

Software Testing and Validation

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Corso di Laurea in Informatica

The NuSMV Model Checker

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CTL (and LTL) Model Checking

- We saw the theoretical algorithm for CTL model checking
 - we said it was not effective, as it required S and R to be in RAM
- Actually, there are methodologies which are able to fit S and R in RAM, also for industrial-sized models
- The “father” of the model checkers using such technologies is SMV
 - Symbolic Model Verifier
 - it has then been refactored as NuSMV
- This set of techniques is referred to as *symbolic model checking*
 - Murphi and SPIN style is dubbed *explicit model checking*



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CTL (and LTL) Model Checking

- In order to understand how symbolic model checking works, we need some preliminaries
- ROBDDs
 - needed to actually fit S and R in RAM
- μ -calculus
 - together with fixpoint computation
 - extension of λ -calculus
 - needed to efficiently implement CTL and LTL model checking using ROBDDs



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- Reduced Ordered (Complemented Edges) Binary Decision Diagrams
 - sometimes called simply OBDDs, and even BDDs
 - here we stick to the precise notation, by also outlining the differences
- Let us start with the basis: BDD
- A BDD is a data structure representing a boolean function
 - of course, OBDDs and ROBDDs are data structures as well
 - we will define them in the following



Boolean Functions

- In our setting a boolean function is $f : \mathbb{B}^n \rightarrow \mathbb{B}$
 - where $\mathbb{B} = \{0, 1\}$ is the set of boolean values
 - 0 stands for false, 1 for true
 - thus, our boolean functions have n boolean variables as arguments
 - and return a single boolean value
- Examples:
 - 0 and 1 are boolean functions with $n = 0$
 - complementation ($f(x) = \neg x$) and identity ($f(x) = x$) are boolean functions with $n = 1$
 - AND ($f(x, y) = x \wedge y$), OR ($f(x, y) = x \vee y$) are boolean functions with $n = 2$
 - generally speaking, there are 2^{2^n} different boolean functions of n boolean variables



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All Boolean Functions of 2 Variables

p	q	F^0	NOR ¹	\leftrightarrow ²	$\neg p$ ³	\leftrightarrow ⁴	$\neg q$ ⁵	XOR ⁶	NAND ⁷	AND ⁸	XNOR ⁹	q ¹⁰	\rightarrow ¹¹	p ¹²	\leftarrow ¹³	OR ¹⁴	T ¹⁵
T	T	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T
T	F	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
F	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
F	F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T



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Boolean Functions Representation

- Roughly speaking, if you have $f(x) = x + 1$ with $x \in \mathbb{R}$, you can only represent f through its computation
 - rules s.t., given x , you compute $x + 1$
- For boolean functions, the explicit tabular representation is also possible (*truth table*)
 - a table with $n + 1$ columns
 - first n columns are for variables values
 - last column is for function value
 - of course, you need 2^n rows



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Boolean Functions Representation

- A truth table must take into account all possible values for all its n arguments
- Which leads to a $O(2^n)$ RAM required
 - even with optimizations (e.g., only 1 column is actually needed)
- This also implies $O(2^n)$ time to compute composition of functions
 - e.g., $f \wedge g$
- One very good thing about truth tables: they are *canonical*
 - for a function f , given the (standard) order of the lines, there is only one truth table



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Boolean Functions Representation

- What about CNF or DNF?
 - CNF: $(x_1 + x_2)(\bar{x}_3 + x_4)$
 - DNF: $x_1\bar{x}_3 + x_1x_4 + x_2\bar{x}_3 + x_2x_4$
 - recall that $+$ is OR, \cdot is AND, $\bar{\cdot}$ is negation
- Difficult to compute function compositions
- Difficult to obtain a minimal representation
- Above all, not canonical: there may be multiple CNFs or DNFs for the same function
 - also if you consider the minimal one



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Boolean Functions for Model Checking

- In Model Checking algorithms, the following operations are needed:
 - compute the returned value for a given tuple of values b_1, \dots, b_n
 - could be ok for truth tables
 - test of equivalence between boolean functions $f_1 \equiv f_2$
 - not ok for truth tables
 - needs canonicity
 - compute the representation of a logical combination of boolean functions
 - e.g.: given the representation of f_1, f_2 , compute the representation of $f_1 \wedge f_2$
 - not ok for truth tables
- Goal: find a representation able to fulfill such requirements
 - while possibly requiring less than $O(2^n)$



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Binary Decision Diagrams

- Roughly speaking, it is a connected DAG (Directed Acyclic Graph), i.e., a tree
 - only one root
 - each internal node has two successors
 - nodes are labeled by boolean variables
 - edges are labeled by boolean values
 - only two leaves, labeled with boolean values

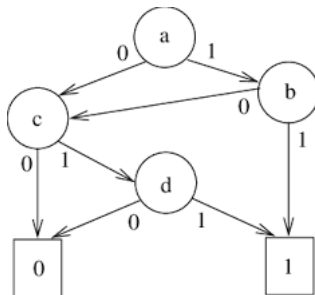


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Binary Decision Diagrams



Represented function: $f(a, b, c, d) = ab + \bar{a}cd + a\bar{b}cd$



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BDDs: Formal Definition

- A BDD is a tuple $\mathcal{B} = \langle V, E, r, \mathcal{V}, \text{var}, \text{low}, \text{high} \rangle$ where:
 - V is a finite set of nodes containing two special nodes **0** and **1**
 - $E \subseteq V \times V$ is a set of edges s.t.:
 - there are no cycles, i.e., for all path $\pi = v_0, \dots, v_n$, where $\forall i = 0, \dots, n. v_i \in V$ and $\forall i = 0, \dots, n-1. (v_i, v_{i+1}) \in E$, we have that $i \neq j$ implies $v_i \neq v_j$
 - let $S(v) = \{w \in V \mid (v, w) \in E\}$ be the set of successors of v
 - each internal node has exactly two successors, i.e., $\forall v \in V \setminus \{0, 1\}. |S(v)| = 2$
 - **0** and **1** are *terminal nodes*, i.e., $\forall v \in \{0, 1\}. |S(v)| = 0$
 - $r \in V$ is the root (i.e., $\forall v \in V. (v, r) \notin E$)
 - $\text{low}, \text{high} : V \rightarrow V$ is the labeling of edges
 - the labeling must be consistent with E , i.e., $\forall v \in V. \text{low}(v), \text{high}(v) \in S(v)$



