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Euler's Königsberg: the explanatory power of mathematics

Tim Räz[1]

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Abstract The present paper provides an analysis of Euler's solutions to the Königsberg bridges problem. Euler proposes three different solutions to the problem, addressing their strengths and weaknesses along the way. I put the analysis of Euler's paper to work in the philosophical discussion on mathematical explanations. I propose that the key ingredient to a good explanation is the degree to which it provides relevant information. Providing relevant information is based on knowledge of the structure in question, graphs in the present case. I also propose computational complexity and logical strength as measures of relevant information.

Keywords Computational complexity · Königsberg bridges problem · Mathematical explanation · Leonhard Euler

1 Introduction

What is a good mathematical explanation? While there is no generally agreed-upon answer to this question, some particular examples of good mathematical explanations have emerged from the recent philosophical discussion. Leonhard Euler's solution to the Königsberg bridges problem is such an example. Philosophers have discussed

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✉ Tim Räz
tim.raez@posteo.de

1 FB Philosophie, Universität Konstanz, 78457, Konstanz, Germany

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the Königsberg case for some time; see, e.g., Penco (1994), Franklin (1994), Wilholt (2004), Pincock (2007, 2012, 2015), Baker (2012), Lyon (2012), Lange (2013). They seem to agree that Euler's Theorem provides a good explanation of why there is no path in Königsberg that crosses every bridge exactly once. However, philosophers disagree about what kind of explanation it is, and about the contribution of mathematics to the explanation.

The present paper provides a new proposal of why Euler's explanation is good. The proposal is based on a close analysis of Euler's original solution to the problem.[1] In his paper, Euler proposes three different solutions to the Königsberg bridges problem. Along the way, Euler addresses the respective strengths and weaknesses of these three solutions. I then put the analysis of Euler's paper to work in the philosophical discussion on mathematical explanations, and I propose that the key ingredient to a good explanation is the degree to which an explanation provides relevant information. Providing relevant information is based on the use of knowledge of the particular structure in question, graph theory in the present case. I also propose computational complexity and logical strength as measures of relevant information.

2 Euler's Königsberg

Euler's paper is entitled "Solutio problematis ad geometriam situs pertinentis" ("The solution of a problem relating to the geometry of position").[2] In the beginning of the paper, Euler states why he is interested in this problem: It is an example of a new, special kind of geometry, which does not involve quantities and measures – it is a "geometry of position", which was previously introduced by Leibniz. This area of mathematics is now called topology.

In paragraph 2, Euler distinguishes two problems. He illustrates the situation in Königsberg using a schematic map, reproduced as Fig. 1. He assigns capital letters A, B, C, D to the areas, and lower case letters a, b, c, ... to the bridges connecting areas. The first problem is to find out whether it is possible to find a path that crosses every bridge of this system exactly once – I will call this the *Königsberg Problem*. Euler notes that there is no definite answer to this problem as yet. He then generalizes the problem, and asks how one can determine the solution, not only for this particular configuration, but for any kind of system, i.e., any kind of branching of the river and any number of bridges – I will call this the *General Problem*. This paragraph is particularly noteworthy because Euler introduces two kinds of letters, one for places, or areas, the other for connections, or bridges. This distinction is key to a graph-theoretic approach to both problems, as we will see in a moment.

In paragraph 3, we learn of a first method for solving the Königsberg Problem: it consists of "tabulating all possible paths" and examining whether one of them uses

[1] See Molinini (2012) for a successful example of analyzing Euler's work in the context of mathematical explanations.

[2] I use the widely available translation (Euler 1956). See Hopkins and Wilson (2004) for a useful overview of Euler's paper. I thank an anonymous referee for his suggestion to consider Euler's paper.

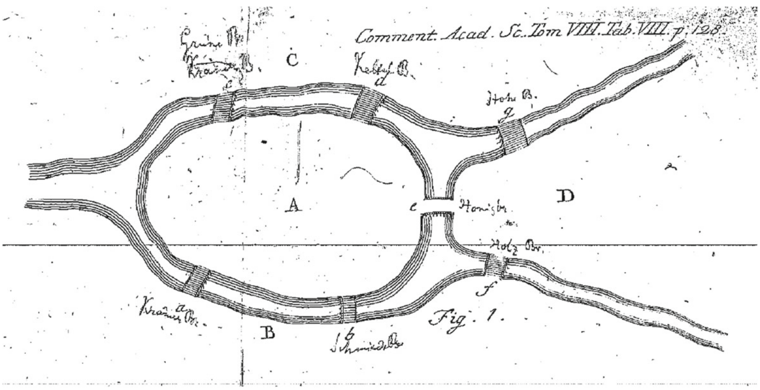

Fig. 1 Euler's Map of Königsberg

every bridge exactly once – I will call this the *Brute Force Method*.[3] Euler rejects this method because it is "too tedious and too difficult": there are too many possible paths, and for bigger systems, this method becomes intractable. The approach generates "details that are irrelevant to the problem". I will return to these suggestive remarks about intractability and irrelevant information below.

