Comments on Minitwistors and the Celestial Supersphere

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Continuing our program of deriving aspects of celestial holography from string theory, we extend the Roiban-Spradlin-Volovich-Witten (RSVW) formalism to celestial amplitudes. We reformulate the tree-level maximally-helicity-violating (MHV) celestial leaf amplitudes for gluons in $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory and for gravitons in $\mathcal{N}=8$ Supergravity in terms of *minitwistor wavefunctions*. These are defined as representatives of cohomology classes on the minitwistor space $\mathbf{MT}$, associated to the three-dimensional Euclidean anti-de Sitter space. In this framework, celestial leaf amplitudes are expressed as integrals over the moduli space of minitwistor lines. We construct a minitwistor generating functional for MHV leaf amplitudes using the Quillen determinant line bundle, extending the approach originally developed by Boels, Mason and Skinner. Building on this formalism, we propose supersymmetric celestial conformal field theories (CFTs) as σ-models, where the worldsheet is given by the celestial supersphere $\mathbf{CP}^{1|2}$, and the target space is the minitwistor superspace $\mathbf{MT}^{2|\mathcal{N}}$. We demonstrate that the semiclassical effective action of these σ-models reproduces the MHV gluonic and gravitational leaf amplitudes in $\mathcal{N}=4$ SYM theory and $\mathcal{N}=8$ Supergravity. This construction provides a concrete realisation of the supersymmetric celestial CFT framework recently introduced by Tropper (2024).

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CONTENTS

I. INTRODUCTION

Tropper [1] proposed a general framework for constructing celestial CFTs that are expected to serve as holographic duals to spacetime *supersymmetric* field theories. In this paper, we provide a concrete realisation of Tropper's proposal by building supersymmetric celestial CFTs on the celestial *supersphere*. These theories are formulated as sigma models, where the target space is the supersymmetric extension of minitwistor space **MT**, associated with three-dimensional Euclidean anti-de Sitter space, H_3^+.

The models we construct are defined by an action functional that ensures a well-posed variational principle, allowing the use of the path-integral formalism to quantise these theories. We demonstrate that, in the semiclassical limit, these minitwistor celestial CFTs reproduce the tree-level maximally-helicity-violating (MHV) celestial leaf amplitudes for gluons in $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory and for gravitons in $\mathcal{N}=8$ Supergravity.

This work builds on the observation of Bu and Seet [2], who demonstrated that the Mellin transform defines a mapping between cohomology classes in projective twistor space **PT** and minitwistor space **MT**. By applying both the Mellin and Penrose transforms, one establishes a correspondence where twistor wavefunctions are mapped to bulk-to-boundary propagators on H_3^+. Building on this, we introduce *minitwistor wavefunctions*, which represent cohomology classes on **MT** and, through the Penrose transform, generate solutions to the covariant wave equation on H_3^+. These wavefunctions satisfy an integral relation, which we refer to as the *celestial Roiban-Spradlin-Volovich-Witten* (RSVW) *identity.*

We also propose a new interpretation of celestial leaf amplitudes. The standard approach involves foliating Klein space $\mathbf{R}^{(2,2)}$ into hyperbolic leaves and expressing physical scattering amplitudes as integrals over these leaves. In our reformulation, we introduce a different basis by expressing celestial amplitudes in terms of minitwistor wavefunctions. Using the celestial RSVW identity, we show that the leaf amplitudes for gluons in $\mathcal{N}=4$ SYM theory and for gravitons in $\mathcal{N}=8$ Supergravity can be written as integrals over the moduli space of minitwistor lines[1]. Moreover, we prove that these leaf amplitudes vanish if the gluon or graviton insertion points do not lie on a common minitwistor line.

It is also shown that minitwistor wavefunctions satisfy another integral relation, which we call the *celestial Boels-Mason-Skinner identity.* This property is derived from an n-fold application of the Penrose transform. Using this result, we construct a generating functional for MHV celestial

[1] See Appendix A for a definition.

amplitudes in terms of the Quillen determinant line bundle, which establishes a direct link between the celestial RSVW formalism and the semiclassical effective action of the minitwistor celestial CFTs developed in this work.

Organisation. In Section II, we begin with a brief review of the Penrose integral-geometric transform on minitwistor space $\mathbf{MT}$, followed by the introduction of minitwistor wavefunctions. We then derive both the celestial RSVW identity and the Boels-Mason-Skinner identity. In Section III, we reformulate the tree-level MHV gluonic celestial leaf amplitudes in $\mathcal{N}=4$ SYM theory using the celestial RSVW formalism and derive the corresponding minitwistor generating functional. Section IV extends these results to $\mathcal{N}=8$ Supergravity. In Section V, we construct the minitwistor celestial CFTs in detail and show that the minitwistor generating functional arises as the semiclassical effective action obtained through the path-integral quantisation of our models. Finally, in Section VI, we outline future directions of our research program aimed at deriving aspects of celestial holography from string theory.

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II. MINITWISTOR WAVEFUNCTIONS

A. Physical Motivation

The main goal of this paper is to formulate a sigma model on the celestial *supersphere*, whose target space is the non-singular quadric $\mathbf{MT}$, representing the *minitwistor space* associated with the three-dimensional hyperboloid H_3^+. This model is designed to reproduce semiclassically the tree-level MHV celestial leaf amplitudes for gluons in $\mathcal{N}=4$ SYM theory and for gravitons in $\mathcal{N}=8$ Supergravity.

A key part of this construction is the introduction of *minitwistor wavefunctions*, which are precisely defined in Subsection II C. These wavefunctions serve as the basic building blocks for the vertex operators in the minitwistor sigma model.

Based on the recent work by Bu and Seet [2], the *celestial leaf amplitudes* may be derived through an alternative, yet mathematically equivalent, formalism. The original aproach of Casali, Melton, and Strominger [4] and Melton *et al.* [5] involves foliating Klein space $\mathbf{R}^{(2.2)}$ by hyperbolic leaves and expressing physical scattering amplitudes as integrals over these leaves. In the alternative approach,

[2] See Kim and Nair [3].

one begins with twistor wavefunctions and subsequently performs a Mellin transform with respect to one of the spinor variables. This is followed by the application of the Penrose transform from minitwistor space **MT** to the hyperboloid H_3^+, thus recovering the bulk-to-boundary propagator $K_\Delta(X; z, \bar{z})$ for the covariant Laplacian $\square_{H_3^+}$.

The minitwistor wavefunctions satisfy two important properties: the *celestial Boels-Mason-Skinner* (BMS) and *Roiban-Spradlin-Volovich-Witten* (RSVW) *integral identities.* The BMS identities (discussed in Subsection II C 2) will be employed in the construction of a generating functional for the celestial leaf amplitudes. On the other hand, the RSVW identity (derived in Subsection II C 3) will be used in reformulating these amplitudes as a Fourier transform in minitwistor space.

In the next subsection, we review the extension of the Penrose integral-geometric transform to the minitwistor space **MT**.

B. Review: Minitwistor Penrose Transform and AdS_3 Wave Equations

The formalisation of the Penrose transform is naturally expressed through the language of integral geometry[3]. The basic structure in this branch of geometric analysis is a *double fibration*,

$$\begin{tikzcd} & \mathscr{Z} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \mathscr{X} & & \mathscr{Y} \end{tikzcd} \tag{1}$$

where $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ denote smooth manifolds, and q_1, q_2 are fibre maps. The diagram (1) qualifies as a double fibration if the product map:

$$q_1 \times q_2 : \mathscr{Z} \longrightarrow \mathscr{X} \times \mathscr{Y} \tag{2}$$

is an embedding of $\mathscr{Z}$ as a submanifold of the product space $\mathscr{X} \times \mathscr{Y}$.

Under the assumption that $q_1 \times q_2$ is indeed an embedding, it follows directly that, for any point $y \in \mathscr{Y}$, the fibre $F(y) := q_2^{-1}(y)$ is smoothly embedded as a submanifold of $\mathscr{X}$. Therefore, the double fibration (1) induces a family of submanifolds $\{F(y)\}_{y \in \mathscr{Y}}$, with each submanifold $F(y)$ smoothly embedded into $\mathscr{X}$ and parametrised by points in $\mathscr{Y}$.

The key idea of integral geometry is that geometric objects on $\mathscr{X}$ can be transported to $\mathscr{Y}$ via the intermediate space $\mathscr{Z}$. For example, given a differential form (or holomorphic section) a defined

[3] A general review of the subject is provided in Guillemin [6]. For a rigorous textbook treatment, we direct the reader to Helgason *et al.* [7], Helgason [8, 9]. Additionally, the text by Quinto, Gonzalez, and Christensen [10] provides an accessible account well-suited for those interested in physical applications.

on $\mathscr{X}$, the pullback $q_1^*(a)$ yields a corresponding form on $\mathscr{Z}$. The *integral-geometric transform* is then constructed by integrating the pullback over the fibres of the map q_2, resulting in a function or section on $\mathscr{Y}$. Formally, the integral-geometric transform is defined as:

$$a \mapsto \widetilde{a}\,(y) := \int\limits_{q_2^{-1}(y)} q_1^*\,(a)\,, \tag{3}$$

where $\widetilde{a}$ denotes the resulting form on $\mathscr{Y}$.

An important feature of this integral transform is that, whenever the pullback $q_1^*(a)$ remains constant along the fibres of q_1, the transformed form $\widetilde{a}$ must satisfy certain differential equations on $\mathscr{Y}$. This observation formalises a result originally noted by Bateman [11] and rediscovered by Penrose [12].

1. Minitwistor Geometry

In Appendix A, we provide a concise discussion of how the hyperbolic geometry of the three-dimensional real manifold H_3^+ can be realised within the projective geometry of $\mathbf{CP}^3$, subject to appropriate reality conditions. Additionally, we give a brief but mathematically rigorous introduction to the minitwistor space[4] $\mathbf{MT}$ associated to the hyperboloid H_3^+. Now, we proceed to specialise the abstract framework of integral geometry, as delineated above, to the geometry of minitwistor space.

Let $\mathbf{H}_3$ denote the complexification of the real hyperboloid H_3^+, and introduce the complexified projective spinor bundle $\mathbf{PS}_3 := \mathbf{CP}^3 \times \mathbf{CP}^1$ as the trivial bundle whose base space is $\mathbf{CP}^3$ and the typical fibre consists of the projectivised space of (undotted) two-component spinors. The relevant double fibration in this geometric setup is given by the diagram:

$$\begin{tikzcd} & \mathbf{PS}_3 \arrow[dl, "\widetilde{q}_1"'] \arrow[dr, "\widetilde{q}_2"] & \\ \mathbf{MT} & & \mathbf{H}_3 \end{tikzcd} \tag{4}$$

The first bundle map $\widetilde{q}_1$ is defined via the minitwistor incidence relation:

$$\widetilde{q}_1 : \left(X_{A\dot{A}}\,, \lambda^A\right) \mapsto \left(\lambda^A,\, \lambda^A X_{A\dot{A}}\right), \tag{5}$$

where $X_{A\dot{A}}$ represents homogeneous coordinates on $\mathbf{CP}^3$, and λ^A denotes homogeneous spinor coordinates in the fibre $\mathbf{CP}^1$. The second bundle map $\widetilde{q}_2$ is the trivial surjection $\widetilde{q}_2 : (X_{A\dot{A}}, \lambda^A) \mapsto X_{A\dot{A}}$.

[4] See also Jones [13], Jones and Tod [14], Hitchin [15, 16] and Honda and Nakata [17].

Having established the double fibration structure, it remains to specify the module of sections of the holomorphic vector bundle on $\mathbf{MT}$ upon which the integral-geometric transform defined in Eq. (3) operates. Therefore, let $\mathscr{C}^{\infty}_{p,q}(\mathbf{MT})$ denote the space of $\mathscr{C}^{\infty}$ complex-valued functions h defined on $(\mathbf{C}^*)^2 \times (\mathbf{C}^*)^2$ that satisfy the following homogeneity property:

$$h\left(a \cdot \lambda^A, b \cdot \mu_{\dot{A}}\right) = a^p b^q \;\; h\left(\lambda^A, \mu_{\dot{A}}\right), \tag{6}$$

for every pair of nonzero complex scalars a and b. In Appendix A 3, we prove that the function space $\mathscr{C}^{\infty}_{p,q}(\mathbf{MT})$ is canonically identified with the module $\Gamma^{\infty}(\mathcal{O}(p,q))$ of smooth sections on the holomorphic vector bundle $\mathcal{O}(p,q) \longrightarrow \mathbf{MT}$.

2. Scalar Representatives

In four-dimensional physics, twistor theory[5] characterises solutions to (linearised) massless field equations on spacetime through equivalence classes in the cohomology group of projective twistor space $\mathbf{PT}$, associated with the module of sections of the holomorphic vector bundle $\mathcal{O}(p) \longrightarrow \mathbf{PT}$. We now extend this framework to the setting of AdS_3.[6] To explain this construction, we restrict our attention to the case of scalar representatives:

$$[\mathsf{f}] \in H^{0,1}\left(\mathbf{MT}, \mathcal{O}(-2,-\Delta)\right). \tag{7}$$

Let $X^{A\dot{A}} \in \mathbf{H}_3$ be a point on the complexified hyperboloid embedded in projective space, $\mathbf{H}_3 \subset \mathbf{CP}^3$, where $X^{A\dot{A}}$ denotes homogeneous coordinates. Define the rational curve $\mathcal{L}(X)$ as the line of incidence in minitwistor space:

$$\mathcal{L}(X) = \left\{ \left(\lambda^A, \mu_{\dot{A}}\right) \in \mathbf{MT} \,\middle|\, \mu_{\dot{A}} = \lambda^A X_{A\dot{A}} \right\}. \tag{8}$$

Employing the sheaf-theoretic notation of Forster [22], let $\rho_{\mathcal{L}(X)}$ denote the restriction homomorphism to the incidence line $\mathcal{L}(X)$, defined by:

$$\rho_{\mathcal{L}(X)}(g)\left(\lambda^A\right) \coloneqq g\left(\lambda^A, \lambda^A X_{A\dot{A}}\right), \;\; \forall\, [g] \in H^{0,1}\left(\mathbf{MT}, \mathcal{O}(-2,-\Delta)\right). \tag{9}$$

Let us restrict our attention to a $(0,1)$-form on $\mathbf{MT}$ valued in $\mathcal{O}(-2,-\Delta)$. The Penrose transform can be introduced as a mapping from representatives of the cohomology class $H^{0,1}\left(\mathbf{MT}, \mathcal{O}(-2,-\Delta)\right)$, given by the Penrose integral:

$$\mathsf{f} \mapsto \int_{\mathcal{L}(X)} \langle \lambda d\lambda \rangle \;\; \rho_{\mathcal{L}(X)}(\mathsf{f})\,(\lambda). \tag{10}$$

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[5] For a modern introduction, see Adamo [18], Atiyah, Dunajski, and Mason [19] and Witten [20, Appendix A].

[6] A similar discussion for the AdS_5 case can be found in Adamo, Skinner, and Williams [21].

Thus, define:

$$F_{\Delta}(X) := |X|^{\Delta} \int_{\mathcal{L}(X)} \langle \lambda d\lambda \rangle \ \ \rho_{\mathcal{L}(X)}\big(\mathsf{f}\big)(\lambda). \tag{11}$$

Accordingly, $F_{\Delta}(X)$ is invariant under the flow of the Euler vector field Υ, such that $\pounds_{\Upsilon} F_{\Delta} = 0$. Consequently, $F_{\Delta}(X)$ is homogeneous of degree zero and, therefore, yields a well-defined *function* on H_3^+.

To verify the consistency of this construction, note that:

$$\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X} \left(\frac{F_{\Delta}(X)}{|X|^{\Delta}} \right) = 0 \iff \bar{\partial}\big|_{\mathcal{L}(X)} \mathsf{f} = 0. \tag{12}$$

Since every f that is $\bar{\partial}$-exact integrates to zero, it follows that $F_{\Delta}(X)$ is determined by the cohomology class $[\mathsf{f}]$ in $H^{0,1}(\mathbf{MT}, \mathcal{O}(-2,-\Delta))$.

Furthermore, from the invariance of $F_{\Delta}(X)$ under the flow of Υ,

$$X^{A\dot{A}} \frac{\partial}{\partial X^{A\dot{A}}} F_{\Delta}(X) = 0, \tag{13}$$

we deduce that:

$$\varepsilon^{AB} \varepsilon^{\dot{A}\dot{B}} \frac{\partial}{\partial X^{A\dot{A}}} \frac{\partial}{\partial X^{B\dot{B}}} \left(\frac{F_{\Delta}(X)}{|X|^{\Delta}} \right) = 0, \tag{14}$$

which is equivalent to:

$$\Box_{\mathbf{H}_3} F_{\Delta}(X) = \Delta(\Delta+2) F_{\Delta}(X). \tag{15}$$

We conclude that the Penrose transform establishes a correspondence between cohomology classes of the non-singular quadric $\mathbf{MT}$ associated to the holomorphic vector bundle $\mathcal{O}(-2,-\Delta) \longrightarrow \mathbf{MT}$, and conformal primaries of the minisuperspace limit of the H_3^+-WZNW model.

C. Minitwistor Wavefunctions

We are now prepared to introduce the notion of a minitwistor wavefunction. This object shows an interesting connection between the Mellin and Penrose transforms. It will be demonstrated that the Mellin transform induces a mapping between cohomology classes on projective twistor space $\mathbf{PT}$, and corresponding cohomology classes on the nonsingular quadric $\mathbf{MT}$. By subsequently applying the Penrose transform, it will become clear that twistor wavefunctions are mapped to bulk-to-boundary Green's functions[7] for the covariant Laplacian $\Box_{H_3^+}$ on the hyperboloid H_3^+.

[7] See Teschner [23, 24, 25, 26], Ribault and Teschner [27].

1. Definitions

Notation. From now on, we shall adopt a simplified notational convention for spinor functions. The explicit display of abstract spinor indices in the arguments of such functions will be reserved exclusively for their initial definition. Thereafter, the spinor type (dotted *vs.* undotted) comprising the domain of each function will remain implicit.

Twistor Scalar Wavefunction. We consider the twistor scalar wavefunction:

$$\mathsf{f}_w \in \Omega^{0,1}\left(\mathbf{PT}, \mathcal{O}\left(-w\right)\right), \tag{16}$$

which admits the following integral representation:

$$\mathsf{f}_w\left(\lambda^A, \mu_{\dot{A}}; \mathrm{m}\right) \coloneqq \int\limits_{\mathbf{C}^*} \frac{dt}{t}\, t^w\ \bar{\delta}^2\left(z^A - t\lambda^A\right) \exp\left(i\,t\left[\mu\bar{z}\right]\right). \tag{17}$$

In the above expression, the spinor delta function is defined as:

$$\bar{\delta}^2\left(\lambda^A\right) \coloneqq \frac{1}{(2\pi i)^2} \bigwedge_{A\in\{1,2\}} \bar{\partial}\left(\frac{1}{\lambda^A}\right). \tag{18}$$

The ordered pair $\mathrm{m} \coloneqq \left(z^A, \bar{z}_{\dot{A}}\right)$ denotes the collection of quantum numbers characterising the state of the particle associated with f_w. For the massless gauge bosons participating in the scattering processes to be considered in the subsequent sections, the spinors are chosen be to normalised as $z^A = (z,1)^T$ and $\bar{z}_{\dot{A}} = (1, -\bar{z})$, where $(z,\bar{z})$ parametrises the insertion points on the celestial sphere. In what follows, we shall omit the explicit dependence on z^A and $\bar{z}_{\dot{A}}$ from the arguments of any wavefunctions to streamline the notation.

The structure of the affine integral (17) implies that the twistor scalar wavefunction satisfies the homogeneity property:

$$\mathsf{f}_w\left(a\lambda, a\mu\right) = a^{-w}\mathsf{f}_w\left(\lambda,\mu\right), \tag{19}$$

for all $a \in \mathbf{C}^*$, as expected from the corresponding cohomology class (16).

Performing the affine integral in Eq. (17) yields the explicit form of the twistor scalar wavefunction:

$$\mathsf{f}_w\left(\lambda,\mu\right) = \bar{\delta}\left(\langle\lambda z\rangle\right)\left(\frac{\langle\lambda\iota\rangle}{\langle z\iota\rangle}\right)^{1-w} \exp\left(i\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\left[\mu\bar{z}\right]\right). \tag{20}$$

We now define the *minitwistor wavefunction* with *celestial conformal weight* Δ as the Mellin transform of f_w with respect to the dotted spinor $\mu_{\dot{A}}$:

$$\widehat{\mathsf{f}}_{\Delta,w}\left(\lambda^A, \mu_{\dot{A}}\right) \coloneqq \int\limits_{\mathbf{R}_+^\times} \frac{ds}{s}\, s^\Delta\ \mathsf{f}_w\left(\lambda^A, s\mu_{\dot{A}}\right), \tag{21}$$

where $\mathbf{R}_{+}^{\times} \coloneqq (\mathbf{R}_{+}, \times)$ denotes the multiplicative group of positive real numbers, and $\frac{ds}{s}$ is the associated Haar measure.

Substituting Eq. (17) into Definition (21), we obtain the complete integral representation of the minitwistor wavefunction:

$$\widehat{\mathsf{f}}_{\Delta,w}(\lambda,\mu) \;=\; \int\limits_{\mathbf{R}_{+}^{\times}} \frac{ds}{s}\, s^{\Delta} \int\limits_{\mathbf{C}^{*}} \frac{dt}{t}\, t^{w}\;\, \bar{\delta}^{2}\left(z^{A} - t\lambda^{A}\right)\, \exp\left(i\, st\, [\mu\bar{z}]\right). \tag{22}$$

The structure of the Mellin and affine integrals in the above expression reveals that the minitwistor wavefunction exhibits the homogeneity property:

$$\widehat{\mathsf{f}}_{\Delta,w}(a\lambda, b\mu) \;=\; a^{\Delta-w} b^{-\Delta}\;\, \widehat{\mathsf{f}}_{\Delta,w}(\lambda,\mu), \tag{23}$$

for all $a, b \in \mathbf{C}^{*}$.

Consequently,

$$\hat{\mathsf{f}}_{\Delta,w} \in \Omega^{0,1}\left(\mathbf{MT}, \mathcal{O}\left(\Delta - w, -\Delta\right)\right). \tag{24}$$

Finally, performing the integrals in the expression (22) for $\mathsf{f}_{\Delta,w}$, we derive the explicit form of the minitwistor scalar wavefunction:

$$\widehat{\mathsf{f}}_{\Delta,w}(\lambda,\mu) \;=\; \bar{\delta}\left(\langle\lambda z\rangle\right) \left(\frac{\langle\lambda\iota\rangle}{\langle z\iota\rangle}\right)^{1+\Delta-w} \frac{\mathcal{C}(\Delta)}{[\mu\bar{z}]^{\Delta}}. \tag{25}$$

2. Celestial Boels-Mason-Skinner Integral Identity

In this subsection, we derive an integral identity that will serve as a key step in constructing the generating functional for celestial leaf amplitudes, analogous to the twistor-space generating functional introduced by Boels, Mason, and Skinner [28]. This identity will henceforth be referred to as the *celestial Boels-Mason-Skinner* (BMS) *identity*[8].

We begin by considering a distinguished representative of minitwistor scalar wavefunctions,

$$\mathcal{F}_{\Delta} \in \Omega^{0,1}\left(\mathbf{MT}, \mathcal{O}\left(-2, -\Delta\right)\right), \tag{26}$$

defined by the expression:

$$\mathcal{F}_{\Delta}\left(\lambda^{A}, \mu_{\dot{A}}\right) \coloneqq \widehat{\mathsf{f}}_{\Delta,\Delta+2}\left(\lambda^{A}, \mu_{\dot{A}}\right), \tag{27}$$

[8] There should be no confusion with the BMS group, as each applies to a different physical context.

and taking the explicit form:

$$\mathcal{F}_{\Delta}(\lambda,\mu) = \bar{\delta}(\langle\lambda z\rangle)\,\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\,\frac{\mathcal{C}(\Delta)}{[\mu\bar{z}]^{\Delta}}. \tag{28}$$

The Penrose transform of $\mathcal{F}_{\Delta}$ is computed by restricting this wavefunction to a specific holomorphic curve in minitwistor space. Let the minitwistor line corresponding to the "spacetime" point $X_{A\dot{A}}$ be defined by the locus of incidence:

$$\mathcal{L}(X) \coloneqq \left\{ \left(\lambda^{A},\mu_{\dot{A}}\right) \in \mathbf{MT} \,\middle|\, \mu_{\dot{A}} = \lambda^{A}X_{A\dot{A}} \right\}. \tag{29}$$

Following the sheaf-theoretic notation of Forster [22], the restriction homomorphism to the conic $\mathcal{L}(X)$ is denoted by ρ_X, which acts on cohomology representatives:

$$[g] \in H^{0,1}\left(\mathbf{MT},\mathcal{O}(-2,-\Delta)\right)$$

according to:

$$\rho_X(g)(\lambda) \coloneqq g\left(\lambda^{A},\lambda^{A}X_{A\dot{A}}\right). \tag{30}$$

Let $\pi : \mathcal{L}(X) \longrightarrow \mathbf{CP}^1$ denote the canonical projection of the conic onto the projective line. We introduce a trivialisation of the fibration π by a choice of homogeneous coordinates λ^A on $\mathbf{CP}^1$. The natural orientation of the fibre is induced by the volume form $D\lambda \coloneqq \varepsilon_{AB}\lambda^A d\lambda^B$.