BDDs: Formal Definition

- A BDD is a tuple $\mathcal{B} = \langle V, E, r, \mathcal{V}, \text{var}, \text{low}, \text{high} \rangle$ where:
 - \mathcal{V} is a finite set of boolean variables
 - thus, the boolean function represented by \mathcal{B} will depend on variables in \mathcal{V}
 - it may be a subset of \mathcal{V}
 - $\text{var} : V \rightarrow \mathcal{V}$ is the labeling of nodes
- A *maximal path* in \mathcal{B} starts from r and ends up either in **0** or **1**
- The *semantics* of \mathcal{B} is the boolean function represented by \mathcal{B}
 - intuitively, we follow all maximal paths which end up in **1**
 - formally: next slide



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BDDs: Semantics

- Given a BDD $\mathcal{B} = \langle V, E, r, \mathcal{V}, \text{var}, \text{low}, \text{high} \rangle$, we recursively define the semantics of each node $v \in V$
 - each node may be seen as the root of a subtree...
 - notation: $\llbracket v \rrbracket_{\mathcal{B}}$, or simply $\llbracket v \rrbracket$ when \mathcal{B} is understood
- Terminal nodes denote the boolean constants:
 $\llbracket 0 \rrbracket = \text{false}$, $\llbracket 1 \rrbracket = \text{true}$
- For internal nodes $v \in V \setminus \{0, 1\}$, semantics is defined as
 $\llbracket v \rrbracket = \text{var}(v) \llbracket \text{high}(v) \rrbracket + \overline{\text{var}(v)} \llbracket \text{low}(v) \rrbracket$
 - recall that $+$ is OR, \cdot is AND, $\bar{\cdot}$ is negation
- The semantics of \mathcal{B} is of course $\llbracket r \rrbracket$



Canonicity of BDDs

- For a given BDD \mathcal{B} , we have a unique represented boolean function
- Given a boolean function f , there is a BDD \mathcal{B} representing f , i.e., $\llbracket r \rrbracket_{\mathcal{B}} = f$
- However, there may be a BDD $\mathcal{B}' \neq \mathcal{B}$ s.t. $\llbracket r' \rrbracket_{\mathcal{B}'} = f$ as well
 - thus, BDDs are not canonical
- Thus, ROBDDs are introduced: by setting limitations, they achieve canonicity
 - for a boolean function f , there exists a *unique* ROBDD representing f
- Furthermore, for increasing efficiency, complemented edges are introduced
 - number of nodes is reduced



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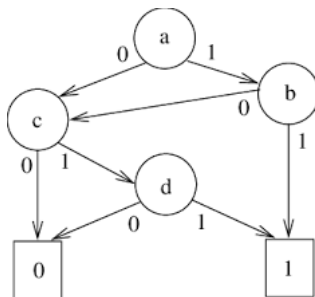


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- An OBDD (Ordered BDD)
 $\mathcal{B} = \langle V, E, r, \mathcal{V}, \text{var}, \text{low}, \text{high}, \text{ord} \rangle$, is a BDD with an additional ord function
- Namely, $\text{ord} : \mathcal{V} \rightarrow \{1, \dots, |\mathcal{V}|\}$
- The following properties must hold
 - ord is injective, i.e., $\forall v, w \in \mathcal{V}. \text{ord}(v) = \text{ord}(w) \rightarrow v = w$
 - note that this implies that ord is indeed bijective...
 - defines an *ordering* on variables in \mathcal{V} , e.g., if $\text{ord}(v) = 10$ then v is the tenth variable
 - given a path π on \mathcal{B} , variables on nodes follow ord
 - i.e., $\forall \pi = v_0, \dots, v_n$ s.t. $\forall i = 0, \dots, n. v_i \in V$ and $\forall i = 0, \dots, n-1. (v_i, v_{i+1}) \in E$ and $v_n \notin \{0, 1\}$, we have that $i < j$ implies $\text{ord}(\text{var}(v_i)) < \text{ord}(\text{var}(v_j))$



OBDDs



Supposing that $V = \mathcal{V}$, a possible ordering is:

$\text{ord}(a) = 1, \text{ord}(b) = 2, \text{ord}(c) = 3, \text{ord}(d) = 4$

If b were connected to d instead of c , also:

$\text{ord}(a) = 1, \text{ord}(b) = 3, \text{ord}(c) = 2, \text{ord}(d) = 4$



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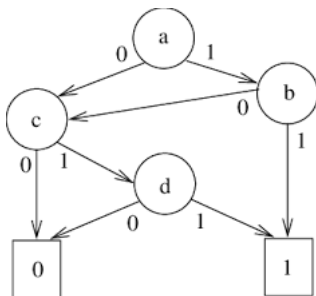
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COBDDs

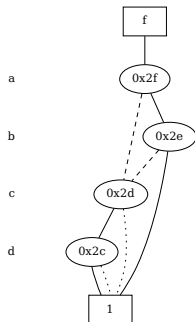
- A COBDD (Complemented edges OBDD)
 $\mathcal{B} = \langle V, E, r, \mathcal{V}, \text{var}, \text{low}, \text{high}, \text{ord}, \text{flip} \rangle$, is an OBDD with an additional $\text{flip} : V \setminus \{\mathbf{1}\} \rightarrow \{0, 1\}$
- For an internal node v , if $\text{flip}(v)$ holds then the *else edge* of v is complemented
- There is now only one terminal node $\mathbf{1}$
 - $\mathbf{0}$ is not needed because of complementation
- Semantics changes, also a flipping bit $b \in \{0, 1\}$ is necessary
- Terminal node denote the boolean constants: $\llbracket \mathbf{1}, b \rrbracket = \bar{b}$
- For internal nodes $v \in V \setminus \{\mathbf{1}\}$, semantics is defined as $\llbracket v, b \rrbracket = \text{var}(v) \llbracket \text{high}(v), b \rrbracket + \overline{\text{var}(v)} \llbracket \text{low}(v), b \oplus \text{flip}(v) \rrbracket$
- Semantics of \mathcal{B} is $\llbracket r, \text{flip}(r) \rrbracket$



COBDDs



Represented function:
 $f(a, b, c, d) = ab + \bar{a}cd + \bar{a}\bar{b}cd$



straight: then, dashed: else,
 dotted: complemented else



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ROBDDs

- A ROBDD (Reduced OBDD) \mathcal{B} is a COBDD with the least number of nodes
 - among the ones representing the same boolean function
- From now on, as usual in the literature, we will use OBDD as synonym for ROBDD
- Efficient algorithms ($O(n)$, being n the number of nodes) exist to compute the AND and the OR of two OBDDs
 - negation is $O(1)$: just complement $\text{flip}(r)$!
- Typically implemented with hash tables of already computed ROBDDs
 - speedup computations, make it easier to find shared subtrees
 - equality check is $O(1)$: just compare r and r'
- Furthermore: multi-rooted DAG can be used to represent multiple functions, sharing some nodes



