Chapter III

Bessel Functions

§ 14 Bessel's differential equation and its solutions

1. Solving the Helmholtz equation in cylindrical coordinates. Bessel functions are perhaps the most frequently used special functions. Typical problems that lead to Bessel functions arise in solving the *Helmholtz equation*

$$\Delta v + \lambda v = 0$$

in cylindrical coordinates. We consider the simplest case, when v is independent of the distance along the axis of the cylinder. Then $v = v(r, \phi)$ and

$$\Delta v + \lambda v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} + \lambda v = 0. \tag{1}$$

So that v will be single-valued, we require that it satisfies the periodicity condition $v(r, \phi + 2\pi) = v(r, \phi)$. Let us expand v in a Fourier series:

$$v(r, \phi) = \sum_{n=-\infty}^{\infty} v_n(r) e^{in\phi},$$

where

$$v_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r, \phi) e^{-in\phi} d\phi. \tag{2}$$

It is easy to obtain a differential equation for $v_n(r)$ by integrating (1) on $(-\pi, \pi)$ with weight $e^{-in\phi}$ and simplifying the terms containing $\partial^2 v/\partial\phi^2$ by integrating twice by parts. Since $v(r,\phi)$ is periodic in ϕ, the integrated terms vanish, and we obtain a differential equation for $u(z) = v_n(r)$ with $z = \sqrt{\lambda}r$:

$$z^2u'' + zu' + (z^2 - n^2)u = 0.$$

We are going to study an equation of somewhat more general form,

$$z^2u'' + zu' + (z^2 - \nu^2)u = 0, \tag{3}$$

where z is a complex variable, and the parameter ν can have any real or complex values.

The solutions of (3) are *Bessel functions of order* ν, or *cylinder functions*, and (3) is *Bessel's equation.*

Many other differential equations can be obtained from Bessel's equation by changes of variable. An example is *Lommel's equation*

$$v'' + \frac{1-2\alpha}{\xi}v' + \left[(\beta\gamma\xi^{\gamma-1})^2 + \frac{\alpha^2 - \nu^2\gamma^2}{\xi^2}\right]v = 0, \tag{4}$$

which is extensively used in applications; its solutions are

$$v(\xi) = \xi^{\alpha}u_\nu(\beta\xi^\gamma)$$

Here $u_\nu(z)$ is a Bessel function of order ν; α, β, γ are constants.

2. Definition of Bessel functions of the first kind and Hankel functions. Bessel's equation (3) is the special case of the generalized equation of hypergeometric type (1.1) for which $\sigma(z) = z$, $\tilde{\tau}(z) = 1$, $\tilde{\sigma}(z) = z^2 - \nu^2$. In reducing (3) to an equation of hypergeometric type, the possible forms of $\phi(z)$ are, as was shown in §1 (see the example given there) $\phi(z) = z^{\pm\nu}e^{\pm iz}$, corresponding to different choices of signs in formula (1.11) for $\pi(z)$ and different values of k. Let us consider, for example, $\phi(z) = z^\nu e^{iz}$. Putting $u(z) = \phi(z)y(z)$, we obtain an equation of hypergeometric type,

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0, \tag{3a}$$

where

$$\sigma(z) = z, \quad \tau(z) = 2iz + 2\nu + 1, \quad \lambda = i(2\nu + 1).$$

By Theorem 1 of §3 a particular solution of (3a) is

$$y(z) = \frac{c_\mu}{\rho(z)}\int\limits_C \frac{\sigma^\mu(s)\rho(s)}{(s-z)^{\mu+1}}ds,$$

where c_μ is a normalizing constant, $\rho(z)$ is a solution of the differential equation

$$[\sigma(z)\rho(z)]' = \tau(z)\rho(z),$$

and μ is a root of the equation

$$\lambda + \mu\tau' + \frac{1}{2}\mu(\mu-1)\sigma'' = 0$$

(we have used formulas (3.2) and (3.3), where in order to avoid confusion we have replaced ν by μ since ν has already been used in the original Bessel equation). The contour C is chosen so that

$$\left.\frac{\sigma^{\mu+1}(s)\rho(s)}{(s-z)^{\mu+2}}\right|_{s_1,s_2} = 0.$$

In the present case,

$$\mu = -\nu - \frac{1}{2}, \quad \rho(z) = z^{2\nu}e^{2iz}.$$

Hence a particular solution of Bessel's equation can be written in the form

$$u_\nu(z) = \phi(z)y(z) = a_\nu z^{-\nu}e^{-iz}\int_C [s(z-s)]^{\nu-\frac{1}{2}}e^{2is}ds, \tag{5}$$

where a_ν is a normalizing constant and C is chosen so that

$$\left. s^{\nu+1/2}(z-s)^{\nu-3/2}e^{2is}\right|_{s_1,s_2} = 0.$$

Let $z > 0$, Re $\nu > 3/2$. Then the ends of the contour can be taken at $s_1 = 0, s_2 = z$. Alternatively, C might go to infinity with $\operatorname{Im} s \to +\infty$. Then C can be one of the contours indicated in Figure 6.

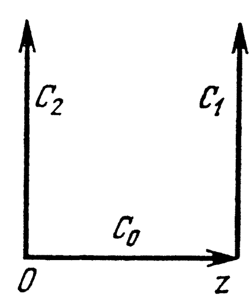

Figure 6.

We thus obtain the following three solutions of Bessel's equation:

$$u_\nu^{(0)}(z) = a_\nu z^{-\nu} e^{-iz} \int\limits_{C_0} [s(z-s)]^{\nu-1/2} e^{2is} ds, \tag{6}$$

$$u_\nu^{(1)}(z) = a_\nu^{(1)} z^{-\nu} e^{-iz} \int\limits_{C_1} [s(z-s)]^{\nu-1/2} e^{2is} ds, \tag{7}$$

$$u_\nu^{(2)}(z) = a_\nu^{(2)} z^{-\nu} e^{-iz} \int\limits_{C_2} [s(z-s)]^{\nu-1/2} e^{2is} ds. \tag{8}$$

In order to have a single-valued branch of the function $[s(z-s)]^{\nu-1/2}$, we take $|\arg s(z-s)| < \pi$. The contours C_0, C_1, C_2 can be parametrized by

$$\begin{aligned} s &= z(1+t)/2 && (-1 \le t \le 1),\\ s &= z(1+it/2) && (0 \le t < \infty),\\ s &= izt/2 && (0 \le t < \infty). \end{aligned}$$

Then (6)–(8) become

$$u_\nu^{(0)}(z) = \frac{a_\nu}{2^{2\nu}} z^\nu \int\limits_{-1}^{1} (1-t^2)^{\nu-1/2} e^{izt} dt = \frac{a_\nu}{2^{2\nu}} z^\nu \int\limits_{-1}^{1} (1-t^2)^{\nu-1/2} \cos zt \, dt, \tag{9}$$

$$u_\nu^{(1)}(z) = -\frac{a_\nu^{(1)}}{\sqrt{2}} \left(\frac{z}{2}\right)^\nu e^{i(z-\pi\nu/2-\pi/4)} \int\limits_0^\infty e^{-zt} t^{\nu-1/2} \left(1+\frac{it}{2}\right)^{\nu-1/2} dt, \tag{10}$$

and

$$u_\nu^{(2)}(z) = \frac{a_\nu^{(2)}}{\sqrt{2}} \left(\frac{z}{2}\right)^\nu e^{-i(z-\pi\nu/2-\pi/4)} \int\limits_0^\infty e^{-zt} t^{\nu-1/2} \left(1-\frac{it}{2}\right)^{\nu-1/2} dt. \tag{11}$$

In accordance with the condition $|\arg s(z-s)| < \pi$, the values of $\arg(1 \pm \frac{1}{2}it)$ in (10) and (11) are taken with the smallest possible absolute values.

If we take the normalizing constants real, and $a_\nu^{(2)} = -a_\nu^{(1)}$, we see from (10) and (11) that when z and ν are real, the functions $u_\nu^{(1)}(z)$ and $u_\nu^{(2)}(z)$ are complex conjugates. It is convenient to introduce a function that is real for real z,

$$u_\nu(z) = \frac{1}{2}[u_\nu^{(1)}(z) + u_\nu^{(2)}(z)]. \tag{12}$$

Let us show that this function is equal to $u_\nu^{(0)}(z)$ if we take

$$a_\nu^{(2)} = -a_\nu^{(1)} = 2a_\nu. \tag{13}$$

To prove this it is enough to apply Cauchy's theorem to the contour C which is the union of C_0, C_1 and C_2 (see Fig. 6). If we close the contour at ∞, Cauchy's theorem yields

$$\int_C [s(z-s)]^{\nu-1/2} e^{2is} ds = -\int_{C_2} [s(z-s)]^{\nu-1/2} e^{2is} ds$$
$$+ \int_{C_0} [s(z-s)]^{\nu-1/2} e^{2is} ds + \int_{C_1} [s(z-s)]^{\nu-1/2} e^{2is} ds = 0$$

(the integral over the part of the contour "at infinity" reduces to zero). Taking account of (13) and using (6)–(8), we obtain

$$u_\nu^{(0)}(z) = \frac{1}{2}[u_\nu^{(1)}(z) + u_\nu^{(2)}(z)], \tag{14}$$

as required.

The function $u_\nu^{(0)}(z)$, with an appropriate choice of a_ν, is the *Bessel function of the first kind*, $J_\nu(z)$; the functions $u_\nu^{(1)}(z)$ and $u_\nu^{(2)}(z)$ with the normalization (13) are the *Hankel functions of the first and second kind*, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. By (14), these functions are connected by the equation

$$J_\nu(z) = \frac{1}{2}[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)]. \tag{15}$$

The integral representations (9)–(11) are useful for investigating the properties of Bessel functions: in particular, the integral representation for $J_\nu(z)$ lets one obtain the power series for $J_\nu(z)$; the integral representations of the Hankel functions are used in obtaining the asymptotic expansions of these functions as $z \to \infty$.

To obtain the power series for $J_\nu(z)$, we replace $\cos zt$ in (9) by its expansion in powers of zt and interchange summation and integration. We obtain

$$J_\nu(z) = \frac{a_\nu}{2^{2\nu}} z^\nu \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \int_{-1}^{1} (1-t^2)^{\nu-1/2} t^{2k} dt.$$

We can evaluate the coefficients in the series:

$$\begin{aligned}
\int_{-1}^{1}(1-t^2)^{\nu-1/2}t^{2k}dt &= 2\int_{0}^{1}(1-t^2)^{\nu-1/2}t^{2k}dt \\
&= \int_{0}^{1}(1-t)^{\nu-1/2}t^{k-1/2}dt = \frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(\nu+k+1)} \\
&= \frac{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}(2k)!}{2^{2k}k!\,\Gamma(\nu+k+1)}.
\end{aligned}$$

Here we used the evenness of the integrand, the relation between the beta and gamma functions, and the duplication formula for the gamma function (see Appendix A). Hence we have

$$J_\nu(z) = \frac{a_\nu}{2^\nu}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}.$$

The series will have a simpler form if we choose a_ν so that

$$\frac{a_\nu}{2^\nu}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right) = 1. \tag{16}$$

Finally we obtain

$$J_\nu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}. \tag{17}$$

Using the value of a_ν given by (16), we can rewrite (9)–(11) as

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\cos zt\,dt, \tag{18}$$

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi}}\frac{z^\nu e^{i(z-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)}\int_{0}^{\infty}e^{-zt}t^{\nu-1/2}\left(1+\frac{it}{2}\right)^{\nu-1/2}dt, \tag{19}$$

and

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi}}\frac{z^\nu e^{-i(z-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)}\int_{0}^{\infty}e^{-zt}t^{\nu-1/2}\left(1-\frac{it}{2}\right)^{\nu-1/2}dt. \tag{20}$$

These are known as *Poisson's integrals* for the Bessel functions.