Consequently, the Penrose transform of $\mathcal{F}_{\Delta}$ is given by the integral-geometric transform:

$$\mathcal{F}_{\Delta} \mapsto \int_{\mathcal{L}(X)} D\lambda\;\rho_X(\mathcal{F}_{\Delta})(\lambda), \tag{31}$$

which, upon evaluation, yields:

$$|X|^{\Delta}\int_{\mathcal{L}(X)} D\lambda\;\rho_X(\mathcal{F}_{\Delta})(\lambda) = \mathcal{C}(\Delta)\,\frac{|X|^{\Delta}}{\langle z|X|\bar{z}]^{\Delta}} = K_{\Delta}(X;z,\bar{z}), \tag{32}$$

where $K_{\Delta}(X;z,\bar{z})$ is identified with the bulk-to-boundary propagator[9] on the hyperboloid H_3^+, with $C(\Delta) \coloneqq i^{-\Delta}\Gamma(\Delta)$.

We now proceed by employing an inductive argument over $n \in \mathbf{N}$ to generalise this result to an n-fold product of Penrose transforms. Recall that $(z,\bar{z})$ label boundary points of H_3^+. Then, the n-fold Penrose transform gives the celestial BMS identity:

$$\int_{\mathcal{L}(X)^{\times n}} \bigwedge_{i=1}^{n} D\lambda_i\;\;|X|^{\Delta_i}\,\rho_X(\mathcal{F}_{\Delta})(\lambda)\,\frac{1}{\lambda_i\cdot\lambda_{i+1}} = \prod_{i=1}^{n}\frac{K_{\Delta}(X;z_i,\bar{z}_i)}{z_i\cdot z_{i+1}}. \tag{33}$$

[9] See Gelfand, Graev, and Vilenkin [29], Teschner [23, 24, 25, 26], Costa, Gonçalves, and Penedones [30] and Penedones [31].

We now proceed to establish that the restriction homomorphism ρ_X, associated with a minitwistor line associated to a "spacetime" point $X_{A\dot{A}}$, may be enforced by means of a *weighted* Dirac delta function on minitwistor space. This will allow us to derive an identity analogous to Eq. (33), expressed as an integral over the minitwistor space $\mathbf{MT}$.

The theory of distributions was extended to include sections of differential forms by De Rham [32], building on earlier work by Schwartz [33, 34, 35]. In this framework, let $\overline{\delta}_\Delta$ denote a distributional form valued in the holomorphic vector bundle $\mathcal{O}(\Delta-2,-2)$, defined by the expression:

$$\overline{\delta}_\Delta\left(\mu_{\dot{A}},\pi_{\dot{A}}\right) \coloneqq \frac{1}{(2\pi i)^2}\int\limits_{\mathbf{C}^*}\frac{dt}{t}\,t^\Delta \bigwedge_{\dot{A}\in\{\dot{1},\dot{2}\}}\overline{\partial}\left(\frac{1}{\mu_{\dot{A}}-t\pi_{\dot{A}}}\right), \tag{34}$$

which imposes the projective coincidence condition $\mu_{\dot{A}}\sim\pi_{\dot{A}}$ in the complex projective line.

To perform the affine integration over t, we invoke the analytic continuation of the Dirac delta function, given by:

$$\overline{\delta}(z) \coloneqq \frac{1}{2\pi i}\,\overline{\partial}\,z^{-1}. \tag{35}$$

Applying this definition to the integrand of Eq. (34), we derive the following explicit form for $\overline{\delta}_\Delta$:

$$\overline{\delta}_\Delta(\mu,\pi) = \overline{\delta}\left([\mu\pi]\right)\left(\frac{[\mu\overline{\iota}]}{[\pi\overline{\iota}]}\right)^{\Delta-1}, \tag{36}$$

where $\overline{\iota}^{\dot{A}}$ denotes a fixed reference spinor that is arbitrarily chosen but non-vanishing.

We now introduce a representative wavefunction on minitwistor space,

$$\widetilde{\mathcal{F}}_\Delta \in \Omega^{0,1}\left(\mathbf{MT},\mathcal{O}\left(\Delta-2,-\Delta\right)\right), \tag{37}$$

defined by the expression:

$$\widetilde{\mathcal{F}}_\Delta\left(\lambda^A,\mu_{\dot{A}}\right) \coloneqq \widehat{\mathsf{f}}_{\Delta,2}\left(\lambda^A,\mu_{\dot{A}}\right), \tag{38}$$

and taking the explicit form:

$$\widetilde{\mathcal{F}}_\Delta(\lambda,\mu) = \overline{\delta}\left(\langle\lambda z\rangle\right)\left(\frac{\langle\lambda\iota\rangle}{\langle z\iota\rangle}\right)^{\Delta-1}\frac{\mathcal{C}(\Delta)}{[\mu\bar{z}]^\Delta}. \tag{39}$$

For each fixed point $X_{A\dot{A}}\in\mathbf{CP}^3$, we define the following distribution on minitwistor space:

$$g_\Delta\left(X_{A\dot{A}};\lambda^A,\mu_{\dot{A}}\right) \coloneqq \overline{\delta}_\Delta\left(\mu_{\dot{A}},\lambda^A\frac{X_{A\dot{A}}}{|X|}\right)\widetilde{\mathcal{F}}_\Delta\left(\lambda^A,\mu_{\dot{A}}\right), \tag{40}$$

which takes values in the holomorphic vector bundle $\mathcal{O}(-2,-2)$. This definition enables us to write the following integral over minitwistor space:

$$\int_{\mathbf{MT}} D\lambda \wedge D\mu \;\; g_{\Delta}(X;\lambda,\mu), \tag{41}$$

where $D\lambda \wedge D\mu$ denotes the canonical volume form on the non-singular quadric $\mathbf{MT}$, ensuring that the integral is projectively well-defined.

Utilising the explicit expressions derived above, we obtain the following form for $g_{\Delta}(X;\lambda,\mu)$:

$$g_{\Delta}(X;\lambda,\mu) = \bar{\delta}(\langle\lambda z\rangle)\,\bar{\delta}\left(\frac{\langle\lambda|X|\mu]}{|X|}\right)\left(\frac{\langle z\iota\rangle}{\langle\lambda\iota\rangle}\frac{\langle\lambda|X|\bar{\iota}]}{|X|\,[\mu\bar{\iota}]}\right)^{1-\Delta}\frac{\mathcal{C}(\Delta)}{[\mu\bar{z}]^{\Delta}}. \tag{42}$$

Introducing projective coordinates on minitwistor space via $\mathsf{Z}^{I} := \left(\lambda^{A}, \mu_{\dot{A}}\right)$, and denoting the canonical volume form by $D^2\mathsf{Z} = D\lambda \wedge D\mu$, direct evaluation of the integral yields:

$$\int_{\mathbf{MT}} D^2\mathsf{Z} \;\; g_{\Delta}(X;\lambda,\mu) = K_{\Delta}(X;z,\bar{z}), \tag{43}$$

which is identified with the bulk-to-boundary propagator on H_3^+.

Finally, by applying an inductive argument on $n \in \mathbf{N}$, we derive the *celestial Roiban-Spradlin-Volovich-Witten* (RSVW) *identity*:

$$\prod_{i=1}^{n}\int_{\mathbf{MT}} D^2\mathsf{Z}_i \;\; \bar{\delta}_{\Delta_i}\left(\mu_{i\dot{A}}, \lambda^{A}\frac{X_{A\dot{A}}}{|X|}\right)\,\widetilde{\mathcal{F}}_{\Delta_i}(\lambda_i,\mu_i;z_i,\bar{z}_i)\,\frac{1}{\lambda_i\cdot\lambda_{i+1}} = \prod_{i=1}^{n}\frac{K_{\Delta_i}(X;z_i,\bar{z}_i)}{z_i\cdot z_{i+1}}. \tag{44}$$

In the subsequent section, we shall employ this identity to reformulate the celestial leaf amplitudes for gluons in $\mathcal{N}=4$ SYM theory, and subsequently extend the analysis to gravitons in $\mathcal{N}=8$ Supergravity.

III. $\mathcal{N}=4$ SUPERSYMMETRIC YANG-MILLS

A. Review

The starting point of our analysis is the Parke-Taylor formula[10]. Consider a scattering process in four-dimensional Yang-Mills theory involving n gluons in the MHV configuration $1^-, 2^-, 3^+, ..., n^+$. The corresponding scattering amplitude, denoted $\mathcal{A}_n^{a_1\cdots a_n}$, is expressed as follows:

$$\mathcal{A}_n^{a_1\dots a_n}(z_i,\bar{z}_i,s_i) \;=\; ig^{n-2}\,\delta^{(4)}\left(\sum_{i=1}^{n} s_i q^{\mu}(z_i,\bar{z}_i)\right)\,\langle\nu_1\nu_2\rangle^4\;\mathsf{Tr}\,\prod_{i=1}^{n}\frac{\mathsf{T}^{a_i}}{\nu_i\cdot\nu_{i+1}}. \tag{45}$$

[10] First introduced by Parke and Taylor [36] and subsequently provided with a rigorous derivation by Berends and Giele [37] and Kim and Nair [3]. For modern introductions, see Elvang and Huang [38], Badger *et al.* [39].

Here, g denotes the Yang-Mills coupling constant, $a_1, ..., a_n$ are the colour indices associated with the external gluons, and T^a are the generators of the gauge group $\mathbf{G}$. These generators satisfy the normalisation condition $\mathsf{Tr}(\mathsf{T}^a\mathsf{T}^b) = \frac{1}{2}\boldsymbol{k}^{ab}$, where $\boldsymbol{k}^{ab}$ denotes the Cartan-Killing form of $\mathbf{G}$, as well as the Lie algebra commutation relations $[\mathsf{T}^a, \mathsf{T}^b] = if^{abc}\mathsf{T}^c$, where f^{abc} are the structure constants of the Lie algebra $\mathfrak{g} \simeq (T_e(\mathbf{G}), [\cdot, \cdot])$. We also adopt the convention $\nu^A_{n+1} := \nu^A_1$, which ensures cyclic symmetry in the denominator of the Parke-Taylor formula.

To proceed, we reformulate the amplitude $\mathcal{A}^{a_1 \dots a_n}_n (z_i, \bar{z}_i, s_i)$ in terms of the frequencies s_i and the normalised spinor basis $z^A_i := (1, z_i)^T$ and $\bar{z}_{i\dot{A}} := (-\bar{z}_i, 1)$, utilising Eq. (45) along with the integral representation of the four-dimensional delta-function,

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4} \int_{\mathbf{R}^4} d^4x \; e^{ip\cdot x}, \;\; \forall\, p^\mu \in \mathbf{R}^4. \tag{46}$$

This yields the expression:

$$\mathcal{A}^{a_1 \dots a_n}_n (z_i, \bar{z}_i, s_i) \; = \; \frac{ig^{n-2}}{(2\pi)^4} \int_{\mathbf{R}^4} d^4x \;\; \mathsf{Tr} \; \prod_{i=1}^{n} s_i^{e_i} \; \exp\left(i\, s_i \, \langle z_i | x | \bar{z}_i]\right) \; \frac{\mathsf{T}^{a_i}}{z_i \cdot z_{i+1}}. \tag{47}$$

1. *Grassmann-valued Spinors; Berezin-de Witt Integral*

The transition to $\mathcal{N} = 4$ SYM theory is achieved via the on-shell superfield formalism[11]. To this end, we introduce Grassmann-valued two-component spinors θ^α_A, where $1 \le \alpha \le 4$ indexes the supersymmetry generators. These variables satisfy the normalisation condition $\int d^{0|2}\theta \;\; \theta^\alpha_A \theta^\alpha_B = \varepsilon_{AB}$, where $d^{0|2}\theta$ denotes the Berezin-de Witt integration measure[12] over the fermionic variables. Then, defining the Grassmann-valued coefficients $\xi^\alpha_i := z^A_i \theta^\alpha_A$, we note the identity:

$$(z_i \cdot z_j)^4 = \int d^{0|8}\theta \;\; \prod_{\alpha=1}^{4} \xi^\alpha_i \xi^\alpha_j, \;\; \text{for all}\, 1 \le i, j \le n. \tag{48}$$

In addition, we must specify both the integration measure and the domain over which the super-amplitudes are defined. Thus, let $\mathbf{R}^{4|8}$ denote the supersymmetric extension of real four-dimensional Euclidean space, augmented with the Grassmann-valued coordinates θ^α_A $(a = 1, ..., 8)$. The Berezin-de Witt integration measure on $\mathbf{R}^{4|8}$ is defined as $d^{4|8}x := d^4x \wedge d^{0|8}\theta$.

Therefore, the amplitude $\mathcal{A}^{a_1 \dots a_n}_n$ obtained in Eq. (47) can be reformulated as:

$$\mathcal{A}^{a_1 \dots a_n}_n (z_i, \bar{z}_i, s_i) \; = \; \frac{ig^{n-2}}{(2\pi)^4} \int_{\mathbf{R}^{4|8}} d^{4|8}x \; \prod_{\alpha=1}^{4} \xi^\alpha_1 \xi^\alpha_2 \;\; \mathsf{Tr} \; \prod_{i=1}^{n} s_i^{e_i} \; \exp\left(i\, s_i \, \langle z_i | x | \bar{z}_i]\right) \; \frac{\mathsf{T}^{a_i}}{z_i \cdot z_{i+1}}. \tag{49}$$

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[11] We follow the techniques developed by Grisaru and Pendleton [40], Brink, Schwarz, and Scherk [41] and Ferber [42], as reviewed by Wess and Bagger [43] and Elvang and Huang [38].

[12] Introduced by Berezin [44] and DeWitt [45]. For a modern and rigorous mathematical exposition, see Manin [46].

2. $\mathcal{N}=4$ SYM Super-amplitude

To extend this to *supersymmetric* amplitudes, we define the superfield encoding the particle spectrum of $\mathcal{N}=4$ SYM theory:

$$\varphi\left(\xi^{\alpha}\right) := \mathsf{a}^{-}+\xi^{\alpha}\,\lambda_{\alpha}+\frac{1}{2!}\,\xi^{\alpha}\xi^{\beta}\;\;\phi_{\alpha\beta}+\frac{1}{3!}\,\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\varepsilon_{\alpha\beta\gamma\delta}\;\;\eta^{\delta}+\mathsf{a}^{+}\prod_{\alpha=1}^{4}\xi^{\alpha}. \tag{50}$$

Here, $\mathsf{a}_i^{\pm}$ represent the classical expectation values associated with the annihilation operators for gluons of positive and negative helicities.

Thus, the gluonic super-amplitude in $\mathcal{N}=4$ SYM theory can be expressed as:

$$\mathscr{A}_n^{a_1\ldots a_n}\left(z_i,\bar{z}_i,s_i\right)=\frac{ig^{n-2}}{(2\pi)^4}\int\limits_{\mathbf{R}^{4|8}}d^{4|8}x\;\;\mathsf{Tr}\;\prod_{i=1}^{n}s_i^{e_i}\,\varphi\left(z_i\cdot\theta^{\alpha}\right)\,\exp\left(i\,s_i\,\langle z_i|x|\bar{z}_i]\right)\,\frac{\mathsf{T}^{a_i}}{z_i\cdot z_{i+1}}. \tag{51}$$

B. Minitwistor Amplitudes for Gluons

We are now positioned to derive the main result of this section. We shall demonstrate that the celestial leaf amplitude for gluons in $\mathcal{N}=4$ SYM theory admits a representation as an integral over the moduli space of minitwistor lines in the non-singular quadric $\mathbf{MT}$.

1. Mellin Transform and Leaf Amplitudes

We begin by recalling the definition of the celestial amplitude[13] as the ε-regulated Mellin transform:

$$\widehat{\mathscr{A}_n^{a_1\ldots a_n}}\left(z_i,\bar{z}_i,\Delta_i\right)\;:=\;\prod_{i=1}^{n}\int\limits_{\mathbf{R}_{+}^{\times}}\frac{ds_i}{s_i}\,s_i^{\Delta_i}\,e^{-\varepsilon s_i}\;\;\mathscr{A}_n^{a_1\ldots a_n}\left(z_i,\bar{z}_i,s_i\right), \tag{52}$$

where $\mathbf{R}_{\times}^{+}:=\left(\mathbf{R}^{+},\cdot\right)$ denotes the multiplicative group of positive real numbers, and $\frac{ds_i}{s_i}$ the Haar measure on $\mathbf{R}_{\times}^{+}$.

Substituting the expression for the gluon super-amplitude, obtained in Eq. (51), into Definition (52) yields:

$$\widehat{\mathscr{A}_n^{a_1\ldots a_n}}\left(z_i,\bar{z}_i,\Delta_i\right)\;=\;\frac{ig^{n-2}}{(2\pi)^4}\;\int\limits_{\mathbf{R}^{4|8}}d^{4|8}x\;\;\mathsf{Tr}\;\prod_{i=1}^{n}\boldsymbol{\varphi}_{2h_i}\left(z_i\cdot\theta^{\alpha}\right)\,\phi_{2h_i}\left(x;z_i,\bar{z}_i\right)\,\frac{\mathsf{T}^{a_i}}{z_i\cdot z_{i+1}}. \tag{53}$$

[13] Cf. Pasterski and Shao [47], Pasterski, Shao, and Strominger [48, 49], Arkani-Hamed *et al.* [50], Banerjee *et al.* [51], Banerjee and Ghosh [52]. For recent pedagogical reviews, see Pasterski [53], Raclariu [54], Strominger [55], Aneesh *et al.* [56], Pasterski [57].

Here, $\phi_{\Delta}(x;z,\bar{z})$ represents the celestial wavefunction[14] for massless scalars, which is defined by:

$$\phi_{\Delta}(x;z,\bar{z}) \coloneqq \frac{\mathcal{C}(\Delta)}{(i\varepsilon+\langle z|x|\bar{z}])^{\Delta}},\ \ \mathcal{C}(\Delta) \coloneqq i^{-\Delta}\Gamma(\Delta). \tag{54}$$

Additionally, $\boldsymbol{\varphi}_{\Delta}(\xi)$ is the superfield expansion for the Mellin-transformed mode coefficients:

$$\boldsymbol{\varphi}_{\Delta}(\xi^{\alpha}) = \widehat{\mathsf{a}}^{-}_{\Delta} + \xi^{\alpha}\,\widehat{\lambda}_{\Delta,\alpha} + \frac{1}{2!}\,\xi^{\alpha}\xi^{\beta}\ \ \widehat{\phi}_{\Delta,\alpha\beta} + \frac{1}{3!}\,\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\,\varepsilon_{\alpha\beta\gamma\delta}\ \widehat{\eta}^{\delta}_{\Delta} + \widehat{\mathsf{a}}^{+}_{\Delta}\ \prod_{\alpha=1}^{4}\xi^{\alpha}. \tag{55}$$

We note that the mode functions $\widehat{\mathsf{a}}^{\pm}_{\Delta} = \widehat{\mathsf{a}}^{\pm}_{\Delta}(z,\bar{z})$, $\widehat{\lambda}_{\Delta,\alpha} = \widehat{\lambda}_{\Delta,\alpha}(z,\bar{z})$ etc. depends on the normalised spinor basis $\{z_i^A, \bar{z}_{i\dot{A}}\}$, parametrising the insertion points of the gluons on the celestial sphere, and the conformal weights Δ_i.

Leaf Representation. We now proceed to implement the formalism of celestial leaf amplitudes[15], requiring a specification of both the integration measure and the domains over which the gluonic amplitudes will be constructed. To that end, we recall that, within the embedding-space formalism, spacetime points are parametrised by homogeneous coordinates $X_{A\dot{A}}$, where the real three-dimensional Euclidean hyperboloid H_3^+ is realised as the projective space $\mathbf{RP}^3$. The geometry of the latter is given by the projectively invariant line-element $ds^2 = \boldsymbol{g}_{A\dot{A}B\dot{B}}dX^{A\dot{A}}\otimes dX^{B\dot{B}}$, as defined in Eq. (A1). The orientation of $\mathbf{RP}^3$ is induced by the natural volume form[16]:

$$D^3X \coloneqq \varepsilon_{A\dot{A}}\,\varepsilon_{B\dot{B}}\,\varepsilon_{C\dot{C}}\,\varepsilon_{D\dot{D}}\ \ X^{A\dot{A}}\,dX^{B\dot{B}}\wedge dX^{C\dot{C}}\wedge dX^{D\dot{D}}. \tag{56}$$

Extending this construction to the supersymmetric case requires promoting the bosonic coordinates $X_{A\dot{A}}$ to their supersymmetric counterparts $\mathbb{X}^{\hat{I}} \coloneqq (X_{A\dot{A}}, \theta^{\alpha}_{A})$, where θ^{α}_{A} $(1\leq\alpha\leq 4)$ are Grassmann-valued spinorial coordinates parametrising the "fermionic dimensions." The corresponding Berezin-de Witt integration measure on the projective superspace $\mathbf{RP}^{3|8}$ is then given by:

$$D^{3|8}\mathbb{X} \coloneqq \frac{D^3X}{|X|^4}\wedge d^{0|8}\theta. \tag{57}$$

With these preliminaries established, the celestial superamplitude in the leaf representation takes the form:

$$\widehat{\mathscr{A}}^{a_1\ldots a_n}_{n}(z_i,\bar{z}_i,\Delta_i) = \frac{ig^{n-2}}{(2\pi)^3}\,\delta(\beta)\,\times\,A^{a_1\ldots a_n}_{n}(z_i,\bar{z}_i,\Delta_i) + \big(\bar{z}_i\to-\bar{z}_i\big), \tag{58}$$

where $\beta \coloneqq 4-2\sum_{i=1}^{n}h_i$ enforces the scaling constraint, and the symbol $(\bar{z}_i\to-\bar{z}_i)$ signifies the repetition of the first term under the implied replacement. The function $A^{a_1\ldots a_n}_{n}(z_i,\bar{z}_i,\Delta_i)$ represents the gluonic leaf super-amplitude, which is defined as the following supersymmetric integral

[14] See Pasterski and Shao [47] and our Appendix A.

[15] See Appendix B or Melton, Sharma, and Strominger [58].

[16] As defined in Gelfand, Graev, and Vilenkin [29], Gel_fand, Gindikin, and Graev [59].

over $\mathbf{RP}^{3|8}$:

$$A_n^{a_1 \ldots a_n}\left(z_i, \bar{z}_i, \Delta_i\right) = \int\limits_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X} \ \mathsf{Tr} \prod_{i=1}^{n} \boldsymbol{\varphi}\left(z_i \cdot \theta^{\alpha}\right) \ K_{2h_i}\left(X; z_i, \bar{z}_i\right) \ \frac{\mathsf{T}^{a_i}}{z_i \cdot z_{i+1}}. \tag{59}$$

The kernel $K_{\Delta}\left(X; z, \bar{z}\right)$ is the bulk-to-boundary Green's function for the covariant Laplacian $\square_{\mathbf{H}_3}$ with conformal weight Δ, given in homogeneous coordinates $X^{A\dot{A}} \in \mathbf{H}_3$ by:

$$K_{\Delta}\left(X; z, \bar{z}\right) = \mathcal{C}\left(\Delta\right) \frac{|X|^{\Delta}}{\langle z|X|\bar{z}]^{\Delta}}. \tag{60}$$

For a concise review of leaf amplitudes, we refer the reader to Appendix B. For further mathematical details, see Gelfand, Graev, and Vilenkin [29], Teschner [24] and Costa, Gonçalves, and Penedones [30], Penedones [31].

