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CTL and LTL Model Checking Algorithms

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Theoretic vs. Practical Algorithms

- Model Checking problem:
 - input: a KS $\mathcal{S} = \langle S, I, R, L \rangle$ and a formula φ
 - output: true iff $S \models \varphi$, $\langle \text{false}, c \rangle$ otherwise, being c a counterexample
- \bullet Depending on φ being LTL or CTL, different algorithms must be provided
- We will first show the "theoretical" algorithm for CTL
 - classical approach: both S and R fit into RAM
 - graph-based: we will see that the one actually used is instead fix-point based
- Then, we will see how they can be efficiently implemented
 - LTL: SPIN and NuSMV
 - CTL: NuSMV





CTL Theoretic Algorithm

- ullet CTL is based on *state* formulas, i.e., φ holds depending on the state we are considering
 - this also holds for subformulas of φ , e.g., **AFAG**p has one subformula **AG**p
- Since we have the full state space S, we label all states $s \in S$ with (sub)formulas holding in s
 - not only the reachable states: all of them
- Then, we use subformulas labeling to decide higher formulas labelling
- ullet Thus, we compute $\lambda:S o 2^{\mathrm{CTL}}$, being CTL the set of all CTL formulas
- At the end, $\mathcal{S} \models \varphi$ iff $\forall s \in I$. $\varphi \in \lambda(s)$





CTL Theoretic Algorithm

- ullet Consider the abstract syntax tree of arphi, call it ϕ
- Start from the leaves in ϕ , which must be an atomic proposition p or true
 - $\forall s \in S. \ p \in \lambda(s) \Leftrightarrow p \in L(s)$
 - $\forall s \in S$. true $\in \lambda(s)$
- ullet Then go upwards in ϕ , using, for each node, the labeling of the sons
 - $\forall s \in S$. $\neg \Phi \in \lambda(s) \Leftrightarrow \Phi \notin \lambda(s)$
 - $\forall s \in S$. $\Phi_1 \land \Phi_2 \in \lambda(s) \Leftrightarrow (\Phi_1 \in \lambda(s) \land \Phi_2 \in \lambda(s))$
 - $\forall s \in S$. **EX** $\Phi \in \lambda(s) \Leftrightarrow (\exists s' : (s,s') \in R \land \Phi \in \lambda(s'))$





CTL Theoretic Algorithm: $\Phi_1 \mathbf{E} \mathbf{U} \Phi_2 \in \lambda(s)$

- We already have $\lambda^{-1}(\{\Phi_1\})$ and $\lambda^{-1}(\{\Phi_2\})$
 - here and in the following, $\lambda^{-1}(\{\Phi\}) = \{s \in S \mid \Phi \in \lambda(s)\}$, for a CTL formula Φ
- All states satisfying Φ_2 are ok, let T be the set of such states
- Then, backward visit of the state space of S, starting from T
- The backward visit stops when Φ_1 does not hold
- Complexity is O(|S| + |R|)





CTL Theoretic Algorithm: $\Phi_1 \mathbf{E} \mathbf{U} \Phi_2 \in \lambda(s)$

}

```
labels CheckEU(KS S, formula \Phi_1 \mathbf{E} \mathbf{U} \Phi_2, labels \lambda)
{
    let S = \langle S, I, R, L \rangle;
    T = \{s \in S \mid \Phi_2 \in \lambda(s)\};
    for each s \in T
        \lambda(s) = \lambda(s) \cup \{\Phi_1 \mathbf{E} \mathbf{U} \Phi_2\};
    while (T \neq \emptyset) {
        let s be s.t. s \in T;
        T = T \setminus \{s\};
        foreach t \in \{t \mid (t,s) \in R\} {
             if \Phi_1 \mathbf{E} \mathbf{U} \Phi_2 \not\in \lambda(t) \land \Phi_1 \in \lambda(t) {
                 /* \Phi_1 \mathbf{E} \mathbf{U} \Phi_2 \notin \lambda(t): visited states check */
                 \lambda(t) = \lambda(t) \cup \{\Phi_1 \mathbf{E} \mathbf{U} \Phi_2\};
                 T = T \cup \{t\};
    return \lambda;
```