Other useful representations for the Hankel functions are obtained from (19) and (20) by replacing t by t/z:

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{z^\nu e^{i(z-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-t} t^{\nu-1/2} \left(1+\frac{it}{2z}\right)^{\nu-1/2} dt, \quad (19a)$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \frac{z^\nu e^{-i(z-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-t} t^{\nu-1/2} \left(1-\frac{it}{2z}\right)^{\nu-1/2} dt. (20a)$$

§ 15 Basic properties of Bessel functions

1. Recursion relations and differentiation formulas. Recursion relations and differentiation formulas for Bessel functions can be found by the method of §4 from the original integral representation of the functions:

$$u_\nu(z) = a_\nu z^{-\nu} e^{-iz} \int_C [s(z-s)]^{\nu-1/2} e^{2is} ds.$$

As an example, let us find a relation of the form

$$A_1(z)u_\nu'(z) + A_2(z)u_\nu(z) + A_3(z)u_{\nu-1}(z) = 0, \quad (1)$$

where $A_i(z)$ are rational functions of z. We have

$$A_1(z)u_\nu'(z) + A_2(z)u_\nu(z) + A_3(z)u_{\nu-1}(z)$$
$$= e^{-iz} z^{-\nu-1} \int_C P(s)[s(z-s)]^{\nu-3/2} e^{2is} ds,$$

where

$$P(s) = A_1 a_\nu \left[(-\nu - iz)s(z-s) + \left(\nu - \frac{1}{2}\right) zs\right] + A_2 a_\nu zs(z-s) + A_3 z^2 a_{\nu-1}.$$

For (1) to be satisfied, $A_1(z)$, $A_2(z)$ and $A_3(z)$ are to be determined by the condition

$$P(s)[s(z-s)]^{\nu-3/2} e^{2is} = \frac{d}{ds}\{Q(s)[s(z-s)]^{\nu-1/2} e^{2is}\},$$

where $Q(s)$ is a polynomial. As shown in §4, one coefficient of $Q(s)$ can be chosen arbitrarily. In the present case, $Q(s)$ is a constant and we can take $Q(s) = a_\nu$. Using the condition on $P(s)$ and $Q(s)$, we obtain the equation

$$\begin{aligned} A_1[(-\nu - iz)s(z-s) + (\nu - 1/2)zs] + A_2 zs(z-s) \\ + A_3 z^2 a_{\nu-1}/a_\nu = 2is(z-s) + (\nu - 1/2)(z - 2s). \end{aligned}$$

If we use the values of a_ν that correspond to $J_\nu(z)$ and $H_\nu^{(1,2)}(z)$, we obtain $a_{\nu-1}/a_\nu = (\nu - 1/2)/2$. The equation that determines A_i is valid for all s. Hence we can find A_i by taking s to have any convenient value. Taking, for example, $s = 0$, we obtain $A_3 = 2/z$. The value $s = z$ yields $A_1 = -2/z$. The coefficient A_2 is easily found by comparing the coefficients of the highest power of s: $A_2 = -2\nu/z^2$. Letting $u_\nu(z)$ stand for either $J_\nu(z)$ or $H_\nu^{(1,2)}(z)$, we obtain the relation

$$\frac{\nu}{z} u_\nu(z) + u_\nu'(z) = u_{\nu-1}(z). \tag{2}$$

By the same method, we can obtain a recursion involving $u_\nu(z), u_{\nu-1}(z)$ and $u_{\nu-2}(z)$. However, it is easier to differentiate (2) and eliminate $u_\nu''(z), u_\nu'(z)$ and $u_{\nu-1}'(z)$ by using Bessel's equation and (2). We find

$$u_\nu(z) - \frac{2(\nu-1)}{z} u_{\nu-1}(z) + u_{\nu-2}(z) = 0. \tag{3}$$

Equations (2) and (3) can be transformed into the equivalent forms

$$\begin{aligned} \frac{1}{z}\frac{d}{dz}[z^\nu u_\nu(z) &= z^{\nu-1} u_{\nu-1}(z), \\ -\frac{1}{z}\frac{d}{dz}[z^{-\nu} u_\nu(z)] &= z^{-(\nu+1)} u_{\nu+1}(z). \end{aligned}$$

Hence

$$\begin{aligned} \left(\frac{1}{z}\frac{d}{dz}\right)^n [z^\nu u_\nu(z)] &= z^{\nu-n} u_{\nu-n}(z), \\ \left(-\frac{1}{z}\frac{d}{dz}\right)^n [z^{-\nu} u_\nu(z)] &= z^{-(\nu+n)} u_{\nu+n}(z). \end{aligned} \tag{4}$$

2. Analytic continuation and asymptotic formulas. We have defined $J_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ only for $z > 0$ and Re $\nu > 3/2$. Now let z be a point of the complex plane cut along $(-\infty, 0)$, i.e. with $|\arg z| < \pi$. This restriction makes z^ν single-valued when ν is not an integer. By using the integral representations (14.18)–(14.20) we can continue $J_\nu(z)$ and $H_\nu^{(1,2)}(z)$ to larger domains for both z and ν.

The integral for $J_\nu(z)$ converges uniformly in z and ν for $\operatorname{Re}\nu \geq -\frac{1}{2}+\delta$, $|z| \leq R$ (δ and R are arbitrary positive numbers), because

$$|(1-t^2)^{\nu-1/2}\cos zt| \leq e^R(1-t^2)^{\delta-1}$$

and the integral $\int_{-1}^{1}(1-t^2)^{\delta-1}dt$ converges. Hence, by Theorem 2, §3, the function $J_\nu(z)$ is an analytic function of z and of ν for $|\arg z| < \pi$, and $\operatorname{Re}\nu > -1/2$.

The integrals for $H_\nu^{(1,2)}(z)$,

$$\int_0^\infty e^{-zt}t^{\nu-1/2}\left(1 \pm \frac{1}{2}it\right)^{\nu-1/2} dt,$$

are the Laplace integrals

$$F(z) = \int_0^\infty e^{-zt}f(t)dt,$$

with $f(t) = t^{\nu-1/2}(1 \pm \frac{1}{2}it)^{\nu-1/2}$. The analytic continuation and asymptotic representation of Laplace integrals of the form

$$F(z,p,q) = \int_0^\infty e^{-zt}t^p(1+at)^q\,dt$$

are discussed in detail in the example for Theorem 1 in Appendix B. In the present case, $p = q = \nu - \frac{1}{2}, a = \pm i/2$, and the results of this example show that the Hankel functions $H_\nu^{(1,2)}(z)$ are analytic in each variable for $z \neq 0$, $|\arg z| < \pi$, $\operatorname{Re}\nu > -1/2$. These functions have asymptotic representations as $z \to \infty$ when $\operatorname{Re}\nu > -1/2$ and $|\arg z| \leq \pi - \epsilon$:

$$H_\nu^{(1,2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm i(z-(\pi\nu/2)-\pi/4)}\left[\sum_{k=0}^{n-1} C_k\left(\pm\frac{i}{z}\right)^k + O\left(\frac{1}{z^n}\right)\right]. \quad (5)$$

Here

$$C_k = \frac{\Gamma\left(\nu+\frac{1}{2}+k\right)}{2^k k!\,\Gamma\left(\nu+\frac{1}{2}-k\right)},$$

and the upper signs apply to $H_\nu^{(1)}(z)$, the lower signs to $H_\nu^{(2)}(z)$. If we use the functional equation $\Gamma(z+1) = z\Gamma(z)$, we can simplify the formula for C_k. We have

$$\Gamma\left(\nu+\frac{1}{2}+k\right) = \left(\nu+\frac{1}{2}\right)\left(\nu+\frac{3}{2}\right)\dots\left(\nu+k-\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right),$$

$$\Gamma\left(\nu+\frac{1}{2}-k\right) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\left(\nu-\frac{1}{2}\right)\left(\nu-\frac{3}{2}\right)\dots\left(\nu-k+\frac{1}{2}\right)}.$$

Hence

$$C_k = \prod_{l=1}^{k}\left[\frac{4\nu^2-(2l-1)^2}{8l}\right], \quad C_0 = 1.$$

Using the equation

$$J_\nu(z) = \frac{1}{2}[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)]$$

we obtain an asymptotic formula for $J_\nu(z)$:

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left\{\sum_{k=0}^{n-1}\frac{C_k}{z^k}\cos\left[z-\frac{\pi}{2}\left(\nu-k+\frac{1}{2}\right)\right] + O\left(\frac{e^{|\operatorname{Im} z|}}{z^n}\right)\right\}. \quad (6)$$

We have considered the analytic continuation of Bessel functions for $z \neq 0, |\arg z| < \pi$ and $\operatorname{Re}\,\nu > -1/2$. The condition $\operatorname{Re}\,\nu > -1/2$ is not essential, since when $\operatorname{Re}\,\nu \leq -1/2$ the analytic continuation can be obtained from the recursion relation (3) with ν decreased by 1. By the differentiation formula (2) the derivatives of $J_\nu(z)$ and $H_\nu^{(1,2)}(z)$ are analytic in z and in ν in the same region as the Bessel functions themselves. By the principle of analytic continuation, the analytic continuations of the Bessel functions still satisfy Bessel's equation.

3. Functional equations. The Bessel equation is not changed by replacing ν by $-\nu$. Therefore it not only has $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ as solutions, but also $H_{-\nu}^{(1)}(z)$ and $H_{-\nu}^{(2)}(z)$. In finding the formulas that connect $H_\nu^{(1,2)}(z)$ with $H_{-\nu}^{(1,2)}(z)$, we shall suppose for the time being that $|\operatorname{Re}\nu| < 1/2$. Then the Hankel functions $H_{\pm\nu}^{(1,2)}(z)$ have the asymptotic representations (5). It is clear from these representations that the functions $H_\nu^{(1,2)}(z)$ have different asymptotic behavior as $z \to \infty$ and therefore are linearly independent solutions of Bessel's equation. Consequently

$$H_{-\nu}^{(1)}(z) = A_\nu H_\nu^{(1)}(z) + B_\nu H_\nu^{(2)}(z), \quad (7)$$

where A_ν and B_ν are constants. If we compare the asymptotic behavior, as $z \to \infty$, of the left-hand and right-hand sides of (7), we find that $A_\nu = e^{i\pi\nu}$ and $B_\nu = 0$, i.e.

$$H_{-\nu}^{(1)}(z) = e^{i\pi\nu} H_\nu^{(1)}(z). \tag{8}$$

Similarly,

$$H_{-\nu}^{(2)}(z) = e^{-i\pi\nu} H_\nu^{(2)}(z). \tag{9}$$

It is easily verified by using (8) and (9) that (5), and therefore also (6), are valid for all values of ν.

We now find the connection between $H_\nu^{(1)}(z), H_\nu^{(2)}(z)$ and $J_\nu(z), J_{-\nu}(z)$. Since

$$\begin{aligned} J_\nu(z) &= \frac{1}{2}[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)], \\ J_{-\nu}(z) &= \frac{1}{2}[H_{-\nu}^{(1)}(z) + H_{-\nu}^{(2)}(z)], \end{aligned} \tag{10}$$

we have

$$\begin{aligned} H_\nu^{(1)}(z) &= \frac{J_{-\nu}(z) - e^{-i\pi\nu} J_\nu(z)}{i \sin \pi\nu}, \\ H_\nu^{(2)}(z) &= \frac{e^{i\pi\nu} J_\nu(z) - J_{-\nu}(z)}{i \sin \pi\nu}. \end{aligned} \tag{11}$$

by (8) and (9).

4. Power series expansions. We have already obtained the power series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \tag{12}$$

for real $z > 0$ and Re $\nu > 3/2$. To establish this expansion for arbitrary values of ν and z we investigate the region of analyticity of (12) by using a theorem of Weierstrass (see [D2], [L3] or [S8]).

Theorem 1. *Let $f_k(z)$ be analytic in a region D and let the series*

$$\sum_{k=0}^{\infty} f_k(z)$$

converge uniformly on every compact subset of D to $f(z)$. Then in D:

1) $f(z)$ is analytic;

2) $f^{(n)}(z) = \sum_{k=0}^{\infty} f_k^{(n)}(z)$;

3) $\sum_{k=0}^{\infty} f_k^{(n)}(z)$ converges uniformly on every compact subset of D.

Remark. The series $\sum_{k=0}^{\infty} f_k(z)$ will converge uniformly in D if there is an m such that for every $z \in D$ and $k > m$ we have

$$\left|\frac{f_k(z)}{f_{k-1}(z)}\right| \leq q < 1,$$

where q is independent of z and $|f_m(z)| \leq C$ for $z \in D$ (C, a constant). This test for the uniform convergence of a series is known as *D'Alembert's test.*

Let us show that (12) converges uniformly for z and ν in the regions $0 < \delta \leq |z| \leq R$, $|\nu| \leq N$, where R and N are arbitrarily large fixed numbers. It will be sufficient to use the following estimate of the ratio of two successive terms of the series:

$$\left|\frac{u_k(z)}{u_{k-1}(z)}\right| = \frac{|z|^2}{4k|k+\nu|} \leq \frac{R^2}{4k(k-N)} \leq \frac{1}{4}$$

where $k \geq \max(R^2, N+1)$. Since the terms of the series are analytic functions of z and ν for $\delta \leq |z| \leq R, |\arg z| < \pi$, and $|\nu| \leq N$, the series (12) represents an analytic function of z and ν for all ν and $|\arg z| < \pi$.

Consequently both sides of (12) are analytic functions of each of z and ν for all ν and $|\arg z| < \pi$. By the principle of analytic continuation, (12) is valid in the specified domain of z and ν.

