A Celestial Description of Planar Yang-Mills Theory

Igor Mol

Federal University of Juiz de Fora – Departament of Mathematics and

Federal University of Minas Gerais – Instituto de Ciências Exatas (ICEx)[*]

Continuing our program of deriving aspects of celestial holography from string theory, we extend the Roiban-Spradlin-Volovich-Witten (RSVW) formalism to celestial amplitudes. We reformulate the tree-level maximally-helicity-violating (MHV) celestial leaf amplitudes for gluons in $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory and for gravitons in $\mathcal{N}=8$ Supergravity in terms of *minitwistor wavefunctions*. These are defined as representatives of cohomology classes on the minitwistor space $\mathbf{MT}$, associated to the three-dimensional Euclidean anti-de Sitter space. In this framework, celestial leaf amplitudes are expressed as integrals over the moduli space of minitwistor lines. We construct a minitwistor generating functional for MHV leaf amplitudes using the Quillen determinant line bundle, extending the approach originally developed by Boels, Mason and Skinner. Building on this formalism, we propose supersymmetric celestial conformal field theories (CFTs) as σ-models, where the worldsheet is given by the celestial supersphere $\mathbf{CP}^{1|2}$, and the target space is the minitwistor superspace $\mathbf{MT}^{2|\mathcal{N}}$. We demonstrate that the semiclassical effective action of these σ-models reproduces the MHV gluonic and gravitational leaf amplitudes in $\mathcal{N}=4$ SYM theory and $\mathcal{N}=8$ Supergravity. This construction provides a concrete realisation of the supersymmetric celestial CFT framework recently introduced by Tropper (2024). Adamo and Groechenig [1]

[*] igormol@ime.unicamp.br

CONTENTS

I. INTRODUCTION

An open problem in high-energy theoretical physics is the dynamical derivation of the celestial holography dictionary, which posits that the physics of asymptotically flat spacetimes can be encoded in a conjectured celestial conformal field theory (CCFT) living on the celestial sphere.

The simplicity of gauge-theory scattering amplitudes for gluons in maximally-helicity-violating (MHV) configurations, described by the Parke-Taylor formula, led Nair to reinterpret these amplitudes as correlation functions of a Wess-Zumino-Novikov-Witten (WZNW) current algebra on the Riemann sphere, $\mathbf{CP}^1$. This viewpoint motivated the development of twistor and ambitwistor string theories and, later, the Cachazo-He-Yuan (CHY) formalism. Here, we ask whether a similar, bottom-up approach can yield concrete dynamical models for CCFTs. Such models may describe realistic physical theories or, at least, specific sectors thereof.

II. MINITWISTOR SUPERWAVEFUNCTIONS

A. Review: Twistor Superwavefunctions

The $\mathcal{N}=4$ supersymmetric twistor wavefunction for gluons, labelled by the scaling weight w, defines a Dolbeault cohomology class:

$$[f^w] \in H^{0,1}\big(\mathbf{PT}^{3|4}, \mathcal{O}\,(-w)\,\big).$$

Introduce homogeneous coordinates:

$$\mathcal{Z}^I := \big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big)$$

on projective twistor superspace $\mathbf{PT}^{3|4} \subset \mathbf{CP}^{3|4}$. The dual (momentum-twistor) coordinates are:

$$\mathcal{W}^I := \big(\nu^A, \bar{\nu}_{\dot{A}}, \eta^\alpha\big).$$

We employ abstract index notation with $I, I', ... \in \{A, \dot{A}, \alpha\}$.

The twistor superwavefunction is explicitly given by:

$$f^w\big(\mathcal{Z}^I; \mathcal{W}^{I'}\big) := \int_{\mathbf{C}^*} \frac{dt}{t}\, t^w \;\bar{\delta}^2\big(\nu^A - t\lambda^A\big)\exp\big(it\,([\mu\bar{\nu}] + \psi\cdot\eta)\big)\,. \tag{1}$$

Here the spinorial delta function is defined by:

$$\bar{\delta}^2\left(\sigma^A\right) := \frac{1}{(2\pi i)^2} \bigwedge_{A\in\{1,2\}} \bar{\partial}\;\frac{1}{\sigma^A}\;\;, \tag{2}$$

where $[\sigma^A]$ denote homogeneous coordinates on the projective line $\mathbf{CP}^1$.

To obtain a more explicit representation, introduce an auxiliary spinor ι^A, chosen arbitrarily but non-vanishing. Using the Green's function for the Dolbeault operator $\overline{\partial}$ on $\mathbf{CP}^1$, one shows:

$$\overline{\delta}^2\left(z^A - t\lambda^A\right) = \frac{1}{(2\pi i)^2} \bigwedge_{A\in\{1,2\}} \overline{\partial}\, \frac{1}{z^A - t\lambda^A} = \overline{\delta}\left(t - \frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\right) \overline{\delta}\left(\langle\lambda z\rangle\right). \tag{3}$$

Substituting this into Eq. (1) gives the explicit gluonic superwavefunction:

$$f^p\left(\mathcal{Z}^I;\mathcal{W}^{I'}\right) = \overline{\delta}\left(\langle\lambda z\rangle\right)\left(\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\right)^{p-1} \exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\;(\,s[\mu\bar{z}] + \psi\cdot\eta\,)\right). \tag{4}$$

Next, define the *projective delta function* $\overline{\delta}_\Delta$ on $\mathbf{CP}^1$ with conformal weight Δ:

$$\overline{\delta}_\Delta\left(z^A, \lambda^A\right) := \frac{1}{(2\pi i)^2}\int_{\mathbf{C}^*} \frac{dt}{t}\; t^\Delta \bigwedge_{A\in\{1,2\}} \overline{\partial}\, \frac{1}{z^A - t\lambda^A} = \overline{\delta}_\Delta\left(\langle\lambda z\rangle\right)\left(\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\right)^{\Delta-1}. \tag{5}$$

Using this identity, Eq. (4) simplifies to:

$$f^p\left(\mathcal{Z}^I;\mathcal{W}^{I'}\right) = \overline{\delta}_p\left(z,\lambda\right)\,\exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\;(s[\mu\bar{z}] + \psi\cdot\eta)\right). \tag{6}$$

B. Minitwistor Superwavefunctions

To construct the celestial superwavefunction for gluons in the minitwistor-superspace formalism, we introduce a normalised spinor basis:

$$z^A := (1, -\zeta)\,,\;\; \bar{z}_{\dot{A}} := (1, -\bar{\zeta}), \tag{7}$$

where ζ and $\bar{\zeta}$ are holomorphic and antiholomorphic coordinates on the celestial sphere $\mathcal{CS} \simeq \mathbf{CP}^1$.

In the gluonic superwavefunction (Eq. (6)), we make the substitutions:

$$\nu^A \mapsto z^A,\;\; \bar{\nu}_{\dot{A}} \mapsto s\cdot\bar{z}_{\dot{A}},$$

where s is a real, nonnegative parameter encoding the gluon's frequency.

From the viewpoint of celestial CFT, a gluon state is entirely specified by three data:

1. The celestial conformal weight Δ.

2. The helicity, encoded by the Grassmann variables η^α, with $\alpha = 1, ..., 4$ labelling the supersymmetry generators.

3. The insertion point on the celestial sphere $\mathcal{CS}$, given by the normalised spinor basis $\{z^A, \bar{z}_{\dot{A}}\}$.

The *minitwistor superwavefunction* associated with a gluon of conformal weight Δ and configuration $\{z^A, \bar{z}_{\dot{A}}, \eta^\alpha\}$ is defined by the half-Mellin transform:

$$\Psi^p_\Delta\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z^A, \bar{z}_A, \eta^\alpha\big) \;:=\; \int\limits_{\mathbf{R}_+} \frac{ds}{s}\; s^\Delta\, f^p\big(\lambda^A, \mu_{\dot{A}}, \eta^\alpha; z^A, s\bar{z}_{\dot{A}}, \eta^\alpha\big). \tag{8}$$

Substituting the definition of f^p from Eq. (1) yields the double-integral representation:

$$\Psi^p_\Delta\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big) \;=\; \int\limits_{\mathbf{R}_+} \frac{ds}{s}\, s^\Delta \int\limits_{\mathbf{C}^*} \frac{dt}{t}\, t^p\; \bar{\delta}^2\big(z^A - t\lambda^A\big)\, \exp\big(it\,(s[\mu\bar{z}] + \psi\cdot\eta)\big) \tag{9}$$

Homogeneity. The affine and Mellin integrals make explicit the homogeneous transformation properties of the superwavefunction Ψ^p_Δ. For any $t_1, t_2 \in \mathbf{C}^*$, one has:

$$\Psi^p_\Delta\big(t_1\cdot\lambda^A, t_2\cdot\mu_{\dot{A}}, t_1\cdot\psi^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big) \;=\; t_1^{\Delta-p}\, t_2^{-\Delta}\, \Psi^p_\Delta\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z^a, \bar{z}_{\dot{A}}, \eta^\alpha\big), \tag{10}$$

$$\Psi^p_\Delta\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; t_1\cdot z^A, t_2\cdot\bar{z}_{\dot{A}}, t_1^{-1}\cdot\eta^\alpha\big) \;=\; t_1^{p-\Delta-2}\, t_2^{-\Delta}\, \Psi^p_\Delta\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big). \tag{11}$$

These homogeneity laws suggest that $z^A, \bar{z}_{\dot{A}}, \eta^\alpha$ may be regarded as dual to $\lambda^A, \mu_{\dot{A}}, \psi^\alpha$. This interpretation will be justified when we introduce the minitwistor transform.

In Eq. (10), the Grassmann-odd coordinate ψ^α scales like λ^A. Hence $\lambda^A, \mu_{\dot{A}}, \psi^\alpha$ transform as:

$$\lambda^A \;\mapsto\; t_1\cdot\lambda^A,\;\; \mu_{\dot{A}} \;\mapsto\; t_2\cdot\mu_{\dot{A}},\;\; \psi^\alpha \;\mapsto\; t_1\cdot\psi^\alpha. \tag{12}$$

By contrast, Eq. (11) shows that η^α transforms inversely to z^A. Thus $z^A, \bar{z}_{\dot{A}}, \eta^\alpha$ obey:

$$z^A \;\mapsto\; t_1\cdot z^A,\;\; \bar{z}_{\dot{A}} \;\mapsto\; t_2\cdot\bar{z}_{\dot{A}},\;\; \eta^\alpha \;\mapsto\; t_1^{-1}\cdot\eta^\alpha. \tag{13}$$

Explicit Form of Ψ^p_Δ. Performing the affine and Mellin integrals in Eq. (9) yields:

$$\Psi^p_\Delta\big(\mathsf{Z}^I; \mathsf{W}^{I'}\big) \;=\; \bar{\delta}\left(\langle\lambda z\rangle\right)\left(\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\right)^{p-\Delta-1}\frac{\mathcal{C}\left(\Delta\right)}{[\mu\bar{z}]^\Delta}\exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\psi\cdot\eta\right). \tag{14}$$

Alternatively, one may invoke the projective delta function $\bar{\delta}_\Delta$ on $\mathbf{CP}^1$ (Eq. (5)). In this notation, the superwavefunction simplifies to:

$$\Psi^p_\Delta\big(\mathsf{Z}^I; \mathsf{W}^{I'}\big) \;=\; \bar{\delta}_{p-\Delta}\left(z, \lambda\right)\frac{\mathcal{C}\left(\Delta\right)}{[\mu\bar{z}]^\Delta}\exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\psi\cdot\eta\right). \tag{15}$$

This form will prove useful when deriving the bulk-to-boundary propagator on the (3|8)-dimensional Euclidean anti-de Sitter space. This derivation proceeds via the Penrose integral-geometric transform of Ψ^p_Δ.

To establish completeness and orthogonality of the family $\{\Psi^p_\Delta\}$, we introduce the conjugate wavefunction:

$$\widetilde{\Psi}^p_\Delta(\mathsf{Z}^I;\mathsf{W}^{I'}) := \bar{\delta}_{p-\Delta}(z,\lambda)\,\frac{\mathcal{C}(\Delta)}{[\bar{z}\mu]^\Delta}\exp\left(-i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\psi\cdot\eta\right). \tag{16}$$

Under conjugation, the phase in the exponential changes sign. The denominator ordering also changes from $[\mu\bar{z}]$ to $[\bar{z}\mu]$. The latter condition simplifies the proportionality factors in the resulting completeness and orthogonality relations.

C. Homogeneous Bundles on Minitwistor Superspace

In the preceding subsection, we derived an explicit formula for the minitwistor superwavefunction $\Psi^p_\Delta(\mathsf{W}^I;\mathsf{Z}^{I'})$ by applying the Mellin transform to the corresponding $\mathcal{N}=4$ supersymmetric twistor wavefunction. We now interpret Ψ^p_Δ geometrically. It defines a section (more precisely a $(0,1)$-current) on the minitwistor superspace $\mathbf{MT}_s$ (see Subsection II C 1).

There is a dual minitwistor superspace $\mathbf{MT}^*_s$, which is canonically isomorphic to the holomorphic celestial supersphere $\mathcal{CS}\simeq\mathbf{CP}^{1|4}$ (Subsection II C 2). Equivalently, $\mathbf{MT}^*_s$ serves as a parameter space for $\mathcal{CS}$. The minitwistor transform then carries sections of holomorphic bundles over $\mathbf{MT}_s$ to sections over $\mathbf{MT}^*_s$.

This transform is our prescription for converting sectional (or leaf) amplitudes into minitwistor amplitudes. It thereby provides a geometric reinterpretation of celestial amplitudes as semiclassical expectation values of Wilson line operators, or alternatively as correlation functions in the minitwistor sigma-model. In this sense, the minitwistor superspace geometry offers a dual description of the flat-space hologram on the celestial supersphere, much as the Fourier transform relates position and momentum representations in elementary quantum mechanics.

1. *Minitwistor Superspace*

Let $\mathbf{MT}\subset\times^2\,\mathbf{CP}^1$ denote the bosonic minitwistor space of three-dimensional Euclidean anti-de Sitter space. We begin by defining the ordinary $(2|4)$-dimensional minitwistor superspace $\mathbf{MT}_s$. This supermanifold is obtained by enforcing the transformation laws of Eq. (12) as a symmetry group on its underlying (projective) geometry.

Definition of $\mathbf{MT}_s$. Introduce the vector superspace $\mathbf{V} \simeq \mathbf{C}^{4|4}$ with Cartesian coordinates $\mathsf{W}^I := \big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big)$. On $\mathbf{U} := \mathbf{V} - \{0\}$, impose the equivalence relation:

$$\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big) \sim \big(t_1 \cdot \lambda^A, t_2 \cdot \mu_{\dot{A}}, t_1 \cdot \psi^\alpha\big), \quad \forall\, t_1, t_2 \in \mathbf{C}^*. \tag{17}$$

Define the $\mathcal{N} = 4$ *minitwistor superspace* $\mathbf{MT}_s$ as the set of equivalence classes $\mathsf{w} := [\mathsf{W}^I] \in \mathbf{U}/\sim$ subject to the non-degeneracy condition:

$$[\lambda^\flat \mu] \neq 0, \quad \big(\lambda^\flat\big)_{\dot{A}} := \big(\lambda^A\big)^*; \tag{18}$$

and equip it with the quotient topology. This condition ensures nontrivial Dolbeault cohomology on $\mathbf{MT}_s$, so that minitwistor superwavefunctions are realised as its cohomology representatives.

Quotient Map. Denote by

$$\pi_0 : \mathbf{M} \longrightarrow \mathbf{MT}_s, \quad \mathsf{W}^I \mapsto \mathsf{w} \tag{19}$$

the natural quotient map, where $\mathsf{w} = [\mathsf{W}^I]$. Here $\mathbf{M} \subset \mathbf{U}$ is the maximal open submanifold on which π_0 is surjective. Its boundary $\partial_{\mathbf{U}}\mathbf{M}$ consists of those equivalence classes $[(\lambda^A, \mu_{\dot{A}}, \psi^\alpha)]$ for which $[\lambda^\flat \mu] = 0$.

Elements of $\mathbf{MT}_s$ will henceforth be called $\mathbf{Z}_2$-*graded minitwistors* and denoted by w, w', etc. For any $\mathsf{w} \in \mathbf{MT}_s$, we refer to a choice of preimage $\mathsf{W}^I \in \pi_0^{-1}(\mathsf{w})$ as a *coordinate representative.*

Orientation. To specify the orientation of $\mathbf{MT}_s$, first introduce the holomorphic and antiholomorphic forms:

$$D\lambda := \varepsilon_{AB}\lambda^A d\lambda^B, \;\; D\mu := \varepsilon^{\dot{A}\dot{B}}\mu_{\dot{A}} d\mu_{\dot{B}}, \tag{20}$$

and let $d^{0|4}\psi$ denote the Berezin measure on $\mathbf{C}^{0|4}$. The natural volume form on $\mathbf{MT}_s$ is the $\mathbf{Z}_2$-graded measure:

$$D^{2|4}\mathsf{W} := D\lambda \wedge D\mu \wedge d^{0|4}\psi. \tag{21}$$

Under the rescaling of Eq. (12), one finds:

$$D^{2|4}\mathsf{W} \mapsto t_1^{-2} t_2^2\, D^{2|4}\mathsf{W}. \tag{22}$$

2. Dual Minitwistor Superspace

The second set of transformation rules, stated in Eq. (13), motivates the introduction of the *dual minitwistor superspace* $\mathbf{MT}_s^*$. We will show in our discussion of scattering amplitudes that this dual space is intimately connected to the holomorphic celestial supersphere. Indeed, each point of the celestial supersphere corresponds to a point in the dual minitwistor superspace.

The minitwistor transform $\mathcal{MT}$ then carries sections over the dual superspace $\mathbf{MT}_s^*$ to sections over the original minitwistor supermanifold $\mathbf{MT}_s$. This operation is *analogous* to the ordinary Fourier transform in quantum mechanics, which maps wavefunctions from the position representation to the momentum representation. In our framework, *the minitwistor superspace geometry provides a dual description of the flat-space hologram.*

Definition of $\mathbf{MT}_s^*$. We begin by defining the dual vector superspace $\mathbf{V}^* \simeq \left(\mathbf{C}^{4|4}\right)^*$ with dual coordinates $\mathsf{Z}^I = (z^A, \bar{z}_{\dot{A}}, \eta^\alpha)$. Here z^A and $\bar{z}_{\dot{A}}$ will serve as the van der Waerden spinors parametrising the holomorphic celestial sphere.

Let $\mathbf{U}^* := \mathbf{V}^* - \{0\}$. On $\mathbf{U}^*$ impose the equivalence relation:

$$\left(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\right) \simeq \left(t_1 \cdot z^A, t_2 \cdot \bar{z}_{\dot{A}}, t_1^{-1} \cdot \eta^\alpha\right), \quad \forall\, t_1, t_2 \in \mathbf{C}^*. \tag{23}$$

The *dual minitwistor superspace* $\mathbf{MT}_s^*$ is then the set of equivalence classes $\mathsf{z} := [\mathsf{Z}^I] \in \mathbf{U}^*/\simeq$ subject to the non-degeneracy condition

$$[z^\flat \bar{z}] \neq 0, \quad \left(z^\flat\right)_{\dot{A}} := \left(z^A\right)^*; \tag{24}$$

and endowed with the quotient topology.

Quotient map. In what follows, we denote by

$$\pi_0^*\colon \mathbf{M}^* \longrightarrow \mathbf{MT}_s^*, \quad \mathsf{Z}^I \mapsto \mathsf{z} \tag{25}$$

the quotient map, where $\mathsf{z} = [\mathsf{Z}^I]$. Here $\mathbf{M}^* \subset \mathbf{U}^*$ is the maximal open submanifold on which π_0^* is surjective. Its boundary $\partial_{\mathbf{U}^*}\mathbf{M}^*$ consists of those equivalence classes $[(z^A, \bar{z}_{\dot{A}}, \eta^\alpha)]$ for which the non-degeneracy condition is violated, i.e. $[z^\flat \bar{z}] = 0$.

Points of $\mathbf{MT}_s^*$ are denoted by $\mathsf{z}, \mathsf{z}', \ldots$ and are called *dual* $\mathbf{Z}_2$*-graded minitwistors.* Any lift $\mathsf{Z}^I \in (\pi_0^*)^{-1}(\mathsf{z})$ is referred to as a *coordinate representative* of the dual minitwistor point z.

Volume superform. To fix the orientation of $\mathbf{MT}_s^*$, first introduce the standard holomorphic and anti-holomorphic measures on $\mathbf{CP}^1$:

$$Dz \coloneqq \varepsilon_{AB} z^A dz^B, \;\; D\bar{z} \coloneqq \varepsilon^{\dot{A}\dot{B}} \bar{z}_{\dot{A}} d\bar{z}_{\dot{B}}. \tag{26}$$

Let $d^{0|4}\eta$ denote the Berezin measure on $\mathbf{C}^{0|4}$. The resulting $\mathbf{Z}_2$-graded volume form is:

$$D^{2|4}\mathsf{Z} \coloneqq Dz \wedge D\bar{z} \wedge d^{0|4}\eta. \tag{27}$$

Under the scaling of Eq. (13), one finds:

$$D^{2|4}\mathsf{Z} \mapsto t_1^6 t_2^2 \, D^{2|4}\mathsf{Z}. \tag{28}$$

3. $\mathcal{O}_A(p,q)$-bundle

We now introduce a two-parameter family of holomorphic vector bundles whose sections model the physical fields and the minitwistor superwavefunctions.

Definition of $\mathcal{O}_A(p,q)$. Fix $p, q \in \mathbf{Z}$ and a normed algebra A selected according to the desired background field or superwavefunction. Define the auxiliary trivial bundle $\mathbf{E} \coloneqq \mathbf{M} \times A$.

On $\mathbf{E}$, impose the following equivalence relation:

$$\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha, |a\rangle\big) \equiv_{p,q} \big(t_1 \cdot \lambda^A, t_2 \cdot \mu_{\dot{A}}, t_1 \cdot \psi^\alpha, t_1^p t_2^q \cdot |a\rangle\big), \quad \forall\, t_1, t_2 \in \mathbf{C}_*. \tag{29}$$

The total space of our bundle is then the quotient:

$$\mathcal{O}_A(p,q) \coloneqq \mathbf{E}/\equiv_{p,q}, \tag{30}$$

with the natural projection $Q : \mathbf{E} \longrightarrow \mathcal{O}_A(p,q)$.

Fibration. Let $\mathrm{pr}_{\mathbf{M}} : \mathbf{E} \longrightarrow \mathbf{M}$ be the projection onto the first factor and define the surjection

$$Q_0 : \mathbf{E} \longrightarrow \mathbf{MT}_s, \quad Q_0 \coloneqq \pi_0 \circ \mathrm{pr}_{\mathbf{M}}. \tag{31}$$

By construction, if $\mathsf{w} \in \mathbf{MT}_s$ is a $\mathbf{Z}_2$-graded minitwistor point and $\mathsf{W}^I \in \pi_0^{-1}(\mathsf{w})$ any coordinate representative, then $Q_0\big(\mathsf{W}^I, |a\rangle\big) = \mathsf{w}$ for all $|a\rangle \in A$.

Next, consider the quotient manifold $\mathcal{O}_A(p,q)$. We endow it with the structure of a holomorphic vector bundle over $\mathbf{MT}_s$ by introducing the projection

$$\pi : \mathcal{O}_A(p,q) \longrightarrow \mathbf{MT}_s. \tag{32}$$

This bundle map is uniquely determined by the condition that the quotient map Q_0 lifts to Q. Equivalently, the following diagram commutes:

$$\begin{tikzcd} & \mathbf{E} \arrow[dl, "Q"'] \arrow[dr, "Q_0"] & \\ \mathcal{O}_A(p,q) \arrow[rr, "\pi"'] & & \mathbf{MT} \end{tikzcd} \tag{33}$$

Hence one has the relation

$$\pi \circ Q = Q_0. \tag{34}$$

Module of Sections. We next characterise the sections of $\mathcal{O}_A(p,q) \xrightarrow{\pi} \mathbf{MT}_s$. A map

$$a : \mathbf{M} \longrightarrow A, \quad \mathsf{W}^I \mapsto |a(\mathsf{W}^I)\rangle \tag{35}$$

is called an *A-valued homogeneous function of bi-degree* (p,q) if the following holds. Write $\mathsf{W}^I = (\lambda^A, \mu_{\dot{A}}, \psi^\alpha)$ and, for any $t_1, t_2 \in \mathbf{C}_*$, define the rescaled coordinates

$$\mathsf{W}'^I := \big(t_1 \cdot \lambda^A, t_2 \cdot \mu_{\dot{A}}, t_1 \cdot \psi^\alpha\big). \tag{36}$$

Then homogeneity demands

$$|a(\mathsf{W}'^I)\rangle \;=\; t_1^p t_2^q \cdot |a(\mathsf{W}^I)\rangle, \quad t_1, t_2 \in \mathbf{C}_*. \tag{37}$$

Since A is a normed algebra, one may equip the space of A-valued functions on $\mathbf{MT}_s$ with the corresponding Fréchet topology. We then define the complex vector space

$$\mathcal{S}_A(p,q) \;:=\; \big\{a : \mathbf{MT}_s \longrightarrow A \big| a \text{ is smooth and homogeneous of bi-degree } (p,q)\big\}\,. \tag{38}$$

The space $\mathcal{S}_A(p,q)$ admits the natural structure of a module over the ring $\mathscr{C}^\infty(\mathbf{MT}_s)$ of complex-valued smooth functions on the minitwistor superspace. If $a \in \mathcal{S}_A(p,q)$ and $\varphi \in \mathscr{C}^\infty(\mathbf{MT}_s)$, we set:

$$(\varphi \cdot a)(\mathsf{W}^I) \;:=\; \varphi\big(\pi_0(\mathsf{W}^I)\big)\, |a(\mathsf{W}^I)\rangle. \tag{39}$$

Main Result. Our principal claim is that the module of smooth sections of $\mathcal{O}_A(p,q) \xrightarrow{\pi} \mathbf{MT}_s$ coincides with $\mathcal{S}_A(p,q)$,

$$\Gamma\big(\mathbf{MT}_s; \mathcal{O}_A(p,q)\big) \;\simeq\; \mathcal{S}_A(p,q). \tag{40}$$

To prove the section/function correspondence, let $s \in \Gamma(\mathbf{MT}_s; \mathcal{O}_A(p,q))$ be a smooth section. For any $\mathsf{W}^I \in \mathbf{M}$ projecting to $\mathsf{w} \in \mathbf{MT}_s$, we have $\pi \circ s \circ \pi_0(\mathsf{W}^I) = \mathsf{w}$. Since $Q\colon \mathbf{E} \longrightarrow \mathcal{O}_A(p,q)$ is the quotient map, there exists a (unique) element $|a(\mathsf{W}^I)\rangle \in A$ such that

$$Q\big(\mathsf{W}^I, |a(\mathsf{W}^I)\rangle\big) = s \circ \pi_0(\mathsf{W}^I). \tag{41}$$

Uniqueness follows because if also $\tilde{a} : \mathsf{W}^I \mapsto |\tilde{a}(\mathsf{W}^I)\rangle$ satisfies $Q(\mathsf{W}^I, \tilde{a}) = s$, then $(\mathsf{W}^I, a) \equiv_{p,q} (\mathsf{W}^I, \tilde{a})$, which forces $a = \tilde{a}$. Thus we obtain a well-defined map

$$a\colon \mathbf{M} \longrightarrow A, \quad \mathsf{W}^I \mapsto |a(\mathsf{W}^I)\rangle. \tag{42}$$

Smoothness of a follows from that of s together with the local triviality of the quotient. It remains to verify homogeneity. Write $\mathsf{W}^I = (\lambda^A, \mu_{\dot{A}}, \psi^\alpha)$ and for $t_1, t_2 \in \mathbf{C}_*$ set $\mathsf{W}'^I = (t_1\lambda^A, t_2\mu_{\dot{A}}, t_1\psi^\alpha)$. Since $\pi_0(\mathsf{W}'^I) = \pi_0(\mathsf{W}^I)$, the section takes the same value, $s(\pi_0(\mathsf{W}'^I)) = s(\pi_0(\mathsf{W}^I))$. Hence

$$Q(\mathsf{W}'^I, |a(\mathsf{W}'^I)\rangle) = Q(\mathsf{W}^I, |a(\mathsf{W}^I)\rangle). \tag{43}$$

By the defining equivalence on $\mathbf{E}$, this implies

$$\big(\mathsf{W}'^I, |a(\mathsf{W}'^I)\rangle\big) \equiv_{p,q} \big(\mathsf{W}'^I, t_1^p t_2^q \cdot |a(\mathsf{W}^I)\rangle\big), \tag{44}$$

and uniqueness then yields

$$|a(\mathsf{W}'^I)\rangle = t_1^p t_2^q \cdot |a(\mathsf{W}^I)\rangle. \tag{45}$$

Thus a is homogeneous of bi-degree (p,q) and so belongs to $\mathcal{S}_A(p,q)$.

Conversely, we construct a section from a homogeneous function $a \in \mathcal{S}_A(p,q)$. For each $\mathbf{Z}_2$-graded minitwistor point $\mathsf{w} \in \mathbf{MT}_s$, choose any coordinate representative $\mathsf{W}^I \in \pi_0^{-1}(\mathsf{w})$. We then define

$$s(\mathsf{w}) := Q\big(\mathsf{W}^I, |a(\mathsf{W}^I)\rangle\big) \in \mathcal{O}_A(p,q). \tag{46}$$

To see that s is well-defined, suppose W'^I is another lift of w. Then there exist $t_1, t_2 \in \mathbf{C}_*$ with $\mathsf{W}'^I = (t_1\lambda^A, t_2\mu_{\dot{A}}, t_1\psi^\alpha)$. Homogeneity of a gives $|a(\mathsf{W}'^I)\rangle = t_1^p t_2^q \cdot |a(\mathsf{W}^I)\rangle$. Hence

$$Q\big(\mathsf{W}'^I, |a(\mathsf{W}'^I)\rangle\big) = Q\big(\mathsf{W}'^I, t_1^p t_2^q \cdot |a(\mathsf{W}^I)\rangle\big) = Q\big(\mathsf{W}^I, |a(\mathsf{W}^I)\rangle\big), \tag{47}$$

so $s(\mathsf{w})$ is independent of the choice of representative. Smoothness of s follows from that of a together with the local triviality of the bundle $\mathcal{O}_A(p,q) \xrightarrow{\pi} \mathbf{MT}_s$. Finally, s is a section because

$$\pi\big(s(\mathsf{w})\big) = \pi\big(Q(\mathsf{W}^I, |a(\mathsf{W}^I)\rangle)\big) = Q_0\big(\mathsf{W}^I, |a(\mathsf{W}^I)\rangle\big) = \pi_0(\mathsf{W}^I) = \mathsf{w}, \tag{48}$$

i.e. $\pi \circ s = \mathrm{id}_{\mathbf{MT}_s}$. Thus $s \in \Gamma(\mathbf{MT}_s; \mathcal{O}_A(p,q))$, completing the correspondence.

4. $\mathcal{O}_A^*(r,s)$-bundle

A full understanding of minitwistor wavefunctions requires the introduction of the dual vector bundle $\mathcal{O}_A^*(r,s) \xrightarrow{\pi^*} \mathbf{MT}_s^*$.

Our correspondence between the holomorphic celestial supersphere $\mathcal{CS}$ and the minitwistor superspace $\mathbf{MT}_s$ is mediated by the minitwistor transform $\mathcal{MT}$. Unlike an ordinary Fourier transform, $\mathcal{MT}$ carries sections of $\mathcal{O}_A(p,q) \xrightarrow{\pi} \mathbf{MT}_s$ to sections of $\mathcal{O}_A^*(r,s) \xrightarrow{\pi^*} \mathbf{MT}_s^*$.

Definition of $\mathcal{O}_A^*(r,s)$. In the previous subsection, we fixed a normed algebra A adapted to the background field theory or wavefunction in question. We now introduce its dual algebra A^*, whose elements we denote by $\langle a|$. We also define the auxiliary trivial bundle $\mathbf{E}^* := \mathbf{M}^* \times A^*$.

On the auxiliary bundle, we impose the equivalence:

$$\big(z^A, \bar{z}_{\dot{A}}, \eta^\alpha, \langle a|\big) \cong_{r,s} \big(t_1 \cdot z^A, t_2 \cdot \bar{z}_{\dot{A}}, t_1^{-1} \cdot \eta^\alpha, t_1^r t_2^s \cdot \langle a|\big), \quad \forall\, t_1, t_2 \in \mathbf{C}^*. \tag{49}$$

The total space of the dual bundle is the quotient

$$\mathcal{O}_A^*(r,s) := \mathbf{E}^* \big/ \cong_{r,s}, \tag{50}$$

with the natural projection $Q^* : \mathbf{E}^* \longrightarrow \mathcal{O}_A^*(r,s)$.

Dual Fibration. Next, let $\mathrm{pr}_{\mathbf{M}^*} : \mathbf{E}^* \longrightarrow \mathbf{M}^*$ be the projection onto the first factor. Define

$$Q_0^* \colon \mathbf{E}^* \longrightarrow \mathbf{MT}_s^*, \quad Q_0^* := \pi_0^* \circ \mathrm{pr}_{\mathbf{M}^*}. \tag{51}$$

By construction, if $\mathsf{z} \in \mathbf{MT}_s^*$ has any lift $\mathsf{Z}^I \in (\pi_0^*)^{-1}(\mathsf{z})$, then $Q_0^*(\mathsf{Z}^I, \langle a|) = \mathsf{z}$ for all $\langle a| \in A^*$.

Finally, we define the dual fibration:

$$\pi^* : \mathcal{O}_A^*(r,s) \longrightarrow \mathbf{MT}_s^*. \tag{52}$$

It is uniquely specified by the requirement that the diagram

$$\begin{tikzcd} & \mathbf{E}^* \arrow[dl, "Q^*"'] \arrow[dr, "Q_0^*"] & \\ \mathcal{O}_A^*(r,s) \arrow[rr, "\pi^*"'] & & \mathbf{MT}_s^* \end{tikzcd} \tag{53}$$

commutes, i.e. $\pi^* \circ Q^* = Q_0^*$.

Sections. The description of sections of the dual bundle $\mathcal{O}_A^*(r,s)$ parallels that of $\mathcal{O}_A(p,q)$. A map

$$a^* : \mathbf{M}^* \longrightarrow A^*, \quad \mathsf{Z}^I \mapsto \langle a(\mathsf{Z}^I)| \tag{54}$$

is called an A^**-valued homogeneous function of bi-degree* (r,s) if the following holds. Write $\mathsf{Z}^I = (z^A, \bar{z}_{\dot{A}}, \eta^\alpha)$ and, for any $t_1, t_2 \in \mathbf{C}_*$, set $\mathsf{Z}'^I = (t_1 z^A, t_2 \bar{z}_{\dot{A}}, t_1^{-1}\eta^\alpha)$. Then homogeneity demands:

$$\langle a(\mathsf{Z}'^I)| = t_1^r t_2^s \cdot \langle a(\mathsf{Z}^I)|. \tag{55}$$

To discuss the section/function correspondence on $\mathcal{O}_A^*(r,s) \xrightarrow{\pi^*} \mathbf{MT}_s^*$, define the dual function space:

$$\mathcal{S}_A^*(r,s) \;\coloneqq\; \big\{\, a^* : \mathbf{M}^* \longrightarrow A^* \,\big|\, a^* \text{ smooth and homogeneous of bi-degree } (r,s) \big\}. \tag{56}$$

Here smoothness is understood in the Fréchet sense, since A (and hence A^*) carries a norm making it a Banach space. We therefore equip $\mathcal{S}_A^*(r,s)$ with the induced Fréchet topology, rendering it a locally convex topological vector space.

By exactly the same arguments as in the previous subsection, one shows that each A^*-valued homogeneous function a^* defines a unique holomorphic section of the dual bundle $\mathcal{O}_A^*(r,s)$, and vice versa. Hence there is a natural isomorphism

$$\Gamma\big(\mathbf{MT}_s^*; \mathcal{O}_A^*(r,s)\big) \;\simeq\; \mathcal{S}_A^*(r,s). \tag{57}$$

5. *Superforms and Currents on Minitwistor Superspace*

The projective delta function $\overline{\delta}_\Delta$ appearing in Eq. (15) indicates that the minitwistor superwavefunction Ψ_Δ^p is most naturally realised as a current[1]. We now review the requisite differential-geometric framework.

Superforms. Fix integers $m, n, p, q \in \mathbf{Z}$ with $0 \leq m, n \leq 2$. Denote by $\bigwedge^{m,n} \mathbf{MT}_s$ the exterior superbundle of $\mathbf{Z}_2$-graded (m,n)-forms on minitwistor superspace. An element of this bundle is called an (m,n)*-superform.*

We consider the sheaf of $\mathcal{O}_{\mathbf{C}}(p,q)$*-valued differential* (m,n)*-superforms.* Explicitly, set

$$\Omega^{m,n}\big(\mathbf{MT}_s; \mathcal{O}_{\mathbf{C}}(p,q)\big) \;\coloneqq\; \Gamma\big(\mathbf{MT}_s; \textstyle\bigwedge^{m,n} \mathbf{MT}_s \otimes \mathcal{O}_{\mathbf{C}}(p,q)\big). \tag{58}$$

[1] The theory of currents was initiated by Schwartz [2, 3, 4] and De Rham [5]. A measure-theoretic framework was developed by Federer [6, 7], Federer and Fleming [8], Federer [9]. The complex-analytic aspects relevant to our discussion were reviewed by King [10]. For a modern treatment emphasising positive line bundles in the context of algebraic geometry, see the review by Demailly *et al.* [11].

This is a module over the ring $\mathscr{C}^\infty(\mathbf{MT}_s)$. Equipped with the Whitney $\mathscr{C}^\infty$-topology[2], this space becomes a locally convex, complete topological vector superspace over $\mathbf{C}$.

Supercurrents. A supercurrent of bi-degree $(2-m, 2-n)$ (equivalently bi-dimension (m,n)) over the bundle $\mathcal{O}_{\mathbf{C}}(2-p, 2-q) \xrightarrow{\pi} \mathbf{MT}_s$ is a continuous, complex-linear functional

$$\mathcal{T}\colon \Omega^{m,n}\big(\mathbf{MT}_s; \mathcal{O}_{\mathbf{C}}(p-2, q-2)\big) \longrightarrow \mathbf{C}. \tag{59}$$

Continuity is understood with respect to the Whitney $\mathscr{C}^\infty$-topology. The space of all such supercurrents is the strong-dual of the corresponding module of superforms:

$$\mathscr{D}'_{2-m,2-n}\big(\mathbf{MT}_s; \mathcal{O}_{\mathbf{C}}(2-p, 2-q)\big) \;\coloneqq\; \big(\Omega^{m,n}\big(\mathbf{MT}_s; \mathcal{O}_{\mathbf{C}}(p-2, q-2)\big)\big)'. \tag{60}$$

One extends the boundary and contraction operators from the exterior algebra of superforms to this space of currents. A natural wedge product exists between superforms and supercurrents. However, an exterior product of two supercurrents is not generically well defined. (See Simon *et al.* [12, Sec. 26].)

Minitwistor Superwavefunctions as Currents. With the preceding framework in place, Ψ^p_Δ is not an ordinary differential $(0,1)$-form on $\mathbf{MT}_s$. Instead, it defines a supercurrent of bi-degree $(0,1)$ over the bundle $\mathcal{O}_{\mathbf{C}}(\Delta - p, -\Delta)$. To state this precisely, fix a dual $\mathbf{Z}_2$-graded minitwistor point $\mathsf{z} \in \mathbf{MT}^*_s$ and choose a coordinate representative $\mathsf{Z}^I \in (\pi^*_0)^{-1}(\mathsf{z})$. Then

$$\mathcal{T}^p_\Delta \in \mathscr{D}'_{0,1}\big(\mathbf{MT}_s; \mathcal{O}_{\mathbf{C}}(\Delta - p, -\Delta)\big), \quad \mathcal{T}^p_\Delta \;\coloneqq\; \Psi^p_\Delta(\,\cdot\,; \mathsf{Z}^I). \tag{61}$$

Next, let

$$\boldsymbol{\alpha} \in \Omega^{2,1}\big(\mathbf{MT}_s; \mathcal{O}_{\mathbf{C}}(p-\Delta, \Delta)\big) \tag{62}$$

be any differential $(2,1)$-superform with values in $\mathcal{O}_{\mathbf{C}}(p-\Delta, \Delta)$. By the section/function correspondence, there exists

$$a \in \mathcal{S}_{\mathbf{C}}(p - \Delta + 2, \Delta - 2) \tag{63}$$

[2] We equip the space $\Omega^{m,n}\big(\mathbf{MT}_s; \mathcal{O}_{\mathbf{C}}(p,q)\big)$ with the following topology. A sequence $(\boldsymbol{\alpha}_i)_{i\in\mathbf{N}}$ of $\mathcal{O}_{\mathbf{C}}(p,q)$-valued differential (m,n)-superforms converges to $\boldsymbol{\alpha}$ if and only if, on each trivialising neighbourhood $U \subset \mathbf{MT}_s$ of the bundle $\mathcal{O}_{\mathbf{C}}(p,q)$, the following holds:

1. Write $(\boldsymbol{\alpha}_i - \boldsymbol{\alpha})_{i\in\mathbf{N}}$ in local coordinates as a finite collection of component functions.
2. For every multi-index k and every compact set $K \subset U$, the derivatives

$$D^k\boldsymbol{\alpha}_i - D^k\boldsymbol{\alpha}$$

converge uniformly to zero on K as $i \to \infty$.

In this way, all component functions of $(\boldsymbol{\alpha}_i - \boldsymbol{\alpha})_{i\in\mathbf{N}}$, together with all their derivatives, vanish uniformly on compact subsets on each trivialising patch.

such that on each trivialising neighbourhood $U \subset \mathbf{MT}_s$ one has

$$\boldsymbol{\alpha}\big|_U \;=\; a(\mathsf{W}^I)\, D^{2|4}\mathsf{W}. \tag{64}$$

Hence the action of the supercurrent $\mathcal{T}^p_\Delta$ on $\boldsymbol{\alpha}$ is given by

$$\langle \mathcal{T}^p_\Delta, \boldsymbol{\alpha}\rangle \;=\; \int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{W}\;\; a(\mathsf{W}^I) \wedge \Psi^p_\Delta(\mathsf{W}^I; \mathsf{Z}^{I'}). \tag{65}$$

In the next subsection, we shall use this pairing to formulate the minitwistor transform.

Currents on the Dual Superspace. We adopt the viewpoint that the minitwistor supergeometry is dual to the flat-space hologram on the holomorphic celestial supersphere. Consistency then demands that the minitwistor transform $\mathcal{MT}$ be invertible. Equivalently, the family of super-wavefunctions $\{\Psi^p_\Delta\}$ must also furnish a corresponding family of supercurrents on the *dual* minitwistor superspace. Accordingly, we extend the above discussion to currents valued in the bundle $\mathcal{O}^*_{\mathbf{C}}(p,q) \xrightarrow{\pi^*} \mathbf{MT}^*_s$.

Differential Superforms on Dual Space. Let $\bigwedge^{m,n}\mathbf{MT}^*_s$ be the exterior superbundle of complex (m,n)-superforms on the dual minitwistor superspace $\mathbf{MT}^*_s$. We consider the module of smooth sections

$$\Omega^{m,n}\big(\mathbf{MT}^*_s; \mathcal{O}^*_{\mathbf{C}}(p,q)\big) \;\coloneqq\; \Gamma\big(\mathbf{MT}^*_s; \textstyle\bigwedge^{m,n}\mathbf{MT}^*_s \otimes \mathcal{O}^*_{\mathbf{C}}(p,q)\big). \tag{66}$$

Endowed with the Whitney $\mathscr{C}^\infty$-topology, this becomes the locally convex, complete superspace of *$\mathcal{O}^*_{\mathbf{C}}(p,q)$-valued differential (m,n)-superforms on $\mathbf{MT}^*_s$.*

Dual Supercurrents. A supercurrent of bi-degree $(2-m, 2-n)$ (equivalently bi-dimension (m,n)) over the bundle $\mathcal{O}^*_{\mathbf{C}}(2-p,2-q) \xrightarrow{\pi^*} \mathbf{MT}^*_s$ is a continuous, $\mathbf{C}$-linear functional

$${}^*\mathcal{T}\colon \Omega^{m,n}\big(\mathbf{MT}^*; \mathcal{O}^*_{\mathbf{C}}(p-2,q-2)\big) \longrightarrow \mathbf{C}. \tag{67}$$

Continuity is again with respect to the Whitney $\mathscr{C}^\infty$-topology.

The space of supercurrents of bi-degree $(2-m,2-n)$ over the bundle $\mathcal{O}^*_{\mathbf{C}}(2-p,2-q)$ is

$$\mathscr{D}'_{2-m,2-n}\big(\mathbf{MT}^*_s; \mathcal{O}^*_{\mathbf{C}}(2-p,2-q)\big) \;\coloneqq\; \big(\Omega^{m,n}\big(\mathbf{MT}^*_s; \mathcal{O}^*_{\mathbf{C}}(p-2,q-2)\big)\big)' \tag{68}$$

endowed with the strong-dual topology.

Supercurrents on the Dual Superspace. We now associate to each minitwistor superwavefunction Ψ^p_Δ a supercurrent on $\mathbf{MT}^*_s$. Fix a $\mathbf{Z}_2$-graded minitwistor point $[\mathsf{W}^I] \in \mathbf{MT}_s$. Then define

$$ {}^*\mathcal{T}^p_\Delta \colon \mathscr{D}'_{0,1}\big(\mathbf{MT}^*_s; \mathcal{O}^*_{\mathbf{C}}(p-\Delta-2, -\Delta)\big), \quad {}^*\mathcal{T}^p_\Delta \;\coloneqq\; \Psi^p_\Delta(\mathsf{W}^I; \,\cdot\,). \tag{69} $$

Let

$$ \boldsymbol{\beta} \in \Omega^{2,1}\big(\mathbf{MT}^*_s; \mathcal{O}^*_{\mathbf{C}}(\Delta-p+2, \Delta)\big) \tag{70} $$

be any differential $(2,1)$-superform. By the section/function correspondence, there exists

$$ b \in \mathcal{S}^*_{\mathbf{C}}(\Delta-p-4, \Delta-2) \tag{71} $$

such that on each trivialising neighbourhood $U^* \subset \mathbf{MT}^*_s$,

$$ \boldsymbol{\beta}\,\big|_{U^*} \;=\; b(\mathsf{W}^I)\, D^{2|4}\mathsf{W}. \tag{72} $$

Hence the action of the supercurrent on $\boldsymbol{\beta}$ is:

$$ \langle {}^*\mathcal{T}^p_\Delta, \boldsymbol{\beta} \rangle \;=\; \int\limits_{\mathbf{MT}^*_s} \Psi^p_\Delta(\mathsf{W}^I; \mathsf{Z}^{I'}) \wedge b(\mathsf{Z}^{I'})\; D^{2|4}\mathsf{Z}. \tag{73} $$

In the following subsection, this pairing will define the *inverse minitwistor transform* $\mathcal{MT}^{-1}$.

D. Minitwistor Transform

We now show that the family $\{\Psi^p_\Delta\}$ of minitwistor wavefunctions is complete and orthogonal. Accordingly, we may interpret

$$ \Psi^p_\Delta\big(\mathsf{Z}^I; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big) $$

as the wavefunction of an external gluon with conformal weight Δ and quantum numbers $z^A, \bar{z}_{\dot{A}}, \eta^\alpha$.

This interpretation follows from the existence of a minitwistor transform $\mathcal{MT}$. The mapping $\mathcal{MT}$ carries holomorphic sections on $\mathbf{MT}_s$ to those on its dual. Moreover, $\mathcal{MT}$ satisfies a Fourier-type inversion theorem.

1. *Completeness Relation*

To derive the completeness relation for the family $\{\Psi^p_\Delta\}$, consider the differential form on the dual superspace $\mathbf{MT}^*_s$:

$$ \boldsymbol{a} \;\coloneqq\; a(z^A, \bar{z}_{\dot{A}}, \eta^\alpha)\;\; D^{2|4}\mathsf{W}, \tag{74} $$

where:

$$a\big(z^A,\bar{z}_{\dot{A}},\eta^\alpha\big) \;\coloneqq\; \Psi^{\tilde{p}}_{\tilde{\Delta}}\big(\lambda^A,\mu_{\dot{A}},\psi^\alpha;z^A,\bar{z}_{\dot{A}},\eta^\alpha\big)\widetilde{\Psi}^p_\Delta\big(\sigma^A,\omega_{\dot{A}},\chi^\alpha;z^A,\bar{z}_{\dot{A}},\eta^\alpha\big). \tag{75}$$

The integral

$$\int\limits_{\mathbf{MT}^*_s} \boldsymbol{a}$$

exists when $\boldsymbol{a}$ is invariant under the transformation of Eq. (13).

Equations (11) and (28) imply that $\boldsymbol{a}$ defines a volume form when:

$$\Delta+\tilde{\Delta}=2, \qquad p+\tilde{p}=0. \tag{76}$$

We therefore set:

$$\boldsymbol{a} \;=\; \Psi^{-p}_{2-\Delta}\big(\lambda^A,\mu_{\dot{A}},\psi^\alpha;z^A,\bar{z}_{\dot{A}},\eta^\alpha\big)\,\widetilde{\Psi}^p_\Delta\big(\sigma^A,\omega_{\dot{A}},\chi^\alpha;z^A,\bar{z}_{\dot{A}},\eta^\alpha\big)\;\; D^{2|4}\mathsf{W}. \tag{77}$$

Substituting the expression for Ψ^p_Δ from Eq. (15) gives:

$$\boldsymbol{a} \;=\; \overline{\delta}_{p-\Delta}\left(z,\sigma\right)\overline{\delta}_{\Delta-p-2}(z,\lambda)\frac{\mathcal{C}\left(\Delta\right)\mathcal{C}\left(2-\Delta\right)}{[\bar{z}\omega]^{\Delta}[\mu\bar{z}]^{2-\Delta}}\exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\left(\psi-\frac{\langle\lambda\iota\rangle}{\langle\sigma\iota\rangle}\chi\right)\cdot\eta\right)\,D^{2|4}\mathsf{W} \tag{78}$$

Finally, integrating over $\mathbf{MT}^*_s$ yields:

$$\int\limits_{\mathbf{MT}^*_s} D^{2|4}\mathsf{W}\;\;\Psi^{-p}_{2-\Delta}\big(\lambda^A,\mu_{\dot{A}},\psi^\alpha;z^A,\bar{z}_{\dot{A}},\eta^\alpha\big)\,\widetilde{\Psi}^p_\Delta\big(\sigma^A,\omega_{\dot{A}},\chi^\alpha;z^A,\bar{z}_{\dot{A}},\eta^\alpha\big) \tag{79}$$

$$=\;\overline{\delta}_{p-\Delta}\left(\lambda,\sigma\right)\,\delta^{0|4}\left(\psi-\frac{\langle\lambda\iota\rangle}{\langle\sigma\iota\rangle}\chi\right)\;\int\limits_{\mathbf{CP}^1} D\bar{z}\;\;\frac{\mathcal{C}\left(\Delta\right)\mathcal{C}\left(2-\Delta\right)}{[\bar{z}\omega]^{\Delta}[\mu\bar{z}]^{2-\Delta}}. \tag{80}$$

We proceed by considering the integral:

$$\mathcal{I}\big(\mu_{\dot{A}},\omega_{\dot{B}}\big) \;\coloneqq\; \int\limits_{\mathbf{CP}^1} D\bar{z}\;\;\frac{\mathcal{C}\left(\Delta\right)\mathcal{C}\left(2-\Delta\right)}{[\bar{z}\omega]^{\Delta}[\mu\bar{z}]^{2-\Delta}}. \tag{81}$$

This expression is well-defined only in a distributional sense. Indeed, if one assumes that $\mathcal{I}$ admits an analytic form, then:

$$\mathcal{I}\big(t_1\cdot\mu_{\dot{A}},t_2\cdot\omega_{\dot{B}}\big) \;=\; t_1^{\Delta-2}\,t_2^{-\Delta}\,\mathcal{I}\big(\mu_{\dot{A}},\omega_{\dot{B}}\big), \qquad \forall\, t_1,t_2\in\mathbf{C}^*. \tag{82}$$

Lorentz invariance, by contrast, requires $\mathcal{I}$ to scale as a power of $[\mu\omega]$. These two requirements are incompatible unless the proportionality factors vanishes or diverges. This is precisely the behaviour of the projective delta function $\overline{\delta}_\Delta$.

In Sharma [13] this is confirmed by explicit integration:

$$\int_{\mathbf{CP}^1} D\bar{z}\; \frac{\mathcal{C}\left(\Delta\right)\mathcal{C}\left(2-\Delta\right)}{[\bar{z}\omega]^{\Delta}[\mu\bar{z}]^{2-\Delta}} \;=\; 4\pi^2\bar{\delta}_{\Delta}\left(\mu,\omega\right). \tag{83}$$

Substitution into Eq. (80) then yields:

$$\int_{\mathbf{MT}_s^*} D^{2|4}\mathsf{W}\;\; \Psi_{2-\Delta}^{-p}\left(\lambda,\mu,\psi;z,\bar{z},\eta\right)\; \widetilde{\Psi}_{\Delta}^{p}\left(\sigma,\omega,\chi;z,\bar{z},\eta\right) \tag{84}$$

$$=\; 4\pi^2\,\bar{\delta}_{p-\Delta}\left(\lambda,\sigma\right)\,\bar{\delta}_{\Delta}\left(\mu,\omega\right)\,\delta^{0|4}\left(\psi^{\alpha}-\frac{\langle\lambda\iota\rangle}{\langle\sigma\iota\rangle}\chi^{\alpha}\right). \tag{85}$$

This establishes the completeness relation for minitwistor wavefunctions.

Minitwistor Delta Function. Equation (84) is rather involved. We seek a concise reformulation that makes its homogeneity properties manifest. To this end, we extend the projective delta function $\bar{\delta}_{\Delta}$ on $\mathbf{CP}^1$ to the minitwistor superspace $\mathbf{MT}_s$. This extension should preserve covariance under the transformation law of Eq. (12).

Define the *minitwistor delta function* with homogeneities Δ_1,Δ_2 by:

$$\bar{\delta}^{2|4}_{\Delta_1,\Delta_2}\big(\mathsf{Z}^I;\mathsf{Z}'^J\big) \;=\; \int_{\mathbf{C}^*}\frac{dt_1}{t_1}\,t_1^{\Delta_1}\int_{\mathbf{C}^*}\frac{dt_2}{t_2}\,t_2^{\Delta_2}\;\bar{\delta}^2\big(\lambda^A-t_1\sigma^A\big)\,\bar{\delta}^2\big(\mu_{\dot{A}}-t_2\omega_{\dot{A}}\big)\,\delta^{0|4}\big(\psi^{\alpha}-t_1\chi^{\alpha}\big), \tag{86}$$

where:

$$\mathsf{Z}^I \;\coloneqq\; \big(\lambda^A,\mu_{\dot{A}},\psi^{\alpha}\big),\;\; \mathsf{Z}'^I \;\coloneqq\; \big(\sigma^A,\omega_{\dot{A}},\chi^{\alpha}\big)\in\mathbf{MT}_s.$$

Now let ι^A be an auxiliary non-vanishing spinor. Using the fundamental solution of the Dolbeault operator $\bar{\partial}$ on $\mathbf{CP}^1$, we find:

$$\bar{\delta}^2\big(\lambda^A-t_1\sigma^A\big) \;=\; \frac{1}{(2\pi i)^2}\bigwedge_{A\in\{1,2\}}\bar{\partial}\;\frac{1}{\lambda^A-t_1\sigma^A} \;=\; \bar{\delta}\left(t_1-\frac{\langle\lambda\iota\rangle}{\langle\sigma\iota\rangle}\right)\bar{\delta}\left(\langle\sigma\lambda\rangle\right), \tag{87}$$

$$\bar{\delta}^2\big(\mu_{\dot{A}}-t_2\omega_{\dot{A}}\big) \;=\; \frac{1}{(2\pi i)^2}\bigwedge_{\dot{A}\in\{\dot{1},\dot{2}\}}\bar{\partial}\;\frac{1}{\mu_{\dot{A}}-t_2\omega_{\dot{A}}} \;=\; \bar{\delta}\left(t_2-\frac{[\mu\bar{\iota}]}{[\omega\bar{\iota}]}\right)\bar{\delta}\left([\omega\mu]\right). \tag{88}$$

By substituting into Eq. (86) and invoking the definition of $\bar{\delta}_{\Delta}$ from Eq. (5), we obtain:

$$\bar{\delta}^{2|4}_{\Delta_1,\Delta_2}\big(\mathsf{Z};\mathsf{Z}'\big) \;=\; \bar{\delta}_{\Delta_1}\big(\lambda^A,\sigma^A\big)\,\bar{\delta}_{\Delta_2}\big(\mu_{\dot{A}},\omega_{\dot{A}}\big)\,\delta^{0|4}\left(\psi^{\alpha}-\frac{\langle\lambda\iota\rangle}{\langle\sigma\iota\rangle}\chi^{\alpha}\right). \tag{89}$$

Canonical Form. Using the minitwistor delta function, the completeness relation in Eq. (84) can be written as:

$$\int\limits_{\mathbf{MT}_s^*} D^{2|4}\mathsf{W}\;\; \Psi_{2-\Delta}^{-p}\left(\mathsf{Z};\mathsf{W}\right)\, \widetilde{\Psi}_{\Delta}^{p}\left(\mathsf{Z}';\mathsf{W}\right) \;=\; 4\pi^2\, \overline{\delta}^{2|4}_{p-\Delta,\Delta}\left(\mathsf{Z};\mathsf{Z}'\right). \tag{90}$$

We now adopt a simple convention for the conjugate wavefunction. Define:

$$\Psi_{\Delta}^{p}\left(\mathsf{W};\mathsf{Z}\right) \;\coloneqq\; \widetilde{\Psi}_{\Delta}^{p}\left(\mathsf{Z};\mathsf{W}\right). \tag{91}$$

With this definition, Eq. (90) takes the canonical form:

$$\int\limits_{\mathbf{MT}_s^*} D^{2|4}\mathsf{W}\;\; \Psi_{2-\Delta}^{-p}\left(\mathsf{Z};\mathsf{W}\right)\, \Psi_{\Delta}^{p}\left(\mathsf{W};\mathsf{Z}'\right) \;=\; 4\pi^2\, \overline{\delta}^{2|4}_{p-\Delta,\Delta}\left(\mathsf{Z};\mathsf{Z}'\right). \tag{92}$$

2. *Orthogonality*

To prove that the minitwistor transform $\mathcal{MT}$ is invertible, we derive an orthogonality relation for the family $\{\Psi_{\Delta}^{p}\}$ of minitwistor wavefunctions. Whereas completeness (Eq. (92)) follows from integrating a differential form over the dual superspace $\mathbf{MT}_s^*$, orthogonality is obtained by integrating over the superspace $\mathbf{MT}_s$.

First, define the $\mathbf{Z}_2$-graded differential form on $\mathbf{MT}_s$,

$$\boldsymbol{b} \;\coloneqq\; b\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big)\;\; D^{2|4}\mathsf{Z}, \tag{93}$$

where:

$$b\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big) \;\coloneqq\; \widetilde{\Psi}_{\tilde{\Delta}}^{\tilde{p}}\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big)\, \Psi_{\Delta}^{p}\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z'^A, \bar{z}'_{\dot{A}}, \eta'^\alpha\big). \tag{94}$$

The integral

$$\int\limits_{\mathbf{MT}_s} \boldsymbol{b}$$

is well-defined only if $\boldsymbol{b}$ is invariant under the transformations of Eq. (12).

Equation (10) implies that $\boldsymbol{b}$ is a volume form on $\mathbf{MT}_s$ when:

$$\Delta + \tilde{\Delta} = 2, \qquad p + \tilde{p} = 0. \tag{95}$$

Accordingly, we set:

$$\boldsymbol{b} \;=\; \widetilde{\Psi}_{2-\Delta}^{-p}\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big)\, \Psi_{\Delta}^{p}\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z'^A, \bar{z}'_{\dot{A}}, \eta'^\alpha\big)\;\; D^{2|4}\mathsf{Z}. \tag{96}$$

Next, substitute the explicit forms of Ψ^p_Δ and $\widetilde{\Psi}^p_\Delta$ from Eqs. (15) and (16). One obtains:

$$\boldsymbol{b} = \overline{\delta}_{p-\Delta}\left(z',\lambda\right)\overline{\delta}_{\Delta-p-2}\left(z,\lambda\right)\frac{\mathcal{C}\left(\Delta\right)\mathcal{C}\left(2-\Delta\right)}{[\mu\bar{z}']^{\Delta}[\bar{z}\mu]^{2-\Delta}}\exp\left(-i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\psi\cdot\left(\eta-\frac{\langle z'\iota\rangle}{\langle z\iota\rangle}\eta'\right)\right) D^{2|4}\mathsf{Z}. \tag{97}$$

We then integrate over the minitwistor superspace:

$$\int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{Z}\;\widetilde{\Psi}^{-p}_{2-\Delta}\left(\lambda,\mu,\psi;z,\bar{z},\eta\right)\Psi^p_\Delta\left(\lambda,\mu,\psi;z',\bar{z}',\eta'\right) \tag{98}$$

$$= \left(\frac{\langle z\iota\rangle}{\langle z'\iota\rangle}\right)^4\overline{\delta}_{\Delta-p-2}\left(z,z'\right)\delta^{0|4}\left(\eta^\alpha-\frac{\langle z'\iota\rangle}{\langle z\iota\rangle}\eta'^\alpha\right)\int\limits_{\mathbf{CP}^1} D\mu\;\frac{\mathcal{C}\left(\Delta\right)\mathcal{C}\left(2-\Delta\right)}{[\mu\bar{z}']^{\Delta}[\bar{z}\mu]^{2-\Delta}}. \tag{99}$$

Using Eq. (5) for the projective delta function on $\mathbf{CP}^1$, we have:

$$\overline{\delta}_{\Delta-p+2}\left(z,z'\right) = \overline{\delta}\left(\langle z'z\rangle\right)\left(\frac{\langle z\iota\rangle}{\langle z'\iota\rangle}\right)^{(\Delta-p+2)-1} = \left(\frac{\langle z\iota\rangle}{\langle z'\iota\rangle}\right)^4\overline{\delta}_{\Delta-p-2}\left(z,z'\right). \tag{100}$$

Substituting into (99) and using the integral identity of Eq. (83) gives the final result:

$$\int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{Z}\;\widetilde{\Psi}^{-p}_{2-\Delta}\left(\lambda,\mu,\psi;z,\bar{z},\eta\right)\Psi^p_\Delta\left(\lambda,\mu,\psi;z',\bar{z}',\eta'\right) \tag{101}$$

$$= 4\pi^2\,\overline{\delta}_{\Delta-p+2}\left(z,z'\right)\overline{\delta}_{\Delta}\left(\bar{z},\bar{z}'\right)\delta^{0|4}\left(\eta^\alpha-\frac{\langle z'\iota\rangle}{\langle z\iota\rangle}\eta'^\alpha\right). \tag{102}$$

Dual Delta Function. Equation (102) is impractical for explicit calculations. To remedy this, we extend the projective delta function $\overline{\delta}_\Delta$ on $\mathbf{CP}^1$ to a dual-minitwistor delta function on $\mathbf{MT}^*_s$. This new distribution must transform homogeneously under rescalings, in accordance with Eq. (13).

Define the *dual-minitwistor delta function* with homogeneity degrees Δ_1 and Δ_2 by:

$$\widetilde{\delta}^{2|4}_{\Delta_1,\Delta_2}\left(\mathsf{W}^I,\mathsf{W}'^J\right) \coloneqq \int\limits_{\mathbf{C}^*}\frac{dt_1}{t_1}\,t_1^{\Delta_1}\int\limits_{\mathbf{C}^*}\frac{dt_2}{t_2}\,t_2^{\Delta_2}\,\overline{\delta}^2\left(z^A-t_1z'^A\right)\overline{\delta}^2\left(\bar{z}_{\dot{A}}-t_2\bar{z}'_{\dot{A}}\right)\delta^{0|4}\left(\eta^\alpha-t_1^{-1}\eta'^\alpha\right), \tag{103}$$

where:

$$\mathsf{W}^I \coloneqq \left(z^A,\bar{z}_{\dot{A}},\eta^\alpha\right),\;\; \mathsf{W}'^I \coloneqq \left(z'^A,\bar{z}'_{\dot{A}},\eta'^\alpha\right)\in\mathbf{MT}^*_s.$$

Substituting Eqs. (87) and (88) into Eq. (103) yields:

$$\widetilde{\delta}^{2|4}_{\Delta_1,\Delta_2}\left(\mathsf{W};\mathsf{W}'\right) = \overline{\delta}_{\Delta_1}\left(z,z'\right)\overline{\delta}_{\Delta_2}\left(\bar{z},\bar{z}'\right)\delta^{0|4}\left(\eta^\alpha-\frac{\langle z'\iota\rangle}{\langle z\iota\rangle}\eta'^\alpha\right). \tag{104}$$

Finally, using the convention:

$$\Psi^{-p}_{2-\Delta}\left(\mathsf{W};\mathsf{Z}\right) = \widetilde{\Psi}^{-p}_{2-\Delta}\left(\mathsf{Z};\mathsf{W}\right), \tag{105}$$

the orthogonality relation becomes:

$$\int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{Z}\;\Psi^{-p}_{2-\Delta}\left(\mathsf{W};\mathsf{Z}\right)\Psi^p_\Delta\left(\mathsf{Z};\mathsf{W}'\right) = 4\pi^2\,\widetilde{\delta}^{2|4}_{\Delta-p+2,\Delta}\left(\mathsf{W};\mathsf{W}'\right). \tag{106}$$

Having established the completeness and orthogonality of the family $\{\Psi^p_\Delta\}$, we now introduce the minitwistor transform $\mathcal{MT}$ and its inverse. Unlike the ordinary Fourier transform, $\mathcal{MT}$ sends sections of holomorphic vector bundles over the minitwistor superspace $\mathbf{MT}_s$ to sections over the dual superspace $\mathbf{MT}_s^*$.

Preliminaries. Let

$$\varphi \coloneqq \varphi\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big) \tag{107}$$

be a holomorphic section of the bundle

$$\mathcal{O}\,(w_1) \oplus \mathcal{O}\,(w_2) \longrightarrow \mathbf{MT}_s. \tag{108}$$

Define the $\mathbf{Z}_2$-graded differential form on $\mathbf{MT}_s$:

$$\boldsymbol{c} \coloneqq \varphi\big(\lambda^A, \mu_{\dot{A}}, \eta^\alpha\big)\, \widetilde{\Psi}^p_\Delta\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big)\;\; D^{2|4}\mathsf{Z}. \tag{109}$$

For the integral

$$\int\limits_{\mathbf{MT}_s} \boldsymbol{c}$$

to be well-defined, $\boldsymbol{c}$ must be invariant under the rescalings of Eq. (12).

Equations (10) and (102) show that $\boldsymbol{c}$ is a volume form precisely when:

$$w_1 = p - \Delta + 2, \qquad w_2 = \Delta - 2. \tag{110}$$

Hence we take[3]:

$$\varphi \in \Gamma\big(\mathcal{O}(p - \Delta + 2) \oplus \mathcal{O}(\Delta - 2); \mathbf{MT}_s\big). \tag{111}$$

We define the *minitwistor transform* of a section φ :

$$\Phi\left(\mathsf{W}^I\right) \coloneqq \mathcal{MT}\big[\varphi(\mathsf{Z}^J)\big](\mathsf{W}^I), \tag{112}$$

where:

$$\Phi\left(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\right) \coloneqq \int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{Z}\;\; \varphi\left(\lambda^A, \mu_{\dot{A}}, \eta^\alpha\right)\, \widetilde{\Psi}^p_\Delta\big(\lambda^A, \mu_{\dot{A}}, \eta^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big). \tag{113}$$

[3] Let $\pi : E \longrightarrow B$ be a holomorphic vector superbundle over the base B. We denote by $\Gamma(E; B)$ the $\mathcal{O}(B)$-module of holomorphic sections of E.

From the homogeneity laws in Eqs. (11) and (13), one finds:

$$\Phi\big(t_1 \cdot z^A, t_2 \cdot \bar{z}_{\dot{A}}, t_1^{-1} \cdot \eta^\alpha\big) \;=\; t_1^{p-\Delta-2}\, t_2^{-\Delta}\, \Phi\big(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big), \qquad \forall\, t_1, t_2 \in \mathbf{C}^*. \tag{114}$$

Thus Φ is a section of:

$$\mathcal{O}\,(p-\Delta-2) \oplus \mathcal{O}\,(-\Delta) \longrightarrow \mathbf{MT}_s^*. \tag{115}$$

Accordingly, the transform acts as:

$$\mathcal{MT} : \Gamma\big(\mathcal{O}(p-\Delta+2) \oplus \mathcal{O}(\Delta-2); \mathbf{MT}_s\big) \longrightarrow \Gamma\big(\mathcal{O}\,(p-\Delta-2) \oplus \mathcal{O}\,(-\Delta)\,; \mathbf{MT}_s^*\big). \tag{116}$$

Inversion Theorem. We now derive the inversion of the minitwistor transform using the completeness relation of Subsec. II D 1 and Fubini's theorem.

Define the superform on $\mathbf{MT}_s^*$:

$$\boldsymbol{d} \;\coloneqq\; \Psi^{\tilde{p}}_{\tilde{\Delta}}\big(\lambda'^A, \mu'_{\dot{A}}, \psi'^\alpha; z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big)\, \Phi\big(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big) \;\; D^{2|4}\mathsf{W}. \tag{117}$$

Equations (10) and (114) imply that $\boldsymbol{d}$ is a volume form when:

$$\tilde{p} = -p, \qquad \tilde{\Delta} = 2 - \Delta. \tag{118}$$

Accordingly, we define the *inverse transform* $\mathcal{MT}^{-1}$ by:

$$\mathcal{MT}^{-1}[\Phi]\big(\lambda'^A, \mu'_{\dot{A}}, \psi'^\alpha\big) \;\coloneqq\; \frac{1}{4\pi^2} \int\limits_{\mathbf{MT}_s^*} D^{2|4}\mathsf{W} \;\; \Psi^{-p}_{2-\Delta}\big(\lambda'^A, \mu'_{\dot{A}}, \psi'^\alpha | z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big)\, \Phi\big(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big). \tag{119}$$

Substitute Φ from Eq. (113) and apply Fubini's theorem. One finds:

$$\mathcal{MT}^{-1}[\Phi]\,\big(\lambda', \mu', \psi'\big) \;=\; \frac{1}{4\pi^2} \int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{Z} \;\; \varphi\,(\lambda, \mu, \eta) \tag{120}$$

$$\int\limits_{\mathbf{MT}_s^*} D^{2|4}\mathsf{W} \;\; \Psi^{-p}_{2-\Delta}\,\big(\lambda', \mu', \psi'; z, \bar{z}, \eta\big)\;\; \widetilde{\Psi}^{p}_{\Delta}\,(\lambda, \mu, \eta; z, \bar{z}, \eta)\,. \tag{121}$$

Invoking the completeness relation (Eq. (92)) yields:

$$\mathcal{MT}^{-1}[\Phi]\,\big(\lambda', \mu', \psi'\big) \;=\; \varphi\,\big(\lambda', \mu', \psi'\big)\,. \tag{122}$$

Hence the inversion formula is:

$$\varphi\big(\lambda^A, \mu_{\dot{A}}, \eta^\alpha\big) \;=\; \frac{1}{4\pi^2} \int\limits_{\mathbf{MT}_s^*} D^{2|4}\mathsf{W} \;\; \Psi^{-p}_{2-\Delta}\,(\lambda, \mu, \psi; z, \bar{z}, \eta)\;\, \Phi\,(z, \bar{z}, \eta)\,. \tag{123}$$

Finally, if one takes this as the defining relation for φ given Φ and applies orthogonality, then Eq. (119) follows from Eq. (123). We summarise our results below.

Summary. Let

$$\mathsf{Z}^I := \left(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\right), \qquad \mathsf{W}^{I'} := \left(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\right)$$

parametrise the minitwistor superspace $\mathbf{MT}_s$ and its dual $\mathbf{MT}_s^*$, respectively. Denote by $D^{2|4}\mathsf{Z}$ and $D^{2|4}\mathsf{W}$ the corresponding measures.

Let

$$\varphi \in \Gamma\big(\mathcal{O}\left(p - \Delta + 2\right) \oplus \mathcal{O}\left(\Delta - 2\right); \mathbf{MT}_s\big)$$

and let

$$\Phi \in \Gamma\big(\mathcal{O}\left(p - \Delta - 2\right) \oplus \mathcal{O}\left(-\Delta\right); \mathbf{MT}_s^*\big).$$

Then φ and Φ are related by the integral transform:

$$\Phi\left(\mathsf{W}\right) = \mathcal{MT}\big[\varphi\left(\mathsf{Z}\right)\big]\left(\mathsf{W}\right) = \int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{Z}\ \varphi\left(\mathsf{Z}\right)\ \widetilde{\Psi}^p_\Delta\left(\mathsf{Z};\mathsf{W}\right), \tag{124}$$

if and only if its inverse holds:

$$\varphi\left(\mathsf{Z}\right) = \mathcal{MT}^{-1}\big[\Phi\left(\mathsf{W}\right)\big]\left(\mathsf{Z}\right) = \int\limits_{\mathbf{MT}_s^*} D^{2|4}\mathsf{W}\ \Psi^{-p}_{2-\Delta}\left(\mathsf{Z};\mathsf{W}\right)\ \Phi\left(\mathsf{W}\right). \tag{125}$$

E. Celestial BMSW Identity

One of the principal results of this work is the geometric and dynamical reformulations of the celestial leaf amplitudes for gluons in $\mathcal{N} = 4$ SYM theory.

In the geometric formulation, we will express the celestial amplitudes as expectation values of holomorphic Wilson loops on minitwistor superspace $\mathbf{MT}_s$. These Wilson loops are supported on a family of nodal minitwistor lines. Expanding the path-ordered exponential that defines the holonomy of the pseudoholomorphic connection will produce the n-fold Penrose transform of the minitwistor superwavefunctions.

In the dynamical formulation, we will realise the celestial amplitudes as semiclassical expectation values of correlators in the minitwistor sigma-model. These correlators will be encoded by the Quillen determinant of a gauge potential on $\mathbf{MT}_s$. Their evaluation again reduces to an n-fold Penrose transform.

A key ingredient in both reformulations is the supersymmetric generalisation of the celestial Boels-Mason-Skinner-Witten (BMSW) identity. The bosonic version was derived in Mol (2025).

This identity states that the n-fold Penrose transform of the minitwistor superwavefunctions Ψ^p_Δ coincides with the integral kernel of the Mellin-transformed Parke-Taylor factors appearing in the gluon amplitudes. We shall now present its derivation.

1. *Preliminaries*

The central result of this subsection rests on the minitwistor Penrose transform[4]. We therefore recall the geometric structures on which it is defined.

The supersymmetric extension of the Hitchin correspondence[5] establishes a bijection between points of the minitwistor superspace $\mathbf{MT}_s$ and totally geodesic null hypersurfaces in an Einstein-Weyl supermanifold[6] $\mathbf{H}_s$. Conversely, each point of $\mathbf{H}_s$ corresponds to a distinguished curve in $\mathbf{MT}_s$, known as a *minitwistor line.*

In our case, $\mathbf{MT}_s$ is the $\mathcal{N} = 4$ supersymmetric extension of an open subset of the quadric $\mathbf{CP}^1 \times \mathbf{CP}^1$. One finds that $\mathbf{H}_s$ is then the complexification of the $(3|8)$-dimensional anti-de Sitter superspace[7]. The precise interplay among $\mathbf{MT}_s$, $\mathbf{H}_s$ and the projective spinor superbundle $\mathbf{P}(\mathcal{S})$ will emerge in our definition of the double fibration below. Before that, however, we review the projective model for the hyperbolic supergeometry of $\mathbf{H}_s$.

Projective Model of Hyperbolic Space. We give a concise construction of the three-dimensional hyperbolic model in $\mathbf{CP}^3$. For the n-dimensional case, see Bailey and Dunne [24].

Let $X_{A\dot{A}}$ be homogeneous coordinates on $\mathbf{CP}^3$. In abstract index notation, the statement $X_{A\dot{A}} \in \mathbf{CP}^3$ is to be interpreted as the equivalence class $[X_{A\dot{A}}]$ in $\mathbf{CP}^3$.

Define the bilinear form and its associated norm by

$$(X, Y) \;:=\; \varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}X^{A\dot{A}}Y^{B\dot{B}}, \quad \|X\|^2 \;:=\; -(X, X), \quad \forall\, X_{A\dot{A}}, Y_{A\dot{A}} \in \mathbf{CP}^3. \tag{126}$$

Let $\mathscr{C}$ denote the complexified null cone,

$$\mathscr{C} \;:=\; \big\{\, X_{A\dot{A}} \in \mathbf{CP}^3 \,\big|\, \|X\| = 0 \,\big\}. \tag{127}$$

The complex hyperbolic space is then the open submanifold $\mathbf{H} := \mathbf{CP}^3 \backslash \mathscr{C}$.

We define a metric tensor $\boldsymbol{g}_{A\dot{A}B\dot{B}}$ on the hyperbolic space $\mathbf{H}$ by requiring that its line element in the dual coordinate basis $\{dX^{A\dot{A}}\}$ takes the form

$$ds^2 \;:=\; \boldsymbol{g}_{A\dot{A}B\dot{B}}\, dX^{A\dot{A}} dX^{B\dot{B}} \;=\; -\frac{1}{\|X\|^2}\left(\|dX\|^2 - \frac{(X, dX)^2}{\|X\|^2}\right). \tag{128}$$

---

[4] Jones [14], Jones and Tod [15].
[5] Hitchin [16, 17].
[6] Leites, Poletaeva, and Serganova [18], DeWitt [19], Rogers [20], Leites [21], Manin [22].
[7] Koning, Kuzenko, and Raptakis [23].

By construction, this metric is invariant under overall rescaling of $X_{A\dot{A}}$, and has no component along the radial (scale) direction.

To formalise these properties, let $\xi \coloneqq X^{A\dot{A}}\nabla_{A\dot{A}}$ be the Euler vector field on $\mathbf{CP}^3$, where

$$\nabla_{A\dot{A}} \ \coloneqq\ \frac{\partial}{\partial X^{A\dot{A}}}. \tag{129}$$

The metric then satisfies:

$$\mathscr{L}_\xi\, \boldsymbol{g}_{A\dot{A}B\dot{B}} \ =\ 0, \quad \xi^{A\dot{A}}\boldsymbol{g}_{A\dot{A}B\dot{B}} \ =\ 0, \tag{130}$$

where $\mathscr{L}_\xi$ denotes the Lie derivative along ξ.

The natural orientation on $\mathbf{H}$ is specified by the $\mathcal{O}_{\mathbf{C}}(4)$-valued differential 3-form:

$$D^3X \ \coloneqq\ \varepsilon_{[AB}\varepsilon_{CD]}\varepsilon_{[\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}]}\ X^{A\dot{A}}\, dX^{B\dot{B}} \wedge dX^{C\dot{C}} \wedge dX^{D\dot{D}}. \tag{131}$$

We assert that the triple $\big(\mathbf{H}, \mathbf{g}_{A\dot{A}B\dot{B}}, D^3X\big)$ realises three-dimensional hyperbolic space. To see this, define the weightless coordinate function:

$$\mathcal{R}_{A\dot{A}} : \mathbf{H} \longrightarrow \mathbf{C}^4, \quad \mathcal{R}_{A\dot{A}} \coloneqq \frac{X_{A\dot{A}}}{\|X\|}. \tag{132}$$

A direct computation shows

$$ds^2 \ =\ \varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\, d\mathcal{R}^{A\dot{A}} d\mathcal{R}^{B\dot{B}} \quad \text{and} \quad \|\mathcal{R}\|^2 \coloneqq -\mathcal{R}_{A\dot{A}}\mathcal{R}^{A\dot{A}} = 1. \tag{133}$$

Thus the map $e\colon X_{A\dot{A}} \mapsto \mathcal{R}_{A\dot{A}}$ embeds $\mathbf{H}$ isometrically onto the hyperboloid:

$$H_3 \ \coloneqq\ \big\{\, \mathcal{R}_{A\dot{A}} \in \mathbf{C}^4 \,\big|\, \|\mathcal{R}\|^2 = 1 \big\}, \tag{134}$$

the standard model of three-dimensional hyperbolic space.

Finally, one checks that the pullback (via e^*) of the standard volume form on H_3 lies in the same orientation class as D^3X. This completes the identification of $\mathbf{H}$ with the classical hyperbolic geometry of H_3.

To formulate $\mathcal{N} = 4$ SYM theory, we must employ the supersymmetric Hitchin correspondence. In the forthcoming discussion of the double fibration on the projective spinor bundle $\mathbf{P}(\mathcal{S})$, we will introduce the minitwistor incidence relations. These relations identify the *minitwistor lines* in $\mathbf{MT}_s$ as distinguished curves. It then follows that *the moduli space of these lines in the* $(2|4)$*-dimensional minitwistor superspace is diffeomorphic to the* $(3|8)$*-dimensional hyperbolic superspace.*

Hyperbolic Superspace. Before presenting the supersymmetric Hitchin correspondence in detail, we extend our hyperbolic model $\mathbf{H}$ by adjoining fermionic directions. The appropriate mathematical framework is that of a vector superbundle, as discussed by Manin [22] and Rogers [20].

We define the $(3|8)$-*dimensional hyperbolic superspace* as the trivial vector superbundle $\mathbf{H}_s := \mathbf{H} \times \mathbf{C}^{0|8}$. Its fibre is the vector superspace $\mathbf{C}^{0|8}$ spanned by the Grassmann-valued van der Waerden spinors θ^{α}_{A}. Each fibre carries the orientation provided by Berezin's measure $d^{0|8}\theta$.

A global trivialisation is provided by the superchart

$$\mathsf{X}^K \colon \mathbf{H}_s \longrightarrow \mathbf{CP}^3 \times \mathbf{C}^{0|8}, \quad \mathsf{X}^K \; := \; \big(X_{A\dot{A}}, \theta^{\alpha}_{A}\big), \tag{135}$$

where K indexes both bosonic and fermionic dimensions.

The canonical orientation measure on $\mathbf{H}_s$ is

$$D^{3|8}\mathsf{X} \; := \; \frac{D^3 X}{\|X\|^4} \wedge d^{0|8}\theta. \tag{136}$$

By defining $\mathbf{H}_s$ as a trivial superbundle, we have imposed projective invariance solely along the bosonic (horizontal) directions. Therefore, under the scale transformation $X_{A\dot{A}}, \theta^{\alpha}_{A} \mapsto t \cdot X_{A\dot{A}}, \theta^{\alpha}_{A}$ the measure $D^{3|8}\mathsf{X}$ remains invariant.

Double Fibration. The Penrose transform is most naturally formulated via a double fibration. We now define the fibration that realises the supersymmetric Hitchin correspondence for the minitwistor superspace.

Let $\mathbf{P}(\mathcal{S}) := \mathbf{CP}^{3|8} \times \mathbf{CP}^1$ be the complex projective spinor superbundle over $\mathbf{CP}^{3|8} := \mathbf{CP}^3 \times \mathbf{C}^{0|8}$. Define the open submanifold

$$\mathbf{P}'(\mathcal{S}) \; := \; \big\{ \big(\mathsf{X}^K, [\lambda^A]\big) \in \mathbf{P}(\mathcal{S}) \,\big|\, \|X\| \neq 0 \big\}. \tag{137}$$

We have two projections from $\mathbf{P}'(\mathcal{S})$:

$$\tau \colon \mathbf{P}'(\mathcal{S}) \longrightarrow \mathbf{H}_s, \quad \tau\big(\mathsf{X}^K, [\lambda^A]\big) := \big(\mathsf{X}^K\big), \tag{138}$$

and

$$\upsilon \colon \mathbf{P}'(\mathcal{S}) \longrightarrow \mathbf{MT}_s \quad \upsilon\big(\mathsf{X}^K, [\lambda^A]\big) := \pi_0\big(\lambda^A, \lambda^A X_{A\dot{A}}, \lambda^A \theta^{\alpha}_{A}\big). \tag{139}$$

The product map

$$\tau \times \upsilon \colon \mathbf{P}'(\mathcal{S}) \longrightarrow \mathbf{H}_s \times \mathbf{MT}_s \tag{140}$$

is an embedding, and we denote its image by $\widetilde{\mathbf{P}'(\mathcal{S})} \subset \mathbf{H}_s \times \mathbf{MT}_s$.

Fix a point $\mathsf{X}^K \in \mathbf{H}_s$. Its τ-fibre is $\mathscr{F}_{\mathsf{X}^K} := \tau^{-1}(\mathsf{X}^K)$. Under the embedding $\tau \times \upsilon$, $\mathscr{F}_{\mathsf{X}^K}$ maps to a submanifold of $\mathbf{MT}_s$. Thus we obtain a family $\{\mathscr{F}_{\mathsf{X}^K}\}_{\mathsf{X}^K \in \mathbf{H}_s}$ of submanifolds in $\mathbf{MT}_s$ parametrised by $\mathbf{H}_s$. Each $\mathscr{F}_{\mathsf{X}^K}$ is precisely the *minitwistor line* corresponding to X^K.

Similarly, fix a minitwistor point $\mathsf{w} \in \mathbf{MT}_s$. Its υ-fibre is $\mathscr{G}_{\mathsf{w}} := \upsilon^{-1}(\mathsf{w})$. Under $\tau \times \upsilon$, $\mathscr{G}_{\mathsf{w}}$ embeds as a submanifold of $\mathbf{H}_s$. Hence there is a family $\{\mathscr{G}_{\mathsf{w}}\}_{\mathsf{w} \in \mathbf{MT}_s}$ of submanifolds in $\mathbf{H}_s$ parametrised by $\mathbf{MT}_s$. In Hitchin's correspondence, $\mathscr{G}_{\mathsf{w}}$ is the totally geodesic null hypersurface associated to w.

Therefore, the correspondence is summarised by the double fibration:

$$\begin{tikzcd} & \mathbf{P}'(\mathcal{S}) \arrow[dl, "\tau"'] \arrow[dr, "\upsilon"] & \\ \mathbf{H}_s & & \mathbf{MT}_s \end{tikzcd} \tag{141}$$

The families of fibres $\{\mathscr{F}_{\mathsf{X}^K}\}$ and $\{\mathscr{G}_{\mathsf{w}}\}$ are related by the *incidence relation*:

$$\big(\mathsf{X}^K, \mathsf{w}\big) \in \widetilde{\mathbf{P}'(\mathcal{S})} \quad \Longleftrightarrow \quad \mathsf{w} \in \mathscr{F}(\mathsf{X}^K) \quad \Longleftrightarrow \quad \mathsf{X}^K \in \mathscr{G}(\mathsf{w}). \tag{142}$$

To describe this explicitly, choose a representative $\mathsf{W}^I = (\lambda^A, \mu_{\dot{A}}, \psi^\alpha) \in (\pi_0)^{-1}(\mathsf{w})$ and write $\mathsf{X}^K = (X_{A\dot{A}}, \theta^\alpha_A)$. Then

$$\big(\mathsf{X}^K, [\mathsf{W}^I]\big) \in \widetilde{\mathbf{P}'(\mathcal{S})} \quad \Longleftrightarrow \quad \mu_{\dot{A}} = \lambda^A X_{A\dot{A}},\ \psi^\alpha = \lambda^A \theta^\alpha_A. \tag{143}$$

These conditions depend only on the projective class $[\mathsf{W}^I]$.

Minitwistor Lines. The planar Yang-Mills amplitudes on minitwistor space localise precisely on certain rational curves called minitwistor lines. Similarly, the worldsheet of our minitwistor sigma-model embeds into $\mathbf{MT}_s$ as such a line. We now characterise these special curves.

Following Hitchin [17] and Jones [14], a *minitwistor line* is defined to be a rational curve whose normal bundle is isomorphic to $\mathcal{O}(2)$. The hyperbolic superspace $\mathbf{H}_s$ parametrises all such lines. To each point $\mathsf{x} \in \mathbf{H}_s$ we associate a unique minitwistor line $\mathcal{L}_{\mathsf{x}} \subset \mathbf{MT}_s$. Denote its normal bundle by $\mathrm{Nor}\,(\mathcal{L}_{\mathsf{x}})$. Kodaira's theorem[8] then identifies the tangent space of $\mathbf{H}_s$ at x with the space of global sections of $\mathrm{Nor}\,(\mathcal{L}_{\mathsf{x}})$:

$$T_{\mathsf{x}}(\mathbf{H}_s) \;\simeq\; \Gamma\big(\mathcal{L}(X,\theta); \mathrm{Nor}\,(\mathcal{L}_{\mathsf{x}})\big). \tag{144}$$

[8] Kodaira [25].