CTL Theoretic Algorithm: **EG** $\Phi \in \lambda(s)$

- We already have $\lambda^{-1}(\{\Phi\})$: this defines a subKS \mathcal{S}' of \mathcal{S}
 - $\lambda^{-1}(\{\Phi\})$ contains all states in which Φ holds
- \bullet Then, compute the strongly connected components (SCCs) of \mathcal{S}'
 - inside such components, Φ holds on all states on all paths
- Finally, label with **EG** Φ all s in such SCCs, plus all backward reachable $t \in S'$
 - so we move on states for which Φ holds forever in at least one path...
- Complexity is again O(|S| + |R|)





CTL Theoretic Algorithm: **EG** $\Phi \in \lambda(s)$

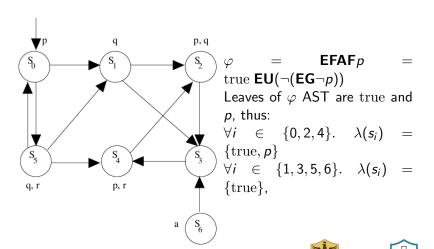
```
labels CheckEG(KS S, formula EG\Phi, labels \lambda)
   let S = \langle S, I, R, L \rangle;
   S' = \{s \in S \mid \Phi \in \lambda(s)\}; R' = \{(s,t) \in R \mid s,t \in S'\};
   A = SCC(S', R'); T = \bigcup_{A \in A} A;
   for each s \in T, \lambda(s) = \lambda(s) \cup \{ EG\Phi \};
   while (T \neq \emptyset) {
       let s be s.t. s \in T;
       T = T \setminus \{s\};
       for each t \in \{t \mid (t,s) \in R'\}
           if \mathbf{EG}\Phi \notin \lambda(t) { /* since (t,s) \in R', \Phi \in \lambda(t) */
              \lambda(t) = \lambda(t) \cup \{ \mathbf{EG} \Phi \};
              T = T \cup \{t\};
   return \lambda;
```

CTL Theoretic Algorithm: Complexity

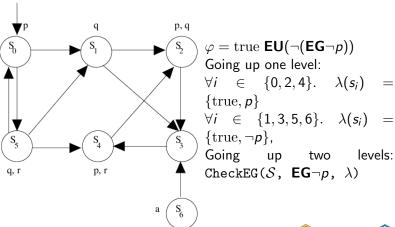
- Complexity is:
 - \circ O(|S|) for boolean combinations and atomic propositions
 - O(|S|) also for **EX** Φ
 - O(|S| + |R|) for **EG** Φ and Φ_1 **EU** Φ_2
- Since this must be done for every subformula of φ , the overall complexity is $O((|S| + |R|)|\varphi|)$
 - ullet |arphi| is the number of nodes of the abstract syntax tree of arphi
- Linear in the size of the input, if one of the two is fixed... is this as good as it seems?
- ullet Alas no: state space explosion hits exactly in |S| and |R|
 - \bullet $|\varphi|$ is typically low for real-world properties to be verified











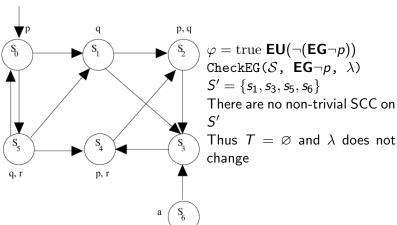




CTL Theoretic Algorithm: **EG** $\Phi \in \lambda(s)$

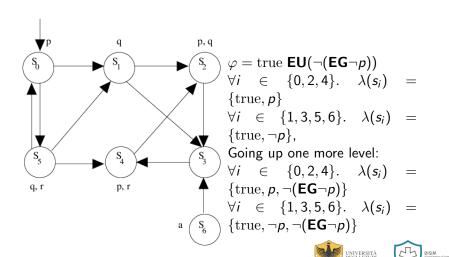
```
labels CheckEG(KS S, formula EG\Phi, labels \lambda)
   let S = \langle S, I, R, L \rangle;
   S' = \{s \in S \mid \Phi \in \lambda(s)\}; R' = \{(s,t) \in R \mid s,t \in S'\};
   A = SCC(S', R'); T = \bigcup_{A \in A \text{ s.t. } |A| > 1} A;
   for each s \in T, \lambda(s) = \lambda(s) \cup \{EG\Phi\};
   while (T \neq \emptyset) {
       let s be s.t. s \in T;
       T = T \setminus \{s\};
       for each t \in \{t \mid (t,s) \in R'\}
           if EG\Phi \notin \lambda(t) {
              \lambda(t) = \lambda(t) \cup \{ \mathbf{EG} \Phi \};
              T = T \cup \{t\};
   return \lambda;
```