The most important innovation of the paper, Euler's graph-theoretical approach, is introduced in paragraph 4. It consists of the use of a particular notation for paths in the bridge system in terms of the crossing of bridges: The crossing of any one of the bridges a and b between A and B can be written as AB. A path from A over B to D is noted as ABD, using any one of the bridges connecting these areas. This notational shift is characteristic of the graph-theoretic nature of Euler's approach to both problems, and marks the beginning of graph theory. In modern graph theory, a multigraph is represented by a set of vertices V, and a multiset of edges E, represented by pairs of vertices.[4] This is exactly what Euler's notation achieves: Two structurally related kinds of objects – bridges and areas in the present case – are brought into correspondence by notating edges (here: bridges) as pairs of vertices (here: areas). There are no longer two separate sets of labels for the two kinds of objects, but one is expressed in terms of the other.[5] The bulk of the paper consists of putting this simple yet powerful idea to work, as Euler notes.

Euler proceeds to derive what I call the *Intermediate Method*. First, he notes that we can represent an Euler path in a bridge system consisting of n bridges by a string of $n + 1$ capital letters (areas). Any path between areas A and B is written AB. We

[3] Euler does not specify how to carry out the Brute Force Method in detail. One way of implementing it is to write down all (finitely many) paths of length seven starting from one of the areas A, B, C, D, and see whether any one of these paths includes just seven different bridges.

[4] A multigraph is a graph where two vertices can be connected by more than one edge. Consequently, the multiset of edges may contain a pair of vertices more than one time.

[5] Arguably, the fact that Euler's paper stands at the beginnings of graph theory is its most important innovation. This supports an observation by Rav (1999) that one of the most important roles of proving theorems lies in the novel insights generated by proof methods beyond determining the truth of particular theorems.

know that if we want to use every bridge exactly once, i.e., find an Euler path in the Königsberg system, the corresponding string has to consist of 8 letters. Next, he determines how many times a particular capital letter (area) has to occur in such a string. Denote the number of bridges connected to an area X, its degree, by $d(X)$.[6] If $d(P)$, i.e., the degree of area P, is odd, then P will have to occur $\frac{d(p)+1}{2}$ times in the string: If three bridges lead to area P, then the letter P will feature twice in the string, whether we start in P or not, and so on. We can now apply this result to the Königsberg system. The letter A has to occur three times, and B, C, D two times. These numbers add up to 9, which is bigger than 8, the length of an Euler path. This shows that it is impossible to find a path that crosses every bridge in Königsberg exactly once. Euler then extends this method to systems with even areas. If the degree of an area is even, there are two possibilities. If area Q is the starting point of the trip and $d(Q)$ is even, then the letter Q will occur $\frac{d(Q)}{2} + 1$ times. If area R is not the starting point and $d(R)$ is even, the letter R will occur $\frac{d(R)}{2}$ times.

Putting together the results for odd and even areas yields the Intermediate Method in the following way. Recall that n designates the total number of bridges in the system. The length of a string representing an Euler path in this system therefore must be of length $n + 1$. I will use the letter P for areas with odd $d(P)$, the letter Q for an area with even $d(Q)$, when starting in Q, and R for areas with even $d(R)$, when not starting in R. We can now sum up how many letters each area contributes to a string that uses every bridge once, and compare the result with the condition for an Euler path:

$$\underbrace{\sum_P \frac{d(P)+1}{2}}_{\text{odd areas}} + \underbrace{\sum_R \frac{d(R)}{2}}_{\text{even nonstart areas}} + \underbrace{\frac{d(Q)}{2} + 1}_{\text{even start area}} = n + 1 \tag{1}$$

On the left-hand-side (LHS), we sum up how many letters each area contributes to a complete string, depending on whether the degree of the areas is odd, even, and whether we start in an area with even or odd degree. If an Euler path exists, this sum has to be equal to $n + 1$. This is the Intermediate Method. It tells us that no Euler path exists if equality is violated. The equation also shows that if there are odd areas, we should start there in order to make the sum on the LHS smaller. If we do so, the contribution of 1 to the even start area drops out.