Let $\mathsf{X}^K = (X_{A\dot{A}}, \theta^\alpha_{\dot{A}})$ be a coordinate representative of a point $\mathsf{x} \in \mathbf{H}_s$. We define its associated minitwistor line by

$$\mathcal{L}(X,\theta) \coloneqq \upsilon\big(\mathscr{F}_{\mathsf{X}^K}\big) = \big\{\, \pi_0\big(\lambda^A, \lambda^A X_{A\dot{A}}, \lambda^A \theta^\alpha_{\dot{A}}\big) \,\big|\, [\lambda^A] \in \mathbf{CP}^1 \,\big\}. \tag{145}$$

By construction, $\mathcal{L}(X,\theta)$ is a rational curve in $\mathbf{MT}_s$. We now check that its normal bundle is $\mathcal{O}(2)$, in accordance with the Hitchin-Jones definition.

First, project onto the bosonic component via

$$p_b : \mathbf{MT}_s \longrightarrow \mathbf{MT} \subset \mathbf{CP}^1 \times \mathbf{CP}^1, \quad p_b(\mathsf{w}) \;\coloneqq\; \big([\lambda^A], [\mu_{\dot{A}}]\big). \tag{146}$$

The image of $\mathcal{L}(X,\theta)$ under p_b is the bosonic minitwistor line $L(X) \coloneqq p_b\big(\mathcal{L}(X,\theta)\big)$.

Consider the Veronese-type embedding

$$V\colon \mathbf{MT} \longrightarrow \mathbf{CP}^3, \quad V\big([\lambda^A], [\mu_{\dot{A}}]\big) \;\coloneqq\; [\lambda^A \mu_{\dot{A}}]. \tag{147}$$

Under V, the curve $L(X)$ becomes a nonsingular conic in $\mathbf{CP}^3$. Any two such conics intersect in precisely two points, which implies that the normal bundle of $L(X)$ is $\mathcal{O}(2)$. It follows that the normal bundle of the full supersymmetric line $\mathcal{L}(X,\theta)$ in $\mathbf{MT}_s$ is also $\mathcal{O}(2)$, in agreement with the Hitchin-Jones definition.

We now demonstrate that $\mathbf{H}_s$ indeed parametrises all minitwistor lines in $\mathbf{MT}_s$. We begin with the bosonic projection. On $\mathbf{CP}^1 \times \mathbf{CP}^1$, the bosonic incidence relation $\mu_{\dot{A}} = \lambda^A X_{A\dot{A}}$ defines the intersection of the quadric $V(\mathbf{MT})$ with a hyperplane in $\mathbf{CP}^3$. A plane section is tangent to $V(\mathbf{MT})$ precisely when $\det(X^{A\dot{A}}) = 0$. If instead the section is non-tangent, then the matrix $X^{A\dot{A}}$ is determined only up to an overall scale.

Let W denote the set of *non-tangent* hyperplane sections of $V(\mathbf{MT})$. Equivalently, W is the space of non-null rays through the origin in complexified Minkowski space $\mathbf{C}^4$. Thus W is diffeomorphic to the projective model $\mathbf{H}$ of complex hyperbolic space. It follows that, upon adjoining the fermionic dimensions, the full moduli superspace of minitwistor lines in $\mathbf{MT}_s$ is precisely $\mathbf{H}_s \simeq \mathbf{H} \times \mathbf{C}^{0|8}$.

The Penrose Integrand. In the minitwistor Penrose transform (Subsection II E 2), one treats the minitwistor line $\mathcal{L}(X,\theta)$ as a fibration over the Riemann sphere $\mathbf{CP}^1$. This formulation simplifies the construction of the top-forms on $\mathcal{L}(X,\theta)$ needed in the Penrose integral formula.

Every point $\mathsf{w} \in \mathcal{L}(X,\theta)$ arises from a unique homogeneous coordinate $[\lambda^A] \in \mathbf{CP}^1$ via $\mathsf{w} = \pi_0(\lambda^A, \lambda^A X_{A\dot{A}}, \lambda^A \theta^\alpha_{\dot{A}})$. Hence there is a natural projection

$$\mathrm{pr}_{\mathcal{L}}\colon \mathcal{L}(X,\theta) \longrightarrow \mathbf{CP}^1, \quad \mathsf{w} \;\longmapsto\; [\lambda^A]. \tag{148}$$

This map realises $\mathcal{L}(X,\theta)$ as a holomorphic fibration over the Riemann sphere. An embedding of the celestial sphere $\mathcal{CS}$ into $\mathcal{L}(X,\theta)$ is then equivalent to a section $s \in \Gamma(\mathbf{CP}^1; \mathcal{L}(X,\theta))$ such that $\mathrm{d}s : T(\mathbf{CP}^1) \to T(\mathcal{L}(X,\theta))$ is an isomorphism.

Fix integers $0 \leq m, n \leq 2$. We define the restriction homomorphism

$$\rho_{\mathcal{L}(X,\theta)}\colon \quad H^{m,n}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(p,q)\big) \;\longrightarrow\; H^{m,n}\big(\mathcal{L}(X,\theta);\, \mathcal{O}_{\mathbf{C}}(p+q)\big) \tag{149}$$

by

$$\rho_{\mathcal{L}(X,\theta)}(\varphi) \coloneqq \varphi|_{\mathcal{L}(X,\theta)} \;\coloneqq\; s^*(\varphi), \tag{150}$$

where $s\colon \mathbf{CP}^1 \to \mathcal{L}(X,\theta)$ is any holomorphic embedding. One checks easily that $\rho_{\mathcal{L}(X,\theta)}$ is independent of the choice of s. Thus $\rho_{\mathcal{L}(X,\theta)}$ carries a Dolbeault class on $\mathbf{MT}_s$ to the corresponding class on the line. Concretely, take the standard parametrisation

$$s\colon \mathbf{CP}^1 \longrightarrow \mathcal{L}(X,\theta), \quad [\lambda^A] \;\mapsto\; \pi_0\big(\lambda^A, \lambda^A X_{A\dot{A}}, \lambda^A \theta^{\alpha}_{A}\big). \tag{151}$$

Then a representative φ restricts as:

$$\varphi|_{\mathcal{L}(X,\theta)}(\lambda^A) \;=\; \varphi(\lambda^A, \lambda^A X_{A\dot{A}}, \lambda^A \theta^{\alpha}_{A}). \tag{152}$$

To construct the Penrose integrand, let φ be a $\mathcal{O}_{\mathbf{C}}(p,q)$-valued differential $(0,1)$-form on $\mathbf{MT}_s$. We wish to build a top-form $\boldsymbol{f}[\varphi] \in \Omega^{1,1}(\mathcal{L}(X,\theta))$ to serve as the integrand in the Penrose formula. Use the homogeneous coordinate $[\lambda^A]$ on each fibre of $\mathcal{L}(X,\theta) \xrightarrow{\mathrm{pr}_{\mathcal{L}}} \mathbf{CP}^1$ to define the holomorphic measure:

$$D\lambda \;\coloneqq\; \varepsilon_{AB}\, \lambda^A d\lambda^B \;\in \Omega^{1,0}\big(\mathcal{L}(X,\theta);\, \mathcal{O}_{\mathbf{C}}(2)\big). \tag{153}$$

Restrict φ to $\mathcal{L}(X,\theta)$ via the restriction homomorphism. Then set:

$$\boldsymbol{f}[\varphi] \;\coloneqq\; D\lambda \wedge \varphi|_{\mathcal{L}(X,\theta)}(\lambda^A). \tag{154}$$

Under the rescaling $\lambda^A \mapsto t\,\lambda^A$ ($t \in \mathbf{C}_*$), one finds $\boldsymbol{f}[\varphi] \mapsto t^{p+q+2}\, \boldsymbol{f}[\varphi]$. Hence $\boldsymbol{f}[\varphi]$ is a volume form precisely when $p+q+2 = 0$.

Finally, define

$$\boldsymbol{f}\colon \Omega^{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big) \;\longrightarrow\; \Omega^{1,1}\big(\mathcal{L}(X,\theta)\big), \quad \varphi \;\longmapsto\; \boldsymbol{f}[\varphi]. \tag{155}$$

This map sends a $\mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)$-valued differential $(0,1)$-form to the Penrose integrand on $\mathcal{L}(X,\theta)$.

2. Minitwistor Correspondence

The minitwistor Penrose transform provides an isomorphism between Dolbeault cohomology classes on the homogeneous bundles over $\mathbf{MT}_s$ and solutions of the covariant wave equation on the hyperbolic superspace $\mathbf{H}_s$. In particular, it encodes bulk-to-boundary propagators on $\mathbf{H}_s$ in terms of cohomology data on $\mathbf{MT}_s$.

Our primary aim in this section is to establish the Penrose machinery on $\mathbf{MT}_s$. This setup leads directly to the celestial BMSW identity. That identity will serve as the bridge between celestial amplitudes and holomorphic Wilson lines on minitwistor superspace. Equivalently, it allows us to generate celestial amplitudes as correlation functions of the minitwistor sigma-model.

As a further application, we employ the minitwistor Penrose transform to construct the celestial superwavefunction $\Phi_\Delta(\mathsf{X}^K;\mathsf{Z}^I)$ for gluons in the spacetime representation. By the minitwistor correspondence, Φ_Δ automatically satisfies the covariant wave equation on the hyperboloid $\mathbf{H}_s$. Importantly, the argument X^K lies in $\mathbf{H}_s$, not in complexified Minkowski superspace $\mathbf{C}^{4|8}$. This reflects the fact that Φ_Δ is defined on the leaves of the hyperbolic foliation of Minkowski superspace used in the leaf amplitude formalism. In other words, Φ_Δ is the *dimensionally reduced* wavefunction.

Definition. The *minitwistor Penrose transform* is the map

$$\mathcal{P}\colon \quad H^{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big) \;\longrightarrow\; \Gamma\big(\mathbf{H}_s;\, \mathcal{O}_{\mathbf{C}}(-\Delta)\big) \tag{156}$$

defined by

$$\mathcal{P}\varphi \;:=\; \int_{\mathcal{L}(X,\theta)} \boldsymbol{f}\,[\varphi]. \tag{157}$$

Here $\boldsymbol{f}[\varphi] \in \Omega^{1,1}(\mathcal{L}(X,\theta))$ is the Penrose integrand introduced in Eq. (154). Equivalently, writing $\mathsf{X}^K = (X_{A\dot{A}}, \theta^\alpha_A)$ and using the explicit form of $\boldsymbol{f}[\varphi]$, one has

$$\mathcal{P}\varphi\left(X_{A\dot{A}}, \theta^\alpha_A\right) \;=\; \int_{\mathcal{L}(X,\theta)} D\lambda \wedge \varphi\,|_{\mathcal{L}(X,\theta)}\,(\lambda^A). \tag{158}$$

Consistency. We now verify consistency of the definition of $\mathcal{P}\varphi$. By construction (cf. end of Subsection II E 1), $\boldsymbol{f}[\varphi]$ is a top-form on $\mathcal{L}(X,\theta)$ that is invariant under the fibre rescaling $\lambda^A \mapsto t\,\lambda^A$. Under the base rescaling $X_{A\dot{A}} \mapsto t\,X_{A\dot{A}}$, one finds $\boldsymbol{f}[\varphi] \mapsto t^{-\Delta}\,\boldsymbol{f}[\varphi]$. This shows

$$\mathcal{P}\varphi\left(t\,X_{A\dot{A}}, \theta^\alpha_A\right) = t^{-\Delta}\,\mathcal{P}\varphi\left(X_{A\dot{A}}, \theta^\alpha_A\right) \tag{159}$$

for all $t \in \mathbf{C}_*$, so $\mathcal{P}\varphi \in \Gamma\big(\mathbf{H}_s;\, \mathcal{O}_{\mathbf{C}}(-\Delta)\big)$.

Next, we check independence of representative. Let φ_1 and φ_2 represent the same class in $H^{0,1}\big(\mathbf{MT}_s;\mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big)$. Then $\varphi_1 = \varphi_2 + \overline{\partial}\Lambda$ for some $\Lambda \in \Omega^{0,0}(\mathbf{MT};\mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta))$. It follows that

$$\mathcal{P}(\varphi_1 - \varphi_2) \;=\; \int_{\mathcal{L}(X,\theta)} D\lambda \wedge \overline{\partial}\Lambda|_{\mathcal{L}(X,\theta)}(\lambda^A) = 0, \tag{160}$$

since $\overline{\partial}\Lambda$ is exact on each fibre. This shows that $\mathcal{P}$ is well-defined on Dolbeault cohomology classes.

Differential Equation. A simple yet important consequence of the definition of $\mathcal{P}$ is the following differential identity. Let $[\varphi] \in H^{0,1}\big(\mathbf{MT}_s;\mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big)$ be any representative. Since φ is holomorphic ($\overline{\partial}\,\varphi = 0$), the chain rule on the restriction to the line $\mathcal{L}(X,\theta)$ gives

$$\nabla_{A\dot{A}}\,\varphi|_{\mathcal{L}(X,\theta)}\,(\lambda^A) \;=\; \lambda_A \left.\frac{\partial\varphi}{\partial\mu^{\dot{A}}}\right|_{\mathcal{L}(X,\theta)}. \tag{161}$$

Moreover, dominated convergence and the mean-value theorem justify exchanging $\nabla_{A\dot{A}}$ with the integral defining the Penrose transform. Hence

$$\nabla_{A\dot{A}}\,\mathcal{P}\varphi \;=\; \int_{\mathcal{L}(X,\theta)} D\lambda \wedge \lambda_A \left.\frac{\partial\varphi}{\partial\mu^{\dot{A}}}\right|_{\mathcal{L}(X,\theta)}. \tag{162}$$

Acting once more with $\nabla_{A\dot{A}}$ then yields the partial differential equation:

$$\nabla^{A\dot{A}}\nabla_{A\dot{A}}\,\mathcal{P}\varphi \;=\; 0. \tag{163}$$

Proper Functions. Our aim is to reformulate Eq. (163) as a covariant wave equation on hyperbolic superspace $\mathbf{H}_s$. To that end, we introduce the following definition. A section Φ of the homogeneous bundle over $\mathbf{H}_s$ is said to define a *proper function* on $\mathbf{H}_s$ iff $\mathscr{L}_\xi\,\Phi = 0$.

The Penrose transform then lifts to a map

$$\mathcal{P}_*\colon \quad H^{0,1}\big(\mathbf{MT}_s;\,\mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big) \;\longrightarrow\; \mathscr{C}^\infty(\mathbf{H}_s) \tag{164}$$

defined by

$$\mathcal{P}_*[\varphi] \;\coloneqq\; \|X\|^{\Delta} \int_{\mathcal{L}(X,\theta)} \boldsymbol{f}[\varphi]. \tag{165}$$

Set $\Phi_\Delta \coloneqq \mathcal{P}_*[\varphi]$. A direct computation shows

$$\nabla^{A\dot{A}}\nabla_{A\dot{A}}\left(\|X\|^{-\Delta}\,\Phi_\Delta(X_{B\dot{B}},\theta^\alpha_C)\right) \;=\; 0. \tag{166}$$

The remaining task is to prove that this equation is equivalent to the eigenvalue problem for the Beltrami-Laplace operator $\Box_{\mathbf{H}}$ on the hyperbolic space $\mathbf{H}$.

A Simple Lemma. The link between Eq. (166) and the spectral theory of the wave operator on hyperbolic space is introduced by the following result. Let

$$\mathcal{J}_{A\dot{A}B\dot{B}} \;\coloneqq\; -i\left(X_{A\dot{A}}\frac{\partial}{\partial X^{B\dot{B}}} - X_{B\dot{B}}\frac{\partial}{\partial X^{A\dot{A}}}\right) \tag{167}$$

be the generator of the Lie algebra of isometries of $\mathbf{H}$. Here $\{\partial/\partial X^{A\dot{A}}\}$ is the coordinate frame. Define the quadratic Casimir operator by:

$$\mathcal{Q} \;\coloneqq\; \frac{1}{2}\mathcal{J}_{A\dot{A}B\dot{B}}\mathcal{J}^{A\dot{A}B\dot{B}}. \tag{168}$$

Then for every proper function $\Phi_\Delta \in \mathscr{C}^\infty(\mathbf{H}_s)$ one has the equivalence:

$$\nabla^{A\dot{A}}\nabla_{A\dot{A}}\left(\|X\|^{-\Delta}\,\Phi_\Delta\right) \;=\; 0 \qquad \Longleftrightarrow \qquad \Box_{\mathbf{H}}\Phi_\Delta \;=\; \Delta(\Delta-2)\Phi_\Delta. \tag{169}$$

To establish this result, we first expand the left-hand side of Eq. (166) as:

$$\|X\|^2\,\nabla^{A\dot{A}}\nabla_{A\dot{A}}\,\Phi_\Delta + 2\Delta\,\mathscr{L}_\xi\Phi_\Delta \;=\; \Delta(\Delta-2)\,\Phi_\Delta. \tag{170}$$

Since Φ_Δ is assumed proper ($\mathscr{L}_\xi\,\Phi_\Delta = 0$), this reduces to:

$$\|X\|^2\,\nabla^{A\dot{A}}\nabla_{A\dot{A}}\Phi_\Delta \;=\; \Delta(\Delta-2)\,\Phi_\Delta. \tag{171}$$

On the other hand, a direct computation of the Casimir operator $\mathcal{Q}$ using Eqs. (167) and (168) gives:

$$\mathcal{Q}\,\Phi_\Delta \;=\; \|X\|^2\,\nabla^{A\dot{A}}\nabla_{A\dot{A}}\,\Phi_\Delta + 2\,X^{A\dot{A}}\nabla_{A\dot{A}}\,\Phi_\Delta + X^{A\dot{A}}\nabla_{A\dot{A}}\big(X^{B\dot{B}}\nabla_{B\dot{B}}\Phi_\Delta\big). \tag{172}$$

It follows that Eq. (171) is equivalent to:

$$\mathcal{Q}\,\Phi_\Delta \;=\; \Delta(\Delta-2)\,\Phi_\Delta. \tag{173}$$

Finally, the lemma follows from the well-known result (e.g. Fronsdal [26]) that on homogeneous spaces $\mathcal{Q} = \Box_{\mathbf{H}}$.

Main Result. The preceding lemma implies that the modified minitwistor Penrose transform $\mathcal{P}_*$ sends $\overline{\partial}$-cohomology classes on $\mathbf{MT}_s$ to solutions of the eigenvalue problem for the Beltrami-Laplace operator on $\mathbf{H}_s$. We now formalise this statement.

Let U be any differentiable manifold and let $\mathcal{T}\colon \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$ be a linear differential operator. We define its kernel by:

$$\ker\left(U;\mathcal{T}\right) \;\coloneqq\; \left\{\,\Phi \in \mathscr{C}^\infty(U)\,\middle|\,\mathcal{T}\Phi = 0\,\right\}. \tag{174}$$

It follows that

$$\mathcal{P}_*\big(H^{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big)\big) \;\subseteq\; \ker\big(\mathbf{H}_s;\, \Box_{\mathbf{H}} - \Delta(\Delta-2)\big). \tag{175}$$

Furthermore, the invertibility of the X-ray transform on projective spaces (Gelfand, Gindikin, and Graev [27]) and a homological argument yield a stronger result[9]:

$$H^{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big) \;\simeq\; \ker\big(\mathbf{H}_s;\, \Box_{\mathbf{H}} - \Delta(\Delta-2)\big). \tag{176}$$

We note that the integral transform in Eq. (157) admits an explicit inversion in the form of a Leray residue formula. This construction builds on Gindikin's analyses of the Radon and John's transforms (Gindikin [28, 29, 30]), the Cauchy-Fantappié formula, and Leray's theory of multidimensional residues.

Celestial Superwavefunction. The simplest non-trivial application of the minitwistor correspondence is the derivation of the celestial superwavefunction $\Phi_\Delta(\mathsf{X}^K;\mathsf{Z}^I)$ from the minitwistor wavefunction $\Psi^p_\Delta(\mathsf{W}^I;\mathsf{Z}^I)$. Here Φ_Δ is a distribution on hyperbolic superspace $\mathbf{H}_s$, and Ψ^p_Δ is a current on minitwistor space $\mathbf{MT}_s$. These are related by a suitable modification of the Penrose transform.

D'Agnolo and Schapira [31] generalised the Penrose transform from its standard double-fibration formulation over flag manifolds to the setting of $\mathscr{D}$-modules. David [32] then applied their theory to currents of differential forms valued in vector bundles. Voronov [33] provided a detailed extension of geometric integration theory to supermanifolds[10]. Using this framework, we lift the minitwistor Penrose transform to the supercurrent category and denote the resulting map by:

$$\mathcal{P}': \quad \mathscr{D}'_{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big) \;\longrightarrow\; \mathscr{D}'(\mathbf{H}_s). \tag{177}$$

Here $\mathscr{D}'(\mathbf{H}_s)$ is the module of distributions on the supermanifold $\mathbf{H}_s$. The covariant wave operator acts on $\mathscr{D}'(\mathbf{H}_s)$ by duality, and we denote its kernel by:

$$\ker'\big(\mathbf{H}_s;\, \Box_{\mathbf{H}} - \Delta(\Delta-2)\big) \;:=\; \big\{\, \omega \in \mathscr{D}'(\mathbf{H}_s) \,\big|\, \Box_{\mathbf{H}}\,\omega \;=\; \Delta(\Delta-2)\,\omega \,\big\}. \tag{178}$$

The minitwistor correspondence in the current-distribution category now reads:

$$H_{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big) \;\simeq\; \ker'\big(\mathbf{H}_s;\, \Box_{\mathbf{H}} - \Delta(\Delta-2)\big). \tag{179}$$

---

[9] See Jones [14] and Bailey and Dunne [24].
[10] For a review, cf. Witten [34].

Fix a dual minitwistor $[\mathsf{Z}^I] \in \mathbf{MT}_s^*$. The representative superwavefunction

$$\Psi^2_\Delta\big(\,\cdot\,;\mathsf{Z}^I\big) \in \mathscr{D}'_{0,1}\big(\mathbf{MT}_s;\,\mathcal{O}_{\mathbf{C}}(\Delta-2,-\Delta)\big) \tag{180}$$

lies in the domain of $\mathcal{P}'$. We therefore define the *celestial superwavefunction* for gluons of conformal weight Δ by:

$$\Phi_\Delta\big(\,\cdot\,;\mathsf{Z}^I\big) \in \mathscr{D}'(\mathbf{H}_s), \quad \Phi_\Delta\big(\,\cdot\,;\mathsf{Z}^I\big) \;\coloneqq\; \mathcal{P}'\big[\Psi^2_\Delta\big(\,\cdot\,;\mathsf{Z}^I\big)\big]. \tag{181}$$

Explicitly, one has:

$$\Phi_\Delta\big(X_{A\dot{A}},\theta^\alpha_{\dot{A}};\mathsf{Z}^I\big) \;=\; \|X\|^\Delta \int_{\mathcal{L}(X,\theta)} D\lambda \wedge \Psi^2_\Delta\,\big|_{\mathcal{L}(X,\theta)}\big(\lambda^A;\mathsf{Z}^I\big). \tag{182}$$

The restriction of a minitwistor superwavefunction Ψ^p_Δ to the line $\mathcal{L}(X,\theta)$ is the supercurrent

$$\Psi^p_\Delta\,\big|_{\mathcal{L}(X,\theta)}\big(\,\cdot\,;\mathsf{Z}^I\big) \in \mathscr{D}'_{0,1}\big(\mathcal{L}(X,\theta);\,\mathcal{O}_{\mathbf{C}}(-p)\big) \tag{183}$$

given by:

$$\Psi^p_\Delta\,\big|_{\mathcal{L}(X,\theta)}\big(\lambda^A;\mathsf{Z}^I\big) \;=\; \bar{\delta}_{p-\Delta}(z^A,\lambda^A)\,\frac{\mathcal{C}(\Delta)}{\langle\lambda|X|\bar{z}]^\Delta}\,\exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\langle\lambda|\theta\cdot\eta\rangle\right). \tag{184}$$

Substituting into the Penrose integral (Eq. (182)) yields:

$$\Phi_\Delta\big(X_{A\dot{A}},\theta^\alpha_{\dot{A}};\mathsf{Z}^I\big) \;=\; K_\Delta\big(X_{A\dot{A}};z^A,\bar{z}_{\dot{A}}\big)\,e^{i\langle z|\theta\cdot\eta\rangle}, \tag{185}$$

where the bulk-to-boundary propagator K_Δ on $\mathbf{H}$ is

$$K_\Delta\big(X_{A\dot{A}};z^A,\bar{z}_{\dot{A}}\big) \;=\; \frac{\mathcal{C}(\Delta)}{\langle z|\mathcal{R}|\bar{z}]^\Delta}, \quad \mathcal{R}_{A\dot{A}} \;=\; \frac{X_{A\dot{A}}}{\|X\|}. \tag{186}$$

The physical interpretation of Φ_Δ proceeds via its relation to the $\mathcal{N}=4$ conformal primary wavefunction ϕ_Δ. This wavefunction is given by:

$$\phi_\Delta(x^\mu,\theta^\alpha_{\dot{A}};z^A,\bar{z}_{\dot{A}},\eta^\alpha) \;=\; \frac{\Gamma(\Delta)}{(\varepsilon - iq(z,\bar{z})\cdot x)^\Delta}\,e^{i\langle z|\theta\cdot\eta\rangle}, \tag{187}$$

where $q^\mu(z,\bar{z}) \coloneqq z^A(\sigma^\mu)_{A\dot{A}}\bar{z}^{\dot{A}}$ is the standard null four-vector. The superwavefunction ϕ_Δ describes a gluon of conformal weight Δ and helicity state η^α. It is obtained by extending the analyses of Banerjee [35, 36], Banerjee, Pandey, and Paul [37], Banerjee and Pandey [38] to the Lorentz *supergroup*.

Dimensional reduction of ϕ_Δ onto the leaves of the hyperbolic foliation of Klein superspace $\mathbf{K}^{4|8} \subset \mathbf{C}^{4|8}$ yields the celestial superwavefunction Φ_Δ. In other words, Φ_Δ is obtained by restricting ϕ_Δ to each hyperbolic slice in the leaf amplitude formalism[11]. An important property of the family $\{\Phi_\Delta\}$ follows from the analysis of the spectral theory for primary fields in the H_3^+-WZNW model[12]. The set $\{\Phi_\Delta\}$ is both complete and δ-function orthonormal.

[11] Banerjee, Gupta, and Misra [39].

[12] Teschner [40, 41, 42, 43], Ribault and Teschner [44].

3. Boels-Mason-Skinner-Witten (BMSW) *Identity*

Building on the constructions of Boels, Mason, and Skinner [45] and Witten [46], we now derive the central result of this subsection. We refer to this as the celestial Boels-Mason-Skinner-Witten (BMSW) identity. It takes the form of an integral formula equating the n-fold minitwistor Penrose transform of the superwavefunctions Ψ^p_Δ, weighted by a Lie-algebra-valued logarithmic differential form, with the kernel of the Mellin-transformed Parke-Taylor factors appearing in the celestial leaf amplitudes for gluonic scattering. This identity will serve as our key formula for both geometric and dynamical interpretations of the $\mathcal{N}=4$ SYM celestial amplitudes.

To derive the integral identity, fix a dual minitwistor point $[\mathsf{Z}^I] \in \mathbf{MT}^*_s$, and let $\mathsf{W}^I = (\lambda^A, \mu_{\dot{A}}, \psi^\alpha)$ parametrise minitwistor superspace. Let $\mathbf{G}$ be a compact, simply-connected and semisimple Lie group with Lie algebra $\mathfrak{g}$. Choose a Lie-algebra-valued section

$$g^{\mathsf{a}} \in \Gamma\big(\mathbf{CP}^1; \mathfrak{g} \otimes \mathscr{O}_{\mathbf{C}}(-2)\big). \tag{188}$$

In the BMSW identity, we use the minitwistor wavefunction

$$\Psi^0_\Delta\big(\,\cdot\,; \mathsf{Z}^I\big) \in \mathscr{D}'_{0,1}\big(\mathbf{MT}_s;\, \mathscr{O}_{\mathbf{C}}(\Delta, -\Delta)\big), \tag{189}$$

which we abbreviate by Ψ_Δ.

Define the Lie-algebra-valued supercurrent

$$\mathcal{K}^{\mathsf{a}}_{\Delta,g} \in \mathscr{D}'_{0,1}\big(\mathbf{MT}_s;\, \mathfrak{g} \otimes \mathscr{O}_{\mathbf{C}}(\Delta-2, -\Delta)\big) \tag{190}$$

by

$$\mathcal{K}^{\mathsf{a}}_{\Delta,g}\big(\mathsf{W}^I\big) \;\coloneqq\; g^{\mathsf{a}}(\lambda^A)\, \Psi_\Delta\big(\mathsf{W}^I; \mathsf{Z}^{I'}\big). \tag{191}$$

Here and below we omit the symbol $\otimes$ for simple juxtaposition.

Since $\mathcal{K}^{\mathsf{a}}_{\Delta,g}$ lies in the domain of $\mathcal{P}'$, we may write the Penrose integral as follows:

$$\mathcal{P}'\big[\mathcal{K}^{\mathsf{a}}_{\Delta,g}\big]\,\big(X_{A\dot{A}}, \theta^\alpha_A\big) \;=\; \int_{\mathcal{L}(X,\theta)} D\lambda \wedge g^{\mathsf{a}}(\lambda^A)\, \Psi_\Delta\,\big|_{\mathcal{L}(X,\theta)}(\lambda^A; \mathsf{Z}^I). \tag{192}$$

Using the restriction formula in Eq. (184), we find that the Penrose integral evaluates to:

$$\int_{\mathcal{L}(X,\theta)} D\lambda \wedge g^{\mathsf{a}}(\lambda^A)\, \Psi_\Delta\,|_{\mathcal{L}(X,\theta)}(\lambda^A; \mathsf{Z}^I) \;=\; \frac{\mathcal{C}(\Delta)}{\langle z|X|\bar{z}]^\Delta}\, e^{i\langle z|\theta\cdot\eta\rangle}\, g^{\mathsf{a}}(z^A). \tag{193}$$

Define the n-fold Cartesian product of the minitwistor line by $\mathcal{L}_n \coloneqq \bigtimes^n \mathcal{L}(X,\theta)$. An inductive argument then shows that:

$$\int\limits_{\mathcal{L}_n} \bigwedge_{i=1}^{n} D\lambda_i \wedge g^{\mathsf{a}_i}(\lambda_i^A)\, \Psi_{\Delta_i}\,\big|_{\mathcal{L}(X,\theta)}\big(\lambda^A; \mathsf{Z}^I\big) \;=\; \prod_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i|X|\bar{z}_i]^{\Delta_i}}\, e^{i\langle z_i|\theta\cdot\eta_i\rangle}\, g^{\mathsf{a}_i}(\lambda_i^A). \tag{194}$$

Next, let $\{\mathsf{T}^{\mathsf{a}}\}$ be a basis for $\mathfrak{g}$. Define

$$g^{\mathsf{a}_i}(\lambda_i^A) \;=\; \frac{\mathsf{T}^{\mathsf{a}_i}}{\lambda_i \cdot \lambda_{i+1}}. \tag{195}$$

Introduce the Lie-algebra-valued logarithmic differential form:

$$\boldsymbol{\omega}^{\mathsf{a}_i}(\lambda_i^A) \;\coloneqq\; \mathsf{T}^{\mathsf{a}_i}\,\frac{D\lambda_i}{\lambda_i \cdot \lambda_{i+1}}. \tag{196}$$

Then Eq. (194) reduces to the *supersymmetric celestial BMSW identity*:

$$\int_{\mathcal{L}_n} \bigwedge_{i=1}^{n} \boldsymbol{\omega}^{\mathsf{a}_i}(\lambda_i^A) \wedge \Psi_{\Delta_i}\big|_{\mathcal{L}(X,\theta)}(\lambda_i^A; \mathsf{Z}_i^I) \;=\; \prod_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i|X|\bar{z}_i]^{\Delta_i}}\, e^{i\langle z_i|\theta\cdot\eta_i\rangle}\,\frac{\mathsf{T}^{\mathsf{a}_i}}{z_i \cdot z_{i+1}}. \tag{197}$$

F. Supersymmetric Celestial RSVW Identity

In the preceding subsection, we derived the integral kernel for the Mellin-transformed Parke-Taylor factors, namely

$$\prod_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i|X|\bar{z}_i]^{\Delta_i}}\, e^{i\langle z_i|\theta\cdot\eta_i\rangle}\,\frac{\mathsf{T}^{\mathsf{a}_i}}{z_i \cdot z_{i+1}}\,, \tag{198}$$

as an n-fold Penrose transform of the minitwistor superwavefunctions Ψ_Δ. We called this the supersymmetric celestial BMSW identity. It provides a direct map from celestial leaf amplitudes to correlation functions in the minitwistor sigma-model, via the power-series expansion of the Quillen determinant line bundle.

We now develop a complementary formulation by translating celestial leaf amplitudes into minitwistor amplitudes. Direct evaluation of the minitwistor transform on distributional data is challenging. Instead, we will derive a convenient rewriting of the identity resolution. Substituting this into the leaf amplitudes then produces their minitwistor counterparts in a systematic manner.

Our approach follows Roiban, Spradlin, and Volovich [47, 48, 49] and Witten [46]. In contrast to the BMSW identity, where one integrates the superwavefunction Ψ_Δ over the minitwistor line $\mathcal{L}(X,\theta)$, we impose the supersymmetric incidence relations directly on Ψ_Δ:

$$\mu_{\dot{A}} = \lambda^A X_{A\dot{A}}, \quad \psi^\alpha = \lambda^A \theta_A^\alpha. \tag{199}$$

These relations are enforced via delta "functions" on $\mathbf{MT}_s$. The resulting multidimensional minitwistor transform reproduces the Parke-Taylor kernel of Eq. (198). We term this the *supersymmetric celestial Roiban-Spradlin-Volovich-Witten* (RSVW) *identity.* Its bosonic version first appeared in Mol (2025).

Imposing the incidence relations on minitwistor superspace $\mathbf{MT}_s$ via distributions (or, more precisely, via currents of differential forms) is subtler than in standard projective twistor superspace $\mathbf{PT}^{3|4}$. In $\mathbf{MT}_s$, the spinor coordinates λ^A and $\mu_{\dot{A}}$ carry independent scaling weights. We must however construct the integrand as a *legitimate* differential form on $\mathbf{MT}_s$, rather than merely as a section of the homogeneous bundle $\mathcal{O}_{\mathbf{C}}(p,q)$. Moreover, the form must have the correct bi-degree in the exterior algebra of $\mathbf{MT}_s$.

We therefore dedicate this first subsection to a careful construction of the integral over $\mathbf{MT}_s$ of the superwavefunction Ψ_Δ that properly implements the incidence relations.

We again consider a compact, simply-connected, semisimple gauge group $\mathbf{G}$ with Lie algebra $\mathfrak{g}$. Choose a Lie-algebra-valued section $g^{\mathsf{a}} \in \Gamma(\mathbf{CP}^1; \mathfrak{g} \otimes \mathcal{O}_{\mathbf{C}}(-2))$. Let $\boldsymbol{k}^{\mathsf{a}}_{\Delta,g}$ be the $(2,1)$-supercurrent on $\mathbf{MT}_s$ defined by:

$$\boldsymbol{k}^{\mathsf{a}}_{\Delta,g} \;\coloneqq\; g^{\mathsf{a}}(\lambda^A)\,\overline{\delta}_\Delta\big(\mu_{\dot{A}}, \lambda^A X_{A\dot{A}}\big) \wedge \delta^{0|4}\big(\psi^\alpha - \lambda^A \theta^\alpha_A\big) \wedge D^{2|4}\mathsf{W}. \tag{200}$$

Under the rescaling

$$\mathsf{W}^I \;\longmapsto\; \mathsf{W}'^I \coloneqq (t_1\,\lambda^A, t_2\,\mu_{\dot{A}}, t_1\,\psi^\alpha), \tag{201}$$

one checks

$$\boldsymbol{k}^{\mathsf{a}}_{\Delta,g} \;\longmapsto\; t_1^{-\Delta} t_2^{\Delta}\,\boldsymbol{k}^{\mathsf{a}}_{\Delta,g}. \tag{202}$$

Hence $\boldsymbol{k}^{\mathsf{a}}_{\Delta,g}$ defines an $\mathcal{O}_{\mathfrak{g}}(-\Delta,\Delta)$-valued supercurrent of bi-degree $(2,1)$.

Now fix a dual minitwistor $[\mathsf{Z}^I] \in \mathbf{MT}^*_s$. The superwavefunction $\Psi_\Delta \coloneqq \Psi^0_\Delta$ transforms under $\mathsf{W}^I \mapsto \mathsf{W}'^I$ as

$$\Psi_\Delta\big(\mathsf{W}^I;\mathsf{Z}^I\big) \;\longmapsto\; t_1^{\Delta} t_2^{-\Delta}\,\Psi_\Delta\big(\mathsf{W}^I;\mathsf{Z}^I\big). \tag{203}$$

Then

$$\mathcal{V}^{\mathsf{a}}_{\Delta,g} \;\coloneqq\; \Psi_\Delta\big(\mathsf{W}^I;\mathsf{Z}^I\big) \wedge \boldsymbol{k}^{\mathsf{a}}_{\Delta,g}\big(\mathsf{W}^I;\mathsf{X}^K\big) \tag{204}$$

defines a bi-degree $(2,2)$ supercurrent on $\mathbf{MT}_s$ that is invariant under $\mathsf{W}^I \mapsto \mathsf{W}'^I$.

Since

$$\mathbf{MT}_s \subset \mathbf{CP}^{1|4} \times \mathbf{CP}^1 \tag{205}$$

and $\mathbf{CP}^{1|4} \times \mathbf{CP}^1$ is compact, any smooth proper function $\Phi\colon \mathbf{MT}_s \to \mathbf{C}$ (i.e. $\mathscr{L}_\xi \Phi = 0$) serves as a test form of bi-degree $(0,0)$. In particular, choosing Φ to be the characteristic function $\boldsymbol{\chi}$ of the quadric, we regard $\mathcal{V}^{\mathsf{a}}_{\Delta,g}$ as a *volume form* on $\mathbf{MT}_s$. Hence the integral

$$\langle \mathcal{V}^{\mathsf{a}}_{\Delta,g}, \boldsymbol{\chi} \rangle = \int\limits_{\mathbf{MT}_s} \mathcal{V}^{\mathsf{a}}_{\Delta,g} \;=\; \int\limits_{\mathbf{MT}_s} \Psi_\Delta\big(\mathsf{W}^I; \mathsf{Z}^{I'}\big) \wedge \boldsymbol{k}^{\mathsf{a}}_{\Delta,g}\big(\mathsf{W}^I; \mathsf{X}^K\big) \tag{206}$$

is well-defined.

Substituting the explicit form of the superwavefunction Ψ_Δ (Eq. (15)) into the definition of $\mathcal{V}^{\mathsf{a}}_{\Delta,g}$ (Eq. (204)), we obtain:

$$\mathcal{V}^{\mathsf{a}}_{\Delta,g} = g^{\mathsf{a}}(\lambda^A)\, \frac{\mathcal{C}(\Delta)}{[\mu\bar{z}]^\Delta}\, \exp\left(i \frac{\langle z\iota\rangle}{\langle \lambda\iota\rangle} \psi \cdot \eta \right)\, \bar{\delta}_{-\Delta}\big(z^A, \lambda^A\big) \wedge \bar{\delta}_{\Delta}\big(\mu_{\dot{A}}, \lambda^A X_{A\dot{A}}\big) \tag{207}$$

$$\wedge\, \delta^{0|4}\big(\psi^\alpha - \lambda^A \theta^\alpha_A\big) \wedge D^{2|4}\mathsf{W}. \tag{208}$$

The integral in Eq. (206) then evaluates to:

$$\int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{W} \quad \Psi_\Delta\big(\mathsf{W}^I; \mathsf{Z}^{I'}\big)\, \bar{\delta}_{\Delta}\big(\mu_{\dot{A}}, \lambda^A X_{A\dot{A}}\big)\, \delta^{0|4}\big(\psi^\alpha - \lambda^A \theta^\alpha_A\big)\, g^{\mathsf{a}}(\lambda^A) \tag{209}$$

$$= \frac{\mathcal{C}(\Delta)}{\langle z|X|\bar{z}]^\Delta}\, e^{i\langle z|\theta\cdot\eta\rangle}\, g^{\mathsf{a}}(z^A). \tag{210}$$

We have denoted wedge products by juxtaposition for compactness. By construction of $\mathcal{V}^{\mathsf{a}}_{\Delta,g}$, this integral is projectively well-defined.

2. A Simple Lemma

The next step is to unify the delta-currents in Eq. (209) into the minitwistor delta $\bar{\delta}^{2|4}_{\Delta_1,\Delta_2}$ introduced in Subsection II D 1 (Eq. (89)). We achieve this via the following result.

Lemma. Let $\mathsf{Y}^I\colon \mathbf{CP}^1 \to \mathcal{L}(X,\theta) \subset \mathbf{MT}_s$ be the embedding of the holomorphic celestial sphere into the minitwistor line, defined by

$$\mathsf{Y}^I\big(\sigma^A\big) \;:=\; \big(\sigma^A, \sigma^A X_{A\dot{A}}, \sigma^A \theta^\alpha_A\big). \tag{211}$$

By construction, Y^I is a global section of $\mathcal{L}(X,\theta) \to \mathbf{CP}^1$, and its differential $\mathrm{d}\mathsf{Y}^I$ is an isomorphism on tangent spaces. Hence each point $\mathsf{w} \in \mathcal{L}(X,\theta)$ corresponds to a unique projective spinor $[\sigma^A] \in \mathbf{CP}^1$. We use these homogeneous coordinates to orient $\mathcal{L}(X,\theta)$. Explicitly, we define the holomorphic measure:

$$D\sigma \;\in\; \Omega^{1,0}\big(\mathcal{L}(X,\theta);\, \mathcal{O}_{\mathbf{C}}(2)\big), \quad D\sigma := \varepsilon_{AB}\, \sigma^A\, d\sigma^B. \tag{212}$$

With these preparations, the following equality of currents holds on $\mathcal{L}(X,\theta)$:

$$\bar{\delta}_\Delta\big(\mu_{\dot{A}}, \lambda^A X_{A\dot{A}}\big)\, \delta^{0|4}\big(\psi^\alpha - \lambda^A \theta^\alpha_A\big)\, g^{\mathsf{a}}(\lambda^A) = \int\limits_{\mathbf{CP}^1} D\sigma\; g^{\mathsf{a}}(\sigma^A) \,\wedge\, \bar{\delta}^{2|4}_{-\Delta,\Delta}\big(\mathsf{W}^I;\, \mathsf{Y}^{I'}(\sigma^A)\big). \tag{213}$$

Proof of Lemma. The first step uses the resolution of identity for the holomorphic delta $\bar{\delta}^2$ on $\mathbf{C}^2$. We rewrite the left-hand side as:

$$\bar{\delta}_\Delta\big(\mu_{\dot{A}}, \lambda^A X_{A\dot{A}}\big)\, \delta^{0|4}\big(\psi^\alpha - \lambda^A \theta^\alpha_A\big)\, g^{\mathsf{a}}(\lambda^A) \tag{214}$$

$$= \int\limits_{\mathbf{C}^2} d^2 s\, \bar{\delta}^2\big(\lambda^A - s^A\big)\, \bar{\delta}_\Delta\big(\mu_{\dot{A}}, s^A X_{A\dot{A}}\big)\, \delta^{0|4}\big(\psi^\alpha - s^A \theta^\alpha_A\big)\, g^{\mathsf{a}}(s^A). \tag{215}$$

Next we parametrise $s^A = t\,\sigma^A$ with $t \in \mathbf{C}_*$ and $[\sigma^A] \in \mathbf{CP}^1$. The measure decomposes as

$$d^2 s \;=\; D\sigma \,\wedge\, \frac{dt}{t}\, t^2, \tag{216}$$

so Eq. (215) becomes:

$$\bar{\delta}_\Delta\big(\mu_{\dot{A}}, \lambda^A X_{A\dot{A}}\big)\, \delta^{0|4}\big(\psi^\alpha - \lambda^A \theta^\alpha_A\big)\, g^{\mathsf{a}}(\lambda^A) \tag{217}$$

$$= \int\limits_{\mathbf{CP}^1} D\sigma \int\limits_{\mathbf{C}_*} \frac{dt}{t}\, t^{-\Delta}\, \bar{\delta}^2\big(\lambda^A - t\sigma^A\big)\, \bar{\delta}_\Delta\big(\mu_{\dot{A}}, \sigma^A X_{A\dot{A}}\big)\, \delta^{0|4}\big(\psi^\alpha - t\sigma^A \theta^\alpha_A\big)\, g^{\mathsf{a}}(\sigma^A). \tag{218}$$

Carrying out the t-integral yields the minitwistor delta $\bar{\delta}^{2|4}_{-\Delta,\Delta}$. Hence

$$\bar{\delta}_\Delta\big(\mu_{\dot{A}}, \lambda^A X_{A\dot{A}}\big)\, \delta^{0|4}\big(\psi^\alpha - \lambda^A \theta^\alpha_A\big)\, g^{\mathsf{a}}(\lambda^A) \tag{219}$$

$$= \int\limits_{\mathbf{CP}^1} D\sigma\; g^{\mathsf{a}}(\sigma^A) \,\wedge\, \bar{\delta}^{2|4}_{-\Delta,\Delta}\big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha \big| \sigma^A, \sigma^A X_{A\dot{A}}, \sigma^A \theta^\alpha_A\big). \tag{220}$$

Finally, substituting $\mathsf{W}^I = (\lambda^A, \mu_{\dot{A}}, \psi^\alpha)$ and $\mathsf{Y}^I(\sigma^A)$ into Eq. (220) completes the proof of the lemma.

3. Main Result

We are now ready to present the main result of this subsection.

Substituting the lemma's identity (Eq. (213)) into the integral of the top-form $\mathcal{V}^{\mathsf{a}}_{\Delta,g}$ (Eq. (209)) yields:

$$\int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{W}\; \Psi_\Delta\big(\mathsf{W}^I; \mathsf{Z}^I\big) \int\limits_{\mathbf{CP}^1} D\sigma\; g^{\mathsf{a}}(\sigma^A) \,\wedge\, \bar{\delta}^{2|4}_{-\Delta,\Delta}\big(\mathsf{W}^I; \mathsf{Y}^{I'}(\sigma^A)\big) \tag{221}$$

$$= \frac{\mathcal{C}(\Delta)}{\langle z|X|\bar{z}]^\Delta}\, e^{i\langle z|\theta\cdot\eta\rangle}\, g^{\mathsf{a}}(z^A). \tag{222}$$

We extend this to n insertions by defining $\mathbf{X}_n := \times^n \mathbf{MT}_s$ and $\mathbf{L}_n := \times^n \mathbf{CP}^1$. An inductive argument then gives:

$$\int_{\mathbf{X}_n} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{W}_i \; \Psi_{\Delta_i}\big(\mathsf{W}_i^I; \mathsf{Z}_i^{I'}\big) \int_{\mathbf{L}_n} \bigwedge_{j=1}^{n} D\sigma_j \; g^{\mathsf{a}_j}(\sigma_j^A) \;\wedge\; \bar{\delta}^{2|4}_{-\Delta_j,\Delta_j}\big(\mathsf{W}_j^J; \mathsf{Y}^{J'}(\sigma_j^A)\big) \tag{223}$$

$$= \prod_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i|X|\bar{z}_i]^{\Delta_i}} \, e^{i\langle z_i|\theta\cdot\eta_i\rangle} \, g^{\mathsf{a}_i}(z_i^A). \tag{224}$$

Finally, choosing a basis $\{\mathsf{T}^{\mathsf{a}_i}\}$ of $\mathfrak{g}$ and setting

$$g^{\mathsf{a}_i}(\sigma^A) \;=\; \frac{\mathsf{T}^{\mathsf{a}_i}}{\sigma_i \cdot \sigma_{i+1}} \tag{225}$$

we find, via the logarithmic form $\boldsymbol{\omega}^{\mathsf{a}_i}(\sigma_i^A)$ of Eq. (196), the $\mathcal{N}=4$ *supersymmetric celestial RSVW identity*:

$$\int_{\mathbf{X}_n} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{W}_i \; \Psi_{\Delta_i}\big(\mathsf{W}_i^I; \mathsf{Z}_i^{I'}\big) \int_{\mathbf{L}_n} \bigwedge_{j=1}^{n} \boldsymbol{\omega}^{\mathsf{a}_j}(\sigma_j^A) \;\wedge\; \bar{\delta}^{2|4}_{-\Delta_j,\Delta_j}\big(\mathsf{W}_j^J; \mathsf{Y}^{J'}(\sigma_j^A)\big) \tag{226}$$

$$= \prod_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i|X|\bar{z}_i]^{\Delta_i}} \, e^{i\langle z_i|\theta\cdot\eta_i\rangle} \, \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i \cdot z_{i+1}}. \tag{227}$$

This formula serves as the key entry in our dictionary between celestial and minitwistor amplitudes for gluons in planar Yang-Mills theory. It provides the link between gluonic amplitudes and the holonomies of the Knizhnik-Zamolodchikov connection.

III. TREE-LEVEL S-MATRIX

The on-shell BCFW recursion relations[13] in $\mathcal{N}=4$ SYM theory admit an explicit solution known as the Drummond-Henn representation[14]. This solution provides a compact, manifestly supersymmetric expression for the tree-level superamplitude. From it one may extract purely gluonic amplitudes valid in any gauge theory. Remarkably, the Drummond-Henn formula is written in terms of nested sums of dual-conformal R-invariants, thus making both standard and dual superconformal symmetries manifest.

In this subsection, we employ the Drummond-Henn solution for tree-level N^k-MHV superamplitudes to derive the corresponding sectional/leaf amplitudes in terms of minitwistor superwavefunctions. Our strategy consists of two main steps:

---

[13] See Britto *et al.* [50], Britto, Cachazo, and Feng [51], Bianchi, Elvang, and Freedman [52], Brandhuber, Heslop, and Travaglini [53], Arkani-Hamed, Cachazo, and Kaplan [54], Elvang, Freedman, and Kiermaier [55].

[14] Cf. Drummond and Henn [56], Dixon *et al.* [57], Korchemsky and Sokatchev [58].

1. We write an integral representation of each R-invariant via the Fadde'ev-Popov method, such that all frequency dependence appears as exponential factors. The resulting representation admits a straightforward Mellin transform.

2. We combine the Fadde'ev-Popov representation of the R-invariant with the celestial RSVW identity. The resulting expression is the tree-level N^k-MHV minitwistor amplitude, given by an integral over the moduli space parametrising configurations of $2k+1$ minitwistor lines.

We begin by deriving in detail the N^1- and N^2-MHV amplitudes. We then extend the construction inductively to obtain the full tree-level celestial $\mathcal{S}$-matrix.

A. N^1-MHV Scattering Amplitude

Consider $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory on four-dimensional Klein space[15] $\mathbf{K}^4 := \left(\mathbf{R}^4, h_{\mu\nu}\right)$. We endow this background with a gauge group $\mathbf{G}$. Its Lie algebra $\mathfrak{g} \simeq \left(T_e\left(\mathbf{G}\right), [\cdot,\cdot]\right)$ is assumed compact and semi-simple. Let $\{\mathsf{T}^{\mathsf{a}}\}$ be a basis of generators satisfying $[\mathsf{T}^{\mathsf{a}}, \mathsf{T}^{\mathsf{b}}] = i f^{\mathsf{abc}}\mathsf{T}^{\mathsf{c}}$, with f^{abc} the structure constants. We choose the trace normalisation $\mathsf{Tr}\left(\mathsf{T}^{\mathsf{a}}\mathsf{T}^{\mathsf{b}}\right) = 2^{-1}\boldsymbol{k}^{\mathsf{ab}}$, where $\boldsymbol{k}^{\mathsf{ab}}$ is the Cartan-Killing form on $\mathfrak{g}$.

We study tree-level scattering of n gluons in a next-to-maximal-helicity-violating (N^1-MHV) configuration. The N^1-MHV superamplitude depends on spinor momenta $\nu_i^A, \bar{\nu}_{i\dot{A}}$ and Grassmann variables η_i^α ($\alpha = 1,2,3,4$) encoding helicity degrees of freedom. It takes the form[16]:

$$\mathcal{A}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\left(\nu_i^A, \bar{\nu}_{i\dot{A}}, \eta_i^\alpha\right) = (2\pi)^4\, \delta^{4|0}\big(P^{A\dot{A}}\big)\delta^{0|8}\big(Q^{\alpha\dot{A}}\big)\, A_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\left(\nu_i^A, \bar{\nu}_{i\dot{A}}, \eta_i^\alpha\right). \tag{228}$$

Here $i = 1, ..., n$ labels the external gluons. The total four-momentum and supercharge are:

$$P^{A\dot{A}} := \sum_{i=1}^n \nu_i^A \bar{\nu}_i^{\dot{A}}, \;\; Q^{\alpha A} := \sum_{i=1}^n \nu_i^A\, \eta_i^\alpha. \tag{229}$$

Dual Coordinates. To express the N^1-MHV superamplitude, we introduce dual (zone) coordinates $y_i^{A\dot{A}}$. These solve the momentum-conserving delta function $\delta^{4|0}\big(P^{A\dot{A}}\big)$ via:

$$y_i^{A\dot{A}} - y_{i+1}^{A\dot{A}} := \nu_i^A \bar{\nu}_i^{\dot{A}}. \tag{230}$$

For $1 \leq i < j \leq n$, set:

$$y_{ij}^{A\dot{A}} := y_i^{A\dot{A}} - y_j^{A\dot{A}}, \;\; y_{ji}^{A\dot{A}} := -y_{ij}^{A\dot{A}}. \tag{231}$$

[15] The Kleinian metric in global rectangular coordinates is:

$$h_{\mu\nu} := \mathrm{diag}\left(-1,-1,+1,+1\right).$$

Klein space is reviewed in Barrett *et al.* [59], Bhattacharjee and Krishnan [60], Crawley *et al.* [61], Duary and Maji [62], Klein [63], Plucker [64], Penrose and Rindler [65].

[16] For a review, see Brandhuber, Spence, and Travaglini [66], Elvang and Huang [67], Badger *et al.* [68].

1. *Dual Conformal Invariant*

The N^1-MHV superamplitude admits a representation in terms of a dual-conformal R-invariant[17]. This invariant takes the form:

$$R_{n;ab}\big(\nu_i^A, \bar{\nu}_{i\dot{A}}, \eta_i^\alpha\big) \;=\; \frac{\langle \nu_{a-1}, \nu_a\rangle\;\langle \nu_{b-1}, \nu_b\rangle\;\delta^{0|4}\big(\varXi^\alpha\big)}{y_{ab}^2\;\langle \nu_n|y_{nb}y_{ba}^{-1}|\nu_{a-1}\rangle\;\langle \nu_n|y_{nb}y_{ba}^{-1}|\nu_a\rangle\;\langle \nu_n|y_{na}y_{ab}^{-1}|\nu_{b-1}\rangle\;\langle \nu_n|y_{na}y_{ab}^{-1}|\nu_b\rangle}, \tag{232}$$

where:

$$\Xi^\alpha\big(\nu_i^A, \bar{\nu}_{i\dot{A}}, \eta_i^\alpha\big) \;\coloneqq\; \sum_{i=1}^{a-1}\langle \nu_n|y_{nb}y_{ba}^{-1}|\nu_i\rangle\,\eta_i^\alpha \;+\; \sum_{i=1}^{b-1}\langle \nu_n|y_{na}y_{ab}^{-1}|\nu_i\rangle\,\eta_i^\alpha. \tag{233}$$

Celestial Description. To pass from momentum-space amplitudes to their celestial-basis form, we introduce a normalised set of van der Waerden spinors $\{z_i^A, \bar{z}_{i\dot{A}}\}$ on the celestial torus $\mathcal{CT}$. These are defined by:

$$z_i^A \;\coloneqq\; (1, -\zeta_i)\,, \;\; \bar{z}_{i\dot{A}} \;\coloneqq\; \big(1, -\bar{\zeta}_i\big)\,, \tag{234}$$

where $\zeta_i, \bar{\zeta}_i$ are planar coordinates on $\mathcal{CT}$.

Denote by $s_i > 0$ the frequency of the i-th gluon. We reparametrise the spinor momenta via:

$$\nu_i^A \;\mapsto\; z_i^A, \;\; \bar{\nu}_{i\dot{A}} \;\mapsto\; s_i\,\bar{z}_{i\dot{A}}. \tag{235}$$

Applying these replacements to the R-invariant (232) yields its celestial form:

$$R_{n;ab} = \frac{\langle z_{a-1}, z_a\rangle\langle z_{b-1}, z_b\rangle\delta^{0|4}\big(\Theta^\alpha\big)}{y_{ab}^2\langle z_n|y_{nb}y_{ba}^{-1}|z_{a-1}\rangle\langle z_n|y_{nb}y_{ba}^{-1}|z_a\rangle\langle z_n|y_{na}y_{ab}^{-1}|z_{b-1}\rangle\langle z_n|y_{na}y_{ab}^{-1}|z_b\rangle}, \tag{236}$$

where:

$$\Theta^\alpha\left(u^A, v^B\right) \;\coloneqq\; \sum_{i=1}^{a-1}\langle u, z_i\rangle\,\eta_i^\alpha \;+\; \sum_{j=1}^{b-1}\langle v, z_j\rangle\,\eta_j^\alpha. \tag{237}$$

2. *Fadde'ev-Popov Representation*

Our objective is to find the Mellin transform of the gluonic superamplitude (Eq. (228)). This yields the celestial amplitude:

$$\widehat{\mathcal{A}}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha\big).$$

[17] For a discussion of dual conformal symmetry, see Drummond *et al.* [69], Mason and Skinner [70], Alday and Roiban [71], Henn [72], Korchemsky and Sokatchev [73], Alday and Maldacena [74], Brandhuber, Heslop, and Travaglini [75].

The R-invariant in Eq. (228) depends nonlinearly on the frequencies s_i through the dual coordinates $y_i^{A\dot{A}}$. This nonlinearity obstructs a direct Mellin transform. To overcome this difficulty, we recast the R-invariant via the Fadde'ev-Popov procedure.

Our aim is an integral representation in which all s_i-dependence is isolated into delta functions. Let $\mathcal{I} := \mathbf{R}^2 \times \mathbf{R}^2$ be the integration domain. We introduce coordinates $U^{A'} := \left(u^A, v^B\right)$ on $\mathcal{I}$, where u^A and v^A are van der Waerden spinors and $A' \in \{A, B\}$. The orientation on $\mathcal{I}$ is given by the Lebesgue measure:

$$d^4U := d^2u \wedge d^2v. \tag{238}$$

Observe that the reality of $\mathcal{I}$ follows from the Kleinian signature.

We localise the Fadde'ev-Popov integral on a constraint subset $\mathscr{C} \subset \mathcal{I}$. To define $\mathscr{C}$, consider the auxiliary spinor functions:

$$f^A_{n;ab}\left(y_i^{B\dot{B}}\right) := z_n^B \, (y_{nb})_{B\dot{B}} \, (y_{ba}^{-1})^{A\dot{B}}, \tag{239}$$

$$g^A_{n;ab}\left(y_i^{B\dot{B}}\right) := z_n^B \, (y_{na})_{B\dot{B}} \, (y_{ab}^{-1})^{A\dot{B}}. \tag{240}$$

Then $\mathscr{C}$ is the set of points $U^{A'} \in \mathcal{I}$ satisfying:

$$u^A = f^A_{n;ab}\left(y_i^{B\dot{B}}\right), \;\; v^A = g^A_{n;ab}\left(y_i^{B\dot{B}}\right). \tag{241}$$

The corresponding constraint delta distribution is:

$$\delta_{\mathscr{C}}\left(U^{A'}\right) := \bar{\delta}^2\left(u^A - f^A_{n;ab}\left(y_i^{B\dot{B}}\right)\right) \bar{\delta}^2\left(v^A - g^A_{n;ab}\left(y_i^{B\dot{B}}\right)\right). \tag{242}$$

With these ingredients, the Fadde'ev-Popov representation of the R-invariant reads:

$$R_{n;ab} = \frac{1}{y_{ab}^2} \int_{\mathcal{I}} d^4U \;\; \mathcal{F}_{ab}\left(U^{A'}\right) \delta^{0|4}\left(\Theta^\alpha\right) \delta_{\mathscr{C}}\left(U^{A'}\right), \tag{243}$$

where:

$$\mathcal{F}_{ab}\left(U^{A'}\right) := \frac{\langle z_{a-1}, z_a\rangle \langle z_{b-1}, z_b\rangle}{\langle z_{a-1}, u\rangle \langle u, z_a\rangle \langle z_{b-1}, v\rangle \langle v, z_b\rangle}. \tag{244}$$

This representation makes the s_i-dependence factor through the constraint delta function $\delta_{\mathscr{C}}$.

3. Delta Functions

We now expand the delta functions in Eq. (243) using their integral representations. We express them in terms of the celestial coordinates z_i^A, $\bar{z}_{i\dot{A}}$ and η_i^α.

Fermionic Delta Function. Let $\alpha = 1, ..., 4$ index the supersymmetry generators, and let ε^α be a Grassmann variable. In superanalysis, one defines:

$$\delta^{0|4}\big(\varepsilon^\alpha\big) \;\coloneqq\; \bigwedge_{\alpha=1}^{4} \varepsilon^\alpha. \tag{245}$$

In terms of a Berezin integral[18],

$$\delta^{0|4}\big(\varepsilon^\alpha\big) \;=\; \int\limits_{\mathbf{R}^{0|4}} d^{0|4}\chi \;\; \exp\left(i\chi\cdot\varepsilon\right), \;\; \chi\cdot\varepsilon \;\coloneqq\; \chi_\alpha\varepsilon^\alpha. \tag{246}$$

Substituting Eq. (237) into Eq. (246) then yields:

$$\delta^{0|4}\left(\Theta^\alpha\right) \;=\; \int\limits_{\mathbf{R}^{0|4}} d^{0|4}\chi \; \bigwedge_{i=1}^{a-1} \; \exp\left(i\left\langle u, z_i\right\rangle \chi\cdot\eta_i\right) \; \bigwedge_{j=a}^{b-1} \; \exp\left(i\left\langle v, z_j\right\rangle \chi\cdot\eta_j\right). \tag{247}$$

Bosonic Delta Function. Let λ^A be a real van der Waerden spinor. The spinor delta function admits the integral representation:

$$\overline{\delta}^2\left(\lambda^A\right) \;=\; \int\limits_{\mathbf{R}^2} \frac{d^2\sigma}{(2\pi)^2} \; \exp\left(i\left\langle\lambda\sigma\right\rangle\right). \tag{248}$$

By substituting Eq. (239) into this representation, one obtains:

$$\overline{\delta}^2\big(u^A - f^A_{n;ab}\big(y_i^{B\dot{B}}\big)\big) \;=\; \int\limits_{\mathbf{R}^2} \frac{d^2\hat{u}}{(2\pi)^2} \;\; \exp\left(i\left\langle z_n|y_{nb}y_{ba}^{-1}|\hat{u}\right\rangle\right) e^{-i\langle u|\hat{u}\rangle}. \tag{249}$$

Similarly, substituting Eq. (240) into Eq. (248) yields:

$$\overline{\delta}^2\big(v^A - g^A_{n;ab}\big(y_i^{B\dot{B}}\big)\big) \;=\; \int\limits_{\mathbf{R}^2} \frac{d^2\hat{v}}{(2\pi)^2} \;\; \exp\left(i\left\langle z_n|y_{na}y_{ab}^{-1}|\hat{v}\right\rangle\right) \; e^{-i\langle v|\hat{v}\rangle}. \tag{250}$$

Next, introduce the change of integration variables:

$$\tilde{u}^{\dot{A}} \;\coloneqq\; \hat{u}_A\left(y_{ba}^{-1}\right)^{A\dot{A}}, \;\; \tilde{v}^{\dot{A}} \;\coloneqq\; \hat{v}_A\left(y_{ab}^{-1}\right)^{A\dot{A}}. \tag{251}$$

It follows that:

$$\overline{\delta}^2\big(u^A - f^A_{n;ab}\big(y_i^{B\dot{B}}\big)\big) \;=\; \left|y_{ba}^2\right| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{u}}{(2\pi)^2} \;\; \exp\left(i\left\langle z_n|y_{nb}|\tilde{u}\right]\right) \exp\left(-i\left\langle u|y_{ba}|\tilde{u}\right]\right), \tag{252}$$

$$\overline{\delta}^2\big(v^A - g^A_{n;ab}\big(y_i^{B\dot{B}}\big)\big) \;=\; \left|y_{ab}^2\right| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{v}}{(2\pi)^2} \;\; \exp\left(i\left\langle z_n|y_{na}|\tilde{v}\right]\right) \exp\left(-i\left\langle v|y_{ab}|\tilde{v}\right]\right). \tag{253}$$

---

[18] See Berezin [76], DeWitt [19], Leites [21] and Manin [22].

Finally, employing the definition of $y_i^{A\dot{A}}$ in Eq. (230), one expands the bosonic delta functions in the celestial parametrisation $z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha$:

$$\overline{\delta}^2\big(u^A - f^A_{n;ab}\big(y_i^{B\dot{B}}\big)\big) \;=\; \big|y^2_{ba}\big| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{u}}{(2\pi)^2} \prod_{i=a}^{b-1} \exp\left(-is_i \left\langle z_i|u\tilde{u}|\bar{z}_i\right]\right) \prod_{i=b}^{n} \exp\left(is_i \left\langle z_i|z_n\tilde{u}|\bar{z}_i\right]\right), \quad (254)$$

$$\overline{\delta}^2\big(v^A - g^A_{n;ab}\big(y_i^{B\dot{B}}\big)\big) \;=\; \big|y^2_{ab}\big| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{v}}{(2\pi)^2} \prod_{i=a}^{b-1} \exp\left(is_i\langle z_i|z_n\tilde{v} + v\tilde{v}|\bar{z}_i]\right) \prod_{i=b}^{n} \exp\left(is_i\langle z_i|z_n\tilde{v}|\bar{z}_i]\right). \quad (255)$$

4. Integral Representation for the R-invariant

In Subsection III A 2 we applied the Fadde'ev-Popov method to the R-invariant, noting that all dependence on the frequency parameters s_i is carried by delta functions. In Subsection III A 3 we then expanded each delta function in the spinor basis $\{z_i^A, \bar{z}_{i\dot{A}}\}$ and the Grassmann variables η_i^α. By combining these two steps, we arrive at the final form of the R-invariant, which we now discuss just before performing the Mellin transform.

Integration Superdomain. We begin by defining the *parameter superspace* as:

$$\mathcal{P} \;\coloneqq\; \mathbf{R}^{8|4}. \quad (256)$$

This supermanifold is globally charted by the coordinates:

$$\tau^M \;\coloneqq\; \big(u^A, v^B, \tilde{u}_{\dot{A}}, \tilde{v}_{\dot{B}}, \chi^\alpha\big), \quad (257)$$

where the abstract index M ranges over $\{A, B, \dot{A}, \dot{B}, \chi^\alpha\}$. We shall refer to τ^M as the *moduli parameters.*

The canonical orientation on $\mathcal{P}$ is provided by the $\mathbf{Z}_2$-graded volume form:

$$d^{8|4}\tau \;\coloneqq\; d^2u \wedge d^2v \wedge d^2\tilde{u} \wedge d^2\tilde{v} \wedge d^{0|4}\chi. \quad (258)$$

Embedding Coordinates. In the following subsections, we shall construct the moduli superspace $\mathscr{M}_3$, which parametrises the configuration of three minitwistor lines on which the amplitude localises. For now, we regard $\mathscr{M}_3$ as an abstract supermanifold.

For each gluon $i = 1, ..., n$ participating in the scattering process, we associate a copy $\mathcal{P}_i$ of the parameter superspace, which may be viewed as a submanifold of $\mathscr{M}_3$. The embedding coordinates adapted to $\mathcal{P}_i$ are defined as follows.

Index ℓ	$\mathcal{Q}_\ell^{A\dot{A}}$	$q_\ell^{\alpha A}$
1	$-\big(u^A+v^A\big)\chi^\alpha$	0
2	$-v^A\chi^\alpha$	$z_n^A\tilde{v}^{\dot{A}}-u^A\tilde{u}^{\dot{A}}+v^A\tilde{v}^{\dot{A}}$
3	0	$z_n^A\big(\tilde{u}^A+\tilde{v}^{\dot{A}}\big)$

Table I. Embedding coordinates on parameter superspace $\mathcal{P}$.

Let the index ℓ run over the set $\{1,2,3\}$. We introduce the family of coordinate functions on the parameter superspace:

$$\mathsf{Q}_\ell^K := \big(\mathcal{Q}_\ell^{A\dot{A}}, q_\ell^{\alpha A}\big) : \mathcal{P} \longrightarrow \mathbf{R}^{4|8}, \tag{259}$$

whose components are listed in Table I.

Next, define the cluster-indicator function $c(i)$ by:

$$c\,(i) \;:=\; \begin{cases} 1, & 1 \le i \le a-1; \\ 2, & a \le i \le b-1; \\ 3, & b \le i \le n. \end{cases} \tag{260}$$

This assigns to the i-th gluon the cluster to which it belongs. The *embedding coordinates* on each copy $\mathcal{P}_i$ are then:

$$\widetilde{\mathsf{Q}}_i^K := \big(\widetilde{\mathcal{Q}}_i^{A\dot{A}}, \tilde{q}_i^{\alpha A}\big) : \mathcal{P}_i \longrightarrow \mathbf{R}^{4|8}, \tag{261}$$

with the identification:

$$\widetilde{\mathcal{Q}}_i^{A\dot{A}} \;:=\; \mathcal{Q}_{c(i)}^{A\dot{A}}, \;\; \tilde{q}_i^{\alpha A} \;:=\; q_{c(i)}^{\alpha A}. \tag{262}$$

Comment. Under the rescaling of the moduli parameters:

$$\tau^M = \big(u^A, v^B, \tilde{u}_{\dot{A}}, \tilde{v}_{\dot{B}}, \chi^\alpha\big) \mapsto \tilde{\tau}^M = \big(u^A, v^B, r\tilde{u}_{\dot{A}}, r\tilde{v}_{\dot{B}}, \chi^\alpha\big), \tag{263}$$

the embedding coordinates transform homogeneously:

$$\widetilde{\mathcal{Q}}_i^{A\dot{A}} \mapsto r\,\widetilde{\mathcal{Q}}_i^{A\dot{A}}, \;\; \tilde{q}_i^{\alpha A} \mapsto \tilde{q}_i^{\alpha A}. \tag{264}$$

Main Result. By substituting Eqs. (247), (254) and (255) into Eq. (236), the R-invariant can be written as:

$$R_{n;ab} \;=\; \mathcal{N}_{ab} \int\limits_{\mathcal{I}} d^{8|4}\tau \;\; \mathcal{F}_{ab}\,(\tau) \bigwedge_{i=1}^{n} \exp\big(i\langle z_i|\tilde{q}_i\cdot\eta_i\rangle + is_i\langle z_i|\widetilde{\mathcal{Q}}_i|\bar{z}_i]\big), \tag{265}$$

with the normalisation factor defined by:

$$\mathcal{N}_{ab} \;\coloneqq\; \frac{1}{(2\pi)^4}\, y_{ab}^2. \tag{266}$$

In the representation (265), the entire dependence on the frequency parameters s_i resides within the exponential factors. This structure renders the formula ideally suited for the Mellin transform needed to obtain the celestial amplitude.

5. *Celestial Superamplitude*

The tree-level N^1-MHV superamplitude in $\mathcal{N} = 4$ SYM theory is obtained by summing its partial amplitudes:

$$\mathcal{A}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\nu_i^A, \bar{\nu}_{i\dot{A}}, \eta_i^\alpha\big) \;=\; \sum_{a,b} \mathcal{A}_{n;ab}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\nu_i^A, \bar{\nu}_{i\dot{A}}, \eta_i^\alpha\big). \tag{267}$$

The sub-amplitudes takes the form:

$$\mathcal{A}_{n;ab}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\nu_i^A, \bar{\nu}_{i\dot{A}}, \eta_i^\alpha\big) \;=\; (2\pi)^4\;\delta^{4|0}\big(P^{A\dot{A}}\big)\,\delta^{0|8}\big(Q^{\alpha A}\big)\,R_{n;ab}\,\mathsf{Tr}\prod_{i=1}^{n}\frac{\mathsf{T}^{\mathsf{a}_i}}{\nu_i\cdot\nu_{i+1}}. \tag{268}$$

To express this amplitude in celestial coordinates, we implement the substitutions $\nu_i^A \mapsto z_i^A$ and $\bar{\nu}_{i\dot{A}} \mapsto s_i\bar{z}_{i\dot{A}}$. Then, using the integral representation of the R-invariant obtained in the preceding section (refer to Eq. (265)), we now proceed to derive the Mellin transform of the partial superamplitude $\mathcal{A}_{n;ab}^{\mathsf{a}_1\ldots\mathsf{a}_n}$.

Preliminaries. We first derive an integral representation for the distributional prefactor:

$$(2\pi)^4\;\delta^{4|0}\big(P^{A\dot{A}}\big)\,\delta^{0|8}\big(Q^{\alpha A}\big).$$

This term enforces four-momentum and supercharge conservation in the scattering process.

The four-momentum delta function is:

$$\delta^{4|0}\big(P^{A\dot{A}}\big) \;=\; \frac{1}{(2\pi)^4}\int_{\mathbf{R}^4} d^4x \;\;\exp\left(i\,x\cdot P\right),\;\;\; x\cdot P \coloneqq x_{A\dot{A}}P^{A\dot{A}}. \tag{269}$$

Now, the supercharge $Q^{\alpha A}$ is a Grassmann-valued van der Waerden spinor. Its fermionic delta function is defined by:

$$\delta^{0|8}\big(Q^{\alpha A}\big) \;\coloneqq\; \frac{1}{2^4}\bigwedge_{\alpha=1}^{4}\varepsilon^{AB}\,Q_A^\alpha\wedge Q_B^\alpha. \tag{270}$$

Equivalently, via a Berezin integral:

$$\delta^{0|8}\big(Q^{\alpha A}\big) \;=\; \int\limits_{\mathbf{R}^{0|8}} d^{0|8}\theta \;\; \exp\big(i\,\theta_{\alpha A}Q^{\alpha A}\big)\,. \tag{271}$$

To combine Eqs. (269) and (271), introduce superspace coordinates $\mathsf{x}^K := \big(x_{A\dot{A}}, \theta^\alpha_A\big)$ on $\mathbf{R}^{4|8}$ with abstract index K. The standard orientation is given by the Berezin-de Witt volume superform:

$$d^{4|8}\mathsf{x} \;:=\; d^4x \wedge d^{0|8}\theta. \tag{272}$$

The explicit forms of $P^{A\dot{A}}$ and $Q^{\alpha A}$ appear in Eqs. (269) and (271). Substituting these equations into the bosonic and fermionic delta functions yields:

$$(2\pi)^4\;\delta^{4|0}\big(P^{A\dot{A}}\big)\,\delta^{0|8}\big(Q^{\alpha A}\big) \;=\; \int\limits_{\mathbf{R}^{4|8}} d^{4|8}\mathsf{x}\;\bigwedge_{i=1}^{n}\exp\big(i\langle z_i|\theta\cdot\eta_i\rangle + is_i\langle z_i|x|\bar{z}_i]\big). \tag{273}$$

Pre-moduli Superspace. We now use the integral formula for the R-invariant in Eq. (265) and the expansion of the distributional prefactor in Eq. (273) to recast the partial amplitude $\mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}$ as follows.

Define the *pre-moduli superspace*:

$$\widehat{\mathscr{M}_3} \;:=\; \mathbf{R}^{4|8}\times\mathcal{P}. \tag{274}$$

Its global chart is given by the moduli coordinates:

$$\hat{\gamma}^Q \;:=\; \big(\mathsf{x}^K,\tau^M\big), \tag{275}$$

with abstract index $Q\in\{K,M\}$. The standard orientation on $\widehat{\mathscr{M}_3}$ is fixed by the $\mathbf{Z}_2$-graded volume form:

$$\mathcal{D}\hat{\gamma} \;:=\; d^{4|8}\mathsf{x}\wedge d^{8|4}\tau. \tag{276}$$

Comment. In the next subsections we will show that the leaf amplitude arises from a dimensional reduction of $\widehat{\mathscr{M}_3}$ to the moduli superspace $\mathscr{M}_3$. The latter parametrises three minitwistor lines on which the amplitude localises. This construction motivates the names “pre-moduli” superspace and “moduli” coordinates.

Finally, substituting Eqs. (265) and (273) into (268) yields:

$$\mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(z^A_i, s_i\bar{z}_{i\dot{A}},\eta^\alpha_i\big) \;=\; \mathcal{N}_{ab}\int\limits_{\widehat{\mathscr{M}_3}}\mathcal{D}\hat{\gamma}\;\;\mathcal{F}_{ab}\,(\tau)\;\mathsf{Tr}\bigwedge_{i=1}^{n} e^{i\langle z_i|(\theta+\tilde{q}_i)\cdot\eta_i\rangle+is_i\langle z_i|x+\widetilde{\mathcal{Q}}_i|\bar{z}_i]}\;\frac{\mathsf{T}^{a_i}}{z_i\cdot z_{i+1}}. \tag{277}$$

Mellin Transform. We conclude by computing the Mellin transform of the amplitude (267). This yields the N^1-MHV *celestial* superamplitude:

$$\widehat{\mathcal{A}}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha\big).$$

Let $\mathscr{R} := \big(\mathbf{R}_+, \cdot\big)$ denote the multiplicative group of positive real numbers. We regard the frequencies s_i as affine coordinates on $\mathscr{R}$. Consider the n-fold product group $\mathscr{R}^n := \bigtimes^n \mathscr{R}$ with global coordinates (s_i) and Haar measure:

$$d\rho_{s_i} \;=\; \bigwedge_{i=1}^{n} d\log s_i. \tag{278}$$

For each external gluon $i = 1, ..., n$, let Δ_i be its celestial conformal weight and ϵ_i its helicity expectation value. Define the scaling dimension:

$$h_i \;:=\; \frac{\Delta_i + \epsilon_i}{2}. \tag{279}$$

Furthermore, denote by:

$$\mathsf{W}_i^I \;:=\; \big(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha\big) \tag{280}$$

the dual real minitwistor encoding the insertion point $\{z_i^A, \bar{z}_{i\dot{A}}\}$ on the celestial torus $\mathcal{CT}$ and the helicity state η_i^α.

The *tree-level N^1-MHV celestial superamplitude* is then defined by the n-dimensional Mellin transform:

$$\widehat{\mathcal{A}}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}_i^I\big) \;:=\; \int\limits_{\mathscr{R}^n} d\rho_{s_i}\;\; \mathcal{A}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(z_i^A, s_i\bar{z}_{i\dot{A}}, \eta_i^\alpha\big) \prod_{i=1}^{n} s_i^{2h_i}. \tag{281}$$

Substituting Eq. (277) into this definition shows that the celestial amplitude decomposes as:

$$\widehat{\mathcal{A}}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}_i^I\big) \;=\; \sum_{a,b} \mathscr{P}_{ab}\, \widehat{\mathcal{A}}_{n;ab}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}_i^I\big), \tag{282}$$

where the *partial celestial superamplitude* takes the form:

$$\widehat{\mathcal{A}}_{n;ab}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}_i^I\big) \;=\; \int\limits_{\widehat{\mathscr{M}}_3} \mathcal{D}\hat{\gamma}\;\; \mathcal{F}_{ab}\,(\tau)\; \mathsf{Tr} \bigwedge_{i=1}^{n} \frac{\mathcal{C}\,(2h_i)}{\langle z_i|x + \widetilde{\mathcal{Q}}_i|\bar{z}_i]^{2h_i}}\; \exp\big(i\langle z_i|\,(\theta + \tilde{q}_i)\cdot\eta_i\rangle\big)\; \frac{\mathsf{T}^{a_i}}{z_i \cdot z_{i+1}}. \tag{283}$$

For simplicity, we henceforth focus on $\widehat{\mathcal{A}}_{n;ab}^{\mathsf{a}_1\ldots\mathsf{a}_n}$ and refer to it simply as the N^1-MHV celestial amplitude.

6. *Sectional Amplitude*

We now dimensionally reduce the integral over $\mathbf{R}^{4|8}$ in Eq. (283). This reduction yields the sectional (or leaf) amplitude

$$\mathcal{M}_{n;ab}^{\mathsf{a}_1\ldots\mathsf{a}_n}(\mathsf{W}_i^I).$$

It is defined by an integral over the real projective superspace $\mathbf{RP}^{3|8}$. We will show below that its minitwistor transform localises on a family of three minitwistor lines in $\mathbf{MT}_s$.

Klein and Projective Spaces. We begin by defining the geometric framework on which the sectional amplitude is built. First, we introduce a coordinate chart on the timelike wedge W^- of Klein space $\mathbf{K}^4$. This chart is related to homogeneous coordinates on $\mathbf{RP}^3$ and is adapted to the standard foliation of $\mathbf{K}^4$ by Lorentzian hyperbolic leaves. We then extend this construction to Klein superspace $\mathbf{K}^{4|8}$ to accommodate the celestial and minitwistor superamplitudes.

Klein space can be partitioned into the lightcone Λ and the timelike and spacelike wedges, denoted W^- and W^+, respectively. The timelike wedge W^- is defined as the set of all $x_{A\dot{A}} \in \mathbf{K}^4$ such that $x^2 := x_{A\dot{A}}x^{A\dot{A}} < 0$.

Let $X_{A\dot{A}}$ be homogeneous coordinates on $\mathbf{RP}^3$. We define the projective coordinates:

$$\mathcal{R}_{A\dot{A}} := |X|^{-1} X_{A\dot{A}}. \tag{284}$$

By construction, $\mathcal{R}_{A\dot{A}}$ is invariant under rescalings $X_{A\dot{A}} \mapsto tX_{A\dot{A}}$ with $t > 0$.

Let r be an affine parameter on the multiplicative group of positive real numbers $\mathscr{R}$. The coordinate system $\mathcal{X} := \left(r, \mathcal{R}_{A\dot{A}}\right)$ charts W^- via the bijection:

$$p \in W^- \mapsto \mathcal{X}(p) = \left(r(p), \mathcal{R}_{A\dot{A}}(p)\right) \in \mathbf{R}_+ \times \mathbf{RP}^3. \tag{285}$$

The map from $\mathcal{X}(p)$ to spacetime coordinates $x_{A\dot{A}}(p)$ is given by:

$$x_{A\dot{A}}\left(p\right) = r\left(p\right) \mathcal{R}_{A\dot{A}}\left(p\right). \tag{286}$$

In terms of the coordinate system $\mathcal{X}$, the Lebesgue measure decomposes on W^- as:

$$d^4x\,\big|_{W^-} = r^4\, d\rho_r \wedge \frac{D^3X}{|X|^4}. \tag{287}$$

Here, D^3X is the canonical volume form on $\mathbf{RP}^3$, and $d\rho_r := d\log r$ is the Haar measure on $\mathscr{R}$.

Klein and Projective Superspaces. We extend the foregoing construction to its supersymmetric analogue. Define the (3|8)-dimensional projective superspace as the trivial superbundle:

$$\mathbf{RP}^{3|8} \simeq \mathbf{RP}^3 \times \mathbf{R}^{0|8}. \tag{288}$$

Its typical fibre is the vector superspace spanned by the Grassmann coordinates θ^α_A, equipped with the Berezin measure $d^{0|8}\theta$. Introduce global coordinates:

$$\mathbb{X}^K \coloneqq \big(X_{A\dot{A}}, \theta^\alpha_A\big), \tag{289}$$

and fix the orientation by the $\mathbf{Z}_2$-graded volume form:

$$D^{3|8}\mathbb{X} \coloneqq \frac{D^3 X}{|X|^4} \wedge d^{0|8}\theta. \tag{290}$$

In complete analogy, define Klein superspace as:

$$\mathbf{K}^{4|8} \simeq \mathbf{K}^4 \times \mathbf{R}^{0|8}. \tag{291}$$

It is charted by $\mathsf{x}^K = \big(x_{A\dot{A}}, \theta^\alpha_A\big)$ and oriented by the Berezin-de Witt volume superform $d^{4|8}\mathsf{x}$ (refer to Eq. (272)).

Finally, let $W_s^- \subset \mathbf{K}^{4|8}$ denote the supersymmetric timelike wedge. By definition,

$$\mathsf{x}^K \in W_s^- \iff x_{A\dot{A}} \in W^-. \tag{292}$$

Upon restriction to W_s^-, the Berezin-de Witt superform decomposes as:

$$d^{4|8}\mathsf{x}\,|_{W_s^-} = r^4\, d\rho_r \wedge D^{3|8}\mathbb{X}. \tag{293}$$

Partial Amplitudes. Having introduced the necessary geometric structures on Klein superspace, we now implement the leaf amplitude formalism.

Recall the normalised basis of van de Waerden spinors $\{z^A_i, \bar{z}_{i\dot{A}}\}$, where $z^A_i = (1, -\zeta_i)$ and $\bar{z}_{i\dot{A}} = (1, -\bar{\zeta}_i)$. Here $\big(\zeta_i, \bar{\zeta}_i\big)$ are planar coordinates on the celestial torus $\mathcal{CT}$. This spinor basis parametrises the insertion point of the i-th gluon on $\mathcal{CT}$.

We introduce an involution $\sharp$ on dotted spinors by:

$$\bar{z}_{i\dot{A}} \mapsto \bar{z}^\sharp_{i\dot{A}} \coloneqq \big(1, \bar{\zeta}_i\big)\,. \tag{294}$$

Equivalently, on planar coordinates it acts as $\big(\zeta_i, \bar{\zeta}_i\big) \mapsto \big(\zeta_i, -\bar{\zeta}_i\big)$. This involution extends to dual minitwistors via:

$$\mathsf{W}^I_i \coloneqq \big(z^A_i, \bar{z}_{i\dot{A}}, \eta^\alpha_i\big) \mapsto \mathsf{W}^{\sharp I}_i \coloneqq \big(z^A_i, \bar{z}^\sharp_{i\dot{A}}, \eta^\alpha_i\big). \tag{295}$$

The first step in the leaf-amplitude algorithm is the decomposition:

$$\widehat{\mathcal{A}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^I_i\big) \;=\; \mathcal{B}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^I_i\big) \;+\; \mathcal{B}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^{\sharp I}_i\big). \tag{296}$$

To write the partial amplitude $\mathcal{B}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}$, we next specify the integration domain. Define the *moduli superspace* for N^1-MHV sectional amplitudes as the supermanifold:

$$\mathscr{M}_3 \;\coloneqq\; \mathbf{RP}^{3|8} \times \mathbf{R}^{8|4}. \tag{297}$$

It is globally charted by the coordinates:

$$\gamma^Q \;\coloneqq\; \big(\mathbb{X}^K, \tau^M\big), \tag{298}$$

and oriented by the measure:

$$\mathcal{D}\boldsymbol{\gamma} \;\coloneqq\; D^{3|8}\mathbb{X} \wedge d^{8|4}\tau. \tag{299}$$

The partial amplitude then takes the form:

$$\mathcal{B}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^I_i\big) \;=\; \int\limits_{\mathscr{R}} d\rho_r\, r^4 \int\limits_{\mathscr{M}_3} \mathcal{D}\boldsymbol{\gamma}\;\; \mathcal{F}_{ab}\left(\tau\right)\, \mathsf{Tr} \bigwedge_{i=1}^{n} \frac{\mathcal{C}\left(2h_i\right)}{\langle z_i|r\mathcal{R} + \widetilde{\mathcal{Q}}_i|\bar{z}_i]^{2h_i}}\, e^{i\langle z_i|(\theta+\tilde{q}_i)\cdot\eta_i\rangle}\, \frac{\mathsf{T}^{a_i}}{z_i\cdot z_{i+1}}. \tag{300}$$

Dimensional Reduction. The final step in reducing the celestial superamplitude to an integral over the moduli superspace $\mathscr{M}_3$ is the integration over the affine parameter r in Eq. (300).