2. *Celestial RSVW Formalism*

Now we reformulate Eq. (59) to derive the *celestial RSVW amplitude*. First, the procedure requires the resolution of the identity for the $\mathcal{N} = 4$ superfield $\boldsymbol{\varphi}$:

$$\boldsymbol{\varphi}\left(z_r \cdot \theta^{\alpha}\right) = \int\limits_{\mathbf{CP}^{0|4}} d^{0|4}\xi_r \ \delta^{(4)}\left(\xi_r^{\alpha} - z_r^{A}\theta_A^{\alpha}\right) \boldsymbol{\varphi}\left(\xi_i^{\alpha}\right). \tag{61}$$

This identity allows us to recast the leaf amplitude $A_n^{a_1 \ldots a_n}\left(z_i, \bar{z}_i, \Delta_i\right)$ as follows:

$$A_n^{a_1 \ldots a_n} = \prod_{r=1}^{n} \int\limits_{\mathbf{CP}^{0|4}} d^{0|4}\xi_r \int\limits_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X} \ \mathsf{Tr} \prod_{i=1}^{n} \delta^{(4)}\left(\xi_i^{\alpha} - z_i^{A}\theta_A^{\alpha}\right) \boldsymbol{\varphi}\left(\xi_i^{\alpha}\right) \ K_{2h_i}\left(X; z_i, \bar{z}_i\right) \ \frac{\mathsf{T}^{a_i}}{z_i \cdot z_{i+1}} \tag{62}$$

Our subsequent task is to define the integration measure over which the gluonic leaf amplitude $A_n^{a_1 \ldots a_n}\left(z_i, \bar{z}_i, \Delta_i\right)$ will be expressed as an integral transform. As will be explained in Section V, the minitwistor superspace[17] $\mathbf{MT}^{2|4}$ may be identified with the trivial vector superbundle[18] $\mathbf{MT}^{2|4} \simeq \mathbf{MT} \times \mathbf{CP}^{0|4}$, where the base manifold $\mathbf{MT}$ corresponds to the bosonic minitwistor space, while the typical fibres are isomorphic to the $\mathbf{Z}_2$-graded vector space spanned by the Grassmann-odd projective coordinates ξ^{α}, with $\alpha = 1, ..., 4$.

The vector superbundle $\mathbf{MT}^{2|4}$ admits a local trivialisation in terms of homogeneous coordinates:

$$\mathsf{Z}^{I} \coloneqq \left(\lambda^{A}, \mu_{\dot{A}}, \xi^{\alpha}\right) \in \mathbf{CP}^{1|0} \times \mathbf{CP}^{1|0} \times \mathbf{CP}^{0|4}. \tag{63}$$

The natural orientation of the total space $\mathbf{MT}^{2|4}$ is induced by the Berezin-de Witt volume superform, given by:

$$D^{2|4}\mathsf{Z} \coloneqq D\lambda \wedge D\mu \wedge d^{0|4}\xi. \tag{64}$$

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[17] Cf. Sämann [60].

[18] See Rogers [61].

Incorporating the celestial RSVW identity established in Eq. (44), the gluonic leaf amplitude $A_n^{a_1 \ldots a_n}(z_i, \bar{z}_i, \Delta_i)$ admis the following representation in terms of an integral transform over minitwistor superspace:

$$A_n^{a_1 \ldots a_n}(z_i, \bar{z}_i, \Delta_i) = \prod_{r=1}^{n} \int_{\mathbf{MT}^{2|4}} D^{2|4}\mathsf{Z}_i \;\; \varphi_{2h_i}(\xi_i^\alpha)\; \widetilde{\mathcal{F}}_{2h_i}(\lambda_i, \mu_i; z_i, \bar{z}_i) \int_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X} \tag{65}$$

$$\mathsf{Tr} \prod_{i=1}^{n} \bar{\delta}_{2h_i}\left(\mu_{i\dot{A}}, \lambda_i^A \frac{X_{A\dot{A}}}{|X|}\right) \delta^{(4)}\left(\xi_i^\alpha - \lambda_i^A \theta_A^\alpha\right) \frac{\mathsf{T}^{a_i}}{\lambda_i \cdot \lambda_{i+1}} \tag{66}$$

Following the key insight of Witten [20, 62], we identify from the preceding expression the *minitwistor gluon wavefunction* with *celestial conformal weight* Δ as:

$$\Phi_\Delta\left(\mathsf{Z}^I; z^A, \bar{z}_{\dot{A}}\right) := \varphi_\Delta(\xi^\alpha)\; \widetilde{\mathcal{F}}_\Delta\left(\lambda^A, \mu_{\dot{A}}; z^A, \bar{z}_{\dot{A}}\right). \tag{67}$$

We thus conclude that the leaf amplitude can be interpreted as a Fourier-transform over $\mathbf{MT}^{2|4}$,

$$A_n^{a_1 \ldots a_n}(z_i, \bar{z}_i, \Delta_i) = \int_{\left(\mathbf{MT}^{2|4}\right)^{\times n}} \bigwedge_{i=1}^{n} D^{2|4}\mathsf{Z}_i \;\; \Phi_{2h_i}(\mathsf{Z}_i; z_i, \bar{z}_i)\; \widetilde{A}_n^{a_1 \ldots a_n}(\mathsf{Z}_i), \tag{68}$$

where the Fourier-transformed amplitude is:

$$\widetilde{A}_n^{a_1 \ldots a_n}\left(\mathsf{Z}_i^I\right) = \int_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X} \;\; \mathsf{Tr} \prod_{i=1}^{n} \bar{\delta}_{2h_i}\left(\mu_{i\dot{A}}, \lambda_i^A \frac{X_{A\dot{A}}}{|X|}\right) \delta^{(4)}\left(\xi_i^\alpha - \lambda_i^A \theta_A^\alpha\right) \frac{\mathsf{T}^{a_i}}{\lambda_i \cdot \lambda_{i+1}}. \tag{69}$$

Finally we can formulate an important geometrical and physical interpretation of the result obtained. The (bosonic) spinor delta-functions appearing in the integrand serve to localise the integration over X to the locus of incidence curves $\mathcal{L}(X)$, which, as previously discussed, correspond to minitwistor lines. These incidence curves are precisely the conic curves in $\mathbf{MT}$ that are associated with a fixed point $X^{A\dot{A}} \in \mathbf{H}_3$ on the hyperboloid, through the incidence relation $\mu_{\dot{A}} = \lambda^A X_{A\dot{A}}$. *The gluonic leaf amplitudes are identically null in all instances wherein the insertion points of the gluons fail to lie on a curve satisfying the defining incidence relation of a minitwistor line.*

Therefore, the leaf amplitudes acquires an elegant geometrical interpretation. *It can be viewed as an integral over the moduli space of minitwistor lines $\mathcal{L}(X)$ within the supersymmetric, non-singular quadric $\mathbf{MT}^{2|4}$.* However, it must be stressed that this integral cannot be interpreted as an ordinary volume form over the moduli space owing to the scaling dimensions associated with the delta functions $\bar{\delta}_{2h_i}$. It would be interesting to find an interpretation for Eq. (69) in the formalism developed by Movshev [63] and Adamo and Groechenig [64].

C. Generating Functional

The geometric reformulation of gluonic leaf amplitudes enables the construction of a generating functional by invoking the Quillen determinant line bundle[19] and the celestial BMS identity.

As we shall explain in Section V, to each point in projective superspace $\mathbb{X}^{\hat{I}} = \left(X_{A\dot{A}}, \theta^{\alpha}_{\dot{A}}\right) \in \mathbf{RP}^{3|8}$, there correspond a *minitwistor superline* embedded in $\mathbf{MT}^{2|4}$, defined by the locus of incidence:

$$\mathcal{L}\left(X,\theta\right) \coloneqq \left\{ \left(\lambda^{A}, \mu_{\dot{A}}, \xi^{\alpha}\right) \in \mathbf{MT}^{2|4} \,\middle|\, \mu_{\dot{A}} = \lambda^{A} \frac{X_{A\dot{A}}}{|X|},\, \xi^{\alpha} = \lambda^{A}\theta^{\alpha}_{\dot{A}} \right\}. \tag{70}$$

In what follows, we shall adopt the conventional notation for the restriction homomorphism associated with cohomological classes on $\mathbf{MT}^{2|4}$. For example, let:

$$[g] \in H^{0,1}\big(\mathbf{MT}^{2|4}, \mathcal{O}\left(-2,-\Delta\right)\big), \tag{71}$$

be a representative of the Dolbeault cohomology class. The restriction of any such representative to the supercurve $\mathcal{L}\left(X,\theta\right)$ will be denoted by:

$$g\,\big|_{\mathcal{L}(X,\theta)}\,\left(\lambda^{A}\right) \coloneqq \rho_{\mathcal{L}(X,\theta)}(g)\left(\lambda^{A}\right) = g\left(\lambda^{A}, \lambda^{A}\frac{X_{A\dot{A}}}{|X|}, \lambda^{A}\theta^{\alpha}_{\dot{A}}\right). \tag{72}$$

We introduce the *minitwistor background potential* $\boldsymbol{\omega}$ as a $(0,1)$-form on the holomorphic vector superbundle $\mathcal{O}\left(-2,-\Delta\right) \longrightarrow \mathbf{MT}^{2|4}$, taking values in the Lie algebra $\mathfrak{g}$ associated with the gauge group $\mathbf{G}$. Formally, this background potential is an element of the space:

$$\boldsymbol{\omega} \in \mathfrak{g} \otimes \Omega^{0,1}\big(\mathbf{MT}^{2|4}, \mathcal{O}\left(-2,-\Delta\right)\big). \tag{73}$$

Our objective is now to demonstrate that:

$$F\left[\boldsymbol{\omega}\right] \coloneqq \int\limits_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X}\;\; \log\det\left(\overline{\partial} + \boldsymbol{\omega}\right)\big|_{\mathcal{L}(X,\theta)}, \tag{74}$$

serves as a *generating functional* for the gluonic leaf amplitudes $A^{a_1\cdots a_n}_{n}\left(z_i,\bar{z}_i,\Delta_i\right)$, when evaluated at the specific background configuration:

$$\boldsymbol{\omega}\left(\lambda^{A}, \mu_{\dot{A}}, \xi^{\alpha}; z^{A}, \bar{z}_{\dot{A}}\right) = \mathsf{T}^{a} \int\limits_{\mathcal{P}\times\mathbf{C}^{2}} d\Delta \wedge dz \wedge d\bar{z}\;\; \varphi_{\Delta}\left(\xi^{\alpha}\right)\mathcal{F}_{\Delta}\left(\lambda^{A}, \mu_{\dot{A}}; z^{A}, \bar{z}_{\dot{A}}\right).$$

In this expression, $\overline{\partial}\,\big|_{\mathcal{L}(X,\theta)}$ denotes the restriction of the Dolbeault operator to the minitwistor superline $\mathcal{L}\left(X,\theta\right)$, T^{a} represents a generator of the Lie algebra $\mathfrak{g}$, the domain of integration $\mathcal{P} \coloneqq$

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[19] Introduced by Quillen [65] and further developed by Biswas and Schumacher [66], Brylinski [67]. For a review emphasising physical applications, see Freed [68].

$1 + i\mathbf{R}$ corresponds to the continuous principal series, where the celestial conformal weight Δ resides, $\boldsymbol{\varphi}_{\Delta}(\xi^{\alpha})$ is the Mellin-transformed $\mathcal{N} = 4$ superfield introduced in Eq. (55), and $\mathcal{F}_{\Delta} \in \Omega^{0,1}(\mathbf{MT}, \mathcal{O}(-2, -\Delta))$ the minitwistor wavefunction defined in Eq. (27).

The first step in our derivation consists in observing that the Penrose integral-geometric transform of the minitwistor background potential $\boldsymbol{\omega}$ is expressed as:

$$\int_{\mathcal{L}(X,\theta)} \boldsymbol{\omega}\big|_{\mathcal{L}(X,\theta)}(\lambda) = \mathsf{T}^{a} \int_{\mathcal{P}\times\mathbf{C}^{2}} d\Delta \wedge dz \wedge d\bar{z}\, \boldsymbol{\varphi}_{\Delta}(\xi^{\alpha})\; K_{\Delta}(X; z, \bar{z}), \tag{75}$$

where $K_{\Delta}(X; z, \bar{z})$ denotes the bulk-to-boundary propagator on the hyperboloid H_{3}^{+} expressed in terms of homogeneous coordinates $X_{A\dot{A}}$ that chart real projective superspace $\mathbf{RP}^{3}$. Accordingly, we can apply the celestial BMS identity established in Eq. (33) to $\boldsymbol{\omega}$.

To proceed, we employ the expansion of the Quillen determinant, as explained by Boels, Mason, and Skinner [28].[20] Upon applying the celestial BMS identity, we find that the generating functional admits the expansion:

$$\digamma[\boldsymbol{\omega}] = \sum_{m\geq 2} \frac{(-1)^{m+1}}{m} \int_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X}\ \mathsf{Tr} \int_{(\mathcal{L}(X,\theta))^{\times n}} \bigwedge_{i=1}^{m} D\lambda_{i}\ \boldsymbol{\omega}\big|_{\mathcal{L}(X,\theta)}\ \frac{1}{\lambda_{i}\cdot\lambda_{i+1}} \tag{76}$$

$$= \sum_{m\geq 2} \frac{(-1)^{m+1}}{m} \int_{(\mathcal{P}\times\mathbf{C}^{2})^{\times m}} \bigwedge_{r=1}^{m} \boldsymbol{\beta}_{r} \int_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X}\ \mathsf{Tr} \prod_{i=1}^{n} \boldsymbol{\varphi}_{\Delta_{i}}(\xi_{i}^{\alpha})\; K_{\Delta_{i}}(X; z_{i}, \bar{z}_{i})\ \frac{\mathsf{T}^{a_{i}}}{z_{i}\cdot z_{i+1}}, \tag{77}$$

where:

$$\boldsymbol{\beta}_{r} := d\Delta_{r} \wedge dz_{r} \wedge d\bar{z}_{r}. \tag{78}$$

To complete the derivation, we perform functional differentiation of $\digamma[\boldsymbol{\omega}]$ with respect to the classical expectation values of the annihilation operators $\mathsf{a}_{\Delta_{i}}^{\ell_{i}}(z_{i}, \bar{z}_{i})$, which correspond to gluon states with helicity ℓ_{i} and celestial conformal weight Δ_{i},

$$\frac{\delta}{\delta \mathsf{a}_{2h_{i}}^{\ell_{1}}(z_{i})} \cdots \frac{\delta}{\delta \mathsf{a}_{\Delta_{n}}^{\ell_{n}}(z_{n}, \bar{z}_{n})} \digamma[\boldsymbol{\omega}]\Bigg|_{\boldsymbol{\varphi}=0} = \int_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X}\ \mathsf{Tr} \prod_{i=1}^{n} \boldsymbol{\varphi}(z_{i}\cdot\theta^{\alpha})\; K_{2h_{i}}(X; z_{i}, \bar{z}_{i})\ \frac{\mathsf{T}^{a_{i}}}{z_{i}\cdot z_{i+1}}. \tag{79}$$

The resulting expression is immediately recognised as the gluonic leaf amplitude $A_{n}^{a_{1}\cdots a_{n}}(z_{i}, \bar{z}_{i}, \Delta_{i})$. We conclude that the formalism constructed in this subsection in terms of the minitwistor background potential $\boldsymbol{\omega}$ provides a generating functional for tree-level MHV scattering amplitudes.

[20] See also Mason and Skinner [69, 70, 71], Bullimore, Mason, and Skinner [72]

Geometrical Interpretation. Let $\pi_{(X,\theta)} : \mathcal{L}(X,\theta) \longrightarrow \mathbf{CP}^1$ be the canonical projection. The embedding of the celestial sphere, modelled by the holomorphic Riemann sphere, into minitwistor superspace $\mathbf{MT}^{2|4}$ as a minitwistor superline $\mathcal{L}(X,\theta)$ can be interpreted as a section $\sigma_{(X,\theta)}$ of the fibration. In fact, by trivialising the fibration $\pi_{(X,\theta)}$ using homogeneous coordinates λ^A, we define the map:

$$\sigma : \mathbf{CP}^1 \longrightarrow \mathcal{L}(X,\theta), \tag{80}$$

which acts as:

$$\sigma : \lambda^A \mapsto \left(\lambda^A, \lambda^A \frac{X_{A\dot{A}}}{|X|}, \lambda^A \theta^{\alpha}_{A}\right). \tag{81}$$

It is obvious that this mapping satisfies the section condition $\pi_{(X,\theta)} \circ \sigma_{(X,\theta)} = id_{\mathbf{CP}^1}$ by construction. However, $\sigma_{(X,\theta)}$ is also an embedding of the celestial sphere into the Hitchin-special supercurve $\mathcal{L}(X,\theta) \subset \mathbf{MT}^{2|4}$.

This simple observation has an important implication for the interpretation of the generating functional $\digamma[\boldsymbol{\omega}]$, which involves an integral of the form:

$$\int\limits_{\mathbf{RP}^{3|8}} D^{3|8}\mathbb{X} \int\limits_{\mathcal{L}(X,\theta)} (...) \ . \tag{82}$$

A careful analysis shows that this expression *should not* be interpreted as integrating over celestial *spheres* at each point $\mathbb{X}^{\hat{I}} \in \mathbf{RP}^{3|8}$. Instead, the correct interpretation is that the integral is taken over the space of all *embeddings* of the celestial sphere into the minitwistor space $\mathbf{MT}^{2|4}$. In other words, (82) should be regarded as an integral over the moduli space of embeddings of the celestial sphere. This interpretation has important consequences for the construction of the on-shell effective action of our sigma model.

IV. $\mathcal{N} = 8$ SUPERGRAVITY

A. Review

Our analysis begins with the Berends-Giele-Kuijf (BGK) formula[21], which describes the tree-level gravitational scattering amplitudes for configurations characterised by maximal helicity violation. Consider a scattering process involving n gravitons in the MHV configuration $1^{--}, 2^{--}, 3^{++}, ..., n^{++}$,

[21] Originally derived in Berends, Giele, and Kuijf [73].

for which the amplitude is expressed as:

$$\mathcal{M}_n\left(z_i, \overline{z}_i, s_i\right) \;=\; \left(\frac{\kappa}{2}\right)^{n-2} \delta^{(4)}\left(\sum_{i=1}^{n} \nu_i^A \bar{\nu}_i^{\dot{A}}\right) \; BGK_n, \tag{83}$$

where the BGK_n factor is defined by:

$$BGK_n \coloneqq \frac{\langle\nu_1\nu_2\rangle^8}{\prod_{r=1}^{n}\nu_r\cdot\nu_{r+1}} \frac{1}{\langle\nu_n\nu_1\rangle\langle\nu_1\nu_{n-1}\rangle\langle\nu_{n-1}\nu_n\rangle} \prod_{i=2}^{n-2} \frac{[\bar{\nu}_i|p_{i+1}+...+p_{n-1}|\nu_n\rangle}{\nu_i\cdot\nu_n}. \tag{84}$$

A specially enlightening modern derivation of this result, using Plebanski's second heavenly equation, was given by Miller [74].

Our objective is to apply the formalism of celestial leaf amplitudes to $\mathcal{M}_n$. Using the minitwistor wavefunctions developed in Subsection (II), we seek to recast the modified Mellin transform of $\mathcal{M}_n$ into a form that admits an interpretation as an integral over a moduli space of minitwistor lines.

1. $\mathbf{CP}^1$ *Fermionic Doublet*

The factorisation procedure to be employed in this section is based on the method developed by Nair [75]. In this formalism, tree-level MHV gravitational scattering amplitudes are expressed in terms of a correlator of “vertex operators.” These operators are constructed with the aid of an auxiliary fermionic doublet $(\hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\chi}}^\dagger)$ defined on $\mathbf{CP}^1$, and their explicit forms are determined by the following mode expansions:

$$\hat{\boldsymbol{\chi}}\left(z\right) \coloneqq \sum_{p\geq 0} \frac{\mathsf{b}_p}{z^{1+p}}, \;\; \hat{\boldsymbol{\chi}}^\dagger\left(z\right) \coloneqq \sum_{p\geq 0} z^p\, \mathsf{b}_p^\dagger. \tag{85}$$

Here, the fermionic annihilation and creation operators b_p and $\mathsf{b}_p^\dagger$ satisfy the anti-commutation relations $\{\mathsf{b}_p, \mathsf{b}_q^\dagger\} = \delta_{pq}$ for all $p, q \geq 0$, and their action on the vacuum state $|0\rangle$ is given by $\mathsf{b}_p|0\rangle = 0$ for all $p \geq 0$.

For notational convenience, we define $\hat{\boldsymbol{\chi}}_i \coloneqq \hat{\boldsymbol{\chi}}\left(z_i\right)$ and $\hat{\boldsymbol{\chi}}_i^\dagger \coloneqq \hat{\boldsymbol{\chi}}^\dagger\left(z_i\right)$, where the sequence $\{z_i\}$ corresponds to the holomorphic coordinates that parametrise the momenta of the gravitons participating in the scattering process under consideration. Boldface symbols are employed for the fermionic fields to emphasise their operator character, and, as will be demonstrated, their correlators will serve to reconstruct the graviton amplitudes.

The anti-commutation relations imply that the two-point correlation function of the doublet $(\hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\chi}}^\dagger)$ assumes the form:

$$\langle 0|\,\hat{\boldsymbol{\chi}}_i\,\hat{\boldsymbol{\chi}}_j^\dagger\,|0\rangle \;=\; \frac{1}{z_{ij}}, \;\; z_{ij} \coloneqq z_i - z_j. \tag{86}$$

In order to introduce the graviton vertex operators, it is useful to reformulate the fermionic system $(\hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\chi}}^\dagger)$ in terms of an alternative representation, denoted $(\boldsymbol{\chi}, \boldsymbol{\chi}^\dagger)$, which is defined on the holomorphic vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ over $\mathbf{CP}^1$ of undotted two-component spinors. The precise correspondence between these two representations is specified as follows. Let ν^A denote a two-component spinor, and consider a local trivialisation $\mathscr{U}$ of the bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ such that $(\nu^A)_{\mathscr{U}} = (\alpha, \beta)^T$, where $\alpha \in \mathbf{C}$ and $\beta \in \mathbf{C}^*$. Within the trivialisation $\mathscr{U}$, it is natural to projectively represent ν^A on the complex projective line by introducing the local affine coordinate z, such that $[z] \coloneqq [\alpha\beta^{-1}] \in \mathbf{CP}^1$. With these preliminary observations in place, we define:

$$\boldsymbol{\chi}(\nu) \coloneqq \beta^{-1} \hat{\boldsymbol{\chi}}(z), \;\; \hat{\boldsymbol{\chi}}^\dagger(\nu) \coloneqq \beta^{-1} \hat{\boldsymbol{\chi}}^\dagger(z). \tag{87}$$

For notational convenience, we introduce the shorthand $\boldsymbol{\chi}_i \coloneqq \boldsymbol{\chi}(z_i)$ and $\boldsymbol{\chi}_i^\dagger \coloneqq \boldsymbol{\chi}^\dagger(z_i)$, where $\{z_i^A, \overline{z}_{i\dot{A}}\}$ constitutes a normalised spinor basis, defined explicitly by:

$$z_i^A \coloneqq \begin{pmatrix} 1 \\ z_i \end{pmatrix}, \;\; \overline{z}_{i\dot{A}} \coloneqq \begin{pmatrix} \overline{z}_i & -1 \end{pmatrix} \tag{88}$$

Similarly, we employ the notation $\boldsymbol{\chi}_{\nu_i} \coloneqq \boldsymbol{\chi}(\nu_i)$ and $\boldsymbol{\chi}_{\nu_i}^\dagger \coloneqq \boldsymbol{\chi}^\dagger(\nu_i)$, where the spinors ν_i^A $(1 \le i \le n)$ parametrise the kinematical data associated with the momenta of the graviton states.

Given these definitions, the two-point correlation function of the fermionic doublet $(\boldsymbol{\chi}, \boldsymbol{\chi}^\dagger)$ may be expressed simply as:

$$\langle 0| \, \boldsymbol{\chi}_{\nu_i} \boldsymbol{\chi}_{\nu_j}^\dagger \, |0\rangle = \frac{1}{\nu_i \cdot \nu_j}. \tag{89}$$

2. BGK_n Formula from $\mathbf{CP}^1$ Correlators

The graviton "vertex operators" are defined as:

$$\mathcal{G}_i \coloneqq \exp\big(\, i \, \langle \nu_i | \, x \, | \overline{\nu}_i] \,\big) \, \boldsymbol{\chi}_{\nu_i}^\dagger \boldsymbol{\chi}_{\nu_i}, \;\; \mathcal{H}_i \coloneqq \frac{[\overline{\nu}_i | \, (-i\boldsymbol{\partial}) \, |\omega\rangle}{\nu_i \cdot \omega} \exp\big(\, i \, \langle \nu_i | \, x \, | \overline{\nu}_i] \,\big). \tag{90}$$

In the above expressions, $(\boldsymbol{\partial})_{A\dot{A}} \coloneqq (\sigma^\mu)_{A\dot{A}} \frac{\partial}{\partial x^\mu}$ and $\omega_A \coloneqq \left(\frac{i}{2\pi}\right)^{1/2} (\, \omega, \, -1 \,)$ is an auxiliary two-component spinor parametrised by $[\omega] \in \mathbf{CP}^1$. This spinor serves as a reference variable over which integration is performed in the final expressions involving the correlation functions of $\mathcal{G}_i$ and $\mathcal{H}_i$. Furthermore, the spinor ω_A defines a state-vector within the $\mathbf{CP}^1$ fermionic system $(\boldsymbol{\chi}, \boldsymbol{\chi}^\dagger)$, which is given by $|\omega\rangle \coloneqq \chi^\dagger(\omega) \, |0\rangle$.