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Other Important OBDD Operations

- Application: given the OBDD for $f(x_1, \dots, x_i, \dots, x_n)$, compute the OBDD for $f(x_1, \dots, 0, \dots, x_n)$ or $f(x_1, \dots, 1, \dots, x_n)$
 - sometimes also written $f(x_1, \dots, x_n)|_{x_i=0}$ or $f(x_1, \dots, x_n)|_{x_i=1}$
 - Shannon expansion: for every boolean function f ,
$$f(x_1, \dots, x_n) = \bar{x}_i f(x_1, \dots, x_n)|_{x_i=0} + x_i f(x_1, \dots, x_n)|_{x_i=1}$$
- Given $f(x, y)$, compute the OBDD for:
 - existentialization: $\exists x : f(x, y) \equiv f(0, y) + f(1, y)$
 - universalization: $\forall x. f(x, y) \equiv f(0, y) \cdot f(1, y)$
 - both generalized to multiple variables x_1, \dots, x_n
- Given $f(x), g(x), h(x)$, compute the OBDD for $ITE(f, g, h)$
 - ITE stands for if-then-else
 - thus, $ITE(f, g, h) = fg + \bar{f}h$



OBDD and Model Checking

- OBDDs extremely good in representing *characteristic functions* of finite sets
 - the characteristic function $\chi : U \rightarrow \{0, 1\}$ of a set $X \subseteq U$ is defined as

$$\chi(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

- If U is finite, then each element $x \in U$ may be encoded using $n = \lceil \log(|U|) \rceil$ boolean variables x_1, \dots, x_n
- Thus, χ may be represented by an OBDD on x_1, \dots, x_n
 - as for Model Checking, we may represent S , $\text{Reach}(S)$, R , ...
 - R will need $2n$ variables!
 - CTL Model Checking algorithm becomes feasible!
 - for many interesting real-sized systems, S , $\text{Reach}(S)$, R will now fit in RAM



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OBDD and Model Checking

- The most difficult part is to derive the OBDD for R directly from the model specification
 - i.e., from the model checker input language
 - it would be rather difficult to do it with SPIN
 - especially because it has a dynamic state space
 - also the one for Murphi would require some effort
 - S is easy, you only have to look at global variables
 - not in SPIN...
- NuSMV input language is tailored to be easily translated into OBDDs
 - also into CNF, as we will see...



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- SMV (Symbolic Model Verifier): McMillan implementation of the ideas in the famous paper “Symbolic model checking: 10^{20} states and beyond”
 - McMillan PhD dissertation about SMV is one of the most important dissertations in Computer Science
- SMV has been then re-written and standardized by the research group in Trento (also Genova and CMU collaborated), thus becoming NuSMV
 - the engine is still McMillan’s work
 - code has been nearly entirely commented, and made more readable
 - some features has been added: interactive mode, bounded model checking
 - OBDDs are handled via the CUDD library (by E. Somenzi at Colorado University)



NuSMV Input Language

Taken from `examples/smv-dist/short.smv`

```
MODULE main
```

```
VAR
```

```
  request : {Tr, Fa}; -- same as saying boolean
                      -- (stand for True and False)
```

```
  state : {ready, busy};
```

```
ASSIGN
```

```
  init(state) := ready;
```

```
  next(state) := case
```

```
    state = ready & (request = Tr): busy;
```

```
    TRUE : {ready, busy};
```

```
  esac;
```

```
SPEC
```

```
  AG((request = Tr) -> AF state = busy)
```



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NuSMV Input Language

- One *module*, there may be more, but one of them must be named `main`
- Module variables are those declared with `VAR`
- Base types are like Murphi ones: enumerations and integer subranges, plus the `word` type (i.e., an array of bits)
- Arrays are possible, but can be indexed only with constants
- Structures are modeled through modules
 - that is, each module has its variables (fields of a structure) and may be instantiated many times



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NuSMV Input Language

- ASSIGN section specifies the set I (via `init`) and the relation R (via `next`)
 - as in Murphi, there expressions which are essentially guard/action
 - differently from Murphi, each action deals with *one variable only*
 - the guard may be defined on any other variable (and it is typically the case)
 - if something is not specified, then it is understood to be non-deterministic
 - indirect specification; also direct specification is allowed, as we will see



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NuSMV Input Language: ASSIGN

- E.g., in `short.smv` initial states are those in which state is ready and request may be either Tr or Fa
- Thus, there are 2 initial states $I = \{\langle \text{ready}, \text{Tr} \rangle, \langle \text{ready}, \text{Fa} \rangle\}$, which may be represented with $\langle \text{ready}, \perp \rangle$
- Also `next(request)` is not specified; before analyzing what does this mean, let us see `next(state)`
- The case expression works as follows: the first condition C which is evaluated to true is fired, other true guards possibly following C are ignored



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NuSMV Input Language: ASSIGN

- This allows to put 1 (i.e., true) as the last guard, representing the “default” case
- NuSMV also checks if a case expression is exhaustive in its conditions, as this allows it to assume that R is total
- Note that the last condition on state leads to a non-deterministic transition: if the first guard is false, then state may take any value between ready e busy, that is any value in its domain
- In general, any subset of the variable domain may be used



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NuSMV Input Language: ASSIGN

- request is completely non-deterministic, as it does not occur in any next
- I.e., if other rules tells that the system may go from s to t and $(\text{request} = \text{Fa}) \in L(t)$, then there exists a transition from s to t' with $(\text{request} = \text{Tr}) \in L(t')$ and $L(t) \setminus \{(\text{request} = \text{Fa})\} = L(t') \setminus \{(\text{request} = \text{Tr})\}$
- Simply stated, if the system may go from s to t and request has a value v in t , then the system may also go from s to t' s.t. t and t' only differ in the value of request, which is different from v
- By combining all non-determinism in this example, the Kripke structure defined here excludes just two transitions

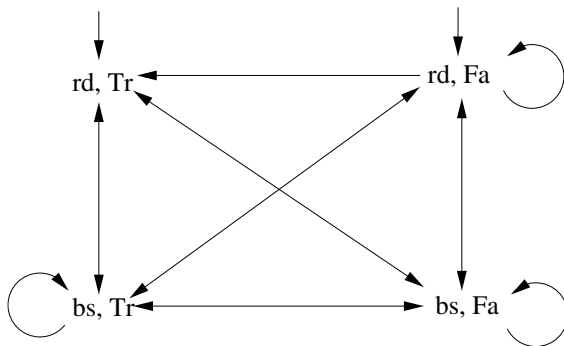


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Automata for short.smv: I and R