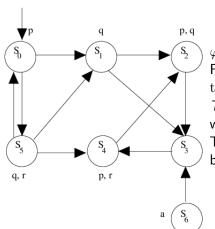












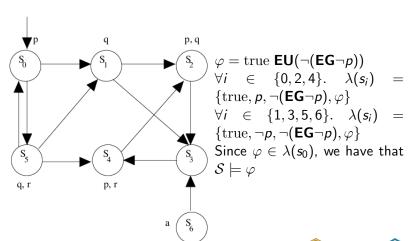
$$\begin{split} \varphi &= \text{true } \mathbf{EU}(\neg(\mathbf{EG} \neg p)) \\ \text{Finally, call } \text{CheckEU}(\mathcal{S}, \\ \text{true } \mathbf{EU}(\neg(\mathbf{EG} \neg p), \ \lambda) \\ T &= S, \text{ as all states are labelled} \\ \text{with } \neg(\mathbf{EG} \neg p) \\ \text{Thus, all states must be labelled with } \varphi \end{split}$$





CTL Theoretic Algorithm: $\Phi_1 \mathbf{E} \mathbf{U} \Phi_2 \in \lambda(s)$

```
labels CheckEU(KS S, formula \Phi_1 \mathbf{E} \mathbf{U} \Phi_2, labels \lambda)
    let S = \langle S, I, R, L \rangle;
    T = \{s \in S \mid \Phi_2 \in \lambda(s)\};
    for each s \in T
        \lambda(s) = \lambda(s) \cup \{\Phi_1 \mathbf{E} \mathbf{U} \Phi_2\};
    while (T \neq \emptyset) {
        let s be s.t. s \in T;
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        for each t \in \{t \mid (t,s) \in R\} {
             if \Phi_1 \mathbf{E} \mathbf{U} \Phi_2 \not\in \lambda(t) \land \Phi_1 \in \lambda(t) {
                 \lambda(s) = \lambda(s) \cup \{\Phi_1 \mathbf{E} \mathbf{U} \Phi_2\};
                 T = T \cup \{t\};
    return \lambda;
```





LTL Model Checking Algorithm

- Many LTL algorithms exist, we will directly see the most efficient one
- Surprising fact: not only LTL is not included inside CTL, it is also more difficult to check!
- Namely, whilst CTL model checking is in P, LTL model checking is PSPACE-complete
 - no, PSPACE is not "good" as P is: NP ⊂ PSPACE
- Efficient algorithms for LTL run in $O((|S| + |R|)2^{|\varphi|})$
- In practice, this is not much worse than CTL model checking
 - the real problem is O(|S| + |R|)
 - φ is usually small, it is difficult to come up with lengthy formulas



LTL Model Checking Algorithm

- The idea is simple: first translate φ into a special automaton $\mathcal{A}(\varphi)$
- ullet Then, perform a DFS visit both ${\mathcal S}$ and ${\mathcal A}(\varphi)$, one step at a time
 - ullet equivalent to verify to Cartesian product $\mathcal{S} imes \mathcal{A}(arphi)$
- If some special node s is found, start from s itself a new nested DFS
- If we are able to come back to s, we have a counterexample for φ
- Otherwise, $\mathcal{S} \models \varphi$
- Such algorithm may be implemented on-the-fly, thus instead of a KS we have an NFSS
 - no need to have S and R in memory before



