IV. NC FUNCTION THEORY

Takesaki[67] : C^*-algebra A = " continuous $\overset{\text{bounded}}{|}$ nc functions
$\mathrm{Rep}(A, \mathcal{H}) \longrightarrow B(\mathcal{H})$ "

$\uparrow$

"nc" Gelfand duality "A" "all nc functions"

Taylor (72-3) nc spectral theory
↳ nc functional calculus
↳ nc power series (Joseph Taylor – Brook Taylor series)

D. Kaliuzhnyi–Verbovetskyi & V. Vinnikov

Voiculescu (2004, 2010) : free analysis w/ applications
in free probability

Popescu et al : nc function theory inspired by
several complex variables & operator theory

NC SETS

$\mathcal{M} = (M_n(\mathbb{C}))_{n\in\mathbb{N}}$, $\mathcal{M}^d = (M_n(\mathbb{C})^d)_n$

- $\mathcal{U} = (\mathcal{U}_n)_n \subseteq \mathcal{M}$ is <u>graded</u> if
$\forall n,m \ \forall S \in \mathcal{U}_n \ \forall T \in \mathcal{U}_m \quad S \oplus T = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \in \mathcal{U}_{n+m}$

- $\mathcal{U}$ is an <u>nc set</u> if it is graded & closed under unitary
conjugation : $\forall S \in \mathcal{U}_n \ \forall n\times n$ unitary U : $U^* S U \in \mathcal{U}_n$

NC FUNCTIONS

An <u>nc function</u> $f : \mathcal{U} \longrightarrow \mathcal{M}^{\tilde{d}}$ is a sequence of functions $f = (f_n)_{n \geq 1}$ where $f_n : \mathcal{U}_n \longrightarrow M_n(\mathbb{C})^{\tilde{d}}$ s.t.

(a) $\forall S \in \mathcal{U}_n \ \forall T \in \mathcal{U}_m$:

$$f_{n+m}\begin{pmatrix} S & \\ & T \end{pmatrix} = \begin{pmatrix} f_n(S) & \\ & f_m(T) \end{pmatrix} \Bigg\} \text{ f respects direct sums}$$

(b) If $S, T \in \mathcal{U}_n$ & $\exists P \in GL_n(\mathbb{C})$: $\underline{T = P^{-1} S P}$, then

$$f_n(T) = P^{-1} f_n(S) \cdot P$$

f respects similarities

Say that $f : \mathcal{U} \longrightarrow \mathcal{M}^{\hat{d}}$ respects intertwining if

$\forall n \ \forall m \quad X \in \mathcal{U}_n \quad Y \in \mathcal{U}_m$ & $\Gamma : \mathbb{C}^m \longrightarrow \mathbb{C}^n$

if $\underset{(X_1\Gamma, \dots, X_d\Gamma)}{X \cdot \Gamma} = \underset{(\Gamma Y_1, \dots, \Gamma Y_d)}{\Gamma \cdot Y} \overset{\text{then}}{\Longrightarrow} f(X) \cdot \Gamma = \Gamma \cdot f(Y)$

<u>Prop</u>: f is an nc map $\Longleftrightarrow$ f respects intertwining

<u>Proof</u>: $f(X)\Gamma = \Gamma f(Y)$ iff

$$\begin{pmatrix} f(X) & f(X)\Gamma \\ 0 & f(Y) \end{pmatrix} = \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix} \cdot \begin{pmatrix} 1 & \Gamma \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} f(X) & \Gamma f(Y) \\ 0 & f(Y) \end{pmatrix} = \begin{pmatrix} 1 & \Gamma \\ & 1 \end{pmatrix} \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix}$$

$\square$

IV. NC FUNCTION THEORY

<u>Example</u>: • $f \in \mathbb{C}\langle x\rangle$ is an nc function!