Once we have established this relationship, we can easily deduce a further theorem, as Euler notes – what is now called *Euler's Theorem*. Note, first, that the number of areas with odd degree has to be even. The sum of the degrees of all areas counts every bridge twice, and therefore has to yield an even number, $2n$. If we multiply

[6]Euler does not employ the notation $d(X)$ for the degree of X; his argument hardly uses any algebraic expressions. The reconstruction given here follows Euler closely, but transforms some of his reasoning into algebra in order to make it more accessible.

Eq. (1) by 2, we can replace the resulting $2n$ on the right-hand-side (RHS) by the sum of the degrees of all areas:

$$\underbrace{\sum_P (d(P)+1)}_{\text{odd areas}} + \underbrace{\sum_R d(R)}_{\text{even nonstart areas}} + \underbrace{d(Q)+2}_{\text{even start area}} = \underbrace{\sum_X d(X)+2}_{\text{all areas}} \qquad (2)$$

From Eq. (2), we can deduce that equality only holds in two cases. First, if there are no odd areas at all, the first summand on the LHS drops out, and equality follows. Second, if there are two odd areas, we have to start in one of them, because otherwise, both the first and the third summand contribute 2 to the LHS, exceeding the RHS by 2. If there are four, six, etc. odd areas, the equality is violated because the first summand contributes at least an additional 4, which exceeds the RHS by at least 2. This is Euler's Theorem. It gives two conditions that are jointly necessary for the existence of an Euler path: it exists only if a) the degree of all areas is even, or if b) the degree of exactly two areas is odd, and we start the journey in one of these areas. Euler's Theorem yields a method for determining whether or not an Euler path exists.[7]

3 Preliminaries

Before I proceed to analyze Euler's work from an explanatory perspective, some preliminary remarks are in order. One important distinction in the debate on mathematical explanations is between mathematical explanations of mathematical facts, so-called "Intra-Mathematical Explanations" (IME), and explanations of empirical phenomena that make use of mathematics, so-called "Scientific Explanations using Mathematics" (SEM).[8] In principle, the three methods can be analyzed as belonging to both IME and SEM: The methods can be used to give explanations that concern certain kinds of graphs, and they can also be used to give explanations that concern real bridge systems, e.g., the historical system in Königsberg. There is much to be said about the two kinds of explanations and how they are related. However, in the present paper, I will analyze the three methods as intra-mathematical explanations of mathematical facts. Euler's paper, the three methods, and the differences between the methods are all purely mathematical. Therefore, the main differences between the three methods are also intra-mathematical. This is not to say that these differences are irrelevant when we apply the methods to explain empirical phenomena, but

[7]Euler's Theorem is different from modern formulations of the theorem in several respects; see, e.g., Diestel (2006, pp. 21) for a modern account. First, Euler asks if it is possible to cross every bridge in Königsberg exactly once, without the assumption that the Euler path should be closed. Second, Euler does not assume that graphs are connected. This is a necessary assumption for the theorem. Third, Euler only proved one direction of the modern version of the theorem, viz. the statement that a closed Euler path exists if, and only if, every area has even number of edges. He did not prove that if a closed Euler path exists, every vertex has an even number of edges; this was only proved 135 years later; see Wilson (1986, p. 270).
[8]The labels IME and SEM are due to Baker (2012); see Mancosu (2011) for a useful overview of the debate on explanations in pure and applied mathematics.

we can understand the explanatory virtues of the methods without taking empirical phenomena into account.

What are the *relata* of the explanations we will consider? In the rest of the paper, I will be somewhat loose when writing about the *explanans* of interest. I will write that a " method explains". This should be read as shorthand for: a method is part of an explanation, or is used in giving an explanation. By this I mean that, if we want to, say, explain why there is no Euler path in a certain graph, we do not draw exclusively on the methods; the structure of the graph in question is also part of the *explanans*.[9]

Turning to the *explanandum*, it is important to keep in mind that the methods operate at different levels. We can ask why an Euler path does or does not exist in a particular graph, such as the Königsberg graph. This would be a single-case *explanandum*. We can also ask why there is an Euler path in some graphs, and not in others. The explanation is also provided by the methods, but the *explanandum* encompasses a whole family of graphs. Finally, we can ask for an explanation of the validity of the methods or theorems themselves. In this case, the answer will consist in a proof that establishes the method's validity. The main focus of the present paper is on the second kind of *explanandum*. The first reason for this choice is that, historically, the *General Problem* is the focus of Euler's paper. The second, systematic reason is that if we want to understand the explanatory power of the methods, we have to understand how they work in general. Of course, once we understand their general features, we also gain a better understanding of how they work in particular cases.