Under the rescaling:

$$\tau^M = \big(u^A, v^B, \tilde{u}_{\dot{A}}, \tilde{v}_{\dot{B}}, \chi^\alpha\big) \;\mapsto\; \tilde{\tau}^M \coloneqq \big(u^A, v^B, r\tilde{u}_{\dot{A}}, r\tilde{v}_{\dot{B}}, \chi^\alpha\big), \tag{301}$$

the measure on moduli superspace transforms as:

$$\mathcal{D}\boldsymbol{\gamma} \;\mapsto\; r^4\,\mathcal{D}\boldsymbol{\gamma}. \tag{302}$$

Similarly, the embedding coordinates of Eq. (262) rescale according to:

$$\widetilde{\mathcal{Q}}^{A\dot{A}}_i \;\mapsto\; r\,\widetilde{\mathcal{Q}}^{A\dot{A}}_i,\;\; \tilde{q}^{\alpha A}_i \;\mapsto\; \tilde{q}^{\alpha A}_i. \tag{303}$$

Substituting these into Eq. (300) shows that all factors of r decouple and can be integrated explicitly. One thus obtains:

$$\mathcal{B}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^I_i\big) \;=\; 2\pi\delta(\beta_1)\,\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^I_i\big), \tag{304}$$

where the scaling parameter:

$$\beta_1 \;\coloneqq\; 8 - 2\sum_{i=1}^{n} h_i, \tag{305}$$

encodes the total scaling dimension of the process.

Finally, the sectional amplitude for n-gluon scattering in an N^1-MHV configuration reads:

$$\mathcal{M}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;ab}\left(\mathsf{W}^I_i\right) = \int\limits_{\mathscr{M}_3} \mathcal{D}\gamma \;\; \mathcal{F}_{ab}\left(\tau\right) \; \mathsf{Tr} \bigwedge_{i=1}^{n} \frac{\mathcal{C}\left(2h_i\right)}{\langle z_i | \mathcal{R} + \widetilde{\mathcal{Q}}_i | \bar{z}_i]^{2h_i}} \; \exp\left(i \langle z_i | \left(\theta + \tilde{q}_i\right) \cdot \eta_i \rangle\right) \; \frac{\mathsf{T}^{a_i}}{z_i \cdot z_{i+1}}. \tag{306}$$

7. *Geometrical Formulation*

Now we derive the main result of this section. By virtue of the celestial RSVW identity, the *minitwistor amplitude*

$$\widetilde{\mathcal{M}}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;ab}\left(\mathsf{Z}^I_i\right)$$

admits an elegant geometric and physical interpretation. It is realised as a volume integral over the moduli superspace $\mathscr{M}_3$. This observation suggests that the minitwistor amplitude may play a more fundamental role than the original celestial amplitude in the construction of a holographic dual to perturbative gauge theory (and perhaps $\mathcal{N} = 8$ Supergravity) in asymptotically flat spacetimes.

Celestial RSVW Identity. The derivation begins by reformulating the celestial RSVW identity in terms of minitwistor geometry. Recall that $\mathbf{RP}^{3|8}$ is the superspace parametrising minitwistor lines in $\mathbf{MT}_s$. For each point $p \in \mathbf{RP}^{3|8}$, the minitwistor line $\mathscr{L}(p)$ is the set of minitwistors:

$$\mathsf{Z}^I \;\coloneqq\; \left(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\right) \in \mathbf{MT}_s$$

obeying the incidence relations:

$$\begin{cases} \mu_{\dot{A}} = \lambda^A \mathcal{R}_{A\dot{A}}(p), \\ \psi^\alpha = \lambda^A \theta^\alpha_A\left(p\right). \end{cases} \tag{307}$$

Now, let $\pi_p : \mathscr{L}(p) \longrightarrow \mathbf{RP}^1$ be the canonical projection and choose homogeneous coordinates $[\sigma^A]$ on $\mathscr{L}(p)$. These coordinates trivialise the fibration π_p. A smooth section of π_p embeds $\mathbf{RP}^1$ into $\mathbf{MT}_s$ as the minitwistor line $\mathscr{L}(p)$. Therefore, define $\mathsf{Y}^I_p : \mathbf{RP}^1 \longrightarrow \mathscr{L}\left(p\right)$ via:

$$\mathsf{Y}^I_p\left(\sigma^A\right) \;\coloneqq\; \left(\sigma^A, \sigma^A \mathcal{R}_{A\dot{A}}(p), \sigma^A \theta^\alpha_A(p)\right). \tag{308}$$

By construction, $\pi \circ \mathsf{Y}^I_p\left(\sigma^A\right) = \sigma^A$. So Y^I_p provides the desired embedding.

We next consider integration over $\mathscr{L}(p)$. In the trivialisation $[\sigma^A]$, the natural measure is $D\sigma \coloneqq \varepsilon_{AB}\sigma^A d\sigma^B$. Let f be a smooth section of $\mathcal{O}(-2)$ on $\mathscr{L}(p)$, so that for any $t > 0$,

$$f\left(t\sigma^A\right) = t^{-2} f\left(\sigma^A\right). \tag{309}$$

Then the differential form:

$$\boldsymbol{\omega}_f\big(\sigma^A\big) \;\coloneqq\; f\big(\sigma^A\big)\, D\sigma, \tag{310}$$

defines a well-posed integration measure on $\mathscr{L}(p)$.

Finally, over the dual minitwistor superspace $\widehat{\mathbf{MT}}^{2|4}$, define the section:

$$\Phi_{\Delta,p}\big(\mathsf{W}^I\big) \;\coloneqq\; \frac{\mathcal{C}\,(\Delta)}{\langle z|\mathcal{R}|\bar{z}]^{\Delta}}\, e^{i\langle z|\theta\cdot\eta\rangle}\, f\big(z^A\big), \tag{311}$$

where $\mathsf{W}^I = \big(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big)$. The celestial RSVW identity then takes the form:

$$\Phi_{\Delta,p}\big(\mathsf{W}^I\big) \;=\; \int\limits_{\mathbf{MT}^{2|4}} D^{2|4}\mathsf{Z}\;\; \Psi_\Delta\big(\mathsf{Z}^I;\mathsf{W}^{I'}\big) \int\limits_{\mathbf{RP}^1} \boldsymbol{\omega}_f\big(\sigma^A\big)\, \overline{\delta}^{2|4}_{(-\Delta,\Delta)}\big(\mathsf{Z}^I;\mathsf{Y}^{I'}\big(\sigma^A\big)\big). \tag{312}$$

Minitwistor Amplitude. Substituting the reformulated celestial RSVW identity (Eq. (312)) into the sectional amplitude (Eq. (306)) yields the following representation.

Let $\ell = 1,2,3$. Recall that $\mathcal{P} \simeq \mathbf{R}^{8|4}$ is the parameter superspace charted by moduli τ^M. The embedding coordinates $\mathcal{Q}_\ell^{A\dot{A}}$ and $q_\ell^{\alpha A}$ are defined in Subsection III A 4.

For each point $\gamma^Q = \big(\mathbb{X}^K, \tau^M\big) \in \mathscr{M}_3$, the minitwistor line $\mathscr{L}_\ell\big(\gamma^Q\big)$ is the set of minitwistors Z^I obeying the incidence relations:

$$\begin{cases} \mu_{\dot{A}} = \lambda^A\big(\mathcal{R}_{A\dot{A}} + \mathcal{Q}_{\ell A\dot{A}}\big), \\ \psi^\alpha = \lambda^A\big(\theta^\alpha_A + q^\alpha_{\ell A}\big). \end{cases} \tag{313}$$

At fixed γ^Q, the conics $\big\{\mathscr{L}_\ell\big(\gamma^Q\big)\big\}_{\ell=1}^3$ form a triplet of minitwistor lines. Varying γ^Q over $\mathscr{M}_3$ sweeps out all such triplets. Thus $\mathscr{M}_3$ serves as the moduli superspace of three-line configurations.

Next, embed $\mathbf{RP}^1$ into each line $\mathscr{L}_\ell\big(\gamma^Q\big)$ via $\mathsf{Y}^I_\ell : \mathbf{RP}^1 \longrightarrow \mathscr{L}_\ell\big(\mathbb{X}^K\big)$, such that:

$$\mathsf{Y}^I_\ell\big(\sigma^A\big) \;\coloneqq\; \big(\sigma^A, \sigma^A\big(\mathcal{R}_{A\dot{A}} + \mathcal{Q}_{\ell A\dot{A}}\big), \sigma^A\big(\theta^\alpha_A + q^\alpha_{\ell A}\big)\big). \tag{314}$$

This map is a smooth section of the fibration $\pi_\ell : \mathscr{L}_\ell\big(\gamma^Q\big) \longrightarrow \mathbf{RP}^1$.

The sectional amplitude then becomes a multi-dimensional minitwistor transform. Writing $\mathbf{M}^n \coloneqq \bigtimes^n \mathbf{MT}_s$, one has:

$$\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^I_i\big) \;=\; \int\limits_{\mathbf{M}^n} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}_i\, \Psi_{2h_i}\big(\mathsf{Z}^I_i;\mathsf{W}^{I'}_i\big)\;\; \widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{Z}^I_i\big). \tag{315}$$

Finally, introduce the Lie-algebra–valued logarithmic one-form on $\mathbf{RP}^1$:

$$\boldsymbol{\omega}^{a_i}\big(\sigma^A_i\big) \;\coloneqq\; \mathsf{T}^{a_i}\, \frac{D\sigma_i}{\sigma_i\cdot\sigma_{i+1}}. \tag{316}$$

With this definition, the N^1-*MHV minitwistor amplitude* is:

$$\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{Z}^I_i\big) \;=\; \int\limits_{\mathscr{M}_3} \mathcal{D}\gamma\;\; \mathcal{F}_{ab}\big(\gamma^Q\big)\,\mathsf{Tr} \bigwedge_{i=1}^{n} \int\limits_{\mathbf{RP}^1} \boldsymbol{\omega}^{a_i}\big(\sigma^A_i\big)\, \overline{\delta}^{2|4}_{(-2h_i,2h_i)}\big(\mathsf{Z}^I;\mathsf{Y}^{I'}_{c(i)}\big(\sigma^A_i\big)\big). \tag{317}$$

Interpretation. The function $\mathcal{F}_{ab}$ (Eq. (260)) lifts to a probability distribution on the moduli superspace. In our discussion of minitwistor celestial CFT, we shall interpret Eq. (317) as the semiclassical expectation value of the observable $\mathcal{F}_{ab}$.

The minitwistor amplitude $\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}$ derived in Eq. (317) computes a volume integral over $\mathscr{M}_3$ weighted by the distribution $\mathcal{F}_{ab}$. The volume form is localised on the minitwistor lines $\mathscr{L}_\ell(\gamma^Q)$ via the delta functions:

$$\bar{\delta}^{2|4}_{(-2h_i,2h_i)}\big(\mathsf{Z}^I;\mathsf{Y}^{I'}_{c(i)}(\sigma_i^A)\big).$$

These factors are supported precisely on the triplet $\{\mathscr{L}_\ell(\gamma^Q)\}_{\ell=1}^{3}$. They are modulated by the celestial scaling dimensions h_i of the external gluons.

Furthermore, the amplitude vanishes whenever the insertion point of the i-th gluon does not lie on the conic $\mathscr{L}_\ell(\gamma^Q)$, for $\ell = c(i)$ its cluster assignment (see Subsection III A 4, Eq. (260)).

B. N^2-MHV Scattering Amplitude

We now construct the celestial and minitwistor superamplitudes for next-to-next-MHV (N^2-MHV) gluon scattering. Our immediate goal is to derive the explicit N^2-MHV *celestial* amplitude. More importantly, we aim to extend this approach to the full tree-level $\mathcal{S}$-matrix of $\mathcal{N} = 4$ SYM theory in Subsection III C.

1. *Order-2 R-Invariant*

To write the tree-level N^2-MHV superamplitude, we introduce the *order-2 R-invariant*, $R_{n;a_1b_1,a_2b_2}$. We begin by defining two auxiliary spinors in terms of the dual coordinates $y_i^{A\dot{A}}$. Let:

$$u_1^A \coloneqq z_n^B\,(y_{nb_1})_{B\dot{B}}\,(y^{-1}_{b_1a_1})^{A\dot{B}},\ \ v_1^A \coloneqq z_n^B\,(y_{na_1})_{B\dot{B}}\,(y^{-1}_{a_1b_1})^{A\dot{B}}. \tag{318}$$

From u_1^A we then define two spinor-valued functions:

$$\tilde{f}^A_{a_1a_2b_2}(u_1^A) \coloneqq u_1^B\,(y_{a_1b_2})_{B\dot{B}}\,(y^{-1}_{b_2a_2})^{A\dot{B}}, \tag{319}$$

$$\tilde{g}^A_{a_1a_2b_2}(u_1^A) \coloneqq u_1^B\,(y_{a_1a_2})_{B\dot{B}}\,(y^{-1}_{a_2b_2})^{A\dot{B}}. \tag{320}$$

Next, introduce a second pair of spinors (u_2^A, v_2^B) to parametrise $\mathcal{I} \coloneqq \mathbf{R}^2\times\mathbf{R}^2$. Their reality follows from the Kleinian signature. The *constraint hypersurface* $\mathscr{C}\subset\mathcal{I}$ is then given by:

$$u_2^A = \tilde{f}^A_{a_1a_2b_2}(u_1^B),\ \ v_2^A = \tilde{g}^A_{a_1a_2b_2}(u_1^B). \tag{321}$$

Finally, in the normalised spinor basis $\{z_i^A, \bar{z}_{i\dot{A}}\}$ that labels insertion points on the celestial torus $\mathcal{CT}$, the order-2 R-invariant is:

$$R_{n;a_1b_1,a_2b_2}\big(u_2^A, v_2^B, y_i^{C\dot{C}}, z_i^D\big) \;\coloneqq\; \frac{\langle z_{a_2-1}, z_{a_2}\rangle\langle z_{b_2-1}, z_{b_2}\rangle\delta^{0|4}\big(\Theta_2^\alpha\big)}{y_{a_2b_2}^2\langle z_{a_2-1}, u_2\rangle\langle u_2, z_{a_2}\rangle\langle z_{b_2-1}, v_2\rangle\langle v_2, z_{b_2}\rangle}, \tag{322}$$

for $\big(u_2^A, v_2^B\big) \in \mathscr{C}$. Here,

$$\Theta_2^\alpha\big(u_2^A, v_2^B\big) \;\coloneqq\; \sum_{i=a_1}^{a_2-1} \langle u_2, z_i\rangle\, \eta_i^\alpha + \sum_{j=a_1}^{b_2-1} \langle v_2, z_j\rangle\, \eta_j^\alpha. \tag{323}$$

2. Fadde'ev-Popov Representation

We now apply the Fadde'ev-Popov method to derive an integral representation of the order-2 R-invariant. This representation is tailored for the subsequent Mellin transform of the N^2-MHV superamplitude.

First, we impose Eq. (321) for the spinor variables u_2^A and v_2^A by inserting delta functions that localise their integration to the constraint hypersurface $\mathscr{C}$. We write:

$$R_{n;a_1b_1,a_2b_2} \;=\; \frac{1}{y_{a_2b_2}^2} \int_{\mathcal{I}} d^2u_2 \wedge d^2v_2 \quad \mathcal{F}_{a_2b_2}\big(u_2^A, v_2^B\big)\, \delta^{0|4}\big(\Theta_2^\alpha\big)\, \delta_{\mathscr{C}}\big(u_2^A, v_2^B\big). \tag{324}$$

Here,

$$\delta_{\mathscr{C}}\big(u_2^A, v_2^B\big) \;\coloneqq\; \overline{\delta}^2\big(u_2^A - \tilde{f}^A_{a_1a_2b_2}\big(u_1^B\big)\big)\, \overline{\delta}^2\big(v_2^A - \tilde{g}^A_{a_1a_2b_2}\big(u_1^B\big)\big) \tag{325}$$

enforces $u_2^A = \tilde{f}^A_{a_1a_2b_2}\big(u_1^B\big)$ and $v_2^A = \tilde{g}^A_{a_1a_2b_2}\big(u_1^B\big)$ on the integration variables. The function appearing under the integral (324) is:

$$\mathcal{F}_{a_2b_2}\big(u_2^A, v_2^B\big) \;\coloneqq\; \frac{\langle z_{a_2-1}, z_{a_2}\rangle\langle z_{b_2-1}, z_{b_2}\rangle}{\langle z_{a_2-1}, u_2\rangle\langle u_2, z_{a_2}\rangle\langle z_{b_2-1}, v_2\rangle\langle v_2, z_{b_2}\rangle}. \tag{326}$$

Comment. We have not imposed the definitions of u_1^A and v_1^A from Eq. (318) as delta-function constraints here. Since in the N^2-MHV superamplitude only the product:

$$R_{n;a_1b_1}R_{n;a_1b_1,a_2b_2},$$

appears, the Fadde'ev-Popov representation of the order-1 R-invariant, $R_{n;a_1b_1}$, already enforces those spinor definitions.

3. Fermionic and Bosonic Delta-functions

Equation (324) expresses the order-2 R-invariant such that all dependence on the frequency parameters s_i factorises through the constraint delta function $\delta_{\mathscr{C}}$. Our next task is to find an integral representation for the delta functions in Eq. (324).

We begin with the Grassmann delta function $\delta^{0|4}(\Theta_2^\alpha)$. Recall from Subsection III A 2 that the Berezin-integral representation of the fermionic delta function takes the form:

$$\delta^{0|4}(\varepsilon^\alpha) \;=\; \int d^{0|4}\chi \;\; \exp\left(i\chi_\alpha \varepsilon^\alpha\right). \tag{327}$$

Substituting Eq. (323) into this form yields:

$$\delta^{0|4}(\Theta_2^\alpha) \;=\; \int d^{0|4}\chi_2 \bigwedge_{i=a_1}^{a_2-1} \exp\left(i\langle u_2, z_i\rangle\, \chi_2 \cdot \eta_i\right) \bigwedge_{i=a_1}^{b_2-1} \exp\left(i\langle v_2, z_i\rangle\, \chi_2 \cdot \eta_i\right). \tag{328}$$

Next, we analyse the holomorphic delta functions in Eq. (325). Substituting Eq. (325) into the integral form of $\overline{\delta}^2$, as defined in Eq. (248), we obtain:

$$\overline{\delta}^2\left(u_2^A - \tilde{f}^A_{a_1a_2b_2}(u_1^A)\right) \;=\; \int\limits_{\mathbf{R}^2} \frac{d^2\hat{u}_2}{(2\pi)^2}\; e^{i\langle u_2 \hat{u}_2\rangle}\, \exp\left(-\,i\langle u_1|y_{a_1b_2}y^{-1}_{b_2a_2}|\hat{u}_2]\right), \tag{329}$$

and similarly,

$$\overline{\delta}^2\left(v_2^A - \tilde{g}^A_{a_1a_2b_2}(u_1^A)\right) \;=\; \int\limits_{\mathbf{R}^2} \frac{d^2\hat{v}_2}{(2\pi)^2}\; e^{i\langle v_2 \hat{v}_2\rangle}\, \exp\left(-\,i\langle u_1|y_{a_1a_2}y^{-1}_{a_2b_2}|\hat{v}_2]\right). \tag{330}$$

We now change integration variables by:

$$\tilde{u}_2^{\dot{A}} \;\coloneqq\; \left(y^{-1}_{b_2a_2}\right)^{A\dot{A}}\, \hat{u}_{2A}, \;\; \tilde{v}_2^{\dot{A}} \;\coloneqq\; \left(y^{-1}_{a_2b_2}\right)^{A\dot{A}}\, \hat{v}_{2A}. \tag{331}$$

In these variables, Eqs. (329) and (330) become:

$$\overline{\delta}^2\left(u_2^A - \tilde{f}^A_{a_1a_2b_2}(u_1^A)\right) \;=\; |y^2_{b_2a_2}| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{u}_2}{(2\pi)^2} \prod_{i=a_2}^{b_2-1} e^{-is_i\langle z_i|u_2\tilde{u}_2|\bar{z}_i]} \prod_{j=a_1}^{b_2-1} e^{-is_i\langle z_j|u_1\tilde{u}_2|\bar{z}_j]}, \tag{332}$$

and:

$$\overline{\delta}^2\left(v_2^A - \tilde{g}^A_{a_1a_2b_2}(u_1^A)\right) \;=\; |y^2_{a_2b_2}| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{v}_2}{(2\pi)^2} \prod_{i=a_2}^{b_2-1} e^{is_i\langle z_i|v_2\tilde{v}_2|\bar{z}_i]} \prod_{j=a_1}^{a_2-1} e^{-is_j\langle z_j|u_1\tilde{v}_2|\bar{z}_j]}. \tag{333}$$

4. Degree-2 R-monomial

Recap. In Subsection III B 2, the Fadde'ev-Popov method was used to derive the integral formula for the order-2 R-invariant (see Eq. (324)). In that formula, the dependence on the frequency parameters s_i is completely factorised into delta functions. In Subsection III B 3, those delta functions were then expanded via their integral representations in terms of the celestial coordinates z_i^A, $\bar{z}_{i\dot{A}}$ and η_i^α.

We next consider the *degree*-2 *R-monomial*, defined by:

$$R_n^{(2)} \;\coloneqq\; R_{n;a_1b_1} R_{n;a_1b_1,a_2b_2}. \tag{334}$$

To represent $R_n^{(2)}$ as an integral, we proceed in two steps. First, substitute Eqs. (328) and (333) into the order-2 R-invariant, Eq. (324). Second, employ the result for $R_{n;a_1b_1}$ from Eq. (265). This then yields the following integral representation.

Integration Superdomain. The parameter superspace is defined by:

$$\mathcal{P} \coloneqq \mathbf{R}^{8|4} \times \mathbf{R}^{8|4}. \tag{335}$$

For each $k = 1, 2$, define the *moduli parameters*:

$$\tau_k^M \;\coloneqq\; \left(u_k^A, v_k^B, \tilde{u}_{k\dot{A}}, \tilde{v}_{k\dot{B}}, \chi_k^\alpha\right) \;\in \mathbf{R}^{8|4}, \tag{336}$$

where the abstract index M ranges over $\{A, B, \dot{A}, \dot{B}, \alpha\}$. Then $\mathcal{P}$ is globally charted by $\boldsymbol{\tau}^P \coloneqq \left(\tau_1^M, \tau_2^{M'}\right)$, with $P \in \{M, M'\}$. In Subsection III B 7 we will identify each component of τ_k^M with the moduli parameters of five minitwistor lines on which the N^2-MHV minitwistor superamplitude localises. This identification justifies reffering to $\mathcal{P}$ as the *parameter superspace* and to τ_k^M as its *moduli coordinates.*

The standard orientation on each copy of $\mathbf{R}^{8|4}$ is provided by the $\mathbf{Z}_2$-graded volume form:

$$d^{8|4}\tau_k \;\coloneqq\; d^2u_k \wedge d^2v_k \wedge d^2\tilde{u}_k \wedge d^2\tilde{v}_k \wedge d^{0|4}\chi_k. \tag{337}$$

Consequently, the integration measure on the parameter superspace $\mathcal{P}$ is:

$$d^{16|8}\boldsymbol{\tau} \;\coloneqq\; d^{8|4}\tau_1 \wedge d^{8|4}\tau_2. \tag{338}$$

Embedding Coordinates. Let $\mathscr{M}_5$ denote the moduli superspace characterising the configuration of a system comprising five minitwistor lines. We will specify $\mathscr{M}_5$ explicitly in the forthcoming subsection. For now, let us regard $\mathscr{M}_5$ as an abstract supermanifold.

For each gluon i participating in the N^2-MHV scattering process, we assign a copy $\mathcal{P}_i$ of the parameter superspace. A natural question then arises: what are the embedding coordinates of $\mathcal{P}_i$ in $\mathscr{M}_5$? The answer is given by Eqs. (328), (332) and (333).

Interval for ℓ	$\mathcal{Q}_\ell^{A\dot{A}}$	$q_\ell^{\alpha A}$
1	0	$-\left(u_1^A + v_1^A\right)\chi_1^\alpha$
2	$\mathcal{Q}^{A\dot{A}} - u_1^A\left(\tilde{u}_2^{\dot{A}} + \tilde{v}_2^{\dot{A}}\right)$	$-v_1^A\,\chi_1^\alpha - \left(u_2^A + v_2^A\right)\chi_2^\alpha$
3	$\mathcal{Q}^{A\dot{A}} - \left(u_1^A + u_2^A\right)\tilde{u}_2^{\dot{A}} + v_2^A\tilde{v}_2^{\dot{A}}$	$-v_1^A\chi_1^\alpha - v_2^A\chi_2^\alpha$
4	$\mathcal{Q}^{A\dot{A}}$	$-v_1^A\chi_1^\alpha$
5	$z_n^A\left(\tilde{u}_1^{\dot{A}} + \tilde{v}_1^{\dot{A}}\right)$	0

Table II. Embedding coordinates $\mathsf{Q}_\ell^K \coloneqq \left(\mathcal{Q}_\ell^{A\dot{A}}, q_\ell^{\alpha A}\right)$ on parameter superspace $\mathcal{P}$.

Let the index ℓ range over $\{1, ..., 5\}$, and define the auxiliary coordinate function:

$$\mathcal{Q}^{A\dot{A}} \;\coloneqq\; z_n^A\tilde{v}_1^{\dot{A}} - u_1^A\tilde{u}_1^{\dot{A}} + v_1^A\tilde{v}_1^{\dot{A}}. \tag{339}$$

Using this, define the coordinate maps:

$$\mathsf{Q}_\ell^K \;\coloneqq\; \left(\mathcal{Q}_\ell^{A\dot{A}}, q_\ell^{\alpha A}\right) : \mathcal{P} \longrightarrow \mathbf{R}^{4|8}, \tag{340}$$

with components given in Table II.

Next, let $c\,(i)$ denote the *indicator function* for the N^2-MHV scattering process. This map assigns to the i-th gluon its corresponding cluster. As we shall demonstrate in the following, each cluster lies within one of the five minitwistor lines described by $\mathscr{M}_5$. The clustering is defined by the prescription:

$$c(i) \;\coloneqq\; \begin{cases} 1, & 1 \le i \le a_1 - 1; \\ 2, & a_1 \le i \le a_2 - 1; \\ 3, & a_2 \le i \le b_2 - 1; \\ 4, & b_2 \le i \le b_1 - 1; \\ 5 & b_1 \le i \le n. \end{cases} \tag{341}$$

Consequently, the embedding coordinates of the copy $\mathcal{P}_i$ of the parameter superspace assigned to the i-th gluon are defined by:

$$\widetilde{\mathcal{Q}}_i^{A\dot{A}} \coloneqq \mathcal{Q}_{c(i)}^{A\dot{A}}, \;\; \tilde{q}_i^{\alpha A} \coloneqq q_{c(i)}^{\alpha A}. \tag{342}$$

Integral Representation. The degree-2 R-monomial admits a representation as an integral over the parameter superspace $\mathcal{P}$ given by:

$$R_n^{(2)} \;=\; \mathcal{N}_{a_1 b_1, a_2 b_2} \int\limits_{\mathcal{I}} d^{16|8}\boldsymbol{\tau}\;\; \mathcal{F}_{a_1 b_1, a_2 b_2}\left(\tau_1^M, \tau_2^{M'}\right) \;\bigwedge_{i=1}^{n}\, e^{is_i\langle z_i|\widetilde{\mathcal{Q}}_i|\bar{z}_i] + i\langle z_i|\tilde{q}_i\cdot\eta_i\rangle}. \tag{343}$$

The normalisation factor is defined by:

$$\mathcal{N}_{a_1b_1,a_2b_2} := \frac{1}{(2\pi)^8}\, y^2_{a_1b_1} y^2_{a_2b_2}, \tag{344}$$

and the integrand $\mathcal{F}_{a_1b_1,a_2b_2}$ takes the form:

$$\mathcal{F}_{a_1b_1,a_2b_2}\big(\tau_1^M, \tau_2^{M'}\big) := \mathcal{F}_{a_1b_1}\big(u_1^A, v_1^B\big)\, \mathcal{F}_{a_2b_2}\big(u_2^A, v_2^A\big). \tag{345}$$

Recall that $\mathcal{F}_{a_1b_1}$ was introduced in Eq. (326).

5. N^2-MHV Celestial Superamplitude

In the preceding subsection, we obtained an integral formula for the degree-2 R-monomial (refer to Eq. (343)). Its frequency dependence is entirely described by exponential functions. This representation is ready for the Mellin transform required to derive the celestial amplitude, which we now proceed to analyse.

Partial Amplitudes. The general solution for the tree-level N^2-MHV scattering amplitude in $\mathcal{N} = 4$ SYM theory takes the form:

$$\mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{2,n}\big(\lambda_i^A, \bar{\lambda}_{i\dot{A}}, \eta_i^\alpha\big) = \sum_{a_1,b_1}\sum_{a_2,b_2} \mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\lambda_i^A, \bar{\lambda}_{i\dot{A}}, \eta_i^\alpha\big). \tag{346}$$

Here, the partial amplitudes corresponding to each sequence of indices:

$$1 \le a_1 \le a_2 \le b_2 \le b_1 \le n, \tag{347}$$

are given by:

$$\mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\lambda_i^A, \bar{\lambda}_{i\dot{A}}, \eta_i^\alpha\big) = (2\pi)^4\, \delta^{4|0}\big(P^{A\dot{A}}\big)\, \delta^{0|8}\big(Q^{\alpha A}\big)\, R_{n;a_1b_1} R_{n;a_1b_1,a_2b_2}\, \mathsf{Tr} \prod_{i=1}^{n} \frac{\mathsf{T}^{a_i}}{\lambda_i \cdot \lambda_{i+1}}. \tag{348}$$

Celestial Parameterisation. The first step in constructing the celestial amplitude consists of expressing the partial amplitudes (348) in terms of celestial coordinates. This parameterisation is obtained by setting $\lambda_i^A = z_i^A$ and $\bar{\lambda}_{i\dot{A}} = s_i \bar{z}_{i\dot{A}}$.

Integral Representation. Our next task is to derive an integral formula for the N^2-MHV scattering amplitude. In Subsection III A 5 we showed that

$$(2\pi)^4\, \delta^{4|0}\big(P^{A\dot{A}}\big)\, \delta^{0|8}\big(Q^{\alpha A}\big)$$

admits a representation as a superspace integral (see Eq. (273)). This distributional factor enforces total momentum and supercharge conservation in the scattering process. By combining that result with the form of the degree-2 R-monomial from Eq. (343), we arrive at the following formulation.

The partial amplitude in Eq. (348) can be written as an integral over the *pre-moduli superspace*:

$$\widehat{\mathscr{M}_5} := \mathbf{R}^{4|8} \times \mathbf{R}^{8|4} \times \mathbf{R}^{8|4}. \tag{349}$$

We chart $\widehat{\mathscr{M}_5}$ by:

$$\hat{\gamma}^Q := \left(\mathsf{x}^K, \tau_1^M, \tau_2^{M'}\right), \tag{350}$$

where the abstract index $Q \in \{K, M, M'\}$ labels the superspace coordinates. In the next subsection, we will apply the leaf-amplitude formalism to reduce $\widehat{\mathscr{M}_5}$ to the moduli superspace $\mathscr{M}_5$. That space governs the configuration of five minitwistor lines on which the (minitwistor) amplitude localises. This justifies calling $\widehat{\mathscr{M}_5}$ the pre-moduli superspace.

The orientation on $\widehat{\mathscr{M}_5}$ is specified by the $\mathbf{Z}_2$-graded volume form:

$$\mathcal{D}\hat{\gamma} := d^{4|8}\mathsf{x} \wedge d^{8|4}\tau_1 \wedge d^{8|4}\tau_2. \tag{351}$$

Here $\mathsf{x}^K = \left(x_{A\dot{A}}, \theta_A^\alpha\right) \in \mathbf{R}^{4|8}$ denotes the standard coordinates on Klein superspace, equipped with the Berezin-de Witt measure $d^{4|8}\mathsf{x}$ defined in Eq. (272).

With these preliminaries in place, the N^2-MHV partial superamplitude becomes:

$$\mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\left(z_i^A, s_i\bar{z}_{i\dot{A}}, \eta_i^\alpha\right) = \int\limits_{\widehat{\mathscr{M}_5}} \mathcal{D}\hat{\gamma}\ \mathcal{F}_{a_1b_1,a_2b_2}\left(\hat{\gamma}^Q\right) \mathcal{T}^{\mathsf{a}_1\ldots\mathsf{a}_n}\left(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha; \hat{\gamma}^Q\right), \tag{352}$$

where the trace factor is given by:

$$\mathcal{T}^{\mathsf{a}_1\ldots\mathsf{a}_n}\left(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha; \hat{\gamma}^Q\right) = \mathsf{Tr} \bigwedge_{i=1}^{n} \exp\left(is_i\langle z_i|x + \widetilde{\mathcal{Q}}_i|\bar{z}_i] + i\langle z_i|(\theta + \tilde{q}_i)\cdot\eta_i\rangle\right) \frac{\mathsf{T}^{a_i}}{z_i \cdot z_{i+1}}. \tag{353}$$

Celestial Superamplitude. We now derive N^2-MHV celestial amplitude. To set notation, define the multiplicative group of positive reals $\mathscr{R} := \left(\mathbf{R}_+, \cdot\right)$, and its n-fold product $\mathscr{R}^n := \bigtimes_{i=1}^n \mathscr{R}$. We regard the frequency parameters (s_i) as affine coordinates on $\mathscr{R}^n$. This space carries the Haar measure $d\rho_{s_i}$, as in Eq. (278).

Next, recall that each gluon insertion on the celestial torus $\mathcal{CT}$ is parametrised by the spinors z_i^A and $\bar{z}_i^A$. The Grassmann variables η_i^α encode the helicity degrees of freedom. Together, these define the dual real minitwistor $\mathsf{W}_i^I := \left(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha\right)$.

For each gluon i, let Δ_i be its celestial conformal weight and ϵ_i its helicity expectation value. We then define the scaling dimension:

$$h_i \coloneqq \frac{\Delta_i + \epsilon_i}{2}. \tag{354}$$

With these preliminaries, we introduce the celestial (partial) superamplitude by a multidimensional Mellin transform:

$$\widehat{\mathcal{A}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}^I_i\big) \coloneqq \int\limits_{\mathscr{R}^n} d\rho_{s_i}\ \mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(z^A_i, s_i\bar{z}_{i\dot{A}}, \eta^\alpha_i\big) \prod_{i=1}^n s_i^{2h_i}. \tag{355}$$

Performing the s_i integrals yields an explicit expression as an integral over the pre-moduli superspace:

$$\widehat{\mathcal{A}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}^I_i\big) = \int\limits_{\widehat{\mathscr{M}}_5} \mathcal{D}\hat{\gamma}\ \mathcal{F}_{a_1b_1,a_2b_2}\big(\hat{\gamma}^Q\big)\ \widehat{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}^I_i;\hat{\gamma}^Q\big), \tag{356}$$

where the new trace factor is:

$$\widehat{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}^I_i;\hat{\gamma}^Q\big) = \mathsf{Tr}\bigwedge_{i=1}^n \frac{\mathcal{C}\big(2h_i\big)}{\langle z_i|x+\widetilde{\mathcal{Q}}_i|\bar{z}_i]^{2h_i}}\, e^{i\langle z_i|(\theta+\tilde{q})\cdot\eta_i\rangle}\,\frac{\mathsf{T}^{a_i}}{z_i\cdot z_{i+1}}. \tag{357}$$

6. *Sectional Amplitude*

Having obtained the celestial amplitude, we now invoke the leaf amplitude formalism. The sectional (or leaf) amplitude arises via a dimensional reduction of $\widehat{\mathscr{M}_5}$ to the moduli superspace $\mathscr{M}_5$, which parametrises a configuration of five minitwistor lines in $\mathbf{MT}_s$.

In this perspective, the minitwistor amplitude:

$$\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{Z}^I_i\big),$$

is viewed as more fundamental than the original celestial amplitude.

Before proceeding, the reader should review Subsection III A 6. That section recapitulates the geometric background on Klein and projective superspaces needed for the leaf amplitude formalism. In particular, one should pay special attention to the measure decomposition stated in Eq. (293).

Leaf Formalism. We now derive the leaf amplitude by decomposing the celestial amplitude (cf. Eq. (356)) into a sum of partial amplitudes.

To this end, we employ the involution operator $\sharp$ introduced in Subsection III A 6. For a van der Waerden spinor $\bar{z}_{\dot{A}} = (1, -\bar{\zeta})$, define its involute by $\bar{z}^\sharp_{\dot{A}} \coloneqq (1, \bar{\zeta})$. Then extend $\sharp$ to dual minitwistor superspace via:

$$\sharp : \mathsf{W}^I = \big(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\big) \mapsto \mathsf{W}^{\sharp I} = \big(z^A, \bar{z}^\sharp_{\dot{A}}, \eta^\alpha\big). \tag{358}$$

To define each partial amplitude, introduce the *moduli superspace*:

$$\mathscr{M}_5 \;\coloneqq\; \mathbf{RP}^{3|8} \times \mathbf{R}^{8|4} \times \mathbf{R}^{8|4}. \tag{359}$$

This supermanifold parametrises the configuration of five minitwistor lines supporting the N^2-MHV amplitude. A natural coordinate chart on $\mathscr{M}_5$ combines the projective superspace coordinates $\mathbb{X}^K$ with the two sets of moduli parameters τ_1^M and $\tau_2^{M'}$. We assemble these into a single coordinate map:

$$\gamma^Q \;\coloneqq\; \big(\mathbb{X}^K, \tau_1^M, \tau_2^{M'}\big), \tag{360}$$

where the abstract index $Q \in \{K, M, M'\}$ labels the superspace coordinates. The canonical orientation on $\mathscr{M}_5$ is then given by the $\mathbf{Z}_2$-graded volume form:

$$\mathcal{D}\boldsymbol{\gamma} \;\coloneqq\; D^{3|8}\mathbb{X} \wedge d^{8|4}\tau_1 \wedge d^{8|4}\tau_2. \tag{361}$$

As established in Melton, Sharma, and Strominger [77], the celestial amplitude admits the decomposition:

$$\widehat{\mathcal{A}}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}_i^I\big) \;=\; \mathcal{B}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}_i^I\big) + \mathcal{B}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}_i^{\sharp I}\big). \tag{362}$$

The partial amplitude is given by:

$$\mathcal{B}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}_i^I\big) \;=\; \int\limits_{\mathbf{R}_+} dH_r \;\, r^4 \int\limits_{\mathscr{M}_5} \mathcal{D}\boldsymbol{\gamma} \;\, \mathcal{F}_{a_1b_1,a_2b_2}\big(\tau_1^M, \tau_2^{M'}\big)\, \widehat{\mathcal{T}}^{\mathsf{a}_1,\ldots,\mathsf{a}_n}\big(\mathsf{W}_i^I; \gamma^Q\big), \tag{363}$$

and the trace factor takes the form:

$$\widehat{\mathcal{T}}^{\mathsf{a}_1,\ldots,\mathsf{a}_n}\big(\mathsf{W}_i^I; \gamma^Q\big) \;=\; \mathsf{Tr} \bigwedge_{i=1}^{n} \frac{\mathcal{C}\,(2h_i)}{\langle z_i | r\mathcal{R} + \widetilde{\mathcal{Q}}_i | \bar{z}_i]^{2h_i}} \, e^{i\langle z_i | (\theta + \tilde{q}_i)\cdot \eta_i\rangle} \, \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i \cdot z_{i+1}}. \tag{364}$$

Consider now the rescaling of the moduli parameters:

$$\tau_\ell^M = \big(u_\ell^A, v_\ell^B, \tilde{u}_{\ell\dot{A}}, \tilde{v}_{\ell\dot{B}}, \chi_\ell^\alpha\big) \;\mapsto\; \tilde{\tau}_\ell^M = \big(u_\ell^A, v_\ell^B, r\tilde{u}_{\ell\dot{A}}, r\tilde{v}_{\ell\dot{B}}, \chi_\ell^\alpha\big). \tag{365}$$

Under this map, the measure on $\mathscr{M}_5$ and the embedding coordinates scale as:

$$\mathcal{D}\boldsymbol{\gamma} \;\mapsto\; r^8\, \mathcal{D}\boldsymbol{\gamma}, \;\; \widetilde{\mathcal{Q}}_i^{A\dot{A}} \;\mapsto\; r\, \widetilde{\mathcal{Q}}_i^{A\dot{A}}, \;\; \tilde{q}_i^{\alpha A} \;\mapsto\; \tilde{q}_i^{\alpha A}. \tag{366}$$

Performing these substitutions in Eq. (356) allows the affine parameter r to factor out and be integrated. One finds:

$$\mathcal{B}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}^I_i\big) \;=\; 2\pi\delta(\beta_2)\,\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}^I_i\big), \tag{367}$$

where the overall conformal weight parameter is defined by:

$$\beta_2 \;\coloneqq\; 12 - 2\sum_{i=1}^{n} h_i. \tag{368}$$

Finally, the tree-level N^2-MHV sectional (or leaf) amplitude assumes the form:

$$\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}\big(\mathsf{W}^I_i\big) \;=\; \int\limits_{\mathscr{M}_5} \mathcal{D}\gamma\;\mathcal{F}_{a_1b_1,a_2b_2}\big(\tau_1^M,\tau_2^{M'}\big)\,\widetilde{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}^I_i;\gamma^Q\big), \tag{369}$$

where the trace factor is given by:

$$\widetilde{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}^I_i;\gamma^Q\big) \;=\; \mathsf{Tr}\bigwedge_{i=1}^{n} \frac{\mathcal{C}\,(2h_i)}{\langle z_i|\mathcal{R}+\widetilde{\mathcal{Q}}_i|\bar{z}_i]^{2h_i}}\,e^{i\langle z_i|(\theta+\tilde{q}_i)\cdot\eta_i\rangle}\,\frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}. \tag{370}$$

Hence, the sectional amplitude reduces to an integral over the moduli superspace $\mathscr{M}_5$.

7. *Geometrical Formulation*

We now turn to the final task. We determine the minitwistor transform of the sectional amplitude (refer to Eq. (369)). Using the celestial RSVW identity, as reformulated in Subsection III A 7, we deduce an expression for the N^2-MHV minitwistor amplitude as a volume integral over the moduli superspace $\mathscr{M}_5$.

Preliminaries. Let the index ℓ range over $\{1,...,5\}$. We work in real minitwistor superspace $\mathbf{MT}_s$. Its homogeneous coordinates are:

$$\mathsf{Z}^I \coloneqq \big(\lambda^A,\mu_{\dot{A}},\psi^\alpha\big).$$

Now consider the family of real minitwistor lines $\big\{\mathscr{L}_\ell\big(\gamma^Q\big)\big\}$, parametrised by the superspace coordinates γ^Q. Each line $\mathscr{L}_\ell\big(\gamma^Q\big)$ is defined by the locus of points Z^I satisfying the supersymmetric incidence relations:

$$\begin{cases} \mu_{\dot{A}} = \lambda^A\big(\mathcal{R}_{A\dot{A}}+\mathcal{Q}_{\ell A\dot{A}}\big), \\ \psi^\alpha = \lambda^A\big(\theta^\alpha_A+q^\alpha_{\ell A}\big). \end{cases} \tag{371}$$

For a fixed point $p\in\mathscr{M}_5$ with coordinates $\gamma^Q_* \coloneqq \gamma^Q\,(p)$, the set $\big\{\mathscr{L}_\ell\big(\gamma^Q_*\big)\big\}$ uniquely determines a configuration of five real minitwistor lines. As p varies over $\mathscr{M}_5$, these configurations sweep out all possible quintets of lines defined by the incidence relations in Eq. (371). Hence, $\mathscr{M}_5$ is identified as the moduli superspace for these quintet families.

Next, let:

$$\pi_\ell : \mathscr{L}_\ell(\gamma^Q) \longrightarrow \mathbf{RP}^1$$

denote the canonical surjection. We trivialise the fibration π_ℓ by introducing homogeneous coordinates $[\sigma^A]$ on $\mathbf{RP}^1$. This trivialisation allows one to define the natural measure on each minitwistor line. We set:

$$D\sigma \coloneqq \varepsilon_{AB}\sigma^A d\sigma^B. \tag{372}$$

An embedding of $\mathbf{RP}^1$ into $\mathbf{MT}_s$ is simply a smooth nonsingular section of π_ℓ. In particular, define $\mathsf{Y}^I_\ell : \mathbf{RP}^1 \longrightarrow \mathscr{L}_\ell(\gamma^Q)$ via:

$$\mathsf{Y}^I_\ell(\sigma^A) \coloneqq \left(\sigma^A, \sigma^A(\mathcal{R}_{A\dot{A}} + \mathcal{Q}_{\ell A\dot{A}}), \sigma^A(\theta^\alpha_A + q^\alpha_{\ell A})\right). \tag{373}$$

By construction, $\pi_\ell \circ \mathsf{Y}^I_\ell(\sigma^A) = \sigma^A$. Therefore, Y^I_ℓ constitutes an embedding of $\mathbf{RP}^1$ into the real minitwistor line $\mathscr{L}_\ell(\gamma^Q)$.

Minitwistor Amplitude. Let $\mathbf{M}^n \coloneqq \bigtimes^n \mathbf{MT}_s$ be our integration superdomain. From the celestial RSVW identity (Eq. (312)), it follows that the sectional amplitude admits an expression as an n-fold minitwistor transform:

$$\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}(\mathsf{W}^I_i) = \int\limits_{\mathbf{M}^n} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}_i\, \Psi_{2h_i}(\mathsf{Z}^I_i; \mathsf{W}^{I'}_i)\ \widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}(\mathsf{Z}^I_i). \tag{374}$$

The N^2-MHV minitwistor superamplitude is given by:

$$\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,a_2b_2}(\mathsf{Z}^I_i) = \int\limits_{\mathscr{M}_5} \mathcal{D}\gamma\ \mathcal{F}_{a_1b_1,a_2b_2}(\gamma^Q)\ \mathsf{Tr} \bigwedge_{i=1}^{n} \int\limits_{\mathbf{RP}^1} \boldsymbol{\omega}^{a_i}(\sigma^A)\, \bar{\delta}^{2|4}_{(-2h_i,2h_i)}(\mathsf{Z}^I_i; \mathsf{Y}^{I'}_i(\sigma^A_i)). \tag{375}$$

The logarithmic form $\boldsymbol{\omega}^{a_i}(\sigma^A_i)$ on the minitwistor line $\mathscr{L}_\ell(\gamma^Q)$ is defined as:

$$\boldsymbol{\omega}^{a_i}(\sigma^A_i) \coloneqq \mathsf{T}^{a_i}\, \frac{D\sigma_i}{\sigma_i \cdot \sigma_{i+1}}. \tag{376}$$

Conclusion. The minitwistor delta-functions under the integral of Eq. (375),

$$\bar{\delta}^{2|4}_{(-2h_i,2h_i)}(\mathsf{Z}^I_i; \mathsf{Y}^{I'}_i(\sigma^A_i)),$$

localise the integration measure over the moduli superspace $\mathscr{M}_5$ onto the support defined by the family of minitwistor lines $\{\mathscr{L}_\ell\}$. The celestial scaling dimensions h_i (associated with the gluons involved in the scattering process) appear as weights in the construction of the volume form on $\mathscr{M}_5$. Thus, the minitwistor amplitude computes a weighted volume on the moduli superspace corresponding to a quintuple of minitwistor lines.

Furthermore, Eq. (375) implies that the minitwistor amplitude vanishes whenever the i-th gluon does not lie on the minitwistor line $\mathscr{L}_{c(i)}$, where $c(i)$ denotes the cluster assignment of the i-th gluon, as defined in Eq. (341).

C. General Case

Let $p \coloneqq 2k+3$. We now extend our analysis to the full tree-level celestial $\mathcal{S}$-matrix. For an N^1-MHV configuration, we will show that the minitwistor amplitude localises on p distinct minitwistor lines. It is then computed as a volume integral over the moduli superspace $\mathscr{M}_p$, which parametrises all admissible configurations of these p lines.

1. *Dual Conformal Invariant*

We begin our analysis by defining the order-$(k+1)$ R-invariant. Fix a family of indices:

$$1 \leq a_1 < a_2 < ... < a_k < a_{k+1} < b_{k+1} < b_k < ... < b_2 \leq b_1 \leq n. \tag{377}$$

Define the sequences of van der Warden spinors $\left\{u_\ell^A\right\}_{1\leq \ell\leq k}$ and $\left\{v_k^A\right\}_{1\leq \ell\leq k}$ inductively. For the first cases, we set:

$$u_1^A \coloneqq z_n^B \left(y_{nb_1}\right)_{B\dot{B}} \left(y^{-1}_{b_1a_1}\right)^{A\dot{B}}, \;\; v_1^A \coloneqq z_n^B \left(y_{na_1}\right)_{B\dot{B}} \left(y^{-1}_{a_1b_1}\right). \tag{378}$$

For all $1 \leq k \leq n-1$, the recursion relations are given by:

$$u_{k+1}^A \coloneqq u_k^B \left(y_{a_kb_{k+1}}\right)_{B\dot{B}} \left(y_{b_{k+1}a_{k+1}}\right)^{A\dot{B}}, \tag{379}$$

$$v_{k+1}^A \coloneqq u_k^B \left(y_{a_ka_{k+1}}\right)_{B\dot{B}} \left(y^{-1}_{a_{k+1}b_{k+1}}\right)^{A\dot{B}}. \tag{380}$$

The *order-$(k+1)$ R-invariant*, expressed in terms of the celestial supercoordinates $z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha$, is defined by:

$$R_{n;a_1b_1,\ldots,a_kb_k} \coloneqq \frac{\langle z_{a_{k+1}-1}, z_{a_{k+1}}\rangle\langle z_{b_{k+1}-1}, z_{b_{k+1}}\rangle\delta^{0|4}\left(\Theta^\alpha_{k+1}\right)}{y^2_{a_{k+1}b_{k+1}}\langle z_{a_{k+1}-1}, u_{k+1}\rangle\langle u_{k+1}, z_{a_{k+1}}\rangle\langle z_{b_{k+1}-1}, v_{k+1}\rangle\langle v_{k+1}, z_{b_{k+1}}\rangle}. \tag{381}$$

Here, Θ^{α}_{k+1} is the Grassmann-valued function entering the fermionic delta distribution, defined as:

$$\Theta^{\alpha}_{k+1}\big(u^{A}_{k+1}, v^{B}_{k+1}\big) \;\coloneqq\; \sum_{i=a_k}^{a_{k+1}-1} \langle u_{k+1}, z_i\rangle\, \eta^{\alpha}_i + \sum_{j=a_k}^{b_{k+1}-1} \langle v_{k+1}, z_j\rangle\, \eta^{\alpha}_j. \tag{382}$$

The order-$(k+1)$ R-invariant is one of the ingredients of the partial amplitude:

$$\mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1b_1,\ldots,a_{k+1}b_{k+1}}\big(\lambda^{A}_i, \bar{\lambda}_{i\dot{A}}, \eta^{\alpha}_i\big), \tag{383}$$

which will be discussed in Subsection III C 5.

To compute the tree-level N^{k+1}-MHV celestial superamplitude, one must perform a half-Mellin transform of the partial amplitude (383). However, the structure of Eq. (381) proves unsuitable for a direct computation of the Mellin transform. To address this difficulty, we invoke the Fadde'ev-Popov procedure, thereby expressing the order-$(k+1)$ R-invariant as an integral over auxiliary spinor variables u^{A}_{k+1} and v^{A}_{k+1}.

In Kleinian signature, we take the integration domain to be $\mathcal{I} \coloneqq \mathbf{R}^2 \times \mathbf{R}^2$, parametrised by $U^{A'}_{k+1} \coloneqq \big(u^{A}_{k+1}, v^{B}_{k+1}\big)$. The standard orientation of $\mathcal{I}$ is given by the Lebesgue measure:

$$d^4U_{k+1} \;\coloneqq\; d^2u_{k+1} \wedge d^2v_{k+1}. \tag{384}$$

Our aim is to factorise all dependence on the s_i into delta-functions. To this end, we define the spinor-valued mappings:

$$f^{A}_{a_ka_{k+1}b_{k+1}}\big(u^{B}_{k}, y^{C\dot{C}}_{i}\big) \;\coloneqq\; u^{B}_{k}\,\big(y_{a_kb_{k+1}}\big)_{B\dot{B}}\,\big(y^{-1}_{b_{k+1}a_{k+1}}\big)^{A\dot{B}}, \tag{385}$$

$$g^{A}_{a_ka_{k+1}b_{k+1}}\big(u^{B}_{k}, y^{C\dot{C}}_{i}\big) \;\coloneqq\; u^{B}_{k}\,\big(y_{a_ka_{k+1}}\big)_{B\dot{B}}\,\big(y^{-1}_{a_{k+1}b_{k+1}}\big)^{A\dot{B}}. \tag{386}$$

The *constraint hypersurface* $\mathscr{C}$ is defined by the locus of points $U^{A'}_{k+1} \in \mathcal{I}$ satisfying:

$$u^{A}_{k+1} \;=\; f^{A}_{a_ka_{k+1}b_{k+1}}\big(u^{B}_{k}, y^{C\dot{C}}_{i}\big),\;\; v^{A}_{k+1} \;=\; g^{A}_{a_ka_{k+1}b_{k+1}}\big(u^{B}_{k}, y^{C\dot{C}}_{i}\big). \tag{387}$$

We define the delta-distribution on $\mathcal{I}$, supported on $\mathscr{C}$, by:

$$\delta_{\mathscr{C}}\big(u^{A}_{k+1}, v^{B}_{k+1}\big) \;\coloneqq\; \overline{\delta}^{2}\big(u^{A}_{k+1} - f^{A}_{a_ka_{k+1}b_{k+1}}\big(u^{B}_{k}, y^{C\dot{C}}_{i}\big)\big)\,\overline{\delta}^{2}\big(v^{A}_{k+1} - g^{A}_{a_ka_{k+1}b_{k+1}}\big(u^{B}_{k}, y^{C\dot{C}}_{i}\big)\big). \tag{388}$$

We may then express Eq. (381) as a Fadde'ev-Popov integral:

$$R_{n;a_1b_1,\ldots,a_{k+1}b_{k+1}} \;=\; \frac{1}{y^2_{a_{k+1}b_{k+1}}} \int_{\mathcal{I}} d^4U_{k+1}\;\, \mathcal{F}_{a_{k+1}b_{k+1}}\big(u^{A}_{k+1}, v^{B}_{k+1}\big)\,\delta^{0|4}\big(\Theta^{\alpha}_{k+1}\big)\,\delta_{\mathscr{C}}\big(u^{A}_{k+1}, v^{B}_{k+1}\big). \tag{389}$$

where:

$$\mathcal{F}_{a_{k+1}b_{k+1}}\big(u^{A}_{k+1}, v^{B}_{k+1}\big) \;\coloneqq\; \frac{\langle z_{a_{k+1}-1}, z_{a_{k+1}}\rangle\langle z_{b_{k+1}-1}, z_{b_{k+1}}\rangle}{\langle z_{a_{k+1}-1}, u_{k+1}\rangle\langle u_{k+1}, z_{a_{k+1}}\rangle\langle z_{b_{k+1}-1}, v_{k+1}\rangle\langle v_{k+1}, z_{b_{k+1}}\rangle}. \tag{390}$$

The next step in deriving the Fadde'ev-Popov representation of the order-k R-invariant is to expand the Grassmann and spinor delta functions in Eq. (389).

Fermionic Delta-Function. Recall that the fermionic delta function $\delta^{0|4}(\varepsilon^\alpha)$ for a Grassmann variable ε^α admits the Berezin integral representation (see Eq. (246)). We introduce:

$$\varepsilon^\alpha = \Theta^\alpha_{k+1}(u^A_{k+1}, v^B_{k+1}),$$

using the definition of Θ^α_{k+1} from Eq. (382). It follows that:

$$\delta^{0|4}(\Theta^\alpha_{k+1}) = \int\limits_{\mathbf{R}^{0|4}} d^{0|4}\chi_{k+1} \bigwedge_{i=a_k}^{a_{k+1}-1} e^{i\langle u_{k+1}, z_i\rangle\, \chi_{k+1}\cdot\eta_i} \bigwedge_{j=a_k}^{b_{k+1}-1} e^{i\langle v_{k+1}, z_j\rangle\, \chi_{k+1}\cdot\eta_j}. \tag{391}$$

Constraint Delta-Function. Consider the constraint delta-function $\delta_{\mathscr{C}}$. For a real van der Waerden spinor λ^A, the two-component delta-distribution $\overline{\delta}^2(\lambda^A)$ is given in Eq. (248). Using the definition of $f^A_{a_k a_{k+1} b_{k+1}}$ from Eq. (385), the u^A_{k+1}-component of $\delta_{\mathscr{C}}$ admits the Fourier representation:

$$\overline{\delta}^2\big(u^A_{k+1} - f^A_{a_k a_{k+1} b_{k+1}}(u^B_k, y^{C\dot{C}}_i)\big) \tag{392}$$

$$= \int\limits_{\mathbf{R}^2} \frac{d^2\hat{u}_{k+1}}{(2\pi)^2}\, e^{i\langle u_{k+1}, \hat{u}_{k+1}\rangle} \exp\big(-iu^B_k\,(y_{a_k b_{k+1}})_{B\dot{B}}\,(y^{-1}_{b_{k+1} a_{k+!}})^{A\dot{B}}\,\hat{u}_{k+1,A}\big). \tag{393}$$

Under the change of variables:

$$\hat{u}_{k+1,A} \mapsto \tilde{u}^{\dot{A}}_{k+1} := (y^{-1}_{b_{k+1} a_{k+1}})^{A\dot{A}}\,\hat{u}_{k+1,A}, \tag{394}$$

one finds:

$$\delta\big(u^A_{k+1} - f^A_{a_k a_{k+1} b_{k+1}}(u^B_k, y^{C\dot{C}}_i)\big) \tag{395}$$

$$= |y^2_{b_{k+1} a_{k+1}}| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{u}_{k+1}}{(2\pi)^2} \prod_{i=a_{k+1}}^{b_{k+1}-1} e^{-is_i\langle z_i|u_{k+1}\tilde{u}_{k+1}|\bar{z}_i]} \prod_{j=a_k}^{b_{k+1}-1} e^{-is_j\langle z_i|u_k\tilde{u}_{k+1}|\bar{z}_i]}. \tag{396}$$

Analogously, with $g^A_{a_k a_{k+1} b_{k+1}}$ as in Eq. (386), the v^A_{k+1}-component is:

$$\overline{\delta}^2\big(v^A_{k+1} - u^B_k\,(y_{a_k a_{k+1}})_{B\dot{B}}\,(y^{-1}_{a_{k+1} b_{k+1}})^{A\dot{B}}\big) \tag{397}$$

$$= \int\limits_{\mathbf{R}^2} \frac{d^2\hat{v}_{k+1}}{(2\pi)^2}\, e^{i\langle v_{k+1}, \hat{v}_{k+1}\rangle} \exp\big(-iu^B_k\,(y_{a_k a_{k+1}})_{B\dot{B}}\,(y^{-1}_{a_{k+1} b_{k+1}})^{A\dot{B}}\,\hat{v}_{k+1,A}\big). \tag{398}$$

With the substitution:

$$\hat{v}_{k+1,A} \mapsto \tilde{v}^{\dot{A}}_{k+1} := (y^{-1}_{a_{k+1} b_{k+1}})^{A\dot{A}}\,\hat{v}_{k+1,A}, \tag{399}$$

we obtain:

$$\overline{\delta}^2\big(v^A_{k+1} - g^A_{a_k a_{k+1} b_{k+1}}\big(u^B_k, y^{C\dot{C}}_i\big)\big) \tag{400}$$

$$= \big|y^2_{a_{k+1}b_{k+1}}\big| \int\limits_{\mathbf{R}^2} \frac{d^2\tilde{v}_{k+1}}{(2\pi)^2} \prod_{i=a_{k+1}}^{b_{k+1}-1} e^{is_i\langle z_i|v_{k+1}\tilde{v}_{k+1}|\bar{z}_i]} \prod_{j=a_k}^{a_{k+1}-1} e^{-is_j\langle z_j|u_k\tilde{v}_{k+1}|\bar{z}_j]}. \tag{401}$$

3. Fadde'ev-Popov Representation

In the preceding subsections, we derived an expression for the order-k R-invariant as an integral over the domain $\mathcal{I}$ (see Eq. (389)). This integral is localised on the constraint hypersurface $\mathscr{C} \subset \mathcal{I}$ via the Dirac delta distribution $\delta_{\mathscr{C}}$. We then expanded the Grassmann and spinor delta functions in the integrand in terms of the celestial coordinates z^A_i, $\bar{z}_{i\dot{A}}$ and η^α_i.

Substituting the expansions of Eqs. (391), (396) and (401) into Eq. (389) for the R-invariant yields the following formulation.

Integration Superdomain. The Fadde'ev-Popov representation of the order-k R-invariant is given by an integral over the parameter superspace:

$$\mathcal{P}_{k+1} \coloneqq \mathbf{R}^{8|4}. \tag{402}$$

Let the abstract index M range over $\{A, B, \dot{A}, \dot{B}, \alpha\}$. The parameter superspace is globally charted by the coordinates:

$$\tau^M_{k+1} \coloneqq \big(u^A_{k+1}, v^B_{k+1}, \tilde{u}_{k+1,\dot{A}}, \tilde{v}_{k+1,\dot{B}}, \chi^\alpha_{k+1}\big). \tag{403}$$

Moreover, the orientation of $\mathcal{P}_{k+1}$ is provided by the measure:

$$d^{8|4}\tau_{k+1} \coloneqq d^2u_{k+1} \wedge d^2v_{k+1} \wedge d^2\tilde{u}_{k+1} \wedge d^2\tilde{v}_{k+1} \wedge d^{0|4}\chi_{k+1}. \tag{404}$$

In subsequent subsections, we will explicitly define the moduli superspace $\mathscr{M}_{2k+3}$ that fully characterises the configuration of a system consisting of $2k+3$ minitwistor lines. For now, we regard $\mathscr{M}_{2k+3}$ as an abstract supermanifold. In this context, the coordinate functions τ^M_{k+1} parametrise a supersymmetric submanifold of $\mathscr{M}_{2k+3}$. Indeed, for each gluon i participating in the N^{k+1}-MHV scattering process, there exists a corresponding copy of this submanifold, denoted by $\mathcal{P}_i$, and parametrised by the coordinate functions $\tau^M_i : \mathcal{P}_i \longrightarrow \mathbf{R}^{8|4}$.

Interval for i	$p_i^{A\dot{A}}$	$\xi_i^{\alpha A}$
$a_k \leq i \leq a_{k+1}-1$	$-u_k^A\left(\tilde{u}_{k+1}^{\dot{A}}+\tilde{v}_{k+1}^{\dot{A}}\right)$	$-\left(u_{k+1}^A+v_{k+1}^A\right)\chi_{k+1}^\alpha$
$a_{k+1} \leq i \leq b_{k+1}-1$	$-\left(u_k^A+u_{k+1}^A\right)\tilde{u}_{k+1}^{\dot{A}}+v_{k+1}^A\tilde{v}_{k+1}^{\dot{A}}$	$-v_{k+1}^A\chi_{k+1}^\alpha$
otherwise	0	0

Table III. Embedding coordinates $\left(p_i^{A\dot{A}}, \xi_i^{\alpha A}\right)$ of the parameter superspace $\mathcal{P}_i$.

Embedding Coordinates. Now, if $\mathscr{M}_{2k+3}$ is regarded as an abstract supermanifold and each parameter superspace $\mathcal{P}_i$ is identified with a submanifold thereof, how does one define the natural embedding coordinates of $\mathcal{P}_i$ in $\mathscr{M}_{2k+3}$? The answer is provided by Eqs. (391), (395) and (400).

Examining the arguments within the exponential functions of these expansions, we introduce the coordinate maps:

$$\left(p_i^{A\dot{A}}, \xi_i^{\alpha A}\right) : \mathcal{P}_i \subset \mathscr{M}_{2k+1} \longrightarrow \mathbf{R}^4 \times \mathbf{R}^{0|8},$$

defined in Table III. The quantities $p_i^{A\dot{A}}$ and $\xi_i^{\alpha A}$ shall henceforth be referred to as the *embedding coordinates* of the parameter superspace $\mathcal{P}_i$ associated with the i-th gluon.

Integral Representation. By substituting Eqs. (391), (395) and (400) into Eq. (389), we obtain the Fadde'ev-Popov representation of the order-$(k+1)$ R-invariant:

$$R_{n;a_1b_1,\ldots,a_{k+1}b_{k+1}} = \mathcal{N}_{a_{k+1}b_{k+1}} \int\limits_{\mathcal{P}} d^{8|4}\tau_{k+1}\ \mathcal{F}_{a_{k+1}b_{k+1}}\left(\tau_{k+1}^M\right) \bigwedge_{i=1}^{n} e^{is_i\langle z_i|p_i|\bar{z}_i]+i\langle z_i|\xi_i\cdot\eta_i\rangle}. \tag{405}$$

The normalisation factor is defined by:

$$\mathcal{N}_{a_{k+1}b_{k+1}} := \frac{1}{(2\pi)^4}\, y_{a_{k+1}b_{k+1}}^2. \tag{406}$$

4. Induction Hypothesis

Our next objective is to generalise the method used for the N^1- and N^2-MHV celestial amplitudes by formulating an induction hypothesis for the N^{k+1}-MHV case. As before, we regard the moduli superspace $\mathscr{M}_{2k+3}$ as an abstract supermanifold; its detailed structure will be specified in the following subsection.

Let the index m range over $1,\ldots,k$. For each m, we postulate a parameter superspace $\mathcal{P}_m$ with global coordinates:

$$\tau_m^M := \left(u_m^A, v_m^B, \tilde{u}_{m\dot{A}}, \tilde{v}_{m\dot{B}}, \chi_m^\alpha\right). \tag{407}$$

In later subsections, we will demonstrate that these τ^M parametrise a supersymmetric submanifold of $\mathscr{M}_{2k+3}$.

We further postulate the existence of embedding coordinates:

$$\mathsf{Q}_i^K := \left(\mathcal{Q}_i^{A\dot{A}}, q_i^{\alpha A}\right) : \mathcal{P}_1 \times ... \times \mathcal{P}_k \longrightarrow \mathbf{R}^{4|8}.$$

These functions depend on the moduli parameters τ^M and are assumed to satisfy the axioms listed below.

Axiom 1. Moduli Reparametrization. Under a reparametrization of the moduli superspaces $\mathcal{P}_m$ given by the transformations:

$$\tau_m^M = \left(u_m^A, v_m^B, \tilde{u}_{m\dot{A}}, \tilde{v}_{m\dot{B}}, \chi_m^\alpha\right) \mapsto \tilde{\tau}_m^M = \left(u_m^A, v_m^B, r\tilde{u}_{m\dot{A}}, r\tilde{v}_{m\dot{B}}, \chi_m^\alpha\right), \tag{408}$$

the embedding coordinates transform according to:

$$\mathcal{Q}_i^{A\dot{A}} \mapsto r\mathcal{Q}_i^{A\dot{A}}, \;\; q_i^{\alpha A} \mapsto q_i^{\alpha A}. \tag{409}$$

Before stating the next axiom, we introduce the integration superdomain:

$$\mathscr{E}^{(k)} := \bigtimes_{m=1}^{k} \mathcal{P}_m. \tag{410}$$

This supermanifold is charted by:

$$\boldsymbol{\tau}^P := \left(\tau_1^{M_1}, \tau_2^{M_2}, ..., \tau_k^{M_k}\right), \tag{411}$$

where the abstract index P runs over $M_1, ..., M_k$. Its canonical $\mathbf{Z}_2$-graded volume form is:

$$\mathcal{D}^{(k)}\boldsymbol{\tau} := \bigwedge_{m=1}^{k} d^{8|4}\tau_m. \tag{412}$$

Axiom 2. Integral Representation. Define the order-k R-monomial by:

$$R_n^{(k)} := \bigwedge_{m=1}^{k} R_{n;a_1b_1,...,a_mb_m}. \tag{413}$$

We postulate that the embedding coordinates $\mathcal{Q}_i^{A\dot{A}}$ and $q_i^{\alpha A}$ are such that:

$$R_n^{(k)} = \int_{\mathscr{E}^{(k)}} \mathcal{D}^{(k)}\boldsymbol{\tau} \;\, \mathcal{F}_{a_1b_1,...,a_kb_k}\left(\tau^P\right) \bigwedge_{i=1}^{n} e^{is_i\langle z_i|\mathcal{Q}_i|\bar{z}_i]+i\langle z_i|q_i\cdot\eta_i\rangle}, \tag{414}$$

where:

$$\mathcal{F}_{a_1 b_1,\ldots,a_k b_k}\big(\tau^P\big) \;\coloneqq\; \prod_{\ell=1}^{k} \mathcal{F}_{a_\ell b_\ell}\big(\tau_\ell^M\big). \tag{415}$$

Axioms 1 and 2 are motivated by our explicit constructions of the N^1- and N^2-MHV celestial amplitudes. In each of these cases, one finds a consistent set of moduli parameters satisfying Eq. (414). For the N^1-MHV case, see Subsection III A 4, especially Eq. (265). Likewise, the N^2-MHV construction is reviewed in Subsection III B 4, especially Eq. (343). Together, these lower-order examples demonstrate that the integral-representation postulate (Axiom 2) naturally extends to the general order-k R-invariant.

5. *Outline of the Argument*

In Subsection III C 3, we applied the Fadde'ev-Popov method to derive an integral representation of the order-$(k+1)$ R-invariant. That representation is written as an integral over the parameter superspace $\mathcal{P} \simeq \mathbf{R}^{8|4}$.

In Subsection III C 4, we introduced our induction hypothesis. We assumed the existence of embedding coordinates Q_i^K that chart each superspace $\mathcal{P}_m$ for $1 \leq m \leq k$. Using these coordinates, we then postulated the integral formula of Eq. (414) for the degree-k R-monomial. This formula is defined over the integration superdomain $\mathscr{E}^{(k)} \coloneqq \bigtimes_{m=1}^{k} \mathcal{P}_m$.

To derive the N^{k+1}-MHV celestial amplitude, we begin by analysing the partial amplitudes. Let $\{a_\ell, b_\ell\}$ be a family of indices satisfying $a_\ell < b_\ell$ for all $\ell = 1, \ldots, k+1$. The corresponding partial amplitude is[19]:

$$\mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1 b_1,\ldots,a_{k+1}b_{k+1}}\big(\lambda_i^A, \bar{\lambda}_{i\dot{A}}, \eta_i^\alpha\big) \;=\; (2\pi)^4\, \delta^{4|0}\big(P^{A\dot{A}}\big)\delta^{0|8}\big(Q^{\alpha A}\big)\, A^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1 b_1,\ldots,a_{k+1}b_{k+1}}\big(\lambda_i^A, \bar{\lambda}_{i\dot{A}}, \eta_i^\alpha\big). \tag{416}$$

The reduced amplitude is defined by:

$$A^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1 b_1,\ldots,a_{k+1}b_{k+1}}\big(\lambda_i^A, \bar{\lambda}_{i\dot{A}}, \eta_i^\alpha\big) \;\coloneqq\; \bigwedge_{\ell=1}^{k+1} R_{n;a_1 b_1,\ldots,a_\ell b_\ell}\, \mathsf{Tr} \prod_{i=1}^{n} \frac{\mathsf{T}^{a_i}}{\lambda_i \cdot \lambda_{i+1}}. \tag{417}$$

The full tree-level N^{k+1}-MHV superamplitude $\mathcal{A}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}$ is obtained by summing two classes of contributions. The first class consists of the partial amplitudes just defined. The second class comprises degenerate cases. A detailed classification appears in Drummond and Henn [56], Korchemsky and Sokatchev [58, 73].

[19] See Drummond *et al.* [78].

To show that the minitwistor amplitude localises on configurations of $2k+3$ minitwistor lines, it suffices to consider the "canonical" partial amplitude in Eq. (416). We assume the index families $\{a_\ell, b_\ell\}$ satisfy the inequalities of Eq. (377). All other configurations then follow by relabelling or by taking degenerate sub-amplitudes.

Our derivation proceeds in four steps:

1. *Integral Representations.* We merge the integral formula for the degree-k R-monomial (Eq. (414)) with the Fadde'ev-Popov representation of the order-$(k+1)$ R-invariant (Eq. (405)). This construction expresses the partial amplitude as an integral over the supermanifold $\widehat{\mathscr{M}_{2k+3}}$, which we term the "pre-moduli" superspace.

2. *Mellin Transform.* We perform a Mellin transform on the resulting integral. This yields the N^{k+1}-MVH celestial amplitude.

3. *Dimensional Reduction.* Invoking the leaf amplitude formalism, we carry out a dimensional reduction of $\widehat{\mathscr{M}_{2k+3}}$. The result is the moduli superspace $\mathscr{M}_{2k+3}$, which parametrises $2k+3$ minitwistor lines.

4. *Minitwistor Amplitude.* Finally, we apply the celestial RSVW identity. This step produces the corresponding minitwistor amplitude.

6. Celestial Amplitude

In this subsection, we first derive an integral representation for the degree-$(k+1)$ R-monomial. We then compute the Mellin transform of the canonical partial amplitude introduced in Eq. (416). The outcome of this computation is the tree-level N^{k+1}-MHV celestial superamplitude.

Notation. For brevity, we let the index ℓ run over $1, ..., k+1$. We introduce the compact label:

$$(ab)_\ell \coloneqq (a_1 b_1, ..., a_{k+1} b_{k+1}),$$

and denote the corresponding N^{k+1}-MHV partial amplitude by:

$$\mathcal{A}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;(ab)_\ell}.$$

This notation is unambiguous: the index structure $(ab)_\ell$ singles out the "canonical" sub-amplitude in Eq. (416) whose sum reproduces the full scattering amplitude.

Interval for i	$\widetilde{\mathcal{Q}}_i^{A\dot{A}}$	$\tilde{q}_i^{\alpha A}$
$a_k \leq i \leq a_{k+1}-1$	$\mathcal{Q}_i^{A\dot{A}} - u_k^A\left(\tilde{u}_{k+1}^{\dot{A}} + \tilde{v}_{k+1}^{\dot{A}}\right)$	$q_i^{\alpha A} - \left(u_{k+1}^A + v_{k+1}^A\right)\chi_{k+1}^\alpha$
$a_{k+1} \leq i \leq b_{k+1}-1$	$\mathcal{Q}_i^{A\dot{A}} - \left(u_k^A + u_{k+1}^A\right)\tilde{u}_{k+1}^{\dot{A}} + v_{k+1}^A\tilde{v}_{k+1}^{\dot{A}}$	$q_i^{\alpha A} - v_{k+1}^A\chi_{k+1}^\alpha$
otherwise	$\mathcal{Q}_i^{A\dot{A}}$	$q_i^{\alpha A}$

Table IV. Embedding coordinates $\widetilde{\mathsf{Q}}_i^K = \left(\widetilde{\mathcal{Q}}_i^{A\dot{A}}, \tilde{q}_i^{\alpha A}\right)$ on the integration superdomain $\mathscr{E}_{k+1}$.

Preliminaries. We begin by introducing the *degree-*$(k+1)$ *R-monomial*, which plays a central role in our construction of the N^{k+1}-MHV celestial amplitude. It is defined by:

$$R_n^{(k+1)} \coloneqq \bigwedge_{\ell=1}^{k+1} R_{n;a_1b_1,\ldots,a_\ell b_\ell}. \tag{418}$$

Our first task is to derive an integral representation for $R_n^{(k+1)}$.

The *integration superdomain* for this representation is defined by:

$$\mathscr{E}_{k+1} \coloneqq \bigtimes_{\ell=1}^{k+1} \mathcal{P}_\ell. \tag{419}$$

We chart this supermanifold by the coordinates:

$$\tau^P \coloneqq \left(\tau_1^{M_1}, \ldots, \tau_{k+1}^{M_{k+1}}\right), \tag{420}$$

where the abstract index P runs over the list $M_1, \ldots, M_{k+1}$.

The canonical $\mathbf{Z}_2$-graded volume form on $\mathscr{E}_{k+1}$ is:

$$\mathcal{D}^{(k+1)}\boldsymbol{\tau} \coloneqq \bigwedge_{\ell=1}^{k+1} d^{8|4}\tau_\ell. \tag{421}$$

Each factor $\mathcal{P}_\ell$ is parametrised by embedding coordinates $\mathsf{Q}_i^K = \left(\mathcal{Q}_i^{A\dot{A}}, q_i^{\alpha A}\right)$. From these, we define embedding coordinates on $\mathscr{E}_{k+1}$:

$$\widetilde{\mathsf{Q}}_i^K \coloneqq \left(\widetilde{\mathcal{Q}}_i^{A\dot{A}}, \tilde{q}_i^{\alpha A}\right) : \mathscr{E}_{k+1} \longrightarrow \mathbf{R}^{4|8}.$$

Explicit expressions for $\widetilde{\mathsf{Q}}_i^K$ appear in Table IV.

We now invoke the second postulate of Subsection III C 4. By multiplying Eqs. (405) and (414), one obtains an integral formula for the degree-$(k+1)$ R-monomial:

$$R_n^{(k+1)} = \mathcal{N}_{(ab)_\ell} \int_{\mathscr{E}_{k+1}} \mathcal{D}^{(k+1)}\boldsymbol{\tau}\ \mathcal{F}_{(ab)_\ell}\left(\tau^{\dot{P}}\right) \bigwedge_{i=1}^{n} \exp\left(is_i\langle z_i|\widetilde{\mathcal{Q}}_i|\bar{z}_i] + i\langle z_i|\tilde{q}_i \cdot \eta_i\rangle\right). \tag{422}$$

Here the *weight function* is:

$$\mathcal{F}_{(ab)_\ell}\big(\tau^{\dot P}\big) \;\coloneqq\; \prod_{\ell=1}^{k+1} \mathcal{F}_{a_\ell b_\ell}\big(\tau_\ell^{\dot P}\big), \tag{423}$$

and the overall normalisation factor is:

$$\mathcal{N}_{a_1 b_1,\dots,a_{k+1}b_{k+1}} \;\coloneqq\; \frac{1}{(2\pi)^{4(k+1)}} \prod_{\ell=1}^{k+1} y^2_{a_\ell b_\ell}. \tag{424}$$

Celestial Reparametrization. We proceed by expressing the partial amplitude (416) in terms of the celestial coordinates $z_i^A, \bar z_{i\dot A}, \eta_i^\alpha$. For each gluon i, the normalised spinor basis $\{z_i^A, \bar z_{i\dot A}\}$ marks its insertion on the celestial torus $\mathcal{CT}$. The Grassmann variables η_i^α encode helicity. Thus we set:

$$\lambda_i^A = z_i^A,\;\; \bar\lambda_{i\dot A} = s_i \bar z_{i\dot A}.$$

Next, we derive an integral representation for the N^{k+1}-MHV partial amplitude in terms of the embedding coordinates $\widetilde{\mathsf{Q}}_i^K$. The integration domain is the *pre-moduli superspace*:

$$\widehat{\mathscr{M}}_{2k+3} \;\coloneqq\; \mathbf{R}^{4|8} \times \mathscr{E}_{k+1}. \tag{425}$$

Under the leaf amplitude formalism, $\widehat{\mathscr{M}}_{2k+3}$ reduces to the moduli superspace parametrising $2k+3$ minitwistor lines.

The supermanifold $\widehat{\mathscr{M}}_{2k+3}$ is globally charted by:

$$\hat\gamma^Q \coloneqq \big(\mathsf{x}^K, \tau_1^{M_1}, \dots, \tau_{k+1}^{M_{k+1}}\big), \tag{426}$$

with abstract index $Q \in \{K, M_1, \dots, M_{k+1}\}$. Its natural $\mathbf{Z}_2$-graded volume form is:

$$\mathcal{D}\hat{\boldsymbol\gamma} \;\coloneqq\; d^{4|8}\mathsf{x} \wedge \mathcal{D}^{(k+1)}\boldsymbol\tau \;=\; d^{4|8}\mathsf{x} \wedge d^{8|4}\tau_1 \wedge \dots \wedge d^{8|4}\tau_{k+1}, \tag{427}$$

where $d^{4|8}\mathsf{x}$ is the Berezin-de Witt measure on $\mathbf{R}^{4|8}$.

Substituting the integral form of the degree-$(k+1)$ R-monomial (Eq. (414)) into the N^{k+1}-MHV partial amplitude yields:

$$\mathcal{A}^{\mathsf{a}_1\dots\mathsf{a}_n}_{n;(a_\ell b_\ell)}\big(z_i^A, s_i\bar z_{i\dot A}, \eta_i^\alpha\big) \;=\; \mathcal{N}_{(a_\ell b_\ell)} \int\limits_{\widehat{\mathscr{M}}_{2k+3}} \mathcal{D}\hat{\boldsymbol\gamma}\;\; \mathcal{F}_{(a_\ell b_\ell)}\big(\hat\gamma^Q\big)\, \mathcal{T}^{\mathsf{a}_1\dots\mathsf{a}_n}\big(\hat\gamma^Q; z_i^A, s_i\bar z_{i\dot A}, \eta_i^\alpha\big). \tag{428}$$

Here, the trace factor is given by:

$$\mathcal{T}^{\mathsf{a}_1\dots\mathsf{a}_n}\big(\hat\gamma^Q; z_i^A, s_i\bar z_{i\dot A}, \eta_i^\alpha\big) \;=\; \mathsf{Tr}\, \bigwedge_{i=1}^{n} \exp\big(is_i\langle z_i|x + \widetilde{\mathcal{Q}}_i|\bar z_i] + i\langle z_i|(\theta + \tilde q_i)\cdot\eta_i\rangle\big)\, \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}. \tag{429}$$

Mellin Transform. We now compute the Mellin transform of the integral formula derived in Eq. (428) for the N^{k+1}-MHV superamplitude. In that formula, all dependence on the frequency parameters s_i appears in exponential factors. The Mellin transform then produces the desired celestial amplitude.

Let $\mathscr{R}$ be the multiplicative group of positive real numbers, and denote its n-fold direct product by $\mathscr{R}^n := \bigtimes^n \mathscr{R}$. We regard the frequency parameters s_i as affine coordinates on $\mathscr{R}$, so that (s_i) defines a Cartesian chart on $\mathscr{R}^n$. The natural orientation on $\mathscr{R}^n$ is given by the Haar measure $d\rho_{s_i}$ (see Eq. (278)).

We combine the normalised spinor basis $\{z_i^A, \bar{z}_{i\dot{A}}\}$, which locates the insertion point of the i-th gluon on the celestial torus, with the Grassmann variables η_i^α encoding its helicity, into the dual real minitwistor:

$$\mathsf{W}^I := \big(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha\big).$$

Therefore, we define the tree-level N^{k+1}-MHV celestial superamplitude as the n-dimensional Mellin transform over $\mathscr{R}^n$:

$$\widehat{\mathcal{A}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) := \int\limits_{\mathscr{R}^n} d\rho_{s_i}\ \mathcal{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(a_\ell b_\ell)}\big(z_i^A, s_i\bar{z}_{i\dot{A}}, \eta_i^\alpha\big) \prod_{i=1}^n s_i^{2h_i}. \tag{430}$$

Substituting Eq. (428) into this definition and performing the integrals over s_i yields:

$$\widehat{\mathcal{A}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) = \boldsymbol{P}_{(ab)_\ell}\, \widehat{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big). \tag{431}$$

The reduced celestial amplitude is given by:

$$\widehat{A}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) = \int\limits_{\widehat{\mathscr{M}}_{2k+3}} \mathcal{D}\hat{\gamma}\ \mathcal{F}_{(ab)_\ell}\big(\hat{\gamma}^Q\big)\, \widehat{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}^I;\hat{\gamma}^Q\big), \tag{432}$$

with the celestial trace factor:

$$\widehat{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}^I;\hat{\gamma}^Q\big) := \mathsf{Tr} \bigwedge_{i=1}^n \frac{\mathcal{C}(2h_i)}{\langle z_i|x+\widetilde{\mathcal{Q}}_i|\bar{z}_i]^{2h_i}}\, e^{i\langle z_i|(\theta+\tilde{q}_i)\cdot\eta_i\rangle}\, \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}. \tag{433}$$

7. *Dimensional Reduction*

We now apply the leaf amplitude formalism to the celestial amplitude obtained in Eq. (432). In this approach, the sectional amplitude arises by reducing the pre-moduli supermanifold $\widehat{\mathscr{M}}_{2k+3}$ down to the moduli superspace $\mathscr{M}_{2k+3}$.

Our construction rests on the correspondence between Klein and projective superspaces reviewed in Subsection III A 6. There we showed that the (supersymmetric) timelike wedge $W_s^- \subset \mathbf{K}^{4|8}$ admits coordinates $\big(r, \mathbb{X}^K\big)$, where r is an affine parameter on $\mathscr{R}$ and $\mathbb{X}^K = \big(X_{A\dot{A}}, \theta^\alpha_A\big)$ are homogeneous coordinates on $\mathbf{RP}^{3|8}$. A key result is the decomposition of the Berezin-de Witt measure on W_s^-:

$$d^{4|8}\mathsf{x}\,\big|_{W_s^-} \;=\; r^4 d\rho_r \wedge D^{3|8}\mathbb{X}, \tag{434}$$

where $d\rho_r \coloneqq d\log r$ is the Haar measure on $\mathscr{R}$.

Let $\mathsf{W}_i^{\sharp I}$ denote the involute dual minitwistor as defined in Eq. (295). The first step in deriving the sectional amplitude is to split the celestial amplitude:

$$\widehat{A}^{\mathsf{a}_1 \ldots \mathsf{a}_\mathsf{n}}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) = \widehat{B}^{\mathsf{a}_1 \ldots \mathsf{a}_\mathsf{n}}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) + \widehat{B}^{\mathsf{a}_1 \ldots \mathsf{a}_\mathsf{n}}_{n;(ab)_\ell}\big(\mathsf{W}_i^{\sharp I}\big). \tag{435}$$

The partial amplitude

$$\widehat{B}^{\mathsf{a}_1 \ldots \mathsf{a}_\mathsf{n}}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big)$$

is obtained by restricting Eq. (432) to the timelike wedge W_s^- and then applying the measure decomposition of Eq. (434).

To express the partial amplitude in closed form, we introduce the *moduli superspace*:

$$\mathscr{M}_{2k+3} \;\coloneqq\; \mathbf{RP}^{3|8} \times \mathscr{E}_{k+1}. \tag{436}$$

This supermanifold is globally charted by:

$$\gamma^Q \;\coloneqq\; \big(\mathbb{X}^K, \tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_{k+1}^{M_{k+1}}\big), \tag{437}$$

where each τ_ℓ^M parametrises the factor superspace $\mathcal{P}_\ell$. The abstract index Q runs over the set $\{K, M_1, \ldots, M_{k+1}\}$. The canonical $\mathbf{Z}_2$-graded volume form on $\mathscr{M}_{2k+3}$ is:

$$\mathcal{D}\boldsymbol{\gamma} \;\coloneqq\; D^{3|8}\mathbb{X} \wedge d^{8|4}\tau_1 \wedge d^{8|4}\tau_2 \wedge \ldots \wedge d^{8|4}\tau_{k+1}. \tag{438}$$

The partial amplitude is then given by:

$$\widehat{B}^{\mathsf{a}_1 \ldots \mathsf{a}_\mathsf{n}}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) \;=\; \int\limits_{\mathscr{R}} d\rho_r\, r^4 \int\limits_{\mathscr{M}_{2k+3}} \mathcal{D}\boldsymbol{\gamma}\;\; \mathcal{F}_{(ab)_\ell}\big(\gamma^Q\big)\, \widehat{\mathcal{T}}^{\mathsf{a}_1 \ldots \mathsf{a}_\mathsf{n}}\big(\mathsf{W}_i^I\big). \tag{439}$$

Here the trace factor is:

$$\widehat{\mathcal{T}}^{\mathsf{a}_1 \ldots \mathsf{a}_\mathsf{n}}\big(\mathsf{W}_i^I\big) \;=\; \mathsf{Tr} \bigwedge_{i=1}^{n} \frac{\mathcal{C}(2h_i)}{\langle z_i | r\mathcal{R} + \widetilde{\mathcal{Q}}_i | \bar{z}_i]^{2h_i}}\, e^{i\langle z_i|(\theta + \tilde{q}_i)\cdot \eta_i\rangle}\, \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i \cdot z_{i+1}}. \tag{440}$$

Under the reparametrization:

$$\tau_\ell^M \;=\; \big(u_\ell^A, v_\ell^B, \tilde{u}_{\ell\dot{A}}, \tilde{v}_{\ell\dot{B}}, \chi_\ell^\alpha\big) \;\mapsto\; \tilde{\tau}_\ell^M \;=\; \big(u_\ell^A, v_\ell^B, r\tilde{u}_{\ell\dot{A}}, r\tilde{v}_{\ell\dot{B}}, \chi_\ell^\alpha\big), \tag{441}$$

the measure (438) transforms as:

$$\mathcal{D}\boldsymbol{\gamma} \;\mapsto\; r^{4(k+1)}\mathcal{D}\boldsymbol{\gamma}. \tag{442}$$

Applying this in the above expression allows the r-integral to decouple and be performed explicitly. One finds:

$$\widehat{B}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) \;=\; 2\pi\delta\left(\beta_{k+1}\right)\, \mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big), \tag{443}$$

where the N^{k+1}*-MHV scaling parameter* is:

$$\beta_{k+1}\big(h_i\big) \;\coloneqq\; 4(k+2) - 2\sum_{i=1}^{n} h_i. \tag{444}$$

Finally, the sectional/leaf amplitude takes the form:

$$\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\big(\mathsf{W}_i^I\big) \;=\; \int\limits_{\mathscr{M}_{2k+3}} \mathcal{D}\boldsymbol{\gamma}\;\, \mathcal{F}_{(ab)_\ell}\big(\gamma^Q\big)\, \widetilde{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}_i^I;\gamma^Q\big), \tag{445}$$

with trace factor:

$$\widetilde{\mathcal{T}}^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{W}_i^I;\gamma^Q\big) \;=\; \mathsf{Tr}\bigwedge_{i=1}^{n} \frac{\mathcal{C}(2h_i)}{\langle z_i|\mathcal{R}+\widetilde{\mathcal{Q}}_i|\bar{z}_i]^{2h_i}}\, e^{i\langle z_i|(\theta+\tilde{q}_i)\cdot\eta_i\rangle}\, \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}. \tag{446}$$

8. Minitwistor Amplitude

Substituting the celestial RSVW identity (Eq. (312)) into the sectional amplitude (Eq. (445)) yields the following representation.

