

Software Testing and Validation

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Corso di Laurea in Informatica

Logics in Model Checking

Igor Melatti

Università degli Studi dell'Aquila

Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica



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Beyond Invariants

- Invariants represent a huge share of properties to be verified on a system
- For many systems, one may be happy with invariants only
 - “nothing bad happens”, that’s all folks
- However, it is not always sufficient: a non-running system of course satisfies invariants
 - no starting states, thus no reachable states...



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Safety vs. Liveness

- **Safety** properties: something bad must never happen
 - example: in the Peterson's protocol, it must not happen that both processes are accessing the resource (L3 in the Murphi model)
- Invariants are a special case of safety properties
 - there are some safety properties which are not invariants
 - however, they can be expressed with invariants by adding variables to the Kripke Structure
 - in the following, we will consider "invariants" and "safety properties" as synonyms
- **Liveness** properties: something good will eventually happen
 - example: in the Peterson's protocol, both processes will eventually access the resource
 - not at the same time!
 - cannot be expressed with invariants



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Safety vs. Liveness

- Notation: let \mathcal{S} be a KS and φ be a formula in any logic
- $\mathcal{S} \models \varphi$ is true iff φ is true in \mathcal{S}
 - what this means depends on the logic, as we will see
- For most properties φ , if $\mathcal{S} \not\models \varphi$ then there exists a path $\pi \in \text{Path}(\mathcal{S})$ which is a *counterexample*
 - by overloading the symbol \models , $\pi \not\models \varphi$
- For safety properties, $|\pi| < \infty$
 - \mathcal{S} arrives to an *unsafe* state and that's it
- For liveness properties, $|\pi| = \infty$
 - since \mathcal{S} is finite, this implies that π contains a loop (*lasso*) in its final part



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Safety vs. Liveness

- Equivalent definition for a safety formula: given a finite counterexample, every extension still contains the error
- There is one formula which is both safety and liveness: the true invariant
 - it cannot have a counterexample...
- There are formulas which are neither safety nor liveness
 - their counterexample is not a path
- For typically used formulas, they are either safety or liveness properties



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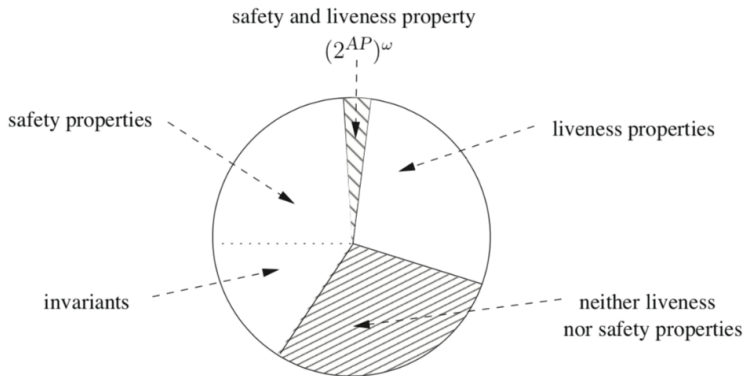
Safety vs. Liveness: Mathematical Definition

- Let a *model* σ be an infinite sequence of truth assignments to all $p \in AP$
 - $\sigma \in (2^{AP})^\omega$
 - could also be seen as a sequence of sets $P \subseteq AP$
 - given a path π of a KS \mathcal{S} , we can always obtain a model from π by replacing each $\pi(i)$ with $L(\pi(i))$
- It is possible to define if $\sigma \models \varphi$, for a given formula φ
- φ is a safety property if, for all σ s.t. $\sigma \not\models \varphi$, there exists j s.t. $\forall \sigma'. \sigma|_j = \sigma'|_j \rightarrow \sigma' \not\models \varphi$
 - i.e., given an (infinite) counterexample σ , there must exist a prefix p of σ s.t. all other models σ' having p as a prefix are again counterexamples
- φ is a liveness property if, for each prefix $w_0 \dots w_i$, there exists σ s.t. $\sigma|_i = w_0 \dots w_i$ and $\sigma \models \varphi$
 - i.e., a (finite) prefix of a model σ cannot be a counterexample, as you may always complete it in a “good” way



Safety vs. Liveness: Mathematical Definition

If we identify a property by the set of its models ($\varphi = \{\sigma \mid \sigma \models \varphi\}$)



Model Checking Logics: Preliminaries

- Model Checking logics are based on the concept of *execution* of a Kripke structure \mathcal{S}
 - thus, on $\pi \in \text{Path}$
- Often, paths are directly viewed as a sequence of atomic propositions, rather than states
 - from $\pi = s_1, s_2, \dots$ to $AP(\pi) = L(s_1), L(s_2), \dots$
- Focusing on executions allows to model *time*
 - time in the sense that we have something coming before of something else (in a path...)
- Trade-off between
 - logics expressiveness: interesting properties can be written
 - logics efficiency: there is an efficient model checking algorithm to compute if $\mathcal{S} \models \varphi$



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Model Checking Logics: Preliminaries

- We will focus on the two leading Model Checking logics: LTL and CTL
 - with some hints on CTL*
 - LTL (Linear-time Temporal Logic) established by Pnueli in 1977
 - CTL (Computation Tree Logic) established by Clarke and Emerson in 1981
 - used for IEEE standards:
 - PSL (Property Specification Language, IEEE Standard 1850)
 - SVA (SystemVerilog Assertions, IEEE Standard 1800).
- We will see syntax and semantics of both logics
 - syntax: how a valid formula is written
 - semantics: what a valid formula “means”
 - that is, when $\mathcal{S} \models \varphi$ holds



