Let

$$I_n \;=\; n^2 \int_0^1 \frac{x^n}{1+2x+\frac{2}{n}}\,dx.$$

Compute

$$L \;=\; \lim_{n\to\infty} \Big(I_n - \tfrac{n}{3}\Big).$$

1. **Split the integrand at $x = 1$:**

$$\frac{1}{1+2x+\frac{2}{n}} \;=\; \frac{1}{3+\frac{2}{n}} \;+\; \left(\frac{1}{1+2x+\frac{2}{n}} - \frac{1}{3+\frac{2}{n}}\right).$$

2. **"Constant" piece:**

$$n^2\int_0^1 \frac{x^n}{3+\frac{2}{n}}\,dx \;=\; \frac{n^2}{3+\frac{2}{n}}\cdot\frac{1}{n+1} \;=\; \frac{n}{3}-\frac{2}{9}+O\Big(\tfrac{1}{n}\Big).$$

3. **Linearize near $x = 1$:**

$$\frac{1}{1+2x+\frac{2}{n}} - \frac{1}{3+\frac{2}{n}} \;=\; -\frac{2(x-1)}{(3+\frac{2}{n})^2} \;+\; O\big((x-1)^2\big).$$

4. **Exact moment:**

$$n^2\int_0^1 x^n(x-1)\,dx \;=\; \frac{n^2}{(n+1)(n+2)} \;=\; 1+O\Big(\tfrac{1}{n}\Big).$$

5. **Remainder contribution:**

$$n^2\int_0^1 x^n\left(\frac{1}{1+2x+\frac{2}{n}} - \frac{1}{3+\frac{2}{n}}\right)dx \;=\; -\frac{2}{(3+\frac{2}{n})^2}\Big[1+O\Big(\tfrac{1}{n}\Big)\Big] \;=\; -\frac{2}{9}+O\Big(\tfrac{1}{n}\Big).$$

6. **Assemble:**

$$I_n \;=\; \Big(\tfrac{n}{3}-\tfrac{2}{9}\Big)+\Big(-\tfrac{2}{9}\Big)+O\Big(\tfrac{1}{n}\Big) \;=\; \tfrac{n}{3}-\tfrac{1}{3}+O\Big(\tfrac{1}{n}\Big).$$

7. **Limit:**

$$L \;=\; \lim_{n\to\infty}\Big(I_n-\tfrac{n}{3}\Big) \;=\; -\tfrac{1}{3}.$$