Through an inductive argument on $n \in \mathbf{N}$, it can be established that the operators $\mathcal{G}_i$ and $\mathcal{H}_i$ satisfy the identity:

$$\mathcal{G}_1 \left(\prod_{i=2}^{n-2} \mathcal{H}_i \right) \mathcal{G}_{n-1} \, \mathcal{G}_n \tag{91}$$

$$= \; e^{i(p_1 + \ldots + p_n)\cdot x} \prod_{i=2}^{n-2} \frac{[\overline{\nu}_i | p_{i+1} + \ldots + p_n | \omega\rangle}{\nu_i \cdot \omega} \; (\boldsymbol{\chi}^\dagger_{\nu_1} \, \boldsymbol{\chi}_{\nu_1}) \, (\boldsymbol{\chi}^\dagger_{\nu_{n-1}} \, \boldsymbol{\chi}_{\nu_{n-1}}) \, (\boldsymbol{\chi}^\dagger_{\nu_n} \, \boldsymbol{\chi}_{\nu_n}). \tag{92}$$

This property implies an integral relation satisfied by the correlation functions of the operators $\mathcal{G}_i$ and $\mathcal{H}_i$, which is expressed as follows:

$$\frac{1}{(2\pi)^4} \int\limits_{\mathbf{R}^4} d^4x \oint\limits_{\mathscr{C}_n} \langle \omega d\omega \rangle \;\; \langle \omega | \, \mathcal{G}_1 \left(\prod_{i=2}^{n-2} \mathcal{H}_i \right) \mathcal{G}_{n-1} \, \mathcal{G}_n \, | \omega \rangle \tag{93}$$

$$= \; \delta^{(4)} \left(\sum_{i=1}^{n} \nu_i^A \bar{\nu}_i^{\dot{A}} \right) \frac{1}{\langle \nu_n \nu_1 \rangle \, \langle \nu_1 \nu_{n-1} \rangle \, \langle \nu_{n-1} \nu_n \rangle} \prod_{j=2}^{n-2} \frac{[\bar{\nu}_j | p_{j+1} + \ldots + p_{n-1} | \nu_n \rangle}{\nu_j \cdot \nu_n}, \tag{94}$$

where $\mathscr{C}_n$ denotes a small contour centred at the insertion point on $\mathbf{CP}^1$ representing the n-th graviton.

Consequently, the scattering amplitude $\mathcal{M}_n$ can be expressed in a form structurally reminiscent of the correlation functions of vertex operators in conventional twistor string theory[22],

$$\mathcal{M}_n \; = \; \left(\frac{\kappa}{2} \right)^2 \;\; \langle \nu_1 \nu_2 \rangle^8 \int\limits_{\mathbf{R}^4} \frac{d^4x}{(2\pi)^4} \oint\limits_{\mathscr{C}_n} \langle \omega d\omega \rangle \;\; \langle \omega | \, \mathcal{G}_1 \left(\prod_{i=2}^{n-2} \mathcal{H}_i \right) \mathcal{G}_{n-1} \, \mathcal{G}_n \, | \omega \rangle \; \prod_{j=1}^{n} \frac{1}{\nu_j \cdot \nu_{j+1}}. \tag{95}$$

The precise nature of this correspondence will be clarified in the subsequent sections.

3. Frequency Dependency

To proceed with our aim of performing the Mellin transform of $\mathcal{M}_n$, thereby yielding the corresponding celestial amplitude, it is necessary to re-express Eq. (95) explicitly in terms of the graviton frequencies s_i and the normalised spinor basis $\{z_i^A, \overline{z}_{i\dot{A}}\}$. This reformulation takes the form:

$$\mathcal{M}_n \; = \; \left(\frac{\kappa}{2} \right)^{n-2} \;\; (z_1 \cdot z_2)^8 \int\limits_{\mathbf{R}^4} \frac{d^4x}{(2\pi)^4} \oint\limits_{\mathscr{C}_n} \langle \omega d\omega \rangle \;\; \langle \omega | \, \mathcal{G}_1 \left(\prod_{i=2}^{n-2} \mathcal{H}_i \right) \mathcal{G}_{n-1} \, \mathcal{G}_n \, | \omega \rangle \; \prod_{j=1}^{n} \frac{s_j^{e_j}}{z_j \cdot z_{j+1}}. \tag{96}$$

In this expression, the sequence e_i $(1 \leq i \leq n)$ denotes the set of exponents characterising the powers in which the frequencies s_i appear in $\mathcal{M}_n$. For the MHV configuration $1^{--}, 2^{--}, 3^{++}, \ldots, n^{++}$, these exponents are given by $e_1 = e_2 = 3$ and $e_3 = \ldots = e_n = -1$.

[22] See, e.g., Abou-Zeid, Hull, and Mason [76], Adamo and Mason [77], Mason and Skinner [78].

As we proceed, the analysis transitions from the MHV amplitude $\mathcal{M}_n$ to the graviton superamplitude $\mathscr{M}_n$ within the onshell superspace formalism of $\mathcal{N} = 8$ Supergravity. To maintain generality throughout the derivation, we shall retain the explicit dependence on the sequence e_i in all subsequent equations.

4. $\mathcal{N} = 8$ *Supergravity*

The transition from Einstein's gravity to $\mathcal{N} = 8$ Supergravity is facilitated by adopting the on-shell superfield formalism, as reviewed in Wess and Bagger [43].

On-shell Superfield Expansion. This formalism introduces Grassmann-valued two-component spinors, denoted η^α_A $(1 \le \alpha \le 8)$ and normalised according to the relation:

$$\int\limits_{\mathbf{R}^{0|2}} d^{0|2}\eta \;\; \eta^\alpha_A \, \eta^\alpha_B = \varepsilon_{AB}. \tag{97}$$

To construct the requisite supermultiplets, we define the set of Grassmann-valued coefficients:

$$\zeta^\alpha_i := z_i \cdot \eta^\alpha \quad (1 \le \alpha \le 8)\,. \tag{98}$$

By employing the properties of the Berezin-de Witt integral, the following identity is established:

$$\left(z_i \cdot z_j\right)^8 \;=\; \int\limits_{\mathbf{R}^{0|16}} d^{0|16}\eta \prod_{\alpha=1}^{8} \zeta^\alpha_i \zeta^\alpha_j. \tag{99}$$

which permits the following reformulation of Eq. (96):

$$\mathcal{M}_n = \left(\frac{\kappa}{2}\right)^{n-2} \int \frac{d^4x}{(2\pi)^4} \wedge d^{0|16}\eta \prod_{\alpha=1}^{8} \zeta^\alpha_1 \zeta^\alpha_2 \oint\limits_{\mathscr{C}_n} \langle \omega d\omega\rangle \, \langle\omega|\, \mathcal{G}_1 \left(\prod_{i=2}^{n-2} \mathcal{H}_i\right) \mathcal{G}_{n-1}\, \mathcal{G}_n \,|\omega\rangle \prod_{j=1}^{n} \frac{s_j^{e_j}}{z_j \cdot z_{j+1}}. \tag{100}$$

The particle spectrum of $\mathcal{N} = 8$ Supergravity is described by the superfield ψ, which parametrises all one-particle states of the theory. This superfield may be expanded as follows:

$$\psi_i(\zeta^\alpha_i) \;=\; \mathsf{h}^- \,+\, \zeta^\alpha_i \,\widetilde{\lambda}_\alpha \,+\, \frac{1}{2!}\, \zeta^\alpha_i \, \zeta^\beta_i \;\, \widetilde{\phi}_{\alpha\beta} \,+\, \ldots \,+\, \mathsf{h}^+_i \;\prod_{\alpha=1}^{8} \zeta^\alpha_i. \tag{101}$$

Here, $\mathsf{h}^-_i := \mathsf{h}^-\left(z, \bar{z}_i, s_i\right)$ represents the classical expectation value associated with the annihilation operator of a graviton of negative helicity, with momentum parametrised by $p^\mu = s_i q\left(z_i, \bar{z}_i\right)$, and $\mathsf{h}^+_i := \mathsf{h}^+\left(z_i, \overline{z}_i, s_i\right)$ denotes the analogous quantity for a graviton of positive helicity. The intermediate terms in this expansion describe the remaining one-particle states in the supermultiplet.

Superspace. At this stage, it is pertinent to specify the structure of the superspace on which the graviton super-amplitude is formulated. Thus, let $\mathbf{R}^{4|16}$ denote the supersymmetric extension of the four-dimensional Euclidean space $\mathbf{R}^4$, obtained by appending to the bosonic coordinates x^μ the set of Grassmann-odd spinorial coordinates θ^α_A $(\alpha = 1, ..., 8)$. These fermionic "dimensions," together with the bosonic ones, are collectively encoded in the supercoordinates[23] $\mathsf{x}^{\hat{\mathrm{I}}} := (x^\mu, \eta^\alpha_A)$. The supermanifold $\mathbf{R}^{4|16}$ is endowed with the canonical orientation defined by the Berezin-de Witt volume superform:

$$d^{4|16}\mathsf{x} := d^4x \wedge d^{0|16}\theta. \tag{102}$$

Accordingly, the graviton super-amplitude $\mathscr{M}_n$ is expressed in the on-shell superfield formalism as:

$$\mathscr{M}_n = \frac{1}{(2\pi)^4}\left(\frac{\kappa}{2}\right)^{n-2} \int\limits_{\mathbf{R}^{4|16}} d^{4|16}\mathsf{x} \oint\limits_{\mathscr{C}_n} \langle\omega d\omega\rangle \, \langle\omega|\, \mathcal{G}_1 \left(\prod_{i=2}^{n-2} \mathcal{H}_i\right) \mathcal{G}_{n-1}\, \mathcal{G}_n \,|\omega\rangle \prod_{j=1}^{n} \psi\left(z_i \cdot \theta^\alpha\right) \frac{s_j^{e_j}}{z_j \cdot z_{j+1}}. \tag{103}$$

B. Celestial Leaf Amplitudes for Gravitons

With the requisite mathematical preliminaries established, we now begin our analysis of graviton celestial amplitudes. The main objective it to demonstrate that the gravitational leaf amplitudes can be interpreted as an integral over the moduli space of minitwistor lines on the $\mathcal{N} = 8$ supersymmetric extension of minitwistor space, $\mathbf{MT}^{2|8}$.

1. *Mellin Transform*

Let us first recall that the celestial superamplitude $\widehat{\mathscr{M}_n}(z_i, \bar{z}_i, \Delta_i)$ for gravitons in $\mathcal{N} = 8$ Supergravity is defined as the ε-regulated Mellin transform of the corresponding scattering amplitude $\mathscr{M}_n(z_i, \bar{z}_i, s_i)$, as follows:

$$\widehat{\mathscr{M}_n}(z_i, \overline{z}_i, \Delta_i) \;:=\; \prod_{i=1}^{n} \int\limits_{\mathbf{R}_+^\times} \frac{ds_i}{s_i}\, s_i^{\Delta_i}\, e^{-\varepsilon s_i} \;\mathscr{M}_n(z_i, \overline{z}_i, s_i), \tag{104}$$

where $\mathbf{R}_+^\times$ denotes the multiplicative group of positive real numbers, and the integral is taken with respect to the Haar measure $\frac{ds_i}{s_i}$.

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[23] We follow standard conventions in abstract index notation. In addition, indices associated with supercoordinates, such as $\hat{\mathrm{I}}, \hat{\mathrm{J}}, ...$, are adorned with a hat to distinguish them from the minitwistor indices $I, J, ...$ appearing in expressions involving the minitwistor representatives $\mathsf{Y}_I, \mathsf{Z}^J$, etc.

We now introduce the *Mellin-transformed graviton vertex operators*:

$$\widehat{\mathcal{G}}_i \coloneqq \int\limits_{\mathbf{R}_+^\times} \frac{ds_i}{s_i^{1-e_i-\Delta_i}}\ e^{-\varepsilon s_i}\,\mathcal{G}_i,\ \ \widehat{\mathcal{H}}_i \coloneqq \int\limits_{\mathbf{R}_+^\times} \frac{ds_i}{s_i^{1-e_i-\Delta_i}}\ e^{-\varepsilon s_i}\,\mathcal{H}_i. \tag{105}$$

In these definitions, e_i ($1 \leq i \leq n$) denotes exponents characterising the frequency dependence of the super-amplitude $\mathscr{M}_n$.

In addition to these vertex operators, we define the *Mellin-transformed* $\mathcal{N} = 8$ *superfield* $\boldsymbol{\psi}_{\Delta_i}(\zeta_i^\alpha)$ as:

$$\boldsymbol{\psi}_{\Delta_i}(\zeta_i^\alpha) \coloneqq \widehat{\mathsf{h}}^-_{\Delta_i}(z_i,\overline{z}_i) + \zeta_i^\alpha\,\widehat{\lambda}_{\alpha,\Delta_i}(z_i,\overline{z}_i) + \zeta_i^\alpha\zeta_i^\beta\,\widehat{\phi}_{\alpha\beta,\Delta_i} + \ldots + \widehat{\mathsf{h}}^+_{\Delta_i}(z_i,\overline{z}_i)\prod_{\alpha=1}^{8}\zeta_i^\alpha. \tag{106}$$

The modes of the superfield expansion now explicitly depend on the celestial conformal weight Δ_i.

By substituting the expressions from Eqs. (90) and (105), and subsequently performing the integral transforms, we obtain the following explicit forms of the Mellin-transformed vertex operators:

$$\widehat{\mathcal{G}}_i = \phi_{2h_i}(x;z_i,\overline{z}_i)\,\hat{\boldsymbol{\chi}}_i^\dagger\,\hat{\boldsymbol{\chi}}_i,\ \ \widehat{\mathcal{H}}_i = \frac{[\overline{z}_i|\,(-i\boldsymbol{\partial})\,|\omega\rangle}{z_i\cdot\omega}\,\phi_{2h_i}(x;z_i,\overline{z}_i), \tag{107}$$

where $\phi_\Delta(x;z,\overline{z})$ is the celestial wavefunction defined in Eq. (54).

The scaling dimensions of these operators are given by:

$$h_i = \frac{\Delta_i + e_i - 1}{2},\ \ h_j \coloneqq \frac{\Delta_j + e_j}{2}. \tag{108}$$

Consequently, the celestial superamplitude for gravitons can be expressed in the following integral form:

$$\widehat{\mathscr{M}_n} = \frac{1}{(2\pi)^4}\left(\frac{\kappa}{2}\right)^{n-2}\int\limits_{\mathbf{R}^{4|16}} d^{4|16}\mathsf{w}\oint\limits_{\mathscr{C}_n}\langle\omega d\omega\rangle\ \langle\omega|\,\widehat{\mathcal{G}}_1\left(\prod_{i=2}^{n-2}\widehat{\mathcal{H}}_i\right)\widehat{\mathcal{G}}_{n-1}\,\widehat{\mathcal{G}}_n\,|\omega\rangle\prod_{j=1}^{n}\frac{\boldsymbol{\psi}_{2h_j}(z_i\cdot\theta^\alpha)}{z_j\cdot z_{j+1}}. \tag{109}$$

2. Gravitational Celestial Leaf Amplitudes

To reformulate the graviton celestial amplitude as an integral over the moduli space of minitwistor lines, we employ the formalism of leaf amplitudes. However, a technical complication immediately presents itself: the celestial wavefunctions $\phi_{2h_i}(x;z_i,\overline{z}_i)$, which appear as multiplicative factors in the vertex operators $\widehat{\mathcal{G}}_i$ and $\widehat{\mathcal{H}}_i$, are currently enclosed within the fixed-ordering correlator $\langle\omega|\cdots|\omega\rangle$. It is necessary to extract these wavefunctions from the correlator, as this step permits the separation of the spacetime integral from the contour integral $\oint_{\mathscr{C}_n}\langle\omega d\omega\rangle$, thus making it possible to apply the leaf amplitude formalism.

We proceed by implementing the following strategy. Let P_i denote the *weight-shifting operator* acting on the Hilbert space of celestial wavefunctions $\mathscr{H}_c$. The action of P_i on a given wavefunction $\phi_{\Delta_j}(x; z_j, \bar{z}_j)$ is:

$$\mathsf{P}_i \cdot \phi_{\Delta_j}(x; z_i, \bar{z}_i) \coloneqq \phi_{\Delta_j + \delta_{ij}}(x; z_j, \bar{z}_j). \tag{110}$$

Consequently, we define a new set of graviton vertex operators:

$$\mathcal{U}_i \coloneqq \exp\big(-\langle z_i|y|\bar{z}_i]\,\mathsf{P}_i\big)\; \hat{\boldsymbol{\chi}}_i^{\dagger}\, \hat{\boldsymbol{\chi}}_i, \;\; \mathcal{V}_i \coloneqq \frac{[\bar{z}_i|\widetilde{\boldsymbol{\partial}}|\omega\rangle}{z_i \cdot \omega}\; \exp\big(-\langle z_i|y|\bar{z}_i]\,\mathsf{P}_i\big). \tag{111}$$

In the above expression, $y^\mu \in \mathbf{R}^4$ denotes an *auxiliary* four-vector that parametrises the operator family $\{\mathcal{U}_i, \mathcal{V}_i\}$. The partial derivative $\widetilde{\boldsymbol{\partial}}$ acts with respect to y^μ, and is defined by:

$$\big(\widetilde{\boldsymbol{\partial}}\big)_{A\dot{A}} \coloneqq (\boldsymbol{\sigma}^\mu)_{A\dot{A}} \frac{\partial}{\partial y^\mu} \;\; (y^\mu \in \mathbf{R}^4), \tag{112}$$

where $(\boldsymbol{\sigma}^\mu)_{A\dot{A}}$ represents the sigma matrices in the Kleinian signature, defined in Appendix B.

It is important to emphasise that the four-vector y^μ should be regarded solely as a continuous label indexing the operator family. It does not carry the interpretation of a spacetime coordinate. Moreover, in all subsequent expressions where y^μ appears, the final evaluation is to be performed at $y^\mu = 0$.

We now establish, through an inductive argument on $n \in \mathbf{N}$, that the vertex operators satisfy the following algebraic identity:

$$\widehat{\mathcal{G}}_1 \left(\prod_{i=2}^{n-2} \widehat{\mathcal{H}}_i\right) \widehat{\mathcal{G}}_{n-1}\, \widehat{\mathcal{G}}_n \;=\; \mathcal{U}_1 \left(\prod_{i=2}^{n-2} \mathcal{V}_i\right) \mathcal{U}_{n-1}\mathcal{U}_n \;\prod_{j=1}^{n} \phi_{2h_j}(x; z_i, \bar{z}_i). \tag{113}$$

From this equality, it follows that the celestial super-amplitude $\widehat{\mathscr{M}_n}(z_i, \bar{z}_i, \Delta_i)$ can be reformulated as a correlator involving the newly introduced graviton vertex operators:

$$\widehat{\mathscr{M}_n}(z_i, \bar{z}_i, \Delta_i) \;=\; \frac{1}{(2\pi)^4} \left(\frac{\kappa}{2}\right)^{n-2} \oint_{\mathscr{C}_n} \langle \omega d\omega\rangle \;\; \langle\omega|\,\mathcal{U}_1 \left(\prod_{r=2}^{n-2} \mathcal{V}_r\right) \mathcal{U}_{n-1}\mathcal{U}_n\,|\omega\rangle \tag{114}$$

$$\times \int_{\mathbf{R}^{4|16}} d^{4|16}\mathsf{x}\; \prod_{i=1}^{n} \boldsymbol{\psi}_{2h_i}(z_i \cdot \theta^\alpha)\; \phi_{2h_i}(x; z_i, \bar{z}_i)\;\; \frac{1}{z_i \cdot z_{i+1}}. \tag{115}$$

The final step required prior to the application of the leaf amplitude formalism is to specify the integration domain over which it is defined. In Appendix A, we review how the Riemannian geometry of the hyperboloid H_3^+ arises from the projective geometry of $\mathbf{RP}^3$. Thus, let $X_{A\dot{A}}$ denote homogeneous coordinates charting the real projective space $\mathbf{RP}^3$. The natural orientation of $\mathbf{RP}^3$ is given by the volume form:

$$D^3X \;=\; \varepsilon_{A\dot{A}}\varepsilon_{B\dot{B}}\varepsilon_{C\dot{C}}\varepsilon_{D\dot{D}}\, X^{A\dot{A}} dX^{B\dot{B}} \wedge dX^{C\dot{C}} \wedge dX^{D\dot{D}}. \tag{116}$$

The $(3, 16)$-dimensional real projective *superspace* $\mathbf{RP}^{3|16}$ is obtained by augmenting the bosonic coordinates $X_{A\dot{A}}$ with Grassmann-valued spinorial variables θ^{α}_{A}, with $\alpha = 1, ..., 8$. Thus, the coordinates of the resulting superchart[24] are given by $\mathbb{X}^{\hat{I}} := \left(X_{A\dot{A}}, \theta^{\alpha}_{A}\right) \in \mathbf{RP}^3 \times \mathbf{R}^{0|16}$. The natural orientation of the supermanifold $\mathbf{RP}^{3|16}$ is defined by the Berezin-de Witt volume superform:

$$D^{3|16}\mathbb{X} := \frac{D^3 X}{|X|^4} \wedge d^{0|16}\zeta. \tag{117}$$

We are now prepared to employ the celestial leaf amplitude formalism, reviewed in Appendix B, which permits to recast $\widehat{\mathscr{M}_n}\left(z_i, \bar{z}_i, \Delta_i\right)$ as:

$$\widehat{\mathscr{M}_n}\left(z_i, \bar{z}_i, \Delta_i\right) = \frac{1}{(2\pi)^3}\left(\frac{\kappa}{2}\right)^{n-2} \delta\left(\beta\right) \times M_n\left(z_i, \bar{z}_i, \Delta_i\right) + (\bar{z}_{i\dot{A}} \longrightarrow -\bar{z}_{i\dot{A}}), \tag{118}$$

where $\beta := 4 - 2\sum_{i=1}^{n} h_i$, and the graviton leaf amplitude $M_n\left(z_i, \bar{z}_i, \Delta_i\right)$ is given by:

$$M_n = \oint_{\mathscr{C}_n} \langle \omega d\omega \rangle \;\; \langle \omega | \mathcal{U}_1 \left(\prod_{r=2}^{n-2} \mathcal{V}_r \right) \mathcal{U}_{n-1}\mathcal{U}_n \left|\omega\right\rangle \int_{\mathbf{RP}^{3|16}} D^{3|16}\,\mathbb{X} \prod_{i=1}^{n} K_{2h_i}\left(X; z_i, \bar{z}_i\right) \; \frac{\psi_{2h_i}\left(z_i \cdot \theta^{\alpha}\right)}{z_i \cdot z_{i+1}} \tag{119}$$

C. Celestial RSVW Formalism of $\mathcal{N} = 8$ Supergravity

We wish to find a geometrical interpretation for the gravitational leaf amplitude $M_n\left(z_i, \bar{z}_i, \Delta_i\right)$. To achieve this, it proves useful to employ the following decomposition:

$$M_n\left(z_i, \bar{z}_i, \Delta_i\right) = ⨍ dh_1 ... dh_n \;\; \mathrm{m}_n\left(z_i, \bar{z}_i, h_i\right), \tag{120}$$

where $⨍ dh_1 ... dh_n$ denotes a weight-shifting "integral" operator, defined by:

$$⨍ dh_1 ... dh_n := \oint_{\mathscr{C}_n} \langle \omega d\omega \rangle \;\; \langle \omega | \mathcal{U}_1 \left(\prod_{r=2}^{n-2} \mathcal{V}_r \right) \mathcal{U}_{n-1}\mathcal{U}_n \left|\omega\right\rangle, \tag{121}$$

and the "reduced" leaf amplitude $\mathrm{m}_n\left(z_i, \bar{z}_i, h_i\right)$ is given by

$$\mathrm{m}_n\left(z_i, \bar{z}_i, h_i\right) := \int_{\mathbf{RP}^{3|16}} D^{3|16}\,\mathbb{X} \prod_{i=1}^{n} \psi_{2h_i}\left(z_i \cdot \theta^{\alpha}\right) \, K_{2h_i}\left(X; z_i, \bar{z}_i\right) \; \frac{1}{z_i \cdot z_{i+1}}. \tag{122}$$

Using Eq. (111), which defines the operators $\mathcal{U}_i$ and $\mathcal{V}_i$, and applying it to the contour integral in Eq. (121), we deduce the following expression for the weight-shifting "integral" operator:

$$⨍ dh_1 ... dh_n = (-1)^{n+1} \left(\prod_{i=2}^{n-2} \sum_{j_i = i+1}^{n-1} \right) \prod_{k=2}^{n-2} \frac{[\bar{z}_k \bar{z}_{j_k}]\langle z_{j_k} z_n \rangle}{z_k \cdot z_n} \, \mathsf{P}_{j_k}. \tag{123}$$

This equation establishes that $⨍ dh_1 ... dh_n$ is a polynomial constructed from a sequence of weight-shifting operators P_i, acting on the Hilbert space $\mathscr{H}_c$ of celestial wavefunctions.