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LTL Syntax

$$\Phi ::= p \mid \Phi_1 \wedge \Phi_2 \mid \neg \Phi \mid (\Phi) \mid \mathbf{X}\Phi \mid \Phi_1 \mathbf{U} \Phi_2$$

- Other derived operators:
 - of course true, false, OR and other propositional logic connectors
 - future (or eventually): $\mathbf{F}\Phi = \text{true} \mathbf{U} \Phi$
 - globally: $\mathbf{G}\Phi = \neg(\text{true} \mathbf{U} \neg\Phi) = \neg\mathbf{F}\neg\Phi$
 - release: $\Phi_1 \mathbf{R} \Phi_2 = \neg(\neg\Phi_1 \mathbf{U} \neg\Phi_2)$
 - weak until: $\Phi_1 \mathbf{W} \Phi_2 = (\Phi_1 \mathbf{U} \Phi_2) \vee \mathbf{G}\Phi_1$
- Other notations:
 - next: $\mathbf{X}\Phi = \bigcirc\Phi$
 - $\mathbf{G}\Phi = \square\Phi$
 - $\mathbf{F}\Phi = \diamond\Phi$
- We are dropping *past operators*, thus this is *pure future LTL*



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LTL Semantics

- Goal: formally defining when $\mathcal{S} \models \varphi$, being \mathcal{S} a KS and φ an LTL formula
 - we say that \mathcal{S} *satisfies* φ , or φ *holds in* \mathcal{S}
- This is true when, for all paths π of \mathcal{S} , π satisfies φ
 - i.e., $\forall \pi \in \text{Path}(\mathcal{S}). \pi \models \varphi$
 - symbol \models is overloaded...
- For a given π , $\pi \models \varphi$ iff $\pi, 0 \models \varphi$
- Finally, to define when $\pi, i \models \varphi$, a recursive definition over the recursive syntax of LTL is provided
 - $\pi \in \text{Path}(\mathcal{S}), i \in \mathbb{N}$



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LTL Semantics for $\pi, i \models \varphi$

- $\pi, i \models p$ iff $p \in L(\pi(i))$
- $\pi, i \models \Phi_1 \wedge \Phi_2$ iff $\pi, i \models \Phi_1 \wedge \pi, i \models \Phi_2$
- $\pi, i \models \neg\Phi$ iff $\pi, i \not\models \Phi$
- $\pi, i \models \mathbf{X}\Phi$ iff $\pi, i+1 \models \Phi$
- $\pi, i \models \Phi_1 \mathbf{U} \Phi_2$ iff $\exists k \geq i: \pi, k \models \Phi_2 \wedge \forall i \leq j < k. \pi, j \models \Phi_1$



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LTL Semantics for Added Operators

- It is easy to prove that:
 - $\forall \pi \in \text{Path}(\mathcal{S}), i \in \mathbb{N}. \pi, i \models \text{true}$
 - $\pi, i \models \mathbf{G}\Phi$ iff $\forall j \geq i. \pi, j \models \Phi$
 - $\pi, i \models \mathbf{F}\Phi$ iff $\exists j \geq i. \pi, j \models \Phi$
 - $\pi, i \models \Phi_1 \mathbf{R} \Phi_2$ iff $\forall k \geq i. \pi, k \models \Phi_2 \vee \exists i \leq j < k : \pi, j \models \Phi_1$
 - i.e., $\forall k \geq i. \pi, k \not\models \Phi_2 \rightarrow \exists i \leq j < k : \pi, j \models \Phi_1$
 - i.e., $\forall k \geq i. \forall i \leq j < k. \pi, j \not\models \Phi_1 \rightarrow \pi, k \models \Phi_2$
 - $\pi, i \models \Phi_1 \mathbf{W} \Phi_2$ iff $(\forall j \geq i. \pi, j \models \Phi_1) \vee (\exists k \geq i : \pi, k \models \Phi_2 \wedge \forall i \leq j < k. \pi, j \models \Phi_1)$
- For many formulas, it is silently required that paths are infinite
- That's why transition relations in KSs must be total



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LTL Semantics: Typical Paths for Common Formulas

- For $p \in AP$, we will also consider p to be any set in $\{P \in 2^{AP} \mid p \in P\}$
 - that is, p is any subset of atomic propositions containing p
 - e.g., p may be any of $\{p\}, \{p, q\}, \{p, r, s\} \dots$
 - furthermore, $\bar{p} = \neg p \in \{P \in 2^{AP} \mid p \notin P\}$
 - e.g., \bar{p} may be any of $\{q\}, \{q, r\}, \{r, s\} \dots$
 - finally, \perp denotes any subset of atomic propositions
- If $\pi \models \mathbf{G}p$, then $\pi = p^\omega$
 - of course, this includes, e.g., $\pi = \{p, q\}\{p, r\}\{p\}\{p, q\}\{p\} \dots$
 - $\pi, 3 \models \mathbf{G}p$: $\pi = \perp \perp \perp p^\omega$
- If $\pi \models \mathbf{F}p$, then $\pi = \perp^* p \perp^\omega$
- If $\pi \models p \mathbf{U} q$, then $\pi = \{p, \bar{q}\}^* q \perp^\omega$
- If $\pi \models p \mathbf{W} q$, then either $\pi = \{p, \bar{q}\}^* q \perp^\omega$ or $\pi = p^\omega$
- If $\pi \models p \mathbf{R} q$, then either $\pi = \{\bar{p}, q\}^\omega$ or $\pi = \{\bar{p}, q\}^* \{p, q\} \perp^\omega$
 - q must be kept holding till when a p appears and releases $q \dots$