Finally, a remark on the term "method". So far, I have adopted Euler's terminology and discussed "methods". In modern terms, we can think of the methods as algorithms, and viewing them as algorithms is of some importance, as we will see below. The terms "method" and "algorithm" can be used interchangeably. Admittedly, it is somewhat unusual to analyze the explanatory merits of algorithms, because the debate on mathematical explanations focuses on the explanatory virtues of proofs. However, the discussion of algorithms can be rephrased in the following manner: The algorithms deductively establish whether or not some graphs have certain properties. In this sense, they are (parts of) proofs.

4 Are the methods explanatory?

The goal of the present paper is to determine just what it is about Euler's three methods that makes them more or less explanatory. However, why should we think that the three methods are more or less explanatory in the first place? This issue is particularly pressing because Euler's discussion is not framed in explanatory terms. I will argue below that some of Euler's remarks are fruitful for our understanding of

[9]To use an imperfect analogy with the deductive-nomological model of explanation, we need both a general law as well as initial conditions to deduce the *explanandum*.

mathematical explanations, but the analysis of the three methods' explanatory virtues should not be based on Euler's remarks exclusively.

There are good reasons for believing that the three methods are, in fact, more or less explanatory. First, there is a consensus in the philosophical and mathematical literature that Euler's Theorem is a good explanation why there is no Euler path in Königsberg; see the references given in Section 1 above. The point of contention in the philosophical debate is not whether this is a good explanation or not, but the reason why it is good. I will simply follow the consensus view and presuppose that Euler's Theorem is a good explanation. Second, Brute Force Methods are considered to have low explanatory power.[10] Brute Force Methods have been called "trivial" by mathematicians; see, e.g., Gowers (2008, p. 580), and they have been characterized as not explanatory by philosophers. According to Colyvan (2012), one of the reasons why proving a theorem by Brute Force Methods is not explanatory is that "it looks as though the theorem itself holds merely by accident" (Ibid., p. 81). If we inspect the Brute Force Method in the Königsberg case, we find that Colyvan's observation is confirmed: The application of the method consists of an exhaustive list of paths of length seven, and the observation that none of these paths contains each edge exactly once. This proof does not tell us whether there is a deep reason for this fact, or if it is merely an accident. Lange (2014) formulates the problem with Brute Force Methods as follows:

> A brute force approach is not selective. It sets aside no features of the problem as irrelevant. [...] In contrast, an explanation must be selective. It must pick out a particular feature of the setup and deem it responsible for (and other features irrelevant to) the result being explained. (Ibid., p. 499)

I agree with Lange's verdict that providing relevant information is at the very core of providing a good explanation. Finally, turning to the Intermediate Method, it is not as explanatory as Euler's Theorem, but it has some explanatory power, because it is an intermediate step towards Euler's Theorem.

I will argue that the explanatory power of the three methods is not categorical – explanatory vs. non-explanatory methods – but that explanatory power is a matter of degree. If explanatory power comes in degrees, this makes it possible to account for a gradual improvement of explanations. We will see an improvement from the Brute Force Method, which has virtually no explanatory power, over the Intermediate Method, which does a lot better, to Euler's Method, which is an improvement on the Intermediate Method.[11]

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[10]Brute Force Methods can be applied whenever a problem is decidable. For example, we may want to find out whether a mathematical structure has a certain property or not, and this can be determined by going through all (finitely many) possible cases. A Brute Force Method then systematically goes through all possible cases, and answers "Yes" if at least one case comes out positive, and "No" otherwise.

[11]We could interpret the Brute Force Method as not being explanatory at all. However, this is, strictly speaking, not true. There are proofs, so-called *zero-knowledge proofs*, that only show that a result is true, without giving us *any* knowledge as to why this is so, in a strict, cryptographic sense of knowledge; see Aaronson (2013, Sec. 9.1). However, these are non-classical, probabilistic proofs, and deductive analogues will always convey *some* information.

5 Explanations in Euler's Königsberg

5.1 Relevant information and graph theory

I have argued above that the three methods we find in Euler's paper differ when it comes to explanatory power. If this is correct, I still have to determine what it is about the three methods that makes them more or less explanatory. I will now analyze the differences between the methods and argue why I believe that these differences are explanatorily relevant.