If $\nu \neq 0, 1, 2, \ldots$, the functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent, since they behave differently as $z \to 0$:

$$J_\nu(z) \approx \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \quad J_{-\nu}(z) \approx \frac{(z/2)^{-\nu}}{\Gamma(-\nu+1)}.$$

It follows that when $\nu \neq n (n = 0, 1, 2, \ldots)$ the general solution of Bessel's equation can be written

$$u(z) = C_1 J_\nu(z) + C_2 J_{-\nu}(z).$$

From (12) and (11) we can obtain the power series for $H_\nu^{(1,2)}(z)$. This presents no difficulty for $\nu \neq n$; therefore we consider only the case $\nu = n$. The points $\nu = n$ are removable singular points on the right-hand sides of (11), since the left-hand sides are analytic functions of ν and consequently approach limits as $\nu \to n$. The denominators in (11) vanish at $\nu = n$; hence if the limits are to exist, the numerators must vanish at $\nu = n$, i.e.

$$J_{-n}(z) = (-1)^n J_n(z).$$

It follows that when $\nu = n$ the solutions $J_n(z)$ and $J_{-n}(z)$ are linearly dependent. Taking limits as $\nu \to n$ and using L'Hospital's rule, we find

$$H_n^{(1,2)}(z) = J_n(z) \pm \frac{i}{\pi}[a_n(z) + (-1)^n a_{-n}(z)], \tag{13}$$

where $a_\nu(z) = \partial J_\nu(z)/\partial\nu$ (the plus sign corresponds to $H_n^{(1)}(z)$).

Since the series for $J_\nu(z)$ converges uniformly in the region we are considering, and its terms are analytic functions of ν, we may, by the theorem of Weierstrass, differentiate the series termwise in order to evaluate $a_\nu(z)$. We obtain

$$a_\nu(z) = J_\nu(z)\ln(z/2) - \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)}\psi(k+\nu+1),$$

where $\psi(z)$ is the logarithmic derivative of the gamma function (see Appendix A). Since

$$\frac{\psi(z)}{\Gamma(z)} \to (-1)^{n+1} n!, \quad z \to -n$$

(see formula (27) in Appendix A), we have

$$\begin{aligned}
(-1)^n a_{-n}(z) &= (-1)^n J_{-n}(z)\ln(z/2) \\
&\quad - (-1)^n \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k (z/2)^{-n+2k}}{k!} (-1)^{n-k}(n-k-1)! \right. \\
&\qquad\qquad \left. + \sum_{k=n}^{\infty} \frac{(-1)^k (z/2)^{-n+2k}\psi(k-n+1)}{k!\Gamma(k-n+1)} \right\} \\
&= J_n(z)\ln(z/2) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} \\
&\qquad\qquad - \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{k!(n+k)!}\psi(k+1).
\end{aligned}$$

Therefore

$$\begin{aligned}
H_n^{(1,2)}(z) = J_n(z) \pm \frac{i}{\pi} &\left\{ 2J_n(z)\ln\frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} \right. \\
&\left. - \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{k!(n+k)!}[\psi(n+k+1) + \psi(k+1)] \right\}.
\end{aligned} \tag{14}$$

When $n = 0$ the first sum is to be taken as zero. The values of $\psi(x)$ for integral x can be found from formula (16), Appendix A.

It follows from (11) and (14) that $H_\nu^{(1,2)}(z)$ have algebraic singular points of type $z^{\pm\nu}$ at $z = 0$ when Re $\nu \neq 0$, and logarithmic singular points when $\nu = 0$.

§ 16 Sommerfeld's integral representations

1. Sommerfeld's integral representation for Bessel functions. The Poisson integral representations of $J_\nu(z)$ and $H_\nu^{(1,2)}(z)$ are useful in discussing properties of the solutions of Bessel's equation. There is a different integral representation which is useful, for example, in diffraction problems. It is obtained in the following way. As we showed in §14, the function

$$u_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r, \phi) e^{-in\phi} d\phi,$$

with $z = \sqrt{\lambda} r$, is a Bessel function of order n if v satisfies the equation $\Delta v + \lambda v = 0$. The simplest solution of $\Delta v + \lambda v = 0$ when $\lambda = k^2 > 0$ is a plane wave $v = e^{i\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{k}$ is the wave vector. If the y axis is taken in the direction of $\mathbf{k}$, then

$$v(r, \phi) = e^{ikr\sin\phi}.$$

Hence we obtain the following integral representation for the Bessel function $u_n(z)$:

$$u_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin\phi - in\phi} d\phi. \tag{1}$$

There is a similar representation for Bessel functions of arbitrary order ν. To obtain it, it is natural to look for a solution of Bessel's equation for arbitrary ν in the form

$$u_\nu(z) = \int_C e^{iz\sin\phi - i\nu\phi} d\phi.$$

We shall show that $u_\nu(z)$ is a solution of Bessel's equation if the contour C is properly chosen. We start, as in the derivation of (1), from the fact that $v(r, \phi) = e^{ikr\sin\phi}$ is a solution of the Helmholtz equation

$$\frac{1}{r}\frac{\partial v}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} + k^2 v = 0. \tag{2}$$

It is easily verified that (2) remains valid for arbitrary complex values of r and ϕ.

We can obtain an equation for $v_\nu(r) = \int_C v(r,\phi)e^{-i\nu\phi}d\phi$ by starting from (2), integrating over C with weight $e^{-i\nu\phi}$, and simplifying the term in $\partial^2 v/\partial\phi^2$ by integrating by parts twice. If we require that the integrated terms, namely

$$e^{-i\nu\phi}\left(\frac{\partial v}{\partial \phi} + i\nu v\right)\Bigg|_{\phi_2}^{\phi_1} = e^{ikr\sin\phi - i\nu\phi}(kr\cos\phi + \nu)\Bigg|_{\phi_1}^{\phi_2}$$

(where ϕ_1 and ϕ_2 are the endpoints of C), are zero, we obtain Bessel's equation for $u_\nu(z) = v_\nu(r)$ with $z = kr$.

We have therefore shown that the function

$$u_\nu(z) = \int\limits_C e^{iz\sin\phi - i\nu\phi}d\phi \tag{3}$$

is actually a solution of Bessel's equation provided that

$$e^{iz\sin\phi - i\nu\phi}(z\cos\phi + \nu)\big|_{\phi_1}^{\phi_2} = 0. \tag{4}$$

Since $\cos\phi = (e^{i\phi} + e^{-i\phi})/2$, condition (4) will evidently be satisfied if

$$e^{iz\sin\phi - i\nu\phi}\big|_{\phi=\phi_1,\phi_2} = 0. \tag{5}$$

for every ν.

A representation of the form (3) is known as a *Sommerfeld representation.*

2. Sommerfeld's integral representations for Hankel functions and Bessel functions of the first kind. The contour C in the integral representation for $u_\nu(z)$ can, for example, be chosen as a contour that extends to infinity in such a way that

$$\mathrm{Re}(iz\,\sin\phi - i\nu\phi) = \mathrm{Re}\left[\frac{1}{2}|z|e^{i\theta}(e^{i\phi} - e^{-i\phi}) - i\nu\phi\right] \to -\infty \tag{6}$$

as $\phi \to \infty$, where $\theta = \arg z$.

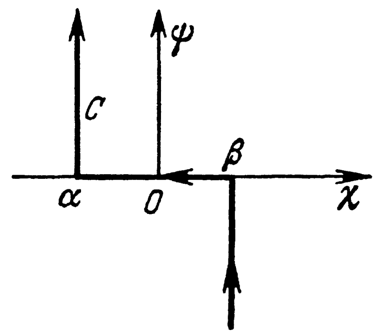

Figure 7.

Let us consider the contour C indicated in Figure 7, where $\phi = \chi + i\psi$. We need to see what conditions on α and β will make the contour have the required properties.

Let $\chi = \alpha$, $\psi \to +\infty$. In this case the terms in ϕ and $e^{i\phi}$ in (6) can be neglected in comparison with $e^{-i\phi}$. The condition on the contour becomes

$$\operatorname{Re} e^{i(\theta-\phi)} \to +\infty, \qquad \psi \to +\infty.$$

It is satisfied if $\cos(\theta - \alpha) > 0$. We may therefore suppose that

$$\theta - \pi/2 < \alpha < \theta + \pi/2. \tag{7}$$

Now let $\chi = \beta$, $\psi \to -\infty$. We find similarly that it is enough to require that $\cos(\theta + \beta) < 0$. This is satisfied if $\beta = -\alpha \pm \pi$. We denote the corresponding contours by C_+ and C_-.

There is a certain amount of freedom in the choice of the contours. Let C' be determined by numbers α' and β' that satisfy

$$\cos(\theta - \alpha') > 0, \quad \cos(\theta + \beta') < 0.$$

We can easily show by Cauchy's theorem that C' can be replaced by any other contour C'' determined by numbers α'' and β'' provided that the inequalities $\cos(\theta - \alpha) > 0$, $\cos(\theta + \beta) < 0$ are satisfied for all $\alpha \in [\alpha', \alpha'']$ and $\beta \in [\beta', \beta'']$. Consequently it is clear, in particular, that the contour C in the Sommerfeld representation can be replaced by a contour that has been shifted by an amount less than π, without affecting the value of the Sommerfeld integral.

Since $u_\nu(z)$ satisfies Bessel's equation, it can be represented in the form

$$u_\nu(z) = C_\nu H_\nu^{(1)}(z) + D_\nu H_\nu^{(2)}(z). \tag{8}$$

We can find C_ν and D_ν by using the asymptotic behavior of $H_\nu^{(1,2)}(z)$. Let us first consider the case when C is taken to be C_+. Let $|z| \to \infty$ and

$\arg z = \pi/2$. Then we can take $\alpha = \beta = \pi/2$; that is, take $\phi = \pi/2 + i\psi$, where $-\infty < \psi < \infty$, in the formula for $u_\nu(z)$. This yields

$$u_\nu(z) = ie^{-i\pi\nu/2} \int_{-\infty}^{\infty} e^{-|z|\cosh\psi} e^{\nu\psi} d\psi$$

$$= 2ie^{-i\pi\nu/2} \int_{0}^{\infty} e^{-|z|\cosh\psi} \cosh\nu\psi d\psi.$$

To find the asymptotic behavior of $u_\nu(z)$ as $z \to \infty$ we can use Watson's lemma (see Appendix B), after first making the change of variable $\cosh\psi = 1 + t$. In fact, after this substitution we obtain

$$u_\nu(z) = 2i\exp(-i\pi\nu/2 - |z|) \int_0^\infty e^{-z|t|} f(t)dt,$$

where

$$f(t) = \frac{1}{\sqrt{t(2+t)}} \cosh[\nu \ln(1 + t + \sqrt{t(2+t)})].$$

Since $f(t) = (2t)^{-1/2}[1 + O(t)]$ as $t \to 0$, Watson's lemma yields

$$u_\nu(z) = 2i\exp(-i\pi\nu/2 - |z|) \frac{\Gamma\left(\frac{1}{2}\right)}{(2|z|)^{1/2}} \left[1 + O\left(\frac{1}{z}\right)\right]$$

$$= i(2\pi/|z|)^{1/2} \exp(-i\pi\nu/2 - |z|) \left[1 + O\left(\frac{1}{z}\right)\right]$$

as $z \to \infty$. Comparing the leading terms on the two sides of (8), we find $D_\nu = 0$, $C_\nu = -\pi$. Therefore

$$H_\nu^{(1)}(z) = -\frac{1}{\pi} \int_{C_+} e^{iz\sin\phi - i\nu\phi} d\phi. \tag{9}$$

In a similar way we obtain

$$H_\nu^{(2)}(z) = \frac{1}{\pi} \int_{C_-} e^{iz\sin\phi - i\nu\phi} d\phi \tag{10}$$

by using the contour C_-. Hence

$$J_\nu(z) = \frac{1}{2}[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)] = \frac{1}{2\pi} \int_{C_1} e^{iz\sin\phi - i\nu\phi} d\phi, \tag{11}$$

where C_1 is indicated in Figure 8. When $\nu = n$, the integral over C_1 reduces to an integral over $(-\alpha - \pi, -\alpha + \pi)$ since the integrand is periodic.

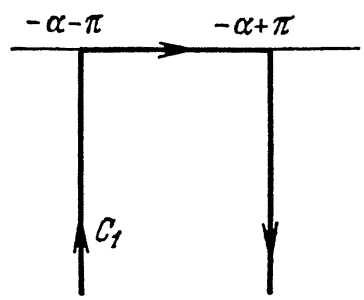

Figure 8.

Since the integral of a periodic function over an interval of length equal to a period is independent of the location of the interval, we have

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin\phi - in\phi} d\phi, \tag{11a}$$

i.e. the functions $J_n(z)$ are the Fourier coefficients of $e^{iz\sin\phi}$. Therefore

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\phi}. \tag{12}$$

By the principle of analytic continuation, (12) is valid for all complex ϕ.