Safety and Liveness Properties in LTL

- Given an LTL formula φ , φ is a safety formula iff
$$\forall \mathcal{S}. (\exists \pi \in \text{Path}(\mathcal{S}) : \pi \not\models \varphi) \rightarrow \exists k : \pi|_k \not\models \varphi$$
- Given an LTL formula φ , φ is a liveness formula iff
$$\forall \mathcal{S}. (\exists \pi \in \text{Path}(\mathcal{S}) : \pi \not\models \varphi) \rightarrow |\pi| = \infty$$
- All LTL formulas are either safety, liveness, or the AND of a safety and a liveness
 - being defined on paths, the counterexample is always a path
- Safety properties are those involving only **G**, **X**, true and atomic propositions
- Liveness are all those involving an **F** or a **U**
 - but beware of negations...
- Some formulas are both safety and liveness, like true, **G** true and so on

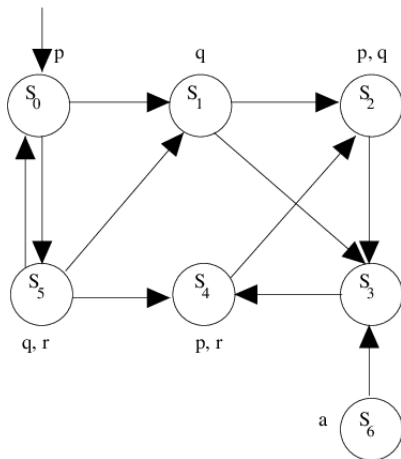


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LTL Examples



$\mathcal{S} \models \mathbf{F}p$ since p holds in the first state

For full: let $\pi \in \text{Path}(\mathcal{S})$

$\pi, 0 \models \mathbf{F}p$ with $j = 0$

recall: $\pi, i \models \mathbf{F}\Phi$ iff

$\exists j \geq i. \pi, j \models \Phi$

$\pi, i \models p$ iff $p \in L(\pi(i))$

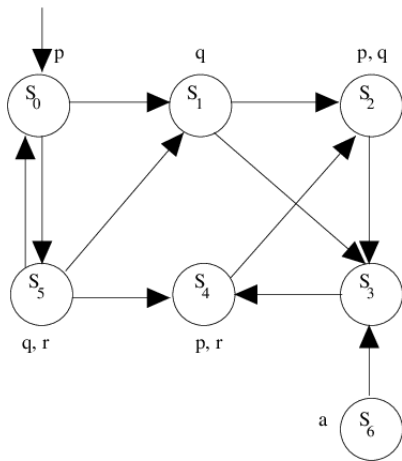


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LTl Examples



$\mathcal{S} \not\models \mathbf{F}a$ since s_6 is not reachable from s_0

counterexample: $\pi = s_0 s_5 s_0 s_5 \dots$

For full: $\pi, 0 \not\models \mathbf{F}a$ as, for all $j \geq 0$, $a \notin L(\pi(j))$

Counterexample is infinite, thus this is a liveness property
Any finite prefix of π is not a counterexample

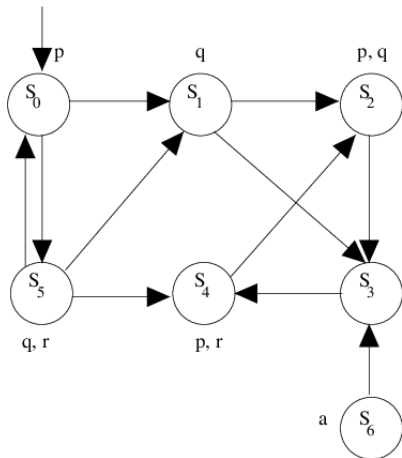


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LTL Examples



$\mathcal{S} \not\models \mathbf{G}p$ since there are many counterexamples, here is one:

$\pi = s_0 s_5 s_0 s_5 \dots$

For full: $\pi, 0 \not\models \mathbf{G}p$ with $j = 1$

recall: $\pi, i \models \mathbf{G}\Phi$ iff

$\forall j \geq i. \pi, j \models \Phi$

$\pi, i \models p$ iff $p \in L(\pi(i))$

Safety property, actually $\pi|_2$ is enough

Every path having $\pi|_2$ as a prefix is a counterexample

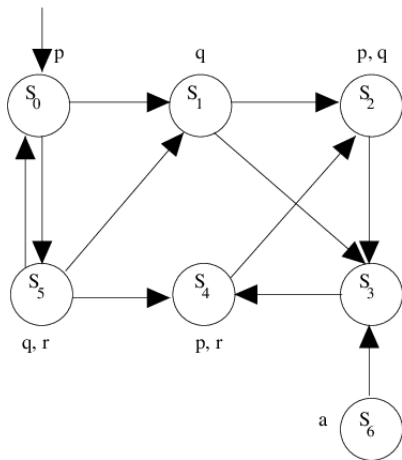


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LTL Examples



$\mathcal{S} \models \mathbf{G}\neg a$ since s_6 is not reachable from s_0

For full: let $\pi \in \text{Path}(\mathcal{S})$
 $\pi, 0 \models \mathbf{G}\neg a$ as the only state s with $a \in L(s)$ is s_6 , which is not reachable from s_0

recall: $\pi \in \text{Path}(\mathcal{S})$ implies $\pi(0) \in I$, thus $\pi(0) = s_0$ here

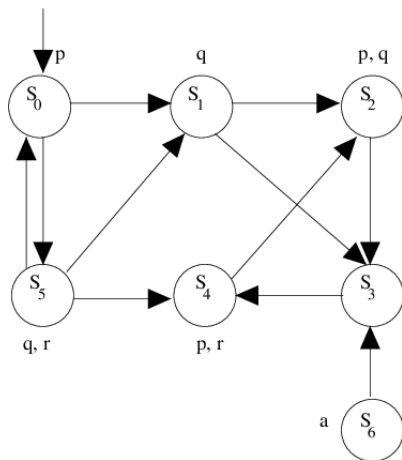


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LTL Examples



$\mathcal{S} \models p \text{ U } q$ since $p \in L(s_0)$,
 $\text{next}(s_0) = \{s_1, s_5\}$ and $q \in L(s_1) \wedge q \in L(s_5)$

