mathtt{x}, \mathtt{z})$ be as in (3.3). Assume that $\mathrm{QM}_t(G)$ is archimedean. If for all $x \in P_G$ there exists $z \in \mathbb{R}^s$ with $Q(x, z) \succ 0$, then there exists a polynomial tuple $p(\mathtt{x}) = (p_1(\mathtt{x}), \ldots, p_s(\mathtt{x}))$ such that $Q(\mathtt{x}, p(\mathtt{x})) \in \mathrm{QM}_t(G)$, i.e., (3.4) holds.*

Proof. Since $\mathrm{QM}_t(G)$ is archimedean, the set S_G is compact, hence so is the projection P_G. For each $x \in P_G$, there exists z (depending on x, that is, $z = z(x)$), such that $Q(x, z(x)) \succ 0$. Let $\delta = \delta(x) > 0$ be such that $Q(w, z(x)) \succ 0$ for all w in the open ball $\mathcal{B}(x, 2\delta)$ centered at x with radius 2δ. Then, $\{\mathcal{B}(x, \delta(x))\}_{x\in P_G}$ is an open covering for P_G. By compactness, there exist finitely many of these open balls covering P_G, say,

$$P_G \subseteq \bigcup_{i=1}^{N} \mathcal{B}(x^i, \delta(x^i)).$$

For each i, there exists $\epsilon_i > 0$ such that $Q(w, z(x^i)) \succeq \epsilon_i I$ for all $w \in \mathcal{B}(x^i, \delta(x^i))$. Hence, we can choose $\epsilon > 0$ small enough such that for all $x \in P_G$ there exists $z \in \mathbb{R}^s$ with $Q(x, z) \succeq \epsilon I$. Define the function

$$\begin{array}{rll} \phi(x) := \arg\min & z^T z \\ s.t. & Q_0(x) + z_1 Q_1(x) + \cdots + z_s Q_s(x) \succeq \frac{\epsilon}{2} I. \end{array} \tag{3.5}$$

From the above, we can see that the feasible set of (3.5) has nonempty interior for all $x \in P_G$. Because of the strict convexity of $\mathtt{z}^T\mathtt{z}$, the minimizer $\phi(x)$ is unique. Further, the objective is a coercive function, that is, for every number $\tau > 0$, the set $\{z : z^T z \leq \tau\}$ is compact. Hence the optimal value function $\phi(x)^T\phi(x)$ is continuous in x. This can be inferred from [Sha97, Theorem 10] or [WSV00, Theorem 4.1.10].

The minimizer function $\phi(x)$ is also a continuous function on P_G, which can be seen as follows. Suppose $\{x^k\} \subseteq P_G$ is a sequence such that $x^k \to x \in P_G$. Then $\|\phi(x^k)\|_2 \to \|\phi(x)\|_2$ by the continuity of the objective function. Clearly, $\{\phi(x^k)\}$ is bounded. Let u be one of its accumulation points. Then $\|u\|_2 = \|\phi(x)\|_2$. Clearly, u is a feasible point corresponding to x. Hence, u is a minimizer for (3.5), and by the uniqueness, $u = \phi(x)$. So $\phi(x)$ is a continuous function on P_G. Note that

$$Q(x, \phi(x)) \succeq \frac{\epsilon}{2} I \quad \text{ on } \quad P_G.$$

By the Stone-Weierstraß theorem (see e.g. [Rud76, Theorem 7.32]), $\phi(x)$ can be approximated arbitrarily well by polynomial functions. In particular, there exists a polynomial $p(\mathtt{x})$ such that

$$Q(x, p(x)) \succ 0 \quad \text{ on } \quad P_G.$$

That is, $Q(\mathtt{x}, p(\mathtt{x}))$ is symmetric matrix polynomial that is positive definite on P_G. By the archimedean property of $\mathrm{QM}_t(G)$, the classical matrix Positivstellensatz (see e.g. [SH06, KS10]) implies that

$$Q(\mathtt{x}, p(\mathtt{x})) = V_0(\mathtt{x}, \mathtt{y})^T V_0(\mathtt{x}, \mathtt{y}) + \sum_i V_i(\mathtt{x}, \mathtt{y})^T G(\mathtt{x}, \mathtt{y}) V_i(\mathtt{x}, \mathtt{y})$$

for some matrix polynomials $V_i(\mathtt{x}, \mathtt{y})$. □

3.2. **The case $Q(\mathtt{x}, \mathtt{z})$ is nonlinear in $\mathtt{z}$.** Denote the set of exponents by

$$\mathbb{N}^s_{2d} := \{\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s_{\geq 0} \mid \alpha_1 + \cdots + \alpha_s \leq 2d\}.$$

We consider the case that $Q(\mathtt{x}, \mathtt{z})$ is polynomial in $\mathtt{z}$, say,

$$Q(\mathtt{x}, \mathtt{z}) := \sum_{\alpha \in \mathbb{N}^s_{2d}} \mathtt{z}_1^{\alpha_1} \cdots \mathtt{z}_s^{\alpha_s} Q_\alpha(\mathtt{x}), \tag{3.6}$$

with each $Q_\alpha(\mathtt{x}) \in \mathcal{S}\mathbb{R}[\mathtt{x}]^{t \times t}$. If we parameterize $\mathtt{z}_i$ by a polynomial $p_i(\mathtt{x})$, a natural generalization of the certificate (3.4) is

$$Q(\mathtt{x}, p(\mathtt{x})) = \sum_{\alpha \in \mathbb{N}^s_{2d}} p_1(\mathtt{x})^{\alpha_1} \cdots p_s(\mathtt{x})^{\alpha_s} Q_\alpha(\mathtt{x}) \in \mathrm{QM}_t(G). \tag{3.7}$$

However, (3.7) is nonlinear in the coefficients of $p = (p_1, \dots, p_s)$. Generally, the existence of p satisfying (3.7) cannot be checked by solving a semidefinite program.

Here we propose a convexification of (3.7). If each product $p_1(\mathtt{x})^{\alpha_1} \cdots p_s(\mathtt{x})^{\alpha_s}$ is replaced by a new polynomial $p_\alpha(\mathtt{x})$, then (3.7) becomes

$$\left\{ \begin{array}{c} \sum_{\alpha \in \mathbb{N}^s_{2d}} p_\alpha(\mathtt{x}) Q_\alpha(\mathtt{x}) = \\ V_0(\mathtt{x}, \mathtt{y})^T V_0(\mathtt{x}, \mathtt{y}) + \sum_{i=1}^{\ell} V_i(\mathtt{x}, \mathtt{y})^T G(\mathtt{x}, \mathtt{y}) V_i(\mathtt{x}, \mathtt{y}), \end{array} \right. \tag{3.8}$$

for some matrix polynomials $V_i(\mathtt{x}, \mathtt{y})$. However, (3.8) does not imply $P_G \subseteq P_Q$ in general. To remedy this, let

$$p := (p_\alpha)_{\alpha \in \mathbb{N}^s_{2d}},$$

and define the matrix polynomial

$$M(p) := (p_{\alpha+\beta})_{\alpha,\beta \in \mathbb{N}^s_d}.$$

In Proposition 3.2 below, under some convexity conditions, we show that (3.8) is a certificate for $P_G \subseteq P_Q$. The matrix polynomial $Q(\mathtt{x}, \mathtt{z})$ is said to be *sos-concave* in $\mathtt{z}$ at a point x if for every $\xi \in \mathbb{R}^t$ the polynomial $\xi^T Q(x, \mathtt{z})\xi$ is sos-concave in $\mathtt{z}$, i.e., its Hessian $\nabla^2(\xi^T Q(x, \mathtt{z})\xi)$ about $\mathtt{z}$ is an sos-matrix polynomial in $\mathtt{z}$. We refer to [Nie11] for more on sos-concavity/convexity of matrix polynomials.