Geometric Background. Label the external gluons by $i = 1, ..., n$. On the moduli superspace $\mathscr{M}_{2k+3}$, we have embedding coordinates $\widetilde{\mathcal{Q}}_i^{A\dot{A}}$, $\tilde{q}_{iA}^\alpha$ which depend on the moduli parameters τ_ℓ^M for $\ell = 1, ..., k+1$ (cf. Subsection III C 3).

The minitwistor superspace $\mathbf{MT}_s$ is charted by homogeneous coordinates:

$$\mathsf{Z}^I \;=\; \big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big).$$

For a fixed moduli point:

$$\gamma^Q \;=\; \big(\mathbb{X}^K, \tau_1^{M_1}, \tau_2^{M_2}, ..., \tau_{k+1}^{M_{k+1}}\big) \in \mathscr{M}_{2k+3},$$

with $\mathbb{X}^K := \left(X_{A\dot{A}}, \theta^\alpha_{\dot{A}}\right) \in \mathbf{RP}^{3|8}$ and $\mathcal{R}_{A\dot{A}} := |X|^{-1} X_{A\dot{A}}$, define the minitwistor line $\mathscr{L}_i(\gamma^Q)$ by the incidence relations:

$$\begin{cases} \mu_{\dot{A}} = \lambda^A \left(\mathcal{R}_{A\dot{A}} + \widetilde{\mathcal{Q}}_{iA\dot{A}}\right), \\ \psi^\alpha = \lambda^A \left(\theta^\alpha_A + \tilde{q}^\alpha_{iA}\right). \end{cases} \tag{447}$$

Varying γ^Q over $\mathscr{M}_{2k+3}$ sweeps out all configurations of $2k+3$ minitwistor lines, justifying the identification of $\mathscr{M}_{2k+3}$ as the corresponding moduli superspace.

Each line $\mathscr{L}_i$ is parametrised by $\mathsf{Y}^I_i : \mathbf{RP}^1 \longrightarrow \mathbf{MT}_s$ such that:

$$\mathsf{Y}^I_i\left(\sigma^A\right) := \left(\sigma^A, \sigma^A\left(\mathcal{R}_{A\dot{A}} + \widetilde{\mathcal{Q}}_{iA\dot{A}}\right), \sigma^A\left(\theta^\alpha_A + \tilde{q}^\alpha_{iA}\right)\right). \tag{448}$$

Here $[\sigma^A]$ are homogeneous coordinates on $\mathbf{RP}^1$. On each line there is a natural Lie-algebra–valued logarithmic form:

$$\boldsymbol{\omega}^{\mathsf{a}_i}\left(\sigma^A\right) := \mathsf{T}^{\mathsf{a}_i} \frac{D\sigma_i}{\sigma_i \cdot \sigma_{i+1}}. \tag{449}$$

Minitwistor Amplitude. Let $\mathbf{M}^n := \bigtimes^n \mathbf{MT}_s$ be the integration superdomain. The celestial RSVW identity then recasts the sectional amplitude (Eq. (445)) as a multidimensional minitwistor transform:

$$\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\left(\mathsf{W}^I_i\right) = \int_{\mathbf{M}^n} \bigwedge_{i=1}^n D^{2|4}\mathsf{Z}_i \, \Psi_{2h_i}\left(\mathsf{Z}^I_i; \mathsf{W}^{I'}_i\right) \; \widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\left(\mathsf{Z}^I_i\right). \tag{450}$$

Moreover, the tree-level N^{k+1}-MHV minitwistor amplitude admits the integral representation:

$$\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}\left(\mathsf{Z}^I_i\right) = \int_{\mathscr{M}_{2k+3}} \mathcal{D}\boldsymbol{\gamma} \;\; \mathcal{F}_{(ab)_\ell}\left(\gamma^Q\right) \mathsf{Tr} \bigwedge_{i=1}^n \int_{\mathbf{RP}^1} \boldsymbol{\omega}^{\mathsf{a}_i}\left(\sigma^A_i\right) \bar{\delta}^{2|4}_{(-2h_i, 2h_i)}\left(\mathsf{Z}^I_i; \mathsf{Y}^{I'}_i\left(\sigma^A_i\right)\right). \tag{451}$$

Conclusion. By lifting $\mathcal{F}_{(ab)_\ell}$ to a probability distribution on $\mathscr{M}_{2k+3}$, the minitwistor amplitude $\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}$ acquires a geometric interpretation as a volume integral over this moduli superspace, weighted by $\mathcal{F}_{(ab)_\ell}$. In the discussion of our minitwistor celestial CFT, we will show that $\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;(ab)_\ell}$ corresponds to the *semiclassical expectation value* of the observable $\mathcal{F}_{(ab)_\ell}$.

The volume form in the integral (451) is localised on the family of minitwistor lines $\mathscr{L}_i$ by the distributions:

$$\bar{\delta}^{2|4}_{(-2h_i, 2h_i)}\left(\mathsf{Z}^I_i; \mathsf{Y}^{I'}_i\left(\sigma^A_i\right)\right).$$

Consequently, the amplitude vanishes whenever the i-th gluon does not lie on its corresponding line $\mathscr{L}_i$. This completes the extension of the N^1- and N^2-MHV superamplitude results to the full tree-level celestial $\mathcal{S}$-matrix.

IV. MINITWISTOR WILSON LINES

We now present a central result of this work: the reformulation of celestial leaf amplitudes in $\mathcal{N} = 4$ SYM theory as expectation values of Wilson line operators on minitwistor superspace. In the preceding sections, we showed that the N^k-MHV celestial amplitudes localise on a specific set Σ of rational curves $\mathcal{L}_1, \ldots, \mathcal{L}_{2k+1}$. This localisation is not a technical detail but a hint toward a geometric description of gluon scattering in asymptotically flat spacetimes. We argue that the language for this description is provided by minitwistor Wilson lines.

To define these operators, two ingredients are required: (i) a path along which to compute holonomy, and (ii) a partial connection whose path-ordered exponential yields the Wilson line. The path is fixed by our localisation result: it is the union of the minitwistor lines $\mathcal{L}_1, \ldots, \mathcal{L}_{2k+1}$ supporting the amplitude. The partial connection is taken to be a pseudoholomorphic structure on a complex vector bundle over $\mathbf{MT}_s$.

For physical insight, we interpret this construction via a minitwistor sigma model, heuristically referred to as a "minitwistor string theory." Although this model is defined only at the semiclassical level, it provides a useful conceptual framework. Here, the sigma-model correlation functions define an effective field theory on $\mathbf{MT}_s$, and one may regard this effective theory (provisionally) as a "string field theory" on minitwistor superspace.

The dynamics of this minitwistor field theory may be understood by *analogy* with Kodaira-Spencer gravity. In this theory, the gauge potential parametrises a deformation of the canonical holomorphic structure on a complex vector bundle. Consequently, the physical degrees of freedom are encoded in its fieldstrength, the pseudocurvature $(0, 2)$-form. The physical observables are therefore the holonomies of the background partial connection, and the central result of this correspondence is that the minitwistor Wilson lines (i.e., the holonomies supported on the set Σ of rational curves where the scattering amplitudes localise) reproduce the gluonic leaf amplitudes.

A. Holomorphic Wilson Lines

Our aim in this section is to construct the gauge-invariant observables of the theory: the holomorphic Wilson lines[20]. We adapt the well-known formalism on projective twistor superspace $\mathbf{PT}^{3|4}$ to minitwistor superspace $\mathbf{MT}_s$. This extension is needed for defining the holonomy operators that compute celestial gluon amplitudes. The main technical problem is to define parallel transport along

[20] See Mason and Skinner [70] and Bullimore and Skinner [79]. For the history of the subject, see Atiyah [80] and Penrose [81].

minitwistor lines in a background where the canonical holomorphic structure has been deformed by a gauge field.

1. *Review: Holomorphic Gauge Theory*

In Appendix A, we provide a concise, pedagogical introduction to the essentials of holomorphic gauge theory. Here, we summarise the core definitions employed in our construction of minitwistor Wilson lines.

Let $\pi\colon E \to \mathbf{MT}_s$ be a complex vector superbundle whose fibres carry the adjoint representation of a gauge Lie superalgebra $\mathfrak{g}$. In the absence of a gauge field, the physical vacuum is specified by the canonical holomorphic structure on E, given by the Dolbeault operator $\overline{\partial}^E$.

Ground States. A matter field $|\psi\rangle \in \Omega^{r,s}(E)$ is called a ground state if it is holomorphic with respect to the vacuum structure:

$$\overline{\partial}^E|\psi\rangle = 0. \tag{452}$$

However, two holomorphic forms that differ by an exact term are gauge-equivalent. Explicitly, if there exists $|\chi\rangle \in \Omega^{r,s-1}(E)$ such that

$$|\psi'\rangle = |\psi\rangle + \overline{\partial}^E|\chi\rangle, \tag{453}$$

then $|\psi\rangle$ and $|\psi'\rangle$ represent the same physical state. Hence the space of distinct ground states is the Dolbeault cohomology of $\mathbf{MT}_s$ with values in E. Defining $\mathcal{H}_{p,q}$ as the Hilbert space of ground states of bi-degree (p, q), we have the isomorphism

$$\mathcal{H}_{p,q} \;\cong\; H^{p,q}(\mathbf{MT}_s; E). \tag{454}$$

Gauge Potential. We now deform the vacuum by introducing a background gauge potential. Let $\boldsymbol{A} \in \Omega^{0,1}\big(\mathbf{MT}_s; \mathrm{End}_{\mathbf{C}}(E)\big)$ be a differential $(0, 1)$-form valued in endomorphisms of E. This defines a new pseudoholomorphic structure $\mathscr{E}$ via the twisted Dolbeault operator

$$\overline{\partial}^{\mathscr{E}} \;:=\; \overline{\partial}^E \;+\; \boldsymbol{A}. \tag{455}$$

Parallel transport along a minitwistor line $\mathcal{L}$ is then given by the operator

$$\langle w'|\mathcal{L}|w\rangle\colon\; E|_w \;\longrightarrow\; E|_{w'}, \tag{456}$$

which transports a vector in the fibre over w to the fibre over w' along $\mathcal{L}$.

Fieldstrength. The failure of $\overline{\partial}^{\mathscr{E}}$ to square to zero defines the fieldstrength (pseudocurvature) of the gauge field:

$$\boldsymbol{F} \;:=\; \overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}} \;=\; \overline{\partial}^{E}\boldsymbol{A} \;+\; \boldsymbol{A} \wedge_{\mathfrak{g}} \boldsymbol{A}. \tag{457}$$

Here $\boldsymbol{F}$ is a $\mathfrak{gl}(r, \mathbf{C})$-valued differential $(0,2)$-form on $\mathbf{MT}_s$. The term $\overline{\partial}^{E}\boldsymbol{A}$ is analogous to the kinetic part of the fieldstrength, while the non-linear piece $\boldsymbol{A} \wedge_{\mathfrak{g}} \boldsymbol{A}$ encodes the self-interactions characteristic of non-Abelian gauge theory.

2. Parallel Transport; Abelian Case

In the preceding subsection, we reviewed the geometric framework for formulating a non-Abelian holomorphic gauge theory on minitwistor superspace. We now pose the following problem. Let $\pi\colon E \to \mathbf{MT}_s$ be a rank-r complex vector bundle endowed with a pseudoholomorphic structure $\mathscr{E}$ and partial connection $\overline{\partial}^{\mathscr{E}}$. Moreover, let $\mathcal{L} \subset \mathbf{MT}_s$ denote a minitwistor line. How does one define parallel transport along $\mathcal{L}$ in the background $\mathscr{E}$?

Restricted Bundle. To answer this question, we first restrict the bundle E to $\mathcal{L}$. Let

$$\pi_{\mathcal{L}}\colon E|_{\mathcal{L}} \;\longrightarrow\; \mathcal{L}, \quad E|_{\mathcal{L}} \;:=\; \pi^{-1}(\mathcal{L}) \tag{458}$$

denote the restricted bundle, where $\pi_{\mathcal{L}} = \pi|_{\pi^{-1}(\mathcal{L})}$. This restriction is analogous to describing a bulk spacetime from the viewpoint of a worldline.

The pseudoholomorphic structure $\mathscr{E}$ on E induces a corresponding structure $\mathfrak{L}$ on $E|_{\mathcal{L}}$. Since $\mathcal{L}$ is a rational curve, choose any nonsingular section $f \in \Gamma(\mathbf{CP}^1; \mathcal{L})$, so that $df\colon T(\mathbf{CP}^1) \to T(\mathcal{L})$ is a bundle morphism. In terms of this section, the induced Dolbeault operator on $E|_{\mathcal{L}}$ is defined by

$$\overline{\partial}^{\mathfrak{L}} \;:=\; \overline{\partial}^{\mathscr{E}}\big|_{\mathcal{L}} \;:=\; f^*\big(\overline{\partial}^{\mathscr{E}}\big). \tag{459}$$

It is straightforward to verify that this definition is independent of the choice of f.

Holomorphic Frame. We next ask whether the restricted bundle $E|_{\mathcal{L}}$ admits a global holomorphic frame over $\mathcal{L}$. Establishing such a frame is the key step in constructing the parallel transport operator along $\mathcal{L}$. To this end, let $\boldsymbol{F}^{\mathfrak{L}} = \overline{\partial}^{\mathfrak{L}} \circ \overline{\partial}^{\mathfrak{L}}$ denote the pseudocurvature of $\mathfrak{L}$. By Lemma 1 of Appendix A, $\boldsymbol{F}^{\mathfrak{L}}$ is $\mathscr{C}^\infty$-linear and hence an element of $\Omega^{0,2}\big(\mathcal{L}; \mathrm{End}_{\mathbf{C}}(E)\big)$. Since $\mathcal{L} \cong \mathbf{CP}^1$ has complex dimension one, all $(0,2)$-forms vanish and thus $\boldsymbol{F}^{\mathfrak{L}} = 0$. It follows that $\overline{\partial}^{\mathfrak{L}}$ is integrable and $\mathfrak{L}$ is holomorphic.

On the other hand, the Birkhoff-Grothendieck theorem[21] states that every holomorphic vector bundle over $\mathbf{CP}^1$ splits as a direct sum of line bundles. Applying this to $E|_{\mathcal{L}}$ equipped with the holomorphic structure $\mathfrak{L}$ gives

$$E|_{\mathcal{L}} \;\cong\; \mathcal{O}_{\mathbf{C}}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbf{C}}(a_r). \tag{460}$$

In particular, $E|_{\mathcal{L}}$ is topologically trivial. Finally, Lemma 2 of Appendix A guarantees that a holomorphic, topologically trivial bundle admits a global holomorphic frame. Hence there exists a frame $H = (H_1, \ldots, H_r)$ trivialising $E|_{\mathcal{L}}$ with

$$\overline{\partial}^{\mathfrak{L}} H_i \;=\; \big(\overline{\partial}^{\mathscr{E}} + \boldsymbol{A}\big)\big|_{\mathcal{L}} H_i \;=\; 0. \tag{461}$$

Now that the existence of a global holomorphic frame H on $E|_{\mathcal{L}}$ has been established, the parallel-transport operator between any two points $w, w' \in \mathcal{L}$ can be written as:

$$\langle w'|\mathcal{L}|w\rangle \;=\; H(w')H^{-1}(w). \tag{462}$$

Abelian Case. To derive the parallel-transport operator for a holomorphic gauge theory on $\mathbf{MT}_s$, we begin with the Abelian case ($r = 1$). Here, the holomorphic frame reduces to a single component $h \in \Gamma(\mathcal{L}; GL(1, \mathbf{C}))$, which we parametrise by a phase function $\phi \in \mathscr{C}^\infty(\mathcal{L};\, \mathfrak{gl}(1, \mathbf{C}))$ via:

$$h = \exp(-\phi). \tag{463}$$

Substituting into Eq. (461) gives

$$\overline{\partial}\big|_{\mathcal{L}}\, \phi \;=\; \boldsymbol{A}\big|_{\mathcal{L}}, \tag{464}$$

where $\overline{\partial}\big|_{\mathcal{L}}$ is the Cauchy-Riemann (CR) operator on the line $\mathcal{L}$.

To solve Eq. (464), we must invert $\overline{\partial}\big|_{\mathcal{L}}$, which requires a fundamental solution of the CR operator. To construct it, let us briefly review the notion of Green differentials on Riemann surfaces.

Let $\Omega^{p,q}_{\mathfrak{m}}\big(\mathcal{L}; \mathcal{O}_{\mathbf{C}}(a)\big)$ denote the space of meromorphic (p, q)-forms on $\mathcal{L}$ valued in the line bundle $\mathcal{O}_{\mathbf{C}}(a)$ (often called *abelian differentials*). Fix simple poles $w_1, \ldots, w_k \in \mathcal{L}$ and assign residues $r_1, \ldots, r_k \in \mathbf{C}$. A *Green differential* $\boldsymbol{\tau} \in \Omega^{1,0}_{\mathfrak{m}}$ with these poles and residues is defined by the partial differential equation[22] (PDE):

$$\overline{\partial}\big|_{\mathcal{L}}\, \boldsymbol{\tau}(\lambda^A) \;=\; \sum_{i=1}^{k} r_i\, \overline{\delta}(w_i \cdot \lambda) \wedge D\lambda, \tag{465}$$

[21] See Birkhoff [82] and Grothendieck [83]. For a pedagogical introduction, cf. Okonek, Spindler, and Schneider [84, Sec. 1.2].

[22] See Demailly [85, Ch. 1, Sec. 2] for a mathematically rigorous discussion of the terms appearing in Eq. (465).

where $\overline{\delta}(w_i\cdot\lambda)$ is the $(0,1)$-current supported at the i-th pole[23]. By Liouville theorem, $\boldsymbol{\tau}$ cannot be holomorphic. A standard existence-and-uniqueness theorem on Riemann surfaces (see Forster [86, Sec. 1.11]) then guarantees a unique solution of Eq. (465). We therefore call $\boldsymbol{\tau}$ the Green differential for this PDE.

To solve Eq. (464), we introduce the Green differential $\boldsymbol{k}_0(w;\lambda^A)$ which satisfies the PDE:

$$\frac{1}{2\pi i}\,\overline{\partial}_\lambda\big|_{\mathcal{L}}\,\boldsymbol{k}_0(w;\lambda)\;+\;\overline{\delta}(w\cdot\lambda)\wedge D\lambda\;=\;0. \tag{466}$$

Having defined $\boldsymbol{k}_0$, we invert the CR operator on ϕ to obtain:

$$\phi(w)\;=\;\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\boldsymbol{k}_0(w;\lambda^A)\wedge\boldsymbol{A}|_{\mathcal{L}}(\lambda^A). \tag{467}$$

Substituting this expression for ϕ into Eq. (462) yields the Abelian parallel-transport operator:

$$\langle w'|\mathcal{L}|w\rangle\;=\;\exp\left(-\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\boldsymbol{k}(w',w;\lambda^A)\wedge\boldsymbol{A}|_{\mathcal{L}}(\lambda^A)\right), \tag{468}$$

where $\boldsymbol{k}(w',w;\lambda^A)\in\Omega^{1,0}_{\mathfrak{m}}(\mathcal{L})$ is the Green differential solving:

$$\frac{1}{2\pi i}\,\overline{\partial}_\lambda\big|_{\mathcal{L}}\,\boldsymbol{k}(w',w;\lambda^A)\;+\;\boldsymbol{J}(w',w;\lambda^A)\wedge D\lambda\;=\;0, \tag{469}$$

with the current:

$$\boldsymbol{J}(w',w;\lambda^A)\;\coloneqq\;\overline{\delta}(w'\cdot\lambda)-\overline{\delta}(w\cdot\lambda)\;\in\;\mathscr{D}'_{0,1}\big(\mathcal{L};\,\mathcal{O}_{\mathbf{C}}(-1)\big). \tag{470}$$

Since the left-hand side of Eq. (469) belongs to $\Omega^{1,1}_{\mathfrak{m}}\big(\mathcal{L};\mathcal{O}_{\mathbf{C}}(1)\big)$, the Green differential $\boldsymbol{k}(w',w;\lambda^A)$ is invariant under the rescalings $w\mapsto t\,w$, $w'\mapsto t\,w'$ and $\lambda^A\mapsto t\,\lambda^A$. This homogeneity condition fixes its analytic form to:

$$\boldsymbol{k}(w',w;\lambda^A)\;=\;\frac{\langle w',w\rangle}{\langle w',\lambda\rangle\langle\lambda,w\rangle}\,D\lambda. \tag{471}$$

3. Non-Abelian Parallel Transport; Wilson Operator

Guided by the Abelian case, we now define the parallel-transport operator along a minitwistor line $\mathcal{L}$ in a non-Abelian background[24]:

$$\langle w'|\mathcal{L}|w\rangle\;=\;\mathrm{Pexp}\left(-\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\boldsymbol{k}(w',w;\lambda^A)\wedge\boldsymbol{A}|_{\mathcal{L}}(\lambda^A)\right). \tag{472}$$

[23] Recall that $2\pi i\,\overline{\delta}(z)=\overline{\partial}\left(\frac{1}{z}\right)\in\mathscr{D}'_{0,1}$.

[24] From now on, we fix $w,w'\in\mathcal{L}$ and, to avoid clutter, write the Green differential $\boldsymbol{k}(w',w;\lambda^A)$ simply as $\boldsymbol{k}(\lambda^A)$.

Using this expression, we will be able to construct a generating functional for leaf-gluon amplitudes in MHV configurations, and later, for N^k-MHV sectors. Our first task is then to clarify the meaning of the path-ordered exponential appearing in Eq. (472).

To simplify notation, let $\{\lambda_i^A\}_{i=1}^n$ denote points along the line $\mathcal{L}$. Define

$$\boldsymbol{k}_i \;\coloneqq\; \boldsymbol{k}(\lambda_i^A), \quad i = 1, \ldots, n. \tag{473}$$

We then introduce the path-ordered, degree-n monomial in $\boldsymbol{k}$:

$$\mathrm{P}\bigwedge_{i=1}^{n} \boldsymbol{k}_i \;\coloneqq\; \frac{\langle w', w\rangle}{\langle w', \lambda_1\rangle\langle\lambda_1, \lambda_2\rangle \ldots \langle\lambda_{n-1}, \lambda_n\rangle\langle\lambda_n, w\rangle} \bigwedge_{i=1}^{n} D\lambda_i. \tag{474}$$

This $(n, 0)$-form is invariant under each scaling $\lambda_i^A \mapsto t\,\lambda_i^A$. Hence it defines a *genuine* differential form on the n-fold product $\mathcal{L}^n \coloneqq \bigtimes^n \mathcal{L}$, not merely a section of a homogeneous bundle.

For any Lie-algebra-valued $(0, 1)$-form $\boldsymbol{B} \in \Omega^{0,1}(\mathcal{L}; \mathfrak{g})$ and parameter g, we define the path-ordered exponential by the formal powerseries:

$$\mathrm{Pexp}\left(g \int_{\mathcal{L}} \boldsymbol{k} \wedge \boldsymbol{B} \right) \;\coloneqq\; \mathbb{I}_{\mathfrak{g}} \;+\; \sum_{n\geq 1} g^n \int_{\mathcal{L}^n} \mathrm{P}\bigwedge_{i=1}^{n} \boldsymbol{k}_i \wedge \boldsymbol{B}(\lambda_i^A). \tag{475}$$

One can make this rigorous via formal distributions, a standard tool in vertex-algebra theory (see Chapter 2, § 1 of Kac [87]). Substituting into Eq. (472) gives the parallel-transport operator:

$$\langle w'|\mathcal{L}|w\rangle^{\boldsymbol{A}} \;=\; \mathbb{I}_{\mathfrak{g}} \;+\; \sum_{n\geq 1} \frac{i^n}{(2\pi)^n} \int_{\mathcal{L}^n} \mathrm{P}\bigwedge_{i=1}^{n} \boldsymbol{k}_i \wedge \boldsymbol{A}|_{\mathcal{L}}(\lambda_i^A). \tag{476}$$

We add the superscript $\boldsymbol{A}$ to the left-hand side of Eq. (476), since this notation makes explicit its dependence on the non-Abelian background gauge potential.

Having completed the geometric preliminaries, we now define the Wilson line operator on minitwistor superspace:

$$\mathbb{W}^{\boldsymbol{A}}_{w',w}[\mathcal{L}] \;\coloneqq\; \mathsf{Tr}_{\mathfrak{g}}\, \langle w'|\mathcal{L}|w\rangle^{\boldsymbol{A}}. \tag{477}$$

This non-local, gauge-invariant observable probes the background configuration induced by the gauge potential $\boldsymbol{A}$.

Physical Interpretation. Recall that we view holomorphic gauge theory on the complex vector bundle $E \to \mathbf{MT}_s$ as a theory of deformations of its complex structure. In this picture, physical states correspond to Dolbeault cohomology classes of E, and the vacuum holomorphic structure is fixed by the standard Dolbeault operator $\overline{\partial}^{E}$. Accordingly, we define the minitwistor Wilson line $\mathbb{W}^{\boldsymbol{A}}_{w',w}[\mathcal{L}]$ to measure the deformation of this vacuum induced by an external potential $\boldsymbol{A}$.

We conclude this subsection by deriving a formal powerseries expression for the minitwistor Wilson line. This result will simplify our derivation of the generating functional for MHV leaf-gluon amplitudes from the expectation values of $\mathbb{W}^{\boldsymbol{A}}_{w',w}[\mathcal{L}]$.

First, recall the path-ordered monomial in $\boldsymbol{k}$ introduced in Eq. (474). One finds

$$\mathrm{P}\bigwedge_{i=1}^{n}\boldsymbol{k}_i \;=\; \mathcal{N}\big(w',w|\lambda_n,\lambda_1\big)\,\bigwedge_{i=1}^{n}\frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}}, \tag{478}$$

where

$$\mathcal{N}\big(w',w|\lambda_n,\lambda_1\big) \;=\; \frac{\langle w',w\rangle\langle\lambda_n,\lambda_1\rangle}{\langle w',\lambda_1\rangle\langle\lambda_n,w\rangle}. \tag{479}$$

The function $\mathcal{N}$ is projectively well-defined for $w,w',\lambda_1,\lambda_n\in\mathbf{CP}^1$, since it is invariant under

$$w\mapsto t\,w,\quad w'\mapsto t\,w',\quad \lambda_1\mapsto t\,\lambda_1,\quad \lambda_n\mapsto t\,\lambda_n. \tag{480}$$

Moreover, it obeys the normalisation

$$\mathcal{N}\big(w',w|w',w\big) \;=\; 1 \quad \text{for all } w,w'\in\mathbf{CP}^1. \tag{481}$$

Substituting Eq. (478) into the powerseries expansion of the parallel-transport operator (Eq. (476)) gives

$$\langle w'|\mathcal{L}|w\rangle^{\boldsymbol{A}} \;=\; \mathbb{I}_{\mathfrak{g}} \;+\; \sum_{n\geq 1}\frac{i^n}{(2\pi)^n}\int\limits_{\mathcal{L}^n}\mathcal{N}\big(w',w|\lambda_n,\lambda_1\big)\bigwedge_{i=1}^{n}\frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}}\wedge\boldsymbol{A}\big|_{\mathcal{L}}(\lambda_i^A). \tag{482}$$

This series provides an expansion of the Wilson line in powers of the gauge potential:

$$\mathbb{W}^{\boldsymbol{A}}_{w',w}[\mathcal{L}] \;=\; r \;+\; \sum_{n\geq 1} g^n\int\limits_{\mathcal{L}^n}\mathcal{N}\big(w',w|\lambda_n,\lambda_1\big)\,\mathsf{Tr}_{\mathfrak{g}}\left(\bigwedge_{i=1}^{n}\frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}}\wedge\boldsymbol{A}\big|_{\mathcal{L}}(\lambda_i^A)\right) \tag{483}$$

where $r\coloneqq\mathrm{rank}_{\mathbf{C}}(\mathfrak{g})$ and $g\coloneqq i/(2\pi)$.

4. MHV Leaf-Gluon Amplitudes

In the preceding subsections, we constructed a theory of minitwistor Wilson lines. These non-local, gauge-invariant observables probe the deformation of the holomorphic structure of a complex vector bundle over minitwistor superspace induced by a background gauge field. Our formalism is now rich enough to derive the MHV leaf-gluon superamplitude from the semiclassical expectation value of the minitwistor Wilson line.

To proceed, we must complete two preliminary tasks. First, we review the minitwistor description of celestial leaf amplitudes. Second, we clarify the meaning of semiclassical expectation values for a non-local operator such as $\mathbb{W}^{\boldsymbol{A}}_{w',w}[\mathcal{L}]$.

Dual Minitwistors and Celestial Supersphere. The celestial description of gluonic scattering in $\mathcal{N}=4$ SYM theory assigns to each of the n external gluons a conformal weight Δ_i and an insertion point on the $\mathcal{N}=4$ celestial supersphere $\mathcal{CS}_s \cong \mathbf{CP}^{1|4}$. Following Tropper [88], we cover $\mathcal{CS}_s$ with coordinates $\mathsf{z} \coloneqq (z, \bar{z}, \eta^\alpha)$, where z and $\bar{z}$ are the holomorphic and antiholomorphic coordinates on $\mathbf{CP}^1$, and η^α are Grassmann variables encoding helicity states. Thus, the i-th gluon's insertion point carries coordinates

$$\mathsf{z}_i \;=\; (z_i, \bar{z}_i, \eta_i^\alpha) \;\in\; \mathcal{CS}_s. \tag{484}$$

From these data, we form the dual minitwistor

$$\mathsf{Z}_i^I \;\coloneqq\; \left(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha\right) \;\in\; \mathbf{MT}_s^*, \tag{485}$$

where the van der Waerden spinors are $z_i^A \coloneqq (1, -z_i)^T$ and $\bar{z}_{i\dot{A}} \coloneqq (1, -\bar{z}_i)^T$. Hence, *the dual minitwistor superspace* $\mathbf{MT}_s^*$ *parametrises the celestial supersphere via*

$$\mathcal{CS}_s \;\longrightarrow\; \mathbf{MT}_s^*, \qquad \mathsf{z}_i \;\longmapsto\; \mathsf{Z}_i^I. \tag{486}$$

We identify the fixed points w and w' on the minitwistor line $\mathcal{L} \subset \mathbf{MT}_s$ with the endpoint gluons: $w = z_1$, $w' = z_n$. This assignment does not restrict the total number of gluons in our generating functional. Here, the subscript n in z_n is merely a label, just as the prime in w' is.

But what is the physical meaning of the minitwistor line $\mathcal{L}$ *on which the Wilson operator* $\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}]$ *is supported?* To answer this, we anticipate some ideas from the minitwistor sigma-model, which we discuss in detail in the next section. We view the sigma-model as a minitwistor "string" with worldsheet the celestial supersphere $\mathcal{CS}_s$ and target manifold the minitwistor superspace $\mathbf{MT}_s$. In this picture, each minitwistor line $\mathcal{L}$ corresponds to one classical configuration of the sigma-model. Remarkably, this interpretation leads directly to the expectation value of the Wilson line operator (see Eq. (489) below), which serves as the generating functional for leaf-gluon superamplitudes.

Semiclassical Expectation Value of $\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}]$. Our physical picture then interprets the holomorphic gauge theory on minitwistor superspace probed by $\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}]$ as the effective field theory induced by the minitwistor sigma-model. When we regard this sigma-model as a minitwistor "string theory," the holomorphic gauge theory plays the role of its "string field theory."

More formally, the sigma-model describes embeddings of the celestial supersphere into minitwistor superspace as a family of minitwistor lines. Its dynamical fields are then holomorphic rational maps

$$\mathbf{CP}^{1|4} \;\longrightarrow\; \mathbf{MT}_s \tag{487}$$

that embed the worldsheet as a minitwistor line.

Moreover, from Section II, the supersymmetric Hitchin correspondence on $\mathbf{MT}_s$ provides a bijection

$$\mathsf{X}^K = \big(X_{A\dot{A}},\, \theta^{\alpha}_{A}\big) \;\in\; \mathbf{H}_s \quad\longleftrightarrow\quad \mathcal{L}(X,\theta) \;\subset\; \mathbf{MT}_s. \tag{488}$$

Thus $\mathbf{H}_s$ is the *moduli superspace* of minitwistor lines. Equivalently, it is the *configuration space* of our minitwistor string.

We then define

$$\mathcal{I}_0[\boldsymbol{A}] \;\coloneqq\; \int\limits_{\mathbf{H}_s} D^{3|8}\mathsf{X}\; \mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}\big[\mathcal{L}(X,\theta)\big], \tag{489}$$

an integral over the moduli superspace of minitwistor lines. Following Feynman's pathintegral formalism and the superposition principle, summing over all embeddings yields an expectation value. In addition, since our sigma-model is defined only semiclassically, $\mathcal{I}_0[\boldsymbol{A}]$ is the *semiclassical expectation value* of the Wilson line operator probing the background gauge potential $\boldsymbol{A}$ in our minitwistor string field theory.

Generating Functional. We now show that $\mathcal{I}_0[\boldsymbol{A}]$ generates the MHV leaf amplitudes for gluons. First, decompose the gauge potential $\boldsymbol{A}$ into Fourier modes $\alpha^{\Delta,\mathsf{a}}$ via the minitwistor transform $\mathcal{MT}$ of Section II. To do so, we project $\boldsymbol{A}$ onto the superwavefunctions $\Psi_\Delta \coloneqq \Psi^0_\Delta$ which satisfy

$$\Psi_\Delta(\mathsf{W}^I;\,\cdot\,) \in \mathscr{D}'_{0,0}\big(\mathbf{MT}^*_s;\, \mathcal{O}_{\mathbf{C}}(-\Delta-2,-\Delta)\big) \quad \text{for fixed } \mathsf{W}^I \in \mathbf{MT}_s, \tag{490}$$

and

$$\Psi_\Delta(\,\cdot\,;\mathsf{Z}^I) \in \mathscr{D}'_{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}_{\mathbf{C}}(\Delta,-\Delta)\big) \quad \text{for fixed } \mathsf{Z}^I \in \mathbf{MT}^*_s. \tag{491}$$

We then expand

$$\boldsymbol{A}\big(\mathsf{W}^I\big) \;=\; 2\pi i \int\limits_{\mathbf{MT}_s^*} \Psi_\Delta\big(\mathsf{W}^I;\mathsf{Z}'^I\big)\,\alpha^{\Delta,\mathsf{a}}\big(\mathsf{Z}'^I\big)\,\mathsf{T}^{\mathsf{a}}\wedge D^{2|4}\mathsf{Z}'. \tag{492}$$

The factor $2\pi i$ is a convenient normalisation that can be absorbed into $\alpha^{\Delta,\mathsf{a}}$, and the prime on Z'^I marks the integration variable.

How do we interpret this expansion? Its key novelty is the use of DeWitt notation for the conformal weight Δ. Let $\mathcal{P}$ denote the domain of Δ, equipped with a Stieltjes measure $d\alpha(\Delta)$. Then, following DeWitt, define

$$\Psi_\Delta\big(\mathsf{W}^I;\mathsf{Z}'^I\big)\,\alpha^{\Delta,\mathsf{a}}\big(\mathsf{Z}'^I\big) \;\coloneqq\; \int\limits_{\mathcal{P}} d\alpha(\Delta)\;\Psi_\Delta\big(\mathsf{W}^I;\mathsf{Z}'^I\big)\,\alpha^{\mathsf{a}}_\Delta\big(\mathsf{Z}'^I\big), \tag{493}$$

so that $\alpha^{\Delta,\mathsf{a}}$ becomes a continuum of mode functions labeled by Δ.

However, the integral over dual minitwistor superspace in Eq. (492) only makes sense if the integrand

$$\Psi_\Delta\big(\mathsf{W}^I;\mathsf{Z}'^I\big)\,\alpha^{\Delta,\mathsf{a}}\big(\mathsf{Z}'^I\big)\,\mathsf{T}^{\mathsf{a}}\wedge D^{2|4}\mathsf{Z}' \;\in\; \Omega^{(2,2)|4}\big(\mathbf{MT}_s^*;\mathfrak{g}\big) \tag{494}$$

defines a $\mathfrak{g}$-valued *top-form* on $\mathbf{MT}_s^*$. In particular, the mode coefficients must lie in

$$\alpha^{\Delta,\mathsf{a}} \;\in\; \Omega^{1,1}\big(\mathbf{MT}_s^*;\,\mathfrak{g}\otimes\mathcal{O}_{\mathbf{C}}(\Delta-4,\Delta-2)\big). \tag{495}$$

It follows that the gauge field itself lives in the direct sum

$$\boldsymbol{A} \;\in\; \bigoplus_{\Delta\in\mathcal{P}} \Omega^{0,1}\big(\mathbf{MT}_s;\,\mathfrak{g}\otimes\mathcal{O}_{\mathbf{C}}(\Delta,-\Delta)\big). \tag{496}$$

This may seem unusual, but note that when we restrict $\boldsymbol{A}$ to any minitwistor line $\mathcal{L}$, we have

$$\boldsymbol{A}\big|_{\mathcal{L}} \;\in\; \Omega^{0,1}(\mathcal{L};\,\mathfrak{g}), \tag{497}$$

a projectively invariant $\mathfrak{g}$-valued $(0,1)$-form on $\mathcal{L}$. This matches our requirement for a gauge potential on a minitwistor line and fits with the definition of the non-Abelian parallel-transport operator (Eq. (472)).

From Eq. (492), the gauge potential induced on the line $\mathcal{L}$ is:

$$\boldsymbol{A}\big|_{\mathcal{L}}(\lambda^A) \;=\; 2\pi i \int\limits_{\mathbf{MT}_s^*} \Psi_\Delta\big|_{\mathcal{L}}\big(\lambda^A;\mathsf{Z}'^I\big)\,\alpha^{\Delta,\mathsf{a}}\big(\mathsf{Z}'^I\big)\,\mathsf{T}^{\mathsf{a}}\wedge D^{2|4}\mathsf{Z}'. \tag{498}$$

Substituting this into the powerseries expansion of the Wilson line operator (Eq. (483)) gives:

$$\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}] \;=\; r \;+\; \sum_{n\geq 1} (-1)^n \int\limits_{\mathcal{L}^n} \mathcal{N}\big(z_n, z_1|\lambda_n, \lambda_1\big)\; \mathsf{Tr}_{\mathfrak{g}} \bigwedge_{i=1}^{n} \frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}} \tag{499}$$

$$\int\limits_{\mathbf{MT}^*_s} \Psi_{\Delta_i}\big|_{\mathcal{L}}\,\big(\lambda^A_i; \mathsf{Z}'^I_i\big)\, \alpha^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}'^I_i\big)\, \mathsf{T}^{\mathsf{a}_i} \wedge D^{2|4}\mathsf{Z}'_i. \tag{500}$$

By Fubini's theorem, we can exchange the integrals. Define $\mathbf{X}^*_n := \times^n \mathbf{MT}^*_s$, oriented by $\bigwedge_{i=1}^n D^{2|4}\mathsf{Z}'_i$. Then Eq. (500) becomes:

$$\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}] \;=\; r \;+\; \sum_{n\geq 1} (-1)^n \int\limits_{\mathbf{X}^*_n} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}'_i \wedge \alpha^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}'_i\big) \int\limits_{\mathcal{L}^n} \mathcal{N}\big(z_n, z_1|\lambda_n, \lambda_1\big) \tag{501}$$

$$\mathsf{Tr}_{\mathfrak{g}} \bigwedge_{j=1}^{n} \frac{D\lambda_j}{\lambda_j\cdot\lambda_{j+1}}\, \mathsf{T}^{\mathsf{a}_j} \wedge \Psi_{\Delta_j}\big|_{\mathcal{L}}\,\big(\lambda^A_j; \mathsf{Z}'^I_j\big). \tag{502}$$

Applying the celestial BMSW identity to the $\mathcal{L}^n$ integral yields:

$$\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}] \;=\; r \;+\; \sum_{n\geq 1} (-1)^n \int\limits_{\mathbf{X}^*_n} \mathcal{N}\big(z_n, z_1|z'_n, z'_1\big)\; \mathsf{Tr}_{\mathfrak{g}} \bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z'_i|X|\bar{z}'_i]^{\Delta_i}} \tag{503}$$

$$e^{i\langle z'_i|\theta\cdot\eta_i\rangle}\, \frac{\mathsf{T}^{\mathsf{a}_i}}{z'_i\cdot z'_{i+1}}\, \alpha^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}'^I_i\big) \wedge D^{2|4}\mathsf{Z}'_i. \tag{504}$$

Hence the integral in Eq. (489) becomes[25]:

$$\mathcal{I}_0[\boldsymbol{A}] \;=\; \sum_{n\geq 1} (-1)^n \int\limits_{\mathbf{X}^*_n} \mathcal{N}\big(z_n, z_1|z'_n, z'_1\big) \int\limits_{\mathbf{H}_s} D^{3|8}\mathsf{X}\; \mathsf{Tr}_{\mathfrak{g}} \bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z'_i|X|\bar{z}'_i]^{\Delta_i}} \tag{505}$$

$$e^{i\langle z'_i|\theta\cdot\eta_i\rangle}\, \frac{\mathsf{T}^{\mathsf{a}_i}}{z'_i\cdot z'_{i+1}}\, \alpha^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}'^I_i\big) \wedge D^{2|4}\mathsf{Z}'_i. \tag{506}$$

As shown in Mol [89], the MHV leaf superamplitude for gluons is given by:

$$M^{\mathsf{a}_1\ldots\mathsf{a}_n}_n\big(\mathsf{Z}^I_i\big) \;=\; \int\limits_{\mathbf{H}_s} D^{3|8}\mathsf{X}\; \mathsf{Tr}_{\mathfrak{g}} \bigwedge_{i=1}^{n} \frac{\mathcal{C}(2h_i)}{\langle z_i|X|\bar{z}_i]^{2h_i}}\; e^{i\langle z_i|\theta\cdot\eta_i\rangle}\; \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}. \tag{507}$$

Here, h_i denotes the scaling dimension of the i-th gluon. The wedge product inside the trace arises because the exponential contains Grassmann-valued spinors θ^α_A.

Consequently, taking functional derivatives of $\mathcal{I}_0[\boldsymbol{A}]$ with respect to the mode coefficients $\alpha^{2h_i,\mathsf{a}_i}\big(\mathsf{Z}^I_i\big)$ and then setting the background field to zero yields:

$$(-1)^n \prod_{i=1}^{n} \frac{\delta}{\delta\alpha^{2h_i,\mathsf{a}_i}\big(\mathsf{Z}^I_i\big)}\, \mathcal{I}_0[\boldsymbol{A}]\Bigg|_{\mathbf{A}=0} \;=\; M^{\mathsf{a}_1\ldots\mathsf{a}_n}_n\big(\mathsf{Z}^I_i\big). \tag{508}$$

[25] The first term in the powerseries expansion of the Wilson line operator $\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}]$ equals the rank r of the Lie algebra $\mathfrak{g}$. Its integral over superspace $\mathbf{H}_s$ vanishes by Berezin integration, $\int d^{0|8}\theta\; r = 0$.

This completes our demonstration that the semiclassical expectation value of the Wilson line operator $\mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}[\mathcal{L}]$ serves as the generating functional for the MHV leaf-gluon superamplitudes.

This result is remarkable: we built the minitwistor Wilson line operator purely from geometric considerations inspired by holomorphic gauge theory on minitwistor superspace, guided by our understanding of topological sigma-models. Importantly, we did not invoke any property specific to Yang-Mills theory. Nevertheless, we recovered the MHV gluonic superamplitudes. This indicates that, from the perspective of celestial CFT, our formulation describes a dual description of gauge theory on asymptotically flat spacetime.

B. Holonomies and N^k-MHV Amplitudes

We extend the holomorphic Wilson line operator $\mathbb{W}[\mathcal{L}]$, originally defined on a single minitwistor line, to an operator supported on an algebraic cycle in the minitwistor superspace. This cycle is built from the family of lines on which the N^k-MHV minitwistor superamplitude localises.

We then show that the semiclassical expectation value of this operator provides a generating functional for the tree-level S-matrix of $\mathcal{N}=4$ SYM.

1. *Summary of Key Steps*

Before proceeding, we review the main steps so far. We studied a holomorphic gauge theory on a complex vector bundle E over the minitwistor superspace $\mathbf{MT}_s$. To give physical intuition, we drew an analogy with topological sigma models and field theory, interpreting the holomorphic gauge theory as the field-theoretic limit of a minitwistor string, which we will describe in the next section.

Next, we introduced the nonlocal, gauge-invariant observable $\mathbb{W}[\mathcal{L}]$ supported on a minitwistor line $\mathcal{L}$. This operator probes deformations of the holomorphic vacuum induced by a background gauge field $\boldsymbol{A}$. By the Hitchin correspondence, the moduli supermanifold of minitwistor lines coincides with hyperbolic superspace $\mathbf{H}_s$, which we identify with the configuration space of the minitwistor strings. Hence we interpret

$$\int_{\mathbf{H}_s} D^{3|8}\mathsf{X}\ \mathbb{W}^{\boldsymbol{A}}_{z_n,z_1}\left[\mathcal{L}(X,\theta)\right] \tag{509}$$

as the semiclassical expectation value of the Wilson line. Physically, this expectation value encodes the state of the gauge-field background and reproduces the generating functional for MHV gluon amplitudes.

Remarkably, aside from choosing the gauge Lie superalgebra $\mathfrak{g}$ and supersymmetry level $\mathcal{N} = 4$, we have not invoked any other super-Yang-Mills input (no Lagrangian density or equations of motion), yet we recover dynamical information about SYM theory. This result supports the conjecture that our CCFT model is dual to four-dimensional flat-space gauge theory.

2. Next Steps; Geometrical Motivation

Now we turn to the problem of identifying an observable in our holomorphic gauge theory whose semiclassical expectation value reproduces the N^k-MHV sectors of leaf-gluon amplitudes. From Section III we learned that the N^k-MHV minitwistor superamplitudes in $\mathcal{N} = 4$ SYM localise on a family Σ of minitwistor lines $\mathcal{L}_1, \ldots, \mathcal{L}_{2k+1}$. But whereas the configuration space of a single minitwistor line $\mathcal{L}$ is the hyperbolic superspace $\mathbf{H}_s$, the configuration space of the entire family Σ is the *moduli superspace*

$$\mathscr{M}_{2k+1} \;=\; \mathbf{H}_s \times \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_k. \tag{510}$$

Each factor $\mathcal{P}_\ell$, for $\ell = 1, \ldots, k$, is called a *parameter space* and corresponds to one next-to-MHV particle. In the split-signature required by the leaf formalism, $\mathcal{P}_\ell \cong \mathbf{R}^{8|4}$, but here we analytically continue these spaces to complex supermanifolds, $\mathcal{P}_\ell \cong \mathbf{C}^{8|4}$.

We chart each parameter space by

$$\tau_\ell^M \;=\; \big(u_\ell^A,\, v_\ell^B,\, \widetilde{u}_{\ell\dot{A}},\, \widetilde{v}_{\ell\dot{B}},\, \chi_\ell^\alpha\big), \tag{511}$$

and we denote homogeneous coordinates on $\mathbf{H}_s$ by X^K. Thus the full moduli superspace $\mathscr{M}_{2k+1}$ admits the global chart:

$$\gamma^Q \;=\; \big(\mathsf{X}^K, \tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_k^{M_k}\big), \tag{512}$$

which we call the *moduli coordinates*. We equip $\mathscr{M}_{2k+1}$ with its standard orientation via the volume superform

$$\mathcal{D}\gamma \;=\; D^{3|8}\mathsf{X} \,\wedge\, d^{8|4}\tau_1 \,\wedge\, d^{8|4}\tau_2 \,\wedge\, \ldots \,\wedge\, d^{8|4}\tau_k, \tag{513}$$

where each $d^{8|4}\tau_\ell$ is the Berezin-DeWitt form on $\mathcal{P}_\ell$. Consequently, when we compute a semiclassical expectation value of a Wilson-like observable supported on Σ, we replace the integral over $\mathbf{H}_s$ by an integral over $\mathscr{M}_{2k+1}$ against the measure $\mathcal{D}\gamma$.

How can we generalise the operator $\mathbb{W}[\mathcal{L}]$, which is supported on a single minitwistor line $\mathcal{L}$, to an observable supported on a family Σ? To address this question, we recall the geometry of the collection of lines $\Sigma = \{\mathcal{L}_1, \ldots, \mathcal{L}_{2k+1}\}$. Each line $\mathcal{L}_m$ carries *moduli functions*

$$\mathcal{Q}_m^{A\dot{A}} = \mathcal{Q}_m^{A\dot{A}}\big(\tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_k^{M_k}\big), \quad q_m^{\alpha A} = q_m^{\alpha A}\big(\tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_k^{M_k}\big), \tag{514}$$

defined on the parameter spaces $\mathcal{P}_1, \ldots, \mathcal{P}_k$ inside the moduli superspace $\mathscr{M}_{2k+1}$. Together with the superspace coordinates $\mathsf{X}^K = \big(X_{A\dot{A}}, \theta^\alpha_A\big)$ charting $\mathbf{H}_s$, these functions determine the i*ncidence maps*

$$Y_m^{A\dot{A}}(\gamma^Q) \;\coloneqq\; X^{A\dot{A}} + \mathcal{Q}_m^{A\dot{A}}(\tau_\ell^M), \quad \xi_m^{\alpha A}(\gamma^Q) \;\coloneqq\; \theta^{\alpha A} + q_m^{\alpha A}(\tau_\ell^M). \tag{515}$$

In these terms, each minitwistor line $\mathcal{L}_m \in \Sigma$ is specified by the incidence relations

$$\mu_{\dot{A}} \;=\; \lambda^A Y_{mA\dot{A}}(\gamma^Q), \quad \psi^\alpha \;=\; \lambda^A \xi^\alpha_{mA}(\gamma^Q). \tag{516}$$

In addition, the family Σ is an ordered set. Its orientation is required by the index structure of the dual conformal R-invariants

$$R_{n;a_1b_1,a_2b_2,\ldots,a_kb_k} \tag{517}$$

which assigns each gluon in the scattering amplitude to a specific line (or cluster) in Σ. We implemented this assignment in Section III via the indicator map $c(i)$, which sends the i-th gluon to its cluster.

Thus, our main problem becomes: how can we formalise a gauge-invariant, Wilson-like operator that depends on an oriented family of minitwistor lines, respects the analytic structure of the moduli superspace $\mathscr{M}_{2k+1}$, and captures the algebraic-geometric character of a minitwistor line, namely, a conic, rational curve with conormal bundle $\mathcal{O}_{\mathbf{C}}(2)$ embedded in the supersymmetric nonsingular quadric $\mathbf{MT}_s$?

We begin with the notion of algebraic cycles from intersection theory. Recall from Chapter 1, § 3 of Fulton [90] that, for an algebraic scheme S, a k-cycle is a finite formal sum $\sum_i a_i[T_i]$, where each T_i is a k-dimensional subvariety of S and each $a_i \in \mathbf{Z}$. The group of all such cycles is $\mathrm{Z}_k(S)$, the free Abelian group generated by the k-dimensional subvarieties of S.

Now, each minitwistor line $\mathcal{L}_m$ is a conic curve on the nonsingular quadric $\mathbf{MT}_s$. Hence the ordered family $\Sigma = \big\{\mathcal{L}_1,\, \ldots,\, \mathcal{L}_{2k+1}\big\}$ defines an algebraic one-cycle

$$\mathscr{S}_{2k+1} \;\coloneqq\; \sum_{m=1}^{2k+1} \big[\mathcal{L}_m\big] \;\in\; \mathrm{Z}_1\big(\mathbf{MT}_s\big). \tag{518}$$

This cycle encodes the orientation of the lines and respects their algebraic-geometric embedding in $\mathbf{MT}_s$. We then introduce the *holonomy operator* supported on $\mathscr{S}_{2k+1}$ as:

$$\mathrm{Hol}\big[\mathscr{S}_{2k+1}\big] \;=\; \mathrm{P}\left(\langle\mathcal{L}_1|u_1][u_1|\mathcal{L}_2|v_1][v_1|\mathcal{L}_3|u_2]\,\ldots\,[u_k|\mathcal{L}_{2k}|v_k][v_k|\mathcal{L}_{2k+1}\rangle\right), \tag{519}$$

where P denotes path ordering. The basic building blocks $\langle\mathcal{L}|u]$, $[u|\mathcal{L}|v]$ and $[v|\mathcal{L}\rangle$ will be defined below. This functional acts on the one-cycles in $\mathrm{Z}_1(\mathbf{MT}_s)$ and on the moduli parameters u_ℓ^A, v_ℓ^A introduced in Eq. (512). Finally, we obtain the *Wilson operator* by tracing over the gauge algebra:

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_{2k+1}\big] \;\coloneqq\; \mathsf{Tr}_{\mathfrak{g}}\,\mathrm{Hol}\big[\mathscr{S}_{2k+1}\big]. \tag{520}$$

This nonlocal, gauge-invariant observable probes the holomorphic gauge theory along the cycle $\mathscr{S}_{2k+1}$. In the sigma-model picture, $\mathscr{S}_{2k+1}$ represents the classical configuration of a system of $2k+1$ minitwistor strings. We will demonstrate that the semiclassical expectation value of $\mathbb{W}[\mathscr{S}]$ serves as the generating functional for the N^k-MHV leaf-gluon amplitudes.

We now proceed to define the building blocks of the holonomy operator.

3. *Building Blocks; Product Rules*

In the CCFT framework, we regard the holonomy $\mathrm{Hol}[\mathscr{S}]$ as a composite operator in the usual CFT sense. We order its elementary building blocks by each line's position in the cycle: beginning, intermediate, or end.

Elementary Blocks. For the initial line $\mathcal{L}_1$, we define the operator:

$$\langle\mathcal{L}_1|u] \;\coloneqq\; \mathrm{Pexp}\left(-\frac{1}{2\pi i}\int\limits_{\mathcal{L}_1}\boldsymbol{a}(u;\lambda^A)\wedge\boldsymbol{A}\big|_{\mathcal{L}_1}(\lambda^A)\right). \tag{521}$$

The Green differential $\boldsymbol{a}(u;\lambda^A)\in\Omega^{1,0}_{\mathfrak{m}}\big(\mathcal{L}_1;\mathcal{O}_{\mathbf{C}}(1)\big)$ satisfies:

$$\frac{1}{2\pi i}\,\overline{\partial}_\lambda\big|_{\mathcal{L}_{in}}\,\boldsymbol{a}\big(u;\lambda^A\big) \;-\; \overline{\delta}(\lambda\cdot u)\wedge D\lambda \;=\; 0, \tag{522}$$

and the path-ordered wedge of $\boldsymbol{a}$ over points $\{\lambda_i^A\}_{i=1}^{a-1}$ is:

$$\mathrm{P}\bigwedge_{i=1}^{a-1}\boldsymbol{a}\big(u;\lambda_i^A\big) \;=\; \frac{D\lambda_1\wedge D\lambda_2\wedge\cdots\wedge D\lambda_{a-1}}{\langle\lambda_1,\lambda_2\rangle\langle\lambda_2,\lambda_3\rangle\ldots\langle\lambda_{a-2},\lambda_{a-1}\rangle\langle\lambda_{a-1},u\rangle}. \tag{523}$$

For a line $\mathcal{L}_m$ in the middle of the cycle, we set:

$$[u|\mathcal{L}_m|v] \;\coloneqq\; \mathrm{Pexp}\left(-\frac{1}{2\pi i}\int\limits_{\mathcal{L}_m}\boldsymbol{b}(u,v;\lambda^A)\wedge\boldsymbol{A}\big|_{\mathcal{L}_m}(\lambda^A)\right). \tag{524}$$

The Green differential $\boldsymbol{b}(u,v;\lambda^A) \in \Omega^{1,0}_{\mathfrak{m}}(\mathcal{L}_m;\mathbf{C})$ satisfies:

$$\frac{1}{2\pi i}\,\overline{\partial}_\lambda\big|_{\mathcal{L}_m}\,\boldsymbol{b}\big(u,v;\lambda^A\big) \;+\; \boldsymbol{K}\big(u,v;\lambda^A\big)\wedge D\lambda = 0, \tag{525}$$

with the $(0,1)$-current:

$$\boldsymbol{K}\big(u,v;\lambda^A\big) \;\coloneqq\; \frac{1}{\langle u,v\rangle}\,\big(\overline{\delta}(u\cdot\lambda) - \overline{\delta}(v\cdot\lambda)\big) \;\in\; \mathscr{D}'_{0,1}\big(\mathcal{L}_m;\,\mathcal{O}_{\mathbf{C}}(-1)\big). \tag{526}$$

The path-ordered wedge of $\boldsymbol{b}$ over points $\{\lambda_i^A\}_{i=a}^{b-1}$ is

$$\mathrm{P}\bigwedge_{i=a}^{b-1}\boldsymbol{b}\big(u,v;\lambda_i^A\big) \;=\; \frac{D\lambda_a\wedge D\lambda_{a+1}\wedge\cdots\wedge D\lambda_{b-1}}{\langle u,\lambda_a\rangle\langle\lambda_a,\lambda_{a+1}\rangle\ldots\langle\lambda_{b-2},\lambda_{b-1}\rangle\langle\lambda_{b-1},v\rangle}. \tag{527}$$

For the final line $\mathcal{L}_f \coloneqq \mathcal{L}_{2k+1}$, we define:

$$[v|\mathcal{L}_f\rangle \;\coloneqq\; \mathrm{Pexp}\left(-\frac{1}{2\pi i}\int\limits_{\mathcal{L}_f}\boldsymbol{c}(v;\lambda^A)\wedge\boldsymbol{A}\big|_{\mathcal{L}_f}(\lambda^A)\right). \tag{528}$$

The Green differential $\boldsymbol{c}(v;\lambda^A) \in \Omega^{1,0}_{\mathfrak{m}}\big(\mathcal{L}_f;\,\mathcal{O}_{\mathbf{C}}(1)\big)$ obeys:

$$\frac{1}{2\pi i}\,\overline{\partial}_\lambda\big|_{\mathcal{L}_f}\,\boldsymbol{c}\big(v;\lambda^A\big) \;+\; \overline{\delta}(\lambda\cdot v)\wedge D\lambda \;=\; 0. \tag{529}$$

Its path-ordered wedge is:

$$\mathrm{P}\bigwedge_{i=b}^{n}\boldsymbol{c}\big(v;\lambda_i^A\big) \;=\; \frac{D\lambda_b\wedge D\lambda_{b+1}\wedge\cdots\wedge D\lambda_n}{\langle v,\lambda_b\rangle\langle\lambda_b,\lambda_{b+1}\rangle\ldots\langle\lambda_{n-1},\lambda_n\rangle\langle\lambda_n,\lambda_{n+1}\rangle}. \tag{530}$$

Combining Eqs. (523), (527) and (530), path ordering yields:

$$\mathrm{P}\left(\bigwedge_{i=1}^{a-1}\boldsymbol{a}\big(u;\lambda_i^A\big)\bigwedge_{i'=a}^{b-1}\boldsymbol{b}\big(u,v;\lambda_{i'}^B\big)\bigwedge_{i''=b}^{n}\boldsymbol{c}\big(v;\lambda_{i''}^C\big)\right) \;=\; \mathcal{F}^\lambda_{ab}(u,v)\bigwedge_{j=1}^{n}\frac{D\lambda_j}{\lambda_j\cdot\lambda_{j+1}}, \tag{531}$$

where:

$$\mathcal{F}^\lambda_{ab}(u,v) \;\coloneqq\; \frac{\langle\lambda_{a-1},\lambda_a\rangle\langle\lambda_{b-1},\lambda_b\rangle}{\langle\lambda_{a-1},u\rangle\langle u,\lambda_a\rangle\langle\lambda_{b-1},v\rangle\langle v,\lambda_b\rangle}. \tag{532}$$

This reproduces the $\mathcal{F}$-function from Section III in terms of the spinors λ_i^A instead of the celestial coordinates z_i.

Product Rules. Here we define how the path-ordering symbol P organises a product of Lie-algebra-valued exponentials. The operator P orders factors according to the index structure of the dual conformal R-invariants. As a result, gluons in the scattering process group into clusters, where each cluster lies on one of the minitwistor lines that support the amplitude.

To define the path-ordering operator P, we first introduce preliminary notation and definitions. In particular, we formalise the concept of a *path-ordered wedge*, which represents the elementary building blocks of the holonomy operator introduced above.

Let $\{\mathscr{L}_i\}_{i=1}^N$ denote a family of rational curves. We do not assume these curves lie in $\mathbf{MT}_s$ or support N^k-MHV minitwistor superamplitudes; they may lie in any algebraic variety. For each line $\mathscr{L}_i$, define its n-fold Cartesian product by:

$$\mathscr{L}_i^n \;\coloneqq\; \underbrace{\mathscr{L}_i \times \ldots \times \mathscr{L}_i}_{n\text{-times}}. \tag{533}$$

Next, introduce Green differentials $\{\varphi_i\}_{i=1}^N$ with $\varphi_i \in \Omega^{1,0}_{\mathfrak{m}}(\mathscr{L}_i)$, and Lie-algebra-valued $(0,1)$-forms $\{\boldsymbol{B}_i\}_{i=1}^N$ with $\boldsymbol{B}_i \in \Omega^{0,1}(\mathscr{L}_i;\mathfrak{g})$. In our application, the φ_i serve as the fundamental solutions $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ of the CR operator $\overline{\partial}\big|_{\mathscr{L}_i}$. The $\boldsymbol{B}_i$ represent the gauge potential induced on each $\mathscr{L}_i$ via pull-back. Finally, introduce a formal coupling constant g.

Then each factor

$$\mathrm{Pexp}\left(g \int_{\mathscr{L}_i} \varphi_i \wedge \boldsymbol{B}_i \right) \tag{534}$$

is called the *path-ordered wedge* of φ_i over $\mathscr{L}_i$. In this terminology, the holonomy operator $\mathrm{Hol}[\mathscr{S}]$ arises by applying the path-ordering symbol P to the product of these wedges over the minitwistor lines that form the cycle $\mathscr{S}$.

We encode the index structure of the order-N dual conformal invariant

$$R_{n;a_1b_1,a_2b_2,\ldots,a_Nb_N}$$

into the path-ordering symbol P via the index family:

$$I_N^n \;\coloneqq\; \left\{\, \vec{a} = (a_0,\ldots,a_N) \in \mathbf{Z}^{N+1} \,\middle|\, a_0 = 1,\, a_N = n{+}1;\; 2 \le a_1 < a_2 < \cdots < a_{N-1} \le n{-}1 \,\right\}. \tag{535}$$

With this notation, the action of P on N wedges is:

$$\mathrm{P}\left(\prod_{k=1}^{N} e^{g\int_{\mathscr{L}_k} \varphi_k \wedge \boldsymbol{B}_k} \right) \;=\; \mathbb{I}_{\mathfrak{g}} + \sum_{n \ge N+1} g^n \sum_{\vec{a}\in I_N^n} \prod_{k=1}^{N} \int_{\Lambda_{k,\vec{a}}} \mathrm{P} \bigwedge_{j \in J_{k,\vec{a}}} \varphi_k(\lambda_j^A) \wedge \boldsymbol{B}_k(\lambda_j^A). \tag{536}$$

Here the integration domain is:

$$\Lambda_{k,\vec{a}} \;\coloneqq\; \mathscr{L}_k^{a_k - a_{k-1}}, \tag{537}$$

and the index set for the exterior product reads:

$$J_{k,\vec{a}} \;\coloneqq\; \big\{\, j \in \mathbf{Z} \,\big|\, a_{k-1} \le j \le a_k - 1 \,\big\}. \tag{538}$$

This completes the specification of the elementary blocks of the holonomy operator and the action of P on its defining wedges. We now turn to the computation of the Wilson line on algebraic one-cycles and its semiclassical expectation value. We will show that this expectation value serves as the generating functional for all tree-level S-matrix elements in $\mathcal{N}=4$ SYM theory.

4. Holonomy, Wilson Lines and Amplitudes

The previous subsections extended our Wilson line operator $\mathbb{W}[\mathcal{L}]$, originally supported on a single minitwistor line $\mathcal{L} \subset \mathbf{MT}_s$, to the observable $\mathbb{W}[\mathscr{S}]$, where $\mathscr{S} \in \mathrm{Z}_1(\mathbf{MT}_s)$ is an algebraic one-cycle. This observable measures how a background gauge field deforms the holomorphic vacuum along $\mathscr{S}$. We now compute its expectation value over the family of minitwistor lines introduced in Section III. From this calculation, we derive a dictionary that maps these holomorphic gauge theory observables on minitwistor superspace to the tree-level $\mathcal{S}$-matrix of $\mathcal{N}=4$ SYM theory.

N^1-MHV *Sector.* In Subsection III.1, we showed that the N^1-MHV minitwistor superamplitudes for gluons decompose into partial amplitudes $\widetilde{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{W}^I_i\big)$. For brevity, we call these the *gluonic minitwistor amplitudes.* The labels a, b arise from the dual conformal invariant $R_{n;ab}$. Without loss of generality, we fix

$$2 \le a < b \le n-2, \tag{539}$$

since all other orderings follow by permutation.

Using the celestial RSVW identity, we found that $\widetilde{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}$ localises on three minitwistor lines $\mathcal{L}_m$ for $m = 1,2,3$. The external gluons then group into clusters: the i-th gluon lies on

$$\begin{cases} \mathcal{L}_1, & \text{if } 1 \le i \le a-1; \\ \mathcal{L}_2, & \text{if } a \le i \le b-1; \\ \mathcal{L}_3, & \text{if } b \le i \le n. \end{cases} \tag{540}$$

We formalise this assignment by the *indicator function*:

$$c_{ab}(i) \;\coloneqq\; \begin{cases} 1, & i \in [1, a-1]; \\ 2, & i \in [a, b-1]; \\ 3, & i \in [b, n]. \end{cases} \tag{541}$$

This function assigns each gluon label i to its corresponding cluster $c_{ab}(i)$.

We associate a set of creation and annihilation operators to each minitwistor line $\mathcal{L}_m$. Denote by $\alpha_m^{\Delta,\mathsf{a}}$ the mode functions for gluons on $\mathcal{L}_m$. Physically, these functions give the classical expectation values of the annihilation operators. Now, combining this observation with Eq. (492), we decompose the gauge potential induced on $\mathcal{L}_m$ as:

$$\boldsymbol{A}\big|_{\mathcal{L}_m}(\lambda_i^A) \;=\; 2\pi i \int\limits_{\mathbf{MT}_s^*} \Psi_{\Delta_i}\big|_{\mathcal{L}_m}(\lambda_i^A;\mathsf{Z}_i'^I)\,\alpha_m^{\Delta_i,\mathsf{a}_i}(\mathsf{Z}_i'^I)\,\mathsf{T}^{\mathsf{a}_i}\wedge D^{2|4}\mathsf{Z}_i'. \tag{542}$$

This organisation of mode coefficients and their associated creation and annihilation operators across the three minitwistor lines provides the physical motivation for interpreting celestial amplitudes in terms of a minitwistor sigma-model. We thus identify each line on which the amplitude localises as a semiclassical configuration of a minitwistor string. Consequently, an N^k-MHV amplitude arises from the correlation functions of a many-body system of $2k+1$ such strings.

To proceed, we briefly review the geometric interpretation of minitwistor amplitudes. The family of lines $\{\mathcal{L}_m\}_{m=1}^3$ is parametrised by the moduli superspace

$$\mathscr{M}_3 \;=\; \mathbf{H}_s\times\mathcal{P}, \tag{543}$$

where $\mathcal{P}$ denotes the parameter space for the single next-to-MHV particle. In Kleinian signature, $\mathcal{P}\cong\mathbf{R}^{8|4}$. Here, we analytically continue to the complex category, so that $\mathcal{P}\cong\mathbf{C}^{8|4}$. We chart $\mathcal{P}$ by coordinates:

$$\tau^M \;=\; \big(u^A,\,v^B,\,\widetilde{u}_{\dot{A}},\,\widetilde{v}_{\dot{B}},\,\chi^\alpha\big), \tag{544}$$

and orient it with the Berezin-DeWitt form $d^{8|4}\tau$. Thus the full moduli space admits coordinates

$$\gamma^Q = \big(\mathsf{X}^K,\tau^M\big):\; \mathscr{M}_3 \;\longrightarrow\; \mathbf{CP}^{3|8}\times\mathbf{C}^{8|4},$$

where $\mathsf{X}^K = (X_{A\dot{A}},\theta_A^\alpha)$ are the standard superspace coordinates on the complexified hyperboloid $\mathbf{H}_s$. We equip $\mathscr{M}_3$ with the canonical orientation given by:

$$\mathcal{D}\gamma \;\coloneqq\; D^{3|8}\mathsf{X}\wedge d^{8|4}\tau. \tag{545}$$

The semiclassical expectation value of the Wilson operator is then expressed as an integral over $\mathscr{M}_3$ against the measure $\mathcal{D}\gamma$.

As a final preparation for computing the holonomy operator, we explain how a point $\gamma^Q\in\mathscr{M}_3$ specifies the family of lines $\{\mathcal{L}_m\}$ via the evaluation map. Each line $\mathcal{L}_m$ is determined by moduli functions

$$\mathcal{Q}_m^{A\dot{A}} = \mathcal{Q}_m^{A\dot{A}}(\tau^M), \quad q_m^{\alpha A} = q_m^{\alpha A}(\tau^M), \tag{546}$$

with domain $\mathcal{P}$. Combining these with the superspace coordinates $\mathsf{X}^K \in \mathbf{H}_s$ defines the incidence maps:

$$Y_m^{A\dot{A}}(\gamma^Q) \;\coloneqq\; X^{A\dot{A}} + \mathcal{Q}_m^{A\dot{A}}(\tau^M), \quad \xi_m^{\alpha A}(\gamma^Q) \;\coloneqq\; \theta^{\alpha A} + q_m^{\alpha A}(\tau^M). \tag{547}$$

Hence $\mathcal{L}_m$ is the locus in $\mathbf{MT}_s$ satisfying $\mu_{m\dot{A}} = \lambda^A Y_{mA\dot{A}}(\gamma^Q)$ and $\psi_m^\alpha = \lambda^A \xi_{mA}^\alpha(\gamma^Q)$. As γ^Q varies over $\mathscr{M}_3$, these relations sweep out all possible configurations of the three-line family.

To compute the holonomy operator for the three-line configuration, we introduce a background gauge potential $\boldsymbol{A}$ on the holomorphic bundle $E \to \mathbf{MT}_s$ and probe how $\boldsymbol{A}$ deforms the vacuum along the algebraic one-cycle:

$$\mathscr{S}_3(\gamma^Q) \;\coloneqq\; \sum_{m=1,2,3} [\mathcal{L}_m] \;\in\; \mathrm{Z}_1(\mathbf{MT}_s). \tag{548}$$

This cycle is parametrised by $\gamma^Q \in \mathscr{M}_3$, just as a single line $\mathcal{L}(X,\theta)$ is parametrised by $\mathsf{X}^K \in \mathbf{H}_s$.

The holonomy along $\mathscr{S}_3(\gamma^Q)$ is:

$$\mathrm{Hol}\big[\mathscr{S}_3(\gamma^Q)\big] \;=\; \mathrm{P}\left(\langle\mathcal{L}_1|u][u|\mathcal{L}_2|v][v|\mathcal{L}_3\rangle\right). \tag{549}$$

Using the elementary blocks in Eqs. (521), (524), (528), we write:

$$\begin{aligned}&\mathrm{Hol}\big[\mathscr{S}_3(\gamma^Q)\big] &(550)\\ &= \mathrm{P}\left(e^{-\frac{1}{2\pi i}\int_{\mathcal{L}_1} \boldsymbol{a}(u;\lambda^A)\wedge \boldsymbol{A}|_{\mathcal{L}_1}(\lambda^A)} e^{-\frac{1}{2\pi i}\int_{\mathcal{L}_2} \boldsymbol{b}(u,v;\lambda^A)\wedge \boldsymbol{A}|_{\mathcal{L}_2}(\lambda^A)} e^{-\frac{1}{2\pi i}\int_{\mathcal{L}_3} \boldsymbol{c}(v;\lambda^A)\wedge \boldsymbol{A}|_{\mathcal{L}_3}(\lambda^A)}\right). &(551)\end{aligned}$$

Applying the product rule (Eq. (536)) and the monomial expansion (Eq. (531)) yields:

$$\begin{aligned}&\mathrm{Hol}\big[\mathscr{S}_3(\gamma^Q)\big] &(552)\\ &= \mathbb{I}_{\mathfrak{g}} + \sum_{n\geq 4} \frac{i^n}{(2\pi)^n} \sum_{2\leq a<b\leq n-1} \int_{\mathbf{L}_n} \mathcal{F}_{ab}^{\lambda}(\gamma^Q) \bigwedge_{i=1}^{n} \frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}} \wedge \boldsymbol{A}\big|_{\mathcal{L}_{c_{ab}(i)}}(\lambda_i^A). &(553)\end{aligned}$$

Here $\mathbf{L}_n \coloneqq \bigtimes^n \mathbf{CP}^1$ and $\mathcal{F}_{ab}^{\lambda}$ is defined as in Eq. (532).

We define the Wilson operator by:

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_3(\gamma^Q)\big] \;\coloneqq\; \mathsf{Tr}_{\mathfrak{g}}\, \mathrm{Hol}[\mathscr{S}_3(\gamma^Q)]. \tag{554}$$

Hence Eq. (552) gives:

$$\begin{aligned}\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_3(\gamma^Q)\big] \;=\; r + \sum_{n\geq 4}(-1)^n \sum_{2\leq a<b\leq n-1} \int_{\mathbf{L}_n} \mathcal{F}_{ab}^{\lambda}(\gamma^Q)\, \mathsf{Tr} \bigwedge_{i=1}^{n} \frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}} &\quad (555)\\ \int_{\mathbf{MT}_s^*} \Psi_{\Delta_i}\Big|_{\mathcal{L}_{c_{ab}(i)}} \left(\lambda_i^A;\mathsf{Z}_i'^I\right) \alpha_{c_{ab}(i)}^{\Delta_i,\mathsf{a}_i}\left(\mathsf{Z}_i'^I\right) \mathsf{T}^{\mathsf{a}_i} \wedge D^{2|4}\mathsf{Z}_i'. &\quad (556)\end{aligned}$$

Define $\mathbf{X}_n^* := \times^n \mathbf{MT}_s^*$ with measure $\bigwedge_i D^{2|4}\mathsf{Z}_i'$. Exchanging the order of integration by Fubini's theorem, Eq. (556) becomes:

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_3(\gamma^Q)\big] \;=\; r \;+\; \sum_{n\geq 4}(-1)^n \sum_{2\leq a<b\leq n-1} \int\limits_{\mathbf{X}_n^*} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}_i' \wedge \alpha^{\Delta_i,\mathsf{a}_i}_{c_{ab}(i)}\big(\mathsf{Z}_i'^{I}\big) \tag{557}$$

$$\int\limits_{\mathbf{L}_n} \mathcal{F}^{\lambda}_{ab}(\gamma^Q)\ \mathsf{Tr}\ \bigwedge_{j=1}^{n} \frac{D\lambda_j}{\lambda_j\cdot\lambda_{j+1}}\mathsf{T}^{\mathsf{a}_j}\wedge \Psi_{\Delta_j}\Big|_{\mathcal{L}_{c_{ab}(j)}}\big(\lambda_j^A;\mathsf{Z}_j'^{I}\big). \tag{558}$$

Finally, applying the celestial BMSW identity gives:

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_3(\gamma^Q)\big] \;=\; r \;+\; \sum_{n\geq 4}(-1)^n \sum_{2\leq a<b\leq n-1} \int\limits_{\mathbf{X}_n^*} \mathcal{F}^{z'}_{ab}(\gamma^Q)\ \mathsf{Tr}\ \bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i'|Y_{c_{ab}(i)}|\bar{z}_i']^{\Delta_i}} \tag{559}$$

$$e^{i\langle z_i'|\xi_{c_{ab}(i)}\cdot\eta_i\rangle}\ \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i'\cdot z_{i+1}'}\ \alpha^{\Delta_i,\mathsf{a}_i}_{c_{ab}(i)}\big(\mathsf{Z}_i'^{I}\big)\wedge D^{2|4}\mathsf{Z}_i'. \tag{560}$$

This completes the explicit expansion of the Wilson operator in our holomorphic gauge theory.

In the minitwistor sigma-model picture, the gauge field configuration $\boldsymbol{A}$ probed by the Wilson operator $\mathbb{W}[\mathscr{S}_3]$ arises from a system of three minitwistor strings. For each point $\gamma^Q \in \mathscr{M}_3$, we associate a semiclassical state $\mathscr{S}_3(\gamma^Q)$, represented by an algebraic cocycle. Hence $\mathscr{M}_3$ serves as the configuration space of these strings. Then, by analogy with Feynman's pathintegral, we define

$$\mathcal{I}_1[\boldsymbol{A}] \;:=\; \int\limits_{\mathscr{M}_3} \mathcal{D}\gamma\ \mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_3(\gamma^Q)\big], \tag{561}$$

which we interpret as the semiclassical expectation value of the non-local observable $\mathbb{W}^{\boldsymbol{A}}[\mathscr{S}_3]$ in our holomorphic gauge theory[26].

Integrating Eq. (559) over $\mathscr{M}_3$ with the measure $\mathcal{D}\gamma$ and using the Berezin integral property $\int d^{0|8}\theta\ r = 0$, we get:

$$\mathcal{I}_1[\boldsymbol{A}] \;=\; \sum_{n\geq 4}(-1)^n \sum_{2\leq a<b\leq n-1} \int\limits_{\mathbf{X}_n^*}\int\limits_{\mathscr{M}_3} \mathcal{D}\gamma\ \mathcal{F}^{z'}_{ab}(\gamma^Q)\ \mathsf{Tr}\ \bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i'|Y_{c_{ab}(i)}|\bar{z}_i']^{\Delta_i}} \tag{562}$$

$$e^{i\langle z_i'|\xi_{c_{ab}(i)}\cdot\eta_i\rangle}\ \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i'\cdot z_{i+1}'}\ \alpha^{\Delta_i,\mathsf{a}_i}_{c_{ab}(i)}\big(\mathsf{Z}_i'^{I}\big)\wedge D^{2|4}\mathsf{Z}_i'. \tag{563}$$

Taking functional derivatives of $\mathcal{I}_1$ with respect to the mode coefficients $\alpha^{2h_i,\mathsf{a}_i}_{c_{ab}(i)}\big(\mathsf{Z}_i^I\big)$ for fixed a,b satisfying $2\leq a<b\leq n-1$, and then setting $\boldsymbol{A}=0$, yields:

$$(-1)^n \prod_{i=1}^{n} \frac{\delta}{\delta\alpha^{2h_i,\mathsf{a}_i}_{c_{ab}(i)}\big(\mathsf{Z}_i^I\big)}\ \mathcal{I}_1[\boldsymbol{A}]\Bigg|_{\boldsymbol{A}=0} \;=\; M^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;ab}\big(\mathsf{Z}_i^I\big). \tag{564}$$

Hence the semiclassical expectation value of the Wilson operator $\mathbb{W}[\mathscr{S}_3]$ serves as the generating functional for the N^1-MHV leaf-gluon amplitudes.

[26] Recall that in the minitwistor-string framework, this gauge theory functions as the string field theory.

N^2-MHV *Sector.* In Subsection III B, we showed that the N^2-MHV minitwistor superamplitude for gluons decomposes into a sum whose individual terms are the partial amplitudes

$$\widetilde{\mathcal{M}}^{\mathsf{a}_1 \ldots \mathsf{a}_n}_{n;a_1 a_2, b_1 b_2}(\mathsf{W}^I_i),$$

as derived in Eq. (375). For brevity, we refer to each $\widetilde{\mathcal{M}}$ simply as a minitwistor amplitude, with the N^2-MHV configuration specified by the subscripts a_ℓ, b_ℓ for $\ell = 1, 2$.

The index structure of each term arises from the order-two dual conformal invariant $R_{n;a_1 a_2, b_1 b_2}$. Henceforth, we restrict to the index family

$$\mathbf{I}_n \;\coloneqq\; \big\{\, \vec{\alpha} = (a_1, a_2, b_1, b_2) \in \mathbf{Z}^4 \,\big|\, 2 \le a_1 < a_2 < b_2 < b_1 \le n-1 \,\big\}. \tag{565}$$

All subsequent results hold equally for any other ordered choice of indices by suitable permutation. Throughout this example we assign the labels $\ell = 1, 2$ to the two next-to-MHV particles.

In Subsection III B, we saw that the minitwistor amplitude localises on a family of five lines, $\{\mathcal{L}_m\}_{m=1}^5 \subset \mathbf{MT}_s$. We label these lines by $m = 1, \ldots, 5$ and group the gluons in the scattering process into clusters, each supported on one line $\mathcal{L}_m$. For a given index vector $\vec{\alpha} \in \mathbf{I}_n$, the i-th gluon resides on line $\mathcal{L}_{c_{\vec{\alpha}}(i)}$, where the indicator function $c_{\vec{\alpha}}(i)$ for an N^2-MHV configuration is given by:

$$c_{\vec{\alpha}}(i) \;\coloneqq\; \begin{cases} 1, & 1 \le i \le a_1 - 1; \\ 2, & a_1 \le i \le a_2 - 1; \\ 3, & a_2 \le i \le b_2 - 1; \\ 4, & b_2 \le i \le b_1 - 1; \\ 5, & b_1 \le i \le n. \end{cases} \tag{566}$$

Accordingly, we introduce annihilation and creation operators for gluons on each line $\mathcal{L}_m$. The classical expectation value of the annihilation operator on line m defines the mode function $\alpha^{\Delta,\mathsf{a}}_m(\mathsf{Z}^I)$. The minitwistor-Fourier expansion of the gauge potential $\boldsymbol{A}\big|_{\mathcal{L}_m}$ appears in Eq. (542).

The clustering of gluons described above provides the physical motivation for interpreting the N^2-MHV amplitudes as correlation functions of a many-body system composed of five minitwistor strings. In this picture, each line $\mathcal{L}_m$ represents a semiclassical configuration of the minitwistor sigma-model.

With this in mind, we ask: what is the classical configuration space of this quintet of minitwistor strings? It is the moduli superspace of the family $\{\mathcal{L}_m\}$, namely the supermanifold:

$$\mathscr{M}_5 \;=\; \mathbf{H}_s \times \mathcal{P}_1 \times \mathcal{P}_2. \tag{567}$$

Here $\mathcal{P}_\ell$ denotes the parameter space associated with the ℓ-th next-to-MHV particle. In this section we analytically continue $\mathcal{P}_\ell$ to the complex category, so that $\mathcal{P}_\ell \cong \mathbf{C}^{8|4}$. Globally, each $\mathcal{P}_\ell$ is charted by the moduli parameters:

$$\tau_\ell^M \;=\; \big(u_\ell^A,\, v_\ell^B,\, \widetilde{u}_{\ell\dot{A}},\, \widetilde{v}_{\ell\dot{B}},\, \chi_\ell^\alpha\big), \tag{568}$$

where u_ℓ^A, v_ℓ^A, $\widetilde{u}_{\ell\dot{A}}$ and $\widetilde{v}_{\ell\dot{A}}$ are bosonic van der Waerden spinors and χ_ℓ^α is a Grassmann variable. Hence the full moduli superspace is globally parametrised by:

$$\gamma^Q \;=\; \big(\mathsf{X}^K,\, \tau_1^{M_1},\, \tau_2^{M_2}\big). \tag{569}$$

To each point $\gamma^Q \in \mathscr{M}_5$ we assign the algebraic one-cycle

$$\mathscr{S}_5(\gamma^Q) \;:=\; \sum_{m=1}^{5}\,[\mathcal{L}_m] \;\in\; \mathrm{Z}_1(\mathbf{MT}_s), \tag{570}$$

which encodes the geometry of the five-line configuration $\{\mathcal{L}_m\}$. When we later reinterpret our construction dynamically via the minitwistor sigma-model, $\mathscr{S}_5(\gamma^Q)$ will describe the classical configuration of the five-string many-body system. In the present context, however, our focus is on probing the holomorphic gauge theory on $\mathbf{MT}_s$ by inserting a Wilson operator supported on the cycle $\mathscr{S}_5(\gamma^Q)$.

How, then, does a point in $\mathscr{M}_5$ map to a one-cycle in $\mathrm{Z}_1(\mathbf{MT}_s)$? The answer follows from the evaluation map. Each line $\mathcal{L}_m$ is encoded by moduli functions

$$\mathcal{Q}_m^{A\dot{A}} = \mathcal{Q}_m^{A\dot{A}}\big(\tau_1^{M_1}, \tau_2^{M_2}\big), \quad q_m^{\alpha A} = q_m^{\alpha A}(\tau_1^{M_1}, \tau_2^{M_2}) \tag{571}$$

defined over the domain $\mathcal{P}_1 \times \mathcal{P}_2$. Their component expressions appear in Table II. Together with the superspace coordinates $\mathsf{X}^K \in \mathbf{H}_s$, they determine the incidence maps:

$$Y_m^{A\dot{A}}\big(\gamma^Q\big) \;=\; X^{A\dot{A}} + \mathcal{Q}_m^{A\dot{A}}\big(\tau_1^{M_1}, \tau_2^{M_2}\big), \quad \xi_m^{\alpha A}\big(\gamma^Q\big) \;=\; \theta^{\alpha A} + q_m^{\alpha A}\big(\tau_1^{M_1}, \tau_2^{M_2}\big). \tag{572}$$

Hence, as in the N^1-MHV case, the line $\mathcal{L}_m$ is the locus in $\mathbf{MT}_s$ satisfying the incidence relations $\mu_{m\dot{A}} = \lambda^A Y_{mA\dot{A}}(\gamma^Q)$ and $\psi_m^\alpha = \lambda^A \xi_{mA}^\alpha(\gamma^Q)$. As γ^Q varies over $\mathscr{M}_5$, these relations for $m = 1, \ldots, 5$ sweep out every configuration of the five-line system.

We now compute the Wilson operator supported on the cycle $\mathscr{S}_5$. Introduce a gauge field $\boldsymbol{A}$ for the holomorphic gauge theory on the bundle E over $\mathbf{MT}_s$. The holonomy along $\mathscr{S}_5(\gamma^Q)$ is:

$$\mathrm{Hol}\big[\mathscr{S}_5(\gamma^Q)\big] \;=\; \mathrm{P}\left(\langle\mathcal{L}_1|u_1][u_1|\mathcal{L}_2|v_1][v_1|\mathcal{L}_3|u_2][u_2|\mathcal{L}_4|v_2][v_2|\mathcal{L}_5\rangle\right), \tag{573}$$

where the moduli u_ℓ^A, v_ℓ^A are all encoded in the superspace coordinates γ^Q.