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[24] We follow the terminology of Rogers [61]. See also Leites [79] and Manin [46].

Celestial RSVW Formula for Gravitons

Our derivation of the celestial RSVW formula for gravitational leaf amplitudes begins with the resolution of the identity for the $\mathcal{N}=8$ superfield:

$$\boldsymbol{\psi}_{2h_i}\left(z_i\cdot\theta^\alpha\right) \;=\; \int\limits_{\mathbf{CP}^{0|8}} d^{0|8}\zeta_i\;\; \delta^{(8)}\left(\zeta_i^\alpha - z_i^A\theta_A^\alpha\right)\, \boldsymbol{\psi}_{2h_i}\left(\zeta_i\right), \tag{124}$$

where $d^{0|8}\zeta$ denotes the Berezin-de Witt integral over the fermionic coordinates ζ^α. This identity allows us to reformulate the "reduced" leaf amplitude $\mathrm{m}_n\left(z_i,\bar{z}_i,\Delta_i\right)$ defined in Eq. (122) as follows:

$$\mathrm{m}_n\left(z_i,\bar{z}_i,\Delta_i\right) \;=\; \prod_{r=1}^{n}\;\int\limits_{\mathbf{CP}^{0|8}} d^{0|8}\zeta_r \int\limits_{\mathbf{RP}^{3|16}} D^{3|16}\mathbb{X} \tag{125}$$

$$\prod_{i=1}^{n}\;\delta^{(8)}\left(\zeta_i^\alpha - z_i^A\theta_A^\alpha\right)\,\boldsymbol{\psi}_{2h_i}\left(\zeta_i^\alpha\right)\,K_{2h_i}\left(X;z_i,\bar{z}_i\right)\;\frac{1}{z_i\cdot z_{i+1}}. \tag{126}$$

To proceed, we introduce local coordinates charting the minitwistor space $\mathbf{MT}$. Let $p\in\mathbf{MT}\mapsto \mathsf{Z}^I\left(p\right) := \left(\lambda^A\left(p\right),\mu_{\dot{A}}\left(p\right)\right)\in\mathbf{CP}^1\times\overline{\mathbf{CP}}^1$, where λ^A and $\mu_{\dot{A}}$ are homogeneous coordinates on the complex projective lines $\mathbf{CP}^1$ and $\overline{\mathbf{CP}}^1$, respectively. Recall that the minitwistor space $\mathbf{MT}$ is a non-singular quadric in $\mathbf{CP}^3$, and its canonical orientation is induced by the volume form $D\lambda\wedge D\mu$, with $D\lambda := \varepsilon_{AB}\lambda^A d\lambda^B$ and $D\mu := \varepsilon^{\dot{A}\dot{B}}\mu_{\dot{A}}d\mu_{\dot{B}}$.

We now invoke the celestial RSVW identity derived in Eq. (44), which allows us to rewrite the expression for the "reduced" leaf amplitude in terms of integrals over minitwistor space:

$$\mathrm{m}_n\left(z_i,\bar{z}_i,\Delta_i\right) = \prod_{r=1}^{n}\;\int\limits_{\mathbf{MT}\times\mathbf{CP}^{0|8}} D\lambda_r\wedge D\mu_r\wedge d^{0|8}\zeta_r\;\;\boldsymbol{\psi}_{2h_r}\left(\zeta_r^\alpha\right)\,\widetilde{\mathcal{F}}_{2h_r}\left(\lambda_r^A,\mu_{r\dot{A}};z_r^A,\bar{z}_{r\dot{A}}\right) \tag{127}$$

$$\int\limits_{\mathbf{RP}^{3|16}} D^{3|16}\mathbb{X}\;\prod_{i=1}^{n}\;\bar{\delta}_{2h_i}\left(\mu_{i\dot{A}},\lambda_i^A\frac{X_{A\dot{A}}}{|X|}\right)\,\delta^{(8)}\left(\zeta_i^\alpha-\lambda_i^A\theta_A^\alpha\right)\;\frac{1}{\lambda_i\cdot\lambda_{i+1}}. \tag{128}$$

The subsequent step in our derivation requires a specification of the domain over which the "reduced" gravitational leaf amplitude is defined. Observe that the first integrals appearing in the above expression are taken over the product space $\mathbf{MT}\times\mathbf{CP}^{0|8}$. As will be discussed in detail in Section V, where our sigma model is introduced, the minitwistor superspace[25] is identified with the trivial superbundle $\mathbf{MT}^{2|8}\simeq\mathbf{MT}\times\mathbf{CP}^{0|8}$. Here, the base manifold is the nonsingular quadric, while the typical fibre isomorphic to $\mathbf{CP}^{0|8}$ follows from the projectivisation of the $\mathbf{Z}_2$-graded vector space spanned by the Grassmann-valued homogeneous coordinates ζ^α, with $\alpha=1,...,8$.

[25] For a rigorous mathematical discussion, see Sämann [60].

The orientation on the total space of this superbundle is defined by the Berezin-de Witt volume superform:

$$D^{2|8}\mathsf{Z} \coloneqq D\lambda \wedge D\mu \wedge d^{0|8}\zeta. \tag{129}$$

Following Witten's original construction[26], the *minitwistor wavefunction* associated with a graviton of *celestial conformal weight* Δ is identified from Eq. (127) as:

$$\Psi_\Delta\left(\mathsf{Z}^I; z^A, \bar{z}_{\dot{A}}\right) \coloneqq \varphi_\Delta\left(\zeta^\alpha\right) \widetilde{\mathcal{F}}_\Delta\left(\lambda^A, \mu_{\dot{A}}; z^A, \bar{z}_{\dot{A}}\right). \tag{130}$$

Consequently, the "reduced" gravitational leaf amplitude can be expressed as a Fourier transform over minitwistor superspace:

$$\mathrm{m}_n\left(z_i, \bar{z}_i, \Delta_i\right) = \int\limits_{\left(\mathbf{MT}^{2|8}\right)^{\times n}} \bigwedge_{i=1}^{n} D^{2|8}\mathsf{Z}_i \;\; \Phi_{2h_i}\left(\mathsf{Z}_i; z_i, \bar{z}_i\right) \widetilde{\mathrm{m}}_n\left(\mathsf{Z}_i\right), \tag{131}$$

where the Fourier-transformed amplitude is given by:

$$\widetilde{\mathrm{m}}_n\left(\mathsf{Z}_i\right) = \int\limits_{\mathbf{RP}^{3|16}} D^{3|16}\mathbb{X} \prod_{i=1}^{n} \bar{\delta}_{2h_i}\left(\mu_{i\dot{A}}, \lambda_i^A \frac{X_{A\dot{A}}}{|X|}\right) \delta^{(8)}\left(\zeta_i^\alpha - \lambda_i^A \theta_A^\alpha\right) \frac{1}{\lambda_i \cdot \lambda_{i+1}}. \tag{132}$$

Thus, the Fourier-transformed leaf amplitudes for gravitons in $\mathcal{N} = 8$ Supergravity[27] is expressed as an integral over the moduli space of minitwistor lines $\mathcal{L}(X)$ embedded in the non-singular quadric **MT**. This interpretation arises from the fact that the integration with respect to the measure D^3X is localised by the weighted distributional forms $\bar{\delta}_{2h_i}$, which impose the incidence relations characterising the curves $\mathcal{L}(X)$. Therefore, the support of the resulting integral is restricted to configurations where the graviton insertion points lie along such minitwistor lines.

It must be emphasised that this integral does not represent a conventional volume over the moduli space of conic curves. The distributional forms $\bar{\delta}_{2h_i}$ possess specific degrees of homogeneity that encode the celestial scaling dimensions Δ_i of the gravitons involved in the scattering process. It would be interesting to reformulate the expression for $\widetilde{\mathrm{m}}_n\left(\mathsf{Z}_i\right)$ using the formalism proposed by Movshev [63] and Adamo and Groechenig [64].

D. Generating Functional

We now proceed to construct a generating functional for gravitational leaf superamplitudes by employing the geometric interpretation derived above. To incorporate the weight-shifting operator $⨍\, dh_1...dh_n$, it is convenient to reformulate the amplitude $M_n\left(z_i, \bar{z}_i, \Delta_i\right)$ as follows.

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[26] See Witten [20, 62].

[27] At tree-level for configurations characterised by maximal helicity violation.

1. Preliminaries: A Celestial Correlator

Let $\boldsymbol{O} \coloneqq \boldsymbol{O}[\omega^A, \mathsf{P}_i, \Psi_{\Delta_i}]$ be an operator depending on the auxiliary spinor ω^A, the weight-shifting operators P_i, and the minitwistor graviton wavefunctions $\Psi_{\Delta_i}(\mathsf{Z}_i; z_i, \bar{z}_i)$. Define the expectation value of $\boldsymbol{O}$ over the auxiliary spinor and the fermionic doublet $(\boldsymbol{\chi}, \boldsymbol{\chi}^\dagger)$ as:

$$\langle \boldsymbol{O} \rangle_{\omega,n} = \oint_{C_n} \langle \omega d\omega \rangle \ \ \langle \omega | \, \boldsymbol{O}[\omega^A, \mathsf{P}_i, \Psi_{\Delta_i}] \, | \omega \rangle, \tag{133}$$

where C_n denotes a small contour centred at z_n, the insertion point of the n-th graviton on the celestial sphere.

Next, introduce a sequence of Grassmann-valued scalars c_i for $i = 1, ..., n$, satisfying the normalisation condition:

$$\int_{\mathbf{CP}^{0|1}} d^{0|1}\mathsf{c}_i \ (\alpha_i + \mathsf{c}_i \beta_i) = \beta_i, \tag{134}$$

for all $\alpha_i, \beta_i \in \mathbf{C}$. Denote by $\Lambda \coloneqq \Lambda[\mathsf{c}_1, ..., c_n]$ the Grassmann algebra generated by the variables c_i, which may be interpreted as *ghost-like fields* corresponding to anti-commuting spinless coordinates.

Let $\mathcal{D}(\mathscr{H}_c)$ be the ring of bounded linear operators on the Hilbert space $\mathscr{H}_c$ of celestial wavefunctions, and extend this structure to the $\mathbf{Z}_2$-graded ring:

$$\mathcal{D}_\Lambda(\mathscr{H}_c) \ \coloneqq \ \Lambda[\mathsf{c}] \otimes \mathcal{D}(\mathscr{H}_c) \, . \tag{135}$$

Using this formalism, the graviton vertex operators $\mathcal{U}_i$ and $\mathcal{V}_i$ can be unified into a single element of $\mathcal{D}_\Lambda(\mathscr{H}_c)$ of the form:

$$\mathcal{U}_i + \mathsf{c}_i \mathcal{V}_i \in \mathcal{D}_\Lambda(\mathscr{H}_c) \, . \tag{136}$$

With these constructions at hand, the gravitational leaf amplitude $M_n(z_i, \bar{z}_i, \Delta_i)$ can be expressed as the correlation function of the following operator:

$$\boldsymbol{M}_n(z_i, \bar{z}_i, \Delta_i) \ = \int_{\mathbf{RP}^{3|16}} D^{3|16}\mathbb{X} \ \prod_{i=1}^{n} \ (\mathcal{U}_i + \mathsf{c}_i \mathcal{V}_i) \ \boldsymbol{\psi}_{2h_i}(z_i \cdot \theta^\alpha) \ K_{2h_i}(X; z_i, \bar{z}_i) \ \ \frac{1}{z_i \cdot z_{i+1}}. \tag{137}$$

In terms of the correlator $\langle \boldsymbol{O} \rangle_{\omega,n}$ previously introduced in Eq. (133), the graviton leaf amplitude admits the following representation:

$$M_n(z_i, \bar{z}_i, \Delta_i) \ = \left\langle \int d\mathsf{c}_2 ... d\mathsf{c}_{n-2} \ \boldsymbol{M}_n(z_i, \bar{z}_i, \Delta_i) \right\rangle_{\omega,n} . \tag{138}$$

We now proceed to construct the generating functional. The first step is to introduce the *minitwistor gravitational background potential* as a representative of a cohomology class:

$$\boldsymbol{\omega}_i \in \mathcal{D}_\Lambda[\mathscr{H}_c] \otimes \Omega^{0,1}\big(\mathbf{MT}^{2|8}, \mathcal{O}(-2,-\Delta)\big). \tag{139}$$

Recall that the minitwistor superline $\mathcal{L}(X,\theta) \subset \mathbf{MT}^{2|8}$, associated with the point $\mathbb{X}^{\hat{I}} = \big(X_{A\dot{A}}, \theta^\alpha_A\big)$ in projective superspace $\mathbf{RP}^{3|16}$, is define by the locus:

$$\mathcal{L}(X,\theta) \coloneqq \left\{ \big(\lambda^A, \mu_{\dot{A}}, \xi^\alpha\big) \in \mathbf{MT}^{2|4} \,\middle|\, \mu_{\dot{A}} = \lambda^A \frac{X_{A\dot{A}}}{|X|},\, \xi^\alpha = \lambda^A \theta^\alpha_A \right\}. \tag{140}$$

In addition, let $\overline{\partial}\,\big|_{\mathcal{L}(X,\theta)}$ denote the restriction of the Dolbeault operator to the supercurve $\mathcal{L}(X,\theta)$. We then define the *generating functional* as:

$$\digamma[\boldsymbol{\omega}_i] \coloneqq \int\limits_{\mathbf{RP}^{3|16}} D^{3|16}\mathbb{X} \;\; \log\det\big(\overline{\partial} + \boldsymbol{\omega}_i\big)\big|_{\mathcal{L}(X,\theta)}. \tag{141}$$

We claim that $\digamma[\boldsymbol{\omega}_i]$ generates $\boldsymbol{M}_n(z_i,\bar{z}_i,\Delta_i)$ upon evaluation at the gravitational background potential:

$$\boldsymbol{\omega}_i = \int\limits_{\mathcal{P}\times\mathbf{C}^2} \boldsymbol{\beta}_i \;\; (\mathcal{U}_i + \mathsf{c}_i\mathcal{V}_i)\; \boldsymbol{\psi}_{\Delta_i}(z_i\cdot\theta^\alpha)\; \mathcal{F}_{\Delta_i}(\lambda_i,\mu_i;z_i,\bar{z}_i)\,. \tag{142}$$

Here, $\mathcal{P} \coloneqq 1 + i\mathbf{R}$ denotes the continuous principal series to which the celestial conformal weight Δ_i belongs, and $\mathcal{F}_{\Delta_i}(\lambda_i,\mu_i;z_i,\bar{z}_i)$ is the minitwistor representative defined in Eq. (27). The volume form on $\mathcal{P}\times\mathbf{C}^2$ is given by:

$$\boldsymbol{\beta}_i \coloneqq d\Delta_i \wedge dz_i \wedge d\bar{z}_i. \tag{143}$$

To establish this claim, we expand the Quillen determinant as:

$$\digamma[\boldsymbol{\omega}_i] = \sum_{m\geq 2} \frac{(-1)^{m+1}}{m} \int\limits_{\mathbf{RP}^{3|16}} D^{3|16}\mathbb{X} \int\limits_{(\mathcal{L}(X,\theta))^{\times m}} \bigwedge_{i=1}^{m} D\lambda_i \;\; \boldsymbol{\omega}_i\big|_{\mathcal{L}(X,\theta)} \;\frac{1}{\lambda_i\cdot\lambda_{i+1}}. \tag{144}$$

Substituting the expression for $\boldsymbol{\omega}_i$ from Eq. (142) into the expansion yields:

$$\digamma[\boldsymbol{\omega}_i] = \sum_{m\geq 2} \frac{(-1)^{m+1}}{m} \int\limits_{\mathbf{RP}^{3|16}} D^{3|16}\mathbb{X} \int\limits_{(\mathcal{P}\times\mathbf{C}^2)^{\times m}} \bigwedge_{i=1}^{m} \boldsymbol{\beta}_i \tag{145}$$

$$(\mathcal{U}_i + \mathsf{c}_i\mathcal{V}_i)\;\; \boldsymbol{\psi}_{2h_i}(z_i\cdot\theta^\alpha)\; K_{2h_i}(X;z_i,\bar{z}_i)\;\; \frac{1}{z_i\cdot z_{i+1}}. \tag{146}$$

Finally, differentiating $\digamma[\boldsymbol{\omega}_i]$ with respect to the classical expectation values $\mathsf{h}^{\ell_i}_{2h_i}(z_i,\bar{z}_i)$ associated with the annihilation operator for a graviton with helicity ℓ_i and celestial scaling dimension $2h_i$, we obtain:

$$\frac{\delta}{\delta \mathsf{h}^{\ell_1}_{2h_1}(z_1,\bar{z}_1)} \cdots \frac{\delta}{\delta \mathsf{h}^{\ell_n}_{2h_n}(z_n,\bar{z}_n)} \digamma[\boldsymbol{\omega}_i] \Bigg|_{\psi=0} = \boldsymbol{M}_n(z_i,\bar{z}_i,\Delta_i)\,, \tag{147}$$

thus completing the proof.

V. MINITWISTOR CELESTIAL CFT

Notation. The integral operator $\int_{\mathbf{CP}^1}$ is understood to act on every term appearing on the right-hand side of an expression, irrespective of whether such terms are enclosed within brackets. For example, if a and b are $(0,1)$-forms on $\mathbf{CP}^1$, then $\int_{\mathbf{CP}^1} a+b := \int_{\mathbf{CP}^1}(a+b)\,.$

Local coordinates on minitwistor space $\mathbf{MT}$ are denoted by $\mathsf{Z}^I := \left(\lambda^A, \mu_{\dot{A}}\right)$, and to avoid unnecessary repetition, spinor indices in the arguments of functions (or functionals) will be shown only at the point of their initial definition.

A. Physical Motivation

Twistor string theory[28] may be formulated as a theory of maps $\left(Z^I,\widetilde{Z}^I\right) : \mathbf{CP}^1 \longrightarrow \mathbf{PT}^s \times \mathbf{PT}^s$, where Z^I and $\widetilde{Z}^I$ denote holomorphic and anti-holomorphic fields, respectively, encoding the embedding of the Riemann sphere into supersymmetric projective twistor space $\mathbf{PT}^s$. The action functional is obtained by gauging the action of a holomorphic $\beta\gamma$-system, where the gauging procedure implements the local rescaling symmetry of the *projective* geometry of twistor space.

Explicitly, the action may be expressed as:

$$\int_{\mathbf{CP}^1} d\sigma \;\; Y_I \overline{\nabla} Z^I + \widetilde{Y}_I \nabla \widetilde{Z}^I + ..., \tag{148}$$

where Y_I and $\widetilde{Y}_I$ are the canonical conjugate fields to Z^I and $\widetilde{Z}^I$, respectively. The ellipsis (...) represents contributions from an auxiliary conformal matter system, which gives rises to worldsheet WZNW currents necessary for anomaly cancellation. The covariant derivatives $\nabla = \partial + \boldsymbol{a}$ and $\overline{\nabla} = \overline{\partial} + \overline{\boldsymbol{a}}$ are Dolbeault operators twisted by gauge fields $\boldsymbol{a}$ and $\overline{\boldsymbol{a}}$ corresponding to the gauging of the rescaling symmetry of projective twistor space, under which the fields transform as:

$$Z^I \mapsto \alpha Z^I, \;\; Y_I \mapsto \alpha^{-1} Y_I, \;\; \overline{\boldsymbol{a}} \mapsto \overline{\boldsymbol{a}} - \overline{\partial} \log \alpha, \tag{149}$$

[28] See Witten [20], Adamo, Skinner, and Williams [21], Abou-Zeid, Hull, and Mason [76], Geyer, Lipstein, and Mason [80, 81], Geyer and Mason [82], Roiban, Spradlin, and Volovich [83], Wolf [84], Berkovits [85].

where $\alpha : \mathscr{U} \subseteq \mathbf{CP}^1 \longrightarrow \mathbf{C}^*$ is a nowhere vanishing holomorphic function, and $\mathscr{U}$ the domain of a local trivialisation. Analogous transformations hold in the anti-holomorphic sector for $\widetilde{Z}^I$, $\widetilde{Y}_I$, and the gauge field $\boldsymbol{a}$.

We propose to follow a similar construction in our theory, with the goal of deriving equations of motion that describe the embedding of the celestial sphere into minitwistor space $\mathbf{MT}$ as a minitwistor line. Unlike the anomaly-free twistor string theory described above, our theory will be defined only at the semiclassical level, due to the presence of quantum anomalies that preclude a full quantum formulation. In the semiclassical limit, we will demonstrate that the on-shell effective action reproduces the celestial leaf amplitudes associated with tree-level scattering processes for gluons in $\mathcal{N} = 4$ SYM theory and for gravitons in $\mathcal{N} = 8$, in configurations characterised by maximal helicity violation.

B. Action Functional

First Step

Recall that the celestial sphere is modelled as the complex projective line. In addition, the minitwistor space $\mathbf{MT}$ admits a holomorphic embedding of the Riemann sphere into minitwistor lines defined by the incidence relation. This embedding is formalised through the following sequence of constructions.

Let $F_{\dot{A}}\left(X_{A\dot{A}}, \lambda^A\right)$ be a holomorphic map[29], possessing homogeneity of degree one with respect to the spinor λ^A and invariant under re-scalings of the "spacetime" coordinate $X_{A\dot{A}} \in H_3^+$, such that:

$$F_{\dot{A}}\left(X_{A\dot{A}}, \lambda^A\right) \coloneqq \lambda^A \frac{X_{A\dot{A}}}{|X|}. \tag{150}$$

The minitwistor line $\mathcal{L}(X)$ in $\mathbf{MT}$ associated with $X_{A\dot{A}}$ is defined as the locus of points satisfying the incidence relation:

$$\mathcal{L}(X) \coloneqq \left\{ \left(\lambda^A, \mu_{\dot{A}}\right) \in \mathbf{MT} \;\middle|\; \mu_{\dot{A}} = F_{\dot{A}}(X, \lambda) \right\}. \tag{151}$$

Following the sheaf-theoretic conventions employed by Forster [22], the restriction homomorphism $\rho_X \coloneqq \rho_{\mathcal{L}(X)}$ associated with the curve $\mathcal{L}(X)$ acts on sections of the holomorphic vector bundle $\mathcal{O}(p,q) \longrightarrow \mathbf{MT}$ by mapping a representative g of a cohomology class as follows:

$$\rho_X(g)\left(\lambda^A\right) \coloneqq g\left(\lambda^A, F_{\dot{A}}(X, \lambda)\right). \tag{152}$$

[29] Rigorously, $F_{\dot{A}}$ is a section of the projective spinor bundle.

Next, consider the canonical surjection $\pi_X : \mathcal{L}(X) \longrightarrow \mathbf{CP}^1$, which projects the curve $\mathcal{L}(X)$ onto the Riemann sphere. The natural embedding of the celestial sphere into the quadric $\mathbf{MT}$ (in the form of a minitwistor line) can be understood as a section of this fibration:

$$\sigma_X : \mathbf{CP}^1 \longrightarrow \mathcal{L}(X), \;\; \sigma_X \left(\lambda^A\right) := \left(\lambda^A, F_{\dot{A}}(X, \lambda)\right), \tag{153}$$

where, by construction, the composition satisfies $\pi_X \circ \sigma_X = id_{\mathbf{CP}^1}$.

On the other hand, the holomorphic function $F_{\dot{A}}(X, \lambda)$, which *defines* the embedding of the Riemann sphere into the minitwistor space through the incidence relation, is *entirely* determined by its homogeneity properties and holomorphicity. Explicitly, it satisfies:

$$F_{\dot{A}}(X, \cdot) \in \Omega^{0,1}(X, \mathcal{O}(1)), \;\; \overline{\partial}\big|_X F_{\dot{A}}(X, \lambda) = 0. \tag{154}$$

Therefore, by analogy with twistor string theory, one might be tempted to introduce an action integral over the Riemann sphere $\mathbf{CP}^1$, in terms of the embedding function $F_{\dot{A}}$, as follows:

$$\int\limits_{\mathbf{CP}^1} D\lambda \;\, F_{\dot{A}} \overline{\partial}\big|_X F^{\dot{A}}. \tag{155}$$

The corresponding equations of motion would yield the constraint $\overline{\partial}\big|_X F_{\dot{A}}(X, \lambda) = 0$, which, as previously noted, completely determines the embedding map $\sigma_X : \mathbf{CP}^1 \longrightarrow \mathcal{L}(X)$.