Proposition 3.2. *Let $G(\mathtt{x}, \mathtt{y}) \in \mathcal{S}\mathbb{R}[x, y]^{k\times k}$ and let $Q(\mathtt{x}, \mathtt{z})$ be as in (3.6). Assume $Q(x, \mathtt{z})$ is sos-concave in $\mathtt{z}$ at every $x \in P_G$. If a polynomial tuple p satisfies (3.8) and $M(p) \succeq 0$ on P_G, then $P_G \subseteq P_Q$.*

Proof. Define a matrix polynomial in $\mathtt{x} = (\mathtt{x}_1, \dots, \mathtt{x}_n)$ and $\mathtt{w} = (\mathtt{w}_\alpha)_{\alpha \in \mathbb{N}^s_{2d}}$ as

$$F(\mathtt{x}, \mathtt{w}) := \sum_{\alpha \in \mathbb{N}^s_{2d}} \mathtt{w}_\alpha Q_\alpha(\mathtt{x}).$$

Pick an arbitrary $x \in P_G$. Let $w_\alpha = p_\alpha(x)$ (note $w_0 = 1$), then

$$F(x, w) \succeq 0, \quad M(w) \succeq 0.$$

For an arbitrary $\xi \in \mathbb{R}^t$, the polynomial $q(\mathtt{z}) := \xi^T Q(x, \mathtt{z})\xi$ is sos-concave in $\mathtt{z}$. Let $u = (w_1, \dots, w_s)$, then one can show that (see e.g. [HN10, Theorem 9] or [Las09b, Theorem 2.6])

$$q(u) \ge \sum_{\alpha \in \mathbb{N}^s_{2d}} \mathtt{w}_\alpha \xi^T Q_\alpha(x)\xi = \xi^T F(x, w)\xi \ge 0.$$

Since $q(u) = \xi^T Q(x, u)\xi \ge 0$ and ξ is arbitrary, we can conclude that $Q(x, u) \succeq 0$, i.e., $x \in P_Q$. The above can also be deduced from the results in [Nie11]. Since $x \in P_G$ was arbitrary, we conclude that $P_G \subseteq P_Q$. $\square$

In the following, we show that (3.8) is almost a necessary certificate for $P_G \subseteq P_Q$ under conditions similar to those in Theorem 3.1 and Proposition 3.2, and under an additional sos-concavity condition.

Theorem 3.3. *Let $G(\mathtt{x}, \mathtt{y}) \in \mathcal{S}\mathbb{R}[x, y]^{k\times k}$ and let $Q(\mathtt{x}, \mathtt{z})$ be as in (3.6). Assume that $\mathrm{QM}_t(G)$ is archimedean. If for every $x \in P_G$, $Q(x, \mathtt{z})$ is sos-concave in $\mathtt{z}$, and there exists z such that $Q(x, z) \succ 0$, then there exist polynomials $p_\alpha \in \mathbb{R}[\mathtt{x}]$ ($\alpha \in \mathbb{N}^s_{2d}$) such that (3.8) holds and $M(p)$ is an sos matrix polynomial.*

Proof. The proof is similar to the one for Theorem 3.1. First, we can similarly prove that there exists $\epsilon > 0$ such that for all $x \in P_G$ there exists z with $Q(x, z) \succeq \epsilon I$. Consider the optimization problem

$$\min \quad z^T z \quad s.t. \quad Q(x,z) \succeq \frac{\epsilon}{2} I. \tag{3.9}$$

For each $x \in P_G$, the feasible set of (3.9) has nonempty interior. It has a unique minimizer, which we also denote by $\phi(x)$. Note that (3.9) is a convex optimization problem and the objective is coercive. Furthermore, $\phi(x)$ is a continuous function on P_G. By the Stone-Weierstraß theorem, there exists a polynomial tuple $q(\mathtt{x}) := (q_1(\mathtt{x}), \dots, q_s(\mathtt{x}))$ such that $Q(x, q(x)) \succeq \frac{\epsilon}{4} I$ on P_G. By the archimedean property and the classical matrix Positivstellensatz (see e.g. [KS10, SH06]), we get

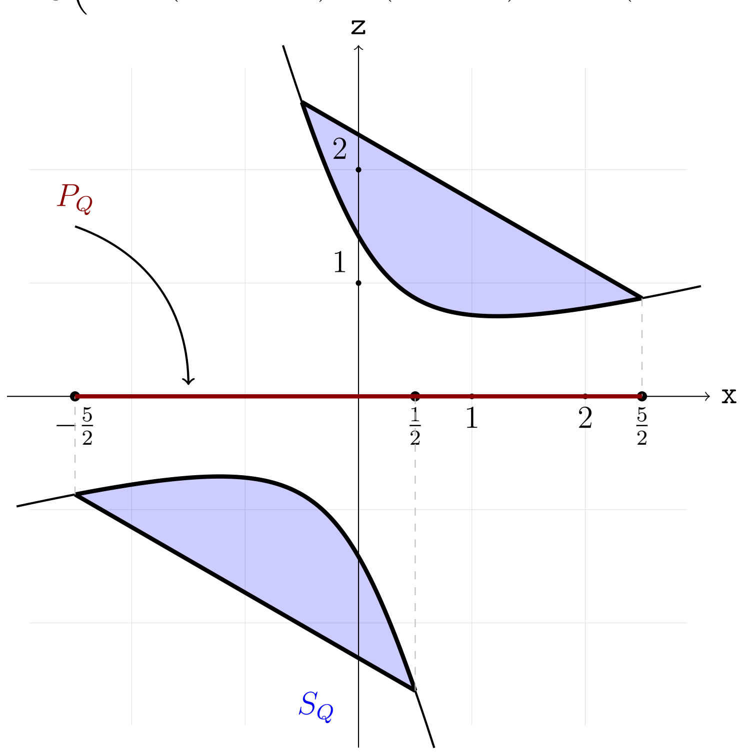
$$Q(\mathtt{x}, q(\mathtt{x})) \in \mathrm{QM}_t(G).$$

For each α, let $p_\alpha = q^\alpha$, then $M(p) = [q]_d [q]_d^T$. In the above, $[q]_d$ is the vector of all monomials in q of degrees $\leq d$. Clearly, $M(p)$ is an sos matrix polynomial and the proof is complete. $\square$

Example 3.4. We want to point out that a lifting continuous map $\phi : P_G \to \mathrm{int}(S_Q)$ need not exist without some convexity assumptions on Q. Hence Theorems 3.1 and 3.3 do not generalize to the non-convex case. Here are two simple examples.

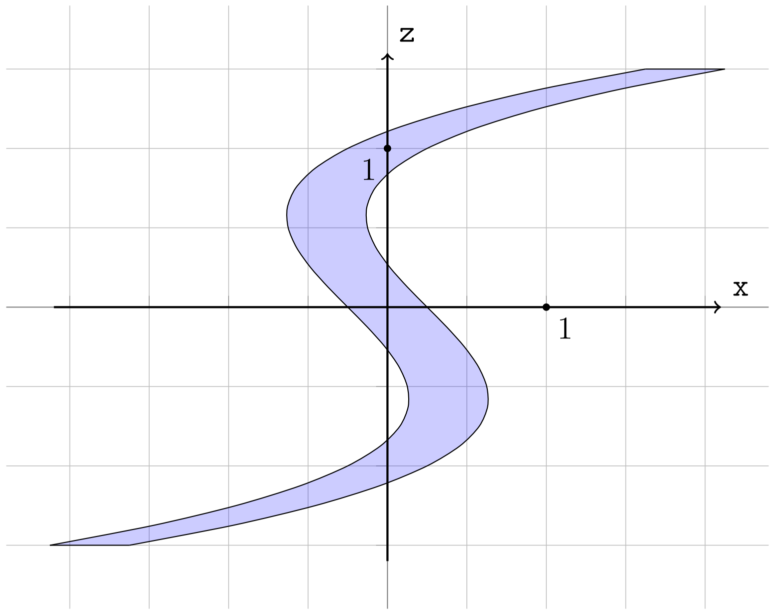
(a) Form S_Q by rotating the semialgebraic set defined as the part of the hyperbola $x^2 - z^2 \geq 1$ lying inside $x^2 \leq 4$ by $60°$ about the origin. That is,

$$Q(\mathtt{x}, \mathtt{z}) := \mathrm{diag}\left(-4 - (-\sqrt{3}\mathtt{x} + \mathtt{z})^2 + (\mathtt{x} + \sqrt{3}\mathtt{z})^2,\ 16 - (\mathtt{x} + \sqrt{3}\mathtt{z})^2\right).$$



Then $P_Q = \left[-\frac{5}{2}, \frac{5}{2}\right]$. The maximal x-coordinate of a point in the bottom component of S_Q is $\frac{1}{2}$, so by letting $P_G = [-1, 1]$ it is clear that each point x in P_G can be lifted to a point $(x, z) \in S_Q$ with $Q(x, z) \succ 0$, but there is no lifting continuous map $P_G \to S_Q$.