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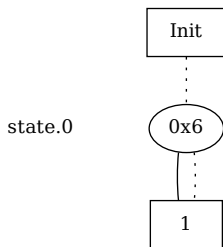
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OBDDs for short.smv: /

Straight lines are then-edges

Dashed lines are else-edges

Dotted lines are complemented-else-edges



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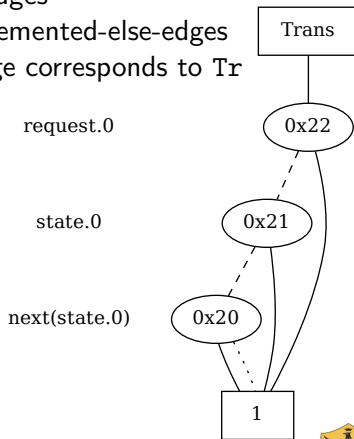
OBDDs for short.smv: R

Straight lines are then-edges

Dashed lines are else-edges

Dotted lines are complemented-else-edges

request.0 “false” edge corresponds to Tr



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NuSMV Input Language

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MODULE main
VAR
  request : {Tr, Fa};
  state : {ready, busy};
ASSIGN
  init(state) := ready;
  next(state) := case
    state = ready & (request = Tr): busy;
    TRUE : {ready, busy};
  esac;
SPEC
  AG((request = Tr) -> AF state = busy)
```



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    TRUE : ready;
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```

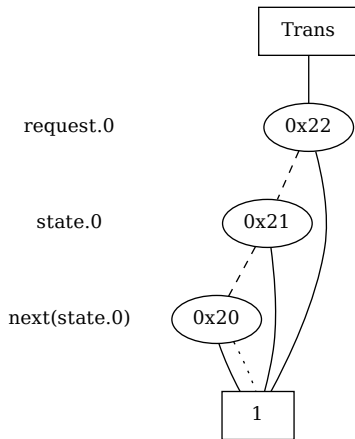


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OBDDs for short.smv: R

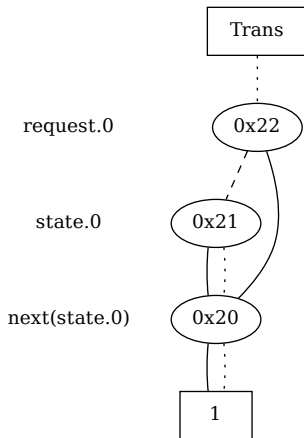


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OBDDs for short.soloready.smv: R



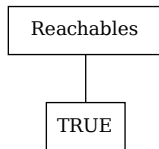
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OBDDs for short.smv: Reach

The one for soloready is the same



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NuSMV Input Language

```
MODULE main
VAR
  request : {Tr, Fa};
  state : {ready, busy};
ASSIGN
  init(state) := ready;
  next(state) := case
    state = ready & (request = Tr): busy;
    TRUE : ready;
  esac;
  next(request) := request;
SPEC
  AG((request = Tr) -> AF state = busy)
```

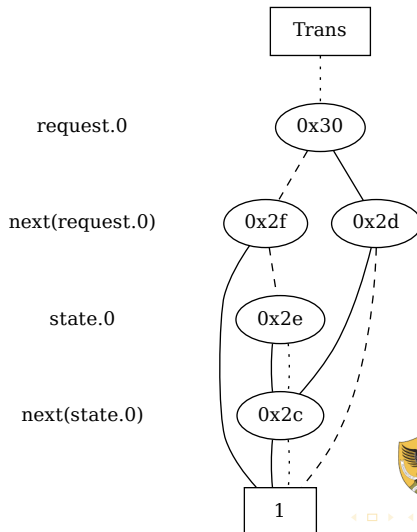


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OBDDs for short.soloready.req_const.smv: R

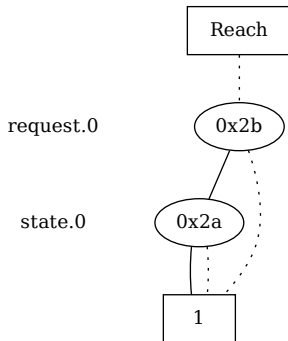


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OBDDs for short.soloready.req_const.smv: Reach



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OBDDs Pros and Cons

```
MODULE main
VAR
  m1 : 0..15; -- m1.0 is MSB!
  m2 : 0..15;
  m3 : 0..30;
ASSIGN
  next(m3) := m1 + m2;

SPEC
  AG(m3 <= 30);
```



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OBDDs Pros and Cons

```
MODULE main
VAR
  m1 : 0..15;
  m2 : 0..15;
  m3 : 0..30;
ASSIGN
  next(m3) := case
    m1*m2 <= 30: m1*m2;
    TRUE: m3;
  esac;

SPEC
  AG(m3 <= 30);
```



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OBDDs for Adder and Multiplier: /

This is a set with $16 \cdot 16 \cdot 31 = 7936$ elements
Just one node to represent it...

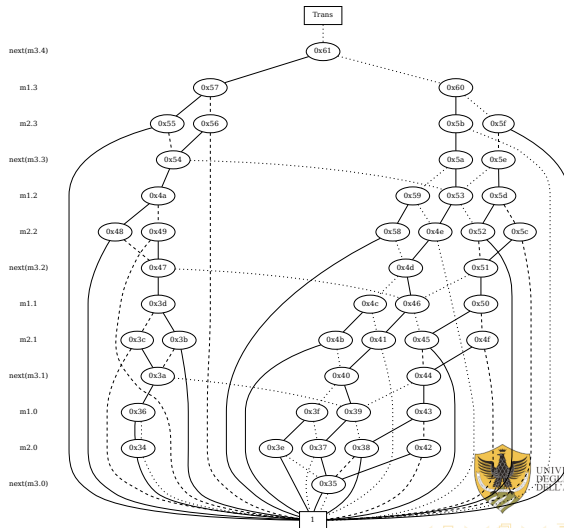


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OBDDs for Adder: R

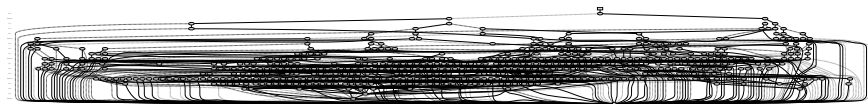


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OBDDs for Multiplier: R



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OBDDs Pros and Cons

- Number of variables is 13 for both models
 - 4 each for m_1 and m_2 , plus 5 for m_3
- Number of BDD nodes:
 - adder: 47
 - multiplier: 538



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OBDDs Pros and Cons

- No magic: SAT could be solved using OBDDs
 - just represent the instance with an OBDD and check if it is different from 0
 - very roughly speaking: if it were possible to solve it “efficiently” in this way, $P=NP$...
- Thus, there are boolean functions for which OBDDs representation is exponential, regardless of variable ordering
 - one example is the multiplier seen above
- It is not possible to say if OBDDs will be a good way to represent a problem, before trying it
 - for the adder, it is much more efficient
- Furthermore, finding a variable order in order to minimize the OBDD representation for a given function is an NP-complete problem



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OBDDs Pros and Cons

- This also holds for Model Checking in general
- Not possible to say a-priori if a system will fit in the available resources when using a model checker
 - RAM and computation time
- Also, it is not possible to decide which model checker is better
 - explicit (Murphi-or-SPIN like) or symbolic (NuSMV like)?
- However, we are going to see some guidelines
 - as for OBDDs: a good ordering is to interleave present and future variables
 - variable ordering: if OBDDs grow, the model checker can try a different variable ordering



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NuSMV Input Language

