Duality between compact matrix cvx sets & operator systems

$\mathcal{R}$ $\overset{\text{fin-dim}}{\text{operator system}}$ (:= closed subspace of $B(H)$ w/ $1\in\mathcal{R}$, $\mathcal{R}^*=\mathcal{R}$)

$UCP(\mathcal{R}) = (UCP_n(\mathcal{R}))_{n\in\mathbb{N}}$

$UCP_n(\mathcal{R}) = \{\varphi : \mathcal{R} \to M_n(\mathbb{R}) \text{ ucp}\}$

Is a matrix convex set, <u>compact</u>

K compact mtx cvx set

W vector space; $\theta : K \to W$ is a matrix affine map if

$\theta = (\theta_n)_n$, $\theta_n : K(n) \to M_n(W)$ satisfies

$$\theta_n\Big(\sum \underset{n_i\times n}{\gamma_i}^* \underset{K_{n_i}}{v_i} \gamma_i\Big) = \sum \gamma_i^* \,\theta_{n_i}(v_i)\, \gamma_i \quad \text{for } \sum \gamma_i^* \gamma_i = I$$

$\rightsquigarrow A(K,W) = \{\text{mtx aff map } K\to W\}$

$A(K,\mathbb{R})$ is an (abstract) operator system [Choi–Effros 77]

$M_r(A(K,\mathbb{R})) \overset{\text{identify}}{\sim} A(K, M_r(\mathbb{R}))$

$F\in A(K,M_r(\mathbb{R}))$ is <u>positive</u> if

$\forall s\ \forall v\in K(s)$ $F_s(v)\geq 0$

The constant function $\mathbb{I}$ is the desired (matrix) order unit.

Then $A(K,\mathbb{R})$ satisfies the C-E77 axioms on an abstract op. sys.

<u>Theorem</u> (Duality [Webster–Winkler 99])

(a) If $\mathcal{R}$ is an operator system, then $UCP(\mathcal{R})$ is a cpt mtx cvx set & $A(UCP(\mathcal{R}),\mathbb{R})$ and $\mathcal{R}$ are isomorphic op. systems.

(b) K cpt mtx cvx set $\Rightarrow$ $A(K,\mathbb{R})$ is an op. system & $UCP(A(K,\mathbb{R}))$ are matrix affinely homeomorphic.

Polars

- $A \in \mathbb{S}_n^d$ $\qquad \underset{\text{matrix cvx hull}}{\operatorname{matco}}(\{A\}) = \left\{ V^*(I_\mu \otimes A)V \;\middle|\; \mu \in \mathbb{N},\ V \text{ isometry} \right\}$

$K = (K_n)_n \subseteq (\mathbb{S}_n^d)_n$

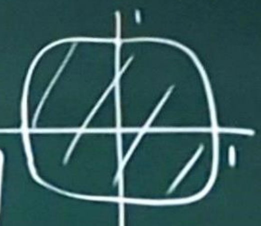
Its polar is $K^\circ = \bigcup_n \left\{ A \in \mathbb{S}_n^d \;\middle|\; L_A|_K \succeq 0 \right\}$

$L_A(X) = I \otimes I - \sum A_j \otimes X_j \succeq 0 \quad \forall X \in K$

Properties:

- K° closed mtx cvx set w/ $0 \in K^\circ$
- $0 \in \operatorname{int} K \Rightarrow K^\circ$ is bounded
- K bdd $\Leftrightarrow 0 \in \operatorname{int} K^\circ$
- $0 \in K \Rightarrow (K^\circ)^\circ = \overline{\operatorname{matco} K}$
- K closed mtx cvx, $0 \in K$ $\Rightarrow K^{\circ\circ} = K$ (Bipolar thm)

Puzzle of the Day #4



$x^4 + y^4 \leq 1$

Is this a spectrahedrop?