(b) The same phenomena can occur even with an S-shape connected $S_Q \subseteq \mathbb{R}^2$. Let S_Q be the band around a cubic curve,



$$S_Q = \left\{(x, z) \in \mathbb{R}^2 \colon \left|x - z(z^2 - 1)\right| \le \frac{1}{4}, \, |z| \le \frac{3}{2}\right\}.$$

We have $P_Q = \left[-\frac{17}{8}, \frac{17}{8}\right]$. As before, each point $x \in P_G := [-1, 1]$ admits a lift to a point $(x, z) \in \operatorname{int}(S_Q)$, but there is no lifting continuous map $P_G \to S_Q$.

3.3. **Some examples.** In the following, we give some examples of the lifted matrix Positivstellensatz proved in Theorems 3.1 and 3.3. The notation e_i denotes the standard ith unit vector, i.e., its ith entry is one and all other entries are zero.

Example 3.5. Consider the matrix polynomials

$$G(\mathtt{x}, \mathtt{y}) = \begin{bmatrix} 1 - \mathtt{y} - \mathtt{x}_1^2 & \mathtt{x}_1\mathtt{x}_2 \\ \mathtt{x}_1\mathtt{x}_2 & \mathtt{y} - \mathtt{x}_2^2 \end{bmatrix}, \quad Q(\mathtt{x}, \mathtt{z}) = \begin{bmatrix} 1 + \mathtt{z}\,\mathtt{x}_2 & \mathtt{z} - 2\,\mathtt{x}_1 \\ \mathtt{z} - 2\,\mathtt{x}_1 & 1 - \mathtt{z}\,\mathtt{x}_2 \end{bmatrix}.$$

Then $P_G = \{(x_1, x_2) \in \mathbb{R}^2 \colon |x_1 \pm x_2| \le 1\}$, and is contained in

$$P_Q = \{(x_1, x_2) \in \mathbb{R}^2 \colon 1 + x_2^2 - 4x_1^2x_2^2 \ge 0\}.$$

The quadratic module $\mathrm{QM}_2(G)$ is archimedean since

$$\begin{aligned} 3 - \mathtt{x}_1^2 - \mathtt{x}_2^2 - 2y^2 = {} & e_1^T G(\mathtt{x}, \mathtt{y}) e_1 + e_2^T G(\mathtt{x}, \mathtt{y}) e_2 + (1 - \mathtt{y})^2 + \\ & \mathtt{x}_2^2(1 - \mathtt{y})^2 + \mathtt{x}_1^2(1 + \mathtt{y})^2 + e_1^T G(\mathtt{x}, \mathtt{y}) e_1 (1 + y)^2 + e_2^T G(\mathtt{x}, \mathtt{y}) e_2 (1 - \mathtt{y})^2. \end{aligned}$$

The polynomial p_1 in Theorem 3.1 can be chosen to be $\mathtt{x}_1$; then

$$Q(\mathtt{x}, p(\mathtt{x})) = \frac{1}{2}\begin{bmatrix} \mathtt{x}_1 + \mathtt{x}_2 & -1 \\ -1 & \mathtt{x}_1 - \mathtt{x}_2 \end{bmatrix}^2 + \frac{1 - \mathtt{x}_1^2 - \mathtt{x}_2^2}{2} I.$$

A certificate of the form (3.4) for $P_G \subseteq P_Q$ is

$$\ell = 4, \quad V_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathtt{x}_1 + \mathtt{x}_2 & -1 \\ -1 & \mathtt{x}_1 - \mathtt{x}_2 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Example 3.6. We present an example where the assumptions of Theorem 3.1 are not met, but the conclusion still holds. Consider the matrix polynomials

$$G(\mathtt{x}, \mathtt{y}) = \begin{bmatrix} 1 - \mathtt{x}_1^2 & \mathtt{x}_1 + \mathtt{x}_2 & \mathtt{x}_2^2 \\ \mathtt{x}_1 + \mathtt{x}_2 & 0 & \mathtt{x}_1 + \mathtt{x}_2 \\ \mathtt{x}_2^2 & \mathtt{x}_1 + \mathtt{x}_2 & \mathtt{y} \end{bmatrix}, \quad Q(\mathtt{x}, \mathtt{z}) = \begin{bmatrix} 1 + 2\epsilon + \mathtt{x}_2 & \mathtt{x}_1^2 \\ \mathtt{x}_1^2 & \mathtt{z} \end{bmatrix},$$

for $\epsilon > 0$. The projection set $P_G = \{(x_1, -x_1) \in \mathbb{R}^2 \colon -1 < x_1 < 1\}$. It is bounded but not closed. The intersection $\mathrm{QM}_2(G) \cap \mathbb{R}[\mathtt{x}]$ is archimedean, because

$$(2 - \mathtt{x}_1^2 - \mathtt{x}_2^2) = e_1^T G(\mathtt{x}, \mathtt{y}) e_1 + \begin{bmatrix} 1 \\ \frac{1}{2}(\mathtt{x}_1 - \mathtt{x}_2) \\ 0 \end{bmatrix}^T G(\mathtt{x}, \mathtt{y}) \begin{bmatrix} 1 \\ \frac{1}{2}(\mathtt{x}_1 - \mathtt{x}_2) \\ 0 \end{bmatrix}.$$

However, the quadratic module $\mathrm{QM}_2(G)$ itself is not archimedean, since S_G is unbounded. The lifting polynomial p_1 can be chosen as $\epsilon^{-1}\mathtt{x}_1^2$, then

$$Q(\mathtt{x}, p(\mathtt{x})) = \mathtt{x}_1^2 \begin{bmatrix} \epsilon & 1 \\ 1 & \epsilon^{-1} \end{bmatrix} + (1 + \mathtt{x}_2 + \epsilon + \epsilon(1 - \mathtt{x}_1^2)) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note the following representations:

$$1 + \mathtt{x}_2 + \epsilon = \begin{bmatrix} \sqrt{\epsilon} \\ \sqrt{4\epsilon}^{-1} \\ 0 \end{bmatrix}^T G(\mathtt{x}, \mathtt{y}) \begin{bmatrix} \sqrt{\epsilon} \\ \sqrt{4\epsilon}^{-1} \\ 0 \end{bmatrix} + 1 - \mathtt{x}_1 + \epsilon \mathtt{x}_1^2,$$

$$1 - \mathtt{x}_1^2 = e_1^T G(\mathtt{x}, \mathtt{y}) e_1, \quad 1 - \mathtt{x}_1 = \frac{1 - \mathtt{x}_1^2}{2} + \frac{(\mathtt{x}_1 - 1)^2}{2}.$$

A certificate of the form (3.4) for $P_G \subseteq P_Q$ is that $\ell = 3$ and

$$V_0 = \begin{bmatrix} \sqrt{\epsilon}\mathtt{x}_1 & \sqrt{\epsilon}^{-1}\mathtt{x}_1 \\ \sqrt{\epsilon}\mathtt{x}_1 & 0 \\ \frac{\mathtt{x}_1 - 1}{\sqrt{2}} & 0 \end{bmatrix}, V_1 = \begin{bmatrix} \sqrt{\epsilon} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} \sqrt{\epsilon} & 0 \\ \sqrt{4\epsilon}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, V_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In Theorem 3.1, if $\mathrm{QM}_t(G)$ is not archimedean, its conclusion might not hold. The following is such an example.

Example 3.7. Consider the matrix polynomial

$$G(\mathtt{x},\mathtt{y}) = \begin{bmatrix} \mathtt{y}^2(1-\mathtt{x}^2)-1 & 0 \\ 0 & 2-\mathtt{x}^2 \end{bmatrix}.$$

Clearly, $P_G = (-1,1)$ is bounded. The intersection $\mathrm{QM}_1(G)\cap\mathbb{R}[\mathtt{x}]$ is archimedean, since $2-x^2 \in \mathrm{QM}_1(G)\cap\mathbb{R}[\mathtt{x}]$. However, the quadratic module $\mathrm{QM}_1(G)$ itself is not archimedean, because S_G is unbounded. We claim that $\mathrm{QM}_1(G)\cap\mathbb{R}[\mathtt{x}]$ is generated by the polynomial $2-\mathtt{x}^2$. For every $g(\mathtt{x}) \in \mathrm{QM}_1(G)\cap\mathbb{R}[\mathtt{x}]$, we can write

$$g(\mathtt{x}) = \sigma_0 + \sigma_1\cdot(\mathtt{y}^2(1-\mathtt{x}^2)-1) + \sigma_2\cdot(2-\mathtt{x}^2) \tag{3.10}$$

for sos polynomials $\sigma_j \in \mathbb{R}[\mathtt{x},\mathtt{y}]$. Note that $g(\mathtt{x})$ does not depend on $\mathtt{y}$. To cancel $\mathtt{y}$ on the right hand side of (3.10), we must have $\sigma_1 = 0$. Similarly, σ_0 and σ_2 cannot depend on $\mathtt{y}$. We can conclude that $g \in \mathrm{QM}_1(2-\mathtt{x}^2) \subseteq \mathbb{R}[\mathtt{x}]$. Finally, for each $\lambda \in (1,2)$, the polynomial $\lambda-\mathtt{x}^2$ is positive on P_G, but it does not belong to $\mathrm{QM}_1(G)\cap\mathbb{R}[\mathtt{x}]$. The conclusion of Theorem 3.1 fails for this example, because $\mathrm{QM}_1(G)$ is not archimedean.