```
MODULE counter_cell(carry_in)
VAR value : boolean;
ASSIGN
    init(value) := 0;
    next(value) := (value + carry_in) mod 2;
DEFINE carry_out := value & carry_in;

MODULE main
VAR
    bit0 : counter_cell(1);
    bit1 : counter_cell(bit0.carry_out);
    bit2 : counter_cell(bit1.carry_out);

SPEC AG(!bit3.carry_out)
```

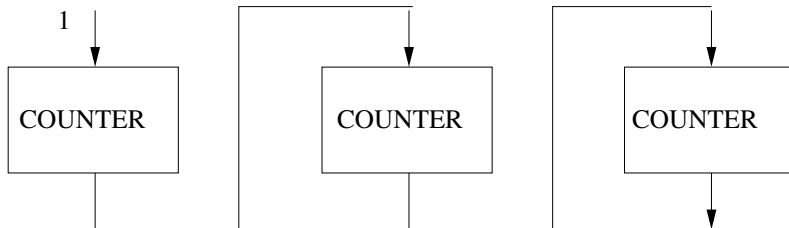


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Counter Cell



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NuSMV Input Language

- 2 modules, `main` and `counter_cell`
- Main *instantiates* the module `counter_cell` for 3 times
- This is an hardware-like instantiation: the `main` module contains 3 equal copies of the `counter_cell` module, the only difference being the lines in input
- Note that this means the module `main` will have 3 copies of variable `value`



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NuSMV Input Language

- Note that `carry_out` (being inside a `DEFINE` section) is *not* a variable, as it is only a shortcut for the expression it defines
 - i.e., there will not be a corresponding variable in the OBDD
 - and indeed, it is not declared as a variable...
- Hence, `bit0` will always sum 1 to its internal variable, and `bit1` will sum 1 only if `bit0` will generate a carry
- The `main` module defines a counter from 0 to 7



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NuSMV Input Language

```
MODULE user(semaphore)
VAR
  state : {idle, entering, critical, exiting};
ASSIGN
  init(state) := idle;
  next(state) :=
    case
      state = idle: entering;
      state = entering & !semaphore: critical;
      state = critical: {critical, exiting};
      state = exiting: idle;
      TRUE : state;
  esac;
```



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NuSMV Input Language

```
next(semaphore) :=  
  case  
    state = entering: TRUE;  
    state = exiting: FALSE;  
    TRUE: semaphore;  
  esac;
```



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NuSMV Input Language

```
MODULE main
```

```
VAR
```

```
    semaphore : boolean;
```

```
    proc1 : process user(semaphore);
```

```
    proc2 : process user(semaphore);
```

```
ASSIGN
```

```
    init(semaphore) := FALSE;
```

```
SPEC
```

```
    AG(!(proc1.state = critical & proc2.state = critical))
```

```
LTLSPEC
```

```
    G F proc1.state = critical
```



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NuSMV Input Language

- In the previous examples, all variables were evolving at the same time
- There is a global clock as in a synchronous digital circuit: given the current value for all variables in the current clock tick, in the next clock tick all variables may change their variables at the same time (synchronously: hardware parallel execution)
- In this example, instead, instantiations are *processes*
- I.e., just one variable at a time may change; other variables are forced to stay fixed
- No dynamic process spawning as in SPIN: the number of processes is known from the beginning



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NuSMV Input Language

- *Synchronous vs. asynchronous* systems
- In asynchronous systems, there is essentially one (implicit) additional module, which acts as a scheduler
- This is indeed what the verification algorithm does
- Each process is automatically provided with an additional variable `running` which is true iff that process is currently running



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NuSMV Input Language

```
MODULE inverter(input)
VAR
  output : boolean;
ASSIGN
  init(output) := 0;
  next(output) := !input;
```

```
MODULE main
VAR
  gate1 : process inverter(gate3.output);
  gate2 : process inverter(gate1.output);
  gate3 : process inverter(gate2.output);
```

```
SPEC
  AG(!gate2.output | !gate3.output)
```

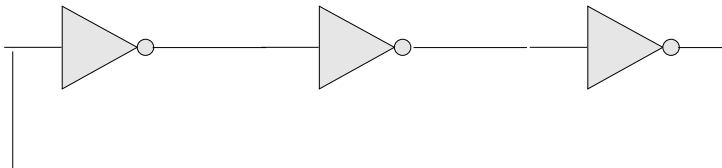


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Inverter Cell



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NuSMV Input Language

Using direct specification it is possible to define non-total transition relations or empty initial states set

```
MODULE inverter(input)
VAR
    output : boolean;
INIT
    output = 0
TRANS
    next(output) = !input
```



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NuSMV Input Language

Without processes, is it equivalent?