We can simplify (11a) by using the formula

$$e^{iz\sin\phi - in\phi} = \cos(z\sin\phi - n\phi) + i\sin(z\sin\phi - n\phi)$$

and the evenness or oddness (in ϕ) of $\cos(z\sin\phi - n\phi)$ and $\sin(z\sin\phi - n\phi)$. Hence we obtain *Bessel's integral*

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z\sin\phi - n\phi) d\phi.$$

§ 17 Special classes of Bessel functions

1. Bessel functions of the second kind. In practice we often deal with solutions of Bessel's equation for real ν and positive z. It is not always convenient to use the Hankel functions since they take complex values. However, $H_\nu^{(2)}(z) = \overline{H_\nu^{(1)}(z)}$ in the present case (the bar denotes the complex conjugate), and

$$J_\nu(z) = \frac{1}{2}[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)] = \operatorname{Re} H_\nu^{(1)}(z).$$

This suggests taking the second linearly independent solution of Bessel's equation to be $\operatorname{Im} H_\nu^{(1)}(z)$, i.e.

$$Y_\nu(z) = \frac{1}{2i}[H_\nu^{(1)}(z) - H_\nu^{(2)}(z)]. \tag{1}$$

The functions $Y_\nu(z)$ are known as *Bessel functions of the second kind.**

We can consider $Y_\nu(z)$ as defined by (1) for arbitrary complex values of ν and z. It is analytic in ν except for $\nu = n (n = 0, \pm1, \pm2, \ldots)$ and analytic in z for $z \neq 0, |\arg z| < \pi$.

We list the basic properties of $Y_\nu(z)$, which follow from the corresponding properties of the Hankel functions.

a) *$Y_\nu(z)$ expressed in terms of $J_\nu(z)$ and $J_{-\nu}(z)$:*

$$Y_\nu(z) = \frac{\cos\pi\nu\, J_\nu(z) - J_{-\nu}(z)}{\sin\pi\nu} \qquad (\nu \neq n).$$

b) *Series expansion of $Y_\nu(z)$ for $\nu = n$:*

$$Y_n(z) = \frac{1}{\pi}\left\{2J_n(z)\ln\frac{z}{2} - \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} \right.$$
$$\left. - \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{n+2k}}{k!(n+k)!}[\psi(n+k+1) + \psi(k+1)]\right\}.$$

c) *Asymptotic formula for $Y_\nu(z)$ as $z \to \infty$:*

$$Y_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left[\sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(\frac{e^{|\operatorname{Im} z|}}{z}\right)\right].$$

* They are also called *Weber functions* or *Neumann functions*, and denoted by $N_\nu(z)$. We note that the Hankel functions are also called *Bessel functions of the third kind.*

d) *Recursion relation and differentiation formula*:

$$Y_{\nu-1}(z) + Y_{\nu+1}(z) = (2\nu/z)Y_\nu(z),$$
$$Y_{\nu-1}(z) - Y_{\nu+1}(z) = 2Y_\nu'(z).$$

Graphs of $J_\nu(x)$ and $Y_\nu(x)$ for some integral values of ν and $x > 0$ are given in Figures 9 and 10.

2. Bessel functions whose order is half an odd integer. Bessel polynomials. The Bessel functions of order half an odd integer* form a distinct class. They are remarkable for being expressible in terms of elementary functions. To establish this, we first use (19a) and (20a) of §14 to show that

$$H_{1/2}^{(1,2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm i(z-\pi/2)},$$

whence

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z, \quad Y_{1/2}(z) = -\left(\frac{2}{\pi z}\right)^{1/2} \cos z.$$

Moreover, according to (15.8) and (15.9),

$$H_{-1/2}^{(1)}(z) = e^{i\pi/2} H_{1/2}^{(1)} = \left(\frac{2}{\pi z}\right)^{1/2} e^{iz},$$
$$H_{-1/2}^{(2)}(z) = e^{-i\pi/2} H_{1/2}^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-iz}.$$

Hence

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z, \quad Y_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z.$$

Taking $\nu = -1/2$ in formulas (15.4), we obtain

$$H_{n-1/2}^{(1,2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n e^{\pm iz}, \tag{2}$$

* These functions include, for example, the solutions of the Helmholtz equation that are obtained by separating variables in spherical coordinates.

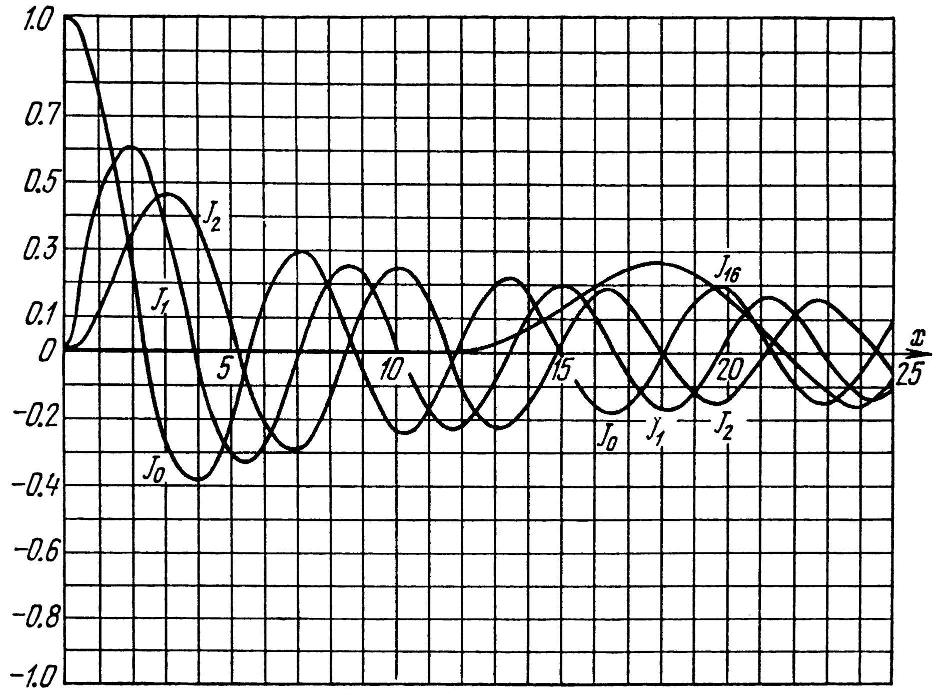

Figure 9.

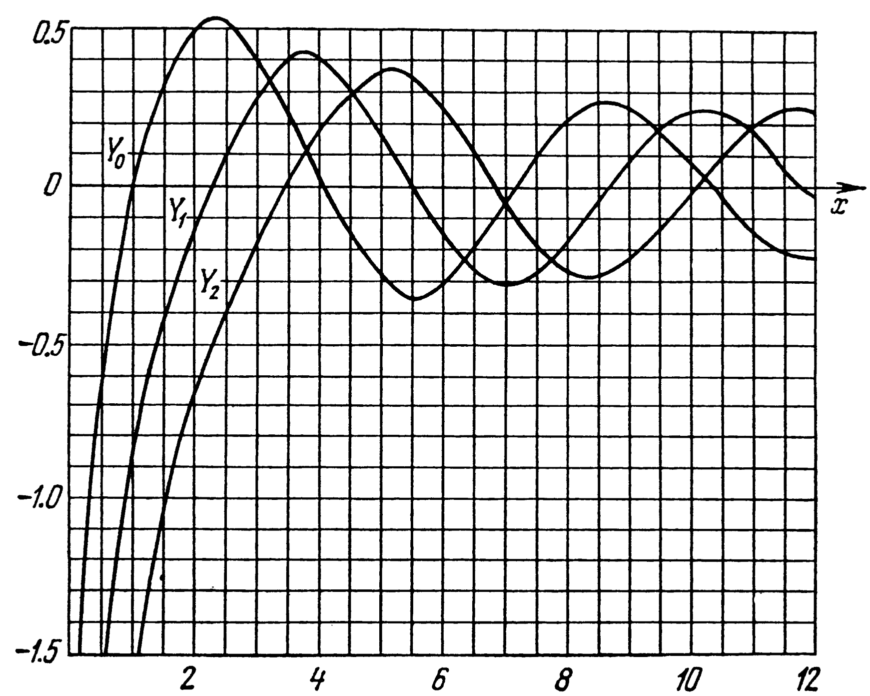

Figure 10.

$$J_{n-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \cos z, \tag{3}$$

$$Y_{n-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \sin z. \tag{4}$$

It was shown by Liouville that halves of odd integers are the only indices for which Bessel functions are elementary.

It follows by induction from (2) that

$$H^{(1)}_{n+1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{iz} p_n\left(\frac{1}{iz}\right),$$

where $p_n(s)$ is a polynomial in s of degree n. From the asymptotic behavior of $H^{(1)}_{n+1/2}(z)$ as $z \to \infty$ it follows that $p_n(0) = (-i)^{n+1}$. Let us show that $p_n(s)$ is a polynomial of hypergeometric type and can be expressed in terms of Bessel polynomials (see §5, Part 1):

$$y_n(z) = 2^{-n} e^{2/z} \frac{d^n}{dz^n}(z^{2n} e^{-2/z}).$$

In fact, from the differential equation for the Hankel functions $H^{(1)}_{n+1/2}(z)$ we can obtain a differential equation for $p_n(s)$:

$$s^2 p_n''(s) + 2(s+1)p_n'(s) - n(n+1)p_n(s) = 0.$$

This is an equation of hypergeometric type, and so the polynomials $p_n(s)$ are polynomials of hypergeometric type. If we express $p_n(s)$ by using the Rodrigues formula, we obtain

$$p_n(s) = B_n e^{2/s} \frac{d^n}{ds^n}(s^{2n} e^{-2/s}).$$

It is then clear that the polynomials $p_n(s)$ are, up to a normalizing factor, the Bessel polynomials $y_n(s)$. Since $p_n(0) = (-i)^{n+1}$, $y_n(0) = 1$, we finally obtain the following formula connecting the Hankel functions $H^{(1)}_{n+1/2}(z)$ with the Bessel polynomials:

$$H^{(1)}_{n+1/2}(z) = (-i)^{n+1} \left(\frac{2}{\pi z}\right)^{1/2} e^{iz} y_n\left(\frac{1}{iz}\right).$$

Similarly,

$$H^{(2)}_{n+1/2}(z) = i^{n+1} \left(\frac{2}{\pi z}\right)^{1/2} e^{-iz} y_n\left(-\frac{1}{iz}\right).$$

3. Modified Bessel functions. We have discussed the Bessel equation

$$z^2u'' + zu' + (z^2 - \nu^2)u = 0$$

for complex z. In the most important applications, z is positive. However, in many problems we are also interested in solutions of the equation

$$z^2u'' + zu' - (z^2 + \nu^2)u = 0 \tag{5}$$

for $z > 0$. This equation is the Bessel equation with z replaced by iz, and hence its solutions are known as *Bessel functions with imaginary argument*, or *modified Bessel functions*. Evidently $J_\nu(iz)$ and $H_\nu^{(1)}(iz)$ are linearly independent solutions of (5). The first solution is bounded as $z \to 0$ if $\nu > 0$, and the second, as $z \to \infty$.

It is customary to use

$$I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz), \tag{6}$$

and

$$K_\nu(z) = \frac{1}{2}\pi e^{i\pi(\nu+1)/2}H_\nu^{(1)}(iz) \tag{7}$$

instead of $J_\nu(iz)$ and $H_\nu^{(1)}(iz)$. These functions are real when $z > 0$ and ν is real, as follows from the formulas (see (14.7) and (14.19a))

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)},$$

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}.$$

These are corollaries of the power series expansion of $J_\nu(iz)$ and the equation (15.11) that connects $H_\nu^{(1)}(iz)$ with $J_\nu(iz)$ and $J_{-\nu}(iz)$. The function $K_\nu(z)$ is known as *Macdonald's function.*

We list the basic properties of $I_\nu(z)$ and $K_\nu(z)$; these follow from their relationship to $J_\nu(iz)$ and $H_\nu^{(1)}(iz)$.

1. *Poisson integral representations.* It follows from (18) and (19a), §14, that

$$I_\nu(z) = \frac{(z/2)^\nu}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\cosh zt\, dt$$

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \frac{\int_0^\infty e^{-t}t^{\nu-1/2}\left(1+t/(2z)\right)^{\nu-1/2} dt}{\Gamma(\nu+1/2)}.$$

2) *Series expansions:*

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)},$$

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu} \quad (\nu \neq n),$$

$$\begin{aligned} K_n(z) = (-1)^{n+1} I_n(z)\ln(z/2) &+ \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} \\ &+ \frac{1}{2}(-1)^n \sum_{k=0}^{\infty} \frac{(z/2)^{2k+n}}{k!(k+n)!}[\psi(n+k+1) + \psi(k+1)] \end{aligned} \tag{8}$$

(when $n = 0$ the first sum is to be taken to be zero).

It is evident from the expansion of $I_\nu(z)$ that when $z > 0$ and $\nu \geq 0$ the function $I_\nu(z)$ is positive and monotone increasing as z increases (see Figure 11).