Büchi Automaton

- A (non-deterministic) Büchi Automaton (BA) is a 5-tuple $\mathcal{A} = \langle \Sigma, Q, \delta, Q_0, F \rangle$ where:
 - ullet is the *alphabet*, i.e., a finite set of symbols
 - Q is the finite set of states
 - $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation
 - $Q_0 \subseteq Q$ are the initial states
 - $F \subseteq Q$ are the final states
- With respect to a KS, we also have final states and edges are labeled with symbols from an alphabet
 - the labeling L is also missing in BAs
 - however, we will see that AP is linked to Σ





Büchi Automaton

- BAs are not different from well-known automata in computational theory
 - finite state automata (FSA) are essentially equal in the definition
- The difference is in the language they accept
 - FSA: a word w is recognized if, by walking inside the FSA through symbols in w, a final state is reached
 - this implies that $|w| < \infty$
 - the set of all recognized w may be infinite, but each w is finite
- A BA recognize a(n infinite) language of infinite words
 - each word w has an infinite number of symbols





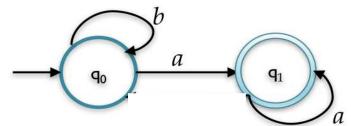
Language Accepted by Büchi Automata

- Let $w = w_0 w_1 \dots$ be an infinite string s.t. $\forall i. \ w_i \in \Sigma$
 - $w \in \Sigma^{\omega}$
- The BA ${\cal A}$ accepts w iff there exists a path $\pi=q_0w_0q_1w_1\dots$ s.t.
 - $\forall i. \ q_i \in Q \land w_i \in w \land (q_i, w_i, q_{i+1}) \in \delta$
 - $q_0 \in Q_0$
 - if $I = \{i \mid q_i \in F\}$, then $|I| = \infty$
 - otherwise stated: π goes through a final state *infinitely often* (or *almost always*)
 - ullet this is where the definition differs from FSAs, where π is finite and its final state must be in F
- $\mathcal{L}(\mathcal{A})$ is the set of infinite words recognized by \mathcal{A}
- Languages recognized by a BA are called ω -regular
 - recall that FSA recognize regular languages





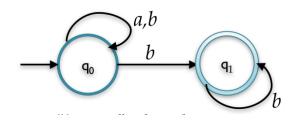
Büchi Automata Examples



- Final states are those with thicker boundaries, initial states are pointed to by an arrow
- ullet This recognizes the language b^*a^ω
- Note that a^* is a language (infinite set of finite words) containing ε , a, aa, aaa, ...
- Note that a^{ω} is a single infinite word aaaaaaaa...
- Thus, $b^*a^\omega = \{a^\omega, ba^\omega, bba^\omega, \ldots\}$
- That is: a finite number of b's, followed by infinite a's



Büchi Automata Examples



- This recognizes the language $(a+b)^*b^\omega$
- That is, $(a+b)^*b^\omega=\{b^\omega,ab^\omega,abab^\omega,abbabbab^\omega,\ldots\}$
- That is: any finite sequence of a and b, followed by infinite b's
- Cannot be recognized by a deterministic BA!
 - instead, deterministic FSAs recognize the same languages of non-deterministic FSAs

Büchi Automata and LTL Properties

- Also LTL properties are related to infinite words
 - recall that a model σ is an infinite sequence of truth assignments to all $p \in AP$
 - by adapting LTL semantics about $\pi \models \varphi$, we can define whether $\sigma \models \varphi$
 - we replace a path state $\pi(i)$ with the set $P_i \subseteq AP$ s.t. $P_i = \{ p \in AP \mid p \in L(\pi(i)) \}$
- Thus, an LTL property recognizes a language $\mathcal{L}(\varphi) = \{ \sigma \in (2^{AP})^{\omega} \mid \sigma \models \varphi \}$
 - ullet sometimes, we use arphi and $P=\mathcal{L}(arphi)$ interchangeably
- Furthermore, the "infinitely often" part recalls the LTL formula GFp
- Also the "eventually forever" FGp is important