Example 3.8. Consider the matrix polynomials

$$G(\mathtt{x},\mathtt{y}) = \begin{bmatrix} \mathtt{x}_1 & \mathtt{y} & \mathtt{x}_1 \\ \mathtt{y} & \mathtt{x}_2 & \mathtt{x}_2 \\ \mathtt{x}_1 & \mathtt{x}_2 & 1 \end{bmatrix}, \quad Q(\mathtt{x},\mathtt{z}) = \begin{bmatrix} \mathtt{x}_1+2\,\mathtt{z}_1-\mathtt{z}_1^2 & \mathtt{z}_1\mathtt{z}_2 & \mathtt{x}_2 \\ \mathtt{z}_1\mathtt{z}_2 & \mathtt{x}_2+2\,\mathtt{z}_2-\mathtt{z}_2^2 & \mathtt{x}_1 \\ \mathtt{x}_2 & \mathtt{x}_1 & 1 \end{bmatrix}.$$

Note that $P_G = [0,1]^2$ and $\mathrm{QM}_3(G)$ is archimedean, because

$$1-\mathtt{x}_1^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} G \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -\mathtt{x}_1 \end{bmatrix} G \begin{bmatrix} 1 \\ 0 \\ -\mathtt{x}_1 \end{bmatrix},$$

$$1-\mathtt{x}_2^2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} G \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -\mathtt{x}_2 \end{bmatrix} G \begin{bmatrix} 0 \\ 1 \\ -\mathtt{x}_2 \end{bmatrix}.$$

As in Example 3.6 this also yields $1-\mathtt{x}_i \in \mathrm{QM}_3(G)$. Hence

$$2-\mathtt{y}^2 = (1-\mathtt{x}_2)+(1-\mathtt{x}_1)\mathtt{y}^2 + \frac{1}{2}\begin{bmatrix} 1 \\ -\mathtt{y} \\ 1 \end{bmatrix}^T G \begin{bmatrix} 1 \\ -\mathtt{y} \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -\mathtt{y} \\ -1 \end{bmatrix}^T G \begin{bmatrix} 1 \\ -\mathtt{y} \\ -1 \end{bmatrix} \in \mathrm{QM}_3(G).$$

The matrix polynomial $Q(\mathtt{x},\mathtt{z})$ is sos-concave in $\mathtt{z}$. The polynomials p_i in Theorem 3.3 can be chosen as

$$p_\alpha = \mathtt{x}_2^{\alpha_1}\mathtt{x}_1^{\alpha_2}, \quad \alpha = (\alpha_1,\alpha_2) \in \mathbb{N}_2^2.$$

Clearly,

$$M(p) = \begin{bmatrix} 1 & \mathtt{x}_2 & \mathtt{x}_1 \\ \mathtt{x}_2 & \mathtt{x}_1^2 & \mathtt{x}_2\mathtt{x}_1 \\ \mathtt{x}_1 & \mathtt{x}_1\mathtt{x}_2 & \mathtt{x}_1^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathtt{x}_2 \\ \mathtt{x}_1 \end{bmatrix} \begin{bmatrix} 1 \\ \mathtt{x}_2 \\ \mathtt{x}_1 \end{bmatrix}^T$$

is sos. We have

$$Q(\mathtt{x}, p(\mathtt{x})) = \begin{bmatrix} \mathtt{x}_2 \\ \mathtt{x}_1 \\ 1 \end{bmatrix} \begin{bmatrix} \mathtt{x}_2 \\ \mathtt{x}_1 \\ 1 \end{bmatrix}^T + \begin{bmatrix} \mathtt{x}_1 + 2(\mathtt{x}_2 - \mathtt{x}_2^2) & 0 & 0 \\ 0 & \mathtt{x}_2 + 2(\mathtt{x}_1 - \mathtt{x}_1^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Observe that

$$\mathtt{x}_1 = e_1^T G(\mathtt{x},\mathtt{y}) e_1, \quad \mathtt{x}_2 = e_2^T G(\mathtt{x},\mathtt{y}) e_2,$$

$$\mathtt{x}_1 - \mathtt{x}_1^2 = \begin{bmatrix} 1 \\ 0 \\ -\mathtt{x}_1 \end{bmatrix}^T G(\mathtt{x},\mathtt{y}) \begin{bmatrix} 1 \\ 0 \\ -\mathtt{x}_1 \end{bmatrix}, \quad \mathtt{x}_2 - \mathtt{x}_2^2 = \begin{bmatrix} 0 \\ 2 \\ -\mathtt{x}_2 \end{bmatrix}^T G(\mathtt{x},\mathtt{y}) \begin{bmatrix} 0 \\ 2 \\ -\mathtt{x}_2 \end{bmatrix}.$$

A certificate of the form (3.8) for $P_G \subseteq P_Q$ is that

$$\ell = 4, \quad V_0(\mathtt{x},\mathtt{y}) = \begin{bmatrix} \mathtt{x}_2 & \mathtt{x}_1 & 1 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\mathtt{x}_1 & 0 & 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, V_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -\mathtt{x}_2 & 0 \end{bmatrix}.$$

4. Containment of Spectrahedrops

In this section, we show how to apply the lifted matrix Positivstellensatz developed in Section 3 to check the containment of spectrahedrops. Recall that a spectrahedrop is the projection of a spectrahedron. Under mild and natural smoothness assumptions on their boundaries, convex semialgebraic sets are spectrahedrops [HN10, Sce11, Las15]. First examples of convex semialgebraic sets that are not spectrahedrops are given by Scheiderer in [Sce18].

Consider two spectrahedrops

$$P_A := \{x : \exists y,\ A(x,y) \succeq 0\}, \quad P_B := \{x : \exists z,\ B(x,z) \succeq 0\},$$

where $A(\mathtt{x},\mathtt{y}) \in \mathcal{S}\mathbb{R}[\mathtt{x},\mathtt{y}]^{k\times k}$, $B(\mathtt{x},\mathtt{z}) \in \mathcal{S}\mathbb{R}[\mathtt{x},\mathtt{z}]^{t\times t}$ are linear pencils as in (1.4). An important question of wide applications is how to check the containment $P_A \subseteq P_B$? When P_A, P_B are spectrahedra (i.e., there are no lifting variables $\mathtt{y},\mathtt{z}$), there exist Positivstellensätze certifying the containment [HKM13, KTT13, KTT15]. In this section, we present a certificate for the containment when there are lifting variables $\mathtt{y},\mathtt{z}$. Here Theorem 3.1 applies. In fact, when the included set is a spectrahedrop, the assumptions in Theorem 3.1 can be weakened. Recall that the intersection $\mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ is archimedean if there exists $f(\mathtt{x}) \in \mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ such that $f(x) \succeq 0$ defines a compact set in $\mathbb{R}^n$. The archimedeanness of $\mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$

implies the boundedness, but not the closedness, of P_A. Clearly, the archimedeanness of $\mathrm{QM}_t(A)$ implies that $\mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ is archimedean and P_A is closed, but not vice versa; cf. Example 3.7.