```
MODULE inverter(input)
VAR
  output : boolean;
ASSIGN
  init(output) := 0;
  next(output) := !input union output;
                -- or {!input, output}
```

```
MODULE main
VAR
  gate1 : inverter(gate3.output);
  gate2 : inverter(gate1.output);
  gate3 : inverter(gate2.output);
```



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NuSMV As A Tool

- NuSMV is provided with an interactive shell, as there are many tasks it may accomplish (simulation, many verification options); see user manual from chapter 3, especially Figure 3.1 at page 87
- Differently from explicit model checkers, no need to give separate commands to generate a file to be compiled and executed: all is represented as OBDDs, you only have to use them properly
- Executing a non-interactive verification in NuSMV is the same as giving the following list of interactive commands
- 1. `read_model` reads and stores the syntactic structure of the input model
 - no OBDDs here: tree-like structure, but representing the syntactic structure of the input (abstract syntax tree)



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NuSMV As A Tool

- 2. `flatten_hierarchy` (recursively) brings inside `main` all modules instantiated by `main`
 - very similar to the unfolding we mentioned for Murphi and SPIN: for such explicit model checkers, this was only needed for theoretical purposes, in order to define the Kripke structure of an input model
 - here, it must be actually performed in the source code of NuSMV, in order to then be able to encode R and I as OBDDs
 - to this aim, there must be only one module, the `main`, containing all variables coming from the modules it instantiates (to be applied recursively)
 - note that, again, this resembles digital circuits, where such a flattening is a natural operation
 - this could entail adding a scheduler module if processes are used



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NuSMV As A Tool

- 3. `encode_variables` for each variable x with domain D s.t. $|D| > 2$, NuSMV defines $x_1 \dots, x_m$ *boolean* variables with $m = \lfloor \log_2 |D| \rfloor + 1$; it also defines the encoding for constants used in the input models
- 4. `build_flat_model` combines the result of the preceding operations to obtain the flattenized and booleanized syntactic structure which represents the Kripke structure defined by the input model
- 5. `build_model` from the syntactic structure to OBDDs for R and I (plus other ones)
- 6. `check_ctlspec` (or `check_ltlspec`, or both, depending on what you have to verify); it starts the actual verification
 - we will be back soon on these last 2 steps



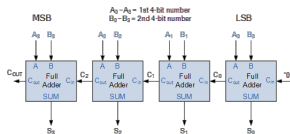
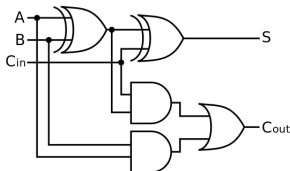
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From Syntactic Structure to OBDDs

- How does `build_model` work?
- All operations must be implemented bitwise (*bit-vector*)
 - this means that we have to build the corresponding digital circuit, remember the Digital Systems Design course?
 - if we have to implement a sum between two variables encoded with maximum 4 bits (note that result is on 5 bits):



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From Syntactic Structure to OBDDs

- Analogously, you can represent other arithmetic operations (subtract, multiply, divide)
- With other simple digital circuits, also equality and ordering can be easily implemented
 - e.g., $\text{next}(a) = b + c$ is translated in this way:
 - multiple OBDDs are used to sum all bits of b and c
 - an OBDD B is created which is true iff all variables of $\text{next}(a)$ are equal to such OBDDs
 - e.g., $\text{next}(a) = \text{case } a < b: b + c; \text{TRUE} : a$ is translated in this way:
 - again we have B as before, plus an OBDD C which is true if $a < b$
 - then, NuSMV computes the OBDD $\text{ITE}(C, B, a)$



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From a NuSMV Description to KS

- From a NuSMV model \mathcal{M} (defined with the ASSIGN section) to the corresponding Kripke structure $\mathcal{S} = (S, I, R, L)$
 - $V = \langle v_1, \dots, v_n \rangle$ is the set of variables defined inside the `main` module of \mathcal{M} , with domains $\langle D_1, \dots, D_n \rangle$
 - note that each D_i may be the instantiation of other modules
 - in which case, again, all variables must be considered as *unfolded*
 - that is, if a variable v is the instantiation of a module with k variables, then v counts as k variables instead of one
 - if one of such k variables is another instantiation, this procedure must be recursively repeated
 - NuSMV calls this operation *hierarchy flattening*
 - essentially, it is the same as for records in Murphi
 - simple types are the recursion base step



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From a NuSMV Description to KS

- $S = D_1 \times \dots \times D_n$ (as in Murphi)
- I is defined by looking at `init` predicates
 - $s \in I$ iff, for all variables $v \in V$, $s(v) \in \text{init}(v)$
 - note that, by NuSMV syntax, each $\text{init}(v)$ is actually a set (possibly a singleton)
 - if $\text{init}(v)$ is not specified in \mathcal{M} , then any value for v is ok: in this case, formally, if $s \in I$, then also $s' \in I$ being $s'(v') = s(v') \forall v' \neq v$



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From a NuSMV Description to KS

- R is defined by looking at `next` predicates
 - we assume all `next` predicates to be defined by the case construct (if not, simply assume it is the case construct with just one TRUE condition)
 - for each (flattened) variable v , we name $g_1(v), \dots, g_{k_v}(v)$ the conditions (guards) of the case for `next(v)`, and $a_1(v), \dots, a_{k_v}(v)$ the resulting values (actions) of the case for `next(v)`
 - note that, by NuSMV syntax, each $a_i(v)$ is actually a set (possibly a singleton)
 - $(s, s') \in R$ iff, for all variables $v \in V$, if $g_i(s(v)) \wedge \forall j < i \neg g_j(s(v))$ then $s'(v) \in a_i(v)$
 - that is, s may go in s' iff, for all variables v , if the values of v in s satisfy the guard g_i (and none of the preceding guards for the same variable), then the value of v in s' is one of the values specified by the case for guard g_i
 - note that, in doing this, you also have to resolve inputs for modules



From a NuSMV Description to KS

- $AP = \{(v = d) \mid v = v_i \in V \wedge d \in D_i\}$
- $(v = d) \in L(s)$ iff variable v has value d in s
- If, instead, the NuSMV model \mathcal{M} is defined with the TRANS section, then
 - $V = \langle v_1, \dots, v_n \rangle$ is the set of variables as above and $S = D_1 \times \dots \times D_n$
 - I is defined by looking at INIT section
 - $s \in I$ iff, for all variables $v \in V$ and for all INIT sections \mathcal{I} , $\mathcal{I}(s(v))$ holds
 - R is defined by looking at TRANS section
 - $(s, s') \in R$ iff, for all variables $v \in V$ and TRANS sections T , $T(s(v), s'(v))$ holds