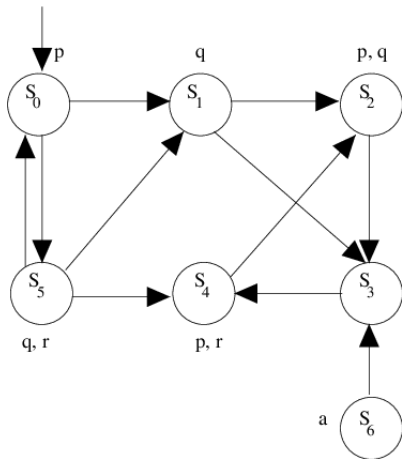


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LTL Examples



$\mathcal{S} \not\models p \mathbf{U} r$, a counterexample is $\pi = s_0 s_1 (s_2 s_3 s_4)$

Again this is a liveness formula, even if $\pi|_1$ would have been enough

In fact, you have to rule out $\{p, \bar{r}\}^\omega \dots$

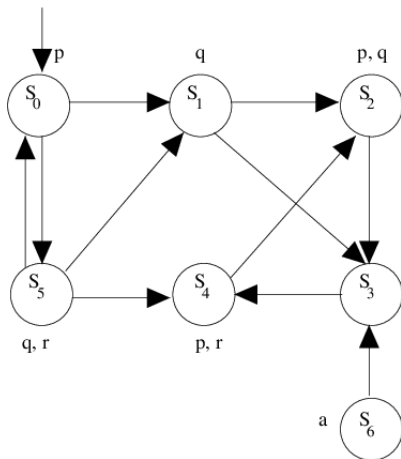


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LTl Examples



$\mathcal{S} \not\models \neg(p \mathbf{U} r)$, a counterexample is $\pi = (s_0 s_5)$

In fact, $(s_0 s_5), 0 \models p \mathbf{U} r$

Thus it may happen that $\mathcal{S} \not\models \Phi$ and $\mathcal{S} \not\models \neg(\Phi)$

Instead, it is impossible that $\mathcal{S} \models \Phi$ and $\mathcal{S} \models \neg(\Phi)$

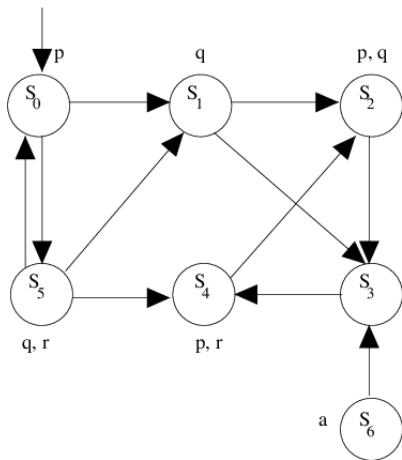


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LTL Examples



$\mathcal{S} \not\models q$, since s_0 is the only initial state and $q \notin L(s_0)$ (all paths in $\text{Path}(\mathcal{S})$ must start from s_0)

$\mathcal{S} \models p$, since $p \in L(s_0)$

$\mathcal{S} \models \mathbf{X}q$, since $q \in L(s_1) \wedge q \in L(s_5)$

$\mathcal{S} \not\models \mathbf{XX}q$, since all states but s_5, s_6 are reachable in exactly 2 steps

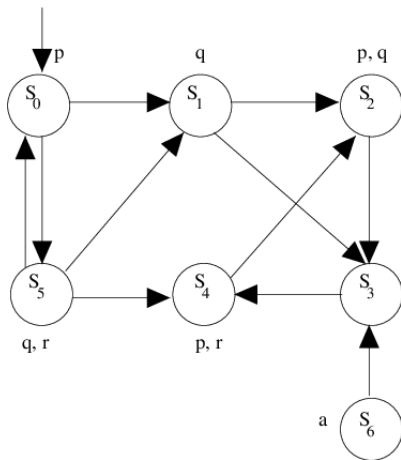


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LTL Examples



$\mathcal{S} \not\models \mathbf{FG}p$, a counterexample is
 $\pi = s_0 s_1 (s_2 s_3 s_4)$
Again this is a liveness formula