• $f(x) = x_1^T$ or x_1^* <u>NOT</u> nc functions

$$f(P^{-1}XP) = (P^{-1}XP)^T = P^{-T} X^T P^{-T}$$

$$P^{-1} f(X) P = P^{-1} X^T P$$

• Formal power series $\mathbb{C}\langle\langle x\rangle\rangle$

$$f = \sum_{w \in \langle x\rangle}^{\text{infinite}} \underset{\underset{\mathbb{C}}{\cap\!\!\!\cap}}{f_w} \cdot w$$

"convergent nc power series"

A natural domain for such f is

$\{$tuples of <u>jointly nilpotent</u> matrices $X\}$

nc rational functions $\mathbb{C}\langle\!\langle x \rangle\!\rangle$

$\mathcal{R}(x)$ ... formal nc rational expressions

= formal expressions built from $\mathbb{C}\langle x\rangle$ using

$(\,,\,)$, ${}^{-1}$, $+$, $\cdot$ that are syntactically valid

e.g. $((x_1 - x_3^3)^{-1} + 7x_2)^{-1}$, $(x - x)^{-1}$

each rat. expression r induces $\forall m$ $M_m(\mathbb{C})^d \xrightarrow{r_m} M_m(\mathbb{C})$

$$\operatorname{dom}_m r = \{X \in M_m(\mathbb{C})^d \mid r_m \text{ is defined at } X\}$$

$$\operatorname{dom} r = \bigcup_{m \in \mathbb{N}} \operatorname{dom}_m r$$

r is degenerate if $\operatorname{dom} r = \emptyset$ $\longrightarrow$ forget these

• r_1, r_2 nondegenerate, $r_1 \sim r_2 :\Leftrightarrow$

$$r_1\big|_{\operatorname{dom} r_1 \cap \operatorname{dom} r_2} = r_2\big|_{\operatorname{dom} r_1 \cap \operatorname{dom} r_2}$$

$\mathbb{r} = [r]_{\sim}$ is an nc rational function.

$\mathbb{C}\langle\!\langle x \rangle\!\rangle = \{ \mathbb{r} \mid r \mid \underset{\text{nondeg.}}{\overset{\text{nc}}{\text{rat}}} \text{ expression} \}$

$\operatorname{dom} \mathbb{r} := \bigcup_{[r]=\mathbb{r}} \operatorname{dom} r$

Theorem: $\mathbb{C}\langle\!\langle x \rangle\!\rangle$ is a skew field.
(Amitsur '67)

Example: $r_1 = (1 - x_1 - x_2 (1-x_1)^{-1} x_2)^{-1}$

$r_2 = -x_2^{-1} (1-x_1) \cdot (x_2 - (1-x_1) x_2^{-1} (1-x_1))^{-1}$

$r_3 = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(I_2 - \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix} - \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \right)^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}\langle\!\langle x \rangle\!\rangle$

Then $r_i \sim r_j \ \forall i \ \forall j$

$(1,1) \in \operatorname{dom} r_2 \setminus \operatorname{dom} r_1$

$(0,0) \in \operatorname{dom} r_1 \setminus \operatorname{dom} r_2$

$\operatorname{dom} r_3 \supsetneq \operatorname{dom} r_1 \cup \operatorname{dom} r_2$

Theorem (Schützenberger '61) $0 \in \operatorname{dom} \mathbb{r}$

Then $\mathbb{r}$ has a linear representation:

$\exists n \in \mathbb{N} \quad \exists b, c \in \mathbb{C}^n \quad \exists A \in M_n(\mathbb{C})^d$ s.t.

$$\mathbb{r} = c^* \cdot \left(I_n - \sum A_j x_j \right)^{-1} \cdot b$$

Theorem (Fliess '74): If n is minimal, then two such linear representations are similar via a unique transition matrix.

$\mathbb{r} \sim (c, A, b) \qquad \mathbb{r} \sim (\tilde{c}, \tilde{A}, \tilde{b})$

$\exists! P \in PGL_n(\mathbb{C})$

$\tilde{A} = P^{-1} A P$

$\tilde{b} = P^{-1} b$

$\tilde{c}^* = c^* P$

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Example: • $f \in \mathbb{C}\langle x\rangle$ is an nc function!