However, the action integral written above is mathematically ill-defined due to a lack of projective invariance. Consider the integration measure $D\lambda := \varepsilon_{AB}\lambda^A d\lambda^B$, which transforms under a rescaling $\lambda^A \mapsto \alpha\lambda^A$ as $D\lambda \mapsto \alpha^2 D\lambda$. Simultaneously, the "kinetic term" transforms as:

$$F_{\dot{A}} \overline{\partial}\big|_X F^{\dot{A}} \mapsto \alpha^2 F_{\dot{A}} \overline{\partial}\big|_X F^{\dot{A}}. \tag{156}$$

Thus, the integrand $D\lambda \;\, F_{\dot{A}} \overline{\partial}\big|_X F^{\dot{A}}$ fails to exhibit the required projective invariance under the transformation $\lambda^A \mapsto \alpha\lambda^A$, precluding its interpretation as a meaningful action integral.

Second Step

Our objective is to construct an action functional whose associated Euler-Lagrange equations precisely reproduce the embedding of the celestial sphere as a minitwistor line in $\mathbf{MT}$. A formulation of such an action can be introduced as follows[30].

[30] A similar solution was formulated by Adamo, Mason, and Sharma [86], Sharma [87] and Chiou *et al.* [88]. For a mathematically rigorous discussion, see Sämann [60, 89], Dunajski [90] and Adamo, Skinner, and Williams [91].

Let (κ_1^A, κ_2^A) denote a normalised spinor basis satisfying the condition $\varepsilon_{AB}\kappa_1^A\kappa_2^B = 1$. The conic curve $\mathcal{L}(X)$ can be parametrised by trivialising the fibration $\pi_X : \mathcal{L}(X) \longrightarrow \mathbf{CP}^1$ through the introduction of local coordinates ω^A, defined by:

$$\omega^A := \omega_1\kappa_1^A + \omega_2\,\kappa_2^A. \tag{157}$$

These coordinates ω^A are postulated to be projectively related to the minitwistor coordinates λ^A through the relation:

$$\lambda^A \equiv \frac{\omega^A}{\omega_1\omega_2} = \frac{\kappa_1^A}{\omega_2} + \frac{\kappa_2^A}{\omega_1}. \tag{158}$$

Within this trivialisation, the holomorphic function $F_{\dot{A}}(X, \lambda)$, which determines the embedding map $\sigma_X : \mathbf{CP}^1 \longrightarrow \mathcal{L}(X)$, yields a representative function:

$$M_{\dot{A}}\left(X_{A\dot{A}}, \omega^A\right) := F_{\dot{A}}\left(X_{A\dot{A}}, \lambda^A(\omega)\right) = \frac{\kappa_1^A}{\omega_2}\frac{X_{A\dot{A}}}{|X|} + \frac{\kappa_2^A}{\omega_1}\frac{X_{A\dot{A}}}{|X|}. \tag{159}$$

Two observations are now in order. First, the projective relation between λ^A and ω^A guarantees that the expression for $M_{\dot{A}}$ can be inverted to recover $F_{\dot{A}}$. Hence, specifying $M_{\dot{A}}$ fully determines the embedding $\sigma_X : \mathbf{CP}^1 \longrightarrow \mathcal{L}(X)$. Second, the function $M_{\dot{A}}$ is uniquely characterised as *the* solution to the following first-order partial differential equation[31]:

$$\overline{\partial}\big|_X M_{\dot{A}}(X, \omega) = 2\pi i\,\overline{\delta}(\omega_2)\; w_{1\dot{A}} + 2\pi i\,\overline{\delta}(\omega_1)\; w_{2\dot{A}}, \tag{160}$$

where the "boundary conditions" are specified by the residues at ω_2 and ω_1, respectively, as:

$$w_{1\dot{A}} = \kappa_1^A\frac{X_{A\dot{A}}}{|X|},\;\; w_{2\dot{A}} = \kappa_2^A\frac{X_{A\dot{A}}}{|X|}. \tag{161}$$

The above discussion implies that, in seeking an action functional for a *minitwistor celestial CFT*, the function $M_{\dot{A}}$ may be regarded as a dynamical variable whose Euler-Lagrange equations yield precisely the differential equation (160), while the boundary conditions are enforced through external source terms.

Further, the function $M_{\dot{A}}$, as opposed to its counterpart $F_{\dot{A}}$, possesses an additional advantage due to its homogeneity properties in the trivialisation ω^A. Specifically, $M_{\dot{A}}$ is homogeneous of degree -1 in ω^A, which renders the kinetic term:

$$M_{\dot{A}}\,\overline{\partial}\big|_X M^{\dot{A}}, \tag{162}$$

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[31] Recall that $\overline{\delta}(z) := (2\pi i)^{-1}\,\overline{\partial}\,z^{-1}$ for all $z \in \mathbf{C}^*$.

suitably weighted for integration against the measure $D\omega := \varepsilon_{AB}\omega^A d\omega^B$. Indeed, under a rescaling $\omega^A \mapsto \alpha\omega^A$, the measure transforms as $D\omega \mapsto \alpha^2 D\omega$, while the kinetic term transforms as:

$$M_{\dot{A}}\,\overline{\partial}\,\big|_X\,M^{\dot{A}} \;\mapsto\; \alpha^{-2}M_{\dot{A}}\overline{\partial}\,\big|_X\,M^{\dot{A}}, \tag{163}$$

ensuring that the integrand:

$$D\omega\;\; M_{\dot{A}}\overline{\partial}\,\big|_X\,M^{\dot{A}}, \tag{164}$$

is weightless under such re-scalings.

Consequently, we are naturally lead to consider the action functional:

$$\mathcal{S}_0\left[M_{\dot{A}}(\omega^A)\right] \;=\; \frac{1}{b}\int\limits_{\mathcal{L}(X)} D\omega\;\; M_{\dot{A}}\,\overline{\partial}\,\big|_X\,M^{\dot{A}} \;+4\pi i\overline{\delta}\,(\omega_2)\,[w_1 M] + 4\pi i\overline{\delta}\,(\omega_1)\,[w_2 M]\,, \tag{165}$$

where we have employed the spinor-helicity bracket notation to express the contraction of $M^{\dot{A}}$ with the external sources as $[w_i M] := w_{i\dot{A}}M^{\dot{A}}$ for $i = 1, 2$. Although the bracket notation is employed for the source terms, we refrain from using it in the kinetic term to avoid notational clutter.

Finally, a more suggestive rewriting of the action is possible by interpreting the sources $w_{i\dot{A}}$ $(i = 1, 2)$ as originating from an external current[32] by defining:

$$J_{\dot{A}} \;\;:=\;\; 4\pi i\,\overline{\delta}\,(\omega_2)\;\kappa_1^A\,\frac{X_{A\dot{A}}}{|X|} \;+\; 4\pi i\,\overline{\delta}\,(\omega_1)\;\kappa_2^A\,\frac{X_{A\dot{A}}}{|X|}. \tag{166}$$

The action functional then takes the form:

$$\mathcal{S}_0 = \frac{1}{b}\int\limits_{\mathcal{L}(X)} D\omega\;\; M_{\dot{A}}\,\overline{\partial}\,\big|_X\,M^{\dot{A}} \;+\; [JM]\,, \tag{167}$$

where $[JM]$ denotes the contraction of $J_{\dot{A}}$ with $M^{\dot{A}}$.

C. Supersymmetry

Our extension of the RSVW formalism to celestial leaf amplitudes in $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ Supergravity shows that the corresponding superamplitudes can be written as a Fourier transform on minitwistor superspace. This implies that any attempt to reproduce these amplitudes using a sigma model on the celestial sphere must include a supersymmetric version of **MT** as the target space.

[32] We follow the terminology of Schwinger [92].

In the subsequent subsections, we shall present an informal discussion of the minitwistor superspace, referring the reader to Rogers [61] for a rigorous mathematical treatment[33]. Thereafter, we proceed to extend our construction of the sigma model action functional by formulating a generalisation in which the domain is the celestial *supersphere* and the corresponding target space is taken to be the minitwistor *superspace.*

1. *Projective Superspace*

We begin by observing that the minitwistor space $\mathbf{MT}$ is the space of oriented geodesics on the hyperboloid H_3^+. A model for the hyperbolic geometry of H_3^+ can be derived from the projective geometry of the three-dimensional real projective space $\mathbf{RP}^3$, as reviewed in the Appendix. Accordingly, our discussion of the minitwistor *superspace* $\mathbf{MT}^{2|\mathcal{N}}$ starts with a preliminary definition of supermanifold $\mathbf{RP}^{3|2\mathcal{N}}$.

We identify the projective superspace $\mathbf{RP}^{3|2\mathcal{N}}$ with the trivial superbundle $\mathbf{RP}^3 \times \mathbf{CP}^{0|2\mathcal{N}}$, in which the three-dimensional real projective space $\mathbf{RP}^3$ serves as the base manifold and the fibre is parametrised by "fermionic degrees of freedom." To be precise, the typical fibre is charted by the introduction of Grassmann-valued *homogeneous* spinorial coordinates θ_A^α, subject to an equivalence relation under projective rescaling by a nonzero complex scalar α, such that $\theta_A^\alpha \sim \alpha\theta_A^\alpha$. The normalisation of the fermionic "dimensions" is imposed through a Berezin integral over the fibre, given by:

$$\int d^{0|2}\theta \;\; \theta_A^\alpha \theta_B^\alpha = \varepsilon_{AB}. \tag{168}$$

The natural orientation of the total space of the superbundle $\mathbf{RP}^{3|2\mathcal{N}}$ is induced by the Berezin-de Witt volume superform, defined as:

$$D^{3|2\mathcal{N}}\mathbb{X} := \frac{D^3X}{|X|^4} \wedge d^{0|2\mathcal{N}}\theta. \tag{169}$$

2. *Minitwistor Superspace*

The minitwistor superspace $\mathbf{MT}^{2|\mathcal{N}}$ (associated with the hyperbolic geometry modelled on $\mathbf{RP}^{3|2\mathcal{N}}$) is constructed by extending the bosonic minitwistor space $\mathbf{MT}$ through the inclusion of Grassmann-odd coordinates that encode the "fermionic dimensions." More precisely, we define

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[33] See also DeWitt [45], Manin [46], Leites [79].

$\mathbf{MT}^{2|\mathcal{N}}$ as the trivial superbundle:

$$\mathbf{MT}^{2|\mathcal{N}} \coloneqq \mathbf{MT} \times \mathbf{CP}^{0|\mathcal{N}}, \tag{170}$$

where the base manifold is the bosonic minitwistor space $\mathbf{MT}$, and the typical fibre is charted by the Grassmann-odd *homogeneous* scalar coordinates ζ^{α}, with $\alpha \in \{1, ..., \mathcal{N}\}$.

A bundle superchart on $\mathbf{MT}^{2|\mathcal{N}}$ is given by a local trivialisation $(\mathscr{U}, \mathsf{Z}^I)$, where $\mathscr{U} \subseteq \mathbf{MT}^{2|\mathcal{N}}$ denotes an open neighbourhood and $\mathsf{Z}^I : \mathscr{U} \longrightarrow U \times \mathbf{CP}^{0|\mathcal{N}}$ is a local coordinate map. The image of Z^I is contained in $U \subset \mathbf{CP}^{1|0} \times \mathbf{CP}^{1|0}$, and the coordinate functions Z^I decompose as:

$$p \in \mathscr{U} \mapsto \mathsf{Z}^I(p) \coloneqq (\lambda^A(p), \mu_{\dot{A}}(p), \zeta^{\alpha}(p)). \tag{171}$$

The canonical orientation of the minitwistor superspace $\mathbf{MT}^{2|\mathcal{N}}$ is specified by the Berezin-de Witt volume superform on the total space of the superbundle, and is defined by:

$$D^{2|\mathcal{N}}\mathsf{Z} \coloneqq D\lambda \wedge D\mu \wedge d^{0|\mathcal{N}}\zeta. \tag{172}$$

3. Minitwistor Superlines

In minitwistor superspace, the minitwistor lines are generalised to *superlines* $\mathcal{L}(X, \theta)$, each associated with a "spacetime" point $(X_{A\dot{A}}, \theta^{\alpha}_{\dot{A}}) \in \mathbf{RP}^{3|2\mathcal{N}}$. The incidence relation defining these supercurves is determined by sections of the *projective spinor superbundle*, $\mathbf{PS}_{3|2\mathcal{N}} \coloneqq \mathbf{RP}^{3|2\mathcal{N}} \times \mathbf{CP}^{1|0}$, which are given by:

$$F_{\dot{A}}(X_{A\dot{A}}, \theta^{\alpha}_{\dot{A}}; \lambda^A) \coloneqq \lambda^A \frac{X_{A\dot{A}}}{|X|}, \;\; G^{\alpha}(X_{A\dot{A}}, \theta^{\alpha}_{\dot{A}}; \lambda^A) \coloneqq \lambda^A \theta^{\alpha}_{\dot{A}}. \tag{173}$$

The supercurve $\mathcal{L}(X, \theta)$, which we shall refer to as the *minitwistor superline* based at (X, θ), is then defined as the locus of points in minitwistor superspace satisfying the incidence relation:

$$\mathcal{L}(X, \theta) \coloneqq \left\{ \mathsf{Z}^I \in \mathbf{MT}^{2|\mathcal{N}} \,\middle|\, \mu_{\dot{A}} = F_{\dot{A}}(X, \theta; \lambda), \, \zeta^{\alpha} = G^{\alpha}(X, \theta; \lambda) \right\}, \tag{174}$$

where $\mathsf{Z}^I = (\lambda^A, \mu_{\dot{A}}, \zeta^{\alpha})$ denotes local coordinates on the minitwistor superspace.

To formalise the restriction of holomorphic sections to a curve $\mathcal{L}(X, \theta)$, let us consider a representative g of a cohomology class associated to the vector bundle $\mathcal{O}(p, q) \longrightarrow \mathbf{MT}$. The restriction homomorphism $\rho_{\mathcal{L}(X,\theta)}$ maps this section to a section of the fibration $\mathcal{L}(X, \theta) \longrightarrow \mathbf{CP}^1$ by evaluating it along the incidence relations defining $\mathcal{L}(X, \theta)$:

$$g\left|_{\mathcal{L}(X,\theta)}\right.(\lambda^A) \coloneqq \rho_{\mathcal{L}(X,\theta)}(g)(\lambda^A) \coloneqq g\left(\lambda^A, F_{\dot{A}}(X, \theta; \lambda), G^{\alpha}(X, \theta; \lambda)\right). \tag{175}$$

Embedding as a Section. We now consider the canonical projection $\pi_{(X,\theta)} : \mathcal{L}(X,\theta) \longrightarrow \mathbf{CP}^1$ which maps each point on the supercurve to its projective coordinate $[\lambda^A] \in \mathbf{CP}^1$. A holomorphic embedding of the celestial sphere onto the minitwistor superline $\mathcal{L}(X,\theta)$ can thus be identified with a section of this fibration. We denote this section by:

$$\sigma_{(X,\theta)} : \mathbf{CP}^{1|0} \longrightarrow \mathcal{L}(X,\theta)\,, \;\; \sigma_{(X,\theta)}\left(\lambda^A\right) := \left(\lambda^A, F_{\dot{A}}(X,\theta;\lambda)\,, G^{\alpha}(X,\theta;\lambda)\right). \tag{176}$$

Hence, $\pi_{(X,\theta)} \circ \sigma_{(X,\theta)} = id_{\mathbf{CP}^1}$ by construction.

4. *Construction of the Action*

First Step: Dynamical Variables. To formulate a theory whose solutions to the equations of motion correspond to embeddings of the celestial sphere into minitwistor superspace $\mathbf{MT}^{2|\mathcal{N}}$, it suffices to specify the holomorphic sections $F_{\dot{A}}(X,\theta;\lambda)$ and $G^{\alpha}(X,\theta;\lambda)$ introduced previously as dynamical variables. However, as observed, both $F_{\dot{A}}(X,\theta;\lambda)$ and $G^{\alpha}(X,\theta;\lambda)$ exhibit homogeneity of degree one in the spinor variable λ^A. Consequently, any attempt to construct a weightless kinetic term using these functions directly would be inconsistent with the homogeneity degree of the measure $D\lambda = \varepsilon_{AB}\lambda^A d\lambda^B$.

To resolve this issue, we employ the same procedure used in the bosonic case: we introduce a normalised spinor basis (κ_1^A, κ_2^A) satisfying the normalisation condition $\varepsilon_{AB}\kappa_1^A\kappa_2^B = 1$, and we define a trivialisation of the fibration $\pi_{(X,\theta)} : \mathcal{L}(X,\theta) \longrightarrow \mathbf{CP}^1$ via a new spinor ω^A expressed as a linear combination of the basis spinors, $\omega^A := \omega_1\kappa_1^A + \omega_2\kappa_2^A$. We also postulate that the spinor ω^A is projectively related to λ^A by Eq. (158).

The next step is to reformulate the section $F_{\dot{A}}(X,\theta;\lambda)$ in terms of this new spinor basis. Specifically, we define the function:

$$M_{\dot{A}}\left(X_{A\dot{A}}, \theta^{\alpha}_A; \omega^A\right) \;:=\; F_{\dot{A}}\left(X_{A\dot{A}}, \theta^{\alpha}_A; \omega^A(\lambda)\right), \tag{177}$$

which contains the same information as $F_{\dot{A}}(X,\theta;\lambda)$ in terms of the new spinor ω^A. The properties of this function have already been analysed in our prior discussion on the bosonic model.

We now focus our attention on $G^{\alpha}(X,\theta;\lambda)$, which we reformulate analogously as a new function:

$$N^{\alpha}\left(X_{A\dot{A}}, \theta^{\alpha}_A; \omega^A\right) \;:=\; G^{\alpha}\left(X_{A\dot{A}}, \theta^{\alpha}_A; \omega^A(\lambda)\right), \tag{178}$$

which can be rewritten as:

$$N^{\alpha}(X,\theta;\omega) \;=\; \frac{\widetilde{w}_1^{\alpha}}{\omega_2} + \frac{\widetilde{w}_2^{\alpha}}{\omega_1}; \;\text{ where }\; \widetilde{w}_i^{\alpha} \;:=\; \kappa_i^A\theta_A^{\alpha}\;(i=1,2)\,. \tag{179}$$

Since λ^A and ω^A are projectively related, the above expression for $N^\alpha(X,\theta;\omega)$ can be inverted to recover $G^\alpha(X,\theta;\lambda)$ if the function $N^\alpha(X,\theta;\omega)$ is given. Therefore, specifying the embedding $\sigma_{(X,\theta)} : \mathbf{CP}^1 \longrightarrow \mathcal{L}(X,\theta)$ into a minitwistor superline is equivalent to specifying the pair of functions $M_{\dot{A}}(X,\theta;\omega)$ and $N^\alpha(X,\theta;\omega)$.

We thus arrive at the following conclusion: the set of functions $\{M_{\dot{A}}, N^\alpha\}$ can be taken as the dynamical fields in the supersymmetric generalisation of the minitwistor celestial CFT.

An additional remark is in order about the function $N^\alpha(X,\theta;\omega)$. It can be seem that this function is uniquely determined as *the* solution to the following partial differential equation:

$$\overline{\partial}\big|_{\mathcal{L}(X,\theta)} N^\alpha(X,\theta;\lambda) \,=\, 2\pi i\,\overline{\delta}(\omega_2)\,\widetilde{w}_1^\alpha \,+\, 2\pi i\,\overline{\delta}(\omega_1)\,\widetilde{w}_2^\alpha, \tag{180}$$

where the boundary conditions are specified by the residues $\widetilde{w}_1^\alpha$ and $\widetilde{w}_2$ at the poles ω_2 and ω_1, respectively.

Second Step: Action Functional We now arrive at the construction of an action that gives rise to a well-posed variational principle, whose associated Euler-Lagrange equations describe the embedding of the celestial sphere into minitwistor superspace as minitwistor supercurves. The proposed action is given by:

$$\mathcal{S}[M_{\dot{A}}, N^\alpha, e_\alpha] \,=\, \int\limits_{\mathcal{L}(X,\theta)} D\omega\;\, M_{\dot{A}}\,\overline{\partial}\big|_{\mathcal{L}(X,\theta)} M^{\dot{A}} \,+\, e_\alpha\,\overline{\partial}\big|_{\mathcal{L}(X,\theta)} N^\alpha \,+\, J_{\dot{A}} M^{\dot{A}} - e_\alpha K^\alpha. \tag{181}$$

The external bosonic current $J_{\dot{A}}$ appearing in the action is identical to the one introduced in Eq. (166), while the newly introduced fermionic current K^α is defined as:

$$K^\alpha \,\coloneqq\, 2\pi i\,\overline{\delta}(\omega_2)\,\kappa_1^A\theta_A^\alpha \,+\, 2\pi i\,\overline{\delta}(\omega_1)\,\kappa_2^A\theta_A^\alpha. \tag{182}$$

Note that e_α is a Lagrange multiplier, and that the equations of motion are given by Eqs. (160) and (180).

Celestial Supersphere. The final level of abstraction is achieved through the extension of the celestial sphere to its $\mathcal{N}=2$ supersymmetric generalisation, the *celestial supersphere.* This extension is obtained by adjoining to the homogeneous coordinates $[\omega^A] \in \mathbf{CP}^1$ two Grassmann-odd variables, η and $\bar{\eta}$, which satisfy the Berezin normalisation condition $\int d^{0|2}\eta\;\,\eta\bar{\eta} = 1$. The celestial supersphere is equipped with a natural orientation given by the Berezin-de Witt superform:

$$d^{1|2}z \,\coloneqq\, D\omega \,\wedge\, d^{0|2}\eta. \tag{183}$$

Hereafter, we denote by $\widetilde{\mathcal{L}}(X,\theta)$ the minitwistor line obtained from the embedding $\mathbf{CP}^{1|2} \longrightarrow \widetilde{\mathcal{L}}(X,\theta)$.

To describe the local geometry of the celestial supersphere, we introduce a *super-vielbein* $E_{\dot{A}}{}^{\alpha}$ that satisfies the orthonormality condition $E_{\dot{A}}{}^{\alpha} E^{\dot{A}}{}_{\beta} = \delta^{\alpha}{}_{\beta}$. Now, note the following *algebraic* identities:

$$M_{\dot{A}}\, \overline{\partial}\,\big|_{\mathcal{L}(X,\theta)}\, M^{\dot{A}} + \, e_{\alpha}\, \overline{\partial}\,\big|_{\mathcal{L}(X,\theta)}\, N^{\alpha} \tag{184}$$

$$= \int d^{0|2}\eta \,\; \big(\, \eta M_{\dot{A}} + \bar{\eta} E_{\dot{A}}{}^{\alpha} e_{\alpha} \,\big)\, \overline{\partial}\,\big|_{\mathcal{L}(X,\theta)} \big(\, \bar{\eta} M^{\dot{A}} - \eta E^{\dot{A}}{}_{\beta} N^{\beta} \,\big), \tag{185}$$

and:

$$J_{\dot{A}} M^{\dot{A}} - e_{\alpha} K^{\alpha} \tag{186}$$

$$= \int d^{0|2}\eta \,\; \big(\, \eta J_{\dot{A}} + \bar{\eta} E_{\dot{A}}{}^{\alpha} e_{\alpha} \,\big)\, \overline{\partial}\,\big|_{\mathcal{L}(X,\theta)} \big(\, \bar{\eta} M^{\dot{A}} + \eta E^{\dot{A}}{}_{\beta} K^{\beta} \,\big). \tag{187}$$

To facilitate a more compact notation, we define the *superfields:*

$$P_{\dot{A}}\,(\omega,\eta,\bar{\eta}) \,\coloneqq\, \eta M_{\dot{A}} + \bar{\eta} E_{\dot{A}}{}^{\alpha} e_{\alpha}, \;\; Q^{\dot{A}}\,(\omega,\eta,\bar{\eta}) \,\coloneqq\, \bar{\eta} M^{\dot{A}} - \eta E^{\dot{A}}{}_{\beta} N^{\beta}, \tag{188}$$

and introduce the *super-currents*:

$$j_{\dot{A}}\,(\omega,\eta,\bar{\eta}) \,\coloneqq\, \eta J_{\dot{A}} + \bar{\eta} E_{\dot{A}}{}^{\alpha} e_{\alpha}, \;\; k^{\dot{A}}\,(\omega,\eta,\bar{\eta}) \,\coloneqq\, \bar{\eta} M^{\dot{A}} + \eta E^{\dot{A}}{}_{\beta} K^{\beta}. \tag{189}$$

Finally, the action for the supersymmetric minitwistor celestial CFT can be expressed in terms of these superfields and super-currents as:

$$\mathcal{S}[P,Q,j,k] \,=\, \frac{1}{b} \int\limits_{\widetilde{\mathcal{L}}(X,\theta)} d^{1|2}z \;\, P_{\dot{A}}\, \overline{\partial}\,\big|_{\mathcal{L}(X,\theta)}\, Q^{\dot{A}} \,+\, [jk], \tag{190}$$

where the bracket $[jk]$ denotes the spinor contraction of the super-currents $j_{\dot{A}}$ and $k^{\dot{A}}$.