What is $\mathcal{D}_{L_A}^\circ$? It is $\operatorname{matco}(\{A, 0\})$
$(\{A \oplus 0\})$

Both free spectrahedra & singly generated mtx cvx sets are examples of free spectrahedrops (:= projections of spectrahedra)

Think "Choi matrix"

LMIs vs cp maps

Example: $\Delta(x) = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix}$ $\qquad \Gamma(x) = \begin{pmatrix} 1+x_1 & x_2 \\ x_2 & 1-x_1 \end{pmatrix}$

$\mathcal{D}_\Delta(1) = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \}$ $\qquad \mathcal{D}_\Gamma(1) = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \}$

$$\Delta\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3/4 \\ 3/4 & 0 \end{pmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 3/4 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 1 & 0 \\ 3/4 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \succeq 0$$

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$$\Gamma\left(\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3/4 \\ 3/4 & 0 \end{pmatrix}\right) = \begin{pmatrix} 3/2 & 0 & 0 & 3/4 \\ 0 & 1 & 3/4 & 0 \\ 0 & 3/4 & 1/2 & 0 \\ \text{[illegible]} & 0 & 0 & 1 \end{pmatrix} \not\succeq 0$$

$\rightarrow \mathcal{D}_\Delta \not\subseteq \mathcal{D}_\Gamma$

$\mathcal{D}_\Gamma \subseteq \mathcal{D}_\Delta$ since $\Delta(x) = V_1^* \Gamma(x) V_1 + V_2^* \Gamma(x) V_2$

for $V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $V_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

<u>Theorem</u>: $L, \tilde{L}$ be monic linear pencils. Assume $\mathcal{D}_L(1)$ is bounded.

(a) $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}} \Longleftrightarrow \tilde{L}(x) = \sum_j V_j^* L(x) V_j$ for $\sum V_j^* V_j = I$

(b) If $L, \tilde{L}$ are minimal, then

$\mathcal{D}_L = \mathcal{D}_{\tilde{L}} \Longleftrightarrow \exists\, U$ orthogonal mtx s.t. $\tilde{L}(x) = U^* L(x) U$.

$\sim$

Lemma: Let $L(x) = I + \overbrace{\sum A_j x_j}^{\Lambda(x)}$ be $\delta \times \delta$.

(a) If T and $X_j \in M_n(\mathbb{R})$ and

$$S := I \otimes T + \sum A_j \otimes X_j \quad \text{is self-adjoint,}$$

then $T = T^*$ & $X_j^* = X_j$ all j.

(b) If $S \succeq 0$, then $T \succeq 0$.

Proof: (a) $S^* - S = I \otimes (T^* - T) + \sum_j A_j \otimes (X_j^* - X_j)$ + prev. lemma imply T, X_j are s.a.

(b) Suppose $T \not\succeq 0$. Pick unit vector v st $\langle Tv, v\rangle < 0$.

$P : \mathbb{R}^\delta \otimes \mathbb{R}^n \longrightarrow \mathbb{R}^\delta \otimes \mathbb{R}v$ be the orthogonal projection &

$y_j := \langle X_j v, v\rangle$. Then

$$PSP = P\left(I \otimes T + \sum A_j \otimes X_j\right)P$$

$\tilde{\mathcal{S}} = \operatorname{span}(1, \tilde{A}_j)$, $\tilde{L}(x) = 1 + \sum \tilde{A}_j x_j$

Consider $\tau : \mathcal{S} \to \tilde{\mathcal{S}}$, $I \mapsto I$, $A_j \mapsto \tilde{A}_j$ linear

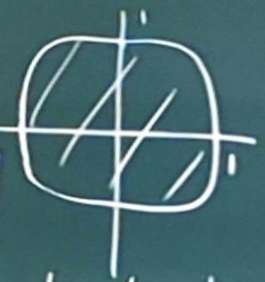
Lemma: $(I, A_1, \ldots, A_d)$ are linearly independent.

Proof: Suppose $\lambda + \sum x_j A_j = 0$. wlog $x_j \neq 0$ for some j. $z = (x_1 \ldots x_d) \in \mathbb{R}^d$. If $\lambda = 0$, then $L(tz) = I$ $\forall t \in \mathbb{R}$.