Using the monomial expansion of the Green differentials $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (Eq. (531)), we write:

$$\mathrm{P}\left(\bigwedge_{i=1}^{a_1-1} \boldsymbol{a}(u_1;\lambda_i^A) \bigwedge_{i=a_1}^{a_2-1} \boldsymbol{b}(u_1,v_1;\lambda_i^A) \bigwedge_{i=a_2}^{b_2-1} \boldsymbol{b}(v_1,u_2;\lambda_i^A) \bigwedge_{i=b_2}^{b_1-1} \boldsymbol{b}(u_2,v_2;\lambda_i^A) \bigwedge_{i=b_1}^{n} \boldsymbol{c}(v_2;\lambda_i^A)\right) \tag{574}$$

$$= \mathcal{F}^{\lambda}_{a_1a_2,b_2b_1}\big(\gamma^Q\big) \bigwedge_{i=1}^{n} \frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}}, \tag{575}$$

with

$$\mathcal{F}^{\lambda}_{a_1a_2,b_2b_1}\big(\gamma^Q\big) \;\coloneqq\; \mathcal{F}^{\lambda}_{a_1a_2}\big(\tau_1^M\big)\,\mathcal{F}^{\lambda}_{b_2b_1}\big(\tau_2^M\big). \tag{576}$$

Applying the product rule (Eq. (536)) gives:

$$\mathrm{Hol}\big[\mathscr{S}_5(\gamma^Q)\big] \tag{577}$$

$$= \mathbb{I}_{\mathfrak{g}} \;+\; \sum_{n\geq 6} \frac{i^n}{(2\pi)^n} \sum_{\vec{\alpha}\in\mathbf{I}_n} \int_{\mathbf{L}_n} \mathcal{F}^{\lambda}_{a_1a_2,b_2b_1}\big(\gamma^Q\big) \bigwedge_{i=1}^{n} \frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}} \mathsf{T}^{\mathsf{a}_i} \wedge \boldsymbol{A}\big|_{c_{\vec{\alpha}}(i)}\big(\lambda_i^A\big). \tag{578}$$

The Wilson operator is its gauge trace,

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_5(\gamma^Q)\big] \;\coloneqq\; \mathsf{Tr}_{\mathfrak{g}}\,\mathrm{Hol}\big[\mathscr{S}_5(\gamma^Q)\big]. \tag{579}$$

Substituting the minitwistor-Fourier expansion of $\boldsymbol{A}$ (cf. Eq. (542)) into the above, one finds:

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_5(\gamma^Q)\big] \;=\; r \;+\; \sum_{n\geq 6} (-1)^n \sum_{\vec{\alpha}\in\mathbf{I}_n} \int_{\mathbf{X}_n^*} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}_i' \wedge \alpha^{\Delta_i,\mathsf{a}_i}_{c_{\vec{\alpha}}(i)}\big(\mathsf{Z}_i'^{I}\big) \tag{580}$$

$$\int_{\mathbf{L}_n} \mathcal{F}^{\lambda}_{a_1a_2,b_2b_1}\big(\gamma^Q\big) \bigwedge_{j=1}^{n} \frac{D\lambda_j}{\lambda_j\cdot\lambda_{j+1}} \mathsf{T}^{\mathsf{a}_j} \wedge \Psi_{\Delta_j}\big|_{\mathcal{L}_{c_{\vec{\alpha}}(i)}}\big(\lambda_j^A;\mathsf{Z}_j'^{I}\big). \tag{581}$$

Invoking the celestial BMSW identity yields the full expansion,

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_5(\gamma^Q)\big] \;=\; r \;+\; \sum_{n\geq 6} (-1)^n \sum_{\vec{\alpha}\in\mathbf{I}_n} \int_{\mathbf{X}_n^*} \mathcal{F}^{z'}_{a_1a_2,b_2b_1}\big(\gamma^Q\big) \tag{582}$$

$$\mathsf{Tr} \bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i'|Y_{c_{\vec{\alpha}}(i)}|\bar{z}_i']^{\Delta_i}}\; e^{i\langle z_i'|\xi_{c_{\vec{\alpha}}(i)}\cdot\eta_i\rangle}\; \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i'\cdot z_{i+1}'}\; \alpha^{\Delta_i,\mathsf{a}_i}_{c_{\vec{\alpha}}(i)}\big(\mathsf{Z}_i'^{I}\big) \wedge D^{2|4}\mathsf{Z}_i'. \tag{583}$$

As in the N^1-MHV case, the semiclassical expectation value is:

$$\mathcal{I}_2[\boldsymbol{A}] \;\coloneqq\; \int_{\mathscr{M}_5} \mathcal{D}\gamma\; \mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_5(\gamma^Q)\big]. \tag{584}$$

Integrating $\mathbb{W}^{\boldsymbol{A}}$ over $\mathscr{M}_5$ and using $\int d^{0|8}\theta\; r = 0$ gives:

$$\mathcal{I}_2[\boldsymbol{A}] \;=\; \sum_{n\geq 6} (-1)^n \sum_{\vec{\alpha}\in\mathbf{I}_n} \int_{\mathbf{X}_n^*} \int_{\mathscr{M}_5} \mathcal{D}\gamma \quad \mathcal{F}^{z'}_{a_1a_2,b_2b_1}\big(\gamma^Q\big) \tag{585}$$

$$\mathsf{Tr}_{\mathfrak{g}} \bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i'|Y_{c_{\vec{\alpha}}(i)}|\bar{z}_i']^{\Delta_i}}\; e^{i\langle z_i'|\xi_{c_{\vec{\alpha}}(i)}\cdot\eta_i\rangle}\; \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i'\cdot z_{i+1}'}\; \alpha^{\Delta_i,\mathsf{a}_i}_{c_{\vec{\alpha}}(i)}\big(\mathsf{Z}_i'^{I}\big) \wedge D^{2|4}\mathsf{Z}_i'. \tag{586}$$

Finally, fixing $\vec{\alpha} \in \mathbf{I}_n$ and $\{\mathsf{Z}_i^I\} \subset \mathbf{MT}_s^*$, we take functional derivatives with respect to the mode coefficients and set $\boldsymbol{A} = 0$:

$$(-1)^n \prod_{i=1}^{n} \frac{\delta}{\delta\alpha^{2h_i,\mathsf{a}_i}_{c_{\vec{\alpha}}(i)}\big(\mathsf{Z}_i^I\big)}\, \mathcal{I}_2[\boldsymbol{A}]\Bigg|_{\boldsymbol{A}=0} \;=\; \mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;a_1a_2,b_2b_1}\big(\mathsf{Z}_i^I\big). \tag{587}$$

This result confirms that the semiclassical limit of the nonlocal gauge-invariant observable $\mathbb{W}[\mathscr{S}]$ in the minitwistor string field theory generates the tree-level N^2-MHV leaf-gluon amplitudes.

5. Tree-level $\mathcal{S}$-Matrix

Building on our results for the N^1- and N^2-MHV gluon amplitudes, we now show how the full tree-level gluon $\mathcal{S}$-matrix in $\mathcal{N}=4$ SYM theory emerges from the semiclassical expectation values of $\mathbb{W}[\mathscr{S}]$. This derivation is central to our work, so we present it in detail. Along the way, we introduce the operator/state correspondence in celestial CFT. In particular, mapping each gluon operator to the minitwistor line that supports its N^k-MHV amplitude motivates interpreting those lines as the classical configurations of minitwistor strings.

From Subsection III C, the N^k-MHV minitwistor superamplitude for n gluons can be written as a sum of partial amplitudes. Each summand has the form $\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{a}}\big(\mathsf{Z}_i^I\big)$, where

$$\vec{a} \;=\; (a_1,\ldots,a_k;\, b_1,\ldots,b_k) \;\in\; \mathbf{I}_k^n, \tag{588}$$

and the index set is

$$\mathbf{I}_k^n \;\coloneqq\; \big\{\, (a_1,\ldots,a_k; b_1,\ldots,b_k) \in \mathbf{Z}^{2k} \,\big|\, 2 \le a_1 < a_2 < \cdots < b_2 < b_1 \le n-1 \,\big\}. \tag{589}$$

Let $N \coloneqq 2k+1$. The ordering of the pairs (a_i, b_j) reflects the structure of the order-N dual conformal invariant

$$R_{n;a_1a_2,a_3a_4,\ldots,b_4b_3,b_2b_1}. \tag{590}$$

Since every term in the full N^k-MHV amplitude arises by permuting the entries of some $\vec{a} \in \mathbf{I}_k^n$, we refer to each $\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{a}}$ as a minitwistor gluon amplitude, with its N^k-MHV configuration specified by $\vec{a}$.

Localisation. Subsection III C established a localisation theorem: each partial amplitude $\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{a}}$ is supported on a family $\{\mathcal{L}_m\}_{m=1}^N \subset \mathbf{MT}_s$ of minitwistor lines. Here $m = 1,\ldots,N$ labels the lines, and $\ell = 1,\ldots,k$ labels one of the next-to-MHV bosons.

The geometric configuration of these lines is parametrised by the moduli superspace:

$$\mathscr{M}_N \;\coloneqq\; \mathbf{H}_s \times \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_k, \tag{591}$$

where each $\mathcal{P}_\ell \cong \mathbf{R}^{8|4}$ is the parameter superspace for one external next-to-MHV gluon. The reality of $\mathcal{P}_\ell$ follows from the fact that we originally defined the leaf-gluon amplitudes in split-signature, which forces the momentum-twistor components to be real.

We then analytically continue to the complex category by setting $\mathcal{P}_\ell \cong \mathbf{C}^{8|4}$. In this way, $\mathscr{M}_N$ becomes the moduli superspace of lines in the complexified minitwistor superspace $\mathbf{MT}_s$. The coordinates τ_ℓ^M on each $\mathcal{P}_\ell$ were introduced in Eq. (511); the full moduli coordinates γ^Q were defined in Eq. (512); and the standard orientation on $\mathscr{M}_N$ is given by the measure $\mathcal{D}\gamma$ in Eq. (513).

Evaluation Map and Incidence Relations. The moduli space $\mathscr{M}_N$ determines the embedding of each minitwistor line $\mathcal{L}_m$ via an evaluation map. In global coordinates, line $\mathcal{L}_m$ carries two sets of moduli functions:

$$\mathcal{Q}_m^{A\dot{A}} = \mathcal{Q}_m^{A\dot{A}}\big(\tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_k^{M_k}\big), \quad q_m^{\alpha A} = q_m^{\alpha A}\big(\tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_k^{M_k}\big) \tag{592}$$

whose domain is the full parameter space $\bigtimes_{\ell=1}^k \mathcal{P}_\ell$. These moduli functions combine with the standard coordinates $\mathsf{X}^K = (X_{A\dot{A}}, \theta_A^\alpha)$ on the complexified hyperbolic superspace $\mathbf{H}_s$ to define the incidence maps:

$$Y_m^{A\dot{A}}\big(\gamma^Q\big) \;=\; X^{A\dot{A}} + \mathcal{Q}_m^{A\dot{A}}\big(\tau_\ell^M\big), \quad \xi_m^{\alpha A}\big(\gamma^Q\big) \;=\; \theta^{\alpha A} + q_m^{\alpha A}\big(\tau_\ell^M\big). \tag{593}$$

Accordingly, the line $\mathcal{L}_m(\gamma^Q)$ is the locus of points $\mathsf{W}_m^I = (\lambda_m^A, \mu_{m\dot{A}}, \psi_m^\alpha)$ in minitwistor superspace satisfying, for all $[\sigma^A] \in \mathbf{CP}^1$,

$$\lambda_m^A = \sigma^A, \quad \mu_{m\dot{A}} = \sigma^A Y_{mA\dot{A}}(\gamma^Q), \quad \psi_m^\alpha = \sigma^A \xi_{mA}^\alpha(\gamma^Q). \tag{594}$$

Operator/State Correspondence. Another consequence of the localisation theorem is that, for each index vector $\vec{a} \in \mathbf{I}_k^n$,

$$\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{a}}\big(\mathsf{W}_i^I\big) \;=\; 0 \tag{595}$$

whenever the minitwistor W_i^I representing the i-th gluon does not lie on the line $\mathcal{L}_{c_{\vec{a}}(i)}$. Here $c_{\vec{a}}(i)$ is the indicator function defined in Subsection III C that assigns each gluon to its cluster (for example, $c_{\vec{a}}(i) = 1$ for $1 \leq i \leq a_1 - 1$, $c_{\vec{a}}(i) = 2$ for $a_1 \leq i \leq a_2 - 1$, and so on).

In celestial CFT, the i-th external gluon is described by a conformal weight Δ_i and an insertion point $\mathsf{z}_i = (z_i, \bar{z}_i, \eta_i^\alpha) \in \mathcal{CS}_s$, where $z_i, \bar{z}_i \in \mathbf{CP}^1$ and η_i^α $(\alpha = 1, \dots, 4)$ encodes the helicity. Since the dual minitwistor superspace $\mathbf{MT}_s^*$ covers $\mathcal{CS}_s$, we represent z_i by a dual minitwistor Z_i^I. We then postulate a field operator

$$O_{\Delta_i}^{\mathsf{a}_i}(\mathsf{Z}_i^I)$$

for the i-th gluon in the dual minitwistor representation. The $\mathcal{MT}$-transform maps this to a minitwistor operator $\widehat{O}_{\Delta_i}^{\mathsf{a}_i}(\mathsf{W}_i^I)$, in direct analogy with the Fourier transform between momentum and position representations in quantum mechanics.

Combining this with the localisation theorem, we attach a family of gluon operators to each line $\mathcal{L}_m$. Fix $\vec{a} \in \mathbf{I}_k^n$. Then for all $i = 1, \dots, n$, the operator $\widehat{O}_{\Delta_i}^{\mathsf{a}_i}(\mathsf{W}_i^I)$ is supported on the line $\mathcal{L}_{c_{\vec{a}}(i)}$. We interpret the classical expectation values of the gluon annihilation operators as the mode coefficients $\alpha_m^{\Delta,\mathsf{a}_i}$ that appear in the gauge field expansion (see Eq. (542)). This assignment *suggests* that each minitwistor line, together with its collection of mode functions, admits a semiclassical interpretation as a minitwistor "string."

Holonomy Operator. The family of lines $\{\mathcal{L}_m\}$ defines an algebraic one-cycle

$$\mathscr{S}_N(\gamma^Q) \;\coloneqq\; \sum_{m=1}^{N} [\mathcal{L}_m] \;\in\; \mathrm{Z}_1(\mathbf{MT}_s), \tag{596}$$

where γ^Q makes its dependence on the moduli superspace $\mathscr{M}_N$ explicit. If each $\mathcal{L}_m$ gives a classical solution of the minitwistor sigma-model, then $\mathscr{S}_N$ describes a many-body state of N such strings in $\mathbf{MT}_s$. In the next section we will show that the correlators of this multi-string system reproduce the full tree-level gluon $\mathcal{S}$-matrix.

As γ^Q varies over $\mathscr{M}_N$, the cycle $\mathscr{S}_N(\gamma^Q)$ sweeps out every possible N-string configuration. This observation motivates defining the semiclassical expectation value of the Wilson operator $\mathbb{W}[\mathscr{S}]$ by integrating over $\mathscr{M}_N$ against the measure $\mathcal{D}\gamma$.

For now, we compute the holonomy operator supported on $\mathscr{S}_N$:

$$\mathrm{Hol}\big[\mathscr{S}_N(\gamma^Q)\big] \;\coloneqq\; \mathrm{P}\left(\langle \mathcal{L}_1|u_1][u_1|\mathcal{L}_2|v_1][v_1|\mathcal{L}_3|u_2]\dots[u_k|\mathcal{L}_{2k}|v_k][v_k|\mathcal{L}_{2k+1}\rangle\right). \tag{597}$$

Here Hol is a functional on $\mathrm{Z}_1(\mathbf{MT}_s)$, and its dependence on $\gamma^Q \in \mathscr{M}_N$ enters through the moduli parameters u_ℓ^A and v_ℓ^A in each elementary block $\langle \mathcal{L}_m|u][u|\mathcal{L}_{m+1}|v]$.

The monomial expansion and the product rule from Subsection IV B 3 imply that the holonomy operator admits the expansion:

$$\mathrm{Hol}\big[\mathscr{S}_N(\gamma^Q)\big] \tag{598}$$

$$= \mathbb{I}_{\mathfrak{g}} + \sum_{n\geq N+1} (-1)^n \sum_{\vec{a}\in\mathbf{I}_k^n} \int\limits_{\mathbf{L}^n} \mathcal{F}^{\lambda}_{\vec{a}}\big(\gamma^Q\big) \bigwedge_{i=1}^{n} \frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}} \wedge \boldsymbol{A}\big|_{\mathcal{L}_{c_{\vec{a}}(i)}}\big(\lambda_i^A\big), \tag{599}$$

where $\mathbf{L}^n \coloneqq \bigtimes^n \mathbf{CP}^1$ and

$$\mathcal{F}^{\lambda}_{\vec{a}}\big(\gamma^Q\big) \coloneqq \mathcal{F}^{\lambda}_{a_1a_2}\big(\tau_1^M\big)\,\mathcal{F}^{\lambda}_{a_3a_4}\big(\tau_2^M\big)\,\ldots\,\mathcal{F}^{\lambda}_{b_4b_3}\big(\tau_{k-1}^M\big)\,\mathcal{F}^{\lambda}_{b_2b_1}\big(\tau_k^M\big). \tag{600}$$

Recall that the $\mathcal{F}$-function first appeared in Subsection III C and was redefined in Eq. (532).

Substituting the gauge potential expansion from Eq. (542) and employing Fubini's theorem, we obtain[27]:

$$\mathrm{Hol}\big[\mathscr{S}_N(\gamma^Q)\big] = \mathbb{I}_{\mathfrak{g}} + \sum_{n\geq N+1} (-1)^n \sum_{\vec{a}\in\mathbf{I}_k^n} \int\limits_{\mathbf{X}_n^*} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}'_i \wedge \alpha^{\Delta_i,\mathsf{a}_i}_{c_{\vec{a}}(i)}\big(\mathsf{Z}'^{I}_i\big) \tag{601}$$

$$\int\limits_{\mathbf{L}^n} \mathcal{F}^{\lambda}_{\vec{a}}\big(\gamma^Q\big) \bigwedge_{j=1}^{n} \frac{D\lambda_j}{\lambda_j\cdot\lambda_{j+1}} \mathsf{T}^{\mathsf{a}_j} \wedge \Psi_{\Delta_j}\Big|_{\mathcal{L}_{c_{\vec{a}}(j)}}\big(\lambda_j^A;\mathsf{Z}'^{I}_j\big). \tag{602}$$

Applying the celestial BMSW identity then yields the final form of this expansion, in terms of the mode coefficients $\alpha^{\Delta,\mathsf{a}}_m$ of the gauge potential and the moduli coordinates γ^Q:

$$\mathrm{Hol}\big[\mathscr{S}_N(\gamma^Q)\big] = \mathbb{I}_{\mathfrak{g}} + \sum_{n\geq N+1} (-1)^n \sum_{\vec{a}\in\mathbf{I}_k^n} \int\limits_{\mathbf{X}_n^*} \mathcal{F}^{z'}_{\vec{a}}\big(\gamma^Q\big) \tag{603}$$

$$\bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z'_i|Y_{c_{\vec{a}}(i)}|\bar{z}'_i]^{\Delta_i}}\, e^{i\langle z'_i|\xi_{c_{\vec{a}}(i)}\cdot\eta_i\rangle}\, \frac{\mathsf{T}^{\mathsf{a}_i}}{z'_i\cdot z'_{i+1}}\, \alpha^{\Delta_i,\mathsf{a}_i}_{c_{\vec{a}}(i)}\big(\mathsf{Z}'^{I}_i\big) \wedge D^{2|4}\mathsf{Z}'_i. \tag{604}$$

Wilson Operator; Semiclassical Expectation Value. The Wilson operator supported on the algebraic one-cycle is defined by:

$$\mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_N(\gamma^Q)\big] \coloneqq \mathsf{Tr}_{\mathfrak{g}}\,\mathrm{Hol}\big[\mathscr{S}_N(\gamma^Q)\big], \tag{605}$$

where the superscript $\boldsymbol{A}$ emphasise its dependence on the background gauge field. Physically, $\mathbb{W}[\mathscr{S}]$ is a nonlocal, gauge-invariant probe of the deformation of the holomorphic vacuum of the gauge theory on $E\to\mathbf{MT}_s$ along $\mathscr{S}_N$.

We define its semiclassical expectation value by integrating over the moduli superspace $\mathscr{M}_N$ with measure $\mathcal{D}\gamma$:

$$\mathcal{I}_k[\boldsymbol{A}] \coloneqq \big\langle \mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_N\big]\big\rangle_0 \coloneqq \int\limits_{\mathscr{M}_N} \mathcal{D}\gamma\, \mathbb{W}^{\boldsymbol{A}}\big[\mathscr{S}_N(\gamma^Q)\big]. \tag{606}$$

[27] Recall that $\mathbf{X}_n^* \coloneqq \bigtimes^n \mathbf{MT}_s^*$.

Substituting the expansion of Hol (see Eq. (604)) and using the Berezin identity $\int d^{0|8}\theta \; r = 0$, one finds:

$$\mathcal{I}_k[\boldsymbol{A}] \;=\; \sum_{n \geq N+1} (-1)^n \sum_{\vec{a} \in \mathbf{I}_k^n} \int\limits_{\mathbf{X}_n^*} \int\limits_{\mathscr{M}_N} \mathcal{F}_{\vec{a}}^{z'}\big(\gamma^Q\big) \tag{607}$$

$$\bigwedge_{i=1}^{n} \frac{\mathcal{C}(\Delta_i)}{\langle z_i' | Y_{c_{\vec{a}}(i)} | \bar{z}_i']^{\Delta_i}} \, e^{i \langle z_i' | \xi_{c_{\vec{a}}(i)} \cdot \eta_i \rangle} \, \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i' \cdot z_{i+1}'} \, \alpha_{c_{\vec{a}}(i)}^{\Delta_i, \mathsf{a}_i}\big(\mathsf{Z}_i'^{I}\big) \wedge D^{2|4}\mathsf{Z}_i'. \tag{608}$$

Finally, fixing $\vec{a} \in \mathbf{I}_k^n$, let h_i denote the scaling dimension of the i-th gluon and let $\{\mathsf{Z}_i^I\} \subset \mathbf{MT}_s^*$ be the insertion points. Functional differentiation of $\mathcal{I}_k[\boldsymbol{A}]$ with respect to the modes $\alpha_{c_{\vec{a}}(i)}^{2h_i, \mathsf{a}_i}\big(\mathsf{Z}_i^I\big)$, followed by setting $\boldsymbol{A} = 0$, yields:

$$(-1)^n \prod_{i=1}^{n} \frac{\delta}{\delta \alpha_{c_{\vec{a}}(i)}^{2h_i, \mathsf{a}_i}\big(\mathsf{Z}_i^I\big)} \, \mathcal{I}_k[\boldsymbol{A}] \;=\; \mathcal{M}_{n;\vec{a}}^{\mathsf{a}_1 \ldots \mathsf{a}_n}\big(\mathsf{Z}_i^I\big). \tag{609}$$

Thus $\mathcal{I}_k[\boldsymbol{A}]$ is the generating functional for the leaf sub-amplitudes in N^k-MHV configurations and hence reproduces the full tree-level gluonic $\mathcal{S}$-matrix in $\mathcal{N} = 4$ SYM theory.

C. Discussion

Now we pause to reflect on the calculations performed so far. In a more conventional approach to flat-space holography, one may begin with the AdS/CFT correspondence and then take the $R \to \infty$ limit of anti-de Sitter space to (hopefully) extract information about the celestial CFT. Here we have adopted a more indirect strategy.

In the AdS/CFT context, Alday and Maldacena [91] examined Wilson loop operators at strong coupling that trace null segments. They studied the classical equations of motion for a string with AdS$_3$ as its target space, subject to boundary conditions that force the worldsheet to end on a null polygon at the conformal boundary of AdS. Using gauge/gravity duality, they showed that this construction computes certain eight-gluon scattering amplitudes.

On the other hand, leaf amplitudes arise by dimensionally reducing the split-signature celestial amplitudes along the standard hyperbolic foliation of Klein space, whose leaves are Lorentzian AdS$_3$. This construction raises the question: if Alday and Maldacena reconstructed gluon scattering amplitudes from null Wilson loops in AdS$_3$, can we likewise reconstruct celestial leaf amplitudes from nonlocal, gauge-invariant observables?

To address this question, we first examined the geometric interpretation of the tree-level N^k-MHV minitwistor amplitudes for gluons discussed in the previous section. We found that these amplitudes localise on a family $\mathcal{L}_1, \ldots, \mathcal{L}_{2k+1}$ of minitwistor lines. Next, we obtained a field-theoretic interpretation of the leaf-gluon amplitudes by formulating holomorphic gauge theory on a complex-vector bundle E over minitwistor superspace $\mathbf{MT}_s$. We then probed the gauge theory by inserting a Wilson-like operator $\mathbb{W}[\mathscr{S}]$ supported on the algebraic one-cycle $\mathscr{S} = \sum_m [\mathcal{L}_m]$. Physically, $\mathbb{W}[\mathscr{S}]$ measures how the background gauge potential $\boldsymbol{A}$ deforms the holomorphic vacuum.

Finally, we demonstrated that the semiclassical expectation value of $\mathbb{W}[\mathscr{S}]$ generates the leaf-gluon amplitudes. We defined this expectation value as an integral over the moduli superspace $\mathscr{M}_N$ (see Eq. (606)). A question then arises: what is the origin of the holomorphic gauge theory on $E \to \mathbf{MT}_s$ whose Wilson operator we have used?

Perhaps our boldest proposal is that the holomorphic gauge theory on $E \to \mathbf{MT}_s$ arises as the string field theory limit of the minitwistor sigma-model introduced in the next section. This picture departs from the usual celestial-holography dictionary. There, the flat-space hologram appears as a CFT on the celestial sphere; here, we instead obtain a holomorphic gauge theory on minitwistor superspace. This theory emerges as the field-theory limit of a sigma-model whose target space is $\mathbf{MT}_s$ and whose worldsheet is the Riemann supersphere.

We then relate our model to the celestial CFT by treating the dual minitwistor superspace $\mathbf{MT}_s^*$ as a covering space of the celestial supersphere $\mathcal{CS}_s$. The $\mathcal{MT}$-transform maps $\mathbf{MT}_s^*$ back to $\mathbf{MT}_s$, thus completing the correspondence between the sigma-model and the celestial CFT.

V. MINITWISTOR STRING THEORY

This section develops the central idea of our work. We propose a semiclassical dynamical model for a celestial CFT. The model is holographically dual to the tree-level gluonic sector of $\mathcal{N} = 4$ SYM. We consider a many-body system of N minitwistor strings, where each string is realised as a topological sigma-model. The worldsheet is the $\mathcal{N} = 4$ celestial supersphere $\mathcal{CS}_s$, and the target is the minitwistor superspace $\mathbf{MT}_s$. We show that, in the leading-trace semiclassical limit, the N-string system reproduces the tree-level N^k-MHV leaf amplitudes for gluons when $N = 2k + 1$.

To briefly recap the developments so far, in Section II we developed the formalism of minitwistor superwavefunctions as a toolkit for studying celestial leaf amplitudes for gluons in $\mathcal{N} = 4$ SYM theory. In Section III we applied this toolkit to the Drummond-Henn solution of the super-BCFW recursion relations; from that analysis we proved the localisation theorem. The theorem states that the minitwistor transform of tree-level leaf-gluon amplitudes in every N^k-MHV sector localises on a family of minitwistor lines.

We interpret those lines as algebraic one-cycles $\mathscr{S}$ on $\mathbf{MT}_s$. By formulating a holomorphic gauge theory on minitwistor superspace, we obtain a geometric interpretation of the localisation theorem: the leaf-gluon amplitudes arise as minitwistor Wilson lines $\mathbb{W}[\mathscr{S}]$ supported on those one-cycles. We assign the Fourier modes $\alpha_m^{\Delta,\mathsf{a}_i}$ of the background gauge potential $\boldsymbol{A}$ to the minitwistor lines in the localisation family $\{\mathcal{L}_m\}$. Interpreting these modes as the classical expectation values of gluon annihilation operators attached to the lines yields a dynamical picture. In this picture, the lines $\mathcal{L}_m \subset \mathbf{MT}_s$ in the localisation family are viewed as minitwistor strings interacting with the background gauge potential.

We now study a semiclassical many-body system of N minitwistor strings propagating on the background defined by the holomorphic gauge theory on $\mathbf{MT}_s$. We show that the partition function of this theory serves as a generating functional for the tree-level leaf-gluon amplitudes in the N^k-MHV sector. From this statement we conclude that the holomorphic gauge theory introduced in Section IV can be identified with the string-field limit of the string theory considered here.

We also define vertex operators that encode the worldsheet interactions of the minitwistor strings. From these vertex operators we construct the corresponding celestial gluon operators. By setting the background gauge potential to zero, we isolate the purely worldsheet interactions. In this limit, the leading-trace, semiclassical correlators of the celestial gluon operators reproduce the tree-level N^k-MHV leaf amplitudes. Finally, we show that the OPEs of the celestial gluon operators close on the S-algebra.

Consequently, we obtain the following picture. Each tree-level N^k-MHV gluonic sector of $\mathcal{N} = 4$ SYM is holographically dual to a semiclassical system of N minitwistor strings, with the integer N related to the MHV level k by $N = 2k + 1$. Together with the other evidence discussed in the final subsection, this observation leads us to the following proposal.

Conjecture. *We conjecture the existence of a* fully quantum-mechanical *topological sigma-model whose worldsheet is the celestial supersphere $\mathcal{CS}_s$ and whose target is the minitwistor superspace $\mathbf{MT}_s$. The Hilbert space of this theory decomposes into sectors labelled by an integer N. In the appropriate semiclassical limit, each such sector is described by the N-string system defined here. Moreover, the full quantum sigma-model realises the celestial CFT dual to the tree-level maximally supersymmetric YM theory.*

The arguments developed in this section are more involved and abstract than those in the preceding discussion. For pedagogical clarity we proceed in two steps. In Subsection A we analyse in detail a single-string system coupled to the holomorphic gauge theory on minitwistor superspace.

We show that this system is dual to the MHV gluonic sector of tree-level $\mathcal{N} = 4$ SYM. The goal of this first discussion is pedagogical rather than fully rigorous: it introduces the principal physical ideas and sets our notation and terminology.

In Subsection B we treat the N-string system in a more mathematically rigorous language. There we demonstrate how the N-string semiclassical dynamics reproduce the N^k-MHV sectors of gauge theory on flat space.

A. Single-String Model

In this subsection, we study a single minitwistor string formulated as a topological sigma-model. Its worldsheet is the $\mathcal{N} = 4$ celestial supersphere, $\mathcal{CS}_s$. Its target is the minitwistor superspace, $\mathbf{MT}_s$. We define this sigma-model at the semiclassical level only. It will likely develop anomalies upon quantisation. A fully rigorous treatment, for example via the BV-BRST formalism, lies beyond our scope. However, treating the sigma-model as a string theory supplies useful intuition. In particular, it provides a dynamical interpretation of the Wilson operators $\mathbb{W}[\mathscr{S}]$ introduced above.

We show in Subsection V A 4 that the semiclassical partition function of a single minitwistor string, interacting with a classical "bath" modelled by the holomorphic gauge theory, reproduces the tree-level MHV leaf superamplitudes for gluons in $\mathcal{N} = 4$ SYM. This match indicates that the field theory studied in Section IV can be seen as a minitwistor string-field limit of the string theory proposed here.

We then construct vertex operators that describe the worldsheet interactions. From those vertex operators we build celestial gluon operators. We then compute the leading-trace semiclassical correlators of these gluon operators and show that they reproduce the tree-level MHV leaf-gluon amplitudes. This result supports our assertion that the semiclassical single-string model realises the celestial CFT dual to the MHV sector of $\mathcal{N} = 4$ SYM at tree level.

1. Formal Preliminaries

We now introduce the essential concepts of the theory. To formalise our sigma-model, we first define the configuration space of the minitwistor string. Using the supersymmetric Hitchin correspondence (Section II), we then define the classical moduli superspace that parameterises the classically allowed configurations of the string. To describe how the worldsheet $\mathcal{CS}$ is mapped into minitwistor superspace $\mathbf{MT}_s$, we introduce the evaluation maps. These maps characterise the minitwistor lines $\mathcal{L} \subset \mathbf{MT}_s$ through the associated incidence relations.

Finally, employing the evaluation maps together with their incidence relations, we construct the embedding maps of $\mathcal{CS}$ into $\mathbf{MT}_s$. In the subsequent subsection, where we define the dynamics of the model, the evaluation maps will serve as the fundamental field variables of the theory.

Configuration Space. The minitwistor sigma-model is a theory of holomorphic rational maps:

$$\phi\colon\ \mathcal{CS}_s \ \longrightarrow\ \mathbf{MT}_s. \tag{610}$$

Algebraic curves in $\mathbf{CP}^1 \times \mathbf{CP}^1$, and hence in its supersymmetric extension $\mathbf{MT}_s$, are classified by a bidegree $\beta = (d_1, d_2)$. For fixed β, let

$$\mathrm{Hol}_\beta\big(\mathcal{CS}_s;\, \mathbf{MT}_s\big) \tag{611}$$

denote the functor of points parametrising holomorphic maps of bidegree β. The automorphism group $\mathrm{Aut}\big(\mathcal{CS}_s\big)$ is the superconformal group of $\mathbf{CP}^{1|4}$. Since two maps (610) differing by a re-parametrization of the worldsheet define the same state, the *physical configuration space* of a single string of bidegree β is:

$$\mathcal{E}_\beta \ \coloneqq\ \mathrm{Hol}_\beta\big(\mathcal{CS}_s; \mathbf{MT}_s\big)/\mathrm{Aut}\big(\mathcal{CS}_s\big). \tag{612}$$

Alternatively, each map $\phi \in \mathcal{E}_\beta$ defines an algebraic one-cycle $[\phi(\mathcal{CS}_s)] \in \mathrm{Z}_1(\mathbf{MT}_s)$. Two cycles that are rationally equivalent describe the same string configuration. Thus, the physical configuration space can also be modelled by the Chow group $\mathrm{A}_{1,\beta}(\mathbf{MT}_s)$ and the natural forgetful functor:

$$\mathcal{E}_\beta \longrightarrow \mathrm{A}_{1,\beta}\big(\mathbf{MT}_s\big), \qquad \phi \ \longmapsto\ \big[\phi(\mathcal{CS}_s)\big]. \tag{613}$$

This algebraic viewpoint makes it easier to connect to our earlier definition of the Wilson operator $\mathbb{W}[\mathscr{S}]$ for algebraic one-cycles.

Classical String Configurations. Since our theory is defined only at the semiclassical level, we must specify which string configurations are classically allowed and in terms of which we define expectation values.

We introduced minitwistor strings to give a dynamical derivation of the localisation theorem. The N^k-MHV gluon amplitudes localise on a family of minitwistor lines $\{\mathcal{L}_m\} \subset \mathbf{MT}_s$. In the previous section, we saw that these amplitudes come from Wilson operators $\mathbb{W}[\mathscr{S}]$ supported on the cycle $\mathscr{S} = \sum_m [\mathcal{L}_m]$. We also showed that gluon creation and annihilation operators attach naturally to each $\mathcal{L}_m$. Hence we interpret each minitwistor line $\mathcal{L} \subset \mathbf{MT}_s$ as a *classical string configuration.*

Each minitwistor line is an irreducible curve of bidegree $\beta = (1,1)$. Translating this geometric fact into dynamics, we define the *classical configuration space* as $\mathcal{E}_c \coloneqq \mathcal{E}_{(1,1)}$, and write $\mathcal{L} \in \mathcal{E}_c$ for any such line. By the supersymmetric Hitchin correspondence (Section II), the hyperbolic superspace $\mathbf{H}_s$ serves as the moduli space of these lines. Hence we identify $\mathscr{M}_c = \mathbf{H}_s$ as the *classical moduli superspace* for minitwistor strings.

Evaluation Maps. A classical string configuration $\mathcal{L} = \mathcal{L}(X,\theta) \in \mathcal{E}_c$ is parameterised by a point $\mathsf{X}^K = \big(X_{A\dot{A}}, \theta^\alpha_{\dot{A}}\big) \in \mathscr{M}_c$ through a pair of evaluation maps. Let Λ denote the Grassmann algebra associated to the vector superspace $\mathbf{C}^{0|4}$ and set $\Lambda[k] \coloneqq \bigwedge^k \mathbf{C}^{0|4}$. Let

$$\Phi_{\dot{A}} \;\in\; \Gamma\big(\mathcal{L}; \mathcal{O}(1) \oplus \mathcal{O}(1)\big), \qquad \varphi^\alpha \;\in\; \Lambda[1] \otimes \Gamma\big(\mathcal{L}; \mathcal{O}(1)\big). \tag{614}$$

Choose homogeneous coordinates $[\lambda^A]$ on $\mathcal{L}$ induced by sections of $H^{0,0}(\mathcal{L};\mathcal{O}(1))$, and let W^I denote the homogeneous coordinates on $\mathbf{MT}_s$ from Section II. The evaluation maps are then:

$$\Phi_{\dot{A}}\big(\lambda^A\big) \;=\; \lambda^A X_{A\dot{A}}, \qquad \varphi^\alpha\big(\lambda^A\big) \;=\; \lambda^A \theta^\alpha_{A}. \tag{615}$$

Hence $\mathcal{L}$ appears as the locus of points $\mathsf{W}^I = \big(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\big) \in \mathbf{MT}_s$ satisfying

$$\mu_{\dot{A}} \;=\; \Phi_{\dot{A}}\big(\lambda^A\big), \qquad \psi^\alpha \;=\; \varphi^\alpha\big(\lambda^A\big). \tag{616}$$

The evaluation maps thus specify how the moduli $\mathsf{X}^K \in \mathscr{M}_c$ determine the configuration $\mathcal{L} \in \mathcal{E}_c$.

This construction suggests a simple approach to defining the sigma-model action. We introduce a Lagrangian in which $\Phi_{\dot{A}}$ and φ^α play the role of fundamental fields. Its Euler-Lagrange equations then reproduce the incidence relations on the minitwistor line. To obtain a well-posed variational principle and to apply a saddle-point approximation in the pathintegral, we seek a Lagrangian quadratic in $\Phi_{\dot{A}}$ and φ^α.

Now, the evaluation maps are homogeneous of degree one in λ^A:

$$\Phi_{\dot{A}}\big(t\,\lambda^A\big) \;=\; t\,\Phi_{\dot{A}}\big(\lambda^A\big), \quad \varphi^\alpha\big(t\,\lambda^A\big) \;=\; t\,\varphi^\alpha\big(\lambda^A\big), \quad \forall\, t \in \mathbf{C}^*. \tag{617}$$

Any quadratic form in these fields thus has degree two in λ^A. But such a form cannot be integrated against the holomorphic measure

$$D\lambda \;\coloneqq\; \langle \lambda\, d\lambda \rangle \;\in\; \Omega^{1,0}\big(\mathcal{L}; \mathcal{O}(2)\big) \tag{618}$$

because the integrand would carry excess homogeneity. A straightforward solution consists in introducing a chart on $\mathcal{L}$ with coordinates σ^B that transform as

$$\lambda^A \;\mapsto\; t\,\lambda^A \qquad \Longrightarrow \qquad \sigma^B \;\mapsto\; t^{-1}\,\sigma^B. \tag{619}$$

In terms of the projective coordinates λ^A, the σ-coordinates are defined by the transition map:

$$\lambda^A = \tau^A(\sigma^B). \tag{620}$$

This transition map specifies how the two patches on $\mathcal{CS}$, given by the domains of the coordinate functions λ^A and σ^B, are glued together. The transition map is a holomorphic section

$$\tau^A \;\in\; \Gamma\big(\mathbf{CP}^1; \mathcal{O}(-1)\oplus\mathcal{O}(-1)\big). \tag{621}$$

The explicit form of τ^A is constructed from the following structures. Let $r = 1, 2$ index a frame field e_r^A trivialising the bundle $\mathcal{O}(1)\oplus\mathcal{O}(1) \to \mathcal{L}$. Hence any undotted two-component spinor field $\boldsymbol{s}^A$ on $\mathcal{L}$ decomposes as

$$\boldsymbol{s}^A \;=\; s^r\,\epsilon_r^A, \qquad s^r \in \mathscr{C}^\infty(\mathcal{L}; \mathbf{C}). \tag{622}$$

Next, let S_+ be the representation space of $\mathrm{SL}(2;\mathbf{C})$ realising the undotted van der Waerden spinors, and choose a basis ι^{rA} for S_+. Define the component functions

$$s^r\big(\sigma^B\big) \;\coloneqq\; (-1)^r\,\langle \iota^r, \sigma\rangle, \tag{623}$$

and let $\|\boldsymbol{\sigma}\| \coloneqq -\prod_{r=1,2} s^r(\sigma^B)$. Accordingly, the transition map (620) is given by

$$\tau^A\big(\sigma^B\big) \;\coloneqq\; \frac{1}{\|\boldsymbol{\sigma}\|}\, s^r\big(\sigma^B\big)\,\epsilon_r^A. \tag{624}$$

From this definition it follows that, under the rescaling $\lambda^A \mapsto t\,\lambda^A$, the σ-coordinates transform as $\sigma^B \mapsto t^{-1}\,\sigma^B$, as required.

Therefore, in terms of the σ-coordinates, the incidence relations become:

$$\lambda^A \;=\; \tau^A\big(\sigma^B\big), \quad \mu_{\dot{A}} \;=\; \Pi_{\dot{A}}\big(\sigma^B\big), \quad \psi^\alpha \;=\; \kappa^\alpha\big(\sigma^B\big). \tag{625}$$

Here the new evaluation maps

$$\Pi_{\dot{A}} \in \Gamma\big(\mathcal{L}; \mathcal{O}(-1)\oplus\mathcal{O}(-1)\big), \qquad \kappa^\alpha \in \Lambda[1]\otimes\Gamma\big(\mathcal{L}; \mathcal{O}(-1)\big) \tag{626}$$

are given by:

$$\Pi_{\dot{A}}\big(\sigma^B\big) \;\coloneqq\; \Phi_{\dot{A}}\big(\tau^A(\sigma^B)\big), \qquad \kappa^\alpha\big(\sigma^B\big) \;\coloneqq\; \varphi^\alpha\big(\tau^A(\sigma^B)\big). \tag{627}$$

Substituting Eq. (616) into these definitions yields the explicit form:

$$\Pi_{\dot{A}}(\sigma^B) \;=\; \frac{\epsilon_1^A X_{A\dot{A}}}{\langle\sigma,\iota^2\rangle} - \frac{\epsilon_2^A X_{A\dot{A}}}{\langle\sigma,\iota^1\rangle}, \qquad \kappa^\alpha(\sigma^B) \;=\; \frac{\epsilon_1^A \theta_A^\alpha}{\langle\sigma,\iota^2\rangle} - \frac{\epsilon_2^A \theta_A^\alpha}{\langle\sigma,\iota^1\rangle}. \tag{628}$$

These maps are homogeneous of degree -1 in σ^B:

$$\Pi_{\dot{A}}(t\,\sigma^B) \;=\; t^{-1}\,\Pi_{\dot{A}}(\sigma^B), \qquad \kappa^\alpha(\sigma^B) \;=\; t^{-1}\,\kappa^\alpha(\sigma^B). \tag{629}$$

In what follows, we call the maps $\Phi_{\dot{A}}$ and φ^α, which depend on the λ-coordinates, the *evaluation maps of the first kind*. Similarly, the maps $\Pi_{\dot{A}}$ and κ^α, which are parameterised by the σ-coordinates, will be called *evaluation maps of the second kind*.

Remark 1. Embedding Maps.

The string configuration is given by an embedding map that sends the worldsheet $\mathcal{CS}$ to a minitwistor line $\mathcal{L}(X,\theta) \subset \mathbf{MT}_s$. We define this embedding map by

$$\mathsf{W}^I(\lambda^A) \;\coloneqq\; \big(\lambda^A,\, \Phi_{\dot{A}}(\lambda^A),\, \varphi^\alpha(\lambda^A)\big), \tag{630}$$

which is parameterised by the λ-coordinates. It is constructed from the first-kind evaluation maps, and so we call the assignment $\lambda^A \mapsto \mathsf{W}^I(\lambda^A)$ the *first-kind parameterisation* of the minitwistor string.

Using the transition map $\lambda^A = \tau^A(\sigma^B)$ defined in Eq. (624), the string configuration can be equivalently characterised in terms of the second-kind evaluation maps via the embedding:

$$\mathsf{Y}^I(\sigma^B) \;\coloneqq\; \big(\tau^A(\sigma^B),\, \Pi_{\dot{A}}(\sigma^B),\, \kappa^\alpha(\sigma^B)\big). \tag{631}$$

We refer to the assignment $\sigma^B \mapsto \mathsf{Y}^I(\sigma^B)$ as the *second-kind parameterisation* of the minitwistor string.

The two parameterisations give equivalent representations of the celestial sphere, and so every physical observable translates between them. From the target-space viewpoint, the first-kind parameterisation is more natural because the undotted spinor components of $\mathsf{W}^I(\lambda^A)$ coincide with the projective coordinates λ^A on $\mathcal{CS}$. Hence we prefer to express the interaction action $\mathcal{U}$ in the λ-coordinates.

By contrast, the second-kind parameterisation simplifies the action $\mathcal{S}_0$ that governs the embedding dynamics (i.e., the *geometric sector*). The evaluation maps $\Pi_{\dot{A}}$ and κ^α carry the homogeneity needed to pair with the holomorphic measure $D\sigma$. This pairing produces a projectively invariant top-form on $\mathcal{CS}$, from which one constructs a Lagrangian quadratic in $\Pi_{\dot{A}}$ and κ^α. Hence the second-kind parameterisation is better suited for studying the semiclassical regime of the model via a saddle-point approximation of the path integral.

Our aim is to formulate a variational principle in which the action $\mathcal{S}$ depends on the fields $\Pi_{\dot{A}}$ and κ^{α} as independent variables. The resulting Euler-Lagrange equations must reproduce the incidence relations in Eq. (625). To this end, we recast the definitions of the evaluation maps as differential equations. We then require that these equations follow from $\delta\mathcal{S} = 0$ and uniquely recover the explicit maps given in Eq. (628).

Partial Differential Equations; Currents. The bosonic evaluation map of the second kind satisfies

$$\frac{1}{2\pi i}\,\overline{\partial}_{\sigma}\,\Pi_{\dot{A}}\big(\sigma^{B}\big) \;+\; \mathcal{J}_{\dot{A}}\big(\sigma^{B};X_{C\dot{C}}\big) \;=\; 0. \tag{632}$$

Here the bosonic current

$$\mathcal{J}_{\dot{A}} \;\in\; \mathscr{D}'_{0,1}\big(\mathcal{L}(X,\theta);\, \mathcal{O}_{\mathbf{C}}(-1)\oplus\mathcal{O}_{\mathbf{C}}(-1)\big) \tag{633}$$

is the distributional $(0,1)$-form on the minitwistor line defined by

$$\mathcal{J}_{\dot{A}}\big(\sigma^{B};X_{C\dot{C}}\big) \;\coloneqq\; \overline{\delta}(\sigma\cdot\iota^{2})\;\epsilon_{1}^{A}X_{A\dot{A}} \;-\; \overline{\delta}(\sigma\cdot\,\iota^{1})\;\epsilon_{2}^{A}X_{A\dot{A}}. \tag{634}$$

The fermionic evaluation map of the second kind obeys

$$\frac{1}{2\pi i}\,\overline{\partial}_{\sigma}\,\kappa^{\alpha}\big(\sigma^{B}\big) \;+\; \mathcal{K}^{\alpha}\big(\sigma^{B};\theta_{C}^{\gamma}\big) \;=\; 0. \tag{635}$$

The fermionic current

$$\mathcal{K}^{\alpha} \;\in\; \mathscr{D}'_{0,1}\big(\mathcal{L}(X,\theta);\; \mathbf{C}^{0|4}\otimes\mathcal{O}_{\mathbf{C}}(-1)\big) \tag{636}$$

is

$$\mathcal{K}^{\alpha}\big(\sigma^{B};\theta_{C}^{\gamma}\big) \;\coloneqq\; \overline{\delta}(\sigma\cdot\iota^{2})\;\epsilon_{1}^{A}\theta_{A}^{\alpha} \;-\; \overline{\delta}(\sigma\cdot\iota^{1})\;\epsilon_{2}^{A}\theta_{A}^{\alpha}. \tag{637}$$

As with the evaluation maps $\Pi_{\dot{A}}$ and κ^{α}, both currents are homogeneous of degree -1 in the spinor coordinates σ^{B}. Explicitly,

$$\mathcal{J}_{\dot{A}}\big(t\,\sigma^{B};X_{C\dot{C}}\big) \;=\; t^{-1}\,\mathcal{J}_{\dot{A}}\big(\sigma^{B};X_{C\dot{C}}\big), \quad \mathcal{K}^{\alpha}\big(t\,\sigma^{B};\theta_{C}^{\gamma}\big) \;=\; t^{-1}\,\mathcal{K}^{\alpha}\big(\sigma^{B};\theta_{C}^{\gamma}\big). \tag{638}$$

The existence and uniqueness theorem for linear PDEs on compact Riemann surfaces (see Forster [86, Sec. 1.11]) guarantees that Eqs. (632) and (635) uniquely determine the evaluation maps. Therefore, any action whose equations of motion reproduce these PDEs yields the incidence relations as its extremal equations. Such an action provides a candidate for the classical dynamics of the minitwistor sigma-model.

Bosonic Sector. To define the action for the bosonic sector, we use the monomials:

$$[\Pi\,\overline{\partial}_\sigma\,\Pi] \quad \text{and} \quad [\Pi\,\mathscr{J}] \;\in\; \Omega^{0,1}\big(\mathcal{L}(X,\theta); \mathcal{O}_{\mathbf{C}}(-2)\big), \tag{639}$$

together with the holomorphic measure:

$$D\sigma \;\coloneqq\; \langle \sigma\, d\sigma \rangle \;\in\; \Omega^{1,0}\big(\mathcal{L}(X,\theta);\, \mathcal{O}_{\mathbf{C}}(2)\big). \tag{640}$$

This yields the top-forms:

$$D\sigma \wedge [\Pi\,\overline{\partial}_\sigma\,\Pi] \quad \text{and} \quad D\sigma \wedge [\Pi\,\mathscr{J}] \;\in\; \Omega^{1,1}\big(\mathcal{L}(X,\theta)\big). \tag{641}$$

We integrate these forms over the minitwistor line $\mathcal{L}(X,\theta)$. The action functional then reads:

$$\mathcal{S}_\Pi(X,\theta) \;\coloneqq\; \frac{1}{b}\int_{\mathcal{L}(X,\theta)} D\sigma \wedge \left(\frac{1}{2\pi i}\,[\Pi\,\overline{\partial}_\sigma\,\Pi] \;+\; [\Pi\,\mathscr{J}]\right). \tag{642}$$

The parameter b plays a role analogous to the Liouville coupling in the semiclassical limit.

The integral in Eq. (642) depends on the moduli $\mathsf{X}^K \in \mathscr{M}_c$ through the integration domain $\mathcal{L}(X,\theta)$ and the current $\mathscr{J}_{\dot{A}}$, which itself depends on the bosonic projection $X_{A\dot{A}}$. Varying $\mathcal{S}_\Pi$ yields the defining PDE for the bosonic evaluation map (Eq. (632)).

Fermionic Sector. To construct the fermionic action, note that the field κ^α has Grassmann degree one, since it is a section of $\mathcal{L}(X,\theta)$ valued in the vector superspace $\mathbf{C}^{0|4}$. The action itself must be a real number. Therefore, we can form a Lagrangian polynomial in κ^α only by pairing it with another field of Grassmann degree three and then performing a Berezin integral over the fermionic directions.

Although the curve $\mathcal{L}(X,\theta)$ lies in the supersymmetric manifold $\mathbf{MT}_s$, it remains bosonic: as a rational curve, it is biholomorphic to the Riemann sphere, $\mathcal{L}(X,\theta) \cong \mathbf{CP}^1$. To incorporate the full $\mathcal{N}=4$ fermionic structure, we extend this curve to a minitwistor superline by adjoining four Grassmann coordinates χ^α. We denote the resulting superspace by $\mathcal{CS}_s(X,\theta)$, interpreting it as the embedding of the celestial supersphere into $\mathbf{MT}_s$ as an irreducible superline of bidegree $(1,1)$:

$$\mathcal{CS}_s(X,\theta) \;\cong\; \mathbf{CP}^{1|4}. \tag{643}$$

We then combine the bosonic coordinates σ^A and the fermionic variables χ^α into the supercoordinates $\mathsf{s} := (\sigma^B, \chi^\beta)$, and define the canonical Berezin-DeWitt volume form[28] on $\mathcal{CS}_s(X,\theta)$ by:

$$D^{1|4}\mathsf{s} \;:=\; D\sigma \,\wedge\, d^{0|4}\chi. \tag{645}$$

We next introduce a Lagrange multiplier

$$e_\alpha \,\in\, \Omega^{0,0}\big(\mathcal{L}(X,\theta);\; \textstyle\bigwedge^3 \mathbf{C}^{0|4}\otimes\mathcal{O}_{\mathbf{C}}(-1)\big). \tag{646}$$

We continue to take $\mathcal{L}(X,\theta)$ (and not $\mathcal{CS}_s(X,\theta)$) as the base for this section because the fermionic directions appear only in the fibre part valued in the exterior superalgebra $\bigwedge^3 \mathbf{C}^{0|4}$. We assume that e_α is homogeneous of degree -1 in the spinor coordinates σ^B, namely

$$e_\alpha\big(t\,\sigma^B, \chi^\beta\big) \;=\; t^{-1}\, e_\alpha\big(\sigma^B, \chi^\beta\big). \tag{647}$$

Consider the monomials

$$e_\alpha \wedge \overline{\partial}_\sigma\, \kappa^\alpha \quad \text{and} \quad e_\alpha \wedge \mathcal{K}^\alpha \,\in\, \Omega^{0,1}\big(\mathcal{L}(X,\theta);\; \textstyle\bigwedge^4 \mathbf{C}^{0|4}\otimes\mathcal{O}_{\mathbf{C}}(-2)\big). \tag{648}$$

Taking the exterior product of these objects with the Berezin-DeWitt measure on the superline produces the differential forms:

$$D^{1|4}\mathsf{s} \,\wedge\, e_\alpha \,\wedge\, \overline{\partial}_\sigma\, \kappa^\alpha \quad \text{and} \quad D^{1|4}\mathsf{s} \,\wedge\, e_\alpha \,\wedge\, \mathcal{K}^\alpha \,\in\, \Omega^{(1,1)|4}\big(\mathcal{CS}_s(X,\theta)\big). \tag{649}$$

Now the base manifold is the full celestial supersphere $\mathcal{CS}_s(X,\theta)$. These expressions are genuine differential forms valued in the Berezinian of $\mathcal{CS}_s(X,\theta)$ and therefore are integrable.

Accordingly, we take the fermionic sector to be governed by the action:

$$\mathcal{S}_{\kappa,e}(X,\theta) \;=\; \frac{1}{b}\int_{\mathcal{CS}_s(X,\theta)} D^{1|4}\mathsf{s} \wedge \left(\frac{1}{2\pi i}\, e_\alpha \wedge \overline{\partial}_\sigma\, \kappa^\alpha \;+\; e_\alpha \wedge \mathcal{K}^\alpha\right). \tag{650}$$

The dependence of the action on the superspace coordinates $\mathsf{X}^K = \big(X_{A\dot{A}}, \theta^\alpha_A\big)$ enters through the integration superdomain and through the current $\mathcal{K}^\alpha$, which itself depends on the Grassmann-valued spinors θ^α_A. Taking the variation of $\mathcal{S}_{\kappa,e}$ with respect to κ^α and e_α yields Eq. (635), which is the defining PDE for the fermionic incidence relations.

---

[28] We record a few remarks on integration over $\mathcal{CS}_s$. The $\mathcal{N}=4$ celestial supersphere is the vector superbundle $\mathcal{CS}_s \cong \mathbf{CP}^1 \times \mathbf{C}^{0|4}$ (see Ch. 12 of Rogers [20]). Projective rescalings act only on the bosonic coordinate σ^A, so that $\sigma^A \sim t\,\sigma^A$ for all $t \in \mathbf{C}^*$. Under this rescaling, the Berezin-DeWitt superform $D^{1|4}\mathsf{s} = D\sigma \wedge d^{0|4}\chi$ transforms as $D^{1|4}\mathsf{s} \mapsto t^2\, D^{1|4}\mathsf{s}$. To obtain a projectively invariant top-form on $\mathcal{CS}_s$, pair $D^{1|4}\mathsf{s}$ with a $(0,1)$-form

$$\boldsymbol{w} \,\in\, \Omega^{0,1}\big(\mathcal{CS};\; \textstyle\bigwedge^4 \mathbf{C}^{0|4}\otimes\mathcal{O}_{\mathbf{C}}(-2)\big). \tag{644}$$

Here $\mathcal{CS} \cong \mathbf{CP}^1$ denotes the bosonic base, and the factor $\bigwedge^4 \mathbf{C}^{0|4}$ denotes the fermionic fibres. The form $\boldsymbol{w}$ has Grassmann degree 4 and homogeneity -2. Hence their wedge product, $D^{1|4}\mathsf{s} \wedge \boldsymbol{w}$, is a Berezinian-valued top-form on $\mathcal{CS}_s$ (cf. § 2.2 of Voronov [33]). Its projective weight vanishes, so it can be integrated over $\mathcal{CS}_s$.

Geometric Sector. The evaluation maps of the second kind, $\Pi_{\dot{A}}$ and κ^{α}, together with the Grassmann-valued Lagrange multiplier e_{α}, constitute the dynamical variables of the *geometric sector*. This sector describes the embedding of the celestial supersphere $\mathcal{CS}_s$ as a minitwistor string $\mathcal{L} \subset \mathbf{MT}_s$. We therefore collect the fundamental fields of the geometric sector into the multiplet:

$$\varDelta \coloneqq \left\{ \Pi_{\dot{A}}(\sigma^B),\, \kappa^{\alpha}(\sigma^B),\, e_{\alpha}(\sigma^B, \chi^{\beta}) \right\}. \tag{651}$$

Combining Eqs. (642) and (650), the *geometric action* reads:

$$\mathcal{S}_0[\Delta|X,\theta] \coloneqq \mathcal{S}_{\Pi}(X,\theta) + \mathcal{S}_{\kappa,e}(X,\theta). \tag{652}$$

To unify the bosonic and fermionic parts, we introduce a pair of conjugate superfields and a supercurrent. First, define a vielbein $E_{\dot{A}}{}^{\alpha}$ on $\mathcal{CS}_s$, normalised by:

$$E_{\dot{A}}{}^{\alpha} E^{\dot{A}}{}_{\beta} = \delta^{\alpha}{}_{\beta}. \tag{653}$$

Next, introduce the *conjugate superfields*:

$$\Sigma_{\dot{A}},\, \Xi^{\dot{A}} \in \Omega^{0,0}\big(\mathcal{L}(X,\theta);\, \textstyle\bigwedge \mathbf{C}^{0|4} \otimes \big(\mathcal{O}_{\mathbf{C}}(-1) \oplus \mathcal{O}_{\mathbf{C}}(-1)\big)\big) \tag{654}$$

with components:

$$\Sigma_{\dot{A}}\big(\sigma^B, \chi^{\beta}\big) \coloneqq \chi^1\chi^2\, \Pi_{\dot{A}}\big(\sigma^B\big) + E_{\dot{A}}{}^{\alpha}\, e_{\alpha}\big(\sigma^B, \chi^{\beta}\big), \tag{655}$$

$$\Xi^{\dot{A}}\big(\sigma^B, \chi^{\beta}\big) \coloneqq \chi^3\chi^4\, \Pi^{\dot{A}}\big(\sigma^B\big) + E^{\dot{A}}{}_{\alpha}\, \kappa^{\alpha}\big(\sigma^B\big). \tag{656}$$

In the inclusion relation (654), we treat the base manifold as the line $\mathcal{L}(X,\theta)$ because the fermionic directions live in fibres valued in the exterior superalgebra $\bigwedge \mathbf{C}^{0|4}$. When we wedge the superfields with the measure $D^{1|4}\mathbf{s}$, the resulting top-forms take values in the Berezinian of $\mathcal{CS}_s(X,\theta)$. Berezin integration then gives:

$$\int d^{0|4}\chi \wedge [\Sigma\, \overline{\partial}_{\sigma}\, \Xi] = [\Pi\, \overline{\partial}_{\sigma}\, \Pi] + \int d^{0|4}\chi \wedge e_{\alpha} \wedge \overline{\partial}_{\sigma}\kappa^{\alpha}. \tag{657}$$

Next, define the *supercurrent*:

$$|X,\theta]^{\dot{A}} \in \mathscr{D}'_{0,1}\big(\mathcal{L}(X,\theta);\, \textstyle\bigwedge \mathbf{C}^{0|4} \otimes \big(\mathcal{O}_{\mathbf{C}}(-1) \oplus \mathcal{O}_{\mathbf{C}}(-1)\big)\big) \tag{658}$$

by:

$$|X,\theta]^{\dot{A}} \coloneqq \chi^3\chi^4\, \mathcal{J}^{\dot{A}}\big(\sigma^B; X_{C\dot{C}}\big) + E^{\dot{A}}{}_{\alpha}\, \mathcal{K}^{\alpha}\big(\sigma^B; \theta^{\gamma}_{C}\big). \tag{659}$$

Berezin integration then yields:

$$\int d^{0|4}\chi \wedge [\Sigma|X,\theta] \;=\; [\Pi\,\mathcal{J}] \;+\; \int d^{0|4}\chi \wedge e_\alpha \wedge \mathcal{K}^\alpha. \tag{660}$$

Combining Eqs. (657) and (660) gives the final form of the geometric action:

$$\mathcal{S}_0[\varDelta|X,\theta] \;\coloneqq\; \frac{1}{b}\int_{\mathcal{CS}_s(X,\theta)} D^{1|4}\mathsf{s} \wedge \left(\frac{1}{2\pi i}\,[\Sigma\,\overline{\partial}_\sigma\,\Xi] \;+\; [\Sigma|X,\theta]\right). \tag{661}$$

3. Classical Theory: Worldsheet CFT

To complete the classical theory, we now specify the auxiliary matter system that defines the worldsheet CFT. In fully quantum-mechanical models (e.g., twistor string theories), this matter system contributes to the total central charge. It also helps cancel or otherwise tame anomalies that arise on quantisation.

In the semiclassical framework adopted here, the reason for introducing the worldsheet CFT is phenomenological: it determines how the minitwistor string couples to the background potential of the holomorphic gauge theory on minitwistor superspace.

We consider two worldsheet fermions, ρ and ρ^*, defined on the celestial sphere $\mathcal{CS}$. Using the embedding maps introduced above, we push these fermions forward to the minitwistor line $\mathcal{L} \subset \mathbf{MT}_s$, which represents the classical string configuration. On $\mathcal{L}$, the fermions couple minimally to the background gauge potential $\boldsymbol{A}$ introduced in the preceding section. We now formalise this physical picture.

Holomorphic Gauge Theory. Consider holomorphic gauge theory formulated on the complex vector bundle $\pi\colon \mathbf{E} \to \mathbf{MT}_s$. Let $\mathbf{G}$ be a semisimple Lie group and denote by $\mathfrak{g}$ its complexified Lie algebra. We assume the fibres of $\mathbf{E}$ are isomorphic to $\mathfrak{g}$.

The classical vacuum of the gauge theory is represented by the canonical holomorphic structure on $\mathbf{E}$ induced by the Dolbeault operator $\overline{\partial}^{\mathbf{E}}$. Nontrivial physical configurations correspond to pseudoholomorphic structures on $\mathbf{E}$, parameterised by a partial connection.

To define a $(0,1)$-connection form $\boldsymbol{A}$, we recall two facts. First, since $\mathfrak{g}$ is semisimple, the adjoint representation yields an isomorphism $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}_{\mathbf{C}}(\mathfrak{g})$. Hence $\boldsymbol{A}$ may be taken to be $\mathfrak{g}$-valued. Second, the Picard group of the bosonic minitwistor space $\mathbf{MT}$ satisfies $\mathrm{Pic}(\mathbf{MT}) \cong \mathbf{Z}\oplus\mathbf{Z}$, so any differential form valued in the natural homogeneous bundle of $\mathbf{MT}$ is characterised by a bidegree $\beta = (\Delta_1, \Delta_2)$.

Combining these observations, the gauge potential on $\mathbf{E}$ can be parameterised by a $(0,1)$-connection form:

$$\boldsymbol{A} \;\in\; \Omega^{0,1}\big(\mathbf{MT}_s;\; \mathcal{O}(\Delta_1,\Delta_2)\big)\otimes\mathfrak{g}. \tag{662}$$

Induced Potential. Let $\mathcal{L}=\mathcal{L}(X,\theta)$ be a minitwistor line representing the classical configuration of the string associated to the moduli $(X,\theta)\in\mathscr{M}_c$. If $\Delta_1+\Delta_2=0$, then the pullback of $\boldsymbol{A}$ to $\mathcal{L}$ via the restriction homomorphism satisfies

$$\boldsymbol{A}\big|_{\mathcal{L}} \;\in\; \Omega^{0,1}(\mathcal{L})\otimes\mathfrak{g}. \tag{663}$$

Thus $\boldsymbol{A}|_{\mathcal{L}}$ is a genuine $(0,1)$-form on the string $\mathcal{L}$, rather than a section of a nontrivial line bundle. From now on, we restrict attention to parameterisations of the gauge potential with bidegree obeying $\Delta_1+\Delta_2=0$, and we define $\Delta\coloneqq\Delta_1$ as the *conformal weight* assigned to the background field $\boldsymbol{A}$. Under these conditions, $\boldsymbol{A}|_{\mathcal{L}}$ is the gauge potential induced on the string $\mathcal{L}$.

Using the first-kind parameterisation (Eq. (630)), the restriction can be written as

$$\boldsymbol{A}\big|_{\mathcal{L}(X,\theta)}\big(\lambda^A\big) \;=\; \boldsymbol{A}\big(\mathsf{W}^I(\lambda^A;X,\theta)\big). \tag{664}$$

Physically, $\boldsymbol{A}|_{\mathcal{L}}$ is the gauge field seen by the string propagating on $\mathbf{MT}_s$.

Celestial Fermions. To couple the string to the background gauge field, we introduce a fermionic matter system on the worldsheet. We realise this system by spinor fields supported on the minitwistor line $\mathcal{L}(X,\theta)$.

On a compact complex manifold $\mathbf{S}$, Atiyah [92, Prop. 3.2] proved that spin structures are in one-to-one correspondence with isomorphism classes of holomorphic line bundles $\mathtt{L}$ satisfying $\mathtt{L}^2\cong\mathtt{K}_{\mathbf{S}}$, where $\mathtt{K}_{\mathbf{S}}$ denotes the canonical line bundle of $\mathbf{S}$. By a slight abuse of notation, we write a choice of such a "square root" simply as $\sqrt{\mathtt{K}_{\mathbf{S}}}$. Using the theory of Leites [21], this statement extends to compact complex supermanifolds; see also Giddings and Nelson [93]. Accordingly, let $\mathtt{K}\cong\mathcal{O}(-2)$ denote the canonical line bundle of the celestial supersphere $\mathcal{CS}_s$.

Now pull back $\mathbf{E}$ to the minitwistor line $\mathcal{L}$ via the restriction homomorphism and denote the restricted bundle by $\mathtt{E}\coloneqq\mathbf{E}|_{\mathcal{L}}$. In the geometric formulation of gauge theory, matter fields are represented by sections of vector bundles associated to $\mathtt{E}$ on which the worldsheet spinors are valued.

So, let V be a complex vector space and let $\mathcal{R}\colon\mathfrak{g}\to\mathfrak{gl}_{\mathbf{C}}(V)$ be a representation. Define the left action ϕ of $\mathfrak{g}$ on $\mathtt{E}\times V$ by:

$$\phi\colon\;\mathfrak{g}\;\longrightarrow\;\mathrm{Aut}_{\mathbf{C}}(\mathtt{E}\times V),\qquad \phi_g(e,v)\;\coloneqq\;\big(\mathrm{ad}_g(e),\mathcal{R}_g(v)\big), \tag{665}$$

for all $g \in \mathfrak{g}$ and $(e,v) \in \mathtt{E}\times V$. Using this action, form the quotient

$$\mathtt{F} := (E\times V)/\mathfrak{g}, \tag{666}$$

and define the surjection $\pi' \colon \mathtt{F} \to \mathcal{L}$ by $\pi'(\mathfrak{g}\cdot(e,v)) := \pi(e)$, for all $e \in \mathtt{E}$ and $v \in V$. Then $\mathtt{F} \xrightarrow{\pi'} \mathcal{L}$ is the vector bundle associated to $\mathtt{E}$ with typical fibre isomorphic to V. We denote its dual bundle by $\mathtt{F}^*$.

Therefore, the matter content of the worldsheet CFT consists of a pair of spinor fields

$$\rho \in \Gamma\big(\mathcal{L};\ \sqrt{\mathtt{K}}\otimes\mathtt{F}\big), \qquad \rho^* \in \Gamma\big(\mathcal{L};\ \sqrt{\mathtt{K}}\otimes\mathtt{F}^*\big). \tag{667}$$

Dynamics. Let $\boldsymbol{a} \in \Omega^{0,1}\big(\mathcal{L};\ \mathfrak{gl}_{\mathbf{C}}(V)\big)$ denote the induced gauge potential on $\mathcal{L}$ acting on the representation space V of the matter sector:

$$\boldsymbol{a} := \mathcal{R}\circ\boldsymbol{A}\big|_{\mathcal{L}}. \tag{668}$$

In the first-kind parameterisation of the string, $\boldsymbol{a}$ is given by

$$\boldsymbol{a}(\lambda^A) \;=\; \mathcal{R}\big[\boldsymbol{A}\big(\mathsf{W}^I(\lambda^A)\big)\big]. \tag{669}$$

In addition, let $\langle\cdot|\cdot\rangle \colon \mathtt{F}^*\otimes\mathtt{F} \to \mathcal{O}_{\mathbf{CP}^1}$ be the canonical pairing. Therefore, we take the dynamics of the matter CFT to be governed by the action:

$$\mathcal{S}_{\mathrm{CFT}}[\Delta,\rho,\rho^*|\boldsymbol{A};X,\theta] \;:=\; \int_{\mathcal{L}(X,\theta)} D\lambda\wedge\big\langle\rho^*\big|\big(\overline{\partial}_\lambda + \boldsymbol{a}(\lambda^A)\big)\rho\big\rangle\,. \tag{670}$$

Here $\overline{\partial}_\lambda$ denotes the CR operator acting on the λ-fibres. We have written the action in the λ-coordinates because the first-kind parameterisation of the string is more natural from the target-space perspective. In this parameterisation, the spinor components of $\mathsf{W}^I(\lambda^A)$ equal λ^A. It is straightforward to reformulate the action in the σ-coordinates using the second-kind parameterisation $\mathsf{Y}^I(\sigma^B)$.

Therefore, the kinetic action is

$$\mathcal{S}_{\mathrm{K}}[\rho,\rho^*] \;=\; \int_{\mathcal{CS}} D\lambda\wedge\big\langle\rho^*\big|\overline{\partial}_\lambda\rho\big\rangle\,. \tag{671}$$

Similarly, the interaction contribution is

$$\mathcal{U}[\Delta,\rho,\rho^*|\boldsymbol{A};X,\theta] \;=\; \int_{\mathcal{L}(X,\theta)} D\lambda\wedge\big\langle\rho^*\big|\boldsymbol{a}(\lambda^A)\rho\big\rangle\,. \tag{672}$$

Observe that the dependence on the evaluation maps contained in the multiplet Δ enters through the embedding map $\mathsf{W}^I(\lambda^A)$ used to define $\boldsymbol{a}(\lambda^A)$ in Eq. (669).

A semiclassical description applies when some degrees of freedom behave classically while others require a quantum treatment. A familiar example is molecular quantum mechanics. There, the centre of mass of the heavy nuclei and the environmental degrees of freedom follow classical mechanics. The lighter, faster electrons require a quantum description.

In our setting, the embedding of the celestial sphere into minitwistor superspace plays the role of the classical centre-of-mass variables. This embedding is defined by evaluation maps and their incidence relations. Similarly, the gauge potential $\boldsymbol{A}$ corresponds to the environmental degrees of freedom and is also treated as classical. By contrast, the worldsheet fermions are intrinsically quantum, and their worldsheet CFT couples minimally to the external classical "bath."

The aim of this subsection is to give a mathematical formulation of this picture.

Notation. Since we employ the path-integral formalism to analyse the semiclassical theory, we must distinguish field variables from classical solutions unambiguously. We adopt the following convention: undecorated symbols denote dynamical variables, while classical solutions carry a tilde.

For example, $\Pi_{\dot{A}}(\sigma^B)$ and $\kappa^\alpha(\sigma^B)$ denote the dynamical variables that define the second-kind evaluation maps. The corresponding classical solutions are $\widetilde{\Pi}_{\dot{A}}(\sigma^B; X, \theta)$ and $\tilde{\kappa}^\alpha(\sigma^B; X, \theta)$. The embedding map that describes the classical configuration $\mathcal{L}(X, \theta)$ in the λ-coordinates is given by[29]:

$$\widetilde{\mathsf{W}}^I\big(\lambda^A; X, \theta\big) = \big(\lambda^A, \lambda^A X_{A\dot{A}}, \lambda^A \theta^\alpha_A\big). \tag{673}$$

Effective Action. We take the dynamics of the minitwistor string propagating on the background gauge potential $\boldsymbol{A}$ to be governed by the action:

$$\mathcal{S}_{\mathrm{I}}[\Delta, \rho, \rho^* | \boldsymbol{A}; X, \theta] \;=\; \mathcal{S}_0[\Delta | X, \theta] + \mathcal{S}_{\mathrm{CFT}}[\Delta, \rho, \rho^* | \boldsymbol{A}; X, \theta]. \tag{674}$$

Here we denote the action by the subscript I, indicating that the sigma-model interacts with the background field $\boldsymbol{A}$. In the next subsection we compute correlators of vertex operators after setting $\boldsymbol{A} = 0$. This choice removes background contributions and isolates the interactions that arise solely from worldsheet insertions.

The onshell effective action describing the worldsheet fermions ρ and ρ^* interacting with a classical background is defined by

$$\mathcal{I}[\rho, \rho^* | \boldsymbol{A}; X, \theta] \;\coloneqq\; \mathcal{S}_{\mathrm{I}}[\Delta, \rho, \rho^* | \boldsymbol{A}; X, \theta]\Big|_{\frac{\delta \mathcal{S}}{\delta \Delta} = 0}. \tag{675}$$

[29] An analogous expression holds for the embedding map $\widetilde{\mathsf{Y}}^I$ in the σ-coordinates; see Eqs. (628) and (631).

The effective action is obtained by substituting the classical solutions that parameterise the string $\mathcal{L}(X,\theta)$ into $\mathcal{S}_{\mathrm{I}}$. To write $\mathcal{I}$ explicitly, let

$$\widetilde{\boldsymbol{a}} \;\in\; \Omega^{0,1}\big(\mathcal{L}(X,\theta);\; \mathrm{GL}_{\mathbf{C}}(V)\big) \tag{676}$$

be the induced potential evaluated at the classical solution:

$$\widetilde{\boldsymbol{a}}(\lambda^A;X,\theta) \coloneqq \mathcal{R}\big[\boldsymbol{A}\big(\widetilde{\mathsf{W}}^I(\lambda^A;X,\theta)\big)\big]. \tag{677}$$

Then the onshell effective action becomes:

$$\mathcal{I}[\rho,\rho^*|\boldsymbol{A};X,\theta] \;=\; \int_{\mathcal{CS}} D\lambda\wedge\big\langle\rho^*\big|\big(\overline{\partial}_\lambda+\widetilde{\boldsymbol{a}}(\lambda^A;X,\theta)\big)\rho\big\rangle\,. \tag{678}$$

Saddle-point Approximation. The idea behind the saddle-point approximation is as follows. Consider the semiclassical limit $b\to 0$ of a path integral. The integral runs over the second-kind evaluation maps $\Pi_{\dot{A}},\kappa^\alpha$ and the Lagrange multiplier e_α, and is weighted by $\exp(-\mathcal{S}_0)$. In this limit the integral is dominated by the saddle point satisfying $\delta\mathcal{S}_0=0$.

At that saddle point, the classical equations of motion, which yield the minitwistor incidence relations that define the line $\mathcal{L}(X,\theta)$, are imposed on the observables appearing in the integrand. We now show how this picture is implemented mathematically.

We denote the Feynman “measure” by

$$[d\Delta] \;\coloneqq\; [d\Pi\, d\kappa\, de], \tag{679}$$

and define the normalisation factor

$$\mathcal{N}_0(X,\theta) \;\coloneqq\; \int [d\Delta]\; e^{-\mathcal{S}_0[\Delta|X,\theta]}. \tag{680}$$

Let $\mathrm{F}[\mathsf{W}^I(\lambda^A)]$ be a c-number functional representing an observable that depends only on the string parameterisation. For simplicity, we use the λ-coordinates because the first-kind parameterisation $\lambda^A\mapsto\mathsf{W}^I(\lambda^A)$ is more natural from the target-space perspective, as noted above.

Applying the saddle-point approximation to the Euclidean path integral (see Zinn-Justin [94, Ch. 5, Sec. 3]) and integrating over $\Pi_{\dot{A}},\kappa^\alpha$ and e_α, we obtain:

$$\lim_{b\to 0^+}\frac{1}{\mathcal{N}_0(X,\theta)}\int [d\Delta]\; e^{-\mathcal{S}_{\mathrm{I}}[\Delta,\rho,\rho^*|\boldsymbol{A};X,\theta]}\;\mathrm{F}\big[\mathsf{W}^I(\lambda^A)\big] \;=\; e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{A};X,\theta]}\;\mathrm{F}\big[\widetilde{\mathsf{W}}^I(\lambda^A;X,\theta)\big]. \tag{681}$$

Hence, Eq. (681) formalises our intuition. In the limit $b\to 0$, the correlation functions are dominated by the equations of motion. These equations impose the restriction homomorphism onto the minitwistor line $\mathcal{L}(X,\theta)$, which describes the string's classical configuration.

Correlation Functions. What is the physical meaning of the right-hand side of Eq. (681)? It evaluates the observable $\mathrm{F}[\mathsf{W}^I(\lambda^A)]$ on the classical string configuration represented by the minitwistor line $\mathcal{L}(X,\theta)$. Note that the result is weighted by the inverse of the exponentiated onshell effective action, $e^{-\mathcal{I}}$. Hence, the semiclassical expectation value of $\mathrm{F}[\mathsf{W}^I(\lambda^A)]$ is obtained by averaging the right-hand side of Eq. (681) over all allowed classical string configurations.

However, because the action $\mathcal{S}_0$ is first-order in the field variables $\Pi_{\dot{A}}, \kappa^\alpha$, each point $(X,\theta) \in \mathscr{M}_c$, which belongs to the classical moduli superspace of the string, completely specifies a classical *state*. Thus we may define

$$d\boldsymbol{v}[\rho,\rho^*;X,\theta] \;\coloneqq\; e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{A};X,\theta]}\, D^{3|8}\mathsf{X}\,[d\rho\, d\rho^*] \tag{682}$$

as a measure on the system's phase space $\varGamma_{\boldsymbol{A}}$. The measure space $(\varGamma_{\boldsymbol{A}}, d\boldsymbol{v})$ can then be identified with the *semiclassical statistical ensemble* of a minitwistor string interacting with the background gauge potential $\boldsymbol{A}$.

It follows that the semiclassical expectation value of the observable $\mathrm{F}[\mathsf{W}^I(\lambda^A)]$ in the celestial CFT defined by our minitwistor string is obtained by integrating the right-hand side of Eq. (681) over the fermions ρ, ρ^* and over the moduli superspace:

$$\lim_{b\to 0} \langle \mathscr{F}[\mathsf{W}^I] \rangle^{\boldsymbol{A}}_{\mathcal{CS}} \;\coloneqq\; \frac{1}{\mathcal{N}_{\mathrm{CFT}}} \int_{\mathscr{M}_c} D^{3|8}\mathsf{X} \int [d\rho\, d\rho^*]\, e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{A};X,\theta]}\, \mathrm{F}\big[\widetilde{\mathsf{W}}^I(\lambda^A;X,\theta)\big]. \tag{683}$$

Here, $\mathscr{F}[\mathsf{W}^I]$ denotes the quantum operator corresponding to the classical observable $\mathrm{F}[\mathsf{W}^I]$. The normalisation factor coming from the worldsheet fermions is

$$\mathcal{N}_{\mathrm{CFT}} \;\coloneqq\; \int [d\rho\, d\rho^*]\, e^{-\mathcal{S}_{\mathrm{K}}[\rho,\rho^*]}. \tag{684}$$

The semiclassical correlation function of the worldsheet CFT is defined by

$$\lim_{b\to 0} \langle \mathscr{F}[\mathsf{W}^I] \rangle^{\boldsymbol{A}}_{\mathrm{WS}(X,\theta)} \;\coloneqq\; \frac{1}{\mathcal{N}(X,\theta)} \int [d\Delta\, d\rho\, d\rho^*]\, e^{-\mathcal{S}_{\mathrm{I}}[\Delta,\rho,\rho^*|\boldsymbol{A};X,\theta]}\, \mathrm{F}[\mathsf{W}^I(\lambda^A)], \tag{685}$$

where the normalisation factor is

$$\mathcal{N}(X,\theta) \;\coloneqq\; \int [d\Delta\, d\rho\, d\rho^*]\, e^{-\mathcal{S}_0[\Delta|X,\theta]-\mathcal{S}_{\mathrm{K}}[\rho,\rho^*]}. \tag{686}$$

When the saddle-point approximation is invoked, Eq. (685) yields:

$$\lim_{b\to 0} \langle \mathscr{F}[\mathsf{W}^I] \rangle^{\boldsymbol{A}}_{\mathrm{WS}(X,\theta)} \;=\; \frac{1}{\mathcal{N}_{\mathrm{CFT}}} \int [d\rho\, d\rho^*]\, e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{A};X,\theta]}\, \mathrm{F}\big[\widetilde{\mathsf{W}}^I(\lambda^A;X,\theta)\big]. \tag{687}$$

Therefore, the semiclassical correlation functions of the celestial CFT induced by the minitwistor string theory are given by an integral over the classical moduli superspace $\mathscr{M}_c$ of the worldsheet

CFT correlators:

$$\lim_{b\to 0}\big\langle \mathscr{F}[\mathsf{W}^I]\big\rangle^{\boldsymbol{A}}_{\mathcal{CS}} \;=\; \lim_{b\to 0}\int_{\mathscr{M}_c} D^{3|8}\mathsf{X}\;\big\langle \mathscr{F}[\mathsf{W}^I]\big\rangle^{\boldsymbol{A}}_{\mathrm{WS}(X,\theta)}\,. \tag{688}$$

Substituting Eq. (685) into this expression gives the full semiclassical correlator of the celestial CFT:

$$\lim_{b\to 0}\big\langle \mathscr{F}[\mathsf{W}^I]\big\rangle^{\boldsymbol{A}}_{\mathcal{CS}} \;\coloneqq\; \lim_{b\to 0}\int_{\mathscr{M}_c}\frac{D^{3|8}\mathsf{X}}{\mathcal{N}(X,\theta)}\int [d\Delta\, d\rho\, d\rho^*]\; e^{-\mathcal{S}_{\mathrm{I}}[\Delta,\rho,\rho^*|\boldsymbol{A};X,\theta]}\;\mathrm{F}[\mathsf{W}^I(\lambda^A)]. \tag{689}$$

Partition Function. Finally, the semiclassical partition function of a minitwistor string propagating on the classical background gauge potential $\boldsymbol{A}$ is defined by

$$\mathscr{Z}[\boldsymbol{A}] \;=\; \lim_{b\to 0}\,\langle 1\rangle^{\boldsymbol{A}}_{\mathcal{CS}}\,. \tag{690}$$

Using Eq. (685) we obtain:

$$\mathscr{Z}[\boldsymbol{A}] \;\coloneqq\; \frac{1}{\mathcal{N}_{\mathrm{CFT}}}\int_{\mathscr{M}_c} D^{3|8}\mathsf{X}\int [d\rho\, d\rho^*]\; e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{A};X,\theta]}. \tag{691}$$

We will shortly demonstrate that $\mathscr{Z}[\mathbf{A}]$ generates the tree-level MHV leaf-gluon amplitudes. This result will then motivate a generalisation to multi-string configurations reproducing the N^k-MHV amplitudes.

5. *Partition Function and MHV Amplitudes*

We now evaluate the semiclassical partition function $\mathscr{Z}[\boldsymbol{A}]$ by integrating over the worldsheet fermions ρ and $\rho^\dagger$. We use the chiral determinant method for this functional integral. For an analytic discussion aimed at string theorists, see Section 7 of Verlinde and Verlinde [95]. For a geometric perspective, see Section 3 of Alvarez and Windey [96].

Applying this method to the action $\mathcal{S}_{\mathrm{CFT}}$, which couples the fermions to the background gauge potential $\boldsymbol{A}$, we obtain:

$$\int [d\rho\, d\rho^*]\; e^{-\mathcal{S}_{\mathrm{CFT}}} \;=\; \mathsf{Tr}\log\big(\mathbb{I}_{\mathfrak{g}} + \boldsymbol{A}\,\overline{\partial}^{-1}\big)\big|_{\mathcal{L}(X,\theta)}. \tag{692}$$

Substituting into Eq. (691) yields:

$$\mathscr{Z}[\boldsymbol{A}] \;=\; \int_{\mathscr{M}_c} D^{3|8}\mathsf{X}\;\mathsf{Tr}\,\log\big(\mathbb{I}_{\mathfrak{g}} + \boldsymbol{A}\,\overline{\partial}^{-1}\big)\big|_{\mathcal{L}(X,\theta)}. \tag{693}$$

Next, we expand the integrand using Quillen's determinant line bundle. As in Subsection 3.3 of Mason [97], one finds[30]:

$$\mathsf{Tr}\,\log\big(\mathbb{I}_{\mathfrak{g}} + \boldsymbol{A}\,\overline{\partial}^{-1}\big)\big|_{\mathcal{L}(X,\theta)} \;=\; \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\,\mathsf{Tr}\int_{\mathbf{L}^n}\bigwedge_{i=1}^{n}\frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}}\wedge \boldsymbol{A}\big|_{\mathcal{L}(X,\theta)}\big(\lambda_i^A\big), \tag{694}$$

[30] For the basic theory, see Quillen [98]. A string theory perspective appears in Freed [99]. For a hands-on review with computational examples, consult Subsection 6.3 of Nair [100].

where $\mathbf{L}^n \coloneqq \bigtimes^n \mathbf{CP}^1$.

We now apply the $\mathcal{MT}$-transform to expand the gauge potential $\boldsymbol{A}(\mathsf{W}^I)$ in terms of the minitwistor superwavefunctions Ψ_Δ:

$$\boldsymbol{A}(\mathsf{W}^I) \;=\; \int_{\mathbf{MT}^*_s} \Psi_\Delta(\mathsf{W}^I;\mathsf{Z}'^I)\,\widetilde{\alpha}^{\Delta,\mathsf{a}}(\mathsf{Z}'^I)\,\mathsf{T}^{\mathsf{a}} \wedge D^{2|4}\mathsf{Z}'. \tag{695}$$

Here, we adopt DeWitt notation for the conformal weight Δ (as defined in Section IV). The new mode coefficients relate to those in the previous section by $\widetilde{\alpha}^{\Delta,\mathsf{a}} = 2\pi i\,\alpha^{\Delta,\mathsf{a}}$. This choice of normalisation makes it more convenient to insert into the powerseries representation of Quillen's determinant.