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λ -calculus: Representing Functions

- In a nutshell: using $f(x)$ has some drawbacks
 - you are forced to name a function (f in the example above)
 - it is not always clear if a letter is a parameter or an argument
 - it is not computationally clear what happens for multiple inputs
 - $f(x, y)$: do you have to provide both x, y , otherwise you get an error?
 - as an alternative, you may provide just one argument, and obtain a new function
 - e.g. $f(x, y) = x + y$, we have that $f(x, 4)$ is a function on x



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λ -calculus: Representing Functions

- Instead of writing $f(x) = E(x)$, for some expression $E(x)$, we write $\lambda x.E(x)$
 - if you *want*, you can name a function $f(x) = \lambda x.E(x)$
- $\lambda(x, y).x + y$: both arguments must be given, otherwise it is an error
- $\lambda x \lambda y.x + y$: if you provide $x = 4$ only, you get a function $\lambda y.4 + y$
- If an OBDD contains variables x_1, \dots, x_n , then it represent some function $\lambda x_1 \dots \lambda x_n. E(x_1, \dots, x_n)$



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μ -calculus: Fixpoints

- In a nutshell: we have a set L with an ordering \leq
 - \leq could be partial, i.e., not defined on some pair $(l_1, l_2) \in L \times L$
 - L, \leq is a *complete lattice* if any subset $A \subseteq L$ has a greatest lower bound and a least upper bound in L
 - $\sup A = \min\{\xi \in L \mid \forall \alpha \in A. \alpha \leq \xi\} \rightarrow \sup A \in L$
 - $\inf A = \max\{\xi \in L \mid \forall \alpha \in A. \xi \leq \alpha\} \rightarrow \inf A \in L$



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μ -calculus: Fixpoints

- Let $I = \{0, \dots, 10\}$, then $2^I, \subseteq$ is a complete lattice
 - e.g., $\{0, 1, 2\} \leq \{0, 1, 2, 3\}$, whilst $\{0, 1, 2\}, \{0, 1, 3\}$ cannot be compared
 - $\sup\{\{0, 1, 2\}, \{0, 1, 3\}\} = \min\{\xi \in 2^I \mid \forall \alpha \in \{0, 1, 2\}, \{0, 1, 3\}. \alpha \subseteq \xi\} = \min\{\{0, 1, 2, 3\}, \dots, I\} = \{0, 1, 2, 3\}$
 - $\inf\{\{0, 1, 2\}, \{0, 1, 3\}\} = \max\{\xi \in 2^I \mid \forall \alpha \in \{0, 1, 2\}, \{0, 1, 3\}. \xi \subseteq \alpha\} = \max\{\{0, 1\}, \dots, \emptyset\} = \{0, 1\}$
- $2^I, \subseteq$ is always a complete lattice, if I is a finite set
 - $\sup J = \bigcup_{\xi \in J} \xi$, $\inf J = \bigcap_{\xi \in J} \xi$
 - at the worst, $\sup J = I$ and $\inf J = \emptyset$



μ -calculus: Fixpoints

- Suppose you have a function $T : L \rightarrow L$. An element $\xi \in L$ is a *fixpoint of T* iff $T(\xi) = \xi$
- Given a T , there may be several fixpoints: we are interested in the maximum or the minimum of such fixpoints
 - notation μT and νT
 - where typically T is expressed with a λ notation
 - $\mu T \equiv \xi$ s.t. $T(\xi) = \xi \wedge \forall \rho \in L. T(\rho) = \rho \rightarrow \xi \leq \rho$
 - $\nu T \equiv \xi$ s.t. $T(\xi) = \xi \wedge \forall \rho \in L. T(\rho) = \rho \rightarrow \rho \leq \xi$



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μ -calculus: Fixpoints

- Let again $I = \{1, \dots, 10\}$
- Let $T : 2^I \rightarrow 2^I$ be defined as $T(\xi) = \xi$, or better $T \equiv \lambda \xi. \xi$
 - we have $\mu T = \emptyset, \nu T = I$
- Let $T \equiv \lambda \xi. \emptyset$
 - we have $\mu T = \nu T = \emptyset$
- Let $T \equiv \lambda \xi. \xi \cup \{10\}$
 - we have $\mu T = \{10\}, \nu T = I$
- Let $T \equiv \lambda \xi. \xi \setminus \{10\}$
 - we have $\nu T = \{0, \dots, 9\}, \mu T = \emptyset$



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μ -calculus: Fixpoints

- We define sets by their characteristic function, thus let us rewrite the previous examples
 - thus the ξ in $\lambda\xi$ is a function $\xi : I \rightarrow \{0, 1\}$
 - it represents a set X , thus $\xi(x) = 1$ iff $x \in X$
- $T \equiv \lambda\xi.\xi$ is ok also if ξ is a characteristic function
- $T \equiv \lambda\xi.\emptyset$ could be rewritten as $T \equiv \lambda\xi.\lambda x.0$
- $T \equiv \lambda\xi.\xi \cup \{10\}$ could be rewritten as
$$T \equiv \lambda\xi.\lambda x.[x = 10 \rightarrow 1] \wedge [x \neq 10 \rightarrow \xi(x)]$$
 - $\mu T \equiv \lambda x.x = 10, \nu T \equiv \lambda x.1$
- $T \equiv \lambda\xi.\xi \setminus \{10\}$ could be rewritten as
$$T \equiv \lambda\xi.\lambda x.[x = 10 \rightarrow 0] \wedge [x \neq 10 \rightarrow \xi(x)]$$
 - $\nu T \equiv \lambda x.x \neq 10, \mu T \equiv \lambda x.0$



μ -calculus: Fixpoints

- We deal with monotonic (i.e., increasing or decreasing) T , thus fixpoints always exists
 - $\xi \leq \rho \rightarrow T(\xi) \leq T(\rho)$, T monotonically increasing
 - $\xi \leq \rho \rightarrow T(\rho) \leq T(\xi)$, T monotonically decreasing
- Previous examples are all monotonic
- By (weak) Knaster-Tarski theorem, $\mu T = \inf\{\xi \mid T(\xi) \leq \xi\}$
 - analogously, $\nu T = \sup\{\xi \mid T(\xi) \geq \xi\}$



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μ -calculus: Fixpoints Computation

- Consequence of Knaster-Tarski: computing μT and νT may be done as follows
- For $k \geq 1$, let $T^k(\xi) = T(T^{k-1}(\xi))$, with $T^1 = T$
- For least fixpoints (μT), start with \emptyset , and apply T since $T^k(\emptyset) = T^{k-1}(\emptyset)$
 - of course, $\emptyset = \lambda x.0$
- For greatest fixpoints (νT), start with U , and apply T since $T^k(U) = T^{k-1}(U)$
 - of course, $U = \lambda x.1$
- At most, $k = |U|$



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Computation of Fixpoints in CTL Model Checking

- Given a KS $\mathcal{S} = (S, I, R, L)$, we want to label states, i.e., to identify subsets of S
 - those for which a given labeling holds
 - labels are CTL/LTL subformulas
- Thus, $L = 2^S$, \leq is \subseteq and $T : 2^S \rightarrow 2^S$
 - in the following, $x = x_1, \dots, x_n$ with $n = \lceil \log |S| \rceil$
 - characteristic functions of subsets of S
 - thus, each subset of S (member of 2^S) is an OBDD
 - hence, a T takes an OBDD and returns another (possibly modified) OBDD
- At most, $k = |S|$
 - usually, much less than that



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μ -calculus: Fixpoints

- The “really interesting” fixpoints are those which are recursively defined
 - typically, basing on some other already defined sets, i.e., characteristic functions
 - e.g., $T \equiv \lambda\xi.\lambda x.f(x) \vee \xi(x)$, where $f : S \rightarrow \{0,1\}$ is known
 - the compactly-written least and greatest fixpoints are $\mu Q.\lambda x.f(x) \vee Q(x)$ and $\nu Q.\lambda x.f(x) \vee Q(x)$
 - e.g., $T \equiv \lambda\xi.\lambda x.f(x) \wedge \xi(x)$
 - e.g., $T \equiv \lambda\xi.\xi(x)$
- By the Knaster-Tarski theorem and the previous reasoning, we may apply the following algorithms
 - least fixpoints μ are computed for increasing T
 - greatest fixpoints ν are computed for decreasing T
 - viceversa are trivial: μT is $\lambda x.0$ for decreasing T and νT is $\lambda x.1$ for increasing T



Computation of Least (Minimum) Fixpoint