So $\mathbb{R} \subseteq \mathcal{D}_L(1)$ ↯ since $\mathcal{D}_L(1)$ bdd.

Hence $\lambda \neq 0$. Then $L(z/\lambda) = 0$ and thus $L(t \cdot z/\lambda) \preceq 0$ $\forall t < 0$. ∎

Puzzle of the Day #4



$x^4 + y^4 \leq 1$

Is this a spectrahedron?

$= I \otimes \langle T v, v \rangle + \Lambda(Y) \succeq 0$

So $\Lambda(Y) \succ 0$

Hence $t \cdot Y \in \mathcal{D}_L(1)$ $\forall t > 0$

contrary to boundedness of $\mathcal{D}_L(1)$. ∎

<u>Theorem</u>: $L, \tilde{L}$ be monic linear pencils. Assume $\mathcal{D}_L(1)$ is bounded.

(a) $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}} \iff \tilde{L}(x) = \sum_j V_j^* L(x) V_j$ for $\sum V_j^* V_j = I$

(b) If $L, \tilde{L}$ are minimal, then

$\mathcal{D}_L = \mathcal{D}_{\tilde{L}} \iff \exists\, U$ orthogonal mtx s.t. $\tilde{L}(x) = U^* L(x) U$.

<u>Proof</u> (Theorem, (a)). Assume $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$. Claim τ is ucp.

Suppose $S = I \otimes T + \sum A_j \otimes X_j \in M_n(\mathcal{Y})$ is PSD.

By lemma, T is PSD. Form $T_\varepsilon = T + \varepsilon I$ $(\varepsilon > 0)$, S_ε from T_ε.

$S_\varepsilon \in M_n(\mathcal{Y})$ is PSD

$$(I \otimes T_\varepsilon^{-1/2})\Big(\underbrace{I \otimes T_\varepsilon + \sum A_j \otimes X_j}_{\substack{\geq \\ 0}}\Big)^{S_\varepsilon\ \text{``}\,=\,\text{''}}(I \otimes T_\varepsilon^{-1/2}) = I \otimes I + \sum A_j \otimes (T_\varepsilon^{-1/2} X_j T^{-1/2})$$

$= L(T_\varepsilon^{-1/2} X T_\varepsilon^{-1/2})$

$\xRightarrow{\text{assumpt}}$ $\tilde{L}(T_\varepsilon^{-1/2} X T_\varepsilon^{-1/2}) \succeq 0$

$$(I \otimes T_\varepsilon^{1/2})\Big(\overbrace{I \otimes I + \sum \tilde{A}_j \otimes T_\varepsilon^{-1/2} X_j T_\varepsilon^{-1/2}}^{\shortparallel}\Big)(I \otimes T_\varepsilon^{1/2})$$

$= I \otimes T_\varepsilon + \sum \tilde{A}_j \otimes X_j \quad \succeq 0.$

Now $\varepsilon \searrow 0$ to deduce τ is ucp.

$\tilde{\mathcal{S}} = \operatorname{span}(1, \tilde{A}_j)$, $\tilde{L}(x) = 1 + \sum \tilde{A}_j x_j$

Consider $\tau : \mathcal{S} \to \tilde{\mathcal{S}}$, $I \mapsto I$, $A_j \mapsto \tilde{A}_j$
linear

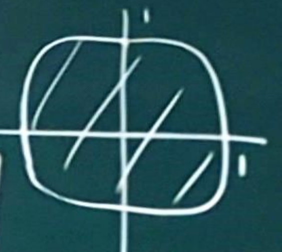
$\underset{\text{Arveson}}{\Longrightarrow}$ τ extends to a ucp $\hat{\tau} : M_\delta \to M_{\tilde{\delta}}$

$\underset{\text{Stinespring}}{\Longrightarrow}$ $\hat{\tau}(y) = V^*(I_\mu \otimes y)V$ for some μ, V isometry

Apply to $y = L(x)$:

$$\tilde{L}(x) = \tau(L(x)) = V^*(I_\mu \otimes L(x))V = \sum_{j=1}^{\mu} V_j^* L(x) V_j \qquad \square$$

Puzzle of the Day #4



$x^4 + y^4 \le 1$

Is this a spectrahedron?