Inserting the expansion of $\boldsymbol{A}$ into Eq. (694) and rearranging the integrals via Fubini's theorem, we find:

$$\mathsf{Tr}\,\log\left(\mathbb{I}_{\mathfrak{g}} + \boldsymbol{A}\,\overline{\partial}^{-1}\right)\big|_{\mathcal{L}(X,\theta)} \;=\; \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\int_{\mathbf{X}^*_n}\bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}'_i \wedge \widetilde{\alpha}^{\Delta_i,\mathsf{a}_i}(\mathsf{Z}'^I_i) \tag{696}$$

$$\mathsf{Tr}_{\mathfrak{g}}\int_{\mathbf{L}^n}\bigwedge_{j=1}^{n}\frac{D\sigma_j}{\lambda_j\cdot\lambda_{j+1}}\mathsf{T}^{\mathsf{a}_j}\wedge\Psi_{\Delta_j}\Big|_{\mathcal{L}(X,\theta)}(\lambda^A_j;\mathsf{Z}'^I_j), \tag{697}$$

where $\mathbf{X}^*_n \coloneqq \bigtimes^n \mathbf{MT}^*_s$. Applying the celestial BMSW identity to this expansion leads to:

$$\mathsf{Tr}\,\log\left(\mathbb{I}_{\mathfrak{g}} + \boldsymbol{A}\,\overline{\partial}^{-1}\right)\big|_{\mathcal{L}(X,\theta)} \tag{698}$$

$$= \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\int_{\mathbf{X}^*_n}\mathsf{Tr}_{\mathfrak{g}}\bigwedge_{i=1}^{n}\frac{\mathcal{C}(\Delta_i)}{\langle z'_i|X|\bar{z}'_i]^{\Delta_i}}\,e^{i\langle z'_i|\theta\cdot\eta_i\rangle}\,\frac{\mathsf{T}^{\mathsf{a}_i}}{z'_i\cdot z'_{i+1}}\widetilde{\alpha}^{\Delta_i,\mathsf{a}_i}(\mathsf{Z}'^I_i)\wedge D^{2|4}\mathsf{Z}'_i. \tag{699}$$

Substituting this result into Eq. (693) yields the final form of the semiclassical partition function:

$$\mathscr{Z}[\boldsymbol{A}] = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\int_{\mathbf{X}^*_n}\int_{\mathscr{M}_c} D^{3|8}\mathsf{X}\;\mathsf{Tr}_{\mathfrak{g}}\bigwedge_{i=1}^{n}\frac{\mathcal{C}(\Delta_i)}{\langle z'_i|X|\bar{z}'_i]^{\Delta_i}}\,e^{i\langle z'_i|\theta\cdot\eta_i\rangle}\,\frac{\mathsf{T}^{\mathsf{a}_i}}{z'_i\cdot z'_{i+1}}\widetilde{\alpha}^{\Delta_i,\mathsf{a}_i}(\mathsf{Z}'^I_i)\wedge D^{2|4}\mathsf{Z}'_i. \tag{700}$$

MHV leaf amplitudes. Having obtained an explicit form of the semiclassical partition function $\mathscr{Z}$, we now show that it generates leaf-gluon amplitudes in MHV configurations.

Consider an n-gluon MHV scattering process with celestial scaling dimensions h_i. Let $\{\mathsf{Z}^I_i\} \subset \mathbf{MT}^*_s$ denote the insertion points. We functionally differentiate with respect to the mode functions $\widetilde{\alpha}^{2h_i,\mathsf{a}_i}(\mathsf{Z}^I_i)$ and then set $\boldsymbol{A}=0$. This gives:

$$\prod_{i=1}^{n}\frac{\delta}{\delta\widetilde{\alpha}^{2h_i,\mathsf{a}_i}(\mathsf{Z}^I_i)}\,\mathscr{Z}_{\text{sc}}\bigg|_{\boldsymbol{A}=0} \;=\; \frac{(-1)^{n-1}}{n}\,\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_n(\mathsf{Z}^I_i). \tag{701}$$

Hence, $\mathscr{Z}$ serves as the generating functional for MHV leaf-gluon amplitudes in $\mathcal{N}=4$ SYM at tree-level.

Conclusion. We showed that the semiclassical partition function $\mathscr{Z}[\boldsymbol{A}]$ of the minitwistor sigma-model, coupled to a background gauge potential $\boldsymbol{A}$, serves as a generating functional for the tree-level MHV leaf-gluon amplitudes. The derivation proceeded by expanding $\mathscr{Z}[\boldsymbol{A}]$ in the Fourier modes $\tilde{\alpha}^{\Delta,\mathsf{a}}$ that parameterise the classical configuration of $\boldsymbol{A}$; by functionally differentiating with respect to those modes; and by finally evaluating the result at $\boldsymbol{A} = 0$.

This result confirmed the physical expectation that the holomorphic gauge field theory on minitwistor superspace (which reproduces the leaf-gluon amplitudes as minitwistor Wilson lines) is the string-field limit of the semiclassical string theory considered here.

6. *Vertex Operators*

The final step in our presentation of the semiclassical system with a single minitwistor string is the construction of the vertex operators $\mathcal{V}^{\mathsf{a}}_{\Delta}$. We set the background gauge potential $\boldsymbol{A} = 0$ to isolate interactions that arise solely from worldsheet insertions. In this trivial background, the leading-trace (large-N_c) semiclassical celestial correlators of $\mathcal{V}^{\mathsf{a}}_{\Delta}$ reproduce the tree-level MHV leaf amplitudes for gluons. Therefore, the semiclassical minitwistor string theory provides a holographic dual to the tree-level MHV gluonic sector of maximally supersymmetric Yang-Mills theory.

In Subsection V C, we present a more detailed discussion of vertex operators in minitwistor string theory and of their algebraic structure. We show that the celestial gluon operators close on the S-algebra, which is a necessary condition for any candidate celestial CFT dual to flat-space gauge theory. The aim of the present section is to introduce the essential concepts involved in the construction of the operator $\mathcal{V}^{\mathsf{a}}_{\Delta}$ in the simpler setting of a single-string system, emphasising physical intuition rather than formal completeness.

Motivation. To motivate the physics, we rewrite the worldsheet CFT action $\mathcal{S}_{\mathrm{CFT}}$ in component form. For this purpose, let $N_c := \dim_{\mathbf{C}}(\mathfrak{g})$ denote the complex dimension of the gauge Lie algebra, and let $r, s = 1, \ldots, N_c$ index a coordinate basis of the representation space V. Introduce a frame field (e_r) trivialising the vector bundle $\mathtt{F} \to \mathcal{L}$ and denote by (e^*_r) the dual frame trivialising $\mathtt{F}^* \to \mathcal{L}$.

The worldsheet fermions ρ and ρ^* decompose in these frames as

$$\rho = \rho^r \otimes e_r, \qquad \rho^* = \bar{\rho}^r \otimes e^*_r, \tag{702}$$

where the coefficients $\rho^r, \bar{\rho}^r$ are $(0,1)$-forms on $\mathcal{L}$.

Let $\{\mathsf{T}^{\mathsf{a}}\}$ be a basis of the Lie algebra $\mathfrak{g}$; the normalisation of this basis will be specified below.

The background gauge potential $\boldsymbol{A}$ on $\mathbf{E} \to \mathbf{MT}_s$ decomposes as

$$\boldsymbol{A}\big(\mathsf{W}^I\big) = A^{\mathsf{a}}\big(\mathsf{W}^I\big) \otimes \mathsf{T}^{\mathsf{a}}, \tag{703}$$

with coefficients $A^{\mathsf{a}} \in \Omega^{0,1}(\mathbf{MT}_s)$.

Next, recall that $\mathcal{R}\colon \mathfrak{g} \to \mathrm{GL}_{\mathbf{C}}(V)$ is the representation of the gauge algebra on V, the space in which the matter fields ρ and ρ^* take values. The induced potential $\boldsymbol{a} \in \Omega^{0,1}(\mathcal{L}; \mathrm{GL}_{\mathbf{C}}(V))$ on the line $\mathcal{L}$, acting on the matter sector, is therefore:

$$\boldsymbol{a}(\lambda^A) \;=\; A^{\mathsf{a}}\big(\mathsf{W}^I(\lambda^A)\big) \otimes \mathcal{R}[\mathsf{T}^{\mathsf{a}}], \tag{704}$$

where $\{\mathcal{R}[\mathsf{T}^{\mathsf{a}_i}]\}$ yields a basis for $\mathrm{GL}_{\mathbf{C}}(V)$. Finally, decomposing $\mathcal{R}[\mathsf{T}^{\mathsf{a}}]$ in the frames (e_r) and (e_r^*) gives the matrix elements

$$\mathrm{T}^{\mathsf{a}}_{rs} \;\coloneqq\; \langle e_r^* | \mathcal{R}[\mathsf{T}^{\mathsf{a}}] e_s \rangle, \tag{705}$$

which are the components of the Lie-algebra generators in the chosen representation and frame.

Accordingly, substituting the decompositions given by Eqs. (702), (704) and (705) into the action $\mathcal{S}_{\mathrm{CFT}}$, we obtain:

$$\mathcal{S}_{\mathrm{CFT}}[\Delta, \rho, \rho^* | \boldsymbol{A}; X, \theta] \;=\; \int_{\mathcal{L}(X,\theta)} D\lambda \wedge \bar{\rho}^r \left(\delta_{rs}\, \overline{\partial}_\lambda + A^{\mathsf{a}}\big(\mathsf{W}^I(\lambda^A)\big)\, \mathrm{T}^{\mathsf{a}}_{rs}\right) \rho^s. \tag{706}$$

From the second term in Eq. (706), we identify the *classical worldsheet current*

$$\jmath^{\mathsf{a}} \;\in\; \Gamma\big(\mathcal{L};\, \mathcal{O}(-2) \otimes \mathfrak{g}\big), \qquad \jmath^{\mathsf{a}} \;\coloneqq\; \bar{\rho}^r\, \mathrm{T}^{\mathsf{a}}_{rs}\, \rho^s. \tag{707}$$

Consequently, the action contribution arising from the coupling of the matter fields to the gauge potential may be written as

$$\mathcal{U}[A^{\mathsf{a}}] \;=\; \int_{\mathcal{CS}} D\lambda \wedge \mathrm{A}^{\mathsf{a}}\big(\mathsf{W}^I(\lambda^A)\big)\, \jmath^{\mathsf{a}}(\lambda^A). \tag{708}$$

This observation motivates the definition of *minitwistor-string vertex operators* supported on the celestial sphere $\mathcal{CS}$. For any $(0,1)$-form

$$\phi \;\in\; \Omega^{0,1}\big(\mathbf{MT}_s;\, \mathcal{O}(\Delta, -\Delta)\big), \tag{709}$$

we define

$$\mathcal{V}^{\mathsf{a}}[\phi] \;\coloneqq\; \int_{\mathcal{CS}} D\lambda \wedge \phi\big(\mathsf{W}^I(\lambda^A)\big)\, \jmath^{\mathsf{a}}(\lambda^A). \tag{710}$$

Observe that the integral in Eq. (710) is well-defined because, by pulling back the form ϕ to the string $\mathcal{L}$ via the restriction homomorphism, one obtains the projectively invariant $(0,1)$-form $\phi|_{\mathcal{L}} \in \Omega^{0,1}(\mathcal{L})$ given by:

$$\phi\big|_{\mathcal{L}(X,\theta)}\big(\lambda^A\big) \;=\; \phi\big(\mathsf{W}^I(\lambda^A;X,\theta)\big). \tag{711}$$

Wedging this form with the holomorphic measure $D\lambda$ and contracting with the current $\jmath^{\mathsf{a}}(\lambda^A)$ produce a $\mathfrak{g}$-valued top form on $\mathcal{L}$.

Recalling the decomposition of the background field $\boldsymbol{A}$ in terms of the family $\{\Psi_\Delta\}$ of $p=0$ minitwistor superwavefunctions (see Eq. (695)), we propose the following candidate vertex operators that generate leaf-gluon amplitudes:

$$\mathcal{V}^{\mathsf{a}}_\Delta\big(\mathsf{Z}^I\big) \;=\; \int_{\mathcal{CS}} D\lambda \wedge \Psi_\Delta\big(\mathsf{W}^I(\lambda^A);\mathsf{Z}^I\big)\,\jmath^{\mathsf{a}}(\lambda^A). \tag{712}$$

Gauge Group. For concreteness, we take the gauge group $\mathbf{G} = SO(N_c)$, where N_c denotes the number of colours of the gauge theory. We impose the standard gauge-theory normalisations on the generators $\{\mathsf{T}^{\mathsf{a}}\}$:

$$\mathsf{Tr}\big(\mathsf{T}^{\mathsf{a}}\mathsf{T}^{\mathsf{b}}\big) = 2\,\delta^{\mathsf{ab}}, \qquad [\mathsf{T}^{\mathsf{a}},\mathsf{T}^{\mathsf{b}}] = i f^{\mathsf{abc}}\mathsf{T}^{\mathsf{c}}. \tag{713}$$

The matter content of the worldsheet CFT is taken to be N_c independent real fermions ρ^r. These fermions transform in the vector representation of $\mathfrak{g} = \mathfrak{so}(N_c)$. We choose the representation space used to construct the associated vector bundle $\mathtt{F}$ to be $V = \mathfrak{so}(N_c)$ and we take the homomorphism $\mathcal{R}$ to be the adjoint representation.

The reason for this choice of gauge group and representation is practical: in the large-N_c limit, one eliminates unwanted multi-trace contributions that arise from current algebra correlators.

WZNW Current Algebra. On an open neighbourhood $\mathscr{U} \subset \mathcal{CS}$ such that $\lambda^1(z) \neq 0$ for all $z \in \mathscr{U}$, let $\lambda := \lambda^2/\lambda^1$ denote the affine coordinate. In the operator formalism, the quantum fields $\hat{\rho}^r$ that represent the worldsheet fermions obey the fundamental OPEs:

$$\hat{\rho}^r(\lambda)\,\hat{\rho}^s(\lambda') \;\sim\; \frac{\delta^{rs}}{\lambda-\lambda'}. \tag{714}$$

Let J^{a} be the quantum operators representing the worldsheet currents. By the correspondence principle and by the form of the classical currents $\jmath^{\mathsf{a}}$ defined in Eq. (707), J^{a} must be proportional to the normally ordered[31] bilinear $\jmath^{\intercal}\mathrm{T}\jmath$:

$$J^{\mathsf{a}}(\lambda) \;=\; \beta\,\big(\hat{\rho}^{r}\,\mathrm{T}^{\mathsf{a}}_{rs}\,\hat{\rho}^{s}\big)(\lambda). \tag{717}$$

Invoking Wick's theorem, we find that the worldsheet currents satisfy the OPEs:

$$J^{\mathsf{a}}(\lambda)\,J^{\mathsf{b}}(\lambda') \;\sim\; 2\beta^{2}\,\frac{\mathsf{Tr}\big(\mathsf{T}^{\mathsf{a}}\mathsf{T}^{\mathsf{b}}\big)}{(\lambda-\lambda')^{2}} + 2\beta\,\frac{if^{\mathsf{abc}}\,J^{\mathsf{c}}(\lambda')}{\lambda-\lambda'}. \tag{718}$$

Consistency of this OPE with the Ward identity requires $2\beta = 1$. Hence the set $\{J^{\mathsf{a}}\}$ generates the level-one $SO(N_c)$ WZNW current algebra on the celestial sphere $\mathcal{CS}$.

WZNW Correlator. The final ingredient of the worldsheet CFT we require is the correlator. Let $\mathbb{G}[J^{\mathsf{a}}]$ be an observable that is polynomial in the worldsheet currents J^{a}, and denote by $\mathrm{G}[\jmath^{\mathsf{a}}]$ the corresponding c-number functional. Because we introduced the semiclassical string using the path-integral formalism, it is convenient to express the correlator as a functional integral.

Take the functional measure to be given by:

$$[d\rho] \;\coloneqq\; \prod_{r=1}^{N_c}\,[d\rho^{r}]. \tag{719}$$

Since the multiplet $\{\rho^{r}\}$ consists of free fermions, the action is purely kinetic:

$$\mathcal{S}_{\mathrm{K}}[\rho^{r}] \;=\; \int_{\mathcal{CS}} D\lambda\wedge\rho^{r}\,\overline{\partial}_{\lambda}\,\rho^{r}. \tag{720}$$

Hence we define the correlator of $\mathbb{G}$ by:

$$\langle\mathbb{G}[J^{\mathsf{a}}]\rangle_{\mathrm{WZNW}} \;\coloneqq\; \frac{1}{\mathcal{N}_{\rho}}\int[d\rho]\;e^{-\mathcal{S}_{\mathrm{K}}[\rho^{r}]}\;\mathrm{G}[\jmath^{\mathsf{a}}], \tag{721}$$

where the normalisation constant is:

$$\mathcal{N}_{\rho} \;\coloneqq\; \int[d\rho]\;e^{-\mathcal{S}_{\mathrm{K}}[\rho^{r}]}. \tag{722}$$

---

[31] Following Francesco, Mathieu, and Sénéchal [101, Sec. 6], we define normal ordering as follows. Let $\widehat{O}_1(\lambda)$ and $\widehat{O}_2(\lambda)$ be a pair of field operators belonging to the worldsheet CFT. Let $\mathscr{C}(\lambda)$ be a small contour centred at λ. The normally ordered operator product of $\widehat{O}_1$ and $\widehat{O}_2$, evaluated at λ, is the quantum observable defined by:

$$\big(\widehat{O}_1\widehat{O}_2\big)(\lambda) \;\coloneqq\; \oint_{\mathscr{C}(\lambda)}\widehat{O}_1(\sigma)\,\boldsymbol{k}(\sigma;\lambda)\,\widehat{O}_2(\lambda). \tag{715}$$

Here the Green differential $\boldsymbol{k}(\sigma;\lambda)$ is the Cauchy kernel:

$$\boldsymbol{k}(\sigma;\lambda) \;\coloneqq\; \frac{d\sigma}{2\pi i}\,\frac{1}{\sigma-\lambda}. \tag{716}$$

Semiclassical Celestial Correlator. Recall that our goal in this subsection is to isolate the interactions arising from worldsheet insertions of vertex operators. To that end, we set the background gauge potential to $\boldsymbol{A} = 0$. The action of the minitwistor sigma-model then reduces to:

$$\mathcal{S}[\Delta, \rho^r | X, \theta] \; = \; \mathcal{S}_0[\Delta | X, \theta] + \mathcal{S}_{\mathrm{K}}[\rho^r]. \tag{723}$$

Let $\mathbb{F} = \mathbb{F}[\mathsf{W}^I; J^{\mathsf{a}}]$ be an observable that depends on the string parameterisation $\mathsf{W}^I(\lambda^A)$ and on the worldsheet currents J^{a}. We assume that $\mathbb{F}$ is polynomial in the currents J^{a}. Denote by $\mathrm{F}[\mathsf{W}^I; \jmath^{\mathsf{a}}]$ the corresponding classical functional. The semiclassical celestial correlator of $\mathbb{F}$ is:

$$\lim_{b\to 0} \langle \mathbb{F} \rangle_{\mathcal{CS}} \; = \; \lim_{b\to 0} \int_{\mathscr{M}_c} \frac{D^{3|8}\mathsf{X}}{\mathcal{N}_0(X,\theta)} \int [d\Delta \, d\rho] \; e^{-\mathcal{S}[\Delta, \rho^r | X, \theta]} \, \mathrm{F}[\mathsf{W}^I(\lambda^A); \jmath^{\mathsf{a}}]. \tag{724}$$

Applying the saddle-point approximation to the Δ-integral yields the reduced expression:

$$\lim_{b\to 0} \langle \mathbb{F} \rangle \; = \; \frac{1}{\mathcal{N}_\rho} \int_{\mathscr{M}_c} D^{3|8}\mathsf{X} \int [d\rho] \; e^{-\mathcal{S}_{\mathrm{K}}[\rho^r]} \; \mathrm{F}[\widetilde{\mathsf{W}}^I(\lambda^A; X, \theta); \jmath^{\mathsf{a}}]. \tag{725}$$

Here $\widetilde{\mathsf{W}}^I(\lambda^A; X, \theta)$ denotes the classical solution of the minitwistor sigma-model equations of motion; it is given by the evaluation maps of the first kind expressed in the λ-coordinates. The map $\widetilde{\mathsf{W}}^I$ describes the embedding of the worldsheet as the minitwistor line $\mathcal{L}(X,\theta) \subset \mathbf{MT}_s$ corresponding to the moduli $(X,\theta) \in \mathscr{M}_c$.

To simplify Eq. (725), we recast it as an integral of the WZNW correlator $\langle \ldots \rangle_{\mathrm{WZNW}}$ defined in Eq. (721). Define the restriction of the observable $\mathbb{F}$ to the minitwistor line $\mathcal{L}(X,\theta)$ by

$$\mathbb{F}\big|_{\mathcal{L}(X,\theta)}[J^{\mathsf{a}}] \; \coloneqq \; \mathbb{F}[\widetilde{\mathsf{W}}^I(\lambda^A; X, \theta); J^{\mathsf{a}}]. \tag{726}$$

We regard $\mathbb{F}|_{\mathcal{L}}$ in two ways. First, it is a quantum operator that is polynomial in the worldsheet current J^{a}. Second, it is the classical functional obtained by evaluating $\mathbb{F}$ on the string parameterisation $\widetilde{\mathsf{W}}^I(\lambda^A; X, \theta)$.

Accordingly, the semiclassical limit of the correlator becomes the moduli-space integral:

$$\lim_{b\to 0} \langle \mathbb{F}[\mathsf{W}^I; J^{\mathsf{a}}] \rangle \; = \; \int_{\mathscr{M}_C} D^{3|8}\mathsf{X} \; \big\langle \mathbb{F}\big|_{\mathcal{L}(X,\theta)}[J^{\mathsf{a}}] \big\rangle_{\mathrm{WZNW}}. \tag{727}$$

Tree-level MHV Amplitudes. With these preparations, we show that the large-N_c semiclassical limit of celestial correlators of the vertex operators $\mathcal{V}^{\mathsf{a}}_{\Delta}$ reproduces the tree-level MHV leaf-gluon amplitudes.

Let $\mathsf{z}_i = (z_i, \bar{z}_i, \eta_i^\alpha) \in \mathcal{CS}_s$ denote the i-th gluon insertion point on the $\mathcal{N} = 4$ celestial supersphere. Recall that the dual minitwistor superspace $\mathbf{MT}^*_s$ may be regarded as a covering space of $\mathcal{CS}_s$. Hence, for the i-th insertion z_i, we may choose a representative

$$\mathsf{Z}^I_i = \left(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^\alpha \right) \in \mathbf{MT}^*_s. \tag{728}$$

To specify the i-th gluon state in the celestial CFT, let Δ_i denote its conformal weight. The scaling dimension h_i of the i-th gluon is related to the conformal weight and to the helicity by $2h_i + |\eta_i| = \Delta_i$, where $|\eta_i|$ denotes the expectation value of the helicity operator.

Now, consider the n-point correlation function:

$$C_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{Z}_i^I;\Delta_i\big) \;\coloneqq\; \lim_{N\to\infty}\lim_{b\to 0}\left\langle \prod_{i=1}^{n} \mathcal{V}^{\mathsf{a}_i}_{2h_i}\big(\mathsf{Z}_i^I\big)\right\rangle_{\mathcal{CS}}. \tag{729}$$

Substitute Eq. (712), which defines the minitwistor-string vertex operator $\mathcal{V}^{\mathsf{a}}_{\Delta}$, into the correlator above. Pull the superwavefunctions Ψ_Δ outside the WZNW correlator and reorganise the integrals. One obtains:

$$C_n^{\mathsf{a}_1\ldots\mathsf{a}_n} = \lim_{N\to\infty}\int\limits_{\mathscr{M}_c} D^{3|8}\mathsf{X}\int\limits_{\mathbf{L}_n}\bigwedge_{i=1}^{n} D\lambda_i\wedge\Psi_{2h_i}\big(\widetilde{\mathsf{W}}^I(\lambda_i^A;X,\theta);\mathsf{Z}_i^I\big)\quad\left\langle\prod_{j=1}^{n} J^{\mathsf{a}_j}(\lambda_j^A)\right\rangle_{\mathrm{WZNW}}. \tag{730}$$

Composing the superwavefunction with the onshell evaluation map $\widetilde{\mathsf{W}}^I$ yields the pullback of Ψ_{2h_i} to the classical string configuration $\mathcal{L}(X,\theta)$. So,

$$\Psi_{2h_i}\big|_{\mathcal{L}(X,\theta)}\big(\lambda_i^A;\mathsf{Z}_i^I\big) \;=\; \Psi_{2h_i}\big(\widetilde{\mathsf{W}}^I(\lambda_i^A;X,\theta);\mathsf{Z}_i^I\big). \tag{731}$$

We now use the observation of Nair [102] that, in the leading-trace (large-N_c) limit, the current-algebra correlator produces the Parke-Taylor factor:

$$\left\langle\prod_{i=1}^{n} J^{\mathsf{a}_i}(\lambda_i^A)\right\rangle_{\mathrm{WZNW}} \sim \mathsf{Tr}\prod_{i=1}^{n}\frac{\mathsf{T}^{\mathsf{a}_i}}{\lambda_i\cdot\lambda_{i+1}} \qquad (N_c\to\infty), \tag{732}$$

where the product is taken cyclically and $\lambda_i\cdot\lambda_{i+1}$ denotes the natural spinor contraction.

Substituting the pullback (Eq. (731)) and the Parke-Taylor factor (Eq. (732)) into the expression for C_n (Eq. (730)) gives the compact form:

$$C_n^{\mathsf{a}_1\ldots\mathsf{a}_n} = \int\limits_{\mathscr{M}_c} D^{3|8}\mathsf{X}\;\mathsf{Tr}\int\limits_{\mathbf{L}_n}\bigwedge_{i=1}^{n}\frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}}\mathsf{T}^{\mathsf{a}_i}\wedge\Psi_{2h_i}\big|_{\mathcal{L}(X,\theta)}\big(\lambda_i^A;\mathsf{Z}_i^I\big). \tag{733}$$

Applying the celestial BMSW identity, we obtain:

$$C_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{Z}_i^I;\Delta_i\big) \;=\; \int\limits_{\mathscr{M}_c} D^{3|8}\mathsf{X}\;\mathsf{Tr}\bigwedge_{i=1}^{n}\frac{\mathcal{C}(2h_i)}{\langle z_i|X|\bar{z}_i]^{2h_i}}\,e^{i\langle z_i|\theta\cdot\eta_i\rangle}\,\frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}. \tag{734}$$

We recognise this expression as the tree-level MHV leaf-gluon superamplitude for n gluons, $\mathcal{M}_n^{\mathsf{a}_1\ldots\mathsf{a}_n}\big(\mathsf{Z}_i^I\big)$.

In our semiclassical minitwistor-string model for the celestial CFT, the *celestial gluon operator* with conformal weight Δ and helicity state η^α is defined by:

$$\mathcal{G}^{\eta,\mathsf{a}}_{\Delta}(z,\bar{z}) \coloneqq \mathcal{V}^{\mathsf{a}}_{\Delta-|\eta|}\big(z^A,\bar{z}_{\dot{A}},\eta^\alpha\big). \tag{735}$$

Thus Eq. (734) can be written as:

$$\lim_{N_c\to\infty}\lim_{b\to 0}\left\langle\prod_{i=1}^{n}\mathcal{G}^{\eta_i,\mathsf{a}_i}_{\Delta_i}(z_i,\bar{z}_i)\right\rangle_{\mathcal{CS}} = \mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_n\big(\mathsf{Z}^I_i\big). \tag{736}$$

Comment. Combining this conclusion with the localisation theorem gives the physical motivation to generalise the model to a many-body system of N minitwistor strings. In the next subsection, we propose this many-body system as the celestial CFT dual to the tree-level N^k-MHV gluonic subsector whenever $N = 2k+1$. In Subsection V C, we also show that the gluon operators close on the S-algebra that any celestial CFT dual to flat-space gauge theory must satisfy.

B. An N-String System Coupled to a Gauge Background

We now present the central result of this work. In Subsections V B 1 and V B 2, we generalise the semiclassical system studied above, which consisted of a single minitwistor string, to a many-body system of N minitwistor strings interacting with a background gauge potential on $\mathbf{MT}_s$.

Our primary aim is to show in Subsection V B 3 that the semiclassical partition function of this N-string system serves as a generating functional for the tree-level leaf-gluon amplitudes in the N^k-MHV sector of $\mathcal{N}=4$ SYM, with $N = 2k+1$. This realises the interpretation of the holomorphic gauge theory on $\mathbf{MT}_s$ as a “minitwistor string field theory” in the semiclassical regime. Hence we obtain a dynamical formulation of the localisation theorem: the minitwistor lines on which the amplitudes localise are identified with the images of the N strings in the system.

Next, in Subsection V C, we analyse the model's vertex operators and use them to construct celestial gluon operators. We show that the leading-trace semiclassical correlators of these gluon operators reproduce the tree-level N^k-MHV leaf-gluon amplitudes.

Finally, we establish that the OPEs of the gluon operators close on the S-algebra of the celestial CFT. This confirms that, in the semiclassical regime, the minitwistor string theory realises the algebraic structure required of *any* proposed holographic dual to the N^k-MHV gluonic sector of $\mathcal{N}=4$ SYM at tree-level.

1. Classical Theory

We now construct a many-body system of N minitwistor strings coupled to a background gauge potential. Its correlation functions reproduce the tree-level leaf amplitudes for gluons in the semiclassical regime. As a preparation, we briefly review the geometric interpretation of the minitwistor amplitudes obtained in the previous sections.

Consider scattering of n gluons in $\mathcal{N} = 4$ SYM theory. Fix an integer k with $1 \le k \le n-1$ and set $N = 2k+1$. Assume the external gluons form an N^k-MHV configuration, and label the next-to-MHV gluons by $\ell = 1, \ldots, k$.

Geometric Formulation. By the localisation theorem of Section III, the minitwistor amplitude localises on a family of minitwistor lines $\{\mathcal{L}_m\} \subset \mathbf{MT}_s$, where $m = 1, \ldots, N$ indexes each line. The *moduli superspace* of the configuration $\{\mathcal{L}_m\}$ is

$$\mathscr{M}_N \;:=\; \mathbf{H}_s \times \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_k. \tag{737}$$

Here $\mathbf{H}_s$ denotes the complexified $(3|8)$-dimensional anti-de Sitter superspace, and $\mathcal{P}_\ell$ is the parameter space for the ℓ-th next-to-MHV gluon. In split signature, all components of the twistor and minitwistor momenta are real, so $\mathcal{P}_\ell \cong \mathbf{R}^{8|4}$. We now perform an analytic continuation to complex parameter spaces, so $\mathcal{P}_\ell \cong \mathbf{C}^{8|4}$.

Next we recall how the moduli superspace $\mathscr{M}_N$ parameterises the geometry of the line family $\{\mathcal{L}_m\}$. This review will clarify how the N^k-MHV leaf-gluon amplitudes arise from correlators of N-string configurations.

Each parameter space $\mathcal{P}_\ell$ carries global coordinates:

$$\tau_\ell^M \;=\; \big(u_\ell^A,\, v_\ell^B,\, \widetilde{u}_{\ell\dot{A}},\, \widetilde{v}_{\ell\dot{B}},\, \chi_\ell^\alpha\big). \tag{738}$$

Hence the full moduli superspace is charted by:

$$\gamma^Q \;=\; \big(\mathsf{X}^K, \tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_k^{M_k}\big), \tag{739}$$

where $\mathsf{X}^K = (X_{A\dot{A}}, \theta_A^\alpha)$ are the standard coordinates on $\mathbf{H}_s$. We orient $\mathscr{M}_N$ using the Berezin-DeWitt form:

$$\mathcal{D}\gamma \;:=\; D^{3|8}\mathsf{X} \wedge d^{8|4}\tau_1 \wedge d^{8|4}\tau_2 \wedge \cdots \wedge d^{8|4}\tau_k. \tag{740}$$

To each line $\mathcal{L}_m$ we assign *moduli functions*:

$$\mathcal{Q}_m^{A\dot{A}} = \mathcal{Q}_m^{A\dot{A}}\big(\tau_1^{M_1}, \tau_2^{M_2}, \ldots \tau_k^{M_k}\big), \qquad q_m^{\alpha\dot{A}} = q_m^{\alpha\dot{A}}\big(\tau_1^{M_1}, \tau_2^{M_2}, \ldots, \tau_k^{M_k}\big). \tag{741}$$

These live on the product superspace $\bigtimes_{\ell=1}^{k} \mathcal{P}_\ell$. Combining them with $(X_{A\dot{A}}, \theta_A^\alpha)$ yields the *characteristic functions* of the m-th line:

$$Y_m^{A\dot{A}}\big(\gamma^Q\big) \;=\; X^{A\dot{A}} + \mathcal{Q}_m^{A\dot{A}}\big(\tau_\ell^M\big), \qquad \xi_m^{\alpha A}\big(\gamma^Q\big) \;=\; \theta^{\alpha A} + q_m^{\alpha A}\big(\tau_\ell^M\big). \tag{742}$$

We then define the *evaluation maps* on each line[32]:

$$\Phi_{m\dot{A}} \in \Gamma\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(1)\oplus\mathcal{O}(1)\big), \quad \varphi_m^\alpha \in \Lambda[1]\otimes\Gamma\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(1)\big), \tag{743}$$

given by:

$$\Phi_{m\dot{A}}\big(\lambda^A;\gamma^Q\big) \;:=\; \lambda^A\, Y_{mA\dot{A}}\big(\gamma^Q\big) \qquad {\varphi_m}^\alpha\big(\lambda^A;\gamma^Q\big) \;:=\; \lambda^A\, \xi^\alpha_{mA}\big(\gamma^Q\big). \tag{744}$$

Here $[\lambda^A]$ are projective coordinates on $\mathcal{L}_m(\gamma^Q)$.

Finally, the minitwistor line $\mathcal{L}_m(\gamma^Q)$ appears as the locus of points $\mathsf{W}_m^I = (\lambda_m^A, \mu_{m\dot{A}}, \psi_m^\alpha)$ satisfying the *incidence relations*:

$$\lambda_m^A = \lambda^A, \quad \mu_{m\dot{A}} = \Phi_{m\dot{A}}\big(\lambda^B;\gamma^Q\big), \quad \psi_m^\alpha = {\varphi_m}^\alpha\big(\lambda^B;\gamma^Q\big). \tag{745}$$

Dynamical Formulation. As before, we derive the dynamics for an N-string system by posing a variational problem. Its solutions reproduce the evaluation maps (Eq. (744)) that encode the incidence relations (Eq. (744)). To apply the saddle-point approximation in Feynman's path integral, we choose a Lagrangian polynomial in the fields.

Recall that $\Phi_{m\dot{A}}$ and φ_m^α are homogeneous of degree one in λ^A. Any quadratic polynomial in these fields then has degree two. Such a term cannot combine with the holomorphic measure $D\lambda = \langle\lambda d\lambda\rangle$ to form a projectively invariant top-form. So, our strategy is to rewrite the evaluation maps in terms of coordinates on $\mathcal{L}_m(\gamma^Q)$ that carry weight -1 under the rescaling $\lambda^A \mapsto t\,\lambda^A$.

We chart $\mathcal{L}_m$ by the coordinate functions σ^B, which are related to λ^A via the transition map $\lambda^A = \tau^A(\sigma^B)$ defined in Eq. (624). In σ-coordinates we define the evaluation maps

$$\Pi_{m\dot{A}} \;\in\; \Gamma\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big), \quad \kappa_m^\alpha \;\in\; \Lambda[1]\otimes\Gamma\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(-1)\big) \tag{746}$$

specified by the relations:

$$\Pi_{m\dot{A}}\big(\sigma^B;\gamma^Q\big) \;:=\; \Phi_{m\dot{A}}\big(\tau^A(\sigma^B);\gamma^Q\big) \;=\; \frac{\epsilon_1^A\, Y_{mA\dot{A}}\big(\gamma^Q\big)}{\langle\sigma,\iota^2\rangle} - \frac{\epsilon_2^A\, Y_{mA\dot{A}}\big(\gamma^Q\big)}{\langle\sigma,\iota^1\rangle}, \tag{747}$$

$$\kappa_m^\alpha\big(\sigma^B;\gamma^Q\big) \;:=\; \varphi_m^\alpha\big(\tau^A(\sigma^B);\gamma^Q\big) \;=\; \frac{\epsilon_1^A\, \xi^\alpha_{mA}\big(\gamma^Q\big)}{\langle\sigma,\iota^2\rangle} - \frac{\epsilon_2^A\, \xi^\alpha_{mA}\big(\gamma^Q\big)}{\langle\sigma,\iota^1\rangle}. \tag{748}$$

Parameterising by the coordinates σ^B, the incidence relations for the minitwistor line $\mathcal{L}_m\big(\gamma^Q\big)$ read:

$$\lambda_m^A \;=\; \tau^A\big(\sigma^B\big), \quad \mu_{m\dot{A}} \;=\; \Pi_{m\dot{A}}\big(\sigma^B;\gamma^Q\big), \quad \psi_m^\alpha \;=\; \kappa_m^\alpha\big(\sigma^B;\gamma^Q\big). \tag{749}$$

[32] Recall that Λ is the Grassmann algebra associated to the vector superspace $\mathbf{C}^{0|4}$ and $\Lambda[k] := \bigwedge^k \mathbf{C}^{0|4}$.

As above, the new evaluation maps arise as the unique solutions of a system of differential equations. To formulate this system, we define the currents

$$\mathscr{J}_{m\dot{A}} \in \mathscr{D}'_{0,1}\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(-1)\otimes\mathcal{O}(-1)\big), \quad \mathscr{K}^\alpha_m \in \Lambda[1]\otimes\mathscr{D}'_{0,1}\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(-1)\big) \tag{750}$$

with local form:

$$\mathscr{J}_{m\dot{A}}\big(\sigma^B;\gamma^Q\big) \;\coloneqq\; \overline{\delta}\big(\sigma\cdot\iota^2\big)\,\epsilon_1^A\,Y_{mA\dot{A}}\big(\gamma^Q\big) \;-\; \overline{\delta}\big(\sigma\cdot\iota^1\big)\,\epsilon_2^A\,Y_{mA\dot{A}}\big(\gamma^Q\big) \tag{751}$$

$$\mathscr{K}^\alpha_m\big(\sigma^B;\gamma^Q\big) \;\coloneqq\; \overline{\delta}\big(\sigma\cdot\iota^2\big)\,\epsilon_1^A\,\xi^\alpha_{mA}\big(\gamma^Q\big) \;-\; \overline{\delta}\big(\sigma\cdot\iota^1\big)\,\epsilon_2^A\,\xi^\alpha_{mA}\big(\gamma^Q\big). \tag{752}$$

Hence the evaluation maps satisfy the linear PDEs:

$$\frac{1}{2\pi i}\,\overline{\partial}_\sigma\,\Pi_{m\dot{A}}\big(\sigma^B;\gamma^Q\big) \;+\; \mathscr{J}_{m\dot{A}}\big(\sigma^B;\gamma^Q\big) \;=\; 0, \tag{753}$$

$$\frac{1}{2\pi i}\,\overline{\partial}_\sigma\,\kappa^\alpha_m\big(\sigma^B;\gamma^Q\big) \;+\; \mathscr{K}^\alpha_m\big(\sigma^B;\gamma^Q\big) \;=\; 0. \tag{754}$$

On the minitwistor line $\mathcal{L}_m(\gamma^Q)$, the Cauchy-Riemann operator $\overline{\partial}_\sigma$ acts only on the σ-fibres.

By the existence and uniqueness theorem for linear PDEs on compact Riemann surfaces (see § 1.11 of Forster [86]), the maps $\Pi_{m\dot{A}}$ and κ^α_m defined in Eqs. (747) and (748) are the unique solutions to these equations. We now seek an action whose stationarity conditions reproduce them.

Bosonic Sector. For the bosonic sector, we define:

$$\mathcal{S}^N_{\Pi_m}\big(\gamma^Q\big) \;\coloneqq\; \frac{1}{b}\sum_{m=1}^{N}\int_{\mathcal{L}_m(\gamma^Q)} D\sigma\wedge\left(\frac{1}{2\pi i}\,[\Pi_m\,\overline{\partial}_\sigma\,\Pi_m] \;+\; [\Pi_m\,\mathscr{J}_m]\right). \tag{755}$$

Varying $\mathcal{S}^N_{\Pi_m}$ with respect to Π_m and setting the variation to zero immediately yields Eq. (753).

Fermionic Sector; Celestial Supersphere. The fermionic sector requires further attention. The field κ^α_m has Grassmann degree one, while the action must be bosonic. Hence we pair κ^α_m with another field of Grassmann degree 3 and carry out a Berezin integral. However, each minitwistor line $\mathcal{L}_m(\gamma^Q)$ is bosonic. In fact $\mathcal{L}_m(\gamma^Q)\cong\mathbf{CP}^1$. So, to define a Berezin integral on this line, we extend its coordinates σ^B by four fermionic directions χ^β associated with $\mathcal{N}=4$ supersymmetry.

We denote the resulting *celestial supersphere* by $\mathcal{CS}_{s,m}(\gamma^Q)$, the m-th copy of the supersymmetric line over which the minitwistor amplitude localises, parametrised by the moduli point $\gamma^Q\in\mathscr{M}_N$. We introduce the $\mathbf{Z}_2$-graded coordinate map:

$$\mathsf{s} \coloneqq (\sigma^B,\chi^\beta)\colon\; \mathcal{CS}_{s,m}\big(\gamma^Q\big) \;\longrightarrow\; \mathbf{CP}^1\times\mathbf{C}^{0|4}. \tag{756}$$

The natural orientation on the celestial supersphere is given by the volume superform:

$$D^{1|4}\mathsf{s} \;:=\; D\sigma \wedge d^{0|4}\chi. \tag{757}$$

Now we can define the action for the fermionic sector. Let

$$e_{m\alpha} \in \Lambda[3]\otimes\Gamma\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(-1)\big) \tag{758}$$

be a Lagrange multiplier of Grassmann degree 3. In this inclusion relation, the base manifold remains the bosonic line $\mathcal{L}_m(\gamma^Q)$ because the fermionic directions lie entirely along its fibres.

Consider the projectively invariant top-forms on the celestial supersphere:

$$D^{1|4}\mathsf{s} \wedge e_{m\alpha} \wedge \overline{\partial}_\sigma\, \kappa_m^\alpha \quad \text{and} \quad D^{1|4}\mathsf{s} \wedge e_{m\alpha} \wedge \mathcal{K}_m^\alpha \;\in\; \Omega^{(1,1)|4}\big(\mathcal{CS}_{s,m}(\gamma^Q)\big), \tag{759}$$

which take values in its Berezinian. We then define the fermionic action as:

$$\mathcal{S}^N_{\kappa_m,e_m}\big(\gamma^Q\big) \;=\; \frac{1}{b}\sum_{m=1}^{N} \int_{\mathcal{CS}_{s,m}(\gamma^Q)} D^{1|4}\mathsf{s} \wedge \left(\frac{1}{2\pi i}\, e_{m\alpha} \wedge \overline{\partial}_\sigma\, \kappa_m^\alpha \;+\; e_{m\alpha} \wedge \mathcal{K}_m^\alpha\right). \tag{760}$$

Varying $\mathcal{S}^N_{\pi_m,e_m}$ with respect to $e_{m\alpha}$ directly yields Eq. (754).

Geometric Sector. We denote by the *geometric sector* of the many-body system of N semiclassical minitwistor strings the sector that governs the embedding of the celestial supersphere $\mathcal{CS}_s$ into minitwistor superspace $\mathbf{MT}_s$. This embedding appears as a family of minitwistor lines $\mathcal{L}_1,\ldots,\mathcal{L}_N$ on which the minitwistor amplitudes localise.

The fundamental field variables that define the geometric sector form the multiplet containing the second-kind evaluation maps together with the Lagrange multipliers:

$$\digamma \;:=\; \big\{\Pi_{m\dot{A}}(\sigma^B),\, \kappa_m^\alpha(\sigma^B),\, e_{m\alpha}(\sigma^B,\chi^\beta)\big\}. \tag{761}$$

Combining Eqs. (755) and (760) yields the total action for the geometric sector:

$$\mathcal{S}^N_0[\digamma|\gamma^Q] \;:=\; \mathcal{S}^N_{\Pi_m}\big(\gamma^Q\big) \;+\; \mathcal{S}^N_{\pi_m,e_m}\big(\gamma^Q\big). \tag{762}$$

We unify the bosonic and fermionic sectors by introducing two conjugate superfields, $\Sigma_{m\dot{A}}$ and $\Xi_m{}^{\dot{A}}$, together with a supercurrent $|m,\gamma^Q]^{\dot{A}}$. The superfields lie in

$$\Lambda\otimes\Gamma\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big). \tag{763}$$

We define:

$$\Sigma_{m\dot{A}}\big(\sigma^B,\chi^\beta\big) \;:=\; \chi^1\chi^2\,\Pi_{m\dot{A}}\big(\sigma^B\big) \;+\; E_{\dot{A}}{}^{\alpha}\, e_{m\alpha}\big(\sigma^B,\chi^\beta\big), \tag{764}$$

$$\Xi_m{}^{\dot A}(\sigma^B,\chi^\beta) \;\coloneqq\; \chi^3\chi^4\,\Pi_m{}^{\dot A}(\sigma^B) \;+\; E^{\dot A}{}_\beta\,\kappa^\beta_m(\sigma^B), \tag{765}$$

where $E_{\dot A}{}^\alpha$ is the rigid vielbein[33] on the celestial supersphere introduced earlier. The supercurrent

$$|m,\gamma^Q]^{\dot A} \in \Lambda\otimes\mathscr{D}'_{0,1}\big(\mathcal{L}_m(\gamma^Q);\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big) \tag{766}$$

has local form:

$$|m,\gamma^Q]^{\dot A} \;\coloneqq\; \chi^3\chi^4\,\mathcal{J}_m{}^{\dot A}(\sigma^B;\gamma^Q) \;+\; E^{\dot A}{}_\alpha\,\mathcal{K}^\alpha_m(\sigma^B;\gamma^Q). \tag{767}$$

Hence the *geometric action* becomes[34]

$$\mathcal{S}_0^N[\Delta|\gamma^Q] \;=\; \frac{1}{b}\sum_{m=1}^{N}\;\int_{\mathcal{CS}_{s,m}(\gamma^Q)} D^{1|4}\mathsf{s}\wedge\left(\frac{1}{2\pi i}\,[\Sigma_m\,\overline{\partial}_\sigma\,\Xi_m] \;+\; [\Sigma_m|m,\gamma^Q]\right). \tag{769}$$

Embedding Maps. In the discussion above, we covered the holomorphic celestial sphere $\mathcal{CS}\cong\mathbf{CP}^1$ by two coordinate systems. The first uses homogeneous coordinates λ^A on $\mathbf{CP}^1$. The second uses coordinates σ^B, which carry homogeneity weight -1. Under the rescaling $\lambda^A\mapsto t\,\lambda^A$, the σ-coordinates transform as $\sigma^B\mapsto t^{-1}\,\sigma^B$. The transition map $\lambda^A=\lambda^A(\sigma^B)$ between these patches is given in Eq. (624).

Both coordinate systems on $\mathcal{CS}$ are useful to describe the field content and dynamics of the minitwistor sigma-model. Accordingly, the evaluation maps admit two distinct representations. The *first-kind evaluation maps* depend on the λ-coordinates and are given by the sections $\Phi_{m\dot A}(\lambda^A;\gamma^Q)$ and $\varphi^\alpha_m(\lambda^A;\gamma^Q)$, introduced in Eqs. (743) and (744). The *second-kind evaluation maps* depend on the σ-coordinates and are given by the sections $\Pi_{m\dot A}(\sigma^B;\gamma^Q)$ and $\kappa^\alpha_m(\sigma^B;\gamma^Q)$, defined in Eqs. (746), (747) and (748).

Hence there are two alternative parameterisations of the m-th string, one associated with each kind of evaluation map. Using the first-kind evaluation maps we parameterise the m-th string by

$$\mathsf{W}^I_m(\lambda^A;\gamma^Q) \;=\; \big(\lambda^A,\,\Phi_{m\dot A}(\lambda^A;\gamma^Q),\,\varphi^\alpha_m(\lambda^A;\gamma^Q)\big). \tag{770}$$

We refer to the assignment $\lambda^A\mapsto\mathsf{W}^I_m(\lambda^A;\gamma^Q)$ as the *first-kind parameterisation* of the string $\mathcal{L}_m(\gamma^Q)$.

Similarly, in terms of the second-kind evaluation maps the m-th string is parameterised by

$$\mathsf{Y}^I_m(\sigma^B;\gamma^Q) \;=\; \big(\lambda^A(\sigma^B),\,\Pi_{m\dot A}(\sigma^B;\gamma^Q),\,\kappa^\alpha_m(\sigma^B;\gamma^Q)\big). \tag{771}$$

---

[33] See § 14.1 of Rogers [20] for a review of the geometric structures on super Riemann surfaces. That section discusses the role of the vielbein $E_{\dot A}{}^\alpha$.

[34] We use the generalised spinor-helicity bracket:

$$[\omega_1\omega_2] \;\coloneqq\; \omega_{1\dot A}\wedge\omega_2{}^{\dot A} \tag{768}$$

for any pair $\omega_{i\dot A}$ of Grassmann-valued dotted van der Waerden spinors.

The assignment $\sigma^B \mapsto \mathsf{Y}^I_m(\sigma^B;\gamma^Q)$ is the *second-kind parameterisation* of the string $\mathcal{L}_m(\gamma^Q)$.

When formulating the geometric action $\mathcal{S}_0^N$, we found it convenient to employ the σ-coordinates. However, to define the worldsheet CFT, it is more practical to use the λ-coordinates. In particular, the parameterisation $\lambda^A \mapsto \mathsf{W}^I_m(\lambda^A;\gamma^Q)$ is natural from the target-space perspective because the first component of W^I_m is the spinor λ^A.

2. Worldsheet CFT

The phenomenology of minitwistor string theory will rely on auxiliary matter systems defined on the worldsheet. To reproduce the tree-level leaf amplitudes for gluons, we will introduce a 2d CFT formed by worldsheet fermions. These fermions will couple to an external gauge potential $\boldsymbol{A}$ on the target superspace $\mathbf{MT}_s$.

Integrating out the fermions will produce a chiral determinant. Evaluating that determinant will yield an effective WZNW action. Consequently, the coupling of the worldsheet fermions to the background gauge field will induce a WZNW current algebra, and the correlators of this algebra will reproduce the Parke-Taylor factors. This mechanism will mirror the corresponding construction in conventional twistor-string models.

Outline. We begin with a pair of worldsheet fermions ρ, ρ^* modelled as spinor fields on the celestial supersphere $\mathcal{CS}_s$. Embedding $\mathcal{CS}_s$ into the target superspace $\mathbf{MT}_s$ produces a family of minitwistor superlines $\{\mathcal{CS}_{s,m}(\gamma^Q)\}_m$. This family is described by the evaluation maps $\Pi_{m\dot{A}}$ and κ^α_m via the incidence relations. Under these evaluation maps, the worldsheet fermions are pushed forward to fermions supported on the lines $\mathcal{CS}_{s,m}(\gamma^Q) \subset \mathbf{MT}_s$. Each such line represents a classical configuration of a minitwistor string in the system. Along every line, the fermions couple minimally to the background gauge field $\boldsymbol{A}$ on $\mathbf{MT}_s$.

Importantly, this construction does not introduce distinct worldsheets for each minitwistor string. The original celestial supersphere $\mathcal{CS}_s$ is a single, fixed object. The different target-space copies of the celestial supersphere arise from the evaluation maps (and their embeddings), which map the worldsheet fermions to different minitwistor superlines in $\mathbf{MT}_s$.

Now, how can we formalise this picture?

A Simple Analogy. As an illustration, consider a nonrelativistic system of N spinless particles. They interact via a potential V. Label the position of the m-th particle by $\vec{x}_m \in \mathbf{R}^3$. Then the full configuration space is $\mathcal{X}_N = \mathbf{R}^{3N}$, and we chart $\mathcal{X}_N$ by the coordinate vector $\vec{X} = (\vec{x}_1, \ldots, \vec{x}_N)$. The potential $V(\vec{X})$ that enters the Schrödinger equation depends on the *full* set of particle coordinates. Hence we may regard the potential as a section and write

$$V \,\in\, \Gamma\big(\mathcal{X}_N;\, \mathcal{X}_N \times \mathbf{R}\big). \tag{772}$$

Embedding Superspace. Proceeding by analogy, we take the *configuration space* of the N-string system to be the *embedding superspace*:

$$\mathbf{X}_N \;\coloneqq\; \bigtimes^N \mathbf{MT}_s. \tag{773}$$

As a supermanifold, $\mathbf{X}_N$ is globally charted by the *embedding coordinates* $\big(\mathsf{W}^I_m\big)_{m=1}^N$. Here W^I_m denotes the embedding coordinates of the string $\mathcal{L}_m$ into $\mathbf{MT}_s$.

Fix a parameter in the moduli superspace $\gamma^Q \in \mathscr{M}_N$. For each m, define $\mathcal{L}_m(\gamma^Q) \subset \mathbf{MT}_s$ to be the minitwistor line representing the classical configuration of the m-th string in the localisation family. Observe that $\mathcal{L}_m(\gamma^Q)$ is the image of $\mathbf{CP}^1$ under the map $\lambda^A \mapsto \mathsf{W}^I_m(\lambda^A; \gamma^Q)$.

With this notation, the classical configuration of the N strings in the embedding superspace $\mathbf{X}_N$ is the Cartesian product of these images. We denote this configuration by:

$$\mathscr{L}(N;\gamma^Q) \coloneqq \mathcal{L}_1(\gamma^Q) \times \mathcal{L}_2(\gamma^Q) \times \cdots \times \mathcal{L}_N(\gamma^Q) \;\subset\; \mathbf{X}_N. \tag{774}$$

The Superpotential. We now introduce the superpotential $\boldsymbol{V} = \boldsymbol{V}(\mathsf{W}^I_1, \ldots, \mathsf{W}^I_N)$ on $\mathbf{X}_N$, which generalises the background gauge field $\boldsymbol{A}$ discussed in the preceding section. In the elementary quantum mechanics analogy, $\boldsymbol{V}$ plays the role of the potential $V(\vec{X})$ in the Schrödinger equation.

We define $\boldsymbol{V}$ by its Fourier expansion, using the $\mathcal{MT}$-transform of Section II. Let $\{\mathsf{T}^{\mathsf{a}}\}$ be a basis of the gauge Lie algebra $\mathfrak{g}$, and let $\widetilde{\alpha}^{\Delta,\mathsf{a}}_m(\mathsf{Z}^I)$ denote the mode functions of the background gauge field assigned to the line $\mathcal{L}_m$. Then we take[35]:

$$\boldsymbol{V}\big(\mathsf{W}^I_1, \ldots, \mathsf{W}^I_N\big) \;=\; \sum_{m=1}^{N} \int_{\mathbf{MT}^*_s} \Psi_\Delta\big(\mathsf{W}^I_m; \mathsf{Z}'^I\big)\, \widetilde{\alpha}^{\Delta,\mathsf{a}}_m\big(\mathsf{Z}'^I\big)\, \mathsf{T}^{\mathsf{a}} \wedge D^{2|4}\mathsf{Z}'. \tag{776}$$

Hence we identify the superpotential $\boldsymbol{V}$ as a Lie-algebra-valued $(0,1)$-form on the natural homogeneous bundle of the embedding superspace $\mathbf{X}_N$. This form extends the gauge potential $\boldsymbol{A}$ on the target superspace $\mathbf{MT}_s$ of a single string to the configuration space $\mathbf{X}_N$ of the N-string system.

---

[35] Here we use DeWitt notation for the conformal weight Δ, so that

$$\Psi_\Delta\big(\mathsf{W}^I_m; \mathsf{Z}'^I\big)\, \widetilde{\alpha}^{\Delta,\mathsf{a}}_m\big(\mathsf{Z}'^I\big) \;=\; \sum_{m\in\mathbf{Z}} \Psi_\Delta\big(\mathsf{W}^I_m; \mathsf{Z}'^I\big)\, \widetilde{\alpha}^{\mathsf{a}}_{\Delta,m}\big(\mathsf{Z}'^I\big). \tag{775}$$

Induced Potential on Celestial Sphere. Applying the restriction homomorphism to the classical configuration $\mathscr{L} = \mathscr{L}(N;\gamma^Q)$ yields the *induced potential* on the holomorphic celestial sphere:

$$\boldsymbol{\upsilon} \in \Omega^{0,1}\big(\mathcal{CS}\big)\otimes\mathfrak{g}, \qquad \boldsymbol{\upsilon} \coloneqq \boldsymbol{V}\big|_{\mathscr{L}}. \tag{777}$$

In terms of the embedding coordinates $(\mathsf{W}^I_m)^N_{m=1}$ and the first-kind parameterisations of the strings, the induced potential takes the form

$$\boldsymbol{\upsilon}(\lambda^A;\gamma^Q) \;=\; \boldsymbol{V}\big(\mathsf{W}^I_1(\lambda^A;\gamma^Q),\ldots,\mathsf{W}^I_N(\lambda^A;\gamma^Q)\big) \tag{778}$$

where $\mathsf{W}^I_m(\lambda^A;\gamma^Q)$ denotes the embedding map of the m-th string.

Celestial Fermions. The coupling of the minitwistor strings to the background gauge field is mediated by the worldsheet fermions ρ and ρ^*. Physically, these fermions are pushed forward to spinor fields living on the lines $\mathcal{L}_1,\ldots,\mathcal{L}_N \subset \mathbf{MT}_s$ of the localisation family, where they couple minimally to the gauge superpotential $\boldsymbol{V}$.

To formalise this intuition, we define ρ and ρ^* as spinor fields on the holomorphic celestial sphere $\mathcal{CS}$ valued in a vector bundle $\mathtt{F}$. The typical fibre of $\mathtt{F}$ is the representation space that models the matter sector of the worldsheet CFT.

Here our discussion parallels that of the single-string system presented in Subsection V A 3. Consider a holomorphic gauge field theory formulated on a rank-N_c complex vector bundle $\mathrm{Pr}\colon E \to \mathbf{X}_N$. Let $\mathbf{G}$ be a semisimple Lie group with Lie algebra $\mathfrak{g}$, and assume $\mathrm{Pr}^{-1}(w) \cong \mathfrak{g}$ for all $w \in \mathbf{X}_N$.

Pulling back E to the configuration $\mathscr{L} = \mathscr{L}(N;\gamma^Q)$ of the N-string system via the restriction homomorphism, we obtain the restricted bundle $\mathtt{E} \coloneqq \mathrm{Pr}^{-1}(\mathscr{L})$ over the holomorphic celestial sphere $\mathcal{CS}$. Since $\mathfrak{g}$ is semisimple, one has $\mathfrak{g} \cong \mathrm{Der}(\mathfrak{g})$. Hence the induced potential $\boldsymbol{\upsilon}$ defined above may be identified with a partial connection on $\mathtt{E} \to \mathcal{CS}$. In particular,

$$\boldsymbol{\upsilon} \in \Omega^{0,1}\big(\mathcal{CS};\mathrm{End}(\mathtt{E})\big). \tag{779}$$

Now let V be the complex vector space that carries the representation of the matter system on the celestial sphere, and let $\mathcal{R}\colon \mathfrak{g} \to \mathrm{GL}(V)$ be a complex representation of the gauge Lie algebra on V. Recall from the previous subsection that we introduced a left action $\varphi\colon \mathfrak{g} \to \mathrm{Aut}(\mathtt{E}\times V)$ defined by $\varphi_g(e,v) \coloneqq (\mathrm{ad}_g(e),\mathcal{R}_g(v))$. Using φ, define an equivalence relation $\simeq$ on $\mathtt{E}\times V$ by $(e,v) \simeq (e',v')$ iff $\varphi_g(e,v) = (e',v')$ for some $g \in \mathfrak{g}$. With respect to this relation, form the quotient $\mathtt{F} \coloneqq (\mathtt{E}\times V)/\mathfrak{g}$. The bundle $\mathtt{F}$ carries the natural structure of the vector bundle associated to $\mathtt{E}$, with typical fibre isomorphic to the representation space V.

Finally, let $\mathtt{K}$ denote the canonical line bundle of the celestial sphere $\mathcal{CS}$. Following Atiyah [92], choose a spin structure $\sqrt{\mathtt{K}}$ on $\mathcal{CS}$. The *celestial fermions* are then sections of the corresponding spinor bundles:

$$\rho \in \Gamma\big(\mathcal{CS}; \sqrt{\mathtt{K}}\otimes\mathtt{F}\big), \qquad \rho^* \in \Gamma\big(\mathcal{CS}; \sqrt{\mathtt{K}}\otimes\mathtt{F}^*\big). \tag{780}$$

The induced potential $\boldsymbol{\upsilon}$ acts on the worldsheet fermions ρ and ρ^* through the partial connection it defines on the associated bundle $\mathtt{F}$. We denote this partial connection by $\boldsymbol{\upsilon}^\sharp$:

$$\boldsymbol{\upsilon}^\sharp \in \Omega^{0,1}\big(\mathcal{CS}; \mathrm{GL}(V)\big), \qquad \boldsymbol{\upsilon}^\sharp \coloneqq \mathcal{R} \circ \boldsymbol{\upsilon}. \tag{781}$$

Using the first-kind parameterisations $\mathsf{W}^I_m(\lambda^A;\gamma^Q)$ of the minitwistor strings, the induced potential on $\mathtt{F}$ can be expressed as

$$\boldsymbol{\upsilon}^\sharp(\lambda^A;\gamma^Q) \coloneqq \mathcal{R}\big[\boldsymbol{V}\big(\mathsf{W}^I_1(\lambda^A;\gamma^Q), \ldots, \mathsf{W}^I_N(\lambda^A;\gamma^Q)\big)\big]. \tag{782}$$

Action. With these formal preparations in place, we take the dynamics of the worldsheet CFT to be governed by the action:

$$\mathcal{S}_{\mathrm{CFT}}[\Delta,\rho,\rho^*|\boldsymbol{V};\gamma^Q] \;\coloneqq\; \int_{\mathcal{CS}} D\lambda \wedge \big\langle \rho^* \big| \big(\overline{\partial}_\lambda + \boldsymbol{\upsilon}^\sharp(\lambda^A;\gamma^Q)\rho\big)\big\rangle. \tag{783}$$

The dependence of the action functional $\mathcal{S}_{\mathrm{CFT}}$ on both the multiplet Δ (which contains the evaluation maps) and on the superpotential $\boldsymbol{V}$ follows from the definition of the induced potential $\boldsymbol{\upsilon}^\sharp$ given in Eq. (782).

From Eq. (783) the kinetic part of the action reads:

$$\mathcal{S}_{\mathrm{K}}[\rho,\rho^*] \;=\; \int_{\mathcal{CS}} D\lambda \wedge \big\langle \rho^* \big| \overline{\partial}_\lambda \rho \big\rangle. \tag{784}$$

The term that governs the interaction with the superpotential $\boldsymbol{V}$ is:

$$\mathcal{U}[\Delta,\rho,\rho^*|\boldsymbol{V};\gamma^Q] \;=\; \int_{\mathcal{CS}} D\lambda \wedge \langle \rho^* | \boldsymbol{\upsilon}^\sharp(\lambda^A;\gamma^Q)\rho\rangle. \tag{785}$$

3. Semiclassical Theory

We now define the semiclassical theory. A full quantum treatment may reveal anomalies, and its detailed analysis lies beyond the scope of this work. Here we adopt the path integral formalism. Because $\mathcal{CS}_s \cong \mathbf{CP}^{1|4}$ carries a natural holomorphic structure, we perform an analytic continuation and work with Euclidean path integrals.

We proceed as follows. We treat the geometric sector of the theory *classically.* This sector describes the immersion of the celestial supersphere $\mathcal{CS}_s$ into the target superspace $\mathbf{X}_N$ as minitwistor strings. The superpotential $\boldsymbol{V}$, which parameterises the configuration of the holomorphic gauge field theory on $\mathbf{X}_N$, is likewise treated as a classical background. The worldsheet fermions ρ and ρ^*, which couple to the external classical "bath" determined by $\boldsymbol{V}$, are retained as fully *quantum* degrees of freedom. As we shall show in the next subsection, the path integral over ρ and ρ^* produces the chiral Dirac determinant that yields the integrand of the generating functional for leaf-gluon amplitudes.

Outline. Our strategy to implement the semiclassical theory is to treat the parameter b that appears in the action $\mathcal{S}_0^N$ (see Eq. (769)) as a Liouville-like coupling. We then evaluate the path integral of an observable $\mathrm{F}[\mathsf{W}_m^I]$, which depends on the parameterisations W_m^I of the strings, by integrating over the embedding maps Δ (see Eq. (761)).

In the limit $b \to 0$, we apply the saddle-point approximation to this path integral. The saddle evaluation yields the observable $\mathrm{F}[\mathsf{W}_m^I]$ evaluated on the classical solutions; these contributions are weighted by the effective action of the worldsheet fermions propagating on the classical background superpotential $\boldsymbol{V}$. This construction defines a measure on the N-string system "phase space" $\mathit{\Gamma}_N$ and hence a corresponding statistical *ensemble.* From that *ensemble* we obtain the semiclassical correlator for the celestial CFT associated with the N-string system.

Notation. The discussion above used the same symbols for two different objects: the field variables in the geometric sector (the evaluation maps) and the classical solutions of the sigma-model equations of motion. In the path-integral formulation that follows, we must distinguish the dynamical variables from the classical solutions unambiguously.

We adopt the convention introduced in Subsection V A 4. Undecorated symbols denote the fundamental fields of the theory. Symbols decorated with a tilde denote the corresponding classical solutions. For example, the evaluation maps of the first kind,

$$\Phi_{m\dot{A}}(\lambda^A), \qquad \varphi_m^\alpha(\lambda^A), \tag{786}$$

and the embedding map associated with the first-kind parameterisation of the m-th string,

$$\mathsf{W}_m^I(\lambda^A) \;\coloneqq\; \big(\lambda^A,\, \Phi_{m\dot{A}}(\lambda^A),\, \varphi_m^\alpha(\lambda^A)\big), \tag{787}$$

refer to field variables expressed in the λ-coordinates.

By contrast, the classical solutions stated in Eqs. (743) and (744) are denoted by

$$\widetilde{\Phi}_{m\dot{A}}(\lambda^A;\gamma^Q), \qquad \widetilde{\varphi}^{\alpha}_m(\lambda^A;\gamma^Q), \tag{788}$$

and the classical embedding map for the first-kind parameterisation of $\mathcal{L}_m(\gamma^Q)$ is

$$\widetilde{\mathrm{W}}^I_m(\lambda^A;\gamma^Q) \;\coloneqq\; \big(\lambda^A,\, \widetilde{\Phi}_{m\dot{A}}(\lambda^A;\gamma^Q),\, \widetilde{\varphi}^{\alpha}_m(\lambda^A;\gamma^Q)\big). \tag{789}$$

The final piece of notation we require is the measure on the moduli superspace $\mathscr{M}_N$ that parameterises the geometric configuration of the localisation family $\{\mathcal{L}_m\}_{m=1}^N \subset \mathbf{MT}_s$. This is the collection of marked, irreducible lines of bidegree $\beta = (1,1)$ on which the N^k-MHV minitwistor sub-amplitude localises. Recall that the MHV level k and the number of strings N are related by $N = 2k+1$.

Fix the multi-index $\vec{\alpha} = (a_\ell, b_\ell)_{\ell=1}^k \in \mathbf{Z}^{2k}$ subject to the ordering

$$2 \le a_1 < a_2 < \cdots < a_k < b_k < \cdots < b_2 < b_1 \le n-1. \tag{790}$$

Let

$$S \;\coloneqq\; \big\{\, z_{a_\ell-1},\, z_{a_\ell},\, z_{b_\ell-1},\, z_{b_\ell} \,\big\}_{\ell=1}^k \tag{791}$$

be the set of marked points on $\{\mathcal{L}_m\}$. We denote by $d\boldsymbol{\Omega}_{\vec{\alpha},\mathcal{S}}(\gamma^Q)$ the standard measure on the moduli superspace $\mathscr{M}_N$ of marked minitwistor lines with special points S.

Actions. The first ingredient required for the semiclassical theory is the full action for the N-string system coupled to the background superpotential $\boldsymbol{V}$. Combining the geometric-sector action with the worldsheet CFT action, we write:

$$\mathcal{S}^N_{\mathrm{I}}[\Delta,\rho,\rho^*|\boldsymbol{V};\gamma^Q] \;\coloneqq\; \mathcal{S}^N_0[\Delta|\gamma^Q] \;+\; \mathcal{S}_{\mathrm{CFT}}[\Delta,\rho,\rho^*|\boldsymbol{V};\gamma^Q]. \tag{792}$$

Let $\mathcal{I}$ denote the effective action that governs the dynamics of the quantum worldsheet fermions ρ and ρ^* propagating on the classical background $\boldsymbol{V}$ which parameterises the configuration of the holomorphic gauge theory on $\mathbf{X}_N$. We define $\mathcal{I}$ by evaluating the full action on the classical solution for the geometric fields Δ:

$$\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q] \;\coloneqq\; \Big(\mathcal{S}^N[\,\Delta,\,\rho,\,\rho^*|\boldsymbol{V};\gamma^Q\,]\Big)_{\delta\mathcal{S}^N/\delta\Delta=0}. \tag{793}$$

We obtain this effective action by substituting into $\mathcal{S}^N$ the classical embedding maps $\widetilde{\mathrm{W}}^I_m$ of the strings, which follow from the equation of motion $\delta\mathcal{S}^N_0/\delta\Delta = 0$. Let $\widetilde{\boldsymbol{v}}^\sharp \in \Omega^{0,1}(\mathcal{CS};\mathrm{GL}(V))$ be the induced partial connection on the vector bundle $\mathtt{F}$ associated to the matter sector $\{\rho, \rho^*\}$, evaluated at those classical string configurations. We set:

$$\widetilde{\boldsymbol{v}}^\sharp(\lambda^A;\gamma^Q) \;:=\; \mathcal{R}\big[\boldsymbol{V}\big(\widetilde{\mathsf{W}}^I_1(\lambda^A;\gamma^Q),\ldots,\widetilde{\mathsf{W}}^I_N(\lambda^A;\gamma^Q)\big)\big]. \tag{794}$$

Hence the effective action takes the explicit form:

$$\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q] \;=\; \int_{\mathcal{CS}} D\lambda\wedge\langle\rho^*|\big(\overline{\partial}_\lambda+\widetilde{\boldsymbol{v}}^\sharp(\lambda^A;\gamma^Q)\big)\rho\rangle. \tag{795}$$

Saddle-Point Approximation; Semiclassical Statistical Ensemble. We now introduce the saddle-point approximation of the Euclidean path integral for the full action $\mathcal{S}^N_{\mathrm{I}}$ with respect to the geometric fields Δ. This approximation motivates the introduction of a measure on the formal phase space of the N-string system coupled to the superpotential $\boldsymbol{V}$. Carrying out the saddle-point analysis leads to a statistical *ensemble* of minitwistor sigma-models. We propose that the semiclassical correlators of the resulting *ensemble* provide the celestial correlators for the minitwistor-string theory.

Treat the parameter b appearing in the geometric-sector action $\mathcal{S}^N_0$ as a Liouville-like coupling that controls the semiclassical expansion. Let $[d\Delta]$ denote the functional "measure" over the evaluation maps of the second kind; we take

$$[d\Delta] \;:=\; \prod_{m=1}^{N}\,[d\Pi_m\,d\kappa_m\,de_m]. \tag{796}$$

Let $\mathrm{F}[\mathsf{W}^I_m]$ be a classical functional representing an observable that depends on the string parameterisations $\mathsf{W}^I_m(\lambda^A)$. Consider the limit $b\to 0$ of the path integral of $\mathrm{F}[\mathsf{W}^I_m]$ over Δ, weighted by $\exp\big(-\mathcal{S}^N_{\mathrm{I}}\big)$. In this limit, the integral is dominated by the stationarity locus of $\mathcal{S}^N_0$, where $\delta\mathcal{S}^N_0/\delta\Delta=0$. Hence, as computed in Ch. 6 of Schulman [103] or § 5.3 of Rivers [104], the saddle-point evaluation yields:

$$\lim_{b\to 0}\frac{1}{\mathcal{N}_0\big(\gamma^Q\big)}\int[d\Delta]\;e^{-\mathcal{S}^N[\Delta,\rho,\rho^*|\boldsymbol{V};\gamma^Q]}\,\mathrm{F}[\mathsf{W}^I_m] \;=\; e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q]}\,F\big[\widetilde{\mathsf{W}}^I_m\big(\lambda^A;\gamma^Q\big)\big]. \tag{797}$$

The normalisation factor $\mathcal{N}_0(\gamma^Q)$ is defined by

$$\mathcal{N}_0\big(\gamma^Q\big) \;:=\; \int[d\Delta]\;e^{-\mathcal{S}^N_0[\Delta|\gamma^Q]}. \tag{798}$$

The physical interpretation of the right-hand side of the saddle-point identity (797) is the following. The observable $\mathrm{F}[\mathsf{W}^I_m]$ is evaluated on the classical configuration $\mathscr{L}(N;\gamma^Q)$ of the N-string system. Using the restriction homomorphism, this term can be written as:

$$\mathrm{F}\big|_{\mathscr{L}(N;\gamma^Q)}\big(\lambda^A\big) \;=\; \mathrm{F}\big[\widetilde{\mathsf{W}}^I_1(\lambda^A;\gamma^Q),\ldots,\widetilde{\mathsf{W}}^I_N(\lambda^A;\gamma^Q)\big]. \tag{799}$$

In Eq. (797), the result (799) is weighted by the inverse of the exponentiated effective action, $e^{-\mathcal{I}}$. Thus, to obtain the semiclassical vacuum expectation value (VEV) of the observable $\mathrm{F}[\mathsf{W}^I_m]$, one must average the right-hand side of Eq. (797) over all classically allowed configurations (each parameterised by a point $\gamma^Q \in \mathscr{M}_N$) and functionally integrate over the worldsheet fermions ρ, ρ^*.

To formalise this picture, let $\mathit{\Gamma}_N$ denote the "formal" phase space[36] of the N-string system coupled to the background gauge field $\boldsymbol{V}$. On $\mathit{\Gamma}_N$ we define the pseudomeasure

$$d\boldsymbol{\mu}_N[\rho, \rho^*; \gamma^Q] \;:=\; \frac{1}{\mathcal{N}_{\mathrm{CFT}}}\, e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q]}\, d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\, [d\rho\, d\rho^*], \tag{800}$$

where the normalisation factor $\mathcal{N}_{\mathrm{CFT}}$ is

$$\mathcal{N}_{\mathrm{CFT}} \;:=\; \int [d\rho\, d\rho^*]\; e^{-\mathcal{S}_{\mathrm{K}}[\rho,\rho^*]}. \tag{801}$$

Equipping $\mathit{\Gamma}_N$ with the pseudomeasure $d\boldsymbol{\mu}_N$ yields the *semiclassical statistical ensemble* of N minitwistor strings coupled to the classical "bath" $\boldsymbol{V}$. Semiclassical correlation functions are then computed as expectation values with respect to $d\boldsymbol{\mu}_N$.

With the structures introduced above, we arrive at the semiclassical celestial correlator. Let $\mathscr{F}[\mathsf{W}^I_m]$ denote the quantum observable corresponding to the classical functional $\mathrm{F}[\mathsf{W}^I_m]$. The existence of $\mathscr{F}$ is guaranteed by the correspondence principle; its semiclassical correlator reads:

$$\lim_{b\to 0} \big\langle \mathscr{F}[\mathsf{W}^I_m]\big\rangle^{\boldsymbol{V}}_{\mathcal{CS}} \;=\; \frac{1}{\mathcal{N}_{\mathrm{CFT}}} \int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big) \int [d\rho\, d\rho^*]\; e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q]}\, \mathrm{F}\big[\widetilde{\mathsf{W}}^I_m(\lambda^A;\gamma^Q)\big]. \tag{802}$$

Define the moduli-dependent, semiclassical vacuum expectation value of the worldsheet CFT by the limit

$$\lim_{b\to 0}\; \big\langle \mathscr{F}[\mathsf{W}^I_m]\big\rangle^{\boldsymbol{V}}_{\mathrm{WS}(\gamma^Q)} \;:=\; \lim_{b\to 0}\, \frac{1}{\mathcal{N}(\gamma^Q)} \int [d\Delta\, d\rho\, d\rho^*]\; e^{-\mathcal{S}^N_{\mathrm{I}}[\Delta,\rho,\rho^*|\boldsymbol{V};\gamma^Q]}\, \mathrm{F}\big[\mathsf{W}^I_m(\lambda^A)\big], \tag{803}$$

with the moduli-dependent normalisation

$$\mathcal{N}\big(\gamma^Q\big) \;:=\; \int [d\Delta\, d\rho\, d\rho^*]\; e^{-\mathcal{S}^N_0[\Delta|\gamma^Q] - \mathcal{S}_{\mathrm{K}}[\rho,\rho^*]}. \tag{804}$$

---

[36] We treat $\mathit{\Gamma}_N$ as a *formal* phase space because we do not commit to a specific topological manifold underlying $\mathit{\Gamma}_N$. Heuristically, one may write $\mathit{\Gamma}_N \cong \mathscr{M}_N \times \mathscr{X}$, where $\mathscr{M}_N$ is the moduli superspace of minitwistor lines (over which the measure $d\boldsymbol{\Omega}_{\vec{\alpha},S}(\gamma^Q)$ is defined) and $\mathscr{X}$ is the function space that models the worldsheet spinor fields ρ and ρ^*. Naively one might take

$$\mathscr{X} = \Gamma\big(\mathbf{CP}^1; \sqrt{\mathtt{K}}\otimes\mathtt{F}\big) \times \Gamma\big(\mathbf{CP}^1; \sqrt{\mathtt{K}}\otimes\mathtt{F}^*\big),$$

since, at the classical level, ρ is a smooth section of the bundle $\sqrt{\mathtt{K}}\otimes\mathtt{F} \to \mathbf{CP}^1$ and ρ^* is a smooth section of $\sqrt{\mathtt{K}}\otimes\mathtt{F}^* \to \mathbf{CP}^1$. Quantum mechanically, however, this identification is problematic. As noted by Feynman and Hibbs [105] in § 7.3, the trajectories that dominate functional integrals (in the measure-theoretic sense) are typically continuous but nowhere differentiable rather than smooth. Hence a choice of topology on $\mathscr{X}$ that would make the path integral mathematically well-posed requires functional-analytic input that goes beyond the present, physics-oriented treatment. For this reason, we continue to regard $\mathit{\Gamma}_N$ as a formal phase space and refrain from specifying a topology on $\mathscr{X}$ here.

Applying the saddle-point approximation in the Δ-sector yields:

$$\lim_{b\to 0}\ \big\langle \mathscr{F}[\mathsf{W}^I_m]\big\rangle^{\boldsymbol{V}}_{\mathrm{WS}(\gamma^Q)} \ = \ \frac{1}{\mathcal{N}_{\mathrm{CFT}}}\int [d\rho\, d\rho^*]\ e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q]}\, \mathrm{F}\big[\widetilde{\mathsf{W}}^I_m(\lambda^A;\gamma^Q)\big]. \tag{805}$$

Hence Eq. (802) may be rewritten as an integral over the moduli superspace:

$$\lim_{b\to 0}\ \big\langle \mathscr{F}[\mathsf{W}^I_m]\big\rangle^{\boldsymbol{V}}_{\mathcal{CS}} \ = \ \lim_{b\to 0}\int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\ \big\langle \mathscr{F}[\mathsf{W}^I_m]\big\rangle^{\boldsymbol{V}}_{\mathrm{WS}(\gamma^Q)}\,. \tag{806}$$

Substituting Eq. (803) into Eq. (806) gives the full semiclassical correlator of the celestial CFT:

$$\lim_{b\to 0}\ \big\langle \mathscr{F}[\mathsf{W}^I_m]\big\rangle^{\boldsymbol{V}}_{\mathcal{CS}} \ = \ \lim_{b\to 0}\int_{\mathscr{M}_N} \frac{d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)}{\mathcal{N}(\gamma^Q)}\int [d\Delta\, d\rho\, d\rho^*]\ e^{-\mathcal{S}^N_{\mathrm{I}}[\Delta,\rho,\rho^*|\boldsymbol{V};\gamma^Q]}\, \mathrm{F}[\mathsf{W}^I_m]. \tag{807}$$

4. Partition Function and the Tree-level Gluon $\mathcal{S}$-Matrix

We now demonstrate that the semiclassical partition function of the N-string system coupled to the classical background superpotential $\boldsymbol{V}$ is a generating functional for the tree-level leaf amplitudes in every N^k-MHV gluonic sector of $\mathcal{N}=4$ SYM theory. This identification supports the claim that the holomorphic gauge theory formulated on minitwistor superspace, studied in Section IV, arises as the string-field-theory limit of the semiclassical strings introduced above.

Our strategy is the following. First, we define the semiclassical partition function of the statistical *ensemble* $\mathit{\Gamma}_N$ with respect to the measure $d\boldsymbol{\mu}_N$, viewing the partition function as a functional of the background superpotential $\boldsymbol{V}$. We denote this functional by $\mathscr{Z}_N[\boldsymbol{V}]$. Second, we recall that $\boldsymbol{V}$ was defined in the previous subsection (see Eq. (776)) via its minitwistor-Fourier decomposition in terms of the classical mode functions $\upsilon^{\Delta,\mathsf{a}}_m(\mathsf{Z}^I)$. Physically, these functions are the expectation values of the gluon annihilation operators, and a set of modes is assigned to each line $\mathcal{L}_m$ in the localisation family. This assignment allows us to expand the partition function in $\upsilon^{\Delta,\mathsf{a}}_m$.

Finally, we show that functional differentiation of $\mathscr{Z}_N[\boldsymbol{V}]$ with respect to $\upsilon^{\Delta,\mathsf{a}}_m$, followed by evaluation at the trivial background $\boldsymbol{V}=0$, reproduces the tree-level leaf amplitude for gluons in every N^k-MHV sector at tree-level.

Partition Function. Let $\mathit{\Gamma}_N$ be the formal phase space of the N-string system introduced in Subsection V B 3. Equip $\mathit{\Gamma}_N$ with the measure $d\boldsymbol{\mu}_N$ given in Eq. (800). The semiclassical statistical *ensemble* of N minitwistor strings interacting with the classical external superpotential $\boldsymbol{V}$ is the pair $(\mathit{\Gamma}_N, d\boldsymbol{\mu}_N)$. Denote the semiclassical partition function of this *ensemble* by $\mathscr{Z}_N[\boldsymbol{V}]$. We regard $\mathscr{Z}_N[\boldsymbol{V}]$ as a functional of the classical mode coefficients $\upsilon^{\Delta,\mathsf{a}}_m$ that parameterise the configuration of the external gauge field $\boldsymbol{V}$.

The semiclassical partition function in the $d\boldsymbol{\mu}_N$-measure obeys the functional relation:

$$\mathscr{Z}_N[\boldsymbol{V}] \;:=\; \lim_{b\to 0} \int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}(\gamma^Q)\;\log\int \frac{[d\digamma\; d\rho\, d\rho^*]}{\mathcal{N}(\gamma^Q)}\; e^{-\mathcal{S}_{\mathrm{I}}^N[\digamma,\rho,\rho^*|\boldsymbol{V};\gamma^Q]}. \tag{808}$$

Employing Eq. (802), this can be written as the integral formula:

$$\mathscr{Z}_N[\boldsymbol{V}] \;=\; \int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\;\log\int \frac{[d\rho\, d\rho^*]}{\mathcal{N}_{\mathrm{CFT}}}\; e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q]}. \tag{809}$$

Next, using Eq. (4) of Witten [106], integrate over the worldsheet fermions ρ and ρ^*. This integration produces the chiral Dirac determinant. Let Z_R denote the renormalisation counter-term that isolates the finite, physically relevant contribution. Hence the chiral determinant admits the path-integral representation:

$$\int [d\rho\, d\rho^*]\; e^{-\mathcal{I}[\rho,\rho^*|\boldsymbol{V};\gamma^Q]} \;=\; \mathrm{Z}_R \det\big(\mathbb{I} \,+\, \boldsymbol{v}^{\mathrm{c}}(\lambda^A;\gamma^Q)\,\overline{\partial}_\lambda^{-1}\big). \tag{810}$$

Henceforth we absorb the counter-term Z_R into the normalisation factor $\mathcal{N}_{\mathrm{CFT}}$. Substituting the identity (810) into Eq. (809) gives the reduced form of the partition function:

$$\mathscr{Z}_N[\boldsymbol{V}] \;=\; \int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\;\mathsf{Tr}\log\big(\mathbb{I} \,+\, \boldsymbol{v}^{\mathrm{c}}(\lambda^A;\gamma^Q)\,\overline{\partial}_\lambda^{-1}\big). \tag{811}$$

Next, following Boels, Mason, and Skinner [45, 107], we *formally* expand the integrand in Eq. (811) as a power series:

$$\mathsf{Tr}\log\big(\mathbb{I} \,+\, \boldsymbol{v}^{\mathrm{c}}(\lambda^A;\gamma^Q)\,\overline{\partial}_\lambda^{-1}\big) \;=\; \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}\;\mathsf{Tr}\int_{(\mathbf{CP}^1)^{\times n}} \bigwedge_{i=1}^{n} \left(\frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}} \wedge \boldsymbol{v}^{\mathrm{c}}(\lambda_i^A;\gamma^Q)\right). \tag{812}$$

We pull back the superpotential $\boldsymbol{V}$ from the embedding superspace $\mathbf{X}_N$ to the celestial sphere $\mathcal{CS}$ via the restriction homomorphism. This induces the partial connection $\boldsymbol{v}^\sharp$ on the associated vector bundle $\mathtt{F} \to \mathcal{CS}$, on which the worldsheet spinors ρ and ρ^* are represented. Using the minitwistor-Fourier expansion given by Eq. (776) and evaluating $\boldsymbol{v}^\sharp$ on the classical solutions of the string equations of motion yields:

$$\boldsymbol{v}^{\mathrm{c}}\big(\lambda_i^A;\gamma^Q\big) \;=\; \sum_{m=1}^{N} \int_{\mathbf{MT}_s^*} \Psi_{\Delta_i}\big|_{\mathcal{L}_m(\gamma^Q)}\big(\lambda_i^A;\mathsf{Z}_i'^{I}\big)\; v_m^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}_i'^{I}\big)\,\mathsf{T}^{\mathsf{a}_i}\wedge D^{2|4}\mathsf{Z}_i'. \tag{813}$$

Substituting the induced potential into Eq. (812) and reorganising the integrals by Fubini's theorem gives the formal expansion:

$$\mathsf{Tr}\log\big(\mathbb{I} \,+\, \boldsymbol{v}^{\mathrm{c}}(\lambda^A;\gamma^Q)\,\overline{\partial}_\lambda^{-1}\big) \;=\; \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{m=1}^{N} \int_{\mathbf{X}_n^*} \bigwedge_{i=1}^{n} \Big(D^{2|4}\mathsf{Z}_i' \wedge v_m^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}_i'^{I}\big)\Big) \tag{814}$$

$$\mathsf{Tr}\int_{(\mathbf{CP}^1)^{\times n}} \bigwedge_{j=1}^{n} \left(\frac{D\lambda_j}{\lambda_j\cdot\lambda_{j+1}}\mathsf{T}^{\mathsf{a}_j} \wedge \Psi_{\Delta_j}\big|_{\mathcal{L}_m(\gamma^Q)}\big(\lambda_j^A;\mathsf{Z}_j'^{I}\big)\right). \tag{815}$$

Invoking the celestial BMSW identity yields:

$$\mathsf{Tr}\log\left(\mathbb{I} \,+\, \boldsymbol{\upsilon}^{\mathrm{c}}(\lambda^A;\gamma^Q)\,\overline{\partial}_\lambda^{-1}\right) \;=\; \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\,\sum_{m=1}^{N}\int_{\mathbf{X}_n^*}\bigwedge_{i=1}^{n}\left(D^{2|4}\mathsf{Z}_i'\wedge\upsilon_m^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}_i'^{I}\big)\right) \tag{816}$$

$$\mathsf{Tr}\,\bigwedge_{j=1}^{n}\left(\frac{\mathcal{C}(\Delta_j)}{\langle z_j'|Y_m|\bar{z}_j']^{\Delta_j}}\exp\left(i\langle z_j'|\xi_m\cdot\eta_j\rangle\right)\;\frac{\mathsf{T}^{\mathsf{a}_j}}{z_j'\cdot z_{j+1}'}\right). \tag{817}$$

Inserting this expression into Eq. (811) finally gives the full semiclassical partition function:

$$\mathscr{Z}_N[\boldsymbol{V}] \;=\; \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\,\sum_{m=1}^{N}\int_{\mathbf{X}_n^*}\bigwedge_{i=1}^{n}\left(D^{2|4}\mathsf{Z}_i'\wedge\upsilon_m^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}_i'^{I}\big)\right) \tag{818}$$

$$\int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\;\mathsf{Tr}\,\bigwedge_{j=1}^{n}\left(\frac{\mathcal{C}(\Delta_j)}{\langle z_j'|Y_m|\bar{z}_j']^{\Delta_j}}\exp\left(i\langle z_j'|\xi_m\cdot\eta_j\rangle\right)\;\frac{\mathsf{T}^{\mathsf{a}_j}}{z_j'\cdot z_{j+1}'}\right). \tag{819}$$

Recovering the N^k-MHV *Leaf-Gluon Amplitudes.* Let $n \geq 4$ and fix an integer $1 \leq k \leq n-1$. Consider a tree-level scattering process involving n gluons in an N^k-MHV configuration. Label the external gluons by $i = 1,\ldots,n$. In celestial CFT, the state of the i-th gluon is specified by its conformal weight Δ_i and by its insertion point on the $\mathcal{N}=4$ celestial supersphere $\mathcal{CS}_s$. We denote the i-th insertion point by

$$\mathsf{z}_i \coloneqq (z_i,\bar{z}_i,\eta_i^\alpha) \in \mathcal{CS}_s. \tag{820}$$

Recall that the dual minitwistor superspace $\mathbf{MT}_s^*$ is a covering space of the celestial supersphere. Accordingly, we represent z_i by a dual minitwistor

$$\mathsf{Z}_i^I \;\coloneqq\; \big(z_i^A,\bar{z}_{i\dot{A}},\eta_i^\alpha\big) \in \mathbf{MT}_s^*. \tag{821}$$

Let h_i denote the scaling dimension of the i-th gluon, and let $|\eta_i|$ denote the expectation value of the helicity operator for that state. Thus the conformal weight Δ_i obeys $2h_i + |\eta_i| = \Delta_i$.