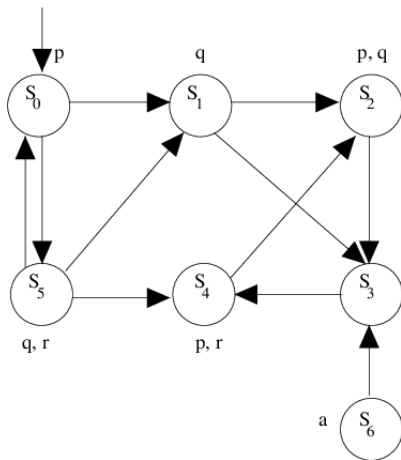


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LTL Examples



$\mathcal{S} \models \mathbf{GF}p$

All lassos are s_0s_5 or $s_2s_3s_4$

In both such lassos, there are states in which p holds

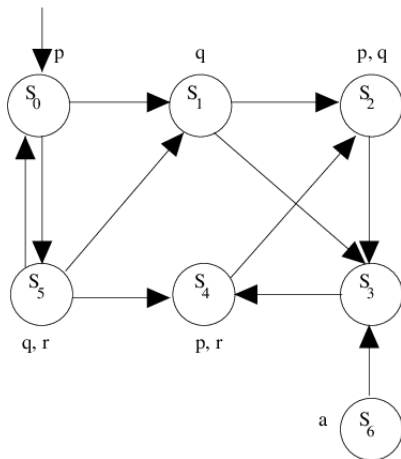


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LTl Examples



$\mathcal{S} \models \mathbf{GF}p \vee \mathbf{FG}p$

Consequence of the two previous slides

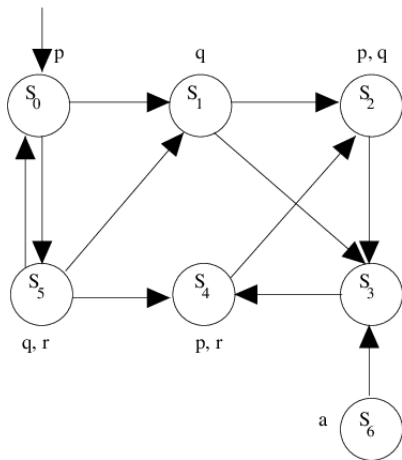


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LTl Examples



$\mathcal{S} \not\models \mathbf{G}(p \mathbf{U} q)$, a counterexample is $\pi = s_0 s_1 (s_2 s_3 s_4)$
 $(p \mathbf{U} q)$ must hold at any reachable state
Ok in s_0, s_1, s_2 , but not in s_3



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LTL Non-Toy Examples

- Recall the Peterson's protocol: checking mutual exclusion is $\mathbf{G}(\neg(p \wedge q))$, being $p = P[1] = L3$, $q = P[2] = L3$
 - all invariants are of the form $\mathbf{G}P$, where P does not contain modal operators \mathbf{X} , \mathbf{U} or \mathbf{F}
- Checking that both processes access to the critical section *infinitely often* is $\mathbf{GF} P[1] = L3 \wedge \mathbf{GF} P[2] = L3$
 - liveness property: no process is infinitely banned to access the critical section
- Even better: $\mathbf{G} (P[1] = L2 \rightarrow \mathbf{F} P[1] = L3)$
 - the same for the other process
 - since it is symmetric, this is actually enough



Equivalence Between LTL Properties

- Definition of equivalence between LTL properties:
 $\varphi_1 \equiv \varphi_2 \text{ iff } \forall \mathcal{S}. \mathcal{S} \models \varphi_1 \Leftrightarrow \mathcal{S} \models \varphi_2$
 - equivalent: $\forall \sigma \dots$
- Idempotency:
 - $\mathbf{FF}p \equiv \mathbf{F}p$
 - $\mathbf{GG}p \equiv \mathbf{G}p$
 - $p \mathbf{U} (p \mathbf{U} q) \equiv (p \mathbf{U} q) \mathbf{U} q \equiv p \mathbf{U} q$
- Absorption:
 - $\mathbf{GFG}p \equiv \mathbf{FG}p$
 - $\mathbf{FGF}p \equiv \mathbf{GF}p$
- Expansion (used by LTL Model Checking algorithms!):
 - $p \mathbf{U} q \equiv q \vee (p \wedge \mathbf{X}(p \mathbf{U} q))$
 - $\mathbf{F}p \equiv p \vee \mathbf{XF}p$
 - $\mathbf{G}p \equiv p \wedge \mathbf{XG}p$



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CTL Syntax

$\Phi ::= p \mid \Phi_1 \wedge \Phi_2 \mid \neg \Phi \mid (\Phi) \mid \mathbf{EX}\Phi \mid \mathbf{EG}\Phi \mid \mathbf{E}\Phi_1 \mathbf{U} \Phi_2$

- Other derived operators (besides true, false, OR, etc):
 - $\mathbf{EF}\Phi = \mathbf{Etrue} \mathbf{U} \Phi$
 - cannot be defined using $\mathbf{E}\neg\mathbf{G}\neg\Phi$, as this is not a CTL formula
 - actually, it is a CTL* formula (see later)
 - in fact, you cannot place a negation between \mathbf{E} and the subformula
 - $\mathbf{AF}\Phi = \neg\mathbf{EG}\neg\Phi$, $\mathbf{AG}\Phi = \neg\mathbf{EF}\neg\Phi$, $\mathbf{AX}\Phi = \neg\mathbf{EX}\neg\Phi$
 - $\mathbf{A}\Phi_1 \mathbf{U} \Phi_2 = (\neg\mathbf{E}\neg\Phi_2 \mathbf{U} (\neg\Phi_1 \wedge \neg\Phi_1)) \wedge \neg\mathbf{EG}\neg\Phi_2$
 - $\Phi_1 \mathbf{AU}\Phi_2 = \mathbf{A}\Phi_1 \mathbf{U}\Phi_2$, $\Phi_1 \mathbf{EU}\Phi_2 = \mathbf{E}\Phi_1 \mathbf{U}\Phi_2$



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Comparison with LTL Syntax

$$\Phi ::= \text{true} \mid p \mid \Phi_1 \wedge \Phi_2 \mid \neg \Phi \mid (\Phi) \mid \mathbf{X}\Phi \mid \Phi_1 \mathbf{U} \Phi_2$$

- Essentially, all temporal operators are preceded by either **E** or **A**
 - with some care for **U**



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CTL Semantics

- Goal: formally defining when $\mathcal{S} \models \varphi$, being \mathcal{S} a KS and φ a CTL formula
- This is true when, for all initial states $s \in I$ of \mathcal{S} , $s \models \varphi$
 - thus, CTL is made of *state* formulas
 - LTL has *path* formulas
- To define when $s \models \varphi$, a recursive definition over the recursive syntax of CTL is provided
 - no need of an additional integer as for LTL syntax



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CTL Semantics for $s \models \varphi$