Corollary: • $\mathcal{D}_L(k) \subseteq \mathcal{D}_{\tilde{L}}(k) \iff \tau$ is k-positive

• $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}} \iff \tau$ is cp

• $\mathcal{D}_L = \mathcal{D}_{\tilde{L}} \iff \tau$ is completely isometric.

$C^*(\mathcal{S}) \subseteq M_\delta$

$$\cong \begin{pmatrix} * & & 0 \\ & \boxed{x} & \\ 0 & & \ddots\ \boxed{x} \end{pmatrix}$$

δ

L minimal
$\Updownarrow$
none of blocks is redundant

Prop: L is minimal $\iff$ Šilov boundary of $\mathcal{S}$ in $C^*(\mathcal{S})$ is $\{0\}$.

(i.e., $\mathcal{S}$ is a reduced op. system)

biggest ideal K in $C^*(\mathcal{S})$ s.t. $C^*(\mathcal{S}) \to C^*(\mathcal{S})/K$ is c.isometric on $\mathcal{S}$.

The other ingredient in (b):

Thm (Arveson): A bijective c. isometry $\tau: \mathcal{S} \to \tilde{\mathcal{S}}$ between reduced op. systems is induced by a $*$-isomorphism $C^*(\mathcal{S}) \to C^*(\tilde{\mathcal{S}})$.

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Detour: invariants

$(d=1)$ $\quad A \underset{u}{\sim} B \overset{\text{Specht}}{\underset{40}{\iff}} \forall$ nc word in x, x^*:

$A, B \in M_n(\mathbb{C})$

$$\operatorname{tr} w(A, A^*) = \operatorname{tr} w(B, B^*)$$

(suffices words of deg ≤ 2

x, x^2, xx^*)

Procesi, Razmyslov 76

$(A_1 \ldots A_d) \underset{u}{\sim} (B_1, \ldots, B_d) \iff \forall$ nc word w in $x_1 \ldots x_d, x_1^* \ldots x_d^*$

$$\operatorname{tr} w(\underline{A}, \underline{A}^*) = \operatorname{tr} w(\underline{B}, \underline{B}^*)$$

(suffices deg $w \leq n^2$)

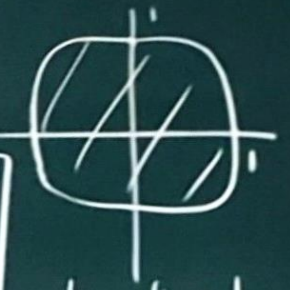
Theorem (Derksen – K – Makam – Volčič '23)

For $\underline{A}, \underline{B} \in M_n^d$:

$\underline{A}$ is similar to $\underline{B}$ $\iff$ $\forall\, T = (T_0, \dots, T_d) \in M_{nd}$

$$\mathrm{rk}\Big(I \otimes T_0 + \sum A_j \otimes T_j\Big) = \mathrm{rk}\Big(I \otimes T_0 + \sum B_j \otimes X_j\Big)$$

Puzzle of the Day #4



$x^4 + y^4 \le 1$

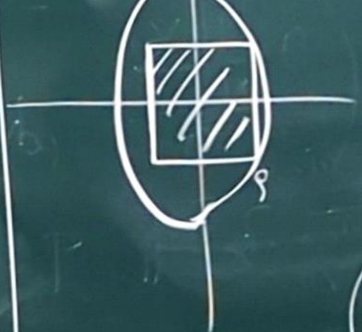
Is this a spectrahedrop?