Theorem 4.1. *Let $A(\mathtt{x},\mathtt{y})$ and $B(\mathtt{x},\mathtt{z})$ be linear pencils as in (1.4). Assume that $\mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ is archimedean. If there is $\epsilon > 0$ such that for each $x \in P_A$ there exists z with $B(x,z) \succeq \epsilon I$, then there exists a tuple $f(\mathtt{x}) := (f_1(\mathtt{x}), \dots, f_s(\mathtt{x}))$ of polynomials in $\mathbb{R}[\mathtt{x}]$ such that*

$$B(\mathtt{x}, f(\mathtt{x})) = B_0 + \sum_{i=1}^{n} \mathtt{x}_i B_i + \sum_{j=1}^{s} f_j(\mathtt{x}) B_{n+j} \in \mathrm{QM}_t(A(\mathtt{x},\mathtt{y})). \tag{4.1}$$

Proof. For brevity, let us write $M := \mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$. We claim that the positivity set of M,

$$S_M := \{x \in \mathbb{R}^n : \forall g \in M,\ g(x) \succeq 0\}$$

equals the closure $\overline{P_A}$. The inclusion $\overline{P_A} \subseteq S_M$ is clear. For the converse, assume $u \in S_M \setminus \overline{P_A}$. Since $\overline{P_A}$ is convex, there is a linear polynomial $\ell(\mathtt{x})$ satisfying $\ell(\mathtt{x}) \geq \alpha > 0$ on $\overline{P_A}$ for some α, and $\ell(u) < 0$. In particular, $\ell(\mathtt{x}) \geq \alpha > 0$ on S_A. So, by the linear Positivstellensatz [KS13, Corollary 4.2.4], $\ell(\mathtt{x}) \in \mathrm{QM}_1(A) \cap \mathbb{R}[\mathtt{x}]$. This implies that $\ell(\mathtt{x})I \in M$, leading to the contradiction $\ell(u) \geq 0$.

The rest of the proof is the same as for Theorem 3.1. We can continuously choose for each $x \in P_A$ a point $z = z(x) \in \mathbb{R}^s$ satisfying $B(x,z) \succeq \frac{\epsilon}{2} I$. By the Stone-Weierstraß theorem, there is a tuple of polynomials $f(\mathtt{x}) := (f_1(\mathtt{x}), \dots, f_s(\mathtt{x}))$ such that $B(\mathtt{x}, f(\mathtt{x})) \succ 0$ on $\overline{P_A} = S_M$. Since M is archimedean, the matrix Positivstellensatz (see e.g. [KS10]) implies $B(\mathtt{x}, f(\mathtt{x})) \in \mathrm{QM}_t(A)$, as desired. □

In Theorem 4.1, we assume the existence of a uniform $\epsilon > 0$ such that for all $x \in P_A$ there exists z with $B(x,z) \succeq \epsilon I$. This is inconvenient to check in applications. However, the condition can be weakened to $B(x,z) \succ 0$ when P_A is closed.

Corollary 4.2. *Let $A(\mathtt{x},\mathtt{y})$ and $B(\mathtt{x},\mathtt{z})$ be linear pencils as in (1.4). Assume that $\mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x}]^{t\times t}$ is archimedean and P_A is closed. If for each $x \in P_A$ there exists z with $B(x,z) \succ 0$, then there exist a tuple $f(\mathtt{x}) := (f_1(\mathtt{x}), \dots, f_s(\mathtt{x}))$ of polynomials in $\mathbb{R}[\mathtt{x}]$ such that (4.1) holds.*

Proof. If $\mathrm{QM}_t(A) \cap \mathcal{S}\mathbb{R}[\mathtt{x},\mathtt{z}]^{t\times t}$ is archimedean, then P_A is bounded. Hence, P_A is compact since it is also closed. An $\epsilon > 0$ satisfying Theorem 4.1 can be found similarly as in the proof of Theorem 3.1. Therefore, the corollary follows from Theorem 4.1. □

Clearly, (4.1) implies that $P_A \subseteq P_B$. Theorem 4.1 essentially says that (4.1) is a necessary certificate when P_A is contained in the interior of P_B, i.e., $P_A \subseteq \mathrm{int}(P_B)$. Note that in (4.1) the polynomials f_i only depend on $\mathtt{x}$.

Example 4.3. Consider the linear pencils

$$A(\mathtt{x},\mathtt{y}) := \operatorname{diag}\left(\begin{bmatrix} \mathtt{y}_1 & \mathtt{x}_1 \\ \mathtt{x}_1 & 1 \end{bmatrix}, \begin{bmatrix} \mathtt{y}_2 & \mathtt{x}_2 \\ \mathtt{x}_2 & 1 \end{bmatrix}, \begin{bmatrix} 1+\mathtt{y}_1 & \mathtt{y}_2 \\ \mathtt{y}_2 & 1-\mathtt{y}_1 \end{bmatrix} \right),$$
$$B(\mathtt{x},\mathtt{y}) := \begin{bmatrix} 1 & \mathtt{x}_1 & \mathtt{z} \\ \mathtt{x}_1 & 1 & \mathtt{x}_2 \\ \mathtt{z} & \mathtt{x}_2 & 1 \end{bmatrix}.$$

The spectrahedrop P_A is the unit 4-norm ball $\{x_1^4 + x_2^4 \leq 1\}$, while P_B is the unit square $[-1,1]^2$. Clearly, $P_A \subseteq P_B$. We give a certificate of the form (4.1) for this inclusion. The polynomial f_1 in Theorem 4.1 can be chosen as $\mathtt{x}_1\mathtt{x}_2$. Note that

$$B(\mathtt{x}, f(\mathtt{x})) = \begin{bmatrix} \mathtt{x}_1 \\ 1 \\ \mathtt{x}_2 \end{bmatrix} \begin{bmatrix} \mathtt{x}_1 \\ 1 \\ \mathtt{x}_2 \end{bmatrix}^T + \begin{bmatrix} 1-\mathtt{x}_1^2 & & \\ & 0 & \\ & & 1-\mathtt{x}_2^2 \end{bmatrix},$$

$$1-\mathtt{x}_1^2 = \begin{bmatrix} 1 \\ -\mathtt{x}_1 \end{bmatrix}^T \begin{bmatrix} \mathtt{y}_1 & \mathtt{x}_1 \\ \mathtt{x}_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\mathtt{x}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1+\mathtt{y}_1 & \mathtt{y}_2 \\ \mathtt{y}_2 & 1-\mathtt{y}_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$1-\mathtt{x}_2^2 = \begin{bmatrix} 1 \\ -\mathtt{x}_2 \end{bmatrix}^T \begin{bmatrix} \mathtt{y}_2 & \mathtt{x}_2 \\ \mathtt{x}_2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\mathtt{x}_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1+\mathtt{y}_1 & \mathtt{y}_2 \\ \mathtt{y}_2 & 1-\mathtt{y}_1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The certificate for the inclusion $P_A \subseteq P_B$ of the form (1.9), or equivalently (4.1), is

$$B(\mathtt{x}, f(\mathtt{x})) = V_0(\mathtt{x})^T V_0(\mathtt{x}) + V_1(\mathtt{x})^T A(\mathtt{x},\mathtt{y}) V_1(\mathtt{x}) + V_2(\mathtt{x})^T A(\mathtt{x},\mathtt{y}) V_2(\mathtt{x}),$$

where the matrix polynomials $V_i(\mathtt{x})$ are:

$$\begin{aligned} V_0(\mathtt{x}) &= \begin{bmatrix} \mathtt{x}_1 & 1 & \mathtt{x}_2 \end{bmatrix}, \\ V_1(\mathtt{x}) &= \begin{bmatrix} 1 & -\mathtt{x}_1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ V_2(\mathtt{x}) &= \begin{bmatrix} 0 & 0 & 1 & -\mathtt{x}_2 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Example 4.4 ([KS13, Example 4.6.3])**.** In this example, we show that the polynomials V_j in the right-hand side of the Positivstellensatz certificate (4.1) might depend on $\mathtt{y}$. This is the case even if there is no lifting variable $\mathtt{z}$. Consider ($n=1$)

$$A(\mathtt{x},\mathtt{y}) := \begin{bmatrix} 0 & \mathtt{x} & 0 \\ \mathtt{x} & \mathtt{y}_1 & \mathtt{y}_2 \\ 0 & \mathtt{y}_2 & \mathtt{x} \end{bmatrix}.$$

Clearly, $P_A = \{0\}$. We claim that $\mathrm{QM}_1(A) \cap \mathbb{R}[\mathtt{x}]$ is archimedean. Obviously, $e_3^T A e_3 = \mathtt{x} \in \mathrm{QM}_1(A)$. Further,

$$-\mathtt{x}^2 = uAu^T \in \mathrm{QM}_1(A) \tag{4.2}$$

for $u = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\mathtt{y}_1 & -\mathtt{x} & 0 \end{bmatrix}$. Hence for each $\lambda > 0$,

$$1 - \lambda \mathtt{x} = \left(1 - \frac{\lambda}{2}\mathtt{x}\right)^2 - \lambda^2 \mathtt{x}^2 \in \mathrm{QM}_1(A).$$

In particular, the assumptions of Theorem 4.1 or Corollary 4.2 are met. However, a certificate of the form

$$-\mathtt{x}^2 = \sum_i V_{0i}^T V_{0i} + \sum_j V_j^T A(\mathtt{x}, \mathtt{y}) V_j \in \mathrm{QM}_1(A)$$

cannot exist for $V_{0i}, V_j \in \mathbb{R}[\mathtt{x}]^3$. Indeed, if $u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \in \mathbb{R}[\mathtt{x}]^3$, then

$$u^T A u = 2u_1 u_2 \mathtt{x} + u_3^2 \mathtt{x} + u_2^2 \mathtt{y}_1 + 2u_2 u_3 \mathtt{y}_2. \tag{4.3}$$

In a sum of terms of the form (4.3), one can eliminate $\mathtt{y}_i$ only if all $u_2 = 0$. But, for $u_2 = 0$, plugging in $\mathtt{x} = 1$ leads to the contradiction $-1 \geq 0$.