- $\forall s \in S. s \models \text{true}$
- $s \models p$ iff $p \in L(s)$
- $s \models \Phi_1 \wedge \Phi_2$ iff $s \models \Phi_1 \wedge s \models \Phi_2$
- $s \models \neg\Phi$ iff $s \not\models \Phi$
- $s \models \mathbf{EX}\Phi$ iff $\exists \pi \in \text{Path}(S, s). \pi(1) \models \Phi$
- $s \models \mathbf{EG}\Phi$ iff $\exists \pi \in \text{Path}(S, s). \forall j. \pi(j) \models \Phi$
- $s \models \mathbf{E}\Phi_1 \mathbf{U} \Phi_2$ iff
 $\exists \pi \in \text{Path}(S, s) \exists k : \pi(k) \models \Phi_2 \wedge \forall j < k. \pi(j) \models \Phi_1$



CTL Semantics for Added Operators

- It is easy to prove that:
 - $s \models \mathbf{AG}\Phi$ iff $\forall \pi \in \text{Path}(\mathcal{S}, s). \forall j. \pi(j) \models \Phi$
 - $s \models \mathbf{AF}\Phi$ iff $\forall \pi \in \text{Path}(\mathcal{S}, s). \exists j. \pi(j) \models \Phi$
 - analogously for **AU**, **AR**, **AW**
 - just replace \forall with \exists for **EF**, **ER**, **EW**
- Analogously to LTL, for many CTL formulas it is silently required that paths are infinite
- So again transition relations in KSs must be total



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Safety and Liveness Properties in CTL

- Some CTL formulas may be neither safety nor liveness
 - being defined on states, the counterexample may be an entire computation tree
- Safety properties are those involving only **AG**, **AX**, true and atomic propositions
- Some formulas are both safety and liveness, like true, **AG** true and so on
- Liveness are formulas like **AF**, **AFAG**, **AU**
- **EF** or **EG** are neither liveness nor safety

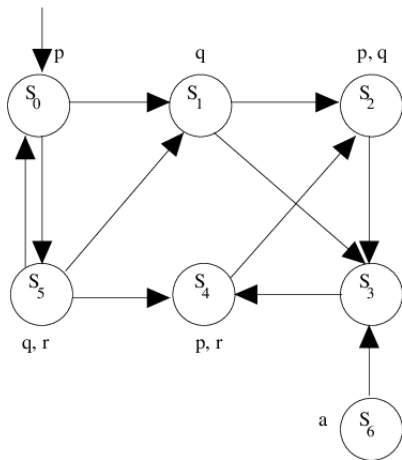


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CTL Examples



$\mathcal{S} \models \mathbf{AF}p$ since p holds in the first state

For full: $s_0 \models \mathbf{F}p$ since $p \in L(s_0)$, thus, for all paths starting in s_0 , p holds in the first state, so it holds eventually

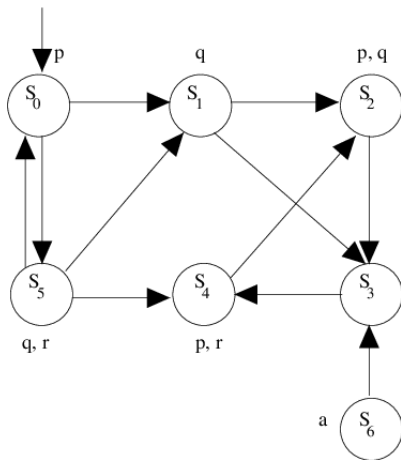


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CTL Examples



$\mathcal{S} \models \mathbf{EF}p$ for the same reason as above

If it holds for all paths, then it holds for one path

$\mathbf{AF}\phi \rightarrow \mathbf{EF}\phi$

The same holds for the other temporal operators **G**, **U** etc

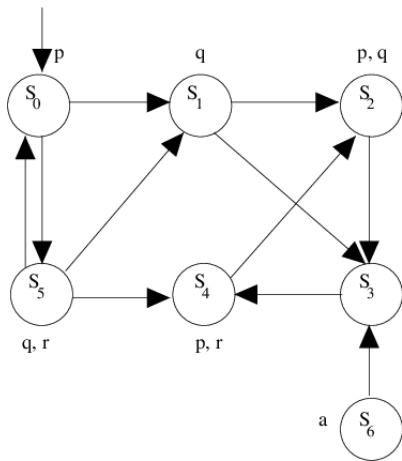


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CTL Examples



$\mathcal{S} \not\models \mathbf{EF}a$ since s_6 is not reachable

Note that the counterexample cannot be a single path

Since it would not enough to disprove existence

The full reachable graph must be provided

One could also show the tree of all paths

Neither safety nor liveness

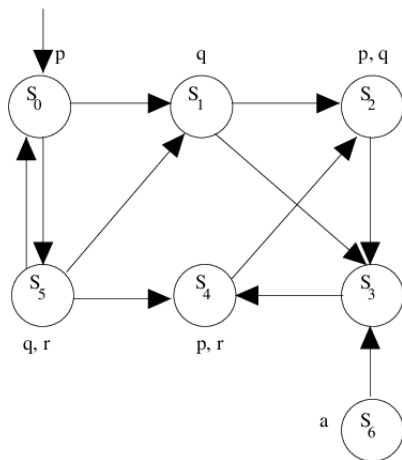


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CTL Examples



$\mathcal{S} \models \mathbf{A}(p \mathbf{U} q)$ since $p \in L(s_0)$,
 $\text{next}(s_0) = \{s_1, s_5\}$ and $q \in L(s_1) \wedge q \in L(s_5)$