Theorem (Convex Positivstellensatz, Helton – K – McCullough 2012)

Suppose L is a monic linear pencil & p is a nc polynomial. Then $\mathcal{D}_L \subseteq \mathcal{D}_p$ (ie, $p|_{\mathcal{D}_L} \ge 0$) iff

$$p = \sum s_j^* s_j + \sum f_j^* L f_j \quad \text{w/ } \deg s_j, \deg f_j \le \lfloor \deg p / 2 \rfloor$$

$\mathcal{D}_L(1)$



Dilations

matrix cube problem: find max ρ s.t. $[-\rho, \rho]^d \subseteq \mathcal{D}_L(1)$

(Nemirovskii 2006 | CM)

(NP-hard)

$[-\rho, \rho]^d = \mathcal{D}_{C_\rho}(1)$

for $C_\rho(x) = \begin{pmatrix} \rho - x_1 & & & & \\ & \rho + x_1 & & & \\ & & \rho - x_2 & & \\ & & & \rho + x_2 & \\ & & & & \ddots \end{pmatrix}$

$\boxed{\mathcal{D}_{C_\rho} \subseteq \mathcal{D}_L \text{ is easy}}$

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(b) If $L, \tilde{L}$ are minimal, then

For $\delta \in \mathbb{N}$ let
$$\frac{1}{\vartheta(\delta)} = \min_{\substack{a \in \mathbb{R}^\delta \\ |a_1| + \cdots + |a_\delta| = \delta}} \int_{S^{\delta-1}} \left|\sum a_i \xi_i^2\right| d\xi = \min_{\substack{B \in \mathbb{S}_\delta \\ \operatorname{tr}|B| = \delta}} \int_{S^{\delta-1}} |\xi^* B \xi| \, d\xi$$

Theorem (Helton–K–McCullough–Schweighofer [18])

Suppose L is a monic $\delta \times \delta$ linear pencil and $[-\rho, \rho]^\delta \subseteq \mathcal{D}_L(1)$

Then $\mathcal{D}_{\varphi_\rho} \subseteq \vartheta(\delta) \cdot \mathcal{D}_L$

and $\vartheta(\delta)$ is the best constant w/ this property.

For even d,
$$\frac{1}{\vartheta(d)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{\delta}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{\delta}{4} + 1\right)} \approx \frac{2}{\sqrt{\pi \cdot d}}$$

Def: C dilates T if $T = \begin{pmatrix} C & * \\ * & * \end{pmatrix}$

Eq: T is a compression of C.

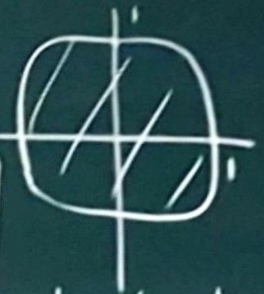
Theorem (Simultaneous dilation): Fix δ, Then $\exists$ Hilbert space H, $\mathcal{C}_\delta$ family of **commuting** s.a. contractions & isometry $V: \mathbb{R}^\delta \to \mathcal{C}_\delta$ s.t.

$\forall$ symmetric $\delta \times \delta$ contraction X $\exists\, T \in \mathcal{C}_\delta$ s.t.

$$X = \vartheta(\delta)\, V^* T V$$

Moreover, $\vartheta(\delta)$ is the best such constant.

Puzzle of the Day #4



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S.D.T $\Rightarrow$ M.C.T

Suppose $[-1,1]^\delta \subseteq \mathcal{D}_L(1)$

Claim $\mathcal{D}_{\mathcal{C}} \subseteq \vartheta(\delta) \cdot \mathcal{D}_L$

Since L is $\delta \times \delta$, map into M_δ is cp iff δ-pos, it suffices to show $\mathcal{D}_{\mathcal{C}}(\delta) \subseteq \vartheta(\delta)\, \mathcal{D}_L(\delta)$

Let $X \in \mathcal{D}_{\mathcal{C}}(\delta)$. Dilate X to $\vartheta(\delta) T \in \mathcal{C}_\delta$

$$\frac{1}{\vartheta(\delta)} X = V^*(T_1 \ldots T_d) V = T$$

Apply spectral thm to T "simult. diag" to get $L(T) \succeq 0$. Then $L\left(\frac{1}{\vartheta(\delta)} X\right) = (I \otimes V)^* L(T)(I \otimes V) \succeq 0$. $\square$

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