```
OBDD lfp(MuFormula T) /*  $\mu Z.T(Z)$  */  
{  
  Q =  $\lambda x.0$ ;  
  Q' = T(Q);  
  /* T clearly says where Q must be replaced */  
  /* e.g.: if  $\mu Z.\lambda x.f(x) \vee Z(x)$ , then  
    Q' =  $\lambda x.f(x) \vee Q(x)$  */  
  while (Q  $\neq$  Q') {  
    Q = Q';  
    Q' = T(Q);  
  }  
  return Q; /* or Q', they are the same... */  
}
```



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Computation of Greatest (Maximum) Fixpoint

```
OBDD gfp(NuFormula T) /*  $\nu Z.T(Z)$  */  
{  
    Q =  $\lambda x. 1$ ;  
    Q' = T(Q);  
    while (Q  $\neq$  Q') {  
        Q = Q';  
        Q' = T(Q);  
    }  
    return Q;  
}
```



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Symbolic Model Checking of $\mathbf{AG}p$

- The idea is to compute the set of reachable states, and check if for all of them p holds
- $\text{Reach} = \mu Z. \lambda x. [I(x) \vee \exists y : (Z(y) \wedge R(y, x))]$
 - of course, we get an OBDD on x as a result
 - recall that x (and y) is a vector of all boolean variables
- $\forall x \in S. \text{Reach}(x) \rightarrow p(x)$
 - computationally easier: check that $\text{Reach}(x) \wedge \neg p(x) = 0$
 - otherwise, we have a reachable state for which p does not hold...



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Symbolic CTL Model Checking

- All CTL formulas can be reduced to 3: **EXf**, **f EU g**, **EGf**
 - all other formulas may be reduced to these three, using negation and other boolean combinations
 - with OBDDs, we can do all such things!
- Given OBDDs for f (and g), we compute the OBDD representing **EXf**, **f EU g**, **EGf**
 - that is, the OBDD for the set $X = \{s \in S \mid \mathcal{S}, s \models \mathbf{EX}f\}$ etc
- Let it be B : then, simply check $\neg B(x) \wedge I(x) = 0$
 - recall that $\mathcal{S} \models \Phi$ iff $\forall s \in I. \mathcal{S}, s \models \Phi$
- **EXf** does not require a fixpoint computation: it is equivalent to (the OBDD representing) $\lambda x. \exists y : R(x, y) \wedge f(y)$



Symbolic CTL Model Checking

- For $f \text{ EU } g$, recall that it is equivalent to the CTL formula $g \vee (f \wedge \text{EX}(f \text{ EU } g))$
- Thus, $f \text{ EU } g = \mu Z. \lambda x. g(x) \vee (f(x) \wedge \text{EX}Z(x)) = \mu Z. \lambda x. g(x) \vee (f(x) \wedge (\exists y : R(x, y) \wedge Z(y)))$
 - note that $g(x) \vee (f(x) \wedge \text{EX}Z(x))$ is increasing, i.e. for $Z_1 \subseteq Z_2$ we have that $(g(x) \vee (f(x) \wedge \text{EX}Z_1(x))) \rightarrow (g(x) \vee (f(x) \wedge \text{EX}Z_2(x)))$
- Analogously: $\text{EG}f = f \wedge \text{EX}(\text{EG}f)$, thus $\text{EG}f = \nu Z. \lambda x. f(x) \wedge \text{EX}Z(x) = \nu Z. \lambda x. f(x) \wedge (\exists y : R(x, y) \wedge Z(y))$
 - note that $f(x) \wedge \text{EX}Z(x)$ is decreasing, i.e. for $Z_1 \subseteq Z_2$ we have that $(f(x) \wedge \text{EX}Z_2(x)) \rightarrow (f(x) \wedge \text{EX}Z_1(x))$



CTL Model Checking

```
bool checkCTL(KS S, CTL  $\varphi$ ) {  
  let  $S = \langle S, I, R, L \rangle$ ;  
   $B = \text{Lb1St}(\varphi)$ ;  
  return  $\lambda x. I(x) \wedge \neg B(x) = \lambda x. 0$ ;  
}  
  
OBDD Lb1St(CTL  $\varphi$ ) { /* also  $S = \langle S, I, R, L \rangle$  */  
  if ( $\exists p \in AP. \varphi = p$ ) return  $\lambda x. p(x)$ ;  
  else if ( $\varphi = \neg\phi$ ) return  $\lambda x. \neg \text{Lb1St}(\phi)(x)$ ;  
  else if ( $\varphi = \phi_1 \wedge \phi_2$ )  
    return  $\lambda x. \text{Lb1St}(\phi_1)(x) \wedge \text{Lb1St}(\phi_2)(x)$ ;  
  else if ( $\varphi = \mathbf{EX}\phi$ )  
    return  $\lambda x. \exists y : R(x, y) \wedge \text{Lb1St}(\phi)(y)$ ;  
  else if ( $\varphi = \mathbf{EG}\phi$ )  
    return  $\text{gfp}(\nu Z. \lambda x. \text{Lb1St}(\phi)(x) \wedge (\exists y : R(x, y) \wedge Z(y)))$ ;  
  else if ( $\varphi = \phi_1 \mathbf{EU} \phi_2$ )  
    return  $\text{lfp}(\mu Z. \lambda x. \text{Lb1St}(\phi_2)(x) \vee$   
       $(\text{Lb1St}(\phi_1)(x) \wedge (\exists y : R(x, y) \wedge Z(y))))$ ;  
}
```