5. Applications

In this section we present two applications of our results. Namely, we show how to solve the matrix cube problem and find the maximum inscribing ellipsoid for spectrahedrops.

5.1. **Matrix cube problem for spectrahedrops.** The matrix cube problem of Ben-Tal and Nemirovski [B-TN02, Nem06] is an important problem in convex geometry and optimization arising from uncertain semidefinite programs in robust control. A natural variant of it asks to find the largest cube that is contained in a spectrahedrop.

Consider the $t \times t$ linear pencil

$$B(\mathtt{x}, \mathtt{z}) := B_0 + \mathtt{x}_1 B_1 + \cdots + \mathtt{x}_n B_n + \mathtt{z}_1 B_{n+1} + \cdots + \mathtt{z}_s B_{n+s}. \tag{5.1}$$

The matrix cube problem is the optimization problem

$$\max \quad \rho \quad \text{s.t.} \quad [-\rho, \rho]^n \subseteq P_B. \tag{5.2}$$

When 0 is in the interior of P_B, we can generally assume $B_0 \succeq 0$. Note that $[-\rho, \rho]^n \subseteq P_B$ if and only if $[-1, 1]^n \subseteq P_{\widetilde{B}}$ with

$$\widetilde{B} := \frac{1}{\rho} B_0 + \mathtt{x}_1 B_1 + \cdots + \mathtt{x}_n B_n + \mathtt{z}_1 B_{n+1} + \cdots + \mathtt{z}_s B_{n+s}.$$

Thus, (5.2) is in turn equivalent to

$$\left\{ \begin{array}{rl} \min & \gamma \\ \text{s.t.} & \gamma B_0 + \sum_{i=1}^n \mathtt{x}_i B_i + \sum_{j=1}^s p_j(\mathtt{x}) B_{n+j} \in \mathrm{QM}_t(D), \\ & \gamma \geq 0, \end{array} \right. \tag{5.3}$$

for scalar polynomials $p_j(\mathtt{x})$. In the above, $D(\mathtt{x})$ is the diagonal matrix

$$D(\mathtt{x}) = \mathrm{diag}(\begin{bmatrix} 1 + \mathtt{x}_1 & 1 - \mathtt{x}_1 & \cdots & 1 + \mathtt{x}_n & 1 - \mathtt{x}_n \end{bmatrix}).$$

One can solve (5.3) as a semidefinite program, when the degrees of the p_j are chosen and a truncation of $\mathrm{QM}_t(D)$ is used.

Remark 5.1. Now that we know how to solve the matrix cube problem for spectrahedrops, we can give an approximation scheme for the matrix cube problem for general convex semialgebraic sets. Namely, we solve the matrix cube problem for each of the Lasserre relaxations [Las09a, Las15] constructively approximating convex semialgebraic sets from above by spectrahedrops.

Example 5.2. Consider the spectrahedrop P_B given by the linear pencil

$$B(\mathtt{x},\mathtt{z}) = \begin{bmatrix} 1 & \mathtt{x}_1 & \mathtt{z}_1 & \mathtt{z}_3 \\ \mathtt{x}_1 & 1 & \mathtt{x}_2 & \mathtt{z}_2 \\ \mathtt{z}_1 & \mathtt{x}_2 & 1 & \mathtt{x}_3 \\ \mathtt{z}_3 & \mathtt{z}_2 & \mathtt{x}_3 & 1 \end{bmatrix}.$$

We want to find the largest square contained in P_B, with a certificate for the inclusion. The positive semidefiniteness of $B(x,z)$ implies that $|x_1|, |x_2|, |x_3| \leq 1$, so P_B is contained in the unit cube $[-1,1]^3$. By solving the optimization problem (5.3), we certify that $[-1,1]^3$ is also the largest cube contained in P_B. The optimal value of γ in (5.3) is 1. The optimal p_j are given as

$$p_1 = \mathtt{x}_1\mathtt{x}_2, \quad p_2 = \mathtt{x}_2\mathtt{x}_3, \quad p_3 = \mathtt{x}_1\mathtt{x}_2\mathtt{x}_3.$$

The certificate for the inclusion $P_B \subseteq [-1,1]^3$ is then

$$B(\mathtt{x}, p(\mathtt{x})) = V_0(\mathtt{x})^T V_0(\mathtt{x}) + \sum_{k=1}^{6} V_k(\mathtt{x})^T D(\mathtt{x}) V_k(\mathtt{x}),$$

where the $V_i(\mathtt{x})$ are

$$\begin{aligned}
V_0(\mathtt{x}) &= \begin{bmatrix} 1 & \mathtt{x}_1 & \mathtt{x}_1\mathtt{x}_2 & \mathtt{x}_1\mathtt{x}_2\mathtt{x}_3 \end{bmatrix}, \\
V_1(\mathtt{x}) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \left(\frac{1-\mathtt{x}_1}{\sqrt{2}}\right) \begin{bmatrix} 0 & 1 & \mathtt{x}_2 & \mathtt{x}_2\mathtt{x}_3 \end{bmatrix}, \\
V_2(\mathtt{x}) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \left(\frac{1+\mathtt{x}_1}{\sqrt{2}}\right) \begin{bmatrix} 0 & 1 & \mathtt{x}_2 & \mathtt{x}_2\mathtt{x}_3 \end{bmatrix}, \\
V_3(\mathtt{x}) &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \left(\frac{1-\mathtt{x}_2}{\sqrt{2}}\right) \begin{bmatrix} 0 & 0 & 1 & \mathtt{x}_3 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
V_4(\mathtt{x}) &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \left(\frac{1+\mathtt{x}_2}{\sqrt{2}}\right) \begin{bmatrix} 0 & 0 & 1 & \mathtt{x}_3 \end{bmatrix}, \\
V_5(\mathtt{x}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \left(\frac{1-\mathtt{x}_3}{\sqrt{2}}\right) \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \\
V_6(\mathtt{x}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \left(\frac{1+\mathtt{x}_3}{\sqrt{2}}\right) \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

We deduce that $P_B = [-1,1]^3$.

5.2. **Maximum inscribing ellipsoid.** Our lifted matrix Positivstellensatz also has applications to finding the largest ellipsoid that is contained in a spectrahedrop. Let $B(\mathtt{x},\mathtt{y})$ be the linear pencil as in (5.1). An ellipsoid is the semialgebraic set

$$\mathcal{E}_P = \{x \in \mathbb{R}^n \colon 1 - \mathtt{x}^T P^{-1}\mathtt{x} \geq 0\}$$

for a positive definite matrix P. The volume of $\mathcal{E}_P$ is measured by the determinant $\det P$. To find the maximum $\mathcal{E}_{\mathcal{P}}$ inscribed in the spectrahedrop P_B, we need to solve the optimization problem:

$$\left\{ \begin{array}{rl} \max & \det P \\ s.t. & \mathcal{E}_P \subseteq P_B, \\ & P \succ 0. \end{array} \right. \tag{5.4}$$