Euler himself dismisses the Brute Force Method early on, in paragraph 3 of the paper. In this passage, he also gives reasons for dismissing the method, and implicit standards for an acceptable method:

> The particular problem of the seven bridges of Königsberg could be solved by carefully tabulating all possible paths, thereby ascertaining by inspection which of them, if any, met the requirement. This method of solution, however, is too tedious and too difficult because of the large number of possible combinations, and in the other problems where many more bridges are involved it could not be used at all. When the analysis is undertaken in the manner just described it yields a great many details that are irrelevant to the problem; undoubtedly this is the reason the method is so onerous. (Euler 1956, p. 574)

These remarks can be fruitfully analyzed from an explanatory perspective; however, they need some unpacking. What does Euler mean when he writes that the Brute Force Method gives us irrelevant details? In one sense, this is wrong: the method does not give us any irrelevant information, because we need a complete list of possible paths of a certain length to establish that there really is no Euler path in a graph. Yet, in another sense, Euler is right. Every time we write down a possible path, we draw on the whole structure, which dictates what sequences are potential Euler paths. In doing this, we use the same structural information more than once, for example if two paths share an initial segment. The Brute Force Method uses the structural information of the graph in a redundant manner.[12]

Compare this to the use of relevant information in the Intermediate Method (and Euler's Theorem, which builds on the Intermediate Method). The main idea behind the Intermediate Method is to determine whether there is an Euler path in a graph on the basis of the degree of vertices. More specifically, if we want to know whether the whole system has the global property of having an Euler path, we only have to determine whether certain parts of the system, the vertices, have certain degrees, which is a local property. This reduces irrelevant information, because each area is used exactly once to determine the degree, and not each time a path passes through it, as in the Brute Force Method. Knowledge about Euler paths is, literally, equal to the sum of knowledge about degrees of vertices.

This brings us to the key idea of Euler's paper: the way in which the property of having an Euler path is calculated on the basis of the degree of vertices. If we

[12]This does not speak against all kinds of algorithms that compile lists of paths; the redundancy could be overcome, to a certain degree, by using a clever search algorithm.

want to compare the sum of degrees of all vertices with the length of Euler paths in Eq. (1), we have to conceptualize bridges as pairs of areas, by notating a connection in terms of the letters for the areas connected. This notational device constitutes the graph-theoretic approach to the Königsberg problem. Establishing a connection between the global property of having an Euler path and the sum of the local properties of degrees of vertices requires graph theory in an essential manner. In contrast, the Brute Force Method does not rely on this notational device. If you want to tabulate all possible paths of a certain length, strings of lower-case letters will do the job. This means that the graph-theoretic representation of the structure is not used in the Brute Force Method. When we apply the Brute Force Method, we use the structure to generate solutions, but we do not make use of graph-theoretical notation. In this sense, graph theory is not necessary for the Brute Force Method, but it is necessary for the Intermediate Method.

I propose that the Intermediate Method is more explanatory because it provides less irrelevant information than the Brute Force Method. It could be asked why we should think that providing irrelevant information yields a less powerful explanation. Irrelevant information makes for an inferior explanation because if we are given an explanation, we presuppose that all the information we are given is explanatorily relevant, which is violated by the irrelevant information. This proposition is not particularly controversial, but it is also not particularly telling. It is much more important to determine what information is in fact relevant. This task is hard, if not impossible, to carry out in general; however, it can be done in particular cases, such as the Königsberg case. The Intermediate Method answers a why-question about the non-existence of an Euler path by examining the degrees of vertices. Thus, relevant information is information about the degree of vertices. Importantly, Euler gets to these properties by using graph theory in an essential manner. This, in turn, means that the explanation is essentially graph-theoretical.

5.2 Computational complexity

Euler states a second reason for rejecting the Brute Force Method in the above quote: The method's intractability. The Brute Force Method yields a "large number of possible combinations", which makes it difficult to carry out in the case of bigger systems, or even impossible. If we examine a naive version of the Brute Force Method, we can see what Euler means: if we construct all possible paths of a certain length starting from one of the V vertices, we can end up with as many as $(|V| - 1)!$ candidate paths: the number of paths grows faster than exponential in the number of vertices in the worst case. For large V, it is impossible to check all possibilities in a reasonable amount of time. I propose that what Euler has in mind here is the *computational complexity* of the Brute Force Method. Computational complexity is a measure of the computational resources that are necessary to solve a problem.[13] Compare this

[13]Note that while the Intermediate Method and Euler's Theorm fare better than the Brute Force Method in terms of time complexity, the Brute Force Method is simpler to state than the other methods, i.e., the program-size complexity of the algorithms may increase, and thus be uncorrelated with, explanatory power.

to the computational complexity of finding an Euler path based on Euler's Theorem: An Euler path can be found in $O(|E|)$ time, i.e., the number of steps it takes to find an Euler path is linear in the number of edges of the graph.[14] Thus, the problem is tractable even for large graphs.