3) *Connections between $K_\nu(z)$ and $K_{-\nu}(z)$, $I_n(z)$ and $I_{-n}(z)$:*

$$\begin{aligned} I_{-n}(z) &= I_n(z), \\ K_{-\nu}(z) &= K_\nu(z). \end{aligned} \tag{9}$$

4) *Asymptotic behavior as $z \to +\infty$:*

$$\begin{aligned} I_\nu(z) &= \frac{e^z}{\sqrt{2\pi z}}[1 + O(1/z)], \\ K_\nu(z) &= \sqrt{\pi/(2z)}e^{-z}[1 + O(1/z)]. \end{aligned}$$

5) *Recursion relations and differentiation formulas:*

$$\begin{aligned} I_{\nu-1}(z) - I_{\nu+1}(z) &= \frac{2\nu}{z} I_\nu(z), \\ I_{\nu-1}(z) + I_{\nu+1}(z) &= 2I_\nu'(z), \\ K_{\nu-1}(z) - K_{\nu+1}(z) &= -\frac{2\nu}{z} K_\nu(z), \\ K_{\nu-1}(z) + K_{\nu+1}(z) &= -2K_\nu'(z), \end{aligned}$$

in particular

$$I_0'(z) = I_1(z), \qquad K_0'(z) = -K_1(z).$$

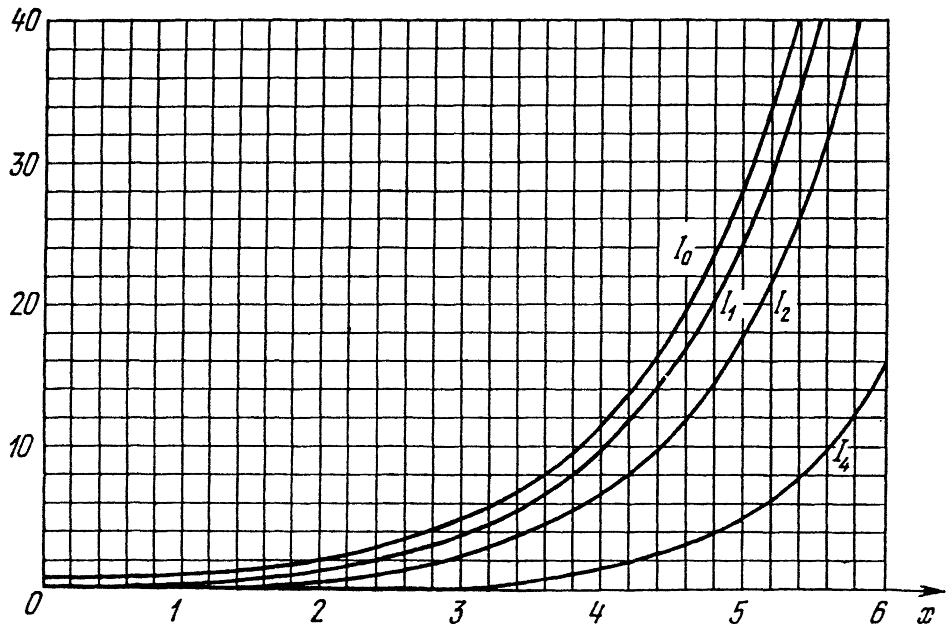

Figure 11.

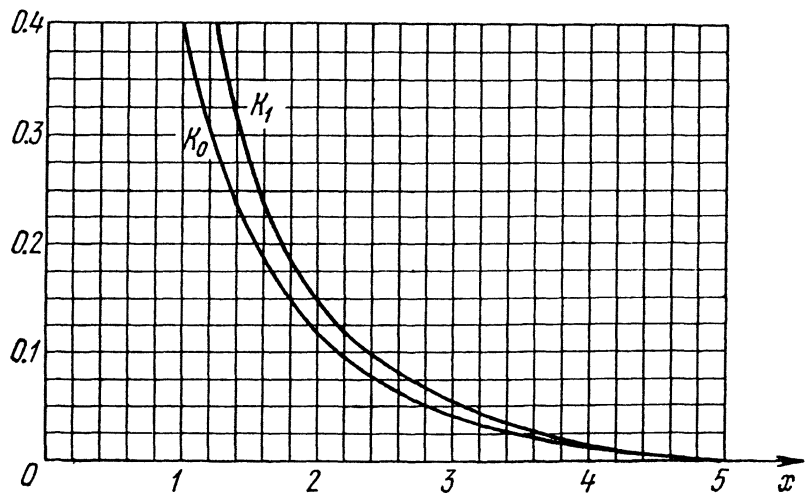

Figure 12.

6) *$I_\nu(z)$ and $K_\nu(z)$ expressed as elementary functions when ν is half an odd integer*:

$$I_{n-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} z^n \left(\frac{1}{z}\frac{d}{dz}\right)^n \cosh z \quad (n = 0, 1, \dots),$$

$$K_{n-1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n e^{-z} \quad (n = 0, 1, \dots).$$

7) *Sommerfeld integral for $K_\nu(z)$ for $z > 0$*:

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh\psi + \nu\psi} d\psi = \int_0^{\infty} e^{-z\cosh\psi}\cosh\nu\psi d\psi. \tag{10}$$

To derive (10) we took $\alpha = \pi/2$, $\phi = \pi/2 + i\psi$, $-\infty < \psi < \infty$, in (16.9). It is evident from (10) that when $z > 0$ and ν is real, $K_\nu(z)$ is positive and monotone decreasing as z increases (see Figure 12, p. 225).

If we make the substitution $\frac{1}{2}ze^{-\psi} = t$ in (10) for $z > 0$, we obtain the Sommerfeld integral:

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^{\infty} \exp\left(-z - \frac{z^2}{4t}\right) t^{-\nu-1} dt. \tag{11}$$

Remark. It follows from the properties of $I_\nu(z)$ and $K_\nu(z)$ that when $\nu \geq 0$ and $z \geq 0$, the general solution of (5) has the form

$$u(z) = AI_\nu(z) + BK_\nu(z),$$

where $B = 0$ if $u(z)$ is bounded at $z = 0$; if $u(z)$ is bounded as $z \to +\infty$ then $A = 0$.

We have considered several useful special classes of Bessel functions. There are other classes of functions that are related to Bessel functions and are convenient in special problems. They include the real and imaginary parts of $u_\nu(z)$ for $\mathrm{Im}\ \nu = 0$, $\arg z = \pm\pi/4$, $\pm 3\pi/4$; and the Airy function

$$Ai(z) = \begin{cases} (|z|/3\pi)^{1/2} K_{1/3}\left(\frac{2}{3}|z|^{3/2}\right) & \text{for } z < 0, \\ \frac{1}{3}(\pi z)^{1/2}\left[I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right] & \text{for } z > 0. \end{cases}$$

The Airy function is a solution of

$$u'' + zu = 0$$

(see (14.4)).

§ 18 Addition theorems

The addition formulas for Bessel functions have the form

$$u_\nu(R) = F(r,\rho,\theta) \sum_{n=0}^{\infty} f_n(r) g_n(\rho) h_n(\theta), \tag{1}$$

where r, ρ, R are the lengths of the sides of a triangle, θ is the angle between r and ρ (Figure 13), and $F(r,\rho,\theta)$ is an elementary function of simple form.

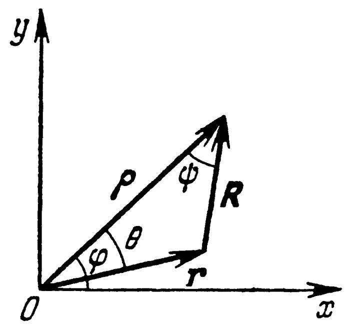

Figure 13.

These formulas provide series expansions of Bessel functions $u_\nu(R)$ of order ν, with terms that are products of the function $F(r,\rho,\theta)$, which is independent of the summation index, by factors each of which depends on only one of r, ρ and θ. Formulas of this kind are important in mathematical physics and other applications of Bessel functions.

1. Graf's addition theorem. Let $u_\nu(z)$ be one of $J_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$. For the simplest addition theorem we use Sommerfeld's integral for $u_\nu(R)$:

$$u_\nu(R) = A \int_C \exp\{iR\sin\phi - i\nu\phi\}\, d\phi \tag{2}$$

(A is a normalizing constant, which is independent of ν in the present case.)

Consider the triangle in Figure 13. Projecting the vector equation $\mathbf{R} = \boldsymbol{\rho} - \mathbf{r}$ on the y axis, we have

$$R\sin(\phi + \psi) = \rho\sin\phi - r\sin(\phi - \theta).$$

It is clear, by the principle of analytic continuation, that this equation remains valid for complex ϕ.

The contour C can, as we showed in §16, be chosen so that a shift by $\psi < \pi$ does not affect the value of the integral. If we replace ϕ in (2) by $\phi+\psi$, we obtain

$$u_\nu(R)e^{i\nu\psi} = A\int_C \exp\{iR\sin(\phi+\psi) - i\nu\phi\}d\phi$$
$$= A\int_C \exp\{i\rho\sin\phi + ir\sin(\theta-\phi) - i\nu\phi\}d\phi.$$

Since

$$e^{ir\sin(\theta-\phi)} = \sum_{n=-\infty}^{\infty} J_n(r)e^{in(\theta-\phi)}$$

by (16.12), we have

$$u_\nu(R)e^{i\nu\psi} = \sum_{n=-\infty}^{\infty} e^{in\theta}J_n(r)A\int_C e^{i\rho\sin\phi - i(\nu+n)\phi}d\phi$$
$$= \sum_{n=-\infty}^{\infty} J_n(r)u_{\nu+n}(\rho)e^{in\theta}.$$

We can interchange summation and integration when $r < \rho$. Hence we obtain

$$u_\nu(R)e^{i\nu\psi} = \sum_{n=-\infty}^{\infty} J_n(r)u_{\nu+n}(\rho)e^{in\theta}.$$

Since the substitutions $R \to kR$, $r \to kr$, $\rho \to k\rho$ do not change θ and ψ, we can write the formula as

$$u_\nu(kR)e^{i\nu\psi} = \sum_{n=-\infty}^{\infty} J_n(kr)u_{\nu+n}(k\rho)e^{in\theta}. \tag{3}$$

This is *Graf's addition theorem.*

2. Gegenbauer's addition theorem. There is a different addition formula when $F(r,\rho,\theta) = R^\nu$. To obtain it, we consider the function

$$R^{-\nu}u_\nu(R) = v(R).$$

Suppose for definiteness that $r < \rho$. In this case $R \neq 0$ and $v(R)$ is bounded as $r \to 0$.

The function $v(R)$ satisfies the equation

$$Rv'' + (2\nu + 1)v' + Rv = 0 \tag{4}$$

(see Lommel's equation, (14.4)). It is easy to deduce a partial differential equation in r and $\mu = \cos\theta$ for fixed ρ. Since $R = (r^2 + \rho^2 - 2r\rho\mu)^{1/2}$, we have

$$\frac{\partial v}{\partial r} = \frac{dv}{dR}\frac{r - \rho\mu}{R}, \quad \frac{\partial v}{\partial \mu} = -\frac{dv}{dR}\frac{r\rho}{R},$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{d^2 v}{dR^2}\left(\frac{r - \rho\mu}{R}\right)^2 + \frac{dv}{dR}\left[\frac{1}{R} - \frac{(r - \rho\mu)^2}{R^3}\right],$$

$$\frac{\partial^2 v}{\partial \mu^2} = \frac{d^2 v}{dR^2}\left(\frac{r\rho}{R}\right)^2 - \frac{dv}{dR}\frac{(r\rho)^2}{R^3}.$$

If we eliminate ρ, we obtain

$$\frac{1}{R}\frac{dv}{dR} = \frac{1}{r}\frac{\partial v}{\partial r} - \frac{\mu}{r^2}\frac{\partial v}{\partial \mu}.$$

Since $R^2 = (r - \rho\mu)^2 + \rho^2(1 - \mu^2)$, we have

$$\frac{d^2 v}{dR^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1 - \mu^2}{r^2}\frac{\partial^2 v}{\partial \mu^2}.$$

Substituting the formulas for dv/dR and d^2v/dR^2 into (4), we obtain the partial differential equation

$$r^2\frac{\partial^2 v}{\partial r^2} + (2\nu + 1)r\frac{\partial v}{\partial r} + r^2 v + (1 - \mu^2) - \frac{\partial^2 v}{\partial \mu^2}(2\nu + 1)\mu\frac{\partial v}{\partial \mu} = 0. \tag{5}$$

In the present case the addition formula (1) has the form

$$v(R) = \sum_{n=0}^{\infty} f_n(r)g_n(\rho)h_n(\mu) \qquad (\mu = \cos\theta). \tag{6}$$

We can determine the forms of $f_n(r)$, $g_n(\rho)$ and $h_n(\mu)$ by the requirement that each term of (6) satisfies (5). For this purpose we look for particular bounded solutions of (5) by the method of separation of variables, taking

$$v = f(r)h(\mu). \tag{7}$$

Substituting (7) into (5), we obtain

$$\frac{r^2 f'' + (2\nu + 1)rf' + r^2 f}{f} = \frac{-(1 - \mu^2)h'' + (2\nu + 1)\mu h'}{h} = \lambda, \tag{5a}$$

where λ is a constant. Hence we obtain an equation of hypergeometric type for $h(\mu)$,

$$(1-\mu^2)h'' - (2\nu+1)\mu h' + \lambda h = 0.$$

Its solutions for $\lambda = n(n+2\nu)$ are the Jacobi polynomials $P_n^{(\nu-1/2,\nu-1/2)}(\mu)$. Hence it is reasonable to take

$$h_n(\mu) = P_n^{(\nu-1/2,\nu-1/2)}(\mu)$$

in (6). Then (6) becomes the expansion of $v(R)$ in a series of Jacobi polynomials:

$$v(R) = \sum_{n=0}^{\infty} a_n(r,\rho) P_n^{(\nu-1/2,\nu-1/2)}(\mu). \tag{8}$$

Since $v(R)$ satisfies the hypotheses of the theorem on expansions in series of the Jacobi polynomials $P_n^{(\nu-1/2,\nu-1/2)}(\mu)$ for $\nu > -1/2$ (see §8), we have

$$\begin{aligned} a_n(r,\rho) &= \frac{1}{d_n^2}\int_{-1}^{1} v(R)(1-\mu^2)^{\nu-1/2} P_n^{(\nu-1/2,\nu-1/2)}(\mu)d\mu \\ &= \frac{1}{d_n^2}\int_{-1}^{1} \frac{u_\nu(R)}{R^\nu}(1-\mu^2)^{\nu-1/2} P_n^{(\nu-1/2,\nu-1/2)}(\mu)d\mu, \end{aligned}$$

where d_n^2 is the squared norm of the Jacobi polynomial.