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$$f(P^{-1}XP) = (P^{-1}XP)^T = P^{T}X^T P^{-T}$$

$$P^{-1}f(X)P = P^{-1}X^T P$$

• Formal power series $\mathbb{C}\langle\langle x\rangle\rangle$

$$f = \sum_{w\in\langle x\rangle}^{\text{infinite}} f_w \cdot w, \quad f_w \in \mathbb{C}$$

"convergent nc power series"

A natural domain for such f is

$\{$tuples of jointly nilpotent matrices $X\}$

$\exists r \in \mathbb{N}$ s.t. $|w| \geq r$, $w(X) = 0$

nc rational functions $\mathbb{C} \langle\!\!\langle x \rangle\!\!\rangle$

(c, A, b) is minimal $\iff$

$$\text{span}\{w(A)b \mid w \in \langle x\rangle\} = \mathbb{C}^n$$

&

$$\text{span}\{c^* w(A) \mid w \in \langle x\rangle\} = \mathbb{C}^n$$

Hankel condition (Fliess 74-5)

$$r = c^*(1 - \sum A_j x_j)^{-1} b$$

formal power series

$$\sum_{w\in\langle x\rangle} \underbrace{c^* w(A) b}_{r_w \in \mathbb{C}} \cdot w$$

$$H_r = (r_{u \cdot v})_{u,v\in\langle x\rangle}$$

Thm (Fliess 74-5)

Nc power series r comes from a nc rational fun

$\iff$ rank $H_r < \infty$

is then the size of the min. realization

Theorem (Schützenberger 61) $0 \in \operatorname{dom} r$

Then r has a linear representation:

$\exists n \in \mathbb{N}$ $\exists b, c \in \mathbb{C}^n$ $\exists A \in M_n(\mathbb{C})^d$ s.t.

$$r = c^* \cdot \left(I_n - \sum A_j x_j\right)^{-1} \cdot b$$

Theorem (Fliess 74): If n is minimal, then two such linear representations are similar via a unique transition matrix.

$r \sim (c, A, b)$ $\quad$ $r \sim (\tilde{c}, \tilde{A}, \tilde{b})$

$\exists!\, P \in PGL_n(\mathbb{C})$

$$\tilde{A} = P^{-1} A P$$

$$\tilde{b} = P^{-1} b$$

$$\tilde{c}^* = c^* P$$

How to find realizations?

$r_j \sim (c_j, A^j, b_j)$

$$r_1 + r_2 \sim \left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} A^1 & \\ & A^2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right)$$

$$r_1 \cdot r_2 \sim \left(\begin{pmatrix} c_1 \\ c_1^* b_1 c_2 \end{pmatrix}, \begin{pmatrix} A^1 & A^1 b_1 c_2^* \\ 0 & A^2 \end{pmatrix}, \begin{pmatrix} 0 \\ b_2 \end{pmatrix} \right)$$

$r_0(0) = a \neq 0$

$$r_0^{-1} \sim \left(\begin{pmatrix} -a^{-1} c_0 \\ a^{-1} \end{pmatrix}, \begin{pmatrix} A^1(1 - b_0 a^{-1} c_1^*) & -A^1 b_1 a^{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Theorem (Kal...-Verb... & Vinnikov 2009, Volčič 2015) If $r \sim (c, A, b)$ is a minimal realization, then

$$\operatorname{dom} r = \left\{ X \mid I \otimes I - \sum A_j \otimes X_j \text{ is invertible} \right\}$$

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Cor: If r satisfies $\operatorname{dom} r = \mathcal{M}^d$, then $r \in \mathbb{C}\langle x\rangle$. Eq., in the minimal realization $r = (c, A, b)$, the A_s are jointly nilpotent.