The difference in computational complexity is closely related to the different ways in which the methods extract information from the graphs. The Brute Force Method extracts the information in a redundant manner; on the other end of the spectrum, Euler's Theorem extracts exactly what we need. In the above quote, Euler himself suggests that this is the case; he writes that irrelevant details are the reason why the Brute Force Method is "onerous". Thus, the high computational complexity of the Brute Force Method is a *consequence* of taking into account irrelevant information.

If there is a correlation between the use of irrelevant information and high computational complexity, it provides us with a useful tool for diagnosing differences in explanatory power: We can use computational complexity, which is an objective property of algorithms, to reason back to differences in the use of irrelevant information, a somewhat vague notion. Concretely, if we have two methods for solving the same problem, and one of them has high computational complexity, while the other has low computational complexity, then the high complexity method is bound to have low explanatory power, because it processes irrelevant information. The decrease in computational complexity from the Brute Force Method to Euler's Theorem aligns with an increase in explanatory power.

Some *caveats* are in order when we use computational complexity to diagnose explanatory power. First, it should be emphasized that the correlation between computational complexity and explanatory power does not imply that the two notions are conceptually the same. Low computational complexity is not to be identified with high explanatory power. I am merely proposing that low computational complexity is a consequence of high explanatory power. The two notions are correlated, but they should not be identified.

Second, computational complexity is a notion that applies to algorithms only; it should not be attributed to the application of an algorithm to a particular case. If a certain method X has lower computational complexity than method Y, the application of X to a particular case a will not always be more efficient than the application Y to the same case a. The notion of computational complexity is usually a statement about how difficult it is to solve problems in the worst case. In instances where the worst case scenario is overly pessimistic, it may be easier to solve a problem based on a method that is of high complexity. Take, for example, a system with a circular graph. If we apply the Brute Force Method to this graph, a list of the possible paths will comprise one item, which is an Euler path, while on Euler's Theorem, we have to check the degree of each vertex.

Finally, we should not use computational complexity to compare explanatory power across different problems. Some problems may have higher complexity because they are just inherently harder, and we do not want to infer that methods

[14]Note that the complexity may depend on the size of both E and V; see, e.g., Gibbons (1985, Ch. 6) for more on issues of complexity.

for hard problems have low explanatory power *simpliciter*. Compare the problem of finding Euler paths with the problem of finding Hamiltonian cycles: The latter problem is known to be NP-complete, i.e., it is inherently computationally hard. However, we do not want to conclude from this that Euler's Method, which only requires linear time, therefore has more explanatory power than any solution to the Hamiltonian cycle problem.

5.3 Logical strength and depth

Let us now compare the Intermediate Method and Euler's Theorem with respect to explanatory power. Euler's Theorem is deduced from the Intermediate Method: Euler's Theorem follows from the Intermediate Method, represented in Eq. (1), via Eq. (2). Euler's Theorem thereby inherits the use of local information and graph-theoretic methods from the Intermediate Method. Thus, the two methods are closely related.

However, there are also relevant differences between the Intermediate Method and Euler's Theorem. First, if we want to apply the Intermediate Method, we have to determine the degree of each area, and carry out the computation on the LHS of (1). The equality can then be violated in different ways: the LHS can exceed the RHS by two, four, six, and so on. Thus, the Intermediate Method still provides too much information, if we only want to know if an Euler path does or does not exist in a particular graph. Euler's Theorem is weaker in that it does not distinguish between these cases; all that matters is whether the degree of the areas is even or odd. Euler's Theorem is maximally informative: it is necessary and sufficient to know whether the degree of certain areas is even or odd to solve the problem.

The difference between the Intermediate Method and Euler's Theorem we have just identified is a difference in logical strength. Euler's Theorem uses a condition that is logically weaker than the Intermediate Method – it is minimal and provides a necessary and sufficient condition for the existence of Euler paths. Thus, we can increase explanatory power if we can weaken the conditions from which we derive a result. It is plausible that those parts of a condition that are not necessary for the derivation of a result provide irrelevant information. The idea that explanatory power is related to the specification of a sufficient and necessary condition, as opposed to a merely sufficient one, has recently been proposed by Pincock (2015). Pincock's conception of "abstract explanation" requires an equivalence. The difference between the Intermediate Method and Euler's Theorem supports the claim that this is in fact explanatorily relevant. Note that the Brute Force Method does not specify any general property that is shared by all, or only, those graphs in which there is an Euler path.