It remains to show that the coefficients $a_n(r,\rho)$ can be represented in the form

$$a_n(r,\rho) = f_n(r)g_n(\rho).$$

To establish this, we integrate (5) over $(-1,1)$ after first multiplying by $(1-\mu^2)^{\nu-1/2}P_n^{(\nu-1/2,\nu-1/2)}(\mu)$, and eliminate the terms in $\partial^2 v/\partial\mu^2$ and $\partial v/\partial\mu$ by integration by parts. Since

$$\left[(1-\mu^2)\frac{\partial^2 v}{\partial\mu^2} - (2\nu+1)\mu\frac{\partial v}{\partial\mu}\right](1-\mu^2)^{\nu-1/2} = \frac{\partial}{\partial\mu}\left[(1-\mu^2)^{\nu+1/2}\frac{\partial v}{\partial\mu}\right],$$

we have

$$\int_{-1}^{1}\left[(1-\mu^2)\frac{\partial^2 v}{\partial\mu^2}-(2\nu+1)\mu\frac{\partial v}{\partial\mu}\right](1-\mu^2)^{\nu-1/2}P_n^{(\nu-1/2,\nu-1/2)}(\mu)d\mu$$

$$= (1-\mu^2)^{\nu+1/2}\frac{\partial v}{\partial\mu}P_n^{(\nu-1/2,\nu-1/2)}(\mu)\Big|_{-1}^{1}$$

$$-\int_{-1}^{1}\frac{\partial v}{\partial\mu}(1-\mu^2)^{\nu+1/2}\frac{d}{d\mu}P_n^{(\nu-1/2,\nu-1/2)}(\mu)d\mu$$

$$= (1-\mu^2)^{\nu+1/2}\left[\frac{\partial v}{\partial\mu}P_n^{(\nu-1/2,\nu-1/2)}(\mu)-v\frac{d}{d\mu}P_n^{(\nu-1/2,\nu-1/2)}(\mu)\right]\Big|_{-1}^{1}$$

$$+\int_{-1}^{1}v\frac{d}{d\mu}\left[(1-\mu^2)^{\nu+1/2}\frac{d}{d\mu}P_n^{(\nu-1/2,\nu-1/2)}(\mu)\right]d\mu.$$

Since $\nu+\frac{1}{2}>0$, the integrated terms vanish. Moreover, it follows from the equation of the Jacobi polynomials that

$$\frac{d}{d\mu}\left[(1-\mu^2)^{\nu+1/2}\frac{d}{d\mu}P_n^{(\nu-1/2,\nu-1/2)}(\mu)\right]$$
$$= -n(n+2\nu)(1-\mu^2)^{\nu-1/2}P_n^{(\nu-1/2,\nu-1/2)}(\mu).$$

Hence we obtain a differential equation for $a_n(r,\rho)$:

$$\frac{\partial^2 a_n}{\partial r^2}+\frac{2\nu+1}{r}\frac{\partial a_n}{\partial r}+\left[1-\frac{n(n+2\nu)}{r^2}\right]a_n=0.$$

As we would expect, this agrees with (5a) when $\lambda=n(n+2\nu)$.

The last equation is a special case of Lommel's equation (14.4). The only solution that is bounded as $r\to 0$ is, up to a factor independent of r, the function $r^{-\nu}J_{\nu+n}(r)$, i.e.

$$a_n(r,\rho)=r^{-\nu}J_{\nu+n}(r)g_n(\rho).$$

Consequently

$$\begin{aligned} a_n(r,\rho) &= r^{-\nu}J_{\nu+n}(r)g_n(\rho)\\ &= \frac{1}{d_n^2}\int_{-1}^{1}R^{-\nu}u_\nu(R)(1-\mu^2)^{\nu-1/2}P_n^{(\nu-1/2,\nu-1/2)}(\mu)d\mu, \end{aligned} \tag{9}$$

where d_n^2 is the squared norm of the Jacobi polynomial. To find $g_n(\rho)$ we calculate the integral on the right-hand side of (9) by using the Rodrigues formula for the Jacobi polynomials,

$$P_n^{(\nu-1/2,\nu-1/2)}(\mu) = \frac{(-1)^n}{2^n n!}\frac{1}{(1-\mu^2)^{\nu-1/2}}\frac{d^n}{d\mu^n}[(1-\mu^2)^{n+\nu-1/2}],$$

and integrating by parts n times:

$$\int_{-1}^{1} R^{-\nu}u_\nu(R)(1-\mu^2)^{\nu-1/2}P_n^{(\nu-1/2,\nu-1/2)}(\mu)d\mu$$

$$= \frac{1}{2^n n!}\int_{-1}^{1}(1-\mu^2)^{n+\nu-1/2}\frac{\partial^n}{\partial\mu^n}[R^{-\nu}u_\nu(R)]d\mu.$$

The integrated terms vanish at ± 1 since the factor $(1-\mu^2)$ enters with positive exponents.

On the other hand,

$$\frac{\partial}{\partial\mu}v(R) = -\frac{r\rho}{R}\frac{dv}{dR}$$

for every function $v(R)$, whence

$$\frac{\partial^n}{\partial\mu^n}\left[\frac{u_\nu(R)}{R^\nu}\right] = (r\rho)^n\left(-\frac{1}{R}\frac{d}{dr}\right)^n\left[\frac{u_\nu(R)}{R^\nu}\right].$$

By the differentiation formula (4) of §15 we have

$$\left(-\frac{1}{R}\frac{d}{dR}\right)^n\left[\frac{u_\nu(R)}{R^\nu}\right] = \frac{u_{\nu+n}(R)}{R^{\nu+n}}.$$

We therefore obtain

$$\int_{-1}^{1} R^{-\nu}u_\nu(R)(1-\mu^2)^{\nu-1/2}P_n^{(\nu-1/2,\nu-1/2)}(\mu)d\mu$$

$$= \frac{1}{2^n n!}(r\rho)^n\int_{-1}^{1} R^{-(\nu+n)}u_{\nu+n}(R)(1-\mu^2)^{n+\nu-1/2}d\mu.$$

Hence, by (9),

$$g_n(\rho)\frac{J_{\nu+n}(r)}{r^{\nu+n}} = \frac{\rho^n}{2^n n! d_n^2}\int_{-1}^{1}\frac{u_{\nu+n}(R)}{R^{\nu+n}}(1-\mu^2)^{n+\nu-1/2}d\mu. \tag{10}$$

Let $r \to 0$. Then $R \to \rho$ and consequently

$$\frac{g_n(\rho)}{2^{\nu+n}\Gamma(\nu+n+1)} = \frac{u_{\nu+n}(\rho)}{\rho^\nu}\frac{1}{2^n n! d_n^2}\int_{-1}^{1}(1-\mu^2)^{n+\nu-1/2}d\mu.$$

Since (see §5,Table 1)

$$d_n^2 = \frac{2^{2\nu-1}\Gamma^2(n+\nu+1/2)}{n!(n+\nu)\Gamma(n+2\nu)},$$

and

$$\int_{-1}^{1}(1-\mu^2)^{n+\nu-1/2}d\mu = 2\int_{0}^{1}(1-\mu^2)^{n+\nu-1/2}d\mu$$

$$= \int_{0}^{1}(1-t)^{n+\nu-1/2}t^{-1/2}dt = \frac{\Gamma(n+\nu+1/2)\Gamma(1/2)}{\Gamma(n+\nu+1)},^*$$

we finally obtain

$$g_n(\rho) = \frac{\sqrt{\pi}}{2^{\nu-1}}\frac{(n+\nu)\Gamma(n+2\nu)u_{\nu+n}(\rho)}{\Gamma(n+\nu+1/2)\rho^\nu}.$$

Then (8) takes the form

$$\frac{u_\nu(R)}{R^\nu} = \frac{\sqrt{\pi}}{2^{\nu-1}}\sum_{n=0}^{\infty}\frac{(n+\nu)\Gamma(n+2\nu)}{\Gamma(n+\nu+1/2)}\frac{J_{\nu+n}(r)}{r^\nu}\frac{u_{\nu+n}(\rho)}{\rho^\nu}P_n^{(\nu-1/2,\nu-1/2)}(\mu). \tag{11}$$

If we use the Gegenbauer polynomials

$$C_n^\nu(\mu) = \frac{(2\nu)_n}{(\nu+1/2)_n}P_n^{(\nu-1/2,\nu-1/2)}(\mu)$$

---

* The integrand is even; the substitution $t=\mu^2$ and the connection between the beta and gamma functions (Appendix A) yield the value of the integral.

instead of the Jacobi polynomials, the expansion (11) takes the simpler form

$$\frac{u_\nu(R)}{R^\nu} = 2^\nu\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)\frac{J_{\nu+n}(r)}{r^\nu}\frac{u_{\nu+n}(\rho)}{\rho^\nu}C_n^\nu(\mu). \tag{12}$$

We recall that the formula was obtained for $\nu > -1/2$, $r < \rho$.

Evidently (12) remains valid if we make the substitutions $R \to kR$, $r \to kr$, $\rho \to k\rho$, i.e.

$$\frac{u_\nu(kR)}{(kR)^\nu} = 2^\nu\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)\frac{J_{\nu+n}(kr)}{(kr)^\nu}\frac{u_{k+n}(k\rho)}{(k\rho)^\nu}C_n^\nu(\mu). \tag{13}$$

Formula (13) is *Gegenbauer's addition formula.*

The Graf and Gegenbauer formulas were obtained under certain restrictions on the parameters. However, (3) and (13) can be extended to a wider range of parameters by the principle of analytic continuation.

3. Expansion of spherical and plane waves in series of Legendre polynomials. Let us consider some corollaries of Gegenbauer's addition formula. These are often applied, for example in quantum-mechanical scattering theory, for solving diffraction problems.

1) In (13) put $\nu = 1/2$, $u_\nu(z) = H_\nu^{(1)}(z)$, and use the explicit expression for $H_{1/2}^{(1)}(z)$. We obtain

$$\frac{e^{ikR}}{r} = i\pi\sum_{n=0}^{\infty}(n+1/2)\frac{J_{n+1/2}(kr)}{r^{1/2}}\frac{H_{n+1/2}^{(1)}(k\rho)}{\rho^{1/2}}P_n(\mu).$$

We have used the identity $C_n^{1/2}(\mu) = P_n(\mu)$, where $P_n(\mu)$ is the Legendre polynomial.

2) An interesting limiting form of the addition theorem is obtained from (13) with $u_\nu(z) = H_\nu^{(2)}(z)$ by letting $\rho \to \infty$. We have

$$R = \rho\left(1 - \frac{2r\mu}{\rho} + \frac{r^2}{\rho^2}\right)^{1/2} = \rho - r\mu + O(1/\rho),$$

$$\lim_{\rho\to\infty}\frac{(k\rho)^{-\nu}H_{\nu+n}^{(2)}(k\rho)}{(kR)^{-\nu}H_\nu^{(2)}(kR)} = \lim_{\rho\to\infty}(R/\rho)^{\nu+1/2}i^n e^{-ik(\rho-R)} = i^n e^{-ikr\mu}.$$

Hence we find from (13) that

$$e^{ikr\mu} = 2^\nu\Gamma(\nu)\sum_{n=0}^{\infty}i^n(\nu+n)\frac{J_{\nu+n}(kr)}{(kr)^\nu}C_n^\nu(\mu).$$

When $\nu = 1/2$, it is then easy to obtain the expansion of a plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ in Legendre polynomials:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \left(\frac{2\pi}{kr}\right)^{1/2} \sum_{n=0}^{\infty} i^n \left(n + \frac{1}{2}\right) J_{n+1/2}(kr) P_n(\mu). \tag{14}$$

Here $\mathbf{k}$ is the wave vector, $\mu = \cos\theta$, and θ is the angle between $\mathbf{k}$ and $\mathbf{r}$.