Our lifted matrix Positivstellensatz certificate for the above inclusion is

$$B_0 + \sum_{i=1}^{n} \mathtt{x}_i B_i + \sum_{j=1}^{s} p_j(\mathtt{x}) B_{n+j} = \sum_{k=1}^{\ell} V_k(\mathtt{x})^T (1 - \mathtt{x}^T P^{-1}\mathtt{x}) V_k(\mathtt{x}) + V_0(\mathtt{x})^T V_0(\mathtt{x}) \tag{5.5}$$

for some matrix polynomials $V_k(\mathtt{x})$. However, (5.5) is nonlinear in P. We apply a change of variables:

$$Q = P^{1/2}, \quad \mathtt{z} = Q^{-1}\mathtt{x}.$$

Then the equation (5.5) becomes

$$B_0 + \sum_{i=1}^{n} (Q\mathtt{z})_i B_i + \sum_{j=1}^{s} q_j(\mathtt{z}) B_{n+j} = \sum_{k=1}^{\ell} U_k(\mathtt{z})^T (1 - \mathtt{z}^T\mathtt{z}) U_k(\mathtt{z}) + U_0(\mathtt{z})^T U_0(\mathtt{z}),$$

for some new lifting polynomials $q_j(\mathtt{z})$. Note that the polynomials $p_j(\mathtt{x})$ and $q_j(\mathtt{z})$ are related by

$$p_j(\mathtt{x}) \; = \; q_j(Q^{-1}\mathtt{x}).$$

Therefore, (5.4) is equivalent to the maximization problem

$$\left\{ \begin{array}{rl} \max & \det Q \\ s.t. & B_0 + \sum_{i=1}^{n} (Q\mathtt{z})_i B_i + \sum_{j=1}^{s} q_j(\mathtt{z}) B_{n+j} \in \mathrm{QM}_t(1 - \mathtt{z}^T\mathtt{z}), \\ & Q \succ 0, \end{array} \right. \tag{5.6}$$

where t is the size of the pencil $B(\mathtt{x},\mathtt{y})$. The determinant maximization problem over a spectrahedron (5.6) can be solved using interior point methods, much like classical semidefinite programs [VBW98], when the degrees of the q_j are chosen and a truncation of $\mathrm{QM}_t(1 - \mathtt{z}^T\mathtt{z})$ is used.

As in the case of the matrix cube problem (cf. Remark 5.1), this solution now leads to an approximation scheme for finding the maximum ellipsoid inscribed in a general convex semialgebraic set.

6. Conclusions and discussion

In this paper, we have proposed a new matrix Positivstellensatz that uses lifting polynomials. It serves as a certificate for containment between projections of two sets defined by polynomial matrix inequalities. The main feature is that the lifting variables can be parameterized by polynomials. Such polynomials are called lifting polynomials. A typical application of this lifted Positivstellensatz is to certify that a spectrahedrop (i.e., projection of a spectrahedron) is contained in another spectrahedrop. Under some mild natural assumptions, we have shown that the proposed lifted matrix Positivstellensatz is a sufficient and necessary certificate for the containment. The certificate can be searched for by solving a semidefinite program.

6.1. **The case of scalar polynomials.** Theorems 3.1 and 3.3 also apply to projections of semialgebraic sets defined by scalar polynomials. We thus obtain a large class of Positivstellensätze for projections of semialgebraic sets.

Let $g_1(\mathtt{x},\mathtt{y}),\dots,g_k(\mathtt{x},\mathtt{y})$ and $q_1(\mathtt{x},\mathtt{z}),\dots,q_t(\mathtt{x},\mathtt{z})$ be scalar polynomials. They give semialgebraic sets

$$\begin{aligned} K_1 &= \{x\in\mathbb{R}^n : \exists y\in\mathbb{R}^r,\ g_1(x,y)\ge 0,\dots,g_k(x,y)\ge 0\},\\ K_2 &= \{x\in\mathbb{R}^n : \exists z\in\mathbb{R}^s,\ q_1(x,z)\ge 0,\dots,q_t(x,z)\ge 0\}. \end{aligned} \tag{6.1}$$

We can get a Positivstellensatz certificate for the containment $K_1\subseteq K_2$.

Corollary 6.1. *Let $g_1,\dots,g_k\in\mathbb{R}[\mathtt{x},\mathtt{y}]$ and $q_1,\dots,q_t\in\mathbb{R}[\mathtt{x},\mathtt{z}]$ be scalar polynomials, and let K_1,K_2 be as in (6.1). Assume the quadratic module of $(g_1,\dots,g_k)$ is archimedean and the degrees of q_j in z are at most $2d$. If for every $x\in K_1$, each $q_j(x,\mathtt{z})$ is sos-concave in $\mathtt{z}$ and there exists z such that $q_j(x,z)>0$, then there exist polynomials $p_\alpha\in\mathbb{R}[\mathtt{x}]$ $(\alpha\in\mathbb{N}^s_{2d})$ and sos polynomials $\sigma_{ij}\in\mathbb{R}[\mathtt{x},\mathtt{y}]$ such that $M(p)$ is an sos matrix polynomial and for each $j=1,\dots,t$,*

$$q_j(\mathtt{x},p(\mathtt{x})) = \sigma_{j0}(\mathtt{x},\mathtt{y}) + \sum_{i=1}^{k} g_i(\mathtt{x},\mathtt{y})\sigma_{ij}(\mathtt{x},\mathtt{y}). \tag{6.2}$$

Proof. By Theorem 3.3, there exists a polynomial tuple $p=(p_1,\dots,p_s)\in\mathbb{R}[\mathtt{x}]^s$ and matrix polynomials $V_i(\mathtt{x},\mathtt{y})$ such that

$$\operatorname{diag}\big(q_1(\mathtt{x},p(\mathtt{x})),\dots,q_t(\mathtt{x},p(\mathtt{x}))\big) = \\ \sum_i V_i(\mathtt{x},\mathtt{y})^T \operatorname{diag}\big(g_1(\mathtt{x},\mathtt{y}),\dots,g_k(\mathtt{x},\mathtt{y})\big)V_i(\mathtt{x},\mathtt{y}) + V_0(\mathtt{x},\mathtt{y})^T V_0(\mathtt{x},\mathtt{y}).$$

Comparing diagonal entries, we see that (6.2) holds for some sos polynomials $\sigma_{ij}(\mathtt{x},\mathtt{y})$. ☐

6.2. **Some open questions.** In future research, the following interesting and important questions should be addressed. They are mostly open to the authors.

Question 6.2. In the certificates (1.9), (3.4), or (3.8), for what kinds of matrix polynomials $G(\mathtt{x},\mathtt{y})$ and $Q(\mathtt{x},\mathtt{z})$, can we choose the polynomials V_j to be independent of $\mathtt{y}$?

The above question is of great interest in computation. If each V_j is independent of $\mathtt{y}$, the semidefinite programs searching for (1.9), (3.4), or (3.8) become much easier to solve. In Example 4.4, the polynomials V_j must depend on $\mathtt{y}$. However, in all the other examples, we can choose V_j to be independent of $\mathtt{y}$.

Convexity is used in a key step in the proofs of Theorems 3.1 and 3.3 to obtain a lifting polynomial map $P_G \to S_Q$. When $Q(\mathtt{x}, \mathtt{z})$ is not convex in $\mathtt{z}$, the lifting polynomials might not exist, cf. Example 3.4. This leads to the following challenging problem:

Question 6.3. In Theorem 3.3, when $Q(\mathtt{x}, \mathtt{z})$ is not sos-concave in $\mathtt{z}$, what is an appropriate certificate for ensuring $P_G \subseteq P_Q$?

Finally, we conclude with the problem of detecting equality of spectrahedrops:

Question 6.4. For two linear pencils $A(\mathtt{x}, \mathtt{y})$ and $B(\mathtt{x}, \mathtt{z})$, what is the appropriate certificate for $P_A = P_B$?

The certificate (4.1) ensures $P_A \subseteq P_B$. To ensure $P_B \subseteq P_A$, one might be tempted to apply a similar certificate again. However, this usually does not work because (4.1) typically requires $P_A \subseteq \mathrm{int}(P_B)$. To get a similar certificate for $P_B \subseteq P_A$, one usually needs $P_B \subseteq \mathrm{int}(P_A)$. Clearly, $P_A \subseteq \mathrm{int}(P_B)$ and $P_B \subseteq \mathrm{int}(P_A)$ generally do not hold simultaneously.