With the preceding remarks we have specified the physics we wish to analyse. We now derive the leaf-gluon amplitude from the semiclassical partition function. Fix a multi-index $\vec{\alpha} = (a_i,b_i) \in \mathbf{Z}^{2k}$ satisfying the inequality (790). Let $i \mapsto c_{\vec{\alpha}}(i)$ be the indicator function that assigns the i-th gluon to a cluster determined by $\vec{\alpha}$. For example, $c_{\vec{\alpha}}(i) = 1$ if $1 \leq i \leq a_1 - 1$, $c_{\vec{\alpha}}(i) = 2$ if $a_1 \leq i \leq a_2 - 1$, and so on.

We functionally differentiate the partition function $\mathscr{Z}_N[\boldsymbol{V}]$ with respect to the modes $\upsilon_{c_{\vec{\alpha}}(i)}^{2h_i,\mathsf{a}_i}\big(\mathsf{Z}_i^I\big)$ and evaluate the result on the trivial background:

$$\left(\prod_{i=1}^{n}\frac{\delta}{\delta\upsilon_{c_{\vec{\alpha}}(i)}^{2h_i,\mathsf{a}_i}\big(\mathsf{Z}_i^I\big)}\,\mathscr{Z}_N[\boldsymbol{V}]\right)_{\boldsymbol{V}=0} \tag{822}$$

$$=\;\frac{(-1)^{n-1}}{n}\int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\;\mathsf{Tr}\,\bigwedge_{i=1}^{n}\left(\frac{\mathcal{C}(2h_i)}{\langle z_i|Y_{c_{\vec{\alpha}}(i)}|\bar{z}_i]^{2h_i}}\,\exp\left(i\langle z_i|\xi_{c_{\vec{\alpha}}(i)}\cdot\eta_i\rangle\right)\;\frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}\right). \tag{823}$$

From Section III we identify the right-hand side as the tree-level N^k-MHV leaf-gluon amplitude $\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i\big)$. Consequently,

$$\left(\prod_{i=1}^{n}\frac{\delta}{\delta v^{2h_i,\mathsf{a}_i}_{c_{\vec{\alpha}}(i)}\big(\mathsf{Z}^I_i\big)}\,\mathscr{Z}_N[\boldsymbol{V}]\right)_{\boldsymbol{V}=0} = \frac{(-1)^{n-1}}{n}\,\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i\big). \tag{824}$$

Discussion. In Section III, we derived a geometric interpretation of the tree-level N^k-MHV leaf-gluon amplitudes $\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}(\mathsf{Z}^I_i)$ for $\mathcal{N}=4$ SYM as a *localisation theorem.* The statement is the following. The minitwistor transform of the leaf-gluon amplitudes, which we denote by $\widetilde{\mathcal{M}}^{\mathsf{a}_1\ldots\mathsf{a}_n}(\mathsf{W}^I_i)$, is given by an integral over the moduli superspace $\mathscr{M}_N$ of marked minitwistor lines $\{\mathcal{L}_m\}_{m=1}^N$, referred to as the *localisation family.* Moreover, the minitwistor amplitude vanishes unless every external gluon participating in the scattering lies on one of the lines $\mathcal{L}_1,\ldots,\mathcal{L}_N$. Finally, the MHV level k and the number N of lines are related by $1+2k-N=0$.

Then, in Section IV, we constructed a field-theory interpretation of tree-level N^k-MHV leaf-gluon amplitudes as semiclassical expectation values of nonlocal observables on minitwistor superspace. These observables are realised as Wilson operators of a holomorphic gauge field theory on $\mathbf{MT}_s$, supported on the minitwistor lines $\mathcal{L}_1,\ldots,\mathcal{L}_N$. We observed that classical modes of the background gauge potential localise on the lines in the family $\{\mathcal{L}_m\}_{m=1}^N$. These modes are then interpreted physically as the expectation values of gluon annihilation operators.

We therefore sought a dynamical interpretation of the leaf-gluon amplitudes in which the lines $\mathcal{L}_m$ for $m=1,\ldots,N$ are realised as minitwistor *strings.* To that end, we implemented a many-body system of semiclassical minitwistor sigma-models. Their worldsheet is the $\mathcal{N}=4$ celestial supersphere, and their target is the embedding superspace $\mathbf{X}_N$. These sigma-models are coupled to a classical background gauge superpotential $\boldsymbol{V}$ via a pair of worldsheet spinor fields ρ and ρ^*.

In this subsection, we demonstrated that the semiclassical partition function of the model serves as the generating functional for the leaf-gluon amplitudes at tree-level. This result is consistent with our picture that the holomorphic gauge theory arises as the string-field-theory limit of the minitwistor strings presented here.

C. Vertex Operators and the S-Algebra

Hollands and Wald [108] take the existence of OPEs as a basic axiom of QFT. They further argue that a theory's essential properties follow from its OPE data. From this perspective, a complete description of a celestial CFT requires three ingredients. One must specify the vertex operators. One must show that their correlators reproduce the celestial amplitudes required by the theory.

One must also show that these operators generate the holographic OPEs that appear, for example, in collinear singularities or as consequences of asymptotic symmetries.

The aim of this subsection is to define the vertex operators associated with the celestial CFT induced by the semiclassical minitwistor strings. We first generalise the statement made at the beginning of this section: the semiclassical correlators of these vertex operators reproduce the tree-level leaf amplitudes for gluons in any N^k-MHV sector. We then verify that the celestial OPEs of these operators close on the gluon S-algebra.

1. *Physical Motivation*

In the preceding discussion, we argued that holomorphic gauge theory on minitwistor superspace arises as the field-theory limit of minitwistor strings. The first step of that argument began by analysing the coupling of the minitwistor sigma-model to a classical background gauge potential. We then showed that the semiclassical partition function of the sigma-model, coupled to this gauge field, provides a generating functional for tree-level gluon amplitudes.

We now present the second step. We identify worldsheet vertex operators that encode string interactions. We then show that the semiclassical celestial correlators of these vertex operators, evaluated in the leading-trace sector, reproduce the tree-level leaf-gluon amplitudes of $\mathcal{N}=4$ SYM theory.

We choose the gauge Lie group to be $\mathbf{G} = \mathrm{SO}(N_c)$, where N_c denotes the number of colours of the gauge theory. This choice is convenient because the leading-trace sector of the celestial correlators is obtained by taking the large-N_c limit, mirroring the familiar limit in conventional gauge/gravity duality.

We take $V = \mathfrak{so}(N_c)$ as the representation space for the matter fields ρ and ρ^*. Accordingly, the vector bundles $\mathtt{F}$ and $\mathtt{F}^*$ over $\mathcal{CS}$, which carry the worldsheet fermions, are associated to the adjoint representation of $\mathfrak{so}(N_c)$. Their construction follows the procedure described in Subsections V A 3 and V B 2.

Henceforth, we index the representation-space components by $r, s = 1, \ldots, N_c$. For concreteness, let e_r be a frame trivialising the vector bundle $\mathtt{F} \to \mathcal{CS}$, and let e^*_r be the dual frame trivialising the bundle $\mathtt{F}^* \to \mathcal{CS}$. It follows that the worldsheet fermions can be written as $\rho = \rho^r \otimes e_r$ and $\rho^* = \rho^r \otimes e^*_r$, where $\rho^r \in \Gamma(\mathcal{CS}; \sqrt{\mathtt{K}})$. Thus, our choice of gauge group and representation space reduces the matter content of the worldsheet CFT to a set of N_c independent real fermions ρ^r

valued in the vector representation of $\mathfrak{so}(N_c)$.

We recall that the action integral $\mathcal{S}_{\text{CFT}}$ (Eq. (783)), which governs the dynamics of the worldsheet CFT, decomposes into a kinetic action $\mathcal{S}_{\text{K}}$ (Eq. (784)) and an interaction term $\mathcal{U}$ (Eq. (785)). Employing the frames introduced above, the kinetic action assumes the familiar form:

$$\mathcal{S}_{\text{K}}[\rho^r] \;=\; \int_{\mathcal{CS}} D\lambda \wedge \rho^r\, \overline{\partial}_\lambda\, \rho^r. \tag{825}$$

We adopt the strong summation convention for representation-space indices.

The physically interesting contribution is given by the interaction term $\mathcal{U}$. To obtain its component form, recall that $\lambda^A \mapsto \mathsf{W}^I_m(\lambda^A)$ represents the embedding map of the m-string $\mathcal{L}_m$ expressed in terms of the first-kind evaluation maps $\Phi_{m\dot{A}}$ and φ^α_m. Let $\boldsymbol{\upsilon}^\sharp$ denote the partial connection in the adjoint representation induced on the celestial sphere by the pull-back of the superpotential $\boldsymbol{V}$. Applying the restriction homomorphism to the minitwistor-Fourier decomposition of $\boldsymbol{V}$ (see Eq. (776)) then gives:

$$\boldsymbol{\upsilon}^\sharp(\lambda^A) \;=\; \sum_{m=1}^{N} \int_{\mathbf{MT}^*_s} \Psi_\Delta\big(\mathsf{W}^I_m(\lambda^A); \mathsf{Z}'^I\big)\, \upsilon^{\Delta,\mathsf{a}}_m\big(\mathsf{Z}'^I\big)\, \mathsf{T}^{\mathsf{a}} \wedge D^{2|4}\mathsf{Z}'. \tag{826}$$

We substitute this representation of the induced potential into the definition of $\mathcal{U}$ given in Eq. (785). To this end, let

$$\jmath^{\mathsf{a}} \;\in\; \Gamma\big(\mathcal{CS};\, \mathtt{K}\otimes\mathfrak{g}\big) \tag{827}$$

denote the *classical worldsheet current*, defined by

$$\jmath^{\mathsf{a}} \;\coloneqq\; \rho^r\, \mathsf{T}^{\mathsf{a}}_{rs}\, \rho^s. \tag{828}$$

Hence the interaction term becomes

$$\mathcal{U}[F, \rho^r|\gamma^Q] \;=\; \sum_{m=1}^{N} \int_{\mathbf{MT}^*_s} \left(\int_{\mathcal{CS}} D\lambda \wedge \Psi_\Delta\big(\mathsf{W}^I_m(\lambda^A); \mathsf{Z}'^I\big)\, \jmath^{\mathsf{a}}(\lambda^A)\right) \upsilon^{\Delta,\mathsf{a}}_m\big(\mathsf{Z}'^I\big) \wedge D^{2|4}\mathsf{Z}'. \tag{829}$$

Here F denotes the geometric fields that enter the embedding maps $\mathsf{W}^I_m(\lambda^A)$, which parameterise the strings in λ-coordinates.

From Eq. (829) we therefore identify the vertex operator attached to the m-th string $\mathcal{L}_m$, carrying conformal weight Δ and associated to the dual minitwistor point Z^I, as

$$\mathcal{V}^{\mathsf{a}}_{\Delta,m}\big(\mathsf{Z}^I\big) \;\coloneqq\; \int_{\mathcal{CS}} D\lambda \wedge \Psi_\Delta\big(\mathsf{W}^I_m(\lambda^A); \mathsf{Z}^I\big)\, \jmath^{\mathsf{a}}(\lambda^A). \tag{830}$$

Consequently, the interaction term reduces to

$$\mathcal{U}[F, \rho^r|\gamma^Q] \;=\; \sum_{m=1}^{N} \int_{\mathbf{MT}^*_s} \mathcal{V}^{\mathsf{a}}_{\Delta,m}\big(\mathsf{Z}'^I\big)\, \upsilon^{\Delta,\mathsf{a}}_m\big(\mathsf{Z}'^I\big) \wedge D^{2|4}\mathsf{Z}'. \tag{831}$$

In this subsection we derive the semiclassical correlation functions of our dynamical model for the celestial CFT. These correlators encode interactions that arise exclusively from worldsheet insertions of the vertex operators $\mathcal{V}^{\mathsf{a}}_{\Delta,m}$. To this end, we set the external gauge potential to $\boldsymbol{V} = 0$.

Hence the action integral governing the dynamics of the celestial CFT becomes:

$$\mathcal{S}^N[F, \rho^r|\gamma^Q] \;=\; \mathcal{S}_0^N[F|\gamma^Q] \,+\, \mathcal{S}_{\mathrm{K}}[\rho^r]. \tag{832}$$

Here $\mathcal{S}_0^N$ (see Eq. (769)) denotes the action of the geometric sector, and $\mathcal{S}_{\mathrm{K}}$ (see Eq. (825)) denotes the kinetic action for the worldsheet fermions. Observe that, with $\boldsymbol{V} = 0$, the matter sector of the worldsheet CFT reduces to a set of N_c independent real *free* fermions ρ^r transforming in the vector representation of $\mathfrak{so}(N_c)$.

Semiclassical Celestial Correlator. The semiclassical correlator associated with the action $\mathcal{S}^N$ is introduced as follows. Let $\widehat{O}_i$ be a collection of quantum observables indexed by $i = 1, \ldots, n$. Suppose each $\widehat{O}_i$ depends only on the parameterisations $\mathsf{W}^I_m(\lambda^A)$ of the strings. By the correspondence principle there then exists a set of classical functionals $\mathrm{O}_i[\mathsf{W}^I_m]$ such that:

$$\lim_{b\to 0} \left\langle \prod_{i=1}^n \widehat{O}_i \right\rangle_{\mathcal{CS}} \;=\; \lim_{b\to 0} \int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}(\gamma^Q) \int \frac{[dF\, d\rho]}{\mathcal{N}(\gamma^Q)}\; e^{-\mathcal{S}^N[F,\rho^r|\gamma^Q]} \prod_{i=1}^n \mathrm{O}_i[\mathsf{W}^I_m(\lambda^A)]. \tag{833}$$

Here the path-integral pseudomeasure of the matter sector is defined by

$$[d\rho] \;\coloneqq\; \prod_{r=1}^{N_c} [d\rho^r], \tag{834}$$

and the normalisation factor by

$$\mathcal{N}(\gamma^Q) \;=\; \int [dF\, d\rho]\; e^{-\mathcal{S}_0^N[F|\gamma^Q] - \mathcal{S}_{\mathrm{K}}[\rho^r]}. \tag{835}$$

Physical Setup. The physical problem we analyse is the tree-level scattering of n gluons in an N^k-MHV configuration. In the celestial-CFT language, let Δ_i denote the conformal weight carried by the i-th gluon. Recall that the dual minitwistor superspace $\mathbf{MT}^*_s$ covers the celestial supersphere $\mathcal{CS}_s$. Choose a dual minitwistor $\mathsf{Z}^I_i = (z^A_i, \bar{z}_{i\dot{A}}, \eta^\alpha_i)$ that parameterises the insertion point of the i-th gluon on $\mathcal{CS}_s$. The scaling dimension h_i of the i-th gluon and the expectation value $|\eta_i|$ of the helicity operator are related to the conformal weight by $2h_i + |\eta_i| = \Delta_i$.

Henceforth we fix a multi-index $\vec{\alpha} \in \mathbf{Z}^{2n}$ as in § V B 4, and let $i \mapsto c_{\vec{\alpha}}(i)$ be the corresponding indicator function that assigns the i-th gluon to its cluster in $\vec{\alpha}$.

Main Result. Consider the semiclassical celestial correlator of the vertex operators $\mathcal{V}^{\mathsf{a}_i}_{2h_i,c_{\vec{\alpha}}(i)}(\mathsf{Z}^I_i)$ in the leading-trace sector:

$$C^{\mathsf{a}_1,\ldots,\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i;\Delta_i\big) \;:=\; \lim_{N_c\to\infty}\,\lim_{b\to 0}\,\Big\langle \prod_{i=1}^{n} \mathcal{V}^{\mathsf{a}_i}_{2h_i,c_{\vec{\alpha}}(i)}\big(\mathsf{Z}^I_i\big)\Big\rangle_{\mathcal{CS}}\,. \tag{836}$$

Using the defining formula for the celestial correlator (see Eq. (833)), we obtain

$$C^{\mathsf{a}_1,\ldots,\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i;\Delta_i\big) \;=\; \lim_{N_c\to\infty}\,\lim_{b\to 0}\int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}(\gamma^Q)\int\frac{[d\digamma\,d\rho]}{\mathcal{N}(\gamma^Q)}\,e^{-\mathcal{S}^N[\digamma,\rho^r|\gamma^Q]}\prod_{i=1}^{n}\mathcal{V}^{\mathsf{a}_i}_{2h_i,c_{\vec{\alpha}}(i)}\big(\mathsf{Z}^I_i\big). \tag{837}$$

Substituting the vertex-operator definition stated in Eq. (830) and reorganising the integrals by Fubini's theorem gives

$$C^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i;\Delta_i\big) \;=\; \lim_{N_c\to\infty}\,\lim_{b\to 0}\int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\int_{(\mathbf{CP}^1)^{\times n}}\bigwedge_{i=1}^{n} D\lambda_i \tag{838}$$

$$\int\frac{[d\digamma\,d\rho]}{\mathcal{N}\big(\gamma^Q\big)}\,e^{-\mathcal{S}^N[\digamma,\rho^r|\gamma^Q]}\bigwedge_{j=1}^{n}\Big(\Psi_{2h_j}\big(\mathsf{W}^I_{c_{\vec{\alpha}}(j)}(\lambda^A_j);\mathsf{Z}^I_j\big)\,\jmath^{\mathsf{a}_j}\big(\lambda^A_j\big)\Big)\,. \tag{839}$$

Performing the path integral over the geometric fields $\digamma$, and then taking the semiclassical limit $b\to 0$, amounts to replacing the embedding maps $\mathsf{W}^I_m(\lambda^A)$ by the classical solutions $\widetilde{\mathsf{W}}^I_m(\lambda^A;\gamma^Q)$ of the sigma-model equations of motion. These classical solutions parameterise the configuration $\mathscr{L}(N;\gamma^Q)\subset\mathbf{X}_N$ of the N-string system represented by the point $\gamma^Q\in\mathscr{M}_N$ in the classical moduli superspace.

Evaluating the minitwistor superwavefunction Ψ_Δ on a classical solution $\widetilde{\mathsf{W}}^I_m$ is equivalent to pulling Ψ_Δ back to the m-th string $\mathcal{L}_m(\gamma^Q)\subset\mathbf{MT}_s$ via the restriction homomorphism, namely

$$\Psi_\Delta\big|_{\mathcal{L}_m(\gamma^Q)}\big(\lambda^A;\mathsf{Z}^I\big) \;=\; \Psi_\Delta\big(\widetilde{\mathsf{W}}^I_m(\lambda^A;\gamma^Q);\mathsf{Z}^I\big). \tag{840}$$

Consequently, letting $\langle\ldots\rangle_{\mathrm{WZNW}}$ denote the current-algebra correlator, the correlation function $C^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}$ can be expressed as

$$C^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i;\Delta_i\big) \;=\; \lim_{N_c\to\infty}\int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big) \tag{841}$$

$$\int_{(\mathbf{CP}^1)^{\times n}}\bigwedge_{i=1}^{n}\Big(D\lambda_i\wedge\Psi_{2h_i}\big|_{\mathcal{L}_{c_{\vec{\alpha}}(i)}}\big(\lambda^A_i;\mathsf{Z}^I_i\big)\Big)\;\;\Big\langle\prod_{j=1}^{n}\jmath^{\mathsf{a}_j}\big(\lambda^A_j\big)\Big\rangle_{\mathrm{WZNW}}\,. \tag{842}$$

Restricting to the leading-trace sector by taking the large-N_c limit yields

$$C^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i;\Delta_i\big)=\int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\;\mathsf{Tr}\int_{(\mathbf{CP}^1)^{\times n}}\bigwedge_{i=1}^{n}\Big(\frac{D\lambda_i}{\lambda_i\cdot\lambda_{i+1}}\mathsf{T}^{\mathsf{a}_i}\wedge\Psi_{2h_i}\big|_{\mathcal{L}_{c_{\vec{\alpha}}(i)}}\big(\lambda^A_i;\mathsf{Z}^I_i\big)\Big)\,. \tag{843}$$

Invoking the celestial BMSW identity then gives

$$C^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i;\Delta_i\big) \;=\; \int_{\mathscr{M}_N} d\boldsymbol{\Omega}_{\vec{\alpha},S}\big(\gamma^Q\big)\ \mathsf{Tr}\bigwedge_{i=1}^{n}\left(\frac{\mathcal{C}(2h_i)}{\langle z_i|Y_{c_{\vec{\alpha}}(i)}|\bar{z}_i]^{2h_i}}\ \exp\big(i\langle z_i|\xi_{c_{\vec{\alpha}}(i)}\cdot\eta_i\rangle\big)\ \frac{\mathsf{T}^{\mathsf{a}_i}}{z_i\cdot z_{i+1}}\right) \tag{844}$$

By the localisation theorem proven in Section III, the right-hand side of Eq. (844) is identified with the tree-level N^k-MHV leaf-gluon amplitude $\mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}$. Therefore we conclude that

$$\lim_{N_c\to\infty}\,\lim_{b\to 0}\,\left\langle\prod_{i=1}^{n}\mathcal{V}^{\mathsf{a}_i}_{2h_i,c_{\vec{\alpha}}(i)}\big(\mathsf{Z}^I_i\big)\right\rangle_{\mathcal{CS}} \;=\; \mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i\big). \tag{845}$$

To conclude, define the celestial gluon operator $\mathcal{G}^{\eta,\mathsf{a}_i}_{\Delta,m}$ with helicity state η^α and conformal weight Δ attached to the m-th string by

$$\mathcal{G}^{\eta,\mathsf{a}_i}_{\Delta}(z,\bar{z}) \;=\; \mathcal{V}^{\mathsf{a}_i}_{\Delta-|\eta|,m}\big(z^A,\bar{z}_{\dot{A}},\eta^\alpha\big), \tag{846}$$

where $|\eta_i|$ denotes the expectation value of the helicity operator and $2h_i = \Delta_i - |\eta_i|$.

Thus, the leading-trace ($k\to 0$), semiclassical ($b\to 0$) correlator of the celestial gluon operators reproduces the leaf-gluon amplitude:

$$\lim_{k\to 0}\,\lim_{b\to 0}\,\left\langle\prod_{i=1}^{n}\mathcal{G}^{\eta_i,\mathsf{a}_i}_{2h_i,c_{\vec{\alpha}}(i)}\big(z_i,\bar{z}_i\big)\right\rangle_{\mathcal{CS}} \;=\; \mathcal{M}^{\mathsf{a}_1\ldots\mathsf{a}_n}_{n;\vec{\alpha}}\big(\mathsf{Z}^I_i\big). \tag{847}$$

> This formula is the central result of the paper: the leading-trace, semiclassical celestial correlators of the gluon vertex operators reproduce the tree-level leaf-gluon amplitudes in every N^k-MHV sector. In this way we have provided a bottom-up realisation of the celestial CFT for $\mathcal{N}=4$ SYM as a many-body system of semiclassical minitwistor strings.

3. *The S-Algebra*

Asymptotic symmetries, together with the structure of gauge-theory collinear singularities, impose a constraint on *any* celestial CFT that is proposed to be holographically dual to gauge theory on asymptotically flat spacetimes. The CFT must contain primary fields that generate the gluon S-algebra[37]. Therefore, to test the consistency of our proposal we verify that the gluon operators $\mathcal{G}^{\mathsf{a}}_{\Delta}$ close on the S-algebra[38].

We restrict our attention to the celestial CFT modelled by a single semiclassical string. As discussed above, the single-string system is dual only to the MHV gluonic subsector of the gauge

---

[37] Cf. Fotopoulos and Taylor [109], Guevara *et al.* [110], Pate *et al.* [111], Himwich, Pate, and Singh [112].
[38] See Banerjee *et al.* [113] for a comprehensive discussion.

theory at tree-level. Nevertheless, this simple model suffices to generate the S-algebra. In practice, working with a single string means that all vertex operators are assigned to the same minitwistor line. Therefore we may omit the index m that labels the vertex operators in Eq. (830).

Current Algebra. The first step in computing the OPE of the string vertex operators is to promote the classical worldsheet current $\jmath^{\mathsf{a}}$ (Eq. (828)) to a quantum operator J^{a}. The correspondence principle implies that J^{a} should be proportional to the normally-ordered product $(\rho^{\mathsf{T}}\mathsf{T}^{\mathsf{a}}\rho)$. Denote the proportionality constant by β.

Applying Wick's theorem yields the OPE for J^{a}, and the result is displayed in Eq. (718) of Subsection V A 6. Requiring consistency of that equation with the Ward identity fixes $2\beta = 1$. Hence J^{a} generates a level-one $\mathrm{SO}(N_c)$ WZNW current algebra on the celestial sphere.

Now, since we are concerned only with the leading-trace sector, we will work with the OPE in the large-N_c limit,

$$J^{\mathsf{a}}(\lambda_1)J^{\mathsf{b}}(\lambda_2) \sim \frac{if^{\mathsf{abc}}J^{c}(\lambda_2)}{\lambda_1-\lambda_2} \qquad (N_c \gg 1). \tag{848}$$

Integrated Vertex Operator. The next step is to derive the integrated form of the vertex operator $\mathcal{V}^{\mathsf{a}}_{\Delta}$. Let $\Phi_{\dot{A}}$ and φ^{α} denote the evaluation maps of the first kind introduced in Eq. (615) of Subsection V A 1. Let $\lambda^{A} \mapsto \mathsf{W}^{I}(\lambda^{A})$ be the embedding map of the minitwistor string $\mathcal{L}$ introduced in Eq. (787) of Remark 1.

Composing the superwavefunction Ψ_{Δ} with this first-kind parameterisation pulls Ψ_{Δ} back to the minitwistor line $\mathcal{L}$. Hence:

$$\Psi_{\Delta}\big(\mathsf{W}^{I}(\lambda^{B});\mathsf{Z}^{I}\big) \;=\; \bar{\delta}_{-\Delta}\big(z^{A},\lambda^{B}\big)\,\frac{\mathcal{C}(\Delta)}{[\Phi(\lambda)\,\bar{z}]^{\Delta}}\,\exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\varphi(\lambda)\cdot\eta\right). \tag{849}$$

Substituting this expression into the definition of $\mathcal{V}^{\mathsf{a}}_{\Delta}$ (see Eq. (830)) and integrating over λ^{A} gives

$$\mathcal{V}^{\mathsf{a}_i}_{\Delta}\big(\mathsf{Z}^{I}\big) \;=\; \frac{\mathcal{C}(\Delta)}{[\Phi(z)\,\bar{z}]^{\Delta}}\,e^{i\varphi(z)\cdot\eta}\,J^{\mathsf{a}_i}(z). \tag{850}$$

Consider now the following operator product:

$$\mathcal{V}^{\mathsf{a}}_{d_1}\big(\mathsf{Z}^{I}_1\big)\,\mathcal{V}^{\mathsf{b}}_{d_2}\big(\mathsf{Z}^{I}_2\big) \;=\; \frac{\mathcal{C}(d_1)}{[\Phi(z_1)\,\bar{z}_1]^{d_1}}\frac{\mathcal{C}(d_2)}{[\Phi(z_2)\,\bar{z}_2]^{d}}\,e^{i(\varphi(z_1)\cdot\eta_1+\varphi(z_2)\cdot\eta_2)}\,J^{\mathsf{a}}(z_1)J^{\mathsf{b}}(z_2). \tag{851}$$

Definition. Let $z_{12} \coloneqq z_1 - z_2$ and $\bar{z}_{12} \coloneqq \bar{z}_1 - \bar{z}_2$. We study the OPE implied by Eq. (851) in the *large-N_c holomorphic collinear limit.* This limit is obtained by holding η^{α}_1, η^{α}_2 and $\bar{z}_{12}$ fixed while $N_c \gg 1$ and $z_{12} \to 0$.

Main Result. Accordingly, Eqs. (848) and (851) imply:

$$\mathcal{V}^{\mathsf{a}}_{d_1}\big(\mathsf{Z}^I_1\big)\,\mathcal{V}^{\mathsf{b}}_{d_2}\big(\mathsf{Z}^I_2\big) \sim \frac{if^{\mathsf{abc}}}{z_{12}}\;\frac{\mathcal{C}(d_1)}{[\Phi(z_2)\,\bar z_1]^{d_1}}\;\frac{\mathcal{C}(d_2)}{[\Phi(z_2)\,\bar z_2]^{d_2}}\;e^{i\varphi(z_2)\cdot(\eta_1+\eta_2)}\,J^{\mathsf{c}}(z_2)\quad (N_c\gg 1). \tag{852}$$

Next, we use the identity

$$\frac{\mathcal{C}(d_1)}{[\mu\bar z_1]^{d_1}}\,\frac{\mathcal{C}(d_2)}{[\mu\,\bar z_2]^{d_2}} \;=\; \sum_{k\geq 0}\frac{\bar z_{12}^k}{k!}\,B(d_1+k,d_2)\,\partial^k_{\bar z_2}\,\frac{\mathcal{C}(d_1+d_2)}{[\mu\bar z_2]^{d_1+d_2}}. \tag{853}$$

Substituting this formula into Eq. (852) gives the OPE of the vertex operators:

$$\mathcal{V}^{\mathsf{a}}_{d_1}\big(\mathsf{Z}^I_1\big)\,\mathcal{V}^{\mathsf{b}}_{d_2}\big(\mathsf{Z}^I_2\big) \sim \frac{if^{\mathsf{abc}}}{z_{12}}\sum_{k\geq 0}\frac{\bar z_{12}^k}{k!}\,B(d_1+k,d_2)\,\partial^k_{\bar z_2}\,\mathcal{V}^{\mathsf{c}}_{d_1+d_2}(z_2,\bar z_2,\eta_1+\eta_2)\quad (N_c\gg 1). \tag{854}$$

Corollary. Recall that the celestial gluon operators $\mathcal{G}^{\eta,\mathsf{a}}_{\Delta}(z,\bar z)$ are defined in terms of the vertex operators $\mathcal{V}^{\mathsf{a}}_{\Delta}(\mathsf{Z}^I)$ (see Eq. (846)). Using Eq. (854) then yields the following OPEs.

For two positive-helicity gluons with conformal weights Δ_i at points $\mathsf{z}_i\in\mathcal{CS}$:

$$\mathcal{G}^{+,\mathsf{a}}_{\Delta_1}(z_1,\bar z_1)\,\mathcal{G}^{+,\mathsf{b}}_{\Delta_2}(z_2,\bar z_2)\;\sim\;\frac{if^{\mathsf{abc}}}{z_{12}}\sum_{k\geq 0}\frac{\bar z_{12}^k}{k!}\,B(\Delta_1+k-1,\Delta_2-1)\,\partial^k_{\bar z_2}\,\mathcal{G}^{+,\mathsf{c}}_{\Delta_1+\Delta_2-1}(z_2,\bar z_2). \tag{855}$$

For one positive-helicity gluon and one negative-helicity gluon:

$$\mathcal{G}^{+,\mathsf{a}}_{\Delta_1}(z_1,\bar z_1)\,\mathcal{G}^{-,\mathsf{b}}_{\Delta_2}(z_2,\bar z_2)\;\sim\;\frac{if^{\mathsf{abc}}}{z_{12}}\sum_{k\geq 0}\frac{\bar z_{12}^k}{k!}\,B(\Delta_1+k-1,\Delta_2+1)\,\partial^k_{\bar z_2}\,\mathcal{G}^{-,\mathsf{c}}_{\Delta_1+\Delta_2-1}(z_2,\bar z_2). \tag{856}$$

Remark 2. We present these OPEs under the simplifying assumption that all gluons are outgoing. This hypothesis is natural in Klein-space kinematics. The corresponding OPEs for all incoming or mixed incoming/outgoing kinematics are obtained by the obvious substitutions.

The primary contributions read:

$$\mathcal{G}^{+,\mathsf{a}}_{\Delta_1}(z_1,\bar z_1)\,\mathcal{G}^{+,\mathsf{b}}_{\Delta_2}(z_2,\bar z_2)\;\sim\;\frac{if^{\mathsf{abc}}}{z_{12}}\,B(\Delta_1-1,\Delta_2-1)\,\mathcal{G}^{+,\mathsf{c}}_{\Delta_1+\Delta_2-1}(z_2,\bar z_2)\;+\;O(\bar z_{12}), \tag{857}$$

$$\mathcal{G}^{+,\mathsf{a}}_{\Delta_1}(z_1,\bar z_1)\,\mathcal{G}^{-,\mathsf{b}}_{\Delta_2}(z_2,\bar z_2)\;\sim\;\frac{if^{\mathsf{abc}}}{z_{12}}\,B(\Delta_1-1,\Delta_2+1)\,\mathcal{G}^{-,\mathsf{c}}_{\Delta_1+\Delta_2-1}(z_2,\bar z_2)\;+\;O(\bar z_{12}), \tag{858}$$

These leading terms reproduce results obtained from asymptotic-symmetry analysis and from the gauge-theory study of collinear singularities. The difference is conceptual: our derivation arises from a dynamical model for the celestial CFT, rather than from a kinematical reparameterisation of gauge theory on flat spacetime[39]. The model is realised as a theory of semiclassical minitwistor strings, and we therefore propose it as a holographic dual to $\mathcal{N}=4$ SYM on flat space at tree level across all N^k-MHV sectors.

[39] See, for a different route, Adamo *et al.* [114].

Remark 3. An alternative approach to extract the gluon S-algebra from string vertex operators dressed by worldsheet currents is to introduce a level-k WZNW current algebra on the celestial sphere $\mathcal{CS}$, and then consider the limit $k \to 0$. However, this limit is ill-defined at the quantum level because k is quantised rather than continuously variable. Thus, strictly speaking, one would have to define the current algebra with $k = 0$ from the outset, which in turn forces the OPE to vanish, $J^{\mathsf{a}} J^{\mathsf{b}} \sim 0$.

A further complication is that the level contributes to the total central charge of the sigma model. Consequently, k is not a freely adjustable parameter that can be sent to zero at will. It is constrained by the requirement of quantum anomaly cancellation.

In principle, these issues can be avoided by noting that our celestial CFT model is defined only at the semiclassical level. Nevertheless, we prefer the approach adopted above: keep $k \neq 0$ and restrict attention to the leading-trace sector by taking the large-N_c limit. This prescription is more physical, as it mirrors the standard large-N limit used in AdS/CFT. Moreover, retaining a nonzero level may prove useful in future work on the fully quantum minitwistor string, where k is expected to enter anomaly-cancellation conditions.

D. Discussion

The preceding sections developed a many-body theory of semiclassical minitwistor strings propagating on a background gauge potential on $\mathbf{MT}_s$. The physical motivation for this theory is the localisation theorem. That theorem states that the minitwistor transform[40] $\mathcal{MT}$ of tree-level N^k-MHV leaf-gluon amplitudes localises on a family of minitwistor lines $\{\mathcal{L}_m\}_{m=1}^N$, where $N = 2k + 1$.

In Section IV we showed that these amplitudes admit a geometric interpretation as semiclassical expectation values of Wilson line operators $\mathbb{W}[\mathscr{S}]$. Here $\mathscr{S}$ denotes an algebraic one-cycle on $\mathbf{MT}_s$ constructed from the localisation family $\{\mathcal{L}_m\}$. Writing the generating functional for leaf-gluon amplitudes as an expectation value of $\mathbb{W}[\mathscr{S}]$ leads to weighted volume integrals over the moduli superspace $\mathscr{M}_N$ of minitwistor lines.

To evaluate those integrals we expanded the background gauge potential $\boldsymbol{A}$ on $\mathbf{MT}_s$ (and subsequently its associated superpotential $\mathbb{A}$ on $\mathbf{X}_N$) in the basis of minitwistor superwavefunctions

[40] Defined in Subsection II D 3.

$\{\Psi_\Delta\}$,

$$\boldsymbol{A}\big|_{\mathcal{L}_m}(\lambda_i^A) \;=\; 2\pi i \int\limits_{\mathbf{MT}_s^*} \Psi_{\Delta_i}\big|_{\mathcal{L}_m}\big(\lambda_i^A;\mathsf{Z}_i'^I\big)\,\alpha_m^{\Delta_i,\mathsf{a}_i}\big(\mathsf{Z}_i'^I\big)\,\mathsf{T}^{\mathsf{a}_i}\wedge D^{2|4}\mathsf{Z}_i'. \tag{859}$$

This expansion follows from the $\mathcal{MT}$ transform introduced in Section II. The result is a collection of modes $\alpha_m^{\Delta,\mathsf{a}}$ labelled by $m = 1,\ldots,N$, so that the modes are naturally associated to the lines $\mathcal{L}_m$ in the localisation family.

Interpreting each $\alpha_m^{\Delta,\mathsf{a}}$ as the classical VEV of a gluon annihilation operator suggested a *dynamical* interpretation of the localisation theorem. In that interpretation, each line $\mathcal{L}_m$ was the image of a "minitwistor string" propagating on the background gauge field on $\mathbf{MT}_s$. The (celestial) gluon operators then arose from the worldsheet vertex-operator algebra of these minitwistor strings. This identification supplied a route from the geometry of localisation to a semiclassical, many-string description of the corresponding N^k-MHV amplitudes.

Vertex Operators. We proceeded by defining worldsheet vertex operators $\mathcal{V}^{\mathsf{a}}_{\Delta,m}$. The Picard group of the bosonic component of the target space,

$$\mathrm{Pic}(\mathbf{MT}) \;\cong\; \mathbf{Z}\otimes\mathbf{Z}, \tag{860}$$

endows each vertex operator with a conformal weight Δ. We identified Δ with the celestial conformal weight of the corresponding primary in the celestial CFT. This construction yielded a denumerable family of vertex operators labelled by the string index $m = 1,\ldots,N$. We therefore regard each $\mathcal{V}^{\mathsf{a}}_{\Delta,m}$ as being "attached" to the minitwistor line $\mathcal{L}_m \subset \mathbf{MT}_s$ that represents the classical configuration of the m-th string. To realise the dynamical picture described above, we built the (celestial) gluon operators $\mathcal{G}^{\eta,\mathsf{a}}_{\Delta,m}$ from the vertex operators $\mathcal{V}^{\mathsf{a}}_{\Delta,m}$.

Setting the background gauge field to zero isolates the interactions that arise from worldsheet insertions. In this background-free limit we computed the leading-trace semiclassical correlators of the $\mathcal{G}^{\eta,\mathsf{a}}_{\Delta,m}$ and reproduced the tree-level N^k-MHV leaf amplitude for gluons. From this result we concluded that, for each MHV level k, the semiclassical system of N minitwistor strings is holographically dual to the tree-level N^k-MHV gluonic sector of $\mathcal{N}=4$ SYM whenever $N = 2k+1$.

S-algebra. Fotopoulos and Taylor [109], Guevara *et al.* [110], Pate *et al.* [111] and Himwich, Pate, and Singh [112] showed that two independent arguments imply that the vertex operator algebra generated by the primary fields of the celestial CFT must obey an algebraic structure

known as the S-algebra[41]. One argument follows from the asymptotic-symmetry analysis of gauge theory on flat space; the other follows from the structure of gluonic collinear singularities studied by Bern *et al.* [116].

So, any dynamical model proposed as the holographic dual to flat-space gauge theory must contain a vertex operator algebra whose primary fields close on the S-algebra, irrespective of model-specific details. In Subsection V C 3 we demonstrated that the celestial gluon operators $\mathcal{G}^{\eta,\mathsf{a}_i}_{\Delta,m}$, constructed from the minitwistor-string vertex operators $\mathcal{V}^{\mathsf{a}_i}_{\Delta,m}$, realise the S-algebra at the level of primary contributions. These operators furthermore contain a tower of corrections; their explicit form is given in Eqs. (855) and (856). These corrections coincide with those obtained by rewriting the celestial leaf amplitudes as Feynman-Witten diagrams for massless scalars propagating on AdS_3, using the formalism of Casali, Melton, and Strominger [117].

Speculations. From these observations it is natural to conjecture the existence of a fully quantum-mechanical minitwistor sigma-model whose Hilbert space decomposes into a denumerable family of sectors labelled by N. We speculate that these sectors may arise either as fundamental eigenstates or, perhaps, as coherent states; semiclassically each such sector is approximated by the N-string system analysed above. In this conjectural theory each sector N of the full quantum sigma-model would provide the holographic dual to the tree-level N^k-MHV gluonic sector of $\mathcal{N}=4$ SYM.

An important feature of the N-string system is that the worldsheet does not split into N distinct components. Instead the worldsheet remains a *single* object, the celestial supersphere $\mathcal{CS}_s$. Different values of N are realised by varying the number of evaluation maps and their associated embeddings into $\mathbf{MT}_s$ obtained via the incidence relations. Thus the multiplicity of minitwistor strings is encoded in the evaluation maps rather than in disconnected worldsheet topologies.

One logical possibility is to introduce an infinite tower of evaluation maps $\Pi_{m\dot{A}}, \kappa^{\alpha}_m$ ($m \in \mathbf{N}$), and to take the full geometric action of the theory to be given by:

$$\mathcal{S}^{\infty}(\gamma^Q) \;=\; \sum_{m=1}^{\infty} \int_{\mathcal{CS}_{s,m}(\gamma^Q)} D^{1|4}\mathsf{s} \wedge \left(\frac{1}{2\pi i}\, [\Sigma_m\, \overline{\partial}_\sigma\, \Xi_m] \;+\; [\Sigma_m | m, \gamma^Q] \right). \tag{861}$$

Such a construction would produce an infinite tower of vertex operators $\mathcal{V}^{\mathsf{a}}_{\Delta,m}$ ($m \in \mathbf{N}$). From these one could then form the gluon vertex operators $\{\mathcal{G}^{\eta,\mathsf{a}}_{\Delta,m}\}_{m=1}^{\infty}$. One may consider the leading-trace

[41] Banerjee *et al.* [115] later classified all S-invariant celestial OPEs for outgoing positive-helicity gluons; they also identified Knizhnik-Zamolodchikov-type null states in theories that obey the S-algebra.

semiclassical correlators of these gluon operators, which have the form:

$$\lim_{k\to 0}\lim_{b\to 0}\left\langle \prod_{i=1}^{n} \mathcal{G}^{\eta_i,\mathsf{a}_i}_{2h_i,c_{\vec{a}}(i)}\big(z_i,\bar{z}_i\big)\right\rangle_{\mathcal{CS}}, \tag{862}$$

and which would generate tree-level gluon amplitudes at MHV level k. In this formulation, those correlators are thought of as exciting only the first N entries of the full, countably-infinite tower of minitwistor-string sectors contained in $\mathcal{S}^{\infty}$.

Equivalently, one can regard the proposal as consisting of a single worldsheet, the celestial supersphere $\mathcal{CS}_s$, together with embedding maps into the target superspace

$$\mathbf{X}_{\infty} = \bigtimes_{m\in\mathbf{N}} \mathbf{MT}_{s,m}. \tag{863}$$

In this picture, a finite-amplitude calculation would correspond to restricting attention to the first N factors of $\mathbf{X}_{\infty}$, while the full action $\mathcal{S}^{\infty}$ encodes the infinite geometric structure of the model.

Applying the BV-BRST formalism to a theory with worldsheet reparameterisation invariance and an infinite family of fields will lead to serious difficulties. This quantisation will require an infinite tower of gauge-fixing ghosts and the corresponding antighosts. The resulting spectrum will likely be non-unitary. Moreover, since each ghost system will contribute to the total central charge, the theory will likely suffer from anomalies. Hence, this renders the construction artificial.

Given these difficulties, a near-term goal is to construct a fully quantum-mechanical, topological sigma-model on $\mathcal{CS}_s$ whose target is simply the minitwistor superspace $\mathbf{MT}_s$. One should then verify whether the model admits a denumerable family of sectors labelled by $N \in \mathbf{N}$, and check whether the sectors of this putative sigma-model map onto the corresponding N^k-MHV sectors of SYM.

There is an indication that this picture can be realised. The celestial correlator obtained above can be written as an integral over the moduli superspace $\mathscr{M}_N$ of N marked minitwistor lines as follows:

$$\left\langle \prod_{i=1}^{n} \mathcal{G}^{\eta_i,\mathsf{a}_i}_{2h_i,c_{\vec{a}}(i)}\big(z_i,\bar{z}_i\big)\right\rangle_{\mathcal{CS}} = \int_{\mathscr{M}_N} d\mathbf{\Omega}_{\vec{\alpha},S}\big(\gamma^{Q}\big)\left\langle \prod_{i=1}^{n} \mathcal{G}^{\eta_i,\mathsf{a}_i}_{2h_i,c_{\vec{a}}(i)}\big(z_i,\bar{z}_i\big)\right\rangle_{\mathrm{WS}(\gamma^{Q})}. \tag{864}$$

Here $\langle\cdot\rangle_{\mathrm{WS}(\gamma^{Q})}$ denotes the correlator of the worldsheet theory, which depends on the moduli $\gamma^{Q} \in \mathscr{M}_N$.

Now, a standard computation of the scattering matrix in bosonic string theory, using the gauge-fixing procedure in the Polyakov pathintegral (see Polchinski [118, Ch. 5]), suggests the following formal analogy. Using Polchinski's notation, one recalls that

$$S(1;\ldots;n) \;\propto\; \sum_{\text{topologies}} e^{-\lambda\chi_E} \int_F \frac{d^m t}{n_R} \left\langle \prod_{k=1}^{m} B_k \prod_{i=1}^{n} \hat{\mathscr{V}}_i \right\rangle_{\mathrm{WS}} . \tag{865}$$

Here B_k denotes the b-ghost insertions, and $\hat{\mathscr{V}}_i$ stands for $\tilde{c}c\mathscr{V}_{\mathrm{m}}$ in the closed-string case and $t_{\mathsf{a}}c^{\mathsf{a}}\mathscr{V}_{\mathrm{m}}$ in the open-string case. Thus the amplitude is written as a moduli-space integral of worldsheet correlators.

There is, therefore, a strong formal analogy between Eqs. (864) and (865). For our purposes this analogy suggests a useful hint: the celestial leaf amplitudes obtained above as moduli-superspace integrals of worldsheet correlators may admit a derivation from a full quantum-mechanical minitwistor-sigma model.

To make this speculation a concrete research direction, we point out the possibility of gauging the original twistor-string theory proposed by Berkovits [119] by means of minitwistor rescaling transformations. To simplify notation, let σ denote an affine coordinate on a local patch of the holomorphic celestial sphere $\mathcal{CS} \cong \mathbf{CP}^1$. Introduce holomorphic, rational maps

$$\mathsf{Y}_I, \mathsf{W}^I : \quad \mathbf{CP}^1 \;\longrightarrow \mathbf{MT}_s \tag{866}$$

which we take to be canonically conjugate field variables. In the coordinate σ these maps are parameterised by their component fields,

$$\mathsf{Y}_I(\sigma) \;=\; \big(\omega_A(\sigma),\, \pi^{\dot{A}}(\sigma),\, \zeta_\alpha(\sigma)\big), \qquad \mathsf{W}^I(\sigma) \;:=\; \big(\lambda^A(\sigma),\, \mu_{\dot{A}}(\sigma),\, \psi^\alpha(\sigma)\big). \tag{867}$$

The kinetic sector of the Berkovits action can then be written, in spinor-helicity notation, as

$$\mathcal{I}_0[\mathsf{Y}_I, \mathsf{W}^I] \;=\; \int_{\mathcal{CS}} d\sigma \wedge \big(\langle\omega\overline{\partial}_\sigma\lambda\rangle + [\pi\overline{\partial}_\sigma\mu] + \zeta\cdot\overline{\partial}_\sigma\psi\big). \tag{868}$$

The minitwistor gauge transformations act on these fields by

$$\lambda^A \longmapsto t_1\,\lambda^A, \qquad \mu_{\dot{A}} \longmapsto t_2\,\mu_{\dot{A}}, \qquad \psi^\alpha \longmapsto t_1\,\psi^\alpha \tag{869}$$

$$\omega_A \longrightarrow t_1^{-1}\,\omega_A, \qquad \pi^{\dot{A}} \longrightarrow t_2^{-1}\,\pi^{\dot{A}}, \qquad \zeta_\alpha \longmapsto t_1^{-1}\,\zeta_\alpha. \tag{870}$$

Thus, to gauge the kinetic action we define the worldsheet $(0,1)$-currents

$$\boldsymbol{j}^\sigma := \langle\omega\lambda\rangle + \zeta\cdot\psi, \qquad \boldsymbol{k}^\sigma := [\pi\mu], \tag{871}$$

and introduce the worldsheet connection $(0,1)$-forms $\boldsymbol{a}_\sigma$ and $\boldsymbol{b}_\sigma$. These gauge potentials transform under a minitwistor gauge transformation as

$$\boldsymbol{a}_\sigma \longmapsto \boldsymbol{a}_\sigma - \overline{\partial}_\sigma \log t_1, \qquad \boldsymbol{b}_\sigma \longmapsto \boldsymbol{b}_\sigma - \overline{\partial}_\sigma \log t_2. \tag{872}$$

Hence we propose that a Berkovits-like minitwistor string theory may be governed by the action

$$\mathcal{I}[\mathsf{Y}_I, \mathsf{W}^I; \boldsymbol{a}_\sigma, \boldsymbol{b}_\sigma] \;=\; \mathcal{I}_0[\mathsf{Y}_I, \mathsf{W}^I] + \int\limits_{\mathcal{CS}} d\sigma \wedge \left(\boldsymbol{a}_\sigma \boldsymbol{j}^\sigma + \boldsymbol{b}_\sigma \boldsymbol{k}^\sigma\right) + \mathcal{I}_{\mathrm{CFT}}. \tag{873}$$

Here $\mathcal{I}_{\mathrm{CFT}}$ denotes an auxiliary matter CFT that models the theory's phenomenology, contributes to the total central charge, and is likely necessary to cancel or tame anomalies arising from quantisation.

We further speculate that this putative celestial CFT, which would serve as the full holographic dual to $\mathcal{N}=4$ SYM, need not be formulated strictly on the celestial supersphere $\mathcal{CS}_s$. Instead, it may be natural to take as worldsheet a supersymmetric generalisation of the weighted projective line[42]. Such a choice would assign to each gluon insertion point $z_i, \bar{z}_i$ a conformal weight Δ_i. We therefore leave to future work the investigation of the full quantum-mechanical treatment of a minitwistor sigma-model governed by the action (873), where the worldsheet is a weighted projective superline[43].

VI. ONE-LOOP GAUGE THEORY AMPLITUDE

A. Introduction

Consider a scattering process with $N \coloneqq m+n$ gluons in an MHV configuration in $\mathcal{N}=4$ SYM theory. We study the one-loop quantum correction to this process. This correction arises from an MHV Feynman constructed via the off-shell celestial CSW prescription.

External Gluons. The external gluons split into two sets: left (L) and right (R).

Left gluons are labelled by $i=1,...,m$. Their spinor momenta are λ_i^A and $\bar{\lambda}_{i\dot{A}}$, and their helicity states are encoded in Grassmann variables η_i^α. Right gluons are labelled by $j=1,...,n$. Their spinor momenta are ν_j^A and $\bar{\nu}_{j\dot{A}}$, and their helicity states are the fermionic variables $\tilde{\eta}_j$

$$P_L^{A\dot{A}} \coloneqq \sum_{i=1}^{m} \lambda_i^A \bar{\lambda}_i^{\dot{A}}, \;\; P_R^{A\dot{A}} \coloneqq \sum_{j=1}^{n} \nu_j^A \bar{\nu}_j^{\dot{A}}. \tag{874}$$

Loop Configuration. To describe the momenta of the virtual gluons circulating in the loop of the MHV Feynman diagram under consideration, we introduce the convention that the two lines joining the MHV vertices are labeled by $k \in \{1,2\}$, with associated four-momenta denoted by

[42] See Dolgachev [120].

[43] Drawing from Ogievetsky, Reshetikhin, and Wiegmann [121] and Schubring and Shifman [122] that sigma-models can be formulated with target spaces given by fibre bundles, it is also possible to consider target spaces beyond the minitwistor superspace. One may consider holomorphic vector superbundles over $\mathbf{MT}_s$ as alternative targets.

$L_k^{A\dot{A}}$. The *celestial CSW prescription* is as follows. Let the fixed auxiliary spinors be defined by $\boldsymbol{n}^A := (0,1)$ and $\bar{\boldsymbol{n}}_{\dot{A}} := (0,1)$. Then, the off-shell four-momentum $L_k^{A\dot{A}}$ is decomposed as:

$$L_k^{A\dot{A}} := s_{\ell_k} \ell_k^A \bar{\ell}_k^{\dot{A}} + r_k \boldsymbol{n}_k^A \bar{\boldsymbol{n}}_k^{\dot{A}}, \tag{875}$$

where $s_{\ell_k}, r_k \in \mathbf{R}_+^n$ and $\{\ell_k^A, \bar{\ell}_{k\dot{A}}\}$ is a normalised spinor basis defined by $\ell_k^A := \big(z_{\ell_k}, -1\big)$ and $\bar{\ell}_{k\dot{A}} := \big(1, -\bar{z}_k\big)$. Physically, the parameters s_{ℓ_k} and r_k play a role analogous to frequency, while the variables z_{ℓ_k} and $\bar{z}_{\ell_k}$ parametrise the insertion point of the virtual gluon on the celestial sphere in the framework of celestial CFT. In addition, we denote by η_{ℓ_k} the helicity state of the k-th virtual gluon.

Phase Superspace. An elementary calculation shows that the Lebesgue measure d^4L_k on the phase space of the k-th virtual gluon, when normalised by the massless scalar propagator $1L_k^2$, may be written in the language of the celestial CSW prescription as:

$$\frac{d^4L_k}{L_k^2} = d^2\ell_k \wedge \frac{dr_k}{r_k} \wedge \frac{ds_{\ell_k}}{s_{\ell_k}}\, s_{\ell_k}^2. \tag{876}$$

Note that the four-momentum $L_k^{A\dot{A}}$ and the helicity state η_{ℓ_k} of the k-th virtual gluon can be combined into a superchart with coordinate functions $\mathbb{L}_k^Q := \big(L_k^{A\dot{A}}, \eta_k^\alpha\big)$, which parametrise the phase superspace $\mathbf{R}^{4|4}$. The orientation on $\mathbf{R}^{4|4}$ is defined by the Berezin-de Witt volume superform:

$$d^{4|4}\mathbb{L}_k := d^4L_k \wedge d^{0|4}\eta_{\ell k}. \tag{877}$$

Scattering Superamplitude. Let $\mathcal{A}$ denote the colour-stripped one-loop gluonic scattering amplitude for the process under consideration. Applying the Feynman rules to the MHV vertices yields:

$$\mathcal{A} = (2\pi)^4\, \delta^{4|0}\big(P_L^{A\dot{A}} + P_R^{A\dot{A}}\big) \int\limits_{\mathbf{R}^{4|4}\times\mathbf{R}^{4|4}} \frac{d^{4|4}\mathbb{L}_1}{L_1^2} \wedge \frac{d^{4|4}\mathbb{L}_2}{L_2^2}\; \delta^{4|0}\big(L_2^{A\dot{A}} - L_1^{A\dot{A}} + P_L^{A\dot{A}}\big)\; \mathcal{A}_L \mathcal{A}_R, \tag{878}$$

where the Parke-Taylor factors associated with the left and right gluons are given by:

$$\mathcal{A}_L := \frac{\delta^{0|8}\big(\mathcal{Q}_L^{\alpha A}\big)}{\langle\lambda_1,\lambda_2\rangle\,\langle\lambda_2,\lambda_3\rangle \ldots \langle\lambda_{m-1},\lambda_m\rangle\,\langle\lambda_m,\ell_2\rangle\,\langle\ell_2,\ell_1\rangle\,\langle\ell_1,\lambda_1\rangle}, \tag{879}$$

$$\mathcal{A}_R := \frac{\delta^{0|8}\big(\mathcal{Q}_R^{\alpha A}\big)}{\langle\nu_1,\nu_2\rangle\,\langle\nu_2,\nu_2\rangle \ldots \langle\nu_{n-1},\nu_n\rangle\,\langle\nu_n,\ell_1\rangle\,\langle\ell_1,\ell_2\rangle\,\langle\ell_2,\nu_1\rangle}. \tag{880}$$

The supercharges for the left and right gluons are defined as:

$$\mathcal{Q}_L^{\alpha A} := \sum_{i=1}^{m} \lambda_i^A \eta_i^\alpha + \ell_2^A \eta_{\ell_2}^\alpha - \ell_1^A \eta_{\ell_1}^\alpha, \tag{881}$$

$$\mathcal{Q}_R^{\alpha A} := \sum_{j=1}^{n} \nu_j^A \tilde{\eta}_j^\alpha - \ell_2^A \eta_{\ell_2}^\alpha + \ell_1^A \eta_{\ell_1}^\alpha. \tag{882}$$

By combining Eqs. (878), (879) and (880), we obtain a single expression for the one-loop MHV amplitude. Expanding the Berezin-de Witt volume superforms into their bosonic and fermionic components, the amplitude takes the form:

$$\mathcal{A} = (2\pi)^4 \, \delta^{4|0}\big(P_L^{A\dot{A}} + P_R^{A\dot{A}}\big) \int\limits_{\mathbf{R}^4\times\mathbf{R}^4} \frac{d^4 L_1}{L_1^2} \wedge \frac{d^4 L_2}{L_2^2} \; \delta^{4|0}\big(L_2^{A\dot{A}} - L_1^{A\dot{A}} + P_L^{A\dot{A}}\big) \tag{883}$$

$$\int\limits_{\mathbf{R}^{0|4}\times\mathbf{R}^{0|4}} d^{0|4}\eta_{\ell_1} \wedge d^{0|4}\eta_{\ell_2} \; \delta^{0|4}\left(\mathcal{Q}_L^{\alpha A}\right) \wedge \delta^{0|4}\left(\mathcal{Q}_R^{\alpha A}\right) \;\; \mathcal{H}\big(\lambda_i^A, \nu_j^B, \ell_k^C\big) \tag{884}$$

Here, the holomorphic function $\mathcal{H} := \mathcal{H}\big(\lambda_i^A, \nu_j^B, \ell_k^C\big)$ is obtained by multiplying the Parke-Taylor factors corresponding to each MHV vertex. This function may be expressed compactly by introducing the notation:

$$\lambda_{m+1}^A := \ell_2^A, \;\; \nu_{n+1}^A := \ell_1^A, \;\; \ell_3^A := \ell_1^A. \tag{885}$$

Thus,

$$\mathcal{H}\big(\lambda_i^A, \nu_j^B, \psi^\alpha\big) = \frac{1}{\langle \lambda_1, \ell_1 \rangle \, \langle \nu_1, \ell_2 \rangle} \prod_{i=1}^{m} \frac{1}{\lambda_i \cdot \lambda_{i+1}} \prod_{j=1}^{n} \frac{1}{\nu_k \cdot \nu_{k+1}}. \tag{886}$$

B. Celestial Amplitude

In the preceding section, we specified the gluon configuration and detailed the structure of the Feynman diagram to be computed. We now translate from the conventional plane-wave representation of scattering amplitudes to their formulation in terms of celestial conformal primaries via the half-Mellin transform. Consequently, we derive an expression for the one-loop celestial superamplitude $\widehat{\mathcal{A}}_{\ell(1)}$ constructed from MHV vertices.

As anticipated from the behaviour of quantum field theories containing massless vector bosons, the amplitude $\widehat{\mathcal{A}}_{\ell(1)}$ exhibits infrared divergences. To isolate the finite contributions, we introduce a regularisation scheme *analogous* to dimensional regularisation, but adapted for celestial amplitudes. In our approach, regularisation is effected by introducing an infinitesimal celestial conformal weight Δ_ε for the virtual gluons circulating in the loop $\ell_1 \cup \ell_2$, rather than by analytically continuing the spacetime dimension. Once the regularised celestial amplitude : $\widehat{\mathcal{A}}_{\ell(1)}$: is obtained, we proceed to apply the leaf amplitude formalism in the subsequent subsection.

Momentum-Conserving Delta-Functions. The first step in our calculation is to invoke the integral representation of the four-dimensional delta-function:

$$\delta^{4|0}\big(P^{A\dot{A}}\big) \;=\; \int\limits_{\mathbf{R}^4} \frac{d^4x}{(2\pi)^4} \;\; \exp\big(ix\cdot P\big), \;\; x\cdot P \;\coloneqq\; x_{A\dot{A}}P^{A\dot{A}}. \tag{887}$$

Using the definitions provided in Eq. (874), we derive the following representation for the bosonic momentum-conserving delta-function for the external gluons appearing in Eq. (883):

$$\delta^{4|0}\big(P_L^{A\dot{A}} + P_R^{A\dot{A}}\big) \;=\; \int\limits_{\mathbf{R}^4} \frac{d^4x}{(2\pi)^4} \; \prod_{i=1}^{m} \exp\big(i\langle\lambda_i|x|\bar{\lambda}_i]\big) \prod_{j=1}^{n} \exp\big(i\langle\nu_j|x|\bar{\nu}_j]\big). \tag{888}$$

Here, the use of the product symbol $\prod_i$ is justified because the arguments of the exponentials are entirely bosonic.

Similarly, recalling Eq. (875) for the loop momenta $L_k^{A\dot{A}}$ ($k \in \{1,2\}$), the momentum-conserving delta-function inside the loop of the MHV Feynman diagram is given by:

$$\delta^{4|0}\big(L_2^{A\dot{A}} - L_1^{A\dot{A}} + P_L^{A\dot{A}}\big) \tag{889}$$

$$= \int\limits_{\mathbf{R}^4} \frac{d^4\tilde{x}}{(2\pi)^4} \;\; \exp\Big(is_{\ell_2}\langle\ell_2|\tilde{x}|\bar{\ell}_2] - is_{\ell_1}\langle\ell_1|\tilde{x}|\bar{\ell}_1] + i\,(r_2 - r_1)\,\langle n|\tilde{x}|\bar{n}]\Big) \prod_{i=1}^{m} \exp\big(i\langle\lambda_i|\tilde{x}|\bar{\lambda}_i]\big)\,. \tag{890}$$

Supercharge Conservation. Next, we address the fermionic delta functions associated with supercharge conservation. Given a Grassmann-valued van der Waerden spinor variable χ_A^α, the fermionic delta function $\delta^{0|8}\big(\chi_A^\alpha\big)$ is formally defined by the expansion:

$$\delta^{0|8}\big(\chi_A^\alpha\big) \;\coloneqq\; \frac{1}{2^4} \bigwedge_{\alpha=1}^{4} \varepsilon^{AB}\,\chi_A^\alpha \,\wedge\, \chi_B^\alpha\,. \tag{891}$$

Because χ_A^α is an element of a $\mathbf{Z}_2$-graded exterior algebra, its multiplication is understood in terms of the wedge product.

By substituting the definitions from Eqs. (881) and (882) for the supercharges $\mathcal{Q}_L^{\alpha A}$ and $\mathcal{Q}_R^{\alpha A}$ into Eq. (891), we deduce:

$$\delta^{0|8}\big(\mathcal{Q}_L^{\alpha A}\big) \;=\; \int d^{0|8}\theta \;\; \exp\left(i\langle\ell_2|\theta\cdot\eta_{\ell_2}\rangle - i\langle\ell_1|\theta\cdot\eta_{\ell_1}\rangle\right) \bigwedge_{i=1}^{m} \exp\big(i\langle\lambda_i|\theta\cdot\eta_i\rangle\big), \tag{892}$$

and:

$$\delta^{0|8}\big(\mathcal{Q}_R^{\alpha A}\big) \;=\; \int d^{0|8}\tilde{\theta} \;\; \exp\big(i\langle\ell_1|\tilde{\theta}\cdot\eta_{\ell_1}\rangle - i\langle\ell_2|\tilde{\theta}\cdot\eta_{\ell_2}\rangle\big) \bigwedge_{j=1}^{n} \exp\big(i\langle\nu_j|\tilde{\theta}\cdot\tilde{\eta}_j\rangle\big). \tag{893}$$

Substituting Eqs. (888)–(893) into Eq. (883) yields an integral representation for $\mathcal{A}_{\ell(1)}$. The resulting expression is lengthy but can be Mellin-transformed, which is required to derive the corresponding celestial superamplitude $\widehat{\mathcal{A}}_{\ell(1)}$. Owing to its complexity, we begin by describing the integration superdomain over which $\mathcal{A}_{\ell(1)}$ is defined, which organises the expression in a more systematic way.

Integration Superdomain. First, observe that the bosonic integration variables $x_{A\dot{A}}$ and the fermionic components θ^{α}_{A} appearing in the momentum- and supercharge-conserving delta functions (Eqs. (888) and (892)) can be combined into a single set of superspace coordinates, $\mathsf{x}^{K} := \big(x_{A\dot{A}}, \theta^{\alpha}_{A}\big)$, which parametrise the integration superdomain $\mathbf{R}^{4|8}$. Here, the abstract index K runs over $\{(A\dot{A}), (\alpha A)\}$. The orientation on $\mathbf{R}^{4|8}$ is given by the Berezin-de Witt volume superform:

$$d^{4|8}\mathsf{x} \,:=\, d^{4}x \wedge d^{0|8}\theta. \tag{894}$$

Similarly, the variables $\tilde{x}_{A\dot{A}}$ and $\tilde{\theta}^{\alpha}_{A}$, which arises from the loop-momentum-conserving and the right-gluon supercharge-conserving delta functions (Eqs. (890) and (893)), are unified into the superspace coordinates $\tilde{\mathsf{x}}^{K} := \big(\tilde{x}_{A\dot{A}}, \tilde{\theta}^{\alpha}_{A}\big)$, with orientation on $\widetilde{\mathbf{R}}^{4|8}$ defined by:

$$d^{4|8}\tilde{\mathsf{x}} \,:=\, d^{4}\tilde{x} \wedge d^{0|8}\tilde{\theta}. \tag{895}$$

As we shall show, performing a dimensional reduction on the coordinates x^{K} and $\tilde{\mathsf{x}}^{K}$ that parametrise the integration superdomain, following the prescriptions of the leaf amplitude formalism, yields the moduli parameters on $\mathbf{RP}^{3|8} \times \widetilde{\mathbf{RP}}^{3|8} \times \mathbf{R}_{+}$ that characterise the minitwistor lines on which the one-loop celestial superamplitudes localise.

To continue, the loop spinor-momenta ℓ^{A}_{k} and $\bar{\ell}_{k\dot{A}}$, together with the helicity state η^{α}_{k} of the k-th virtual gluon, may be assembled into a dual minitwistor variable, $\mathsf{W}^{I}_{\ell_k} := \big(\ell^{A}_{k}, \bar{\ell}_{k\dot{A}}, \eta^{\alpha}_{k}\big)$, which is associated with the $\mathbf{Z}_2$-graded volume form:

$$d^{2|4}\mathsf{W}_{\ell_k} \,=\, d\ell_k \wedge d\bar{\ell}_k \wedge d^{0|4}\eta_{\ell_k}. \tag{896}$$

Note that $d\ell_k \wedge d\bar{\ell}_k$ is the Lebesgue measure on $\mathbf{R}^2$, rather than the canonical holomorphic measure on $\mathbf{RP}^1$. This definition is necessary because we integrate over all *off-shell* loop momenta $L^{A\dot{A}}_{1}$ and $L^{A\dot{A}}_{2}$ in accordance with our celestial version of the CSW prescription. Moreover, our integration domain is real rather than complex since, in preparing for the application of the leaf amplitude formalism, we have assumed Kleinian signature instead of Minkowski signature.

Finally, the volume superform used to express the amplitude $\mathcal{A}_{\ell(1)}$ is obtained by combining all the previously defined measures (Eqs. (894) and (896)) into:

$$d\mathcal{V} \;\coloneqq\; d^{4|8}\mathsf{x} \wedge d^{4|8}\tilde{\mathsf{x}} \wedge d^{2|4}\mathsf{W}_{\ell_1} \wedge d^{2|4}\mathsf{W}_{\ell_2}, \tag{897}$$

which defines the orientation of the integration superdomain:

$$\mathscr{D} \coloneqq \mathbf{R}^{4|8} \times \widetilde{\mathbf{R}}^{4|8} \times \mathbf{R}^{2|4} \times \mathbf{R}^{2|4}. \tag{898}$$

Explicit Expression for $\mathcal{A}_{\ell(1)}$. With these structures in place, we find that the superamplitude can be formulated as:

$$\mathcal{A}_{\ell(1)} \;=\; \frac{1}{(2\pi)^4} \int\limits_{\mathscr{D}} d\mathcal{V}\;\; \Sigma\left(\tilde{x}\right)\, \mathcal{G}\big(\tilde{x}|\ell_k, \bar{\ell}_k\big)\, \mathcal{H}\big(\lambda_i, \nu_j, \ell_k\big)\, \exp\left(i\langle \ell_1|\theta_{\ell_1} \cdot \eta_{\ell_1}\rangle + i\langle \ell_2|\theta_{\ell_2} \cdot \eta_{\ell_2}\rangle\right) \tag{899}$$

$$\bigwedge_{i=1}^{m}\; \exp\big(i\langle\lambda_i|x+\tilde{x}|\bar{\lambda}_i] + i\langle\lambda_i|\theta\cdot\eta_i\rangle\big) \;\; \bigwedge_{j=1}^{n}\; \exp\big(i\langle\nu_j|x|\bar{\nu}_j] + i\langle\nu_j|\tilde{\theta}_j\cdot\tilde{\eta}_j\rangle\big). \tag{900}$$

In this expression, the function $\mathcal{H}$ of the spinor momenta is as defined in Eq. (886). The new objects are introduced as follows.

First, the helicity state $\eta^{\alpha}_{\ell_1}$ of the virtual gluon associated with the line ℓ_1 couples to the Grassmann-valued two-component spinor variable defined by $\theta^{\alpha A}_{\ell_1} \coloneqq \tilde{\theta}^{\alpha A} - \theta^{\alpha A}$. Similarly, the helicity state $\eta^{\alpha}_{\ell_2}$ of the virtual gluon circulating along the line ℓ_2 couples to $\theta^{\alpha A}_{\ell_2} \coloneqq -\theta^{\alpha A}_{\ell_1}$.

The quantity Σ is defined as a function of the coordinates $\tilde{x}_{A\dot{A}}$ by:

$$\Sigma\big(\tilde{x}_{A\dot{A}}\big) \;\coloneqq\; \int\limits_{\mathbf{R}^2_+} \frac{dr_1}{r_1} \wedge \frac{dr_2}{r_2}\;\; \exp\big(i\,(r_2 - r_1)\,\langle \boldsymbol{n}|\tilde{x}|\bar{\boldsymbol{n}}]\big)\,, \tag{901}$$

and this integral *diverges.* In fact, Σ contains the infrared divergence of the amplitude $\mathcal{A}_{\ell(1)}$ that is anticipated from the behaviour of gauge field theories. In conventional approaches, such divergences are isolated from the finite contributions via dimensional regularisation. In our scheme, which may be better suited for celestial holography, we introduce an infinitesimal celestial conformal weight Δ_ε (with the understanding that $\Delta_\varepsilon \to 0^+$ as $\varepsilon \to 0$) associated with the virtual gluons circulating in the loop $\ell_1 \cup \ell_2$, so that Σ is replaced by the ε-regularised expression:

$$\Sigma_\varepsilon\big(\tilde{x}\big) \;\coloneqq\; \int\limits_{\mathbf{R}^2_+} \frac{dr_1}{r_1} \wedge \frac{dr_2}{r_2}\;\; r_1^{\Delta_\varepsilon} r_2^{\Delta_\varepsilon}\, \exp\left(i\,(r_2 - r_1)\,\langle \boldsymbol{n}|\tilde{x}|\bar{\boldsymbol{n}}]\right). \tag{902}$$

A straightforward evaluation of this integral yields:

$$\Sigma_\varepsilon\big(\tilde{x}\big) \;=\; \frac{\Gamma\left(\Delta_\varepsilon\right)^2}{\langle \boldsymbol{n}|\tilde{x}|\bar{\boldsymbol{n}}]^{2\Delta_\varepsilon}}. \tag{903}$$

In what follows, we denote by $\mathcal{A}^{\varepsilon}_{\ell(1)}$ the ε-regularised one-loop gluonic superamplitude, obtained by replacing $\Sigma(\tilde{x})$ with $\Sigma_{\varepsilon}(\tilde{x})$ in Eq. (899).

The function $\mathcal{G}$ is defined by:

$$\mathcal{G}\big(\tilde{x}_{A\dot{A}}|\ell_k^A, \bar{\ell}_{k\dot{A}}\big) := \int_{\mathbf{R}_+^2} \frac{ds_{\ell_1}}{s_{\ell_1}} \wedge \frac{ds_{\ell_2}}{s_{\ell_2}}\ s_{\ell_1}^2 s_{\ell_2}^2 \exp\big(is_{\ell_2}\langle\ell_2|\tilde{x}|\bar{\ell}_2] - is_{\ell_1}\langle\ell_1|\tilde{x}|\bar{\ell}_2]\big). \tag{904}$$

Note that $\mathcal{G}$ assumes the standard form of a Mellin transform when the variables s_{ℓ_1} and s_{ℓ_2} are interpreted as frequency-like parameters. In this case, the integral can be evaluated straightforwardly, yielding:

$$\mathcal{G}\left(\tilde{x}|\ell_k, \bar{\ell}_k\right) = \frac{\mathcal{C}\,(2)}{\langle\ell_1|\tilde{x}|\bar{\ell}_1]^2}\frac{\mathcal{C}\,(2)}{\langle\ell_2|\tilde{x}|\bar{\ell}_2]^2}. \tag{905}$$

3. Half-Mellin Transform

We now proceed to derive the ε-regularised one-loop celestial superamplitude $\widehat{\mathcal{A}}^{\varepsilon}_{\ell(1)}$ by applying the Mellin transform with respect to the dotted spinor momenta $\bar{\lambda}_{i\dot{A}}$ (for $1 \leq i \leq n$) and $\bar{\nu}_{i\dot{A}}$ (for $1 \leq j \leq m$).

Preliminaries. To this end, consider the i-th gluon in the left subset L. Let $\{z_i^A, \bar{z}_{i\dot{A}}\}$ denote a normalised basis of van der Waerden spinors that parametrises the insertion point of the i-th gluon on the holomorphic celestial sphere $\mathcal{CS} \simeq \mathbf{CP}^1$. Furthermore, let $s_i \in \mathbf{R}_+$ represent the frequency associated with this gluon. Thus, we perform the substitution on $\mathcal{A}^{\varepsilon}_{\ell(1)}$ (as given in Eq. (899)):

$$\lambda_i^A \mapsto z_i^A,\ \ \bar{\lambda}_{i\dot{A}} \mapsto s_i \bar{z}_{i\dot{A}}. \tag{906}$$

Recall that, in the context of celestial CFT, the quantum state of the gluon $i \in L$ is completely specified by z_i^A, $\bar{z}_{i\dot{A}}$, and the Grassmann variable η_i^{α} encoding its helicity. These quantum numbers are naturally assembled into the dual minitwistor:

$$\mathsf{W}^I_{\mathrm{L},i} := \big(z_i^A, \bar{z}_{i\dot{A}}, \eta_i^{\alpha}\big). \tag{907}$$

Analogously, consider the j-th gluon in the right subset R. We denote by $\{w_j^A, \bar{w}_{j\dot{A}}\}$ the normalised spinor basis that characterises the j-th insertion point on the celestial sphere $\mathcal{CS}$, and let $t_j \in \mathbf{R}_+$ denote the frequency of the j-th gluon. We then substitute in $\widehat{\mathcal{A}}^{\varepsilon}_{\ell(1)}$:

$$\nu_j^A \mapsto w_j^A,\ \ \bar{\nu}_{j\dot{A}} \mapsto t_j \bar{w}_{j\dot{A}}. \tag{908}$$

Moreover, in celestial CFT, the configuration of the gluon $j \in R$ is specified by the ordered pair $\big(w_j^A, \bar{w}_{j\dot{A}}\big)$ of van der Waerden spinors, and the fermionic variable $\tilde{\eta}_j^{\alpha}$ describing its helicity, which defines the dual minitwistor:

$$\mathsf{W}^{I}_{\mathrm{R},j} := \left(w^{A}_{i}, \bar{w}_{j\dot{A}}, \eta^{\alpha}_{j}\right). \tag{909}$$

Finally, to fully characterise a gluon in celestial CFT, one must specify its celestial conformal weight. For a left-gluon $i \in L$, let $\Delta_{\mathrm{L},i}$ denote its conformal weight and $\epsilon_{\mathrm{L},i}$ its helicity expectation value. The scaling dimension of the i-th left-gluon is defined as:

$$h_{\mathrm{L},i} := \frac{\Delta_{\mathrm{L},i} + \epsilon_{\mathrm{L},i}}{2}. \tag{910}$$

Similarly, for a right-moving gluon $j \in R$, let $\Delta_{\mathrm{R},j}$ and $\epsilon_{\mathrm{R},j}$ represent its conformal weight and helicity expectation value, respectively. The scaling dimension of the j-th right-gluon is given by:

$$h_{\mathrm{R},j} := \frac{\Delta_{\mathrm{R},j} + \epsilon_{\mathrm{R},j}}{2}. \tag{911}$$

Celestial Amplitude. Accordingly, the ε-regularised one-loop celestial superamplitude $\widehat{\mathcal{A}}^{\varepsilon}_{\ell(1)}$ is defined via the half-Mellin transform as:

$$\widehat{\mathcal{A}}^{\varepsilon}_{\ell(1)}\left(\mathsf{W}^{I}_{L,i};\mathsf{W}^{J}_{R,j}\right) := \int\limits_{\mathbf{R}^{N}_{+}} d\boldsymbol{\mu}(s_i,t_j)\ \mathcal{A}^{\varepsilon}_{\ell(1)}\left(z^{A}_{i}, s_i\bar{z}_{i\dot{A}}, \eta^{\alpha}_{i}; w^{A}_{j}, t_j\bar{w}_{j\dot{A}}, \tilde{\eta}^{\alpha}_{j}\right), \tag{912}$$

where the integration measure over $\mathbf{R}^{N}_{+}$ (with $N = n + m$) is specified by:

$$d\boldsymbol{\mu}(s_i,t_j) := \bigwedge_{i=1}^{n} \frac{ds_i}{s_i}\, s_i^{2h_{\mathrm{L},i}} \bigwedge_{j=1}^{m} \frac{dt_j}{t_j}\, t_i^{2h_{\mathrm{R},j}}. \tag{913}$$

Notice that the factor $\bigwedge_{i=1}^{n} \frac{ds_i}{s_i}$ defines the Haar measure on the direct product group $\prod_{i=1}^{n} \mathbf{R}^{\times}_{+}$, where $\mathbf{R}^{\times}_{+} := (\mathbf{R}_{+}, \cdot)$ denotes the multiplicative group of positive real numbers.

Additionally, the renormalised celestial amplitude is defined as the asymptotic limit:

$$:\widehat{\mathcal{A}}^{\varepsilon}_{\ell(1)}:\left(\mathsf{W}^{I}_{\mathrm{L},i};\mathsf{W}^{J}_{\mathrm{R},j}\right) = \lim_{\varepsilon\to 0^{+}} \frac{\widehat{\mathcal{A}}^{\varepsilon}_{\ell(1)}\left(\mathsf{W}^{I}_{\mathrm{L},i};\mathsf{W}^{J}_{\mathrm{R},j}\right)}{\Gamma(\Delta_{\varepsilon})^{2}}. \tag{914}$$

Integral Representation. By performing the integral transform specified in Eq. (912) and substituting the explicit expressions for Σ_{ε}, $\mathcal{G}$ and $\mathcal{H}$ from Eqs. (902), (904) and (886) into Eq. (914), we obtain an integral representation for the celestial amplitude $\widehat{\mathcal{A}}_{\ell(1)}\left(\mathsf{W}_{\mathrm{L},i};\mathsf{W}_{\mathrm{R},j}\right)$ that we now describe.

To define the integration superdomain, we introduce the normalised $\mathbf{Z}_2$-graded volume forms:

$$\mathcal{D}^{2|4}\mathsf{W}_{\ell_1} := \frac{1}{z_1\cdot\ell_1}\, d^{2|4}\mathsf{W}_{\ell_1},\ \ \mathcal{D}^{2|4}\mathsf{W}_{\ell_2} := \frac{1}{w_1\cdot\ell_2}\, d^{2|4}\mathsf{W}_{\ell_2}. \tag{915}$$

Notice that these measures are invariant under the rescalings $\ell_1^A \mapsto t_1 \ell_1^A$ and $\ell_2^A \mapsto t_2 \ell_2^A$, respectively. Then, we redefine the volume element on the integration superdomain $\mathscr{D}$ as:

$$d\boldsymbol{V} \coloneqq d^{4|8}\mathsf{x} \wedge d^{4|8}\tilde{\mathsf{x}} \wedge \mathcal{D}^{2|4}\mathsf{W}_{\ell_1} \wedge \mathcal{D}^{2|4}\mathsf{W}_{\ell_2}. \tag{916}$$

Consequently, the renormalised celestial superamplitude $:\widehat{\mathcal{A}}_{\ell(1)}:$ can be written as:

$$:\widehat{\mathcal{A}}_{\ell(1)}:\big(\mathsf{W}^I_{\mathrm{L},i};\mathsf{W}^J_{\mathrm{R},j}\big) = \frac{1}{(2\pi)^4}\int\limits_{\mathscr{D}} d\boldsymbol{V}\;\; \mathcal{E}_L\big(\mathsf{W}^I_{\mathrm{L},i}\big)\mathcal{E}_R\big(\mathsf{W}^J_{\mathrm{R},j}\big)\mathcal{E}_\ell\big(\mathsf{W}^K_{\ell_k}\big). \tag{917}$$

The integrand is expressed in terms of the following auxiliary functions:

$$\mathcal{E}_L\big(\mathsf{W}^I_{\mathrm{L},i}\big) \coloneqq \bigwedge_{i=1}^{m} \frac{\mathcal{C}\left(2h_{\mathrm{L},i}\right)}{\langle z_i|x+\tilde{x}|\bar{z}_i]^{2h_{\mathrm{L},i}}}\, e^{i\langle z_i|\theta\cdot\eta_i\rangle}\,\frac{1}{z_i\cdot z_{i+1}}, \tag{918}$$

$$\mathcal{E}_R\big(\mathsf{W}^I_{\mathrm{R},j}\big) \coloneqq \bigwedge_{j=1}^{n} \frac{\mathcal{C}(2h_{\mathrm{R},j})}{\langle w_j|x|\bar{w}_j]^{2h_{\mathrm{R},j}}}\, e^{i\langle w_j|\tilde{\theta}\cdot\tilde{\eta}_j\rangle}\,\frac{1}{w_j\cdot w_{j+1}}, \tag{919}$$

$$\mathcal{E}_\ell\big(\mathsf{W}^I_{\ell_k}\big) \coloneqq \bigwedge_{k\in\{1,2\}} \frac{\mathcal{C}\left(2\right)}{\langle \ell_k|\tilde{x}|\bar{\ell}_k]^{2}}\, e^{i\langle \ell_k|\theta_{\ell_k}\cdot\eta_{\ell_k}\rangle}\,\frac{1}{\ell_k\cdot \ell_{k+1}}. \tag{920}$$

In other words, each auxiliary function $\mathcal{E}$ is the product of an $\mathcal{N}=4$ supersymmetric gluonic celestial wavefunction and the corresponding Parke-Taylor factor.

C. Leaf Amplitude Formulation

Finally, we apply the leaf amplitude formalism to the renormalised celestial superamplitude $\widehat{\mathcal{A}}_{\ell(1)}\big(\mathsf{W}_{\mathrm{L},i};\mathsf{W}_{\mathrm{R},j}\big)$. From a physical perspective, this procedure is equivalent to performing a dimensional reduction on the superspace coordinates x^K and $\tilde{\mathsf{x}}^K$ that parametrise the integration superdomain $\mathbf{R}^{4|8}\times\widetilde{\mathbf{R}}^{4|8}$. This reduction yields an integral representation for the celestial amplitude in terms of coordinates $\mathbb{X}^K$, $\widetilde{\mathbb{X}}^K$ and v that parametrise the integration superspace:

$$\mathscr{M}_{\ell(1)} \coloneqq \mathbf{RP}^{3|8}\times\widetilde{\mathbf{RP}}^{3|8}\times\mathbf{R}_+. \tag{921}$$

Moreover, when the amplitude is expressed in terms of minitwistor superwavefunctions $\Psi^p_\Delta\big(\mathsf{Z};\mathsf{W}\big)$, the superspace $\mathscr{M}_{\ell(1)}$ is identified with the moduli space of a closed configuration of minitwistor superlines, over which the celestial leaf amplitude localises. Thus, the transition from physical celestial amplitudes to leaf amplitudes corresponds to a dimensional reduction of the moduli space of nodal twistor lines, resulting in a closed configuration of minitwistor lines.

Before applying the leaf formalism, it is important to define precisely the geometric structures on the supermanifolds upon which the celestial leaf superamplitudes are to be constructed.

Thus, let $X_{A\dot{A}}$ denote homogeneous coordinates on the three-dimensional real projective space $\mathbf{RP}^3$, and let u denote an affine parameter on the multiplicative group of positive reals $\mathbf{R}_+^\times$. In addition, define the "projective" coordinates on $\mathbf{RP}^3$ by:

$$\mathcal{R}_{A\dot{A}} \coloneqq \frac{X_{A\dot{A}}}{|X|}, \tag{922}$$

which are invariant under rescalings $X_{A\dot{A}} \mapsto tX_{A\dot{A}}$.