Büchi Automata and LTL Properties

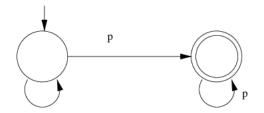
- Let φ be an LTL formula, and let $\mathcal{L}(\varphi)$ be the set of models of φ . Then, there exists a BA \mathcal{A}_{φ} s.t. $\mathcal{L}(\mathcal{A}_{\varphi}) = \mathcal{L}(\varphi)$
 - it is easy to show that the vice versa does not hold
- We skip the proof, but:
 - of course, we have $\Sigma = 2^{AP}$
 - the size of \mathcal{A}_{φ} , i.e., the number of states, is $2^{O(|\varphi|)}$
 - since we typically verify small properties, this is ok
- There exist tools performing such translation
 - inside SPIN model checker, using option -f



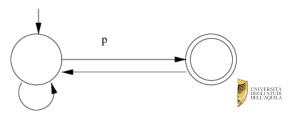


Büchi Automata Examples

Büchi automaton for **FG***p*:



Büchi automaton for **GF**p:







LTL Model Checking: Automata-Theoretic Solution

- Given S, φ decide if $S \models \varphi$
- Consider S as a BA where F = S
- Then, $\mathcal{S} \models \varphi \equiv \mathcal{L}(\mathcal{S}) \subseteq \mathcal{L}(\varphi)$
- Furthermore, $\equiv \mathcal{L}(\mathcal{S}) \cap \mathcal{L}(\neg \varphi) = \varnothing$
- ullet Finally, $\equiv \mathcal{L}(\mathcal{S} imes \mathcal{A}(
 eg arphi)) = arphi$
- The last step is the one which is actually computed
- Complexity is $O((|\mathcal{S}| \cdot |\mathcal{A}(\neg \varphi)|)^2) = O((|\mathcal{S}| \cdot 2^{|\varphi|})^2)$





On-the-Fly LTL Model Checking for $\mathcal{L}(\mathcal{S} imes\mathcal{A}(egarphi))=arphi$

- The graph to be visited is defined as G = (V, E) where:
 - $V = S \times Q$
 - thus, each state is a pair with a state from ${\mathcal S}$ and a state from ${\mathcal A}(\neg\varphi)$
 - $((s,q),(s',q')) \in E$ iff $(s,s') \in R$ and $\exists p \in L(s') : \delta(q,p,q')$ • thus, $\Sigma = AP$
- On such G, we must find acceptance cycles
 - an acceptance state is (s, q) s.t. $q \in F$
 - we have an acceptance cycle if (s, q) is an acceptance state and it is reachable from itself
- If an acceptance cycle is found, we have a counterexample and $\mathcal{S}\not\models\varphi$
- If the visit of G terminates without finding one $\mathcal{S} \models \varphi$





On-the-Fly LTL Model Checking

- No need for S, Q, R, δ to be in RAM from the beginning
 - similar to Murphi: we have a next function directly derived from the input model
 - also $\mathcal{A}(\varphi)$ is described by a suitable language
- Depth-First Visit, easily and efficiently adaptable for finding acceptance cycles
- Namely, *Nested* Depth-First Visit: one for exploring $\mathcal{S} \times \mathcal{A}(\varphi)$, the other to detect cycles
 - the two searches are interleaved
- If an acceptance cycle is found, the DFS stack contains the counterexample





Nested DFS for LTL Model Checking

```
DFS(KS_BA SA, state (s,q), bool n, state a) {
   let \mathcal{SA} = \langle S_A, I_A, R_A, L_A \rangle;
   for each (s', q') \in S_A s.t. ((s, q), (s', q')) \in R_A {
      if (n \wedge (s', q') == a)
         exit reporting error;
      if ((s', q', n) \notin T) {
         T = T \cup \{(s', q', n)\};
         DFS (SA, (s', q'), n, a);
          if (\neg n \land (s', q')) is accepting) {
            DFS (SA, (s', q'), \text{ true}, (s', q'));
1 1 1 1
LTLMC(KS S, LTL \varphi) {
   \mathcal{A} = BA_from_LTL(\varphi); T = \varnothing;
   let S = \langle S, I, R, L \rangle, A = \langle \Sigma, Q, \delta, Q_0, F \rangle;
   for each s \in I, q \in Q_0
      DFS(\mathcal{S} \times \mathcal{A}, (s,q), false, null);
```