There is a second difference between Euler's Theorem and the Intermediate Method that is indicative of their different explanatory power. Euler's Theorem can not only be deduced from the Intermediate Method, it also provides an explanation of one aspect of the Intermediate Method. Why does the sum on the LHS of Eq. (1) exceed $n + 1$ in some cases? Euler's Theorem shows that this is so because in these cases, there are more than two areas with odd degree, as we can infer from Eq. (2). This might indicate that Euler's Theorem has more explanatory power. The idea that the explanatory power of proofs is responsible for the "depth" of theorems has recently been proposed by Lange (2015). One way in which a proof can be deeper than another is if it answers more why-questions than a more shallow theorem. In the present case, we would say that Euler's Theorem is "deeper" because it explains a fact that the more "shallow" Intermediate Method leaves open. Note that this is an explanatory relation between methods, not to be confused with the explanatory relation between methods and graphs.

In sum, I have identified various differences between the three methods, and I have argued that these differences are responsible for the difference in explanatory power between the methods. How do these differences fit together, and what is the general picture of explanatory power that emerges from this analysis? These are the questions to which I now turn.

6 The proposal

The most important difference between the three methods is the degree to which they provide relevant information in their answer as to why there is an Euler path in some graphs and not in others. I propose that providing relevant information is directly related to explanatory power in the following manner: Explanation X has greater explanatory power than explanation Y if the answer provided by X is more relevant, or less redundant, than the answer provided by Y.

The idea that good explanations provide relevant, or salient, information, is not new. It fits nicely with a recent account of mathematical explanations by Lange (2014). Lange focuses on the question whether proofs of a theorem are explanatory or not. Lange finds that the difference between a proof that explains as opposed to a non-explanatory proof has to do with "differences in *the way* they extract the theorem from the axioms" (Ibid., p. 487, emphasis in original). In particular, an explanation "must pick out a particular feature of the setup and deem it responsible for (and other features irrelevant to) the result being explained" (Ibid., p. 499). In a nutshell, the difference between explanatory and non-explanatory proof has to do with the particular path between assumptions and result. This also applies to the Königsberg case: All three methods take a structure, or certain kinds of structures, as a starting point, and arrive at the same conclusion, viz. the existence or non-existence of Euler paths. But they arrive at this conclusion in very different ways. My analysis thus aligns nicely with Lange's account.

The idea that an explanation is better if it provides more relevant information is intuitively appealing, but it prompts further questions. Can we give a general account of what it means to provide relevant information? And: Can we specify a general measure of relevant information? It will be hard to give a satisfactory answer to the first question. To see why, it is helpful to return to Brute Force Methods for a moment. Brute Force Methods do generally provide lots of irrelevant information, as I argued in Section 4. However, Brute Force Methods also have an advantage: They can be applied to many problems without deep knowledge of the mathematical structure in question. All that is necessary is knowledge of how to exhaust the (finite) search space of possible answers. If we turn this on its head, we see that a good explanation depends on a deep understanding of the mathematical structure in question. The tradeoff that we observe in Brute Force Methods suggests that providing relevant information depends on the specifics of a particular problem – which, in turn, makes it hard to say what relevant information is in general.

However, following Lange (2014), we can still systematize different kinds of problems at various levels of specificity. Lange discusses cases where the symmetry of a particular problem is the feature to be explained, and so the explanation should exploit that symmetry. In other cases, such as the Königsberg problem, other aspects are relevant. I argued that in the Königsberg case, it was important that Euler conceptualized the problem as graph-theoretical. This conceptualization made it possible to explain a global property of the graph in terms of local properties of vertices. This strategy to extract relevant information is not specific to the Königsberg case, but applies more widely to graph-theoretic problems, many of which follow the same pattern as the Königsberg case. Gowers (2008, p. 215) writes: "Many questions in graph theory take the form of asking what some structural property of a graph can tell you about its other properties." The idea to explain some global property of a structure in terms of properties of local constituents of that same structure generalizes even further; it is at the core not only of many algebraic structures, such as groups, but also of, say, differential geometry. Thus, there is no all-encompassing answer to the question of what information is relevant, but there are certain strategies that apply to a certain range of problems and kinds of mathematical structures.

Turning to the question of how to measure relevant information, I have identified some features of the three methods that may serve this purpose. First, there is computational complexity. The idea is that a method which is based on a deeper understanding of a structure will extract information in an economic manner, which, in turn, yields a low computational complexity. Second, I proposed that the logical strength of the properties used in a method indicate explanatory power: Weaker conditions that still yield the same result are to be preferred, because they get to the heart of a problem. This echoes a proposal by Pincock (2015). Third, I concurred with Lange (2015) that the depth of a theorem, or method, aligns with explanatory power, in that a deeper theorem or method explains mathematical facts that a more shallow theorem leaves open.[15]

7 The proposal in perspective

In order to further clarify what the present proposal amounts to, it may be helpful to contrast it with another recent account of the Königsberg case. Pincock (2007) proposes interpreting Euler's Theorem as an instance of "abstract explanations". These are explanations that pick out certain relations of a physical system, while other aspects of the system are ignored. Euler's solution to the Königsberg Problem relies on the abstract structure depicted in Fig. 2.