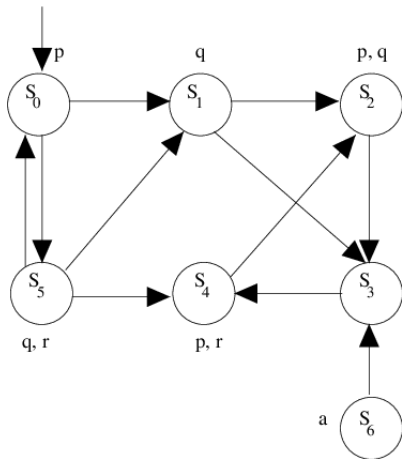


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CTL Examples



$\mathcal{S} \not\models \mathbf{A}(p \mathbf{U} r)$, a counterexample is $\pi = s_0 s_1 (s_2 s_3 s_4)$

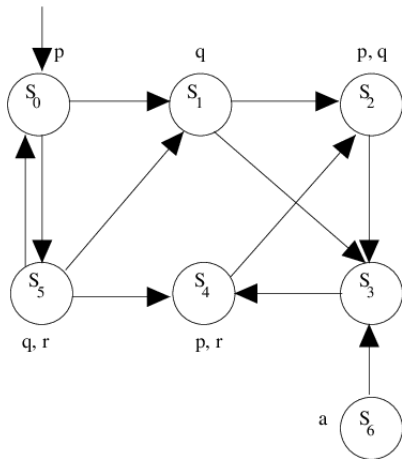


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CTL Examples



$\mathcal{S} \models \mathbf{E}(p \mathbf{U} r)$, an example is
 $\pi = (s_0 s_5)$

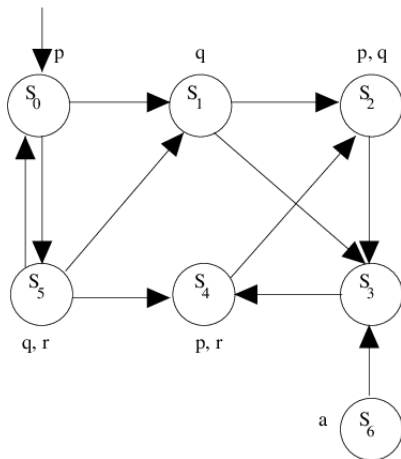


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CTL Examples



$\mathcal{S} \not\models \neg \mathbf{E}(p \mathbf{U} r)$, a counterexample is $\pi = (s_0 s_5)$

In fact, $\mathcal{S} \not\models \Phi$ iff $\mathcal{S} \models \neg(\Phi)$ whenever $|I| = 1$

In fact, the implicit for all is on initial states only, whilst it is on all paths for LTL...

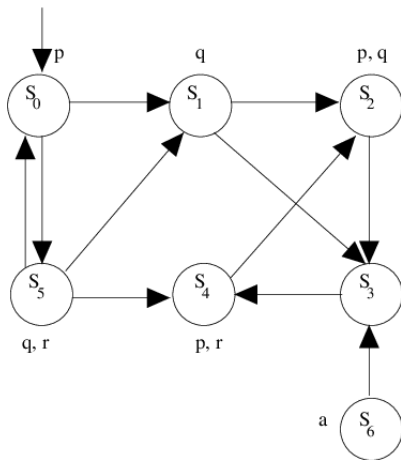


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CTL Examples



$\mathcal{S} \not\models \mathbf{AFAG}p$, a counterexample is $\pi = s_0s_1(s_2s_3s_4)$
This is a liveness formula

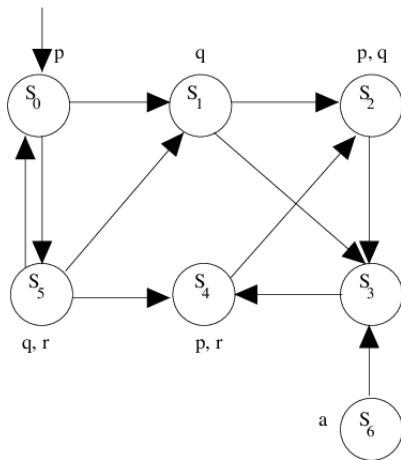


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CTL Examples



$\mathcal{S} \not\models \mathbf{EFEG}p$, a counterexample is again a computation tree

All lassos are s_0s_5 or $s_2s_3s_4$

In both such lassos, there are states in which p does not hold

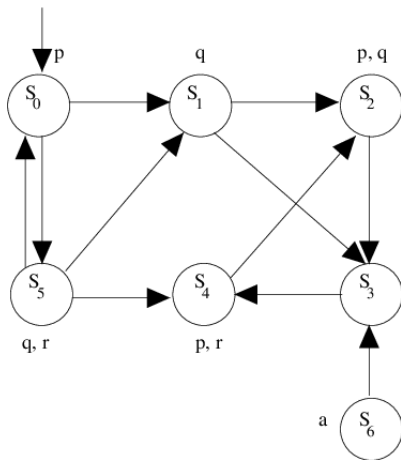


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CTL Examples



$\mathcal{S} \not\models \mathbf{AFEG}p$, a counterexample is again a computation tree
Since $\mathcal{S} \not\models \mathbf{EFEG}p \dots$

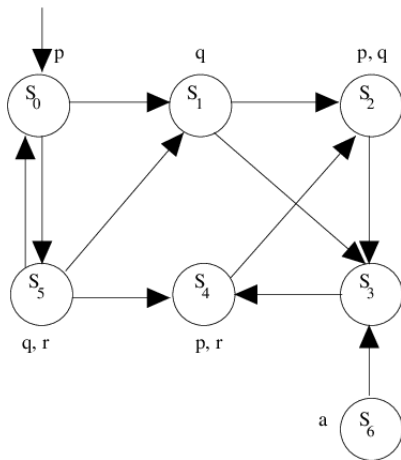


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CTL Examples



$\mathcal{S} \not\models \mathbf{EFAG}p$, a counterexample is again a computation tree
 Since $\mathcal{S} \not\models \mathbf{EFEG}p...$