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References

[B-TN02] A. Ben-Tal, A. Nemirovski: *On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty,* SIAM J. Optim. 12 (2002) 811–833. 1, 17

[BPR13] G. Blekherman, P. Parrilo, R. Thomas (editors): *Semidefinite optimization and convex algebraic geometry*, MOS-SIAM Series on Optimization 13, SIAM, 2013. 6

[BCR98] J. Bochnak, M. Coste, M. F. Roy: *Real algebraic geometry*, Results in Mathematics and Related Areas (3) 36, Springer-Verlag, Berlin, 1998. 3

[Cim12] J. Cimprič: *Real algebraic geometry for matrices over commutative rings*, J. Algebra 359 (2012) 89–103. 6

[deK02] E. de Klerk: *Aspects of semidefinite programming: Interior point algorithms and selected applications*, Applied Optimization, 65. Kluwer Academic Publishers, Dordrecht, 2002. 1, 6

[deKL11] E. de Klerk, M. Laurent: *On the Lasserre hierarchy of semidefinite programming relaxations of convex polynomial optimization problems*, SIAM J. Optim. 21 (2011) 824–832. 6

[FO85] R. Freund and J. Orlin: *On the complexity of four polyhedral set containment problems*, Math. Program. 33 (1985) 139–145. 1

[FNT17] T. Fritz, T. Netzer, A. Thom: *Spectrahedral Containment and Operator Systems with Finite-Dimensional Realization*, SIAM J. Appl. Algebra Geom. 1 (2017) 556–574. 6

[GN11] J. Gouveia and T. Netzer: *Positive polynomials and projections of spectrahedra*, SIAM J. Optim. 21 (2011) 960–976. 3

[GPT13] J. Gouveia, P.A. Parrilo, R.R. Thomas: *Lifts of convex sets and cone factorizations*, Math. Oper. Res. 38 (2013) 248–264. 6

[GT13] J. Gouveia, R.R. Thomas: *Spectrahedral approximations of convex hulls of algebraic sets*, Semidefinite optimization and convex algebraic geometry, 293-–340, MOS-SIAM Series on Optimization 13, SIAM, 2013. 6

[GK94] P. Gritzmann and V. Klee: *On the complexity of some basic problems in computational convexity: I. Containment problems.*, Discrete Math. 136 (1994) 129–174. 1

[HKM12] J. W. Helton, I. Klep and S. McCullough: *The convex Positivstellensatz in a free algebra*, Adv. Math. 231 (2012) 516–534. 2

[HKM13] J. W. Helton, I. Klep and S. McCullough: *The matricial relaxation of a linear matrix inequality*, Math. Program. 138 (2013) 401–445. 2, 14

[HKM17] J. W. Helton, I. Klep and S. McCullough: *The tracial Hahn-Banach theorem, polar duals, matrix convex sets, and projections of free spectrahedra*, J. Eur. Math. Soc. 19 (2017) 1845–1897. 3

[HN09] J. W. Helton and J. Nie: *Sufficient and necessary conditions for semidefinite representability of convex hulls and sets,* SIAM J. Optim. 20 (2009) 759–791. 2

[HN10] J. W. Helton and J. Nie: *Semidefinite representation of convex sets*, Math. Program. 122 (2010) 21–64. 2, 6, 9, 14

[HG05] D. Henrion, A. Garulli (editors): *Positive polynomials in control*, Lecture Notes in Control and Information Sciences 312, Springer-Verlag, 2005. 6

[HL06] D. Henrion and J. B. Lasserre: *Convergent relaxations of polynomial matrix inequalities and static output feedback*, IEEE Trans. Automat. Control 51 (2006) 192-–202. 6

[HNS16] D. Henrion, S. Naldi, M. Safey El Din: *Exact algorithms for linear matrix inequalities*, SIAM J. Optim. 26 (2016) 2512–2539. 6

[KTT13] K. Kellner, T. Theobald and C. Trabandt: *Containment problems for polytopes and spectrahedra*, SIAM J. Optim. 23 (2013) 1000–1020. 2, 14

[KTT15] K. Kellner, T. Theobald and C. Trabandt: *A semidefinite hierarchy for containment of spectrahedra*, SIAM J. Optim. 25 (2015) 1013–1033. 2, 14

[KS10] I. Klep and M. Schweighofer: *Pure states, positive matrix polynomials and sums of hermitian squares*, Indiana Univ. Math. J. 59 (2010) 857–874. 2, 3, 6, 8, 10, 15

[KS13] I. Klep and M. Schweighofer: *An exact duality theory for semidefinite programming based on sums of squares*, Math. Oper. Res. 38 (2013) 569–590. 2, 15, 16

[Las01] J. B. Lasserre: *Global optimization with polynomials and the problem of moments*, SIAM J. Optim., 11 (2001) 796–817. 6

[Las09a] J. B. Lasserre: *Convex sets with semidefinite representation*, Math. Program. 120 (2009) 457–477. 2, 18

[Las09b] J. B. Lasserre: *Convexity in semialgebraic geometry and polynomial optimization*, SIAM J. Optim. 19 (2009) 1995–2014. 9

[Las09c] J. B. Lasserre: *Moments, Positive Polynomials and Their Applications*, Imperial College Press, 2009. 6

[Las15] J. B. Lasserre: *An introduction to polynomial and semi-algebraic optimization*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2015. 2, 6, 14, 18

[Lau09] M. Laurent: *Sums of squares, moment matrices and optimization over polynomials*, Emerging Applications of Algebraic Geometry (M. Putinar, S. Sullivant, eds.), IMA Volumes Math. Appl. 149, Springer, 157–270, 2009. 6

[Lau14] M. Laurent: *Optimization over polynomials: Selected topics*, Proceedings of the International Congress of Mathematicians (ICM 2014), 843–869. 2014. 6

[Mar08] M. Marshall: *Positive polynomials and sums of squares*, Mathematical Surveys and Monographs 146, American Mathematical Society, 2008. 4

[Nem06] A. Nemirovski: *Advances in convex optimization: conic programming,* plenary lecture, International Congress of Mathematicians (ICM), Madrid, Spain, 2006. 1, 17

[NPS10] T. Netzer, D. Plaumann, M. Schweighofer: *Exposed faces of semidefinitely representable sets*, SIAM J. Optim. 20 (2010) 1944–1955. 2

[Nie11] J. Nie: *Polynomial matrix inequality and semidefinite representation*, Math. Oper. Res. 36 (2011) 398–415. 9

[PS03] P. Parrilo and B. Sturmfels: *Minimizing polynomial functions*, Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science (S. Basu and L. Gonzalez-Vega, eds.), DIMACS Series in Discrete Mathematics and Computer Science 60, 83–99, AMS, 2003. 6

[Put93] M. Putinar: *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J. 42 (1993) 969–984. 6

[PV99] M. Putinar, F.H. Vasilescu: *Solving moment problems by dimensional extension*, C. R. Math. Acad. Sci. Paris 328 (1999) 495–499. 6

[RT08] C. Riener, T. Theobald: *Positive Polynome und semidefinite Optimierung.* (German), Jahresber. Deutsch. Math.-Verein. 110 (2008) 57-–74. 6

[Rud76] W. Rudin: *Principles of mathematical analysis*, Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. 8

[Sha97] A. Shapiro: *First and second order analysis of nonlinear semidefinite programs*, Math. Program. 77 (1997) 301–320. 8

[Sce09] C. Scheiderer: *Positivity and sums of squares: A guide to recent results*, Emerging Applications of Algebraic Geometry (M. Putinar, S. Sullivant, eds.), IMA Volumes Math. Appl. 149, 271–324, Springer, 2009. 6

[Sce11] C. Scheiderer: *Convex hulls of curves of genus one*, Adv. Math. 228 (2011) 2606–2622. 2, 14

[Sce18] C. Scheiderer: *Spectrahedral shadows*, SIAM J. Appl. Algebra Geometry 2 (2018) 26–44. 2, 14

[SH06] C. Scherer, C. Hol: *Matrix sum-of-squares relaxations for robust semi-definite programs*, Math. Program., Ser. B, 107 (2006) 189–211. 2, 3, 6, 8, 10

[Scm09] K. Schmüdgen: *Noncommutative real algebraic geometry – some basic concepts and first ideas*, Emerging applications of algebraic geometry, 325–350, IMA Vol. Math. Appl. 149, Springer, 2009. 6

[Scw05] M. Schweighofer: *Optimization of polynomials on compact semialgebraic sets*, SIAM J. Optim. 15 (2005) 805–825. 6

[VBW98] L. Vandenberghe, S. Boyd, Shao-Po Wu: *Determinant maximization with linear matrix inequality constraints*, SIAM J. Matrix Anal. Appl. 19 (1998) 499–533. 19

[WSV00] H. Wolkowicz, R. Saigal and L. Vandenberghe (eds.): *Handbook of Semidefinite Programming: Theory, Algorithms and Applications*, International Series in Operations Research & Management Science, Kluwer Academic Publishers, 2000. 1, 6, 8

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