§ 19 Semiclassical approximation (WKB method)

The transition from the classical physics of the late nineteenth century to the quantum mechanics of the early twentieth century is exemplified by the problem of finding uniformly asymptotic solutions of differential equations of the form

$$[k(x)y']' + \lambda r(x)y = 0 \tag{1}$$

as $\lambda \to +\infty$. We describe such approximate solutions as *semiclassical approximations* [L1]. The initial investigations of Wentzel, Kramers and Brillouin were subsequently extended by Langer and many others. Semiclassical approximations are useful in many problems of mathematical physics.

1. Semiclassical approximation for the solutions of equations of second order. Let us study the behavior of the solutions of an equation of the form

$$[k(x)y']' + \lambda r(x)y = 0 \tag{1}$$

as $\lambda \to +\infty$. We see from simple examples with $k(x) =$ const. or $r(x) =$ const. that the behavior of the solutions depends in an essential way on the signs of $k(x)$ and $r(x)$. Consequently we are going to discuss (1) in regions where $k(x)$ and $r(x)$ have constant signs. We first consider the case when $k(x)$ and $r(x)$ have the same sign, say $k(x) > 0$ and $r(x) > 0$, and we suppose that these functions have continuous first and second derivatives.

1°. Let us try to reduce (1) to a simpler form by the substitutions

$$y(x) = \phi(x)u(s), \qquad s = s(x) \tag{2}$$

If we substitute (2) into (1), we see that (1) becomes

$$u'' + f(s)u' + [\lambda g(s) - q(s)]u = 0, \tag{1a}$$

where

$$f(s) = \frac{2\,k(x)s'(x)\,\phi'(x) + [k(x)s'(x)]'\phi(x)}{k(x)\,\phi(x)[s'(x)]^2},$$

$$g(s) = \frac{r(x)}{k(x)[s'(x)]^2}, \qquad q(s) = -\frac{[k(x)\,\phi'(x)]'}{k(x)\,\phi(x)[s'(x)]^2}.$$

It will be convenient to choose the functions $s(x)$ and $\phi(x)$ so that $g(s) \to 1$ and $f(s) \to 0$ as $\lambda \to +\infty$, i.e.

$$[s'(x)]^2 = \frac{r(x)}{k(x)},$$
$$\frac{\phi'(x)}{\phi(x)} = -\frac{1}{2}\frac{[k(x)s'(x)]'}{k(x)s'(x)} = -\frac{1}{4}\frac{[k(x)r(x)]'}{k(x)r(x)}. \tag{3}$$

Then equation (1) assumes the standard form

$$u'' + [\lambda - q(s)]u = 0. \tag{4}$$

Here

$$q(s) = -\frac{[k(x)\phi'(x)]'}{r(x)\phi(x)}.$$

If we use (3) for $\phi(x)$, we can write

$$q(s) = \frac{k}{4r}\left[\left(\frac{k'}{k} + \frac{r'}{r}\right)' + \left(\frac{3}{4}\frac{k'}{k} - \frac{1}{4}\frac{r'}{r}\right)\left(\frac{k'}{k} + \frac{r'}{r}\right)\right]. \tag{5}$$

If $k(x)$ and $r(x)$ have different signs on (a, b), (2) takes (1) to a form similar to (4), namely

$$u''(s) - [\lambda + q(s)]u(s) = 0. \tag{6}$$

Since the behavior of the solutions of (6) as $\lambda \to +\infty$ can be studied by the same methods as for (4), we shall consider only the case when (1) has been carried into (4) by the substitutions (2).

In (3) we have

$$s(x) = \int_{x_0}^{x} [r(t)/k(t)]^{1/2} dt \quad (a < x_0 < b), \quad \phi(x) = [k(x)r(x)]^{-1/4}.$$

Let $s(a) = c$ $(c < 0)$ and $s(b) = d$ $(d > 0)$. Then $s(x)$ is continuous and monotone increasing on (a, b). Hence it has an inverse $x = x(s)$ which is monotone increasing and continuous on (c, d), and $q(s)$ is continuous on (c, d).

2°. It is natural to expect that as $\lambda \to +\infty$ the solutions of (4) will agree in the limit with the solutions of the simplified equation

$$u'' + \lambda u = 0,$$

i.e. that as $\lambda \to +\infty$ we will have the approximate equation

$$u(s) \approx A \cos \mu s + B \sin \mu s,$$

where $\mu = \sqrt{\lambda}$ and A and B are constants.

We can verify this conjecture by a method that was proposed by Steklov [S5]. We solve the equation

$$u'' + \mu^2 u = q(s)u \tag{3a}$$

by variation of parameters, treating the right-hand side as known. We obtain

$$u(s) = \bar{u}(s) + R_\mu(s), \tag{7}$$

where

$$\bar{u}(s) = A \cos \mu s + B \sin \mu s,$$

$$R_\mu(s) = \frac{1}{\mu} \int_0^s \sin \mu(s - s') q(s') u(s') ds'.$$

Let us show that when $c < c_1 \le s \le d_1 < d$ $(c_1 < 0, d_1 > 0)$ we can neglect $R_\mu(s)$ in (7) as $\mu \to \infty$, i.e.

$$\lim_{\mu \to \infty} \frac{R_\mu(s)}{\bar{M}(\mu)} = 0, \tag{8}$$

where $\bar{M}(\mu) = \max\limits_{c_1 \le s \le d_1} |\bar{u}(s)|$. It is clear from the formula for $R_\mu(s)$ that

$$|R_\mu(s) \le \frac{1}{\mu} L M(\mu), \tag{9}$$

where

$$L = \int_{c_1}^{d_1} |q(s')| ds', \quad M(\mu) = \max_{c_1 \le s \le d_1} |u(s)|.$$

We estimate $M(\mu)$ as $\mu \to \infty$. From (7) and (9) we have

$$|u(s)| \leq \bar{M}(\mu) + \mu^{-1} L M(\mu),$$

whence

$$M(\mu) \leq \bar{M}(\mu) + \mu^{-1} L M(\mu).$$

If we solve this inequality for $M(\mu)$ and use (9) for $\mu > L$, we obtain

$$\frac{R_\mu(s)}{\bar{M}(\mu)} \leq \frac{L}{\mu - L},$$

which establishes (8).

Returning to the original variables, we see that when $k(x) > 0$ and $r(x) > 0$ on (a, b) the solutions of (1) have the representation

$$y(x) \approx \frac{1}{\sqrt{k(x)p(x)}}[A \cos \xi(x) + B \sin \xi(x)], \tag{10}$$

as $\lambda \to +\infty$, on every interval $[a_1, b_1] \subset (a, b)$; here

$$p(x) = \left(\lambda \frac{r(x)}{k(x)}\right)^{1/2}, \quad \xi(x) = \int_{x_0}^{x} p(t)dt.$$

The semiclassical method of solving (1) consists of replacing the solution of (1) by the approximate solution (10).

When $k(x) > 0$ and $r(x) < 0$, we find similarly that

$$y(x) \approx \frac{1}{\sqrt{k(x)p(x)}}[Ae^{\xi(x)} + Be^{-\xi(x)}], \tag{10a}$$

where

$$p(x) = \left(\lambda \left|\frac{r(x)}{k(x)}\right|\right)^{1/2}, \quad \xi(x) = \int_{x_0}^{x} p(t)dt.$$

In replacing the exact solution by an approximate solution, what is significant is only the inequality $|q(s)| \ll \mu$ in (4). Consequently the approximate solutions (10) and (10a) can be used not only in cases when λ is large, but also when $\lambda \sim 1$ provided that $|q(s)| \ll 1$. As we see from (5), this is the case when $k(x)$ and $r(x)$ have small derivatives, i.e. when the coefficients of (1) vary slowly and smoothly. We note that logarithmic derivatives of the functions $k(x)$ and $r(x)$, as well as derivatives of the corresponding logarithmic derivatives, enter into the right-hand side of formula (5). Here the summand

$$\frac{k}{4r}\left(\frac{3}{4}\frac{k'}{k} - \frac{1}{4}\frac{r'}{r}\right)\left(\frac{k'}{k} + \frac{r'}{r}\right)$$

will seem to dominate. Hence the smallness condition

$$\left|\frac{k}{4r}\left(\frac{3}{4}\frac{k'}{k} - \frac{1}{4}\frac{r'}{r}\right)\left(\frac{k'}{k} + \frac{r'}{r}\right)\right| \ll 1 \tag{5a}$$

must first be fulfilled. Condition (5a) is somewhat cruder than the condition $|q(s)| \ll 1$. However, if we consider the Schrödinger equation in the form

$$\frac{d^2\psi}{dx^2} + p^2(x)\psi = 0, \quad p^2(x) = 2[E - U(x)],$$

then the condition (5a) for this equation will coincide with that for applicability of the semiclassical approximation $|p'(x)/p^2(x)| \ll 1$, which is widely used in quantum mechanics. For this it is sufficient to set $k(x) = 1$ and $r(x) = p^2(x)$ in (5a) provided that $\lambda \approx 1$.

3°. It is also of practical interest to find an approximate solution of (1) that is valid up to the endpoints of (a, b) as $\lambda \to +\infty$. Consider, for example, the problem of an approximate representation of (1) for $a \le x < b$. If $k(a) > 0$ and $r(a) > 0$, all the reasoning that led to (10) remains valid. Hence we need to consider the case when at least one of the functions $k(x)$ and $r(x)$ is zero at $x = a$. Let, for example,

$$k(x) = (x-a)^m\, k_0(x), \quad r(x) = (x-a)^l r_0(x),$$

where $k_0(a) > 0$, $r_0(a) > 0$ and $k_0(x)$ and $r_0(x)$ have continuous second derivatives for $a \le x < b$. We assume that $(l-m)/2 > -1$ so that $s(a)$ will be finite. In this case the expressions for $s(x)$ and $q(s)$ are

$$s(x) = \int_a^x \left(\frac{r_0(t)}{k_0(t)}\right)^{1/2} (t-a)^{(l-m)/2} dt, \tag{11}$$

$$\begin{aligned} q(s) = (x-a)^{m-l-2}\frac{k_0(x)}{4r_0(x)}\bigg\{&\frac{(l+m)(3m-l-4)}{4} \\ &+ \frac{x-a}{2}\left[(3m+l)\frac{k_0'}{k_0} + (m-l)\frac{r_0'}{r_0}\right] \\ &+ (x-a)^2\left[\left(\frac{k_0'}{k_0} + \frac{r_0'}{r_0}\right)' + \left(\frac{3}{4}\frac{k_0'}{k_0} - \frac{1}{4}\frac{r_0'}{r_0}\right)\left(\frac{k_0'}{k_0} + \frac{r_0'}{r_0}\right)\right]\bigg\}. \end{aligned} \tag{12}$$

If $x \approx a$, we have

$$s(x) \approx \left(\frac{r_0(a)}{k_0(a)}\right)^{1/2} \frac{(x-a)^{(l-m+2)/2}}{\frac{1}{2}(l-m+2)}$$

and consequently we can represent $q(s)$ in the form

$$q(s) = \frac{\nu^2 - \frac{1}{4}}{s^2} + s^{\gamma-2} f(s),$$

where

$$\gamma = \frac{2}{l-m+2} > 0, \qquad \nu = \frac{|m-1|}{l-m+2},$$

and $f(s)$ is continuous for $0 \leq s < s(b)$. We see that in the present case $q(s)$ has a singular point at $s = 0$. Hence in order to apply Steklov's method we have to separate out the singularity of $q(s)$, i.e. we write (3) in the form

$$u'' + \left(\mu^2 - \frac{\nu^2 - \frac{1}{4}}{s^2}\right) u = s^{\gamma-2} f(s) u \quad (\mu = \sqrt{\lambda}) \tag{13}$$

and solve this equation by variation of parameters, treating the right-hand side of (13) as known. Since the equation

$$u'' + \left(\mu^2 - \frac{\nu^2 - \frac{1}{4}}{s^2}\right) u = 0$$

is a Lommel equation (14.4) with solution

$$u = A v_\nu(\mu s) + B v_{-\nu}(\mu s),$$

where $v_\nu(z) = \sqrt{z} J_\nu(z)$ and A and B are constants, we obtain a solution of (13) in the form

$$u(s) = A v_\nu(\mu s) + B v_{-\nu}(\mu s) + R_\mu(s), \tag{14}$$

where

$$R_\mu(s) = \int_{s_0}^{s} K_\mu(s, s')(s')^{\gamma-2} f(s') u(s') ds',$$

$$K_\mu(s, s') = \frac{\pi}{2\mu \sin \pi\nu} [v_\nu(\mu s) v_{-\nu}(\mu s') - v_\nu(\mu s') v_{-\nu}(\mu s)].$$

It can be shown that $R_\mu(s)$ can be neglected in (14) as $\mu \to +\infty$. In estimating $R_\mu(s)$ it is convenient to take $s_0 > 0$ when $B \neq 0$ and $s_0 = 0$ when $B = 0$. The estimates can be carried out along the same lines as before, but they are rather more complicated technically because to estimate the functions $v_{\pm\nu}(\mu s)$, which appear in place of $\cos \mu s$ and $\sin \mu s$, it is necessary to consider small and large values of μs separately:

$$|v_{\pm\nu}(\mu s)| \leq \begin{cases} C(\mu s)^{\pm\nu+1/2} & \text{for } \mu s \leq 1, \\ C & \text{for } \mu s > 1 \end{cases}$$

(where C is a constant).