D. Phenomenology

In conventional superstring theory, phenomenological considerations arise from the study of Calabi-Yau compactifications or D-brane configurations. In twistor and ambitwistor string theories, such considerations are incorporated within an auxiliary conformal matter system, which we briefly discussed in Subsection V A. The minitwistor celestial CFTs constructed in this work follow an analogous procedure: the relevant physical features are encoded in an auxiliary matter model, which we shall now proceed to define.

1. Fermionic System

We begin with the following physical motivations. First, the generating functional $\mathcal{F}[\boldsymbol{\omega}]$, derived in Subsection III C for $\mathcal{N} = 4$ SYM theory and in Subsection IV D for $\mathcal{N} = 8$ Supergravity, is expressed as an integral over projective superspace $\mathbf{RP}^{3|2\mathcal{N}}$ of the Quillen determinant.

On the other hand, let $\mathbf{G}$ be a gauge Lie group with Lie algebra $\mathfrak{g} \simeq (T_e(\mathbf{G}), [\cdot,\cdot])$, Denote by $\{\mathsf{T}^a\}$ a basis of generators of $\mathfrak{g}$ in the fundamental representation π, satisfying the normalisation condition $\mathsf{Tr}\left(\mathsf{T}^a\mathsf{T}^b\right) = \frac{1}{2}\boldsymbol{k}^{ab}$, where $\boldsymbol{k}^{ab}$ denotes the Cartan-Killing form of $\mathbf{G}$. The generators obey the commutation relations $[\mathsf{T}^a, \mathsf{T}^b] = if^{abc}\mathsf{T}^c$, where f^{abc} are the structure constants of $\mathfrak{g}$.

Consider a $\mathfrak{g}$-valued connection one-form $\boldsymbol{\omega} = \boldsymbol{\omega}^a\mathsf{T}^a$ on a principal $\mathbf{G}$-bundle over the Riemann sphere $\mathbf{CP}^1$. In addition, let $(q,\bar{q})$ denote a fermionic system defined on the associated vector bundle, whose typical fibre is isomorphic to the representation space of π. The dynamics of the fermionic system $(q,\bar{q})$ is governed by the action functional:

$$\mathcal{S}_{(q,\bar{q})} := \int\limits_{\mathbf{CP}^1} d\sigma \;\; \bar{q}^{\mathrm{i}}\left(\overline{\partial} + \boldsymbol{\omega}^a\mathsf{T}^a_{\mathrm{ij}}\right)q^{\mathrm{j}}, \tag{191}$$

where $d\sigma$ denotes the "volume" form on $\mathbf{CP}^1$ and $\overline{\partial}$ is the Dolbeault operator.

Now, it is well-known[34] that the corresponding quantum effective action for this system is given by the chiral determinant of the associated Dirac operator (or twisted Dolbeault operator):

$$\mathcal{W}_{(q,\bar{q})} \,\propto\, \mathsf{Tr}\log\left(\overline{\partial} + \boldsymbol{\omega}\right). \tag{192}$$

Keeping these remarks in mind, we proceed by considering the principal $\mathbf{G}$-bundle over a minitwistor supercurve $\mathcal{L}(X,\theta)$, equipped with a $\mathfrak{g}$-valued connection one-form $\boldsymbol{\omega} = \boldsymbol{\omega}^a\mathsf{T}^a$. Introduce the fermionic system $(q,\bar{q})$ defined on the associated vector bundle over $\mathcal{L}(X,\theta)$, and define the superfields:

$$\psi(\omega,\eta,\overline{\eta}) := \overline{\eta}\,q, \;\; \bar{\psi}(\omega,\eta,\overline{\eta}) := \eta\,\bar{q}. \tag{193}$$

We propose that the *interaction term* in the action functional is given by:

$$\mathcal{S}_{int} := \int\limits_{\mathcal{L}(X,\theta)} d^{1|2}z \;\; \bar{\psi}\left(\overline{\partial} + \boldsymbol{\omega}\right)\big|_{\mathcal{L}(X,\theta)}\,\psi. \tag{194}$$

Therefore, the complete action describing the minitwistor celestial CFT is:

$$\mathcal{I} = \frac{1}{b}\int\limits_{\widetilde{\mathcal{L}}(X,\theta)} d^{1|2}z \;\; P_{\dot{A}}\,\overline{\partial}\big|_{\mathcal{L}(X,\theta)}\,Q^{\dot{A}} \;+\; [jk] \;+\; b\,\bar{\psi}\left(\overline{\partial} + \boldsymbol{\omega}\right)\big|_{\mathcal{L}(X,\theta)}\,\psi, \tag{195}$$

where the parameter b controls the semiclassical approximation, and is analogous to the Liouville coupling constant[35].

[34] See Nair [93].

[35] Cf. Ribault and Teschner [27].

2. Semiclassical Analysis

We turn now to the task of demonstrating that the semiclassical effective action arising from the theory described by Eq. (195) yields the generating functional $\mathcal{F}[\boldsymbol{\omega}]$.

Prior to presenting the path-integral formulation of the effective action, we recall the key observation made in the concluding remarks of Section III. The dynamical fields in the present theory correspond to the embedding maps of the celestial sphere into minitwistor superspace $\mathbf{MT}^{2|\mathcal{N}}$, realised as minitwistor superlines $\mathcal{L}\,(X,\theta)$. Formally, these embeddings are represented by sections:

$$\sigma_{(X,\theta)} : \mathbf{CP}^1 \longrightarrow \mathcal{L}\,(X,\theta)\,, \tag{196}$$

of the canonical fibration:

$$\pi_{(X,\theta)} : \mathcal{L}\,(X,\theta) \longrightarrow \mathbf{CP}^1. \tag{197}$$

Thus, an integral such as:

$$\int\limits_{\mathbf{RP}^{3|2\mathcal{N}}} D^{3|2\mathcal{N}}\mathbb{X} \int\limits_{\mathcal{L}(X,\theta)} (\ldots) \quad , \tag{198}$$

must be interpreted as an integral over the moduli space of *embeddings* of the celestial sphere into $\mathbf{MT}^{2|\mathcal{N}}$, rather than an integral over multiple distinct celestial spheres parametrised by coordinates $\mathbb{X}^{\hat{I}} = \left(X_{A\dot{A}}, \theta^{\alpha}_{A}\right)$ in projective superspace $\mathbf{RP}^{3|2\mathcal{N}}$. This is physically important, because semiclassically the path integral is not summing over disconnected celestial spheres but rather over all possible embeddings of a single celestial sphere as a minitwistor superline.

Consequently, the semiclassical effective action $\mathcal{W}$ must account for contributions from all possible embeddings of the celestial sphere into $\mathbf{MT}^{2|\mathcal{N}}$, and is given by:

$$\mathcal{W} \;=\; -\lim_{b\to 0^+} b\;\; \log \int\limits_{\mathbf{RP}^{3|2\mathcal{N}}} D^{3|2\mathcal{N}}\mathbb{X} \int \left[dMdNd\psi d\bar{\psi}\right]\;\; \exp\left(-\mathcal{I}\right). \tag{199}$$

In the semiclassical limit $b \to 0^+$, the dominant contribution to the path integral over the embedding fields $M_{\dot{A}}$ and N^{α} arises from the saddle-point approximation, which enforces the classical equations of motion, given by Eqs. (160) and (180), and are precisely those that define the embedding maps $\mathbf{CP}^1 \longrightarrow \mathcal{L}\,(X,\theta)$. Accordingly, the path integral over $M_{\dot{A}}$ and N^{α} in the limit $b \to 0^+$ reduces to an evaluation of the remaining terms in the action at the locus defined by the incidence relation $\mathcal{L}\,(X,\theta)$, which is achieved by the application of the restriction homomorphism $\rho_{\mathcal{L}(X,\theta)}$.

Having restricted the bosonic path integral to the moduli space of embeddings of the celestial sphere, we now consider the fermionic contributions. The path integral over the fermionic system

$(\psi, \bar{\psi})$ results in the chiral determinant, which, as shown in Eq. (180), is the supersymmetric extension of the Quillen determinant. Consequently, the semiclassical effective action $\mathcal{W}$ becomes:

$$\mathcal{W} = \int\limits_{\mathbf{RP}^{3|2\mathcal{N}}} D^{3|2\mathcal{N}}\mathbb{X} \;\; \log\det\left(\overline{\partial} + \boldsymbol{\omega}\right) \big|_{\mathcal{L}(X,\theta)}. \tag{200}$$

The above expression recovers the minitwistor generating functional $\mathcal{F}[\boldsymbol{\omega}]$ for the background potential $\boldsymbol{\omega}$ appropriate to the spacetime theory under consideration. This demonstrates that the minitwistor sigma model reproduces semiclassically the tree-level MHV celestial leaf superamplitudes for gluons in $\mathcal{N} = 4$ SYM theory and for gravitons in $\mathcal{N} = 8$ Supergravity.

VI. DISCUSSION

The supersymmetric minitwistor celestial CFTs developed in the previous section reproduce the tree-level celestial leaf superamplitudes in $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ Supergravity for MHV configurations. Banerjee and Ghosh [52] proposed that the MHV sector of celestial CFT might serve as a minimal model for celestial holography. The minitwistor celestial CFTs introduced here could contribute to addressing this conjecture for the following reason.

The formalism presented in this paper relies on minitwistor wavefunctions, defined as cohomology classes in minitwistor space. This approach extends the RSVW prescription to celestial amplitudes and offers a new perspective on the celestial leaf amplitudes. In this setup, tree-level MHV leaf amplitudes are expressed as integrals over the moduli space of minitwistor lines. It is therefore reasonable to suggest that celestial amplitudes corresponding to next-to-MHV (NMHV) configurations could be described by integrals over the moduli space of higher-degree curves in minitwistor space.

We propose that future work should focus on extending the minitwistor generating functional developed here to systematically incorporate NMHV celestial amplitudes. Additionally, it would be worthwhile to study a celestial version of the Cachazo-Svrcek-Witten (CSW) expansion for leaf amplitudes. We expect that such an extension could offer a concrete path toward proving the Banerjee-Gosh conjecture.

The integral over the moduli space of special curves in minitwistor superspace that arises from our extension of the RSVW formalism to celestial amplitudes is not a standard volume integral. This is because the delta functions $\overline{\delta}_{\Delta_i}$, which localise the superspace integral to the locus of incidence in minitwistor space, have non-trivial homogeneity properties given by the celestial conformal weights Δ_i. It would be interesting to reformulate these integrals using the formalism introduced by Movshev [63] and further elaborated by Adamo and Groechenig [64].

In the semiclassical limit $b \to 0^+$ of our minitwistor celestial CFTs, where the parameter b plays a role similar to the Liouville coupling constant, we have been able to reproduce celestial amplitudes at tree level. An important direction for future research is to explore whether extending beyond the semiclassical limit could produce loop corrections, similar to those studied in the context of celestial Liouville theory in Mol [94]. Another promising approach would be to consider a complex scaling reduction of Berkovits' original twistor string theory to minitwistor space[36]. Investigating whether the resulting minitwistor string theory can generate NMHV celestial leaf amplitudes as integrals over the moduli space of higher-degree algebraic curves in minitwistor superspace $\mathbf{MT}^{2|\mathcal{N}}$ would be an important step forward.

Appendix A: Minitwistor Geometry

The non-singular quadric $\mathbf{MT} \subset \mathbf{CP}^3$ upon which our celestial CFTs will be defined corresponds to the minitwistor space associated with the hyperboloid $\mathbf{H}_3$. Consequently, we shall succinctly review how the hyperbolic geometry of $\mathbf{H}_3$ arises from the projective geometry of $\mathbf{CP}^3$. Then, we recapitulate the Hitchin construction[37] of minitwistor space, and discuss the mapping from representatives of cohomology classes in $\mathbf{MT}$ to conformal primaries of the H_3^+-WZNW model.

Here, we follow the notation of Kobayashi and Nomizu [95].

Note. The present subsection adopts a more mathematical style compared to the remainder of this manuscript, and it is intended to acquaint the reader with basic notions of minitwistor geometry.

1. Hyperbolic Space from Projective Geometry

Let the four-vector $X^{A\dot{A}}$ be homogeneous coordinates on $\mathbf{CP}^3$, subject to the equivalence relation $X^{A\dot{A}} \sim a \cdot X^{A\dot{A}}$, where a is any non-zero complex scalar. A necessary and sufficient condition for a set of components of a holomorphic metric (given in the chart $X^{A\dot{A}}$) to be well-defined on $\mathbf{CP}^3$, is that such components must be invariant under the "gauge transformations" $X^{A\dot{A}} \mapsto a \cdot X^{A\dot{A}}$ $(a \in \mathbf{C}^*)$, and they must not possess any components along the scaling dimension defined by this equivalence relation.

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[36] See Berkovits [85].
[37] As developed in Hitchin [16].

A first fundamental form satisfying these conditions is given by:

$$ds^2 = -\frac{dX^2}{X^2} + \left(\frac{X \cdot dX}{X^2}\right)^2 . \tag{A1}$$

The invariance of ds^2 under rescalings is manifest. Introducing the metric tensor $\boldsymbol{g}_{A\dot{A}B\dot{B}}$ associated with ds^2, for which $ds^2 = \boldsymbol{g}_{A\dot{A}B\dot{B}} dX^{A\dot{A}} \otimes dX^{B\dot{B}}$, we observe that:

$$\pounds_{\Upsilon} \boldsymbol{g} = 0, \tag{A2}$$

where $\Upsilon := X^{A\dot{A}} \frac{\partial}{\partial X^{A\dot{A}}} \in \mathscr{X}\left(\mathbf{CP}^3\right)$ is the Euler vector field, and $\pounds_{\Upsilon}$ is the Lie derivative along the flow of Υ.

Consequently, ds^2 is devoid of components along the scaling dimension, and defines a first fundamental form on the open submanifold $\mathbf{CP}^3 - \mathcal{B}$. Here, $\mathcal{B}$ denotes the closed submanifold on which $\boldsymbol{g}_{A\dot{A}B\dot{B}}$ becomes singular, and is defined as:

$$\mathcal{B} := \left\{ \left(X^{A\dot{A}}\right) \in \mathbf{CP}^3 \,|\, \langle X, X \rangle = 0 \right\} . \tag{A3}$$

Rewriting ds^2 in terms of normalised coordinates, $X^{A\dot{A}} / |X|$, we find:

$$ds^2 = -\varepsilon_{A\dot{A}} \varepsilon_{B\dot{B}} d\left(\frac{X^{A\dot{A}}}{|X|}\right) d\left(\frac{X^{B\dot{B}}}{|X|}\right) , \quad |X|^2 := \langle X, X \rangle . \tag{A4}$$

It follows that the submanifold $\mathbf{CP}^3 - \mathcal{B}$, endowed with the geometry induced by $\boldsymbol{g}_{A\dot{A}B\dot{B}}$, is isometric to complexified AdS_3, with $\mathcal{B} \simeq \partial AdS_3$ corresponding to its conformal boundary.

To obtain either the Kleinian hyperboloid $\mathbf{H}_3$ or the Lorentzian AdS_3, we restrict the metric tensor $\boldsymbol{g}_{AB\dot{A}\dot{B}}$ to an appropriate slice of $\mathbf{CP}^3$. This is achieved by imposing suitable reality conditions on the components of $X^{A\dot{A}}$.

2. Minitwistor Space

A minitwistor space[38] $\mathcal{M}$ is defined as any two-dimensional complex manifold containing a rational curve $\mathscr{C}$ (a holomorphic embedding of $\mathbf{CP}^1$ into $\mathcal{M}$) with a normal bundle isomorphic to $\mathcal{O}(2)$. Any such curve $\mathscr{C}$ is referred to as a *minitwistor line* or a *Hitchin special curve*. By virtue of a theorem due to Kodaira[39], the manifold $\mathcal{M}$ admits a three-parameter family $\mathscr{F}$ of such special curves. Denoting the corresponding parameter space by $\mathbf{W}(\mathscr{F})$, a corollary of Kodaira's theorem establishes the existence of an isomorphism between the tangent space $T_x\big(\mathbf{W}(\mathscr{F})\big)$, at any point

[38] See Hitchin [15] and Jones and Tod [14].
[39] Cf. Kodaira [96] or Kobayashi and Nomizu [95]

$x \in \mathbf{W}(\mathscr{F})$, and the space of globally defined smooth sections of the normal bundle $\mathcal{N}_x$ of the Hitchin special curve $\mathscr{C}_x \subset \mathcal{M}$ corresponding to x, given by $\Gamma(\mathscr{C}_x, \mathcal{N}_x)$. Furthermore, it has been demonstrated by Hitchin (1982) that $\mathbf{W} := \mathbf{W}(\mathscr{F})$, constructed in this manner, can be endowed with the structure of a Weyl manifold.

For our purposes, we specialise to the non-singular quadric $\mathbf{MT} := \mathbf{CP}^1 \times \mathbf{CP}^1$. Let $\left(\lambda^A, \mu_{\dot{A}}\right)$ denote homogeneous coordinates on $\mathbf{CP}^1 \times \mathbf{CP}^1$. The quadric $\mathbf{MT}$ is realised as an embedding into $\mathbf{CP}^3$ through the mapping:

$$\left(\lambda^A, \mu_{\dot{A}}\right) \in \mathbf{MT} \mapsto [\lambda^A \mu_{\dot{A}}] \in \mathbf{CP}^3. \tag{A5}$$

For any point $X^{A\dot{A}} \in \mathbf{CP}^3$, the locus of incidence:

$$\mathcal{L}(X) := \left\{\left(\lambda^A, \mu_{\dot{A}}\right) \in \mathbf{MT} \,\middle|\, \mu_{\dot{A}} = \lambda^A X_{A\dot{A}}\right\}, \tag{A6}$$

defines a conic contained within $\mathbf{MT}$. This conic satisfies the following property: there exists a hyperplane $\mathcal{H} \subset \mathbf{CP}^3$ such that $\mathcal{L}(X) = \mathbf{MT} \cap \mathcal{H}$. Any such conics intersect in precisely two points. As a consequence, these conics have a normal bundle isomorphic to $\mathcal{O}(2)$ and thus constitute special curves in the sense of Hitchin. Moreover, the condition $\det\left(X^{A\dot{A}}\right) = 0$ is both necessary and sufficient for a plane section to be tangent to $\mathbf{MT}$. Any such plane section determines $X^{A\dot{A}}$ up to a proportionality factor. Accordingly, the space $\mathbf{W}$ of plane sections that are not tangent to $\mathbf{MT}$ may be identified with the set of non-null rays emanating from the origin in Minkowski space $\mathbf{R}^{(1,3)}$, thereby realising $\mathbf{W}$ as a hyperboloid embedded into $\mathbf{R}^{(1,3)}$.

Now consider two points $X^{A\dot{A}}, Y^{A\dot{A}} \in \mathbf{W}$ that are null-separated in the Weyl conformal structure. In this case, either the corresponding conics in $\mathbf{MT}$ intersect, or the associated planes in $\mathbf{CP}^3$ intersect along a line $\mathcal{L}$ that is tangent to $\mathbf{MT}$. Consequently, there exists a unique plane passing through $\mathcal{L}$ and tangent to $\mathbf{MT}$. This implies that the algebraic equation:

$$\det\left(X^{A\dot{A}} + tY^{A\dot{A}}\right) = 0, \tag{A7}$$

admits solutions with multiplicity greater than one. By restricting $X^{A\dot{A}}$ and $Y^{A\dot{A}}$ to be unit vectors in Minkowski space and expanding the preceding equation in t, it follows that $X^{A\dot{A}}$ and $Y^{A\dot{A}}$ are null-separated in $\mathbf{W}$ if and only if:

$$X^{A\dot{A}} Y_{A\dot{A}} = \varepsilon_{A\dot{A}} \varepsilon_{B\dot{B}} X^{A\dot{A}} Y^{B\dot{B}} = 1. \tag{A8}$$

However, it is well known that the distance δ between the points $X^{A\dot{A}}$ and $Y^{A\dot{A}}$ in the first fundamental form induced on the hyperboloid is given by $\delta = \cosh^{-1}\left(X^{A\dot{A}} Y_{A\dot{A}}\right)$. Thus, the conformal metric associated with Hitchin's construction is equivalent to the class of metrics conformally related to the standard hyperbolic metric on $\mathbf{H}_3$.

We now establish that the Weyl conformal structure determined via the Hitchin correspondence from $\mathbf{MT}$ is equivalent to the conformal structure obtained from the hyperboloid $\mathbf{H}_3$, modulo conformal rescalings of the metric; consequently, $\mathbf{W}$ and $\mathbf{H}_3$ are conformally related. In fact, let $X^{A\dot{A}}, Y^{A\dot{A}} \in \mathbf{W}$ be two points, and let $\Pi_X, \Pi_Y \subset \mathbf{CP}^3$ denote the planes corresponding to these points. Consider the line of intersection $\mathcal{L} \coloneqq \Pi_X \cap \Pi_Y$. Let $\mathscr{F}$ denote the one-parameter family of planes passing through $\mathcal{L}$. The family $\mathscr{F}$ defines a set of conics contained within $\mathbf{MT}$ that intersect at $\mathcal{L}$ and defines a Weyl geodesic γ in $\mathbf{W}$. However, γ is precisely the intersection of the hyperboloid $\mathbf{H}_3$ with the two-dimensional subspace spanned by $X^{A\dot{A}}$ and $Y^{A\dot{A}}$ in Minkowski space. Since $\mathbf{H}_3$ is endowed with a hyperbolic metric, the curve γ is a geodesic of the metric induced on $\mathbf{H}_3$. This establishes the equivalence of the conformal structure on $\mathbf{W}$ with that on $\mathbf{H}_3$, thereby completing the proof that the non-singular quadric $\mathbf{MT}$ is the minitwistor space of $\mathbf{H}_3$ via the Hitchin correspondence.

3. The Holomorphic Vector Bundle $\mathcal{O}(p,q) \longrightarrow \mathbf{MT}$

In this subsection, we shall define the holomorphic vector bundle $\mathcal{O}(p,q) \longrightarrow \mathbf{MT}$ which serves as the domain upon which the minitwistor Penrose transform is defined. We establish that this bundle can be identified with the infinite-dimensional function space $\mathscr{C}^{\infty}_{p,q}(\mathbf{MT})$, consisting of smooth complex-valued functions:

$$h : (\mathbf{C}^*)^2 \times (\mathbf{C}^*)^2 \longrightarrow \mathbf{C}, \tag{A9}$$

satisfying the homogeneity property:

$$h\left(a \cdot \lambda^A, b \cdot \mu_{\dot{A}}\right) = a^p b^q \; h\left(\lambda^A, \mu_{\dot{A}}\right), \tag{A10}$$

for every pair of nonzero complex scalars a and b.

Definition. Recall that the space of dotted two-component spinors forms a holomorphic vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ fibered over $\mathbf{CP}^1$. Let $\mu_{\mathscr{U}} \coloneqq (\mathscr{U}, \vec{\mu})$ denote a local trivialisation of this bundle, where $\mathscr{U} \subset \mathbf{CP}^1$ is an open neighbourhood and $\vec{\mu} : \mathscr{U} \longrightarrow \mathbf{C}^2$ is a coordinate map, which associates each dotted spinor $\mu_{\dot{A}}$ with its coordinate representation:

$$(\mu)_{\mathscr{U}} \coloneqq (\mu_{\dot{1}}, \mu_{\dot{2}}). \tag{A11}$$

Similarly, the space of undotted two-component spinors can be described by a corresponding holomorphic vector bundle. Let $\lambda_{\mathscr{V}} := (\mathscr{V}, \vec{\lambda})$ be an analogous local trivialisation, where $\mathscr{V}$ is an open neighbourhood, and the map $\vec{\lambda} : \mathscr{V} \longrightarrow \mathbf{C}^2$ assigns each undotted spinor λ^A to its coordinate representation:

$$\lambda_{\mathscr{V}} := (\lambda^1, \lambda^2). \tag{A12}$$

Next, define the set:

$$\mathcal{P} := \left\{ \left(\vec{\lambda}, \vec{\mu}, s\right) \middle| \, \vec{\lambda} \in (\mathbf{C}^*)^2 \,,\, \vec{\mu} \in (\mathbf{C}^*) \,,\, s \in \mathbf{C} \right\}, \tag{A13}$$

and introduce an equivalence relation $\simeq$ on $\mathcal{P}$ as follows. For any pair of nonzero complex scalars $a, b \in \mathbf{C}^*$, we impose the condition:

$$\left(\vec{\lambda}, \vec{\mu}, s\right) \simeq \left(a \cdot \vec{\lambda},\, b \cdot \vec{\mu},\;\, a^p b^q \cdot s\right). \tag{A14}$$

Taking the quotient of $\mathcal{P}$ by $\simeq$ yields a new manifold,

$$\mathcal{O}\,(p, q) := \mathcal{P}/ \simeq, \tag{A15}$$

which is endowed with the quotient topology.