Prop (continuous nc functions are analytic)
Suppose $\mathcal{U} \subseteq \mathcal{M}^d$ is open (each $\mathcal{U}_n \subseteq M_n(\mathbb{C})^d$ is open) and $f : \mathcal{U} \to \mathcal{M}^{\tilde{d}}$ is continuous nc map
each f_n is continuous

nc rational functions $\mathbb{C} \langle\!\!\langle x \rangle\!\!\rangle$

Then f is analytic (i.e., each f_n is analytic)

Def (Directional derivatives)
f nc map, $X, H \in M_n(\mathbb{C})^d$

$$f'(X)[H] := \lim_{t \to 0} \frac{f(X+tH) - f(X)}{t}$$

Ex: $f = x^3$

$$\begin{aligned}(X+tH)^3 - X^3 &= X^3 + tX^2H + tXHX + tHX^2 \\ &\quad + t^2XH^2 + t^2HXH + t^2H^2X + t^3H^3 - X^3 \\ &= t(X^2H + XHX + HX^2) + t^2 \cdot \text{stuff}\end{aligned}$$

$$\Rightarrow f'(X)[H] = X^2H + XHX + HX^2$$

"$X^2 \otimes 1 + X \otimes X + 1 \otimes X^2$"

$g = x^{-1}$

$(X+tH)^{-1} - X^{-1}$

$[X(1+tX^{-1}H)]^{-1} - X^{-1}$

$(1+tX^{-1}H)^{-1}X^{-1} - X^{-1}$

$= [(1+tX^{-1}H)^{-1} - 1]X^{-1}$

$$[1 - tX^{-1}H + t^2(X^{-1}H)^2 - t^3 \cdots - 1]X^{-1} = (-tX^{-1}H + t^2 \cdot \text{stuff})X^{-1}$$

$$\longrightarrow -X^{-1}HX^{-1}$$

The main trick to prove proposition is to show $f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & f'(X)[H] \\ 0 & f(X) \end{pmatrix}$

(Elementary 2x2 matrix lin. alg. calculation)

- Continuous nc function

$\Longrightarrow$ analytic nc function

$\Longrightarrow$ convergent nc power series expansion (Taylor-Taylor series)

Grothendieck theorem

Theorem (Grothendieck 66, Ax 68)

Let $P : \mathbb{C}^d \to \mathbb{C}^d$ be a polynomial map.

If P is injective, then P is bijective.

Moreover, P^{-1} is also a polynomial map.

Theorem (Bass–Connell–Wright[82])

Suppose $f: \mathbb{C}^d \to \mathbb{C}^d$ is a polynomial map w/ inverse f^{-1}.

Then $\deg(f^{-1}) \leq (\deg f)^{d-1}$

Jacobian conjecture: If $f: \mathbb{C}^d \to \mathbb{C}^d$ poly map has $|Jf| = $ constant, then f is invertible

$$Jf = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$$

Theorem (NC inverse function theorem, Pascoe[14])

Suppose $f: \mathcal{U} \to \mathcal{M}^d$ is a nc function. TFAE

(a) $f'(X)$ is nonsingular for all $X \in \mathcal{U}$

(b) f is injective

(c) f^{-1} exists and is an nc function.

IV. NC FUNCTION THEORY

(b) $\Rightarrow$ (a) Suppose $f'(X)$ is singular, i.e., $f'(X)[H] = 0$ for some nonzero H. Then

$$f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & f'(X)[H] \\ 0 & f(X) \end{pmatrix} = \begin{pmatrix} f(X) & 0 \\ 0 & f(X) \end{pmatrix} = f\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$$

(a) $\Rightarrow$ (b) is a 4×4 block matrix calculation $\square$

Corollary (nc Grothendieck–Ax thm): If $P: \mathcal{M}^d \to \mathcal{M}^d$ is an nc poly map, and $\underbrace{P \text{ is injective}}_{\text{eg: } P' \text{ is nonsingular everywhere}}$, then P is bijective & $P^{-1}\big|_{M_n(\mathbb{C})^d}$ is given by an nc poly.

- Continuous nc function

$\Longrightarrow$ analytic nc function

$\Longrightarrow$ convergent nc power series expansion (Taylor-Taylor series)

Grothendieck theorem

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If P is injective, then P is bijective.

Moreover, P^{-1} is also a polynomial map.

Theorem (Augat '18) ProcLMS

If $P: M^d \to M^d$ is an nc polynomial & is bijective, then P^{-1} is also an nc poly.

"hyporational functions"

$\sum a_i z b_i = C$

$x_1 \mapsto x_1 + x_2^2$

$x_2 \mapsto x_2$