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CTL Non-Toy Examples

- Recall the Peterson's protocol: checking mutual exclusion is **AG**($\neg(p \wedge q)$), being $p = P[1] = L3, q = P[2] = L3$
 - equivalent to LTL **G** p
- It is always possible to restart:
AGEF $P[1] = L0 \wedge \mathbf{AGEF} P[2] = L0$



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CTL vs. LTL: a Comparison

- Recall that $\varphi_1 \equiv \varphi_2$ iff $\forall \mathcal{S}. \mathcal{S} \models \varphi_1 \Leftrightarrow \mathcal{S} \models \varphi_2$
 - also holds (w.l.g.) when φ_1 is LTL and φ_2 is CTL
- Of course, some CTL formulas cannot be expressed in LTL
 - it is enough to put an **E**, since LTL always universally quantifies paths
 - so, there is not an LTL φ s.t. $\varphi \equiv \mathbf{EG}p$
 - no, $\mathbf{F}\neg p$ is not the same, why?
- So, one might think: LTL is contained in CTL
 - in the sense, for each LTL formula, there is a CTL equivalent formula
 - simply replace each temporal operator **O** with **AO**, that's it
 - let \mathcal{T} be a translator doing this
 - for any LTL formula φ , $\varphi \equiv \mathcal{T}(\varphi)$
 - actually, $\mathbf{G}p \equiv \mathcal{T}(\mathbf{G}p) = \mathbf{AG}p$



CTL vs. LTL: a Comparison

- Theorem. Let φ be an LTL formula. Then, either i) $\varphi \equiv \mathcal{T}(\varphi)$ or ii) there does not exist a CTL formula ψ s.t. $\varphi \equiv \psi$
 - idea of proof: replacing with **E** is of course not correct, and temporal operators on paths are the same
- Corollary. There exists an LTL formula φ s.t., for all CTL formulas ψ , $\varphi \not\equiv \psi$
- Proof of corollary:
 - by the theorem above and the definitions, we need to find
 - 1 an LTL formula φ
 - 2 a KS \mathcal{S}
 - where $\mathcal{S} \models \varphi$ and $\mathcal{S} \not\models \mathcal{T}(\varphi)$
 - viceversa is not possible



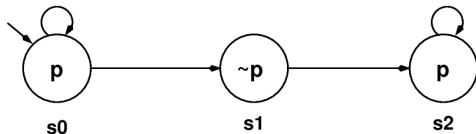
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CTL vs. LTL: a Comparison

- For example, as for the LTL formula, we may take $\varphi = \mathbf{FG}p$
 - note instead that $\mathbf{GF}p \equiv \mathbf{AGAF}p$
- For example, as for the KS \mathcal{S} , we may take



- We have that $\mathcal{S} \models \mathbf{FG}p$, but $\mathcal{S} \not\models \mathbf{AFAG}p$
- Thus, CTL requires “more” than the corresponding LTL

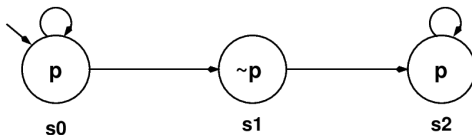


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CTL vs. LTL: a Comparison



- $\mathcal{S} \not\models \mathbf{AFAG}p$ means that
$$\neg(\forall \pi \in \text{Path}(\mathcal{S}). \exists j : \forall \rho \in \text{Path}(\mathcal{S}, \pi(j)). \forall k. p \in \rho(k))$$
$$= \exists \pi \in \text{Path}(\mathcal{S}). \forall j : \exists \rho \in \text{Path}(\mathcal{S}, \pi(j)). \exists k. p \notin \rho(k)$$
- In our \mathcal{S} , $\pi = s_0^\omega$: in fact, at any point of π , you may branch and go through $\neg p$ instead...
- $\mathcal{S} \models \mathbf{FG}p$ means that $\forall \pi \in \text{Path}(\mathcal{S}). \exists j : \forall k \geq j. p \in \pi(k)$
- Thus, there is not a CTL formula equivalent to $\mathbf{FG}p$
- Furthermore, there is not an LTL formula equivalent to $\mathbf{AFAG}p$

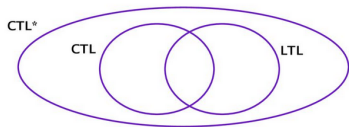


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CTL, LTL and CTL*



- CTL* introduced in 1986 (Emerson, Halpern) to include both CTL and LTL
- No restrictions on path quantifiers to be 1-1 with temporal operators, as in CTL
- State formulas: $\Phi ::= \text{true} \mid p \mid \Phi_1 \wedge \Phi_2 \mid \neg \Phi \mid \mathbf{A}\Psi \mid \mathbf{E}\Psi$
- Path formulas: $\Psi ::= \Phi \mid \Psi_1 \wedge \Psi_2 \mid \neg \Psi \mid \Psi_1 \mathbf{U} \Psi_2 \mid \mathbf{F}\Psi \mid \mathbf{G}\Psi$

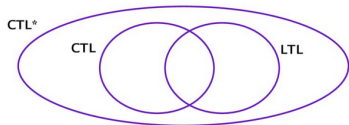


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CTL, LTL and CTL*



- The intersection between CTL and LTL is both syntactic and “semantic”
- Some formulas are both CTL and LTL in syntax: all those involving only boolean combinations of atomic propositions
- “Semantic” intersection: some LTL formulas may be expressed in CTL and vice versa, using different syntax
 - **AGAF** p and **GF** p
 - **AG** p and **G** p
 - etc



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