To establish the vector bundle structure of $\mathcal{O}\,(p, q) \longrightarrow \mathbf{MT}$, consider the following diagram:

$$\begin{tikzcd} \mathbf{C}^2 \times \mathbf{C}^2 \arrow[dr, "\pi"'] & & \mathcal{O}(p,q) \arrow[dl, "\mathcal{Q}"] \\ & \mathbf{MT} & \end{tikzcd} \tag{A16}$$

with the mappings in the diagram defined as follows:

$$\pi(\vec{\lambda}, \vec{\mu}) := ([\lambda^A], [\mu_{\dot{A}}]), \;\; \mathcal{Q}(\lambda^A, \mu_{\dot{A}}) := \pi(\vec{\lambda}, \vec{\mu}). \tag{A17}$$

Consequently, $\mathcal{Q}$ is a surjection with fibres isomorphic to $\mathbf{C}$, and the transition functions between local trivialisations satisfy the holomorphic cocycle condition, thereby showing that $\mathcal{Q}$ defines a holomorphic bundle over $\mathbf{MT}$.

Proposition. Let $\mathscr{C}^\infty_{p,q}\,(\mathbf{MT})$ denote the space of $\mathscr{C}^\infty$ complex-valued functions h defined on $(\mathbf{C}^*)^2 \times (\mathbf{C}^*)^2$ that satisfy the homogeneity property:

$$h\left(a \cdot \lambda^A, b \cdot \mu_{\dot{A}}\right) = a^p b^q \;\, h\left(\lambda^A, \mu_{\dot{A}}\right), \tag{A18}$$

for every pair of nonzero complex scalars a and b. The function space $\mathscr{C}^\infty_{p,q}\,(\mathbf{MT})$ can be canonically identified with the module $\Gamma^\infty(\mathcal{O}(p, q))$ of smooth sections on the holomorphic vector bundle $\mathcal{O}\,(p, q) \longrightarrow \mathbf{MT}$.

Proof. First, consider $h \in \mathscr{C}^{\infty}_{p,q}(\mathbf{MT})$. For each minitwistor $\widetilde{\mathsf{Z}}^I \coloneqq (\widetilde{\lambda}^A, \widetilde{\mu}_{\dot{A}}) \in \mathbf{MT}$, define the set:

$$\digamma \coloneqq \left\{ (Z, h(Z)) \;\middle|\; Z = (\vec{\lambda}, \vec{\mu}) \in \mathbf{C}^2 \times \mathbf{C}^2,\, \pi(Z) = \widetilde{\mathsf{Z}}^I \right\}. \tag{A19}$$

By the homogeneity condition satisfied by h, the set $\digamma$ consists of equivalence classes under $\simeq$. Consequently, this set determines a unique point $\sigma(\widetilde{\mathsf{Z}}^I)$ in the fibre of $\mathcal{O}(p,q)$ over $\widetilde{\mathsf{Z}}^I$. Thus, the function h defines a section $\widetilde{\mathsf{Z}}^I \mapsto \sigma(\widetilde{\mathsf{Z}}^I)$ contained in $\Gamma^{\infty}(\mathcal{O}(p,q))$.

Conversely, suppose $\sigma \in \Gamma^{\infty}(\mathcal{O}(p,q))$ is a smooth section. For each minitwistor $\widetilde{\mathsf{Z}}^I \in \mathbf{MT}$, there exists a class $\mathsf{C} = \{(Z,s)\}$ of equivalent pairs such that $\pi(Z) = \widetilde{\mathsf{Z}}^I$. Choosing a representative $Z = (\vec{\lambda}, \vec{\mu}) \in \mathbf{C}^2 \times \mathbf{C}^2$, we find a unique pair $(Z,s) \in \mathsf{C}$ in the class. The section σ therefore determines a function $f \in \mathscr{C}^{\infty}_{p,q}(\mathbf{MT})$ defined by $Z \mapsto f(Z) \coloneqq s$, which satisfies the homogeneity property by construction. Hence, the spaces $\mathscr{C}^{\infty}_{p,q}(\mathbf{MT})$ and $\Gamma^{\infty}(\mathcal{O}(p,q))$ are isomorphic, as claimed.

Appendix B: Leaf Amplitudes Review

1. Klein Space

The *leaf representation* of celestial amplitudes is our main motivation for introducing multi-gluon and multi-graviton wavefunctions using minitwistor variables. This formalism requires the analytic continuation of Minkowski spacetime $\mathbf{R}^{(1,3)}$ from a Lorentzian $(-+++)$ to a Kleinian $(--++)$ signature. For this purpose, we briefly review the relevant aspects of Kleinian geometry, which have been investigated in detail by Barrett *et al.* [97], Bhattacharjee and Krishnan [98], Crawley *et al.* [99], Cheung, Oz, and Yin [100] and Duary and Maji [101].

We begin by considering Cartesian coordinates X^{μ} $(0 \leq \mu, \nu, ... \leq 3)$ on $\mathbf{R}^4$, and define the Kleinian metric tensor $h_{\mu\nu}$ as:

$$h_{\mu\nu} \coloneqq \mathrm{diag}(-1,-1,+1,+1). \tag{B1}$$

The resulting vector space $\mathbf{R}^{(2,2)} \coloneqq \left(\mathbf{R}^4, \langle\cdot,\cdot\rangle\right)$, equipped with the inner product:

$$\langle X, Y\rangle \coloneqq h_{\mu\nu} X^{\mu} Y^{\nu}, \;\; \text{for}\, X^{\mu}, Y^{\mu} \in \mathbf{R}^{(2,2)}, \tag{B2}$$

is known as the *four-dimensional Klein space.*

We identify three distinguished submanifolds contained in $\mathbf{R}^{(2,2)}$: the *null cone* Λ, the *time-like* *wedge* $\mathbf{W}^-$, and the *space-like wedge* $\mathbf{W}^+$, defined (respectively) as:

$$\Lambda \coloneqq \{(X^\mu) \in \mathbf{R}^{(2,2)} \mid \langle X, X\rangle = 0\}, \tag{B3}$$

$$\mathbf{W}^- \coloneqq \{(X^\mu) \in \mathbf{R}^{(2,2)} \mid \langle X, X\rangle < 0\}, \tag{B4}$$

$$\mathbf{W}^+ \coloneqq \{(X^\mu) \in \mathbf{R}^{(2,2)} \mid \langle X, X\rangle > 0\}. \tag{B5}$$

Time-like Wedge. To chart the time-like wedge $\mathbf{W}^-$, we employ the coordinate system $X^\mu_- :\mathbf{W}^- \longrightarrow \mathbf{R}^4$, defined as:

$$X^0_- \coloneqq \tau \cos(\psi) \cosh(\rho), \quad X^1_- \coloneqq \tau \sin(\psi) \cosh(\rho), \tag{B6}$$

$$X^2_- \coloneqq \tau \cos(\varphi) \sinh(\rho), \quad X^3_- \coloneqq \tau \sin(\varphi) \sinh(\rho), \tag{B7}$$

where $\tau, \rho \in (0, \infty)$ and $(\psi, \varphi) \in S^1 \times S^1$.

With respect to this parametrisation, the induced first fundamental form on $\mathbf{W}^-$ takes the form:

$$\left(ds^2\right)_{\mathbf{W}^-} = -d\tau^2 + \tau^2 \left(d\rho^2 - \cosh^2(\rho)\, d\psi^2 + \sinh^2(\rho)\, d\phi^2\right). \tag{B8}$$

It follows that the hypersurfaces of constant τ contained in $\mathbf{W}^-$ are diffeomorphic to the three-dimensional Lorentzian anti-de Sitter space with periodic time, denoted $AdS_3/\mathbf{Z}$. Furthermore, the integration measure on $\mathbf{W}^-$ in the coordinate chart X^μ_- is given by the volume form:

$$\left(d^4X\right)_{\mathbf{W}^-} \coloneqq \frac{1}{2}\tau^3 \sinh(2\rho)\, d\tau d\rho d\psi d\varphi, \tag{B9}$$

where the juxtaposition of differentials is understood as the exterior product of differential forms, $d\tau d\rho d\psi d\varphi = d\tau \wedge d\rho \wedge d\psi \wedge d\varphi$.

A closed submanifold of $\mathbf{W}^-$, which plays a central role in the subsequent discussion, is the *standard Kleinian hyperboloid* $\mathbf{H}_3$, defined as:

$$\mathbf{H}_3 \coloneqq \{(X^\mu) \in \mathbf{R}^{(2,2)} \mid \langle X, X\rangle = -1\}. \tag{B10}$$

We chart $\mathbf{H}_3$ using the coordinate system $x^\mu : \mathbf{H}_3 \longrightarrow \mathbf{R}^4$, given by the functions:

$$x^0 \coloneqq \cos(\psi) \cosh(\rho), \quad x^1 \coloneqq \sin(\psi) \cosh(\rho), \tag{B11}$$

$$x^2 \coloneqq \cos(\varphi) \sinh(\rho), \quad x^3 \coloneqq \sin(\varphi) \sinh(\rho). \tag{B12}$$

The restriction of the first fundamental form of $\mathbf{W}^-$ to $\mathbf{H}_3$, obtained via the pull-back ι^*_- of the inclusion map $\iota_- : \mathbf{H}_3 \longrightarrow \mathbf{W}^-$, endows $\mathbf{H}_3$ with a Lorentzian metric. This structure implies the existence of an isometry from $\mathbf{H}_3$ onto $AdS_3/\mathbf{Z}$. The corresponding integration measure on $\mathbf{H}_3$ is given by the volume form:

$$d^3x := \frac{1}{2}\sinh(2\rho)\, d\rho d\psi d\varphi. \tag{B13}$$

Combining Eqs. (B9) and (B13) for the volume forms on $\mathbf{W}^-$ and $\mathbf{H}_3$, respectively, we deduce:

$$\left(d^4X\right)_{\mathbf{W}^-} = \tau^3 d\tau d^3x. \tag{B14}$$

Space-like Wedge. The space-like wedge $\mathbf{W}^+$ is parametrised by the coordinate system $X_+^\mu : \mathbf{W}^+ \longrightarrow \mathbf{R}^4$, defined by:

$$X_+^0 := \tau\cos(\psi)\sinh(\rho)\,, \quad X_+^1 := \tau\sin(\psi)\sinh(\rho)\,, \tag{B15}$$

$$X_+^2 := \tau\cos(\varphi)\cosh(\rho)\,, \quad X_+^3 := \tau\sin(\varphi)\cosh(\rho)\,. \tag{B16}$$

The first fundamental form induced on $\mathbf{W}^+$ from the Kleinian metric $h_{\mu\nu}$ is given by:

$$\left(ds^2\right)_{\mathbf{W}^+} = d\tau^2 - \tau^2\left(d\rho^2 + \sinh^2(\rho)\, d\psi^2 - \cosh^2(\rho)\, d\varphi^2\right). \tag{B17}$$

The integration measure on $\mathbf{W}^+$ is represented by the volume form:

$$\left(d^4X\right)_{\mathbf{W}^+} = -\frac{1}{2}\tau^3\sinh(2\rho)\, d\tau d\rho d\psi d\varphi. \tag{B18}$$

Contained in $\mathbf{W}^+$ is the *unit hyperboloid* $\mathbf{H}_3^+ := \{X_+^\mu \in \mathbf{R}^{(2,2)} \mid \langle X_+, X_+\rangle = 1\}$, which is charted by the coordinate system $y^\mu : \mathbf{H}_3^+ \longrightarrow \mathbf{R}^4$ defined by the functions:

$$y^0 := \cos(\psi)\sinh(\rho)\,, \quad y^1 = \sin(\psi)\sinh(\rho)\,, \tag{B19}$$

$$y^2 := \cos(\varphi)\cosh(\rho)\,, \quad y^3 := \sin(\varphi)\cosh(\rho)\,. \tag{B20}$$

The first fundamental form on $\mathbf{H}_3^+$, induced by the pull-back of the inclusion map $\iota_+ : \mathbf{H}_3^+ \longrightarrow \mathbf{W}^+$, is given in these coordinates by:

$$\left(ds^2\right)_{\mathbf{H}_3^+} = d\rho^2 + \sinh^2(\rho)\, d\psi^2 - \cosh^2(\rho)\, d\varphi^2. \tag{B21}$$

The geometric distinction between $\mathbf{H}_3$ (from the time-like wedge) and $\mathbf{H}_3^+$ lies in the interchange of the time-like and space-like orientations. Despite this difference, $\mathbf{H}_3^+$ remains diffeomorphic to $AdS_3/\mathbf{Z}$. The volume form on $\mathbf{H}_3^+$ is expressed as:

$$d^3y = -\frac{1}{2}\sinh(2\rho)\, d\rho d\psi d\varphi. \tag{B22}$$

Finally, from Eqs. (B18) and (B22), we deduce the following decomposition of the integration measure:

$$\left(d^4X\right)_{\mathbf{W}^+} = \tau^3 d^3y. \tag{B23}$$

2. Spinor Algebra

Employing the Van der Waerden formalism, as reviewed by Veblen [102] and Penrose and Rindler [103], the inner product of two undotted two-component spinors, denoted μ^A and ν^A, is defined as:

$$\langle \mu\nu \rangle := \mu \cdot \nu := \varepsilon_{AB}\mu^A\nu^B = \mu^A\nu_A, \tag{B24}$$

where ε_{AB} is the Levi-Civita symbol, antisymmetric under index exchange, and normalised according to $\varepsilon_{12} = -\varepsilon_{21} = 1$. The lowering and raising of spinor indices adhere to the standard convention, according to which $\mu_A := \varepsilon_{AB}\mu^B$ and $\nu^A = \varepsilon^{AB}\nu_B$ with ε^{AB} satisfying $\varepsilon^{AC}\varepsilon_{CB} = \delta^A{}_B$.

Similarly, the inner product of two dotted spinors, $\bar{\mu}_{\dot{A}}$ and $\bar{\nu}_{\dot{A}}$, is defined as:

$$[\bar{\mu}\bar{\nu}] := \varepsilon^{\dot{A}\dot{B}}\bar{\mu}_{\dot{A}}\bar{\nu}_{\dot{B}}, \tag{B25}$$

where $\varepsilon^{\dot{A}\dot{B}}$ is the antisymmetric Levi-Civita symbol for the dotted spinor space. Index manipulation for dotted spinors follows analogous rules, employing the conventions $\bar{\mu}^{\dot{A}} := \varepsilon^{\dot{A}\dot{B}}\bar{\mu}_{\dot{B}}$ and $\bar{\nu}_{\dot{A}} := \varepsilon_{\dot{A}\dot{B}}\bar{\nu}^{\dot{B}}$, where $\varepsilon_{\dot{A}\dot{B}}$ satisfies $\varepsilon_{\dot{A}\dot{C}}\varepsilon^{\dot{C}\dot{B}} = \delta_{\dot{A}}{}^{\dot{B}}$.

The *Kleinian Pauli matrices*, which establish a correspondence between vector and spinor indices in Kleinian signature, are defined as follows:

$$\boldsymbol{\sigma}^0 = \sigma^3, \;\; \boldsymbol{\sigma}^1 = \sigma^1, \;\; \boldsymbol{\sigma}^2 = \mathbb{I}, \;\; \boldsymbol{\sigma}^3 = i\sigma^2, \tag{B26}$$

where $(\sigma^\mu)_{A\dot{A}}$ denotes the Lorentzian Pauli matrices, defined in the $(-+++)$ signature.

The reason for this choice becomes apparent through the following construction. Consider the map from $\mathbf{R}^{(2,2)}$ into the space of real 2×2 matrices, given by:

$$X^\mu \mapsto (\underline{X})_{A\dot{A}} := \sum_{\mu=0}^{3} X^\mu(\boldsymbol{\sigma}^\mu)_{A\dot{A}} = \begin{pmatrix} X^0+X^2 & X^1+X^3 \\ X^1-X^3 & X^2-X^0 \end{pmatrix}. \tag{B27}$$

In this expression, the Einstein summation convention (according to which indices are raised and lowered using the Kleinian metric $h_{\mu\nu}$) is temporarily suspended, and ordinary summation is employed instead.

The matrix $\underline{X}$ satisfies two important properties:

$$\underline{X}^* = \underline{X}, \;\; \det \underline{X} = \langle X, X\rangle\,. \tag{B28}$$

The first condition imposes a reality constraint, while the second relates the determinant of $\underline{X}$ to the Kleinian inner product of the four-vector X^μ.

For a transformation of the form:

$$\underline{X} \mapsto U\underline{X}\widetilde{U}^{-1}, \tag{B29}$$

to preserve the conditions given by Eq. (B28), it is both necessary and sufficient that U and $\widetilde{U}$ take the forms:

$$U = \exp\left(\sum_{i=1}^{3} \lambda_i \boldsymbol{\sigma}^i\right), \quad \widetilde{U} = \exp\left(\sum_{i=1}^{3} \widetilde{\lambda}_i \boldsymbol{\sigma}^i\right), \tag{B30}$$

with $\lambda_i, \widetilde{\lambda}_i \in \mathbf{R}^3$.

This construction realises the $(2,2)$ representation of $SL(2,\mathbf{R}) \times \widetilde{SL(2,\mathbf{R})}$ within the vector representation of $O(2,2)$, as anticipated by the well-known isomorphism:

$$Spin\,(2,2) \simeq SL(2,\mathbf{R}) \times \widetilde{SL(2,\mathbf{R})}. \tag{B31}$$

Thus, the mapping defined by Eq. (B27) allows the conversion between vector and spinor indices in Kleinian signature, as we claimed.

Standard Null-vector. To facilitate the parametrisation of the celestial sphere at null infinity, we introduce the pair of two-component spinors η^A and $\bar{\eta}_{\dot{A}}$. These spinors are defined in terms of the angular coordinates $\bar{\zeta}, \zeta \in \mathbf{CP}^1$, arising from the stereographic projection of the celestial sphere onto the complex plane, and are given by:

$$\eta^A := \begin{pmatrix} \zeta \\ 1 \end{pmatrix}, \quad \bar{\eta}_{\dot{A}} := \begin{pmatrix} 1 & -\bar{\zeta} \end{pmatrix}. \tag{B32}$$

The *standard null four-vector* $q^\mu = q^\mu(\zeta, \bar{\zeta})$ is defined as:

$$q^\mu(\zeta,\bar{\zeta}) := \eta^A (\boldsymbol{\sigma}^\mu)_{A\dot{A}} \bar{\eta}^{\dot{A}} = \left(\zeta\bar{\zeta} - 1, \zeta + \bar{\zeta}, 1 + \zeta\bar{\zeta}, \zeta - \bar{\zeta}\right). \tag{B33}$$

3. Celestial Wavefunctions and the Leaf Amplitude Representation

The leaf representation of celestial amplitudes, introduced by Melton, Sharma, and Strominger [58], arises from the consideration of the following integral over spacetime:

$$\mathcal{I}(\pi_i, \bar{\pi}_i) := \int_{\mathbf{R}^{(2,2)}} d^4X \prod_{i=1}^{n} \phi_{2h_i}\left(X \middle| \pi_i, \bar{\pi}_i\right). \tag{B34}$$

Here, $\phi_\Delta\left(X \middle| \pi_i, \bar{\pi}_i\right)$ denotes the *celestial conformal primary wavefunction* for massless scalars with conformal weight Δ, defined by:

$$\phi_\Delta\left(X \middle| \pi_i, \bar{\pi}_i\right) := \frac{\mathcal{C}(\Delta)}{\left(i\varepsilon + \langle \pi_i | X | \bar{\pi}_i]\right)^\Delta}, \quad \mathcal{C}(\Delta) := i^{-\Delta}\Gamma(\Delta). \tag{B35}$$

The spacetime integral in Eq. (B34) contains the key features of the leaf representation, which we subsequently employ to reduce from twistor to minitwistor variables. Thus, we briefly review its computation.

Since the null cone Λ in $\mathbf{R}^{(2,2)}$ has zero measure, the spacetime integral can be decomposed as the sum of contributions from the time-like and space-like wedges:

$$\mathcal{I} = \int_{\mathbf{W}^-} d^4X \prod_{i=1}^{n} \phi_{2h_i} \left(X \middle| \pi_i, \bar{\pi}_i\right) + \int_{\mathbf{W}^+} d^4X \prod_{i=1}^{n} \phi_{2h_i} \left(X \middle| \pi_i, \bar{\pi}_i\right). \tag{B36}$$

Employing the decomposition of the measures for $\mathbf{W}^-$ and $\mathbf{W}^+$, given (respectively) by Eqs. (B14) and (B23), the above expression can be reformulated as:

$$\mathcal{I} = \int_{(0,\infty)} d\tau\, \tau^3 \int_{\mathbf{H}_3} d^3x \prod_{i=1}^{n} \phi_{2h_i} \left(\tau x^\mu \middle| \pi_i, \bar{\pi}_i\right) + \int_{(0,\infty)} d\tau\, \tau^3 \int_{\mathbf{H}_3^+} d^3x \prod_{i=1}^{n} \phi_{2h_i} \left(\tau y^\mu \middle| \pi_i, \bar{\pi}_i\right). \tag{B37}$$

An important property of these integrals follows from the parametrisation of the null four-vector introduced in Eq. (B33). In fact, it can be shown that:

$$\int_{\mathbf{H}_3^+} d^3x \prod_{i=1}^{n} \phi_{2h_i} \left(\tau y^\mu \middle| \pi_i, \bar{\pi}_i\right) = \int_{\mathbf{H}_3} d^3x \prod_{i=1}^{n} \phi_{2h_i} \left(\tau x^\mu \middle| \pi_i, -\bar{\pi}_i\right). \tag{B38}$$

This result allows one to replace the integration over the unit hyperboloid in the space-like wedge, $\mathbf{H}_+^3$, with an integration over the standard Kleinian hyperboloid in the time-like wedge, $\mathbf{H}_3$, provided the substitution $\bar{\pi}_{i\dot{A}} \mapsto -\bar{\pi}_{i\dot{A}}$ is made for all $1 \leq i \leq n$.

Accordingly, the integral $\mathcal{I}$ can be expressed as:

$$\mathcal{I} = \int_{(0,\infty)} d\tau\, \tau^3 \int_{\mathbf{H}_3} d^3x \prod_{i=1}^{n} \phi_{2h_i} \left(\tau x^\mu \middle| \pi_i, \bar{\pi}_i\right) + (\bar{\pi}_{i\dot{A}} \to -\bar{\pi}_{i\dot{A}}), \tag{B39}$$

where $(\bar{\pi}_{i\dot{A}} \to -\bar{\pi}_{i\dot{A}})$ signifies the repetition of the first term with the indicated substitution.

Factorising the τ-dependence, we rewrite the integrand as:

$$\prod_{i=1}^{n} \phi_{2h_i} \left(\tau x^\mu \middle| \pi_i, \bar{\pi}_i\right) = \tau^{-2\sum_{i=1}^{n} h_i} \prod_{i=1}^{n} \frac{\mathcal{C}\left(2h_i\right)}{\left(i\varepsilon' + \langle \pi_i | x | \bar{\pi}_i]\right)^{2h_i}}, \tag{B40}$$

where ε' is a redefined infinitesimal regulator.

Substituting this expression into Eq. (B39), we obtain:

$$\mathcal{I} = \int_{(0,\infty)} d\tau\, \tau^{3-2\sum_{i=1}^{n} h_i} \int_{\mathbf{H}_3} d^3x \prod_{i=1}^{n} \frac{\mathcal{C}\left(2h_i\right)}{\left(i\varepsilon' + \langle \pi_i | x | \bar{\pi}_i]\right)^{2h_i}} + (\bar{\pi}_{i\dot{A}} \to -\bar{\pi}_{i\dot{A}}). \tag{B41}$$

Finally, using the generalised Dirac delta function, analytically continued to the complex domain as explained in Donnay, Pasterski, and Puhm [104], we express this integral as:

$$\mathcal{I} = 2\pi\delta\left(\beta\right) \int_{\mathbf{H}_3} d^3x \prod_{i=1}^{n} G_{2h_i} \left(x \middle| \pi_i, \bar{\pi}_i\right) + (\bar{\pi}_{i\dot{A}} \to -\bar{\pi}_{i\dot{A}}), \tag{B42}$$

where $\beta := 4 - 2\sum_{i=1}^{n} h_i$, and G_Δ is the bulk-to-boundary Green's function[40] for the covariant Laplacian on $\mathbf{H}_3$, given by:

$$G_\Delta\left(x\middle|\pi,\bar{\pi}\right) := \frac{\mathcal{C}\left(\Delta\right)}{\left(i\varepsilon + \langle\pi|x|\bar{\pi}]\right)^{\Delta}}. \tag{B43}$$

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