Volume Forms. Recall that Klein space $\mathbf{R}^{(2,2)}$ is naturally partitioned into the lightcone Λ and the timelike and spacelike wedges, denoted by W^- and W^+, respectively. In particular, W^- is the set of all points $x_{A\dot{A}} \in \mathbf{K}^4$ such that $x^2 \coloneqq x_{A\dot{A}}x^{A\dot{A}} < 0$. We now introduce a coordinate system on W^- by defining $\mathcal{X} \coloneqq \big(u, \mathcal{R}_{A\dot{A}}\big)$, so that for every point $p \in W^-$ we have:

$$\mathcal{X}(p) \coloneqq \big(u\,(p)\,, \mathcal{R}_{A\dot{A}}\,(p)\,\big) \in \mathbf{R}_+ \times \mathbf{RP}^3. \tag{923}$$

The coordinates $\mathcal{X}$ are related to the global rectangular coordinates $x_{A\dot{A}}$ on $\mathbf{K}^4$ via:

$$x_{A\dot{A}}(p) \;=\; u(p)\,\mathcal{R}_{A\dot{A}}(p), \;\text{ for all }\; p \in W^-. \tag{924}$$

In terms of $\mathcal{X}$, the restriction of the Lebesgue measure d^4x to W^- decomposes as:

$$d^4x\big|_{W^-} \;=\; \frac{D^3X}{|X|^4} \wedge \frac{du}{u}\,u^4. \tag{925}$$

Moreover, since the standard measure on $\mathbf{R}^{4|8}$ is given by $d^{4|8}\mathsf{x} = d^4x \wedge d^{0|8}\theta$, the volume element on the supersymmetric extension of the timelike wedge $\mathbf{W}^- \subset \mathbf{R}^{4|8}$ takes the form:

$$d^{4|8}\mathsf{x}|_{\mathbf{W}^-} \;=\; \frac{D^3X}{|X|^4} \wedge d^{0|8}\theta \wedge \frac{du}{u}\,u^4. \tag{926}$$

Projective Superspace. Now, let $\mathbf{RP}^{3|8}$ denote the $(3|8)$-dimensional real projective superspace, which we identify with the trivial vector superbundle $\mathbf{RP}^3 \times \mathbf{R}^{0|8}$, with typical fibre isomorphic to the vector superspace spanned by the Grassmann-valued van der Waerden spinors θ_A^α $(1 \leq \alpha \leq 4)$. A trivialisation of this fibration is achieved by introducing the homogeneous coordinates $\mathbb{X}^K \coloneqq \big(X_{A\dot{A}}, \theta_A^\alpha\big)$ that chart the total space $\mathbf{RP}^{3|8}$. The canonical orientation on $\mathbf{RP}^{3|8}$ is then defined by the volume superform:

$$D^{3|8}\mathbb{X} \coloneqq \frac{D^3X}{|X|^4} \wedge d^{0|8}\theta. \tag{927}$$

Note that, under the superspace rescaling transformation:

$$\mathbb{X}^K = \big(X_{A\dot{A}}, \theta^\alpha_A\big) \mapsto \mathbb{X}'^K = \big(sX_{A\dot{A}}, t\theta^\alpha_A\big),$$

this measure transforms as:

$$D^{3|8}\mathbb{X} \mapsto D^{3|8}\mathbb{X}' = t^{-8} D^{3|8}\mathbb{X}.$$

Moreover, the supersymmetric extension of Eq. (925) becomes:

$$d^{4|8}\mathsf{x}|_{\mathbf{W}^+} = D^{3|8}\mathbb{X} \wedge \frac{du}{u} u^4. \tag{928}$$

2. Pre-Leaf Amplitude

Performing the dimensional reduction on the supercoordinates x^K according to the procedure developed by Sharma et. al., we find that the renormalised celestial superamplitude may be expressed as a sum of partial amplitudes:

$$: \widehat{\mathcal{A}}_{\ell(1)} : \big(\mathsf{W}_{\mathrm{L},i}; \mathsf{W}_{\mathrm{R},j}\big) = \mathcal{B}_0\big(z_i, \bar{z}_i, \eta_i; w_j, \bar{w}_j, \tilde{\eta}_j\big) + \mathcal{B}_0\big(z_i, i\bar{z}_i, \eta_i; w_j, i\bar{w}_j; \tilde{\eta}_j\big). \tag{929}$$

Notice that the second term is obtained from the first by the replacements:

$$\bar{z}_{i\dot{A}} \mapsto i\bar{z}_{i\dot{A}}, \ \ \bar{w}_{i\dot{A}} \mapsto i\bar{w}_{i\dot{A}}. \tag{930}$$

The partial superamplitude $\mathcal{B}_0\big(\mathsf{W}^I_{\mathrm{L},i}; \mathsf{W}^J_{\mathrm{R},j}\big)$ is obtained by replacing the full integral over the supercoordinates x^K, which parametrise Klein superspace, with an integral over the coordinates $\mathbb{X}^K$ that chart the supersymmetric extension of the timelike wedge, $\mathbf{W}^- \subset \mathbf{R}^{4|8}$. In particular, one has:

$$\mathcal{B}_0\big(\mathsf{W}^I_{\mathrm{L},i}; \mathsf{W}^{I'}_{\mathrm{R},j}\big) = \frac{1}{(2\pi)^4} \int \mathcal{D}^{2|4}\mathsf{W}_{\ell_1} \wedge \mathcal{D}^{2|4}\mathsf{W}_{\ell_2} \int_{\mathscr{D}_0} D^{3|8}\mathbb{X} \wedge d^{4|8}\tilde{\mathsf{x}} \ \ \mathcal{I}\big(\mathsf{W}^I_{\mathrm{L},i}; \mathsf{W}^{I'}_{\mathrm{R},j}; \mathsf{W}^{I''}_{\ell_k}\big), \tag{931}$$

where the integrals are taken over the dual minitwistors $\mathsf{W}^I_{\ell_1}$ and $\mathsf{W}^I_{\ell_2}$, which parametrise the internal (virtual) gluon states, and over the superdomain:

$$\mathscr{D}_0 \coloneqq \mathbf{RP}^{3|8} \times \mathbf{R}^{4|8}. \tag{932}$$

The function $\mathcal{I}$ is represented by an integral over the multiplicative group of positive reals $\mathscr{R} \simeq \big(\mathbf{R}_+, \cdot\big)$, endowed with the Haar measure $dH_u \coloneqq d\log u$, and is defined as:

$$\mathcal{I}\big(\mathsf{W}^I_{\mathrm{L},i}; \mathsf{W}^{I'}_{\mathrm{R},j}; \mathsf{W}^{I''}_{\ell_k}\big) = \int_{\mathscr{R}} dH_u \ \ u^4\, \mathcal{E}_L\big(\mathsf{W}^I_{\mathrm{L},i}\big)\, \mathcal{E}_R\big(\mathsf{W}^{I'}_{\mathrm{R},j}\big)\, \mathcal{E}_\ell\big(\mathsf{W}^{I''}_{\ell_k}\big). \tag{933}$$

Expressed in terms of the new integration variables, the auxiliary functions $\mathcal{E}_L$ and $\mathcal{E}_R$ take the forms:

$$\mathcal{E}_L\big(\mathsf{W}^I_{\mathrm{L},i}\big) \;=\; \bigwedge_{i=1}^{m} \frac{\mathcal{C}\big(2h_{\mathrm{L},i}\big)}{\langle z_i|u\mathcal{R}+\tilde{x}|\bar{z}_i]^{2h_{\mathrm{L},i}}}\, e^{i\langle z_i|\theta\cdot\eta_i\rangle}\, \frac{1}{z_i\cdot z_{i+1}}, \tag{934}$$

and:

$$\mathcal{E}_{\mathrm{R}}\big(\mathsf{W}^I_{\mathrm{R},j}\big) \;=\; \bigwedge_{j=1}^{n} \frac{\mathcal{C}\big(2h_{\mathrm{R},j}\big)}{\langle w_j|u\mathcal{R}|\bar{w}_j]^{2h_{\mathrm{R},j}}}\, e^{i\langle w_j|\tilde{\theta}\cdot\tilde{\eta}_j\rangle}\, \frac{1}{w_j\cdot w_{j+1}}. \tag{935}$$

Meanwhile, $\mathcal{E}_\ell$ remains as specified in Eq. (920).

Next, consider the superspace rescaling:

$$\tilde{\mathsf{x}}^K = \big(\tilde{x}_{A\dot{A}}, \tilde{\theta}^\alpha_A\big) \;\mapsto\; \big(u\tilde{x}_{A\dot{A}}, \tilde{\theta}^\alpha_A\big).$$

Under this transformation, the Berezin-de Witt volume superform transforms as:

$$d^{4|8}\tilde{\mathsf{x}} \;\mapsto\; u^4\, d^{4|8}\tilde{\mathsf{x}}.$$

Substituting these transformations into Eqs. (931)–(935) allows the affine parameter u to factorise and be integrated, yielding:

$$\mathcal{B}_0\big(\mathsf{W}^I_{\mathrm{L},i};\mathsf{W}^{I'}_{\mathrm{R},j}\big) \;=\; \frac{1}{(2\pi)^3}\,\delta\big(\beta_{\ell(1)}\big)\,\mathcal{B}_1\big(\mathsf{W}^I_{\mathrm{L},i};\mathsf{W}^{I'}_{\mathrm{R},j}\big), \tag{936}$$

where the quantity $\beta_{\ell(1)}$, which encodes the *total scaling dimension* of the scattering process, takes the form:

$$\beta_{\ell(1)} \;\coloneqq\; 4 - 2\sum_{i=1}^{m} h_{\mathrm{L},i} - 2\sum_{j=1}^{n} h_{\mathrm{R},j}. \tag{937}$$

The new partial amplitude $\mathcal{B}_1 \coloneqq \mathcal{B}_1\big(\mathsf{W}^I_{\mathrm{L},i};\mathsf{W}^J_{\mathrm{R},j}\big)$, which we term the *pre-leaf amplitude*, is defined by:

$$\mathcal{B}_1\big(\mathsf{W}^I_{\mathrm{L},i};\mathsf{W}^{I'}_{\mathrm{R},j}\big) \;=\; \int\limits_{\mathbf{R}^{4|8}} \mathcal{D}^{2|4}\mathsf{W}_{\ell_1}\wedge\mathcal{D}^{2|4}\mathsf{W}_{\ell_2} \int\limits_{\mathscr{D}_0} D^{3|8}\mathbb{X}\wedge d^{4|8}\tilde{\mathsf{x}}\;\, \mathcal{F}_{\mathrm{L}}\big(\mathsf{W}^I_{\mathrm{L},i}\big)\,\mathcal{F}_{\mathrm{R}}\big(\mathsf{W}^{I'}_{\mathrm{R},j}\big)\,\mathcal{E}_\ell\big(\mathsf{W}^{I''}_{\ell_k}\big). \tag{938}$$

Here, the new auxiliary functions are represented by:

$$\mathcal{F}_{\mathrm{L}}\big(\mathsf{W}^I_{\mathrm{L},i}\big) \;\coloneqq\; \bigwedge_{i=1}^{m} \frac{\mathcal{C}\big(2h_{\mathrm{L},i}\big)}{\langle z_i|\mathcal{R}+\tilde{x}|\bar{z}_i]^{2h_{\mathrm{L},i}}}\, e^{i\langle z_i|\theta\cdot\eta_i\rangle}\, \frac{1}{z_i\cdot z_{i+1}}, \tag{939}$$

and:

$$\mathcal{F}_{\mathrm{R}}\big(\mathsf{W}^I_{\mathrm{R},j}\big) \;\coloneqq\; \bigwedge_{j=1}^{n} \frac{\mathcal{C}\big(2h_{\mathrm{R},j}\big)}{\langle w_j|\mathcal{R}|\bar{w}_j]^{2h_{\mathrm{R},j}}}\, e^{i\langle w_j|\tilde{\theta}\cdot\eta_j\rangle}\, \frac{1}{w_j\cdot w_{j+1}}. \tag{940}$$

These expressions complete the dimensional reduction from integration over the supercoordinates x^K, which parametrise $\mathbf{R}^{4|8}$, to the coordinates $\mathbb{X}^K$ that parametrise $\mathbf{RP}^{3|8}$. The final step in obtaining the leaf amplitude is to carry out the reduction with respect to the remaining integration supervariables, namely $\tilde{\mathsf{x}}^K \in \widetilde{\mathbf{R}}^{4|8}$.

Next, we address the final step required to derive the (renormalised, one-loop) *leaf* superamplitude for gluons in celestial CFT.

Notation. For clarity, we denote by $\widetilde{\mathbf{R}}^{4|8}$ the integration superdomain parametrised by $\tilde{\mathsf{x}}^K$, thus distinguishing it from the domain over which x^K is defined. Similarly, the projective superspace obtained by dimensionally reducing $\widetilde{\mathbf{R}}^{4|8}$ is denoted by $\widetilde{\mathbf{RP}}^{3|8}$.

Preliminaries. Proceeding analogously to the argument developed in the preceding section, let $\widetilde{X}_{A\dot{A}}$ denote homogeneous coordinates on $\widetilde{\mathbf{RP}}^{3|8}$, and define the corresponding projective coordinates by:

$$\widetilde{\mathcal{R}}_{A\dot{A}} := \frac{\widetilde{X}_{A\dot{A}}}{|\widetilde{X}|}. \tag{941}$$

Next, on the timelike wedge $\widetilde{W}^-$ of $\widetilde{\mathbf{R}}^{4|8}$, we introduce the coordinates $\widetilde{\mathcal{X}} := \big(v, \widetilde{\mathcal{R}}_{A\dot{A}}\big)$, which are related to the global rectangular coordinates $\tilde{x}_{A\dot{A}}$ via:

$$\tilde{x}_{A\dot{A}}(p) \;=\; v(p)\,\widetilde{\mathcal{R}}_{A\dot{A}}(p), \;\; \text{for all}\, p \in \widetilde{W}^-. \tag{942}$$

Thus, the Lebesgue measure $d^4\tilde{x}$ restricted to $\widetilde{W}^-$ decomposes as:

$$d^4\tilde{x}\,|_{\widetilde{W}^-} \;=\; \frac{D^3\widetilde{X}}{|\widetilde{X}|^4} \wedge \frac{dv}{v}\, v^4. \tag{943}$$

Turning to the supersymmetric extension of the above structures, let $\widetilde{\mathbb{X}}^K := \big(\widetilde{X}_{A\dot{A}}, \tilde{\theta}^{\alpha}_{\dot{A}}\big)$ denote a global trivialisation of the vector superbundle $\widetilde{\mathbf{RP}}^{3|8} \simeq \widetilde{\mathbf{RP}}^{3} \times \mathbf{R}^{0|8}$. The total space is endowed with the orientation defined by the volume superform:

$$D^{3|8}\widetilde{\mathbb{X}} := \frac{D^3\widetilde{X}}{|\widetilde{X}|^4} \wedge d^{0|8}\tilde{\theta}. \tag{944}$$

Recall that the standard measure on Klein superspace is given by $d^{4|8}\tilde{\mathsf{x}} = d^4\tilde{x} \wedge d^{0|8}\tilde{\theta}$, and let $\widetilde{\mathbf{W}}^-$ denote the supersymmetric extension of the timelike wedge. Hence, Eq. (943) generalises to:

$$d^{4|8}\tilde{\mathsf{x}}\,|_{\widetilde{\mathbf{W}}^-} \;=\; D^{3|8}\widetilde{\mathbb{X}} \wedge \frac{dv}{v}\, v^4. \tag{945}$$

Dimensional Reduction. We now apply the leaf formalism to dimensionally reduce the integral over $\tilde{\mathsf{x}}^K$ in Eq. (938). In this procedure, we find that the function $\mathcal{B}_1$ decomposes into a sum of partial amplitudes:

$$\mathcal{B}_1\big(\mathsf{W}^I_{\mathrm{L},i}; \mathsf{W}^J_{\mathrm{R},j}\big) \;=\; \mathcal{M}_{\ell(1)}\big(z_i, \bar{z}_i, \eta_i; w_j, \bar{w}_j, \tilde{\eta}_j\big) - i\mathcal{M}_{\ell(1)}\big(z_i, i\bar{z}_i, \eta_i; w_j, i\bar{w}_j, \tilde{\eta}_j\big). \tag{946}$$

Here, the arguments of the second term are obtained from those of the first by the replacements $\bar{z}_{i\dot{A}} \mapsto i\bar{z}_{i\dot{A}}$ and $\bar{w}_{i\dot{A}} \mapsto i\bar{w}_{i\dot{A}}$.

The partial superamplitude $\mathcal{M}_{\ell(1)} := \mathcal{M}_{\ell(1)}\big(\mathsf{W}^{I}_{\mathrm{L},i};\mathsf{W}^{I'}_{\mathrm{R},j}\big)$ is precisely the renormalised, one-loop corrected celestial *leaf* superamplitude describing the scattering of gluons in configurations characterised by maximal-helicity-violation in $\mathcal{N} = 4$ SYM theory. We now describe the explicit formulation of $\mathcal{M}_{\ell(1)}$.

Let the *moduli space* be defined by the supermanifold:

$$\mathscr{M}_{\ell(1)} := \mathbf{RP}^{3|8} \times \widetilde{\mathbf{RP}}^{3|8} \times \mathbf{R}_{+}, \tag{947}$$

which is charted by the ($\mathbf{Z}_2$-graded) coordinate map:

$$p \in \mathscr{M}_{\ell(1)} \mapsto \tau^{M}(p) := \big(\mathbb{X}^{K}(p), \widetilde{\mathbb{X}}^{K'}(p), v(p)\big), \tag{948}$$

with M serving as the abstract index for the supercoordinates. The standard orientation on $\mathscr{M}_{\ell(1)}$ is provided by the volume superform:

$$d\boldsymbol{\tau} := D^{3|8}\mathbb{X} \wedge D^{3|8}\widetilde{\mathbb{X}} \wedge \frac{dv}{v}. \tag{949}$$

Thus, the leaf superamplitude is expressed as:

$$\mathcal{M}_{\ell(1)}\big(\mathsf{W}^{I}_{\mathrm{L},i};\mathsf{W}^{I'}_{\mathrm{R},j}\big) = \int \mathcal{D}^{2|4}\mathsf{W}_{\ell_1} \wedge \mathcal{D}^{2|4}\mathsf{W}_{\ell_2} \int\limits_{\mathscr{M}_{\ell(1)}} d\boldsymbol{\tau}\; \mathcal{F}_{\mathrm{L}}\big(\mathsf{W}^{I}_{\mathrm{L},i}\big)\mathcal{F}_{\mathrm{R}}\big(\mathsf{W}^{I'}_{\mathrm{R},j}\big)\mathcal{F}_{\ell}\big(\mathsf{W}^{I''}_{\ell_k}\big), \tag{950}$$

where the auxiliary functions are given by:

$$\mathcal{F}_{\mathrm{L}}\big(\mathsf{W}^{I}_{\mathrm{L},i}\big) = \bigwedge_{i=1}^{m} \frac{\mathcal{C}\big(2h_{\mathrm{L},i}\big)}{\langle z_i|\mathcal{R} + v\widetilde{\mathcal{R}}|\bar{z}_i]^{2h_{\mathrm{L},i}}}\, e^{i\langle z_i|\theta\cdot\eta_i\rangle}\, \frac{1}{z_i \cdot z_{i+1}}, \tag{951}$$

$$\mathcal{F}_{\mathrm{R}}\big(\mathsf{W}^{I}_{\mathrm{R},j}\big) = \bigwedge_{j=1}^{n} \frac{\mathcal{C}\big(2h_{\mathrm{R},j}\big)}{\langle w_j|\mathcal{R}|\bar{w}_j]^{2h_{\mathrm{R},j}}}\, e^{i\langle w_j|\tilde{\theta}\cdot\tilde{\eta}_j\rangle}\, \frac{1}{w_j \cdot w_{j+1}}, \tag{952}$$

$$\mathcal{F}_{\ell}\big(\mathsf{W}^{I}_{\ell_k}\big) = \bigwedge_{k=1}^{2} \frac{\mathcal{C}(2)}{\langle \ell_k|\widetilde{\mathcal{R}}|\bar{\ell}_k]^{2}}\, e^{i\langle \ell_k|\theta_{\ell_k}\cdot\eta_{\ell_k}\rangle}\, \frac{1}{\ell_k \cdot \ell_{k+1}}. \tag{953}$$

D. One-loop Minitwistor Superamplitude

Celestial/Minitwistor Correspondence. The renormalised one-loop celestial leaf amplitude $\mathcal{M}_{\ell(1)}$ was derived in the preceding section (cf. Eq. (950)). This amplitude is a function of the dual minitwistors:

$$\mathsf{W}^{I}_{\mathrm{L},i} = \big(z^{A}_{i}, \bar{z}_{i\dot{A}}, \eta^{\alpha}_{i}\big) \text{ and } \mathsf{W}^{I'}_{\mathrm{R},j} = \big(w^{A}_{j}, \bar{w}_{j\dot{A}}, \tilde{\eta}^{\alpha}_{j}\big),$$

which encode the quantum states of the left-moving gluon $i \in L$ and the right-moving gluon $j \in R$ from the perspective of celestial CFT. In this context, the ordered pairs $\left(z_i^A, \bar{z}_{i\dot{A}}\right)$ and $\left(w_j^A, \bar{w}_{j\dot{A}}\right)$ of van der Waerden spinors parametrise the insertion points of the i-th and j-th gluons on the celestial sphere $\mathcal{CS}$, while the Grassmann variables η_i^α and $\tilde{\eta}_j^\alpha$ encode their helicity degrees of freedom. In other words, describing the state of a gluon using the dual minitwistors $\mathsf{W}_{\mathrm{L},i}^I$ and $\mathsf{W}_{\mathrm{R},j}^{I'}$ is formally *analogous* to describing the state of a nonrelativistic particle in the *position* basis in elementary quantum mechanics.

Recall that, in elementary quantum mechanics, one may equivalently describe the state of a particle in momentum space by means of the Fourier transform. In our setting, we transition from the description in terms of celestial supercoordinates $\left(z^A, \bar{z}_{\dot{A}}, \eta^\alpha\right)$ to a representation in terms of minitwistor superspace coordinates $\mathsf{Z}^I = \left(\lambda^A, \mu_{\dot{A}}, \psi^\alpha\right)$ via the minitwistor transform.

Celestial RSVW Representation. To that end, we begin with the celestial RSVW identity. Let $f := f\left(\sigma^A\right)$ be a homogeneous holomorphic function of an undotted van der Waerden spinor σ^A that satisfies:

$$f\left(t\sigma^A\right) = t^{-2} f\left(\sigma^A\right), \;\; \text{for all}\, t \in \mathbf{C}^*. \tag{954}$$

Furthermore, let $\mathcal{L} := \mathcal{L}\left(X, \theta\right) \subset \mathbf{MT}_s$ denote the minitwistor line defined by the embedding:

$$\mathsf{Z}_{\mathcal{L}}^I : \mathbf{CP}^1 \longrightarrow \mathbf{MT}_s,$$

whose parametrisation in terms of homogeneous coordinates σ^A is given by:

$$\mathsf{Z}_{\mathcal{L}}^I\left(\sigma^A\right) \;:=\; \left(\sigma^A, \sigma^A X_{A\dot{A}}, \sigma^A \theta_A^\alpha\right). \tag{955}$$

Then, the celestial RSVW formula may be expressed as:

$$\frac{\mathcal{C}(\Delta)}{\langle z|X|\bar{z}]^\Delta}\, e^{i\langle z|\theta\cdot\eta\rangle} f\left(z\right) \;=\; \int\limits_{\mathbf{MT}_s} D^{2|4}\mathsf{Z} \;\; \Psi_\Delta\left(\mathsf{Z}^I; \mathsf{W}^{I'}\right) \int\limits_{\mathbf{CP}^1} D\sigma \;\; \overline{\delta}^{2|4}_{(-\Delta,\Delta)}\left(\mathsf{Z}^I; \mathsf{Z}_{\mathcal{L}}^{I'}\left(\sigma\right)\right) f\left(\sigma\right). \tag{956}$$

Substituting this identity into Eq. (950), we obtain the following result. Let:

$$\mathsf{Z}_{\mathrm{L}}^I, \mathsf{Z}_{\mathrm{R}}^I, \mathsf{Z}_{\ell}^I : \mathbf{CP}^1 \longrightarrow \mathbf{MT}_s,$$

be a family of embeddings of minitwistors lines, defined respectively by:

$$\mathsf{Z}_{\mathrm{L}}^I\left(\sigma^A\right) \;:=\; \left(\sigma^A, \sigma^A\left(\mathcal{R}_{A\dot{A}} + v\widetilde{\mathcal{R}}_{A\dot{A}}\right), \sigma^A \theta_A^\alpha\right), \tag{957}$$

$$\mathsf{Z}_{\mathrm{R}}^I\left(\sigma^A\right) \;:=\; \left(\sigma^A, \sigma^A \mathcal{R}_{A\dot{A}}, \sigma^A \tilde{\theta}_A^\alpha\right), \tag{958}$$

$$\mathsf{Z}^I_\ell(\sigma^A) \;\coloneqq\; \left(\sigma^A, \sigma^A\widetilde{\mathcal{R}}_{A\dot{A}}, \sigma^A\theta^\alpha_{\ell_k A}\right). \tag{959}$$

Here, the bosonic variables $\mathcal{R}_{A\dot{A}}$, $\widetilde{\mathcal{R}}_{A\dot{A}}$, and v, along with the fermionic coordinates $\theta^{\alpha A}$, $\tilde{\theta}^{\alpha A}$, and $\theta^{\alpha A}_{\ell_k}$, serve as the moduli parameters that characterise the system (957)–(959) of minitwistor lines. This observation justifies our interpretation of the supervariables τ^M charting $\mathscr{M}_{\ell(1)}$ as the moduli parameters and of the superspace $\mathscr{M}_{\ell(1)}$ as the corresponding moduli space.

Minitwistor Amplitude. Consequently, the leaf superamplitude becomes a *minitwistor transform*:

$$\mathcal{M}_{\ell(1)}\left(\mathsf{W}^I_{\mathrm{L},i};\mathsf{W}^{I'}_{\mathrm{R},j}\right) \;=\; \int \mathcal{D}^{2|4}\mathsf{W}_{\ell_1}\wedge\mathcal{D}^{2|4}\mathsf{W}_{\ell_2}\int\limits_{\mathscr{E}^M} d\boldsymbol{\nu}\;\widetilde{\mathcal{M}}_{\ell(1)}\left(\mathsf{Z}^I_{\mathrm{L},i};\mathsf{Z}^{I'}_{\mathrm{R},j};\mathsf{Z}^{I''}_{\ell_k}\right), \tag{960}$$

where the integration superdomain:

$$\mathscr{E}^M \;\coloneqq\; \prod_{i=1}^{M}\mathbf{MT}_s,\;\; M\coloneqq m+n+2,$$

is endowed with an orientation determined by the volume superform:

$$d\boldsymbol{\nu} = \bigwedge_{i=1}^{m} D^{2|4}\mathsf{Z}_{\mathrm{L},i}\Psi_{2h_{\mathrm{L},i}}\left(\mathsf{Z}_{\mathrm{L},i};\mathsf{W}_{\mathrm{L},i}\right)\bigwedge_{j=1}^{n} D^{2|4}\mathsf{Z}_{\mathrm{R},j}\Psi_{2h_{\mathrm{R},j}}\left(\mathsf{Z}_{\mathrm{R},j};\mathsf{W}_{\mathrm{R},j}\right)\bigwedge_{k=1}^{2} D^{2|4}\mathsf{Z}_{\ell_k}\Psi_2\left(\mathsf{Z}_{\ell_k};\mathsf{W}_{\ell_k}\right). \tag{961}$$

Accordingly, the (renormalised, one-loop) *minitwistor superamplitude* encoding the scattering of $m+n$ gluons in $\mathcal{N}=4$ SYM theory is given by:

$$\widetilde{\mathcal{M}}_{\ell(1)}\left(\mathsf{Z}^I_{\mathrm{L},i};\mathsf{Z}^{I'}_{\mathrm{R},j};\mathsf{Z}^{I''}_{\ell_k}\right) \;=\; \int\limits_{\mathscr{M}_{\ell(1)}} d\boldsymbol{\tau}\;\mathcal{G}_{\mathrm{L}}\left(\mathsf{Z}^I_{\mathrm{L},i}\right)\mathcal{G}_{\mathrm{R}}\left(\mathsf{Z}^{I'}_{\mathrm{R},j}\right)\mathcal{G}_{\ell(1)}\left(\mathsf{Z}^{I''}_{\ell_k}\right), \tag{962}$$

where the integrand functions are defined as:

$$\mathcal{G}_{\mathrm{L}}\left(\mathsf{Z}^I_{\mathrm{L},i}\right) \;=\; \int\limits_{\mathscr{L}^m}\bigwedge_{i=1}^{m}\boldsymbol{\omega}\left(\sigma^A_i\right)\;\bar{\delta}^{2|4}_{\left(-2h_{\mathrm{L},i},2h_{\mathrm{L},i}\right)}\left(\mathsf{Z}^I_{\mathrm{L},i};\mathsf{Z}^{I'}_{\mathrm{L}}\left(\sigma^A_i\right)\right), \tag{963}$$

$$\mathcal{G}_{\mathrm{R}}\left(\mathsf{Z}^I_{\mathrm{R},j}\right) \;=\; \int\limits_{\mathscr{L}^n}\bigwedge_{j=1}^{n}\boldsymbol{\omega}\left(\sigma^A_j\right)\;\bar{\delta}^{2|4}_{\left(-2h_{\mathrm{R},j},2h_{\mathrm{R},j}\right)}\left(\mathsf{Z}^I_{\mathrm{R},j};\mathsf{Z}^{I'}_{\mathrm{R}}\left(\sigma^A_j\right)\right), \tag{964}$$

$$\mathcal{G}_{\ell}\left(\mathsf{Z}^I_{\ell_k}\right) \;=\; \int\limits_{\mathscr{L}^2}\bigwedge_{k=1}^{2}\boldsymbol{\omega}\left(\sigma^A_k\right)\;\bar{\delta}^{2|4}_{(-2,2)}\left(\mathsf{Z}^I_{\ell_k};\mathsf{Z}^{I'}_{\ell}\left(\sigma^A_k\right)\right). \tag{965}$$

Here, the integration domain is:

$$\mathscr{L}^m \;\coloneqq\; \prod_{i=1}^{m}\mathbf{CP}^1,$$

and the logarithmic differential form is given by:

$$\boldsymbol{\omega}\left(\sigma^A_i\right) \;\coloneqq\; \frac{D\sigma_i}{\sigma_i\cdot\sigma_{i+1}}. \tag{966}$$

Geometric Interpretation. It follows that the integrand of the minitwistor superamplitude is localised on the system (957)–(959) of minitwistor lines. Consequently, the full superamplitude $\mathcal{M}_{\ell(1)}$ admits a geometric interpretation as an integral over the moduli space parametrising this system of lines in minitwistor superspace. Furthermore, the celestial amplitude vanishes whenever the configuration $\{\mathsf{W}^{I}_{\mathrm{L},i}, \mathsf{W}^{I'}_{\mathrm{R},j}\}$ of external gluons does not lie along the minitwistor lines:

$$\mathscr{L}_{\mathrm{L}}(\mathcal{R}, \widetilde{\mathcal{R}}, \theta) \coloneqq \{(\lambda^A, \mu_{\dot{A}}, \psi^\alpha) \in \mathbf{MT}_s \,|\, \mu_{\dot{A}} = \lambda^A(\mathcal{R}_{A\dot{A}} + v\widetilde{\mathcal{R}}_{A\dot{A}}),\ \ \psi^\alpha = \lambda^A \theta^\alpha_A\}, \tag{967}$$

$$\mathscr{L}_{\mathrm{R}}(\mathcal{R}, \tilde{\theta}) \coloneqq \{(\lambda^A, \mu_{\dot{A}}, \psi^\alpha) \in \mathbf{MT}_s \,|\, \mu_{\dot{A}} = \lambda^A \mathcal{R}_{A\dot{A}},\ \ \psi^\alpha = \lambda^A \tilde{\theta}^\alpha_A\}. \tag{968}$$

Appendix A: Mini-Introduction to Holomorphic Gauge Theory

As shown in Section IV, every tree-level celestial gluon amplitude in $\mathcal{N} = 4$ SYM theory, including all N^k-MHV sectors, can be reconstructed from holomorphic Wilson line operators on minitwistor superspace. We interpret this result as evidence that these Wilson lines are gauge-invariant observables of a “minitwistor string field theory.” Such a field theory arises as the effective description of a minitwistor sigma-model.

This geometric perspective on scattering amplitudes suggests a new mathematical framework for gauge theory in asymptotically flat spacetimes. To prepare the reader, we now give a concise introduction to holomorphic gauge theory (HGT) on minitwistor superspace. We emphasise physical intuition over mathematical rigour and explain how familiar gauge-theory concepts emerge from *complex* differential geometry.

We focus on the concept of a *pseudoholomorphic structure* along with its related notions of partial connections and pseudocurvature. The idea of a partial connection was introduced by Bott [123] and Rawnsley [124] in the context of holomorphic foliations. Our exposition follows the pedagogical treatments of Donaldson and Kronheimer [125] and the reviews by Donaldson [126] and Guichard [127]. For a full, rigorous account, see Kobayashi [128], Chern *et al.* [129] and Moroianu [130].

1. Basic Structures

In this subsection, we introduce the framework for constructing minitwistor Wilson lines. We begin with an algebraic formulation. Then we incorporate the analytic structure of a holomorphic gauge theory, reviewing the necessary concepts as we proceed.

Notation 4. Let $\pi\colon E \to \mathbf{MT}_s$ be a complex vector bundle. We describe physical fields as sections of E, possibly tensored with exterior powers of the cotangent bundle. We define the space of smooth E-valued differential forms of bidegree (r, s) as:

$$\Omega^{r,s}(\mathbf{MT}_s; E) \;\coloneqq\; \Gamma\big(\mathbf{MT}_s; \textstyle\bigwedge^{r,s} T^*(\mathbf{MT}_s)\otimes E\big). \tag{A1}$$

We denote the space of smooth vector fields by:

$$\mathscr{X}(\mathbf{MT}_s) \;\coloneqq\; \Gamma\big(\mathbf{MT}_s;\, T(\mathbf{MT}_s)\big). \tag{A2}$$

In particular, the dual space satisfies:

$$\mathscr{X}^*(\mathbf{MT}_s) \;\cong\; \Omega^1(\mathbf{MT}_s; \mathbf{C}). \tag{A3}$$

To introduce a gauge theory, we require that each bundle fibre carry a gauge Lie-algebra structure. Specifically, for any point $\mathsf{w} \in \mathbf{MT}_s$, we demand $E|_{\mathsf{w}} \cong \mathfrak{g}$, where $\mathfrak{g}$ is a complexified Lie algebra. This identification endows each fibre with the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$.

We then unify this bracket with the exterior calculus of differential forms by defining a generalised wedge product $\wedge_{\mathfrak{g}}$. For two $\mathfrak{g}$-valued forms α and β, $\alpha \wedge_{\mathfrak{g}} \beta$ computes their pointwise Lie bracket in the fibre and antisymmetrises their form components. Later, we shall express the fieldstrength tensor using $\wedge_{\mathfrak{g}}$.

Definition 5. Let $\alpha \in \Omega^{r,s}(\mathbf{MT}_s; E)$ and $\beta \in \Omega^{k,\ell}(\mathbf{MT}_s; E)$ be two $\mathfrak{g}$-valued forms. We define their generalised wedge product

$$\alpha \wedge_{\mathfrak{g}} \beta \;\in\; \Omega^{r+k,s+\ell}(\mathbf{MT}_s; E) \tag{A4}$$

by its contraction with vector fields. For any $X_i \in \mathscr{X}(\mathbf{MT}_s)$, set $p = r+s$ and $q = k+\ell$ and write:

$$i_{X_1\wedge\cdots\wedge X_{p+q}}\big(\alpha \wedge_{\mathfrak{g}} \beta\big) \;\coloneqq\; \sum_{\upsilon\in S_{p+q}} \frac{(-1)^{|\nu|}}{p!\,q!}\,\big[\alpha(X_{\upsilon(1)},\ldots,X_{\upsilon(p)}),\beta(X_{\upsilon(p+1)},\ldots,X_{\upsilon(p+q)})\big]_{\mathfrak{g}}, \tag{A5}$$

where i_X denotes the interior product with X, the sum runs over all $\upsilon \in S_{p+q}$, and $|\upsilon|$ is the parity of υ.

The basic object in HGT is the gauge potential (partial connection) $\boldsymbol{A}\big(\mathsf{W}^I\big)$. It is a $(0,1)$-form whose values are linear operators on the fibres of E. Hence

$$\boldsymbol{A} \;\in\; \Omega^{0,1}\big(\mathbf{MT}_s;\, \mathrm{End}_{\mathbf{C}}(E)\big) \;\cong\; \Omega^{0,1}\big(\mathbf{MT}_s;\, \mathfrak{gl}(r,\mathbf{C})\big). \tag{A6}$$

This partial connection deforms the canonical holomorphic structure of the bundle. Its fieldstrength $\boldsymbol{F}$ captures the pseudocurvature of the theory.

Our final algebraic tool defines how the connection $\boldsymbol{A}$ acts on E-valued fields. Since $\boldsymbol{A}$ is an $\mathrm{End}_{\mathbf{C}}(E)$-valued form, we build its action from the natural evaluation map:

$$\varepsilon\colon\ \mathrm{End}_{\mathbf{C}}(E)\otimes E \longrightarrow E, \qquad \varepsilon\big(T\otimes|e\rangle\big) \coloneqq T|e\rangle, \tag{A7}$$

for any operator $T \in \mathrm{End}_{\mathbf{C}}(E)$ and vector $|e\rangle \in E$. We then promote this to a second exterior product $\wedge_\varepsilon$.

Definition 6. Let

$$\mathcal{T} \in \Omega^{r,s}\big(\mathbf{MT}_s;\, \mathrm{End}_{\mathbf{C}}(E)\big) \quad \text{and} \quad |\boldsymbol{e}\rangle \in \Omega^{k,\ell}\big(\mathbf{MT}_s; E\big), \tag{A8}$$

and set $p = r+s$, $q = k+\ell$. For any vector fields $X_i \in \mathscr{X}(\mathbf{MT}_s)$,

$$i_{X_1\wedge\cdots\wedge X_{p+q}}\big(\mathcal{T}\wedge_\varepsilon \boldsymbol{e}\big) \coloneqq \sum_{\upsilon\in S_{p+q}} \frac{(-1)^{|\nu|}}{p!\,q!}\,\varepsilon\big(\mathcal{T}(X_{\upsilon(1)},\dots,X_{\upsilon(p)})\otimes\big|\boldsymbol{e}(X_{\upsilon(p+1)},\dots,X_{\upsilon(p+q)})\big\rangle\big). \tag{A9}$$

This construction defines the covariant action of $\boldsymbol{A}$. For an E-valued field $|\psi\rangle$, the gauge interaction term is:

$$\boldsymbol{A}\wedge_\varepsilon|\psi\rangle, \tag{A10}$$

so the pseudoholomorphic covariant derivative reads:

$$\overline{\partial}^{\mathscr{E}}|\psi\rangle = \overline{\partial}^{E}|\psi\rangle + \boldsymbol{A}\wedge_\varepsilon|\psi\rangle. \tag{A11}$$

2. Partial Connections and Holomorphic Gauge Potentials

We identify the vacuum of HGT with the canonical holomorphic structure on the bundle $E \to \mathbf{MT}_s$. Nontrivial field configurations, such as semiclassical states, arise from deformations of this vacuum. To make this precise, we now define a pseudoholomorphic structure.

This concept generalises the Cauchy-Riemann operator $\overline{\partial}$ on a complex manifold to the bundle E. A *pseudoholomorphic structure* (PHS) $\mathscr{E}$ on E is given by a $\mathbf{C}$-linear operator:

$$\overline{\partial}^{\mathscr{E}}:\quad \Omega^{r,s}(\mathbf{MT}_s;E) \longrightarrow \Omega^{r,s+1}(\mathbf{MT}_s;E). \tag{A12}$$

This operator satisfies the graded Leibniz rule:

$$\overline{\partial}^{\mathscr{E}}(\alpha\wedge|\boldsymbol{\psi}\rangle) = \overline{\partial}\alpha\wedge|\boldsymbol{\psi}\rangle + (-1)^{r+s}\,\alpha\wedge\overline{\partial}^{\mathscr{E}}|\boldsymbol{\psi}\rangle, \tag{A13}$$

for any $\alpha \in \Omega^{r,s}(\mathbf{MT}_s;\mathbf{C})$ and $|\boldsymbol{\psi}\rangle \in \Omega^{p,q}(\mathbf{MT}_s;E)$.

Physically, a section $|\boldsymbol{\psi}\rangle \in \Omega^{p,q}(\mathbf{MT}_s; E)$ represents a matter field. We call $|\boldsymbol{\psi}\rangle$ holomorphic (a BPS-like sate) with respect to $\mathscr{E}$ if

$$\overline{\partial}^{\mathscr{E}}|\boldsymbol{\psi}\rangle = 0. \tag{A14}$$

A Technical Aside. In practice, one often defines a PHS by first specifying its action on $\Gamma(\mathbf{MT}_s; E)$ and then extending to all forms. Let

$$\mathscr{D}: \quad \Gamma(\mathbf{MT}_s; E) \longrightarrow \Omega^{0,1}(\mathbf{MT}_s; E) \tag{A15}$$

be a $\mathbf{C}$-linear operator satisfying the Leibniz rule:

$$\mathscr{D}(f \cdot |\boldsymbol{\phi}\rangle) \;=\; \overline{\partial} f \otimes |\boldsymbol{\phi}\rangle \;+\; f \cdot \mathscr{D}|\boldsymbol{\phi}\rangle, \tag{A16}$$

for any $f \in \mathscr{C}^\infty(\mathbf{MT}_s)$ and section $|\boldsymbol{\phi}\rangle \in \Gamma(\mathbf{MT}_s; E)$. Then $\mathscr{D}$ extends *uniquely* to a pseudoholomorphic operator (partial connection) $\overline{\partial}^{\mathscr{D}}$ on all E-valued (r, s)-forms by:

$$\overline{\partial}^{\mathscr{D}}(\alpha \otimes |\boldsymbol{\phi}\rangle) \;\coloneqq\; \overline{\partial}\alpha \otimes |\boldsymbol{\phi}\rangle \;+\; (-1)^{r+s} \alpha \wedge \mathscr{D}|\boldsymbol{\phi}\rangle. \tag{A17}$$

To verify that this extension obeys the general Leibniz rule (Eq. (A13)), consider a decomposable form:

$$|\boldsymbol{\psi}\rangle = \beta \otimes |\boldsymbol{\phi}\rangle, \tag{A18}$$

with $\beta \in \Omega^{p,q}(\mathbf{MT}_s; \mathbf{C})$ and $|\boldsymbol{\phi}\rangle \in \Gamma(\mathbf{MT}_s; E)$. Then

$$\begin{aligned}
\overline{\partial}^{\mathscr{D}}\big(\alpha \wedge |\boldsymbol{\psi}\rangle\big) \;&=\; \overline{\partial}^{\mathscr{D}}(\alpha \wedge (\beta \otimes |\boldsymbol{\phi}\rangle)) && \text{(A19)}\\
&=\; \overline{\partial}(\alpha \wedge \beta) \otimes |\boldsymbol{\phi}\rangle \;+\; (-1)^{p+q}\, (\alpha \wedge \beta) \otimes \mathscr{D}|\boldsymbol{\phi}\rangle && \text{(A20)}\\
&=\; \overline{\partial}\alpha \wedge |\boldsymbol{\psi}\rangle \;+\; (-1)^p\, \alpha \wedge \big(\overline{\partial}\beta \otimes |\boldsymbol{\phi}\rangle + (-1)^q \beta \otimes \mathscr{D}|\boldsymbol{\phi}\rangle\big) && \text{(A21)}\\
&=\; \overline{\partial}\alpha \wedge |\boldsymbol{\psi}\rangle \;+\; (-1)^p\, \alpha \wedge \mathscr{D}|\boldsymbol{\phi}\rangle. && \text{(A22)}
\end{aligned}$$

Here $p \coloneqq \deg(\alpha)$ and $p \coloneqq \deg(\beta)$. This computation shows that defining $\mathscr{D}$ on sections suffices to extend it uniquely to all E-valued forms.

Semiclassical States. With these definitions, we obtain a simple physical result. It formalises our view of the canonical holomorphic structure on E as the classical vacuum. Deforming this ground state by adiabatically turning on a background gauge potential produces semiclassical states. The point is that these deformations lie in a moduli space.

The above result can be stated as follows. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two pseudoholomorphic structures on the same bundle E. Then they differ by an $\mathrm{End}_{\mathbf{C}}(E)$-valued $(0,1)$-form:

$$\overline{\partial}^{\mathscr{E}_2} - \overline{\partial}^{\mathscr{E}_1} \;\in\; \Omega^{0,1}\big(\mathbf{MT}_s; \mathrm{End}_{\mathbf{C}}(E)\big). \tag{A23}$$

Conversely, fix a state $\mathscr{E}$. Let

$$\boldsymbol{A} \;\in\; \Omega^{0,1}\big(\mathbf{MT}_s;\, \mathrm{End}_{\mathbf{C}}(E)\big), \tag{A24}$$

and let $\hbar$ be a (formal) deformation parameter. We then define

$$\overline{\partial}^{\mathscr{E}(\hbar)} \;=\; \overline{\partial}^{\mathscr{E}} + \hbar\, \boldsymbol{A} \wedge_{\varepsilon} . \tag{A25}$$

This operator defines a new, deformed PHS. Physically, $\boldsymbol{A}$ represents the background gauge field being turned on.

The proofs are straightforward. First, define the difference operator

$$\mathcal{T} \coloneqq \overline{\partial}^{\mathscr{E}_1} - \overline{\partial}^{\mathscr{E}_2}. \tag{A26}$$

For any smooth function f and section $|\boldsymbol{\psi}\rangle$, the Leibniz rule for each $\overline{\partial}^{\mathscr{E}_i}$ gives

$$\mathcal{T}(f \cdot |\boldsymbol{\psi}\rangle) = f \cdot \mathcal{T}|\boldsymbol{\psi}\rangle. \tag{A27}$$

Thus $\mathcal{T}$ is $\mathscr{C}^\infty$-linear. Hence it defines an $\mathrm{End}_{\mathbf{C}}(E)$-valued $(0,1)$-form.

Next, set:

$$\overline{\partial}^{\mathscr{E}(\hbar)} = \overline{\partial}^{\mathscr{E}} + \hbar\, \boldsymbol{A} \wedge_{\varepsilon} . \tag{A28}$$

We check the Leibniz rule on sections $|\boldsymbol{\psi}\rangle \in \Gamma(\mathbf{MT}_s; E)$:

$$\begin{aligned}
\overline{\partial}^{\mathscr{E}(\hbar)}\big(f \cdot |\boldsymbol{\psi}\rangle\big) \;&=\; \overline{\partial}^{\mathscr{E}}\big(f \cdot |\boldsymbol{\psi}\rangle\big) \;+\; \hbar\, f \cdot \big(\boldsymbol{A} \wedge_{\varepsilon} |\boldsymbol{\psi}\rangle\big) && \text{(A29)}\\
&=\; \overline{\partial} f \otimes |\boldsymbol{\psi}\rangle \;+\; f \cdot \big(\overline{\partial}^{\mathscr{E}} |\boldsymbol{\psi}\rangle + \hbar\, \boldsymbol{A} \wedge_{\varepsilon} |\boldsymbol{\psi}\rangle\big) && \text{(A30)}\\
&=\; \overline{\partial} f \otimes |\boldsymbol{\psi}\rangle \;+\; f \cdot \overline{\partial}^{\mathscr{E}(\hbar)} |\boldsymbol{\psi}\rangle. && \text{(A31)}
\end{aligned}$$

By our earlier argument, this operator extends uniquely to all E-valued forms. **QED.**

The conceptual upshot is that the set of all pseudoholomorphic structures on the bundle $E \to \mathbf{MT}_s$ forms an affine space modelled on

$$\Omega^{0,1}\big(\mathbf{MT}_s;\, \mathrm{End}_{\mathbf{C}}(E)\big). \tag{A32}$$

Physically, this means any semiclassical state can be reached from any other by adding a suitable background field configuration $\boldsymbol{A}$. Thus the space of semiclassical states is realised geometrically as the moduli of deformations of the vacuum holomorphic structure.

3. Pseudocurvature and Fieldstrength

The state of HGT is encoded by the geometry of a complex vector bundle $E \to \mathbf{MT}_s$. The vacuum is the state with no field "excitations." We say a nondegenerate classical vacuum exists when E admits a holomorphic structure. In that case, the vacuum corresponds to the canonical holomorphic structure on E, defined by the Dolbeault operator:

$$\overline{\partial}^E : \quad \Omega^{p,q}(\mathbf{MT}_s; E) \longrightarrow \Omega^{p,q+1}(\mathbf{MT}_s; E). \tag{A33}$$

Its integrability condition,

$$\overline{\partial}^E \circ \overline{\partial}^E = 0, \tag{A34}$$

implies the fieldstrength tensor vanishes. Physically, this condition signifies a particle-free state. In this case, we call the potential $\boldsymbol{A}$ pure gauge. A key question is when such a vacuum state exists.

In supersymmetric theories, BPS states play a central role. These states correspond to a PHS $\mathscr{E}$ on the bundle E. This structure comes with a partial connection $\overline{\partial}^{\mathscr{E}}$, whose pseudocurvature

$$\boldsymbol{F}^{\mathscr{E}} := \overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}} \tag{A35}$$

represents the *fieldstrength tensor* of the theory.

Physical Motivation. To understand the physical meaning of identifying the fieldstrength with the pseudocurvature of a deformed holomorphic structure $\mathscr{E}$, we may contrast it to curvature in Einstein's gravity.

In pseudo-Riemannian geometry, curvature manifests as tidal accelerations. If the Riemann tensor vanishes on a neighbourhood U, spacetime is locally flat there. Physically, this flatness implies no tidal acceleration between any two nearby freely falling bodies. Experimentally, one tests this by establishing the existence of a local inertial frame. Mathematically, one introduces frame fields $\boldsymbol{e}^a{}_\mu$ on U that satisfy $\nabla_\mu \boldsymbol{e}^a{}_\nu = 0$ under the Levi-Civita connection ∇.

The gauge-theoretic analogue of flatness follows the same pattern as in gravity. The key geometric objects are replaced by their gauge theory counterparts:

1. The spacetime worldvolume U is replaced by a trivialising neighbourhood $\mathscr{U} \subset \mathbf{MT}_s$. Over $\mathscr{U}$, the bundle E becomes locally trivial:

$$E\big|_{\mathscr{U}} \cong U \times \mathbf{C}^r. \tag{A36}$$

2. The Levi-Civita connection ∇, which defines parallel transport, is replaced by the twisted Dolbeault operator:

$$\overline{\partial}^{\mathscr{E}} = \overline{\partial}^{E} + \boldsymbol{A}. \tag{A37}$$

This operator measures how the canonical holomorphic structure $\overline{\partial}^{E}$ is deformed by the background gauge potential $\boldsymbol{A}$.

3. The inertial frame fields $\boldsymbol{e}^{a}{}_{\mu}$ correspond to a local holomorphic frame $\boldsymbol{s} = \{\boldsymbol{s}_i\}_{i=1}^{r}$. These are local holomorphic sections of E over the patch $\mathscr{U}$ that trivialise the bundle.

A nonzero pseudocurvature $\boldsymbol{F}^{\mathscr{E}}$ obstructs the existence of a frame whose sections remain holomorphic under the deformed structure. Conversely, if $\boldsymbol{F}^{\mathscr{E}} = 0$, then there exists a frame $\{\boldsymbol{s}_i\}_{i=1}^{r}$ on an open set $\mathscr{U} \subset \mathbf{MT}_s$ satisfying:

$$\overline{\partial}^{\mathscr{E}}\boldsymbol{s}_i\big|_{\mathscr{U}} \;=\; \big(\overline{\partial}^{E} + \boldsymbol{A}\big)\boldsymbol{s}_i\big|_{\mathscr{U}} \;=\; 0, \qquad \text{for } i = 1, \ldots, r. \tag{A38}$$

This condition is the direct analogue of setting up an inertial frame in a region of vanishing gravitational field.

a. Holomorphic Frames

To analyse the local physics of a gauge theory on a rank-r complex vector bundle E, we introduce a local frame field over an open set $\mathscr{U} \subset \mathbf{MT}_s$. A local frame $\boldsymbol{s}$ is an ordered collection of r sections,

$$\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_r), \qquad \boldsymbol{s}_i \in \Gamma\big(\mathscr{U};\, E\big|_{\mathscr{U}}\big). \tag{A39}$$

At each point $\mathsf{w} \in \mathscr{U}$, the vectors $\{\boldsymbol{s}_i(\mathsf{w})\}$ form a basis of the fibre $E\big|_{\mathsf{w}}$.

Physically, choosing a frame is equivalent to selecting a local basis for the internal degrees of freedom.

Holomorphicity. When does a gauge theory on the vector bundle E admit a classical vacuum? In our geometric formulation, a vacuum is a field configuration with vanishing pseudocurvature. Hence E admits a classical vacuum when it admits a holomorphic structure.

A local criterion is the following:

Lemma 7. *The bundle $E \to \mathbf{MT}_s$ is holomorphic if and only if each open patch $\mathscr{U}$ of a chosen trivialising cover admits a local holomorphic frame.*

Proof. Assume the theory admits a nondegenerate classical vacuum. By definition, this means that the bundle E is holomorphic. Let $\mathscr{U} \subset \mathbf{MT}_s$ be a trivialising neighbourhood. Then there exists a holomorphic trivialisation map

$$\tau_{\mathscr{U}}\colon\ \pi^{-1}(\mathscr{U}) \ \longrightarrow\ \mathscr{U} \times \mathbf{C}^r. \tag{A40}$$

Choose the standard basis $\{\boldsymbol{e}_i\}_{i=1}^r$ of the typical fibre $\mathbf{C}^r$. Pull this basis back by the inverse trivialisation $\tau_{\mathscr{U}}^{-1}$. For each $\mathsf{w} \in \mathscr{U}$ define the local sections:

$$\boldsymbol{s}_i(\mathsf{w}) := \tau_{\mathscr{U}}^{-1}(\mathsf{w}, \boldsymbol{e}_i). \tag{A41}$$

The differential of $\tau_{\mathscr{U}}$ is an isomorphism on tangent spaces. Hence $\{\boldsymbol{s}_i(\mathsf{w})\}$ forms a basis of the fibre $E\big|_{\mathsf{w}}$ at every $\mathsf{w} \in \mathscr{U}$. Since $\tau_{\mathscr{U}}$ is biholomorphic, each map $\mathsf{w} \mapsto \boldsymbol{s}_i(\mathsf{w})$ is holomorphic. Thus the sections $\{\boldsymbol{s}_i\}$ provide a local holomorphic frame on $\mathscr{U}$.

To prove the converse, we examine the consistency conditions satisfied by the transition functions that glue local patches. Let $\mathscr{U}$ and $\mathscr{U}'$ be two overlapping trivialising neighbourhoods. Let $\boldsymbol{s} = (\boldsymbol{s}_i)$ and $\boldsymbol{s}' = (\boldsymbol{s}'_i)$ be local frames on $\mathscr{U}$ and $\mathscr{U}'$, respectively. Assume each frame is holomorphic with respect to the partial connection $\overline{\partial}^{\mathscr{E}}$. In physical terms, on each patch we choose a basis of field sections with no antiholomorphic dependence relative to the background complex structure.

On the overlap $\mathscr{U} \cap \mathscr{U}'$ the frames are related by a smooth, invertible transition map:

$$g_i{}^j\colon\ \mathscr{U} \cap \mathscr{U}' \ \longrightarrow\ GL(r, \mathbf{C}), \qquad \boldsymbol{s}'_i = g_i{}^j \boldsymbol{s}_j. \tag{A42}$$

Apply the Leibniz rule for $\overline{\partial}^{\mathscr{E}}$ on $\mathscr{U} \cap \mathscr{U}'$. This gives

$$\overline{\partial}^{\mathscr{E}} \boldsymbol{s}'_i \ = \ \overline{\partial} g_i{}^j \otimes \boldsymbol{s}_j \ + \ g_i{}^j\, \overline{\partial}^{\mathscr{E}} \boldsymbol{s}_j. \tag{A43}$$

By hypothesis $\overline{\partial}^{\mathscr{E}} \boldsymbol{s}_i = 0$ and $\overline{\partial}^{\mathscr{E}} \boldsymbol{s}'_i = 0$. Hence $\overline{\partial} g_i{}^j = 0$ on $\mathscr{U} \cap \mathscr{U}'$. Thus each transition function $g_i{}^j$ is holomorphic on the overlap.

Holomorphic transition functions give a holomorphic atlas for E. By definition, this equips the bundle $E \to \mathbf{MT}_s$ with a global holomorphic structure. Therefore E is a holomorphic vector bundle. $\square$

b. Existence of a Classical Vacuum

We are now in a position to state an important physical observation. For a holomorphic gauge theory defined on a complex vector bundle $E \to \mathbf{MT}_s$, the bundle is holomorphic if and only if there exists a partial connection $\overline{\partial}^{\mathscr{E}}$ on E whose pseudocurvature, $\boldsymbol{F} := \overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}}$, vanishes. In more physical terms, this asserts that the gauge theory admits a classical vacuum if and only if one can find a gauge potential for which the associated field-strength tensor is zero everywhere.

To understand why this must be true, we first consider the case where E is holomorphic. We can simply choose our pseudoholomorphic structure $\mathscr{E}$ to be the canonical holomorphic structure that is naturally induced by the Dolbeault operator on E. In this case, $\overline{\partial}^{\mathscr{E}} = \overline{\partial}^{E}$, and the integrability condition $\overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}} = 0$ holds by definition, guaranteeing a vanishing field strength.

For the more involved converse, let us assume we have a pseudoholomorphic structure $\mathscr{E}$ on our vector bundle whose partial connection $\overline{\partial}^{\mathscr{E}}$ is integrable, meaning its pseudocurvature vanishes. Our goal is to demonstrate that this implies $\pi\colon E \to \mathbf{MT}_s$ is holomorphic. To do this, we will explicitly construct a local holomorphic frame on an arbitrary trivialising neighbourhood $\mathscr{U} \subset \mathbf{MT}_s$ and then invoke our previous result. Let $\boldsymbol{s} := (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_r)$ be any local frame on $\mathscr{U}$; we do not assume this frame is holomorphic, so in general $\overline{\partial}^{\mathscr{E}} \boldsymbol{s}_i \neq 0$.

The key observation is that for each basis section $\boldsymbol{s}_i$, the quantity $\overline{\partial}^{\mathscr{E}} \boldsymbol{s}_i$ is a section of the bundle of $(0,1)$-forms on $\mathscr{U}$ with values in E; that is, $\overline{\partial}^{\mathscr{E}} \boldsymbol{s}_i \in \Omega^{0,1}(\mathscr{U}; E|_{\mathscr{U}})$. Since the set $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_r\}$ is complete, we can expand the differential $\overline{\partial}^{\mathscr{E}} \boldsymbol{s}_i$ in terms of the frame itself, which means there must exist a set of complex-valued differential $(0,1)$-forms, $\alpha_i{}^j \in \Omega^{0,1}(\mathscr{U})$, such that $\overline{\partial}^{\mathscr{E}} \boldsymbol{s}_i = \alpha_i{}^j \otimes \boldsymbol{s}_j$. Now, we apply the operator $\overline{\partial}^{\mathscr{E}}$ again and use the Leibniz rule:

$$(\overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}}) \boldsymbol{s}_i \;=\; \big(\overline{\partial} \alpha_i{}^j - \alpha_i{}^k \wedge \alpha_k{}^j\big) \otimes \boldsymbol{s}_j. \tag{A44}$$

Our initial assumption was that the partial connection is "flat," meaning the field-strength vanishes, so the left-hand side is zero. This forces the term in the parenthesis to be zero as well:

$$\overline{\partial} \alpha_i{}^j \;=\; \alpha_i{}^k \wedge \alpha_k{}^j. \tag{A45}$$

This is precisely the Maurer-Cartan equation. Therefore, the Frobenius integrability condition guarantees the existence of a set of complex-valued differentiable functions $f_i{}^j$ on some open subset $\mathscr{U}' \subseteq \mathscr{U}$ that satisfy $f^{ik} f_{jk} = \delta^i{}_j$ and, most importantly, solve the differential equation $\alpha_i{}^j + f_i{}^k \overline{\partial} f_k{}^j = 0$, where we use the Kronecker delta $\delta^i{}_j$ to lower and raise the internal indices $i, j, \ldots$

Finally, we can use these functions to define a new frame field $\boldsymbol{s}'_i = f_i{}^j \boldsymbol{s}_j$ on the sub-patch $\mathscr{U}'$. A straightforward calculation shows that this new frame is indeed holomorphic:

$$\overline{\partial}^{\mathscr{E}} \boldsymbol{s}'_i \;=\; \big(\overline{\partial} f_i{}^k + f_i{}^j \alpha_j{}^k\big) \otimes \boldsymbol{s}_k. \tag{A46}$$

Contracting this expression with $f_\ell{}^i$ yields:

$$f_\ell{}^i \overline{\partial}^{\mathscr{E}} \boldsymbol{s}'_i \;=\; \big(\alpha_\ell{}^k + f_\ell{}^i \overline{\partial} f_i{}^k\big) \otimes \boldsymbol{s}_k \;=\; 0. \tag{A47}$$

This vanishes because of the way we defined the functions $f_i^{\ j}$ as solutions to the Frobenius integrability condition. We have thus successfully constructed a local holomorphic frame, $\boldsymbol{s}'_i$. Since we can do this for any trivialising neighbourhood, our preceding result implies that the bundle E must be holomorphic, which completes the argument.

This theorem provides the rigorous justification for our initial definition, identifying the canonical holomorphic structure of the vector bundle with the classical vacuum of the gauge theory. The key insight is that, for a generic pseudoholomorphic connection $\overline{\partial}^{\mathscr{E}}$, there might not be a full basis of solutions to the equation $\overline{\partial}^{\mathscr{E}}\boldsymbol{s} = 0$. The integrability condition, $\overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}} = 0$, which physically signifies the vanishing of the field-strength tensor, $\boldsymbol{F} = 0$, is the necessary and sufficient condition for the existence of a maximal number of independent, locally holomorphic sections, which in turn defines the nondegenerate classical vacuum of the theory.

c. Properties of Pseudocurvature

Let us begin the final part of our review by recapitulating the essential features of the formalism developed in the preceding subsections. We have formulated a holomorphic gauge theory on minitwistor superspace as a theory of pseudoholomorphic structures, $\mathscr{E}$, on a complex vector bundle E. In this setting, each physical state of the theory corresponds to a particular choice of structure $\mathscr{E}$. The associated partial connection, $\overline{\partial}^{\mathscr{E}}$, is identified with the gauge potential, and its field-strength tensor is given by the pseudocurvature, a $(0,2)$-form defined as $\boldsymbol{F} \coloneqq \overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}}$.

We then established that such a theory admits a nondegenerate classical vacuum if and only if the underlying vector bundle, $\pi \colon E \to \mathbf{MT}_s$, is holomorphic. The vacuum state was identified with the canonical holomorphic structure $\overline{\partial}^{E}$ that exists on any such bundle. The consistency of this framework was confirmed by demonstrating that a configuration of vanishing field strength ($\boldsymbol{F} = 0$) exists precisely when the bundle E is holomorphic, which aligns with the physical requirement of a vanishing field strength in vacuum.

While this geometric formulation is elegant, it is instructive to connect it explicitly to the familiar language of conventional gauge theory. To this end, we now derive some properties of the field-strength tensor (e.g., the Bianchi identity) directly from the definition of the pseudocurvature. This exercise will serve to solidify the identification of $\boldsymbol{F}$ as the field-strength tensor of the theory and demonstrate the utility of the formalism.

Smooth Linearity. Our first task is to clarify the analytic nature of the pseudocurvature $\boldsymbol{F}$, which we have identified with the field-strength tensor. A genuine classical field-strength should act locally on matter fields; that is, its value at a point $\mathsf{w} \in \mathbf{MT}_s$ should depend only on the value of the matter field at w. Mathematically, this property is captured by the condition of $\mathscr{C}^\infty$-linearity.

Thus, we now show that the pseudocurvature $\boldsymbol{F}$ is indeed a $\mathscr{C}^\infty(\mathbf{MT}_s)$-linear operator. This means that $\boldsymbol{F}$ can be regarded as an *operator-valued differential form* that acts pointwise on the fibres of the vector bundle E,

$$\boldsymbol{F} \;\in\; \Omega^{0,2}\big(\mathbf{MT}_s;\, \mathrm{End}_{\mathbf{C}}(E)\big). \tag{A48}$$

To see why this must be true, we examine how $\boldsymbol{F}$ acts on a matter field $|\phi\rangle \in \Gamma(\mathbf{MT}_s; E)$ that has been multiplied by an arbitrary complex-valued smooth function $f \in \mathscr{C}^\infty(\mathbf{MT}_s)$. Applying the definition of the pseudocurvature and repeatedly using the Leibniz rule for the partial connection $\overline{\partial}^{\mathscr{E}}$, we find:

$$\boldsymbol{F}\big(f\cdot|\phi\rangle\big) \;=\; \overline{\partial}^{\mathscr{E}}\big(\overline{\partial}^{\mathscr{E}}(f\cdot|\phi\rangle)\big) \;=\; \overline{\partial}^{\mathscr{E}}\big(\overline{\partial}f\otimes|\phi\rangle\big) + \overline{\partial}^{\mathscr{E}}\big(f\cdot\overline{\partial}^{\mathscr{E}}|\phi\rangle\big) \;=\; f\cdot\boldsymbol{F}|\phi\rangle. \tag{A49}$$

This result confirms that $\boldsymbol{F}$ is indeed $\mathscr{C}^\infty$-linear. Physically, this is the precise condition ensuring that the field-strength acts *locally*, as a well-behaved classical observable.

This pointwise, linear action on the fibres motivates a convenient and physically suggestive notation. For any operator-valued form $\boldsymbol{T} \in \Omega^{r,s}\big(\mathbf{MT}_s; \mathrm{End}_{\mathbf{C}}(E)\big)$ and any matter field $|\phi\rangle \in \Gamma(\mathbf{MT}_s; E)$, we will denote their action as

$$\boldsymbol{T}|\phi\rangle \;\coloneqq\; \boldsymbol{T}\wedge_\varepsilon|\phi\rangle. \tag{A50}$$

This notation is deliberately reminiscent of the action of an observable on a state-vector in quantum mechanics, reinforcing the picture of $\boldsymbol{F}$ as the field-strength observable acting on a matter field.

Structure Equation. A central question in any gauge theory is to understand how the field content of the theory changes when the system is perturbed. Let us consider a holomorphic gauge theory formulated on a complex vector bundle E over minitwistor superspace. The states of this theory, $\mathscr{E}$, are described by pseudoholomorphic structures, which are specified by a partial connection $\overline{\partial}^{\mathscr{E}}$ and an associated pseudocurvature $(0,2)$-form $\boldsymbol{F}^{\mathscr{E}}$. Suppose we begin with a known state $\mathscr{E}_1$ and perturb it to a new state $\mathscr{E}_2$ by adiabatically turning on a background gauge potential, which we represent by an endomorphism-valued $(0,1)$-form $\boldsymbol{A} \in \Omega^{0,1}\big(\mathbf{MT}_s;\, \mathrm{End}_{\mathbf{C}}(E)\big)$. How is the field-strength $\boldsymbol{F}^{\mathscr{E}_2}$ of the new state related to the original field-strength $\boldsymbol{F}^{\mathscr{E}_1}$ and the perturbing field $\boldsymbol{A}$? It turns out that the new field-strength is completely determined by the initial state and the perturbation via the well-known decomposition:

$$\boldsymbol{F}^{\mathscr{E}_2} \;=\; \boldsymbol{F}^{\mathscr{E}_1} + \overline{\partial}^{\mathscr{E}_1}\boldsymbol{A} + \boldsymbol{A} \wedge_\varepsilon \boldsymbol{A}. \tag{A51}$$

To see why this relation holds, we begin by recalling that the set of all pseudoholomorphic structures on the bundle E forms an affine space over the infinite-dimensional vector space $\Omega^{0,1}\big(\mathbf{MT}_s;\, \mathrm{End}_{\mathbf{C}}(E)\big)$. This affine structure implies that the difference between any two partial connections, $\overline{\partial}^{\mathscr{E}_2} - \overline{\partial}^{\mathscr{E}_1}$, is a $\mathscr{C}^\infty$-linear operator. As such, it can be identified with the endomorphism-valued $(0,1)$-form $\boldsymbol{A}$ that we have interpreted physically as the background gauge potential. This allows us to write the "perturbed" partial connection directly in terms of the original one,

$$\overline{\partial}^{\mathscr{E}_2} = \overline{\partial}^{\mathscr{E}_1} + \boldsymbol{A}. \tag{A52}$$

Physically, this equation states that the new gauge derivative is simply the old one plus a term accounting for the newly introduced background field.

To find the corresponding field strength, we examine its action on an arbitrary matter field, which we represent as a section $|\phi\rangle \in \Gamma(\mathbf{MT}_s; E)$. A direct computation using Eq. (A52), the $\mathbf{C}$-linearity of $\overline{\partial}^{\mathscr{E}_2}$ and its graded Leibniz rule yields:

$$\begin{aligned}
\boldsymbol{F}^{\mathscr{E}_2}|\phi\rangle \;&=\; \overline{\partial}^{\mathscr{E}_2}\big(\overline{\partial}^{\mathscr{E}_2}|\phi\rangle\big) \;=\; \overline{\partial}^{\mathscr{E}_2}\big(\overline{\partial}^{\mathscr{E}_1}|\phi\rangle\big) \;+\; \overline{\partial}^{\mathscr{E}_2}\big(\boldsymbol{A}|\phi\rangle\big) && \text{(A53)}\\
&=\; \overline{\partial}^{\mathscr{E}_1}\big(\overline{\partial}^{\mathscr{E}_1}|\phi\rangle\big) + \boldsymbol{A}\big(\overline{\partial}^{\mathscr{E}_1}|\phi\rangle\big) + \overline{\partial}^{\mathscr{E}_1}\big(\boldsymbol{A}|\phi\rangle\big) + \boldsymbol{A} \wedge_\varepsilon \big(\boldsymbol{A} \wedge_\varepsilon |\phi\rangle\big) && \text{(A54)}\\
&=\; \big(\boldsymbol{F}^{\mathscr{E}_1} + \overline{\partial}^{\mathscr{E}_1}\boldsymbol{A} + \boldsymbol{A} \wedge_\varepsilon \boldsymbol{A}\big)|\phi\rangle. && \text{(A55)}
\end{aligned}$$

A brief technical remark is in order regarding the derivation of the final line. In manipulating expressions involving both differential forms and sections, we adopted a quantum mechanics-like notation (refer to Eq. (A50)) where the wedge product, $\wedge_\varepsilon$, is implicitly understood. However, one must be careful to respect the graded nature of the exterior algebra. For example, the term $\overline{\partial}^{\mathscr{E}_1}(\boldsymbol{A}|\phi\rangle)$ is shorthand for $\overline{\partial}^{\mathscr{E}_1}(\boldsymbol{A} \wedge_\varepsilon |\phi\rangle) = \overline{\partial}^{\mathscr{E}_1}\boldsymbol{A} \wedge_\varepsilon |\phi\rangle - \boldsymbol{A} \wedge_\varepsilon \overline{\partial}^{\mathscr{E}_1}|\phi\rangle$. Rearranging this gives the operator relation:

$$\big(\overline{\partial}^{\mathscr{E}_1}\boldsymbol{A}\big)|\phi\rangle = \overline{\partial}^{\mathscr{E}_1}\big(\boldsymbol{A}|\phi\rangle\big) + \boldsymbol{A}\big(\overline{\partial}^{\mathscr{E}_1}|\phi\rangle\big), \tag{A56}$$

which was used in Eq. (A54). Since the final expression, Eq. (A55), holds for an arbitrary matter field $|\phi\rangle$, we can abstract it to the operator equation given in Eq. (A51).

An immediate consequence arises when we apply this result to a *supersymmetric* minitwistor gauge theory. Such theories admit a natural, nondegenerate classical vacuum state, which corresponds to the canonical holomorphic structure on the vector superbundle $E \to \mathbf{MT}_s$ induced by the standard Dolbeault operator, $\overline{\partial}^E$. In this vacuum state, the field-strength is, by definition, zero: $\boldsymbol{F} = \overline{\partial}^E \circ \overline{\partial}^E = 0$. Let us now consider a semiclassical BPS-state $\mathscr{E}$ that arises from adiabatically perturbing this vacuum by turning on a background gauge potential $\boldsymbol{A}$. By specialising Eq. (A51) to this case (setting $\mathscr{E}_1 = E$ and $\mathscr{E}_2 = \mathscr{E}$) , we obtain the *structure equation* for the field-strength tensor:

$$\boldsymbol{F}^{\mathscr{E}} \ = \ \overline{\partial}^E \boldsymbol{A} \ + \ \boldsymbol{A} \wedge_{\varepsilon} \boldsymbol{A}. \tag{A57}$$

This result is the analogue of the familiar equation $F = dA + A \wedge A$ in standard YM theory, adapted to the holomorphic setting of minitwistor superspace.

Bianchi Identity. Having established how the field-strength tensor transforms under perturbations, we now ask if there are any intrinsic constraints it must satisfy. Indeed, the field-strength $\boldsymbol{F}$ of any pseudoholomorphic structure $\mathscr{E}$ on E must obey a consistency condition known as the Bianchi (differential) identity. To formulate this, we first note that the partial connection $\overline{\partial}^{\mathscr{E}}$ on E naturally induces a pseudoholomorphic structure on the associated vector bundle of endomorphisms, $\mathrm{End}_{\mathbf{C}}(E) \to \mathbf{MT}_s$. This allows us to act with $\overline{\partial}^{\mathscr{E}}$ on $\mathrm{End}_{\mathbf{C}}(E)$-valued differential forms, such as the pseudocurvature $\boldsymbol{F}$ itself,

$$\overline{\partial}^{\mathscr{E}} \boldsymbol{F} \ \in \ \Omega^{0,3}\big(\mathbf{MT}_s; \, \mathrm{End}_{\mathbf{C}}(E)\big). \tag{A58}$$

The Bianchi identity is the statement that this action yields identically zero,

$$\overline{\partial}^{\mathscr{E}} \boldsymbol{F} \ = \ 0. \tag{A59}$$

In our geometric framework, the argument for this identity is quite simple. We again consider the action of the differential operators $\overline{\partial}^{\mathscr{E}}$ and $\boldsymbol{F}$ on an arbitrary matter field $|\phi\rangle \in \Gamma(\mathbf{MT}_s; E)$. Using the graded Leibniz rule, we find:

$$\overline{\partial}^{\mathscr{E}}\big(\boldsymbol{F}|\phi\rangle\big) \ = \ \big(\overline{\partial}^{\mathscr{E}} \boldsymbol{F}\big)|\phi\rangle + \boldsymbol{F}\big(\overline{\partial}^{\mathscr{E}}|\phi\rangle\big), \tag{A60}$$

where, as before, we use a compact notation that implicitly includes the necessary anti-symmetrisation. On the other hand, we can evaluate the left-hand side directly using the definition of the field-strength and the associativity of operator composition:

$$\overline{\partial}^{\mathscr{E}}\big(\boldsymbol{F}|\phi\rangle\big) \ = \ \overline{\partial}^{\mathscr{E}}\big((\overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}})|\phi\rangle\big) \ = \ \big(\overline{\partial}^{\mathscr{E}} \circ \overline{\partial}^{\mathscr{E}}\big)\overline{\partial}^{\mathscr{E}}|\phi\rangle \ = \ \boldsymbol{F}\big(\overline{\partial}^{\mathscr{E}}|\phi\rangle\big). \tag{A61}$$

Comparing Eqs. (A60) and (A61), we see that $(\overline{\partial}^{\mathscr{E}} \boldsymbol{F})|\phi\rangle = 0$. Since this holds for any matter field $|\phi\rangle$, the operator $\overline{\partial}^{\mathscr{E}}\mathbf{F}$ must be zero, which is the Bianchi identity given in Eq. (A59).

For the state of a theory defined by a perturbation $\boldsymbol{A}$ around a trivial vacuum, with partial connection $\overline{\partial}^{E}$, it is often more useful to express the Bianchi identity in terms of these fundamental fields. By substituting $\overline{\partial}^{\mathscr{E}} = \overline{\partial}^{E} + \boldsymbol{A}$ into Eq. (A59), we arrive at an equivalent form of the Bianchi identity:

$$\overline{\partial}^{E}\boldsymbol{F} \,+\, \boldsymbol{A} \wedge_{\varepsilon} \boldsymbol{F} \;=\; 0. \tag{A62}$$

From a physical standpoint, the Bianchi identity plays the role of a local conservation law. This is analogous to other areas of field theory. In gravitation, for example, the contracted Bianchi identity on the Einstein tensor, $\nabla_{\mu} G^{\mu\nu} = 0$, leads directly to the statement of local energy-momentum conservation, $\nabla_{\mu} T^{\mu\nu} = 0$, through the Einstein field equation. Similarly, in the context of holomorphic gauge theory, Eq. (A62) can be cast in the form of a conservation law, $\overline{\partial}^{E}\boldsymbol{F} = \star\boldsymbol{J}$, where the $\mathrm{End}_{\mathbf{C}}(E)$-valued $(0,3)$-form $\star\boldsymbol{J} \coloneqq -\boldsymbol{A} \wedge_{\varepsilon} \boldsymbol{F}$ can be interpreted as the conserved "charge" current density. This current arises, via Noether's theorem, from the underlying gauge symmetry of the theory, that is, the invariance of the physics under changes of local trivialisation of the vector bundle E.

---

[1] T. Adamo and M. Groechenig, (2013).

[2] L. Schwartz, Journal d?Analyse Mathématique **4**, 88 (1954).

[3] L. Schwartz, in *Annales de l'institut Fourier*, Vol. 7 (1957) pp. 1–141.

[4] L. Schwartz, in *Annales de l'institut Fourier*, Vol. 8 (1958) pp. 1–209.

[5] G. De Rham, *Differentiable manifolds: forms, currents, harmonic forms*, Vol. 266 (Springer Science & Business Media, 2012).

[6] H. Federer, Transactions of the American Mathematical Society **117**, 43 (1965).

[7] H. Federer, Transactions of the American Mathematical Society **93**, 418 (1959).

[8] H. Federer and W. H. Fleming, Annals of Mathematics **72**, 458 (1960).

[9] H. Federer, *Geometric measure theory* (Springer, 2014).

[10] J. R. King, (1971).

[11] J.-P. Demailly, T. Peternell, G. Tian, A. N. Tyurin, and J.-P. Demailly, Transcendental Methods in Algebraic Geometry: Lectures given at the 3rd Session of the Centro Internazionale Matematico Estivo (CIME) Held in Cetraro, Italy, July 4–12, 1994 , 1 (1996).

[12] L. Simon *et al.* (Centre for Mathematical Analysis, Australian National University Canberra, 1984).

[13] A. Sharma, Journal of High Energy Physics **2022**, 1 (2022).

[14] P. E. Jones, *Minitwistors*, Ph.D. thesis, University of Oxford (1984).

[15] P. Jones and K. Tod, Classical and Quantum Gravity **2**, 565 (1985).

[16] N. J. Hitchin, Communications in Mathematical Physics **83**, 579 (1982).

[17] N. Hitchin, Lecture Notes in Mathematics **970** (1982).

[18] D. Leites, E. Poletaeva, and V. Serganova, Journal of Nonlinear Mathematical Physics **9**, 394 (2002).

[19] B. S. DeWitt, *Supermanifolds* (Cambridge University Press, 1992).

[20] A. Rogers, *Supermanifolds: theory and applications* (World Scientific, 2007).

[21] D. A. Leites, Russian Mathematical Surveys **35**, 1 (1980).

[22] Y. I. Manin, Gauge Field Theory and Complex Geometry , 181 (1997).

[23] N. E. Koning, S. M. Kuzenko, and E. S. Raptakis, Journal of High Energy Physics **2025**, 1 (2025).

[24] T. Bailey and E. Dunne, Proceedings of the American Mathematical Society **126**, 1245 (1998).

[25] K. Kodaira, Proceedings of the National Academy of Sciences **50**, 218 (1963).

[26] C. Fronsdal, Physical Review D **10**, 589 (1974).

[27] I. M. Gelfand, S. G. Gindikin, and M. I. Graev, *Selected topics in integral geometry*, Vol. 220 (American Mathematical Soc., 2003).

[28] S. Gindikin, Amer. Math. Soc. Transl. Ser. 2 **146**, 23 (1990).

[29] S. Gindikin, Journal of Algebra **313**, 199 (2007).

[30] S. Gindikin, Journal of Geometry and Physics **78**, 127 (2014).

[31] A. D'Agnolo and P. Schapira, journal of functional analysis **139**, 349 (1996).

[32] L. David, Differential Geometry and its Applications **18**, 113 (2003).

[33] T. Voronov, *Geometric integration theory on supermanifolds*, Vol. 1 (CRC Press, 1991).

[34] E. Witten, arXiv preprint arXiv:1209.2199 (2012).

[35] S. Banerjee, Journal of High Energy Physics **2019**, 1 (2019).

[36] S. Banerjee, General Relativity and Gravitation **51**, 1 (2019).

[37] S. Banerjee, P. Pandey, and P. Paul, Physical Review D **101**, 106014 (2020).

[38] S. Banerjee and P. Pandey, Journal of High Energy Physics **2020**, 1 (2020).

[39] S. Banerjee, N. Gupta, and S. Misra, arXiv preprint arXiv:2506.14891 (2025).

[40] J. Teschner, arXiv preprint hep-th/9705214 (1997).

[41] J. Teschner, Nuclear Physics B **546**, 369 (1999).

[42] J. Teschner, Nuclear Physics B **546**, 390 (1999).

[43] J. Teschner, Nuclear Physics B **571**, 555 (2000).

[44] S. Ribault and J. Teschner, Journal of High Energy Physics **2005**, 014 (2005).

[45] R. Boels, L. Mason, and D. Skinner, Journal of High Energy Physics **2007**, 014 (2007).

[46] E. Witten, Communications in Mathematical Physics **252**, 189 (2004).

[47] R. Roiban, M. Spradlin, and A. Volovich, Journal of High Energy Physics **2004**, 012 (2004).

[48] R. Roiban, M. Spradlin, and A. Volovich, Physical Review D **70**, 026009 (2004).

[49] R. Roiban, M. Spradlin, and A. Volovich, Physical review letters **94**, 102002 (2005).

[50] R. Britto, F. Cachazo, B. Feng, and E. Witten, Physical review letters **94**, 181602 (2005).

[51] R. Britto, F. Cachazo, and B. Feng, Nuclear Physics B **715**, 499 (2005).

[52] M. Bianchi, H. Elvang, and D. Z. Freedman, Journal of High Energy Physics **2008**, 063 (2008).

[53] A. Brandhuber, P. Heslop, and G. Travaglini, Physical Review D?Particles, Fields, Gravitation, and Cosmology **78**, 125005 (2008).

[54] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, Journal of High Energy Physics **2010**, 1 (2010).

[55] H. Elvang, D. Z. Freedman, and M. Kiermaier, Journal of High Energy Physics **2009**, 009 (2009).

[56] J. Drummond and J. Henn, Journal of High Energy Physics **2009**, 018 (2009).

[57] L. J. Dixon, J. M. Henn, J. Plefka, and T. Schuster, Journal of High Energy Physics **2011**, 1 (2011).

[58] G. Korchemsky and E. Sokatchev, Nuclear Physics B **829**, 478 (2010).

[59] J. Barrett, G. Gibbons, M. Perry, C. Pope, and P. Ruback, International Journal of Modern Physics A **9**, 1457 (1994).

[60] B. Bhattacharjee and C. Krishnan, Physical Review D **106**, 106018 (2022).

[61] E. Crawley, A. Guevara, N. Miller, and A. Strominger, Journal of High Energy Physics **2022**, 1 (2022).

[62] S. Duary and S. Maji, arXiv preprint arXiv:2406.02342 (2024).

[63] F. Klein, Mathematische Annalen **2**, 198 (1870).

[64] J. Plucker, Philosophical Transactions of the Royal Society of London , 725 (1865).

[65] R. Penrose and W. Rindler, *Spinors and space-time*, Vol. 1 (Cambridge university press, 1984).

[66] A. Brandhuber, B. Spence, and G. Travaglini, Journal of Physics A: Mathematical and Theoretical **44**, 454002 (2011).

[67] H. Elvang and Y.-t. Huang, arXiv preprint arXiv:1308.1697 (2013).

[68] S. Badger, J. Henn, J. C. Plefka, and S. Zoia, *Scattering Amplitudes in Quantum Field Theory* (Springer Nature, 2024).

[69] J. Drummond, J. Henn, G. Korchemsky, and E. Sokatchev, Nuclear physics B **795**, 52 (2008).

[70] L. Mason and D. Skinner, Journal of High Energy Physics **2010**, 1 (2010).

[71] L. F. Alday and R. Roiban, Physics reports **468**, 153 (2008).

[72] J. Henn, Fortschritte der Physik **57**, 729 (2009).

[73] G. Korchemsky and E. Sokatchev, Nuclear Physics B **839**, 377 (2010).

[74] L. F. Alday and J. Maldacena, Journal of High Energy Physics **2007**, 064 (2007).

[75] A. Brandhuber, P. Heslop, and G. Travaglini, Nuclear Physics B **794**, 231 (2008).

[76] F. A. Berezin, *Introduction to superanalysis*, Vol. 9 (Springer Science & Business Media, 2013).

[77] W. Melton, A. Sharma, and A. Strominger, arXiv preprint arXiv:2312.07820 (2023).

[78] J. Drummond, J. Henn, G. Korchemsky, and E. Sokatchev, Nuclear Physics B **828**, 317 (2010).

[79] M. Bullimore and D. Skinner, arXiv preprint arXiv:1101.1329 (2011).

[80] M. Atiyah, Mathematical analysis and applications , 129 (1981).

[81] R. Penrose, Twistor Newsletter **27**, 1 (1988).

[82] G. D. Birkhoff, Transactions of the American Mathematical Society **10**, 436 (1909).

[83] A. Grothendieck, American Journal of Mathematics **79**, 121 (1957).

[84] C. Okonek, H. Spindler, and M. Schneider, *Vector Bundles on Complex Projective Spaces*, Progress

in Mathematics (Springer US, 2013).

[85] J.-P. Demailly, *Complex analytic and differential geometry* (Université de Grenoble I Grenoble, 1997).

[86] O. Forster, Lectures on Riemann Surfaces , 96 (1981).

[87] V. G. Kac, *Vertex algebras for beginners*, Vol. 10 (American Mathematical Soc., 1998).

[88] A. Tropper, (2024), arXiv:2412.13113 [hep-th].

[89] I. Mol, arXiv preprint arXiv:2410.02620 (2024).

[90] W. Fulton, *Intersection theory*, Vol. 2 (Springer Science & Business Media, 2013).

[91] L. F. Alday and J. Maldacena, Journal of High Energy Physics **2009**, 082 (2009).

[92] M. F. Atiyah, in *Annales scientifiques de l'École Normale Supérieure*, Vol. 4 (1971) pp. 47–62.

[93] S. B. Giddings and P. Nelson, Communications in mathematical physics **116**, 607 (1988).

[94] J. Zinn-Justin, *Quantum field theory and critical phenomena*, Vol. 171 (Oxford university press, 2021).

[95] E. Verlinde and H. Verlinde, Nuclear Physics B **288**, 357 (1987).

[96] O. Alvarez and P. Windey, in *Mathematical Aspects of String Theory* (World Scientific, 1987) pp. 76–94.

[97] L. J. Mason, Journal of High Energy Physics **2005**, 009 (2005).

[98] D. Quillen, Functional Analysis and Its Applications **19**, 31 (1985).

[99] D. S. Freed, Mathematical aspects of string theory **1**, 189 (1987).

[100] V. Nair, Notes for lectures at BUSSTEPP (2005).

[101] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer, 1997).

[102] V. P. Nair, Physics Letters B **214**, 215 (1988).

[103] L. S. Schulman, *Techniques and applications of path integration* (Courier Corporation, 2012).

[104] R. J. Rivers, *Path integral methods in quantum field theory* (Cambridge University Press, 1988).

[105] R. Feynman and A. Hibbs, McGraw-Hiill, New York (1965).

[106] E. Witten, Physics Letters B **117**, 324 (1982).

[107] R. Boels, L. Mason, and D. Skinner, Physics Letters B **648**, 90 (2007).

[108] S. Hollands and R. M. Wald, arXiv preprint arXiv:2312.01096 (2023).

[109] A. Fotopoulos and T. R. Taylor, Journal of High Energy Physics **2019**, 1 (2019).

[110] A. Guevara, E. Himwich, M. Pate, and A. Strominger, Journal of High Energy Physics **2021**, 1 (2021).

[111] M. Pate, A.-M. Raclariu, A. Strominger, and E. Y. Yuan, Reviews in Mathematical Physics **33**, 2140003 (2021).

[112] E. Himwich, M. Pate, and K. Singh, Journal of High Energy Physics **2022**, 1 (2022).

[113] S. Banerjee, R. Mandal, S. Misra, S. Panda, and P. Paul, Physical Review D **110**, 026020 (2024).

[114] T. Adamo, W. Bu, E. Casali, and A. Sharma, Journal of High Energy Physics **2022**, 1 (2022).

[115] S. Banerjee, R. Mandal, S. Misra, S. Panda, and P. Paul, arXiv preprint arXiv:2311.16796 (2023).

[116] Z. Bern, L. Dixon, M. Perelstein, and J. Rozowsky, Nuclear Physics B **546**, 423 (1999).

[117] E. Casali, W. Melton, and A. Strominger, Journal of High Energy Physics **2022**, 1 (2022).

[118] J. Polchinski, *String Theory*, Cambridge Monographs on Mathematical Physics No. v. 2 (Cambridge University Press, 1998).

[119] N. Berkovits, Physical review letters **93**, 011601 (2004).

[120] I. Dolgachev, in *Group Actions and Vector Fields: Proceedings of a Polish-North American Seminar Held at the University of British Columbia January 15–February 15, 1981* (Springer, 2006) pp. 34–71.

[121] E. Ogievetsky, N. Reshetikhin, and P. Wiegmann, Nuclear Physics B **280**, 45 (1987).

[122] D. Schubring and M. Shifman, Physical Review D **101**, 045003 (2020).

[123] R. Bott, in *Lectures on Algebraic and Differential Topology: Delivered at the II. ELAM* (Springer, 2006) pp. 1–94.

[124] J. Rawnsley, Proceedings of the American Mathematical Society **73**, 391 (1979).

[125] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds* (Oxford university press, 1997).

[126] S. K. Donaldson, A, A **1**, 2 (2006).

[127] O. Guichard, The Geometry, Topology and Physics of Moduli Spaces of Higgs Bundles, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap **36**, 1 (2018).

[128] S. Kobayashi, *Differential geometry of complex vector bundles* (Princeton University Press, 2014).

[129] S.-s. Chern, S.-s. Chern, S.-s. Chern, and S.-s. Chern, *Complex manifolds without potential theory*, Vol. 18 (Springer, 1967).

[130] A. Moroianu, *Lectures on Kähler geometry*, Vol. 69 (Cambridge University Press, 2007).