[15]Note that there may be other fruitful ways of measuring relevant information. To give an example, Mark Colyvan (2012, pp. 83) briefly discusses the idea of using so-called relevant logic to distinguish between explanatory and non-explanatory proofs. I think that this idea is compatible with what I have proposed here.

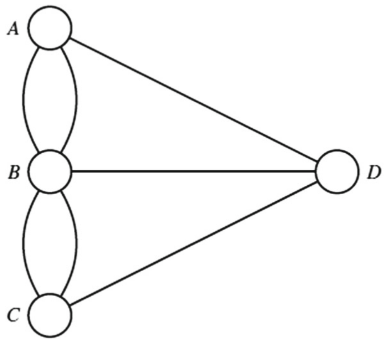

Fig. 2 Königsberg Graph

This is the graph that captures all the structural features of the city of Königsberg that are necessary to solve the problem. Abstract explanations can rely on mathematics, by using a structure-preserving mapping between the physical system and a mathematical domain, but this mapping does not depend on an arbitrary choice of units, or a coordinate system – it captures an intrinsic feature of the system. In his 2015 paper, Pincock characterizes abstract explanations as falling under the "ontic conception" of explanations, following the classic distinction between ontic, epistemic and modal conceptions by Salmon (1984). Abstract explanations are ontic because an objective, abstract explanatory relation between the *explanandum* and the abstract structure obtains. Applied to Euler's explanation, this means that the existence or non-existence of Euler paths in graphs depends objectively and abstractly on the corresponding graph.

I agree with Pincock's point that the abstraction from the city of Königsberg to the Königsberg graph is an important step towards a successful solution to the Königsberg problem. Characterizing the problem based on the abstract graph is certainly part of the explanation. However, Pincock's account misses a different aspect of the explanatory power of Euler's Theorem. Pincock focuses on the relation between the world – the city of Königsberg – and the mathematical structure – the Königsberg graph – which captures what is relevant about the world. My proposal, on the other hand, focuses on the different ways in which we extract information from the graph on the level of pure mathematics. In particular, I argued that one of the core innovations of Euler's paper is the notational trick of writing edges in terms of pairs of vertices, which makes it possible to determine the property of having an Euler path in terms of degrees of vertices. However, whether or not this trick is used is independent of the graph being an abstract structure. Rather, the trick has to do with using a particular representation of the graph in order to extract exactly the information we need. Both the Brute Force Method and Euler's Theorem are based on the abstract graph structure, but only Euler's Theorem uses the graph-theoretic nature of the structure, viz. writing edges as pairs of vertices. What Pincock's account misses is that the way in which the abstract structure is *represented* is a key part of Euler's explanation.

Returning to the distinction between ontic and epistemic conceptions of explanation, how should the present proposal be classified? According to the present proposal, Euler's work has both ontic and epistemic aspects. On the one hand, the fact that we can determine the existence of Euler paths in terms of degrees of vertices is an objective, abstract fact about graphs. This aspect of the explanation is captured by Pincock's account. On the other hand, in order to get to this fact, it is indispensable to use the graph-theoretic *representation* of the structure. The epistemic notions of representation and notation are not adequately captured if we focus on the ontic aspects of the explanation. Thus, Pincock's proposal, or the ontic conception of explanations, is not wrong. Rather, mathematics makes both ontic and epistemic contributions to one and the same explanation.[16]

8 Conclusion

In the present paper, I analyzed Euler's solution to the Königsberg bridges problem in order to better understand why his solution provides a good explanation of the fact that there is no Euler path in Königsberg. I proposed that the main contribution factor to explanatory power is that an explanation with high explanatory power provides us with more relevant information than an explanation with low explanatory power. Euler invented graph theory to achieve the goal of determining those properties that are relevant for the question at hand. Based on a close reading of Euler's paper, I also proposed that computational complexity, logical strength, and the depth of a result, can be indicative of explanatory power.

How should we proceed from here? The present proposal can be extended and tested in various ways. The analysis of explanatory power should, first, be transferred to other problems in graph theory to see whether there is a "local mode" of explanation in this area of mathematics. Then, the measures of explanatory power that I proposed can also be tested. In particular, the idea that computational complexity goes down as explanatory power goes up should be applied to other cases. If the present outline of a proposal is right, then the right approach to explanatory power will require a "disciplined pluralism": We will have to dive into the details of mathematics to describe the many modes of explanations that are currently employed in mathematical practice.

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