Returning to the original variables, we find that, in the case when $k(x) = (x-a)^m k_0(x)$ and $r(x) = (x-a)^l r_0(x)$, with $l - m + 2 > 0$, and $k_0(x)$ and $r_0(x)$ are positive and have continuous second derivatives for $a \leq x < b$, the solutions of (1) can be approximately represented, as $\lambda \to +\infty$, $a \leq x \leq b_1 < b$, in the form

$$\begin{gathered} y(x) \approx \left(\frac{\xi(x)}{k(x)p(x)}\right)^{1/2} \{AJ_\nu[\xi(x)] + BJ_{-\nu}[\xi(x)]\}, \\ p(x) = \left(\lambda \frac{r(x)}{k(x)}\right)^{1/2}, \quad \xi(x) = \int_a^x p(t)dt; \quad \nu \neq 0, 1, \ldots \end{gathered} \tag{15}$$

When ν is an integer we have to replace $J_{-\nu}(\xi)$ in (15) by $Y_\nu(\xi)$. We may observe that when $\xi(x) \gg 1$, replacing the Bessel functions in (15) by the first terms of their asymptotic expansions leads to a formula equivalent to (10). If $k(x) > 0$ and $r(x) > 0$, (15) is replaced by

$$\begin{gathered} y(x) \approx \left(\frac{\xi(x)}{k(x)p(x)}\right)^{1/2} \{AI_\nu[\xi(x)] + BK_\nu[\xi(x)]\}, \\ p(x) = \left(\lambda \left|\frac{r(x)}{k(x)}\right|\right)^{1/2}, \quad \xi(x) = \int_a^x p(t)dt. \end{gathered} \tag{16}$$

Similar formulas can be obtained for $a < x \leq b$ if $k(x)$ and $r(x)$ have the forms

$$\begin{gathered} k(x) = (b-x)^m k_0(x), \quad r(x) = (b-x)^l r_0(x), \\ k_0(x) > 0, \quad r_0(x) > 0. \end{gathered}$$

We have described a method of obtaining an asymptotic formula for the solutions of (1) as $\lambda \to +\infty$. We now turn to applications of this formula to the problems with which we are directly concerned.

2. Asymptotic formulas for classical orthogonal polynomials for large values of n. Let us obtain an approximate formula for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for large n when $\alpha \geq 0$, $\beta \geq 0$ and $x \in [-1,1]$. The function $y(x) = P_n^{(\alpha,\beta)}(x)$ satisfies the differential equation (1) with

$$k(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}, \quad r(x) = (1-x)^{\alpha}(1+x)^{\beta},$$
$$\lambda = n(n+\alpha+\beta+1).$$

In the present case $m = \beta+1$, $l = \beta$, $\nu = \beta$. If $n \to \infty$, we have $\lambda \to +\infty$. When $-1 \leq x \leq 1-\delta$, we have

$$y(x) \approx \frac{\sqrt{\xi}}{(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}}[AJ_\beta(\xi) + BJ_{-\beta}(\xi)],$$

where

$$\xi = \xi(x) = \mu \int_{-1}^{x} \frac{dt}{\sqrt{1-t^2}} = \mu \arccos(-x), \quad \mu = \sqrt{\lambda}.$$

Since the limit

$$\lim_{x \to -1} y(x) = P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)n!}$$

exists, we have

$$B = 0,$$
$$A = \lim_{x \to -1} \frac{(1-x)^{(\alpha/2)+(1/4)}(1+x)^{(\beta/2)+(1/4)}y(x)}{\sqrt{\xi}J_\beta(\xi)}$$
$$= 2^{(2\alpha+1)/4} P_n^{(\alpha,\beta)}(-1)\, 2^{\beta}\Gamma(\beta+1) \lim_{x \to -1} \left(\frac{\sqrt{1+x}}{\xi}\right)^{\beta+1/2}.$$

By L'Hospital's rule,

$$\lim_{x \to -1} \frac{\sqrt{1+x}}{\xi(x)} = \lim_{x \to -1} \frac{1}{2\sqrt{1+x}\,\xi'(x)} = \frac{1}{\sqrt{2}}.$$

Therefore

$$A = (-1)^n \frac{2^{(\alpha+\beta)/2}\Gamma(n+\beta+1)}{n!\mu^{\beta+1/2}}, \quad \mu = \sqrt{n(n+\alpha+\beta+1)}.$$

Putting $x = -\cos\theta$, we have

$$P_n^{(\alpha,\beta)}(-\cos\theta) \approx \frac{(-1)^n \Gamma(n+\beta+1)\sqrt{\theta/2}}{n!\mu^\beta (\cos(\theta/2))^{\alpha+1/2}(\sin(\theta/2))^{\beta+1/2}} J_\beta(\mu\theta) \tag{17}$$

for $0 \le \theta \le \pi - \delta$.

From (17) we can easily deduce an approximate formula for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for $-1 < x \le 1$ if we use the equation

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x).$$

If $x \in [-1+\delta, 1-\delta]$, an approximate formula for $P_n^{(\alpha,\beta)}(x)$ can be found by using the asymptotic representation of $J_\beta(\mu\theta)$ as $\mu\theta \to +\infty$ and the asymptotic representation of $\Gamma(z)$ as $z \to \infty$ (see Appendix A):

$$P_n^{(\alpha,\beta)}(\cos\theta) \approx \frac{\cos\{[n+(\alpha+\beta+1)/2]\theta - (2\alpha+1)\pi/4\}}{\sqrt{\pi n}(\sin(\theta/2))^{\alpha+1/2}(\cos(\theta/2))^{\beta+1/2}} \qquad (0 < \delta \le \theta \le \pi - \delta). \tag{18}$$

When $\alpha = \beta = 0$, this becomes an asymptotic formula for the Legendre polynomials:

$$P_n(\cos\theta) \approx \left(\frac{2}{\pi n}\right)^{1/2} \frac{\cos[(n+1/2)\theta - \pi/4]}{\sqrt{\sin\theta}}.$$

Similarly we can obtain an approximate formula for the Laguerre polynomials $L_n^\alpha(x)$ for $x > 0$ and large n. In particular, we have

$$L_n^\alpha(x) \approx \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \cos\left[2\sqrt{nx} - (2\alpha+1)\frac{\pi}{4}\right] \tag{19}$$

for $0 < \delta \le x \le N < \infty$. If $\alpha = \pm 1/2$, formula (19) is valid down to $x = 0$, since in this case $\nu = \pm 1/2$ and (13) does not have a singular point at $s = 0$.

A corresponding formula for the Hermite polynomials $H_n(x)$ can be obtained from (19) by using formulas (6.14) and (6.15), which express the Hermite polynomials in terms of Laguerre polynomials:

$$H_n(x) \approx \sqrt{2}\left(\frac{2n}{e}\right)^{n/2} e^{x^2/2} \cos\left(\sqrt{2n}\,x - \frac{1}{2}\pi n\right) \qquad (|x| \le N < \infty). \tag{20}$$

Remark 1. Inequalities (20a), (27a) and (28a) of §7 (p. 54), which were obtained there by rather complicated calculations, are easily deducible from the estimates (18)–(20).

Remark 2. We derived (18) for $\alpha \geq 0$ and $\beta \geq 0$. However, it remains valid for all α and β. We can prove this by induction. Suppose that (18) holds for $P_n^{(\alpha+1,\beta+1)}(\cos\theta)$ and $P_n^{(\alpha+2,\beta+2)}(\cos\theta)$. Applying the differential equation of the Jacobi polynomials and the differentiation formula (5.6), we obtain

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) = -\frac{1}{\lambda_n}\Bigg[&\tau(x)\frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x) \\ &+\sigma(x)\frac{(n+\alpha+\beta+1)(n+\alpha+\beta+2)}{4}P_{n-2}^{(\alpha+2,\beta+2)}(x)\Bigg], \end{aligned}$$

where

$$\lambda_n = (n+\alpha+\beta+1), \quad \tau(x) = \beta-\alpha-(\alpha+\beta+2)x, \quad \sigma(x) = 1-x^2.$$

Hence

$$\begin{aligned} &P_n^{(\alpha,\beta)}(\cos\theta) \\ &\quad = -\Bigg[\frac{\beta-\alpha-(\alpha+\beta+2)\cos\theta}{2n}P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta) \\ &\qquad +\frac{\sin^2\theta}{4}\left(1+\frac{\alpha+\beta+2}{n}\right)P_{n-2}^{(\alpha+2,\beta+2)}(\cos\theta)\Bigg]. \end{aligned}$$

Substitute the asymptotic representations for $P_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)$ and $P_{n-2}^{(\alpha+2,\beta+2)}(\cos\theta)$, as obtained from (18), into the right-hand side, retaining only the principal terms, and we find

$$\begin{aligned} P_n^{(\alpha,\beta)}(\cos\theta) &\approx -\frac{\sin^2\theta}{4}\frac{\cos\{[n-2+(\alpha+\beta+5)/2]\theta-(2\alpha+5)\pi/4\}}{\sqrt{\pi n}(\sin(\theta/2))^{\alpha+5/2}(\cos(\theta/2))^{\beta+5/2}} \\ &= \frac{\cos\{[n+(\alpha+\beta+1)/2]\theta-(2\alpha+1)\pi/4\}}{\sqrt{\pi n}(\sin(\theta/2))^{\alpha+1/2}(\cos(\theta/2))^{\beta+1/2}}. \end{aligned}$$

This agrees with (18). Similarly we can establish the validity of (19) for all real α.

3. Semiclassical approximation for equations with singular points. The central field. In discussing the motion of a particle in a central field it is useful to obtain a semiclassical approximation for an equation of the form

$$u'' + r(x)u = 0, \tag{21}$$

where $x^2r(x)$ is continuous together with its first and second derivatives for $0 \le x \le b$. The previous approximation cannot be used for (21) in a neighborhood of $x = 0$. However, the change of variables $x = e^z, u = e^{z/2}v(z)$ transforms the equation into

$$v''(z) + r_1(z)v = 0, \tag{22}$$

where

$$r_1(z) = -\frac{1}{4} + x^2r(x)\Big|_{x\,=\,e^z}.$$

As $z \to -\infty$ (which corresponds to $x \to 0$), $r_1(z)$ differs little from a constant, namely $-\frac{1}{4}+\lim_{x\to0} x^2r(x)$. Moreover, $\lim_{z\to-\infty} r_1^{(k)}(z) = 0$ $(k = 1, 2)$. Hence $r_1(z)$ and its derivatives vary slowly for negative z of large modulus, and we can apply the semiclassical approximation to (22). If the conditions of applicability of this method are satisfied for all required values of z, we can return to the original variables and obtain an approximate solution of (21) of the previous form, but with $r(x)$ replaced by $r(x) - 1/(4x^2)$.

For example, consider the solution of the Schrödinger equation in spherical coordinates for the radial part of the wave function $R(r)$,

$$-\frac{1}{2}R'' + \left[U(r) + \frac{l(l+1)}{2r^2}\right] R = ER,$$

where r is distance from the origin, $U(r)$ is the potential energy, E is the total energy of the particle, and $l = 0, 1, 2, \ldots$ are the orbital quantum numbers. In the semiclassical approximation we obtain

$$R(r) = \begin{cases} (\xi/p)^{1/2}[AJ_{1/3}(\xi) + BJ_{-1/3}(\xi)] & (r \ge r_0), \\ (\xi/p)^{1/2}[CI_{1/3}(\xi) + DK_{1/3}(\xi)] & (r \le r_0), \end{cases}$$

where

$$p = p(r) = \left|2[E - U(r)] - \frac{(l+1/2)^2}{r^2}\right|^{1/2},$$

$$\xi = \xi(r) = \left|\int_{r_0}^{r} p(r')dr'\right|,$$

and r_0 is a root (supposed simple) of the equation $p(r) = 0$.

Since $R(r)$ must be bounded as $r \to 0$, i.e. as $\xi \to \infty$, we have $C = 0$. Since $R(r)$ and $R'(r)$ must agree at $r = r_0$, we can express A and B in terms of D. If we expand the function under the square root sign in the formula for $p(r)$ in powers of $(r - r_0)$, we easily see that $p(r)/|r - r_0|^{1/2}$ and $\xi(r)/|r - r_0|^{3/2}$ and their first derivatives are continuous at $r = r_0$. Hence the joining conditions for $R(r)$ and $R'(r)$ at $r = r_0$ imply similar joining conditions for

$$\Phi(r) = (\xi/2)^{1/3}(p/\xi)^{1/2}R(r)$$

and its derivative. We have

$$\Phi(r) = \begin{cases} \dfrac{A(\xi/2)^{2/3}}{\Gamma(4/3)} + \dfrac{B}{\Gamma(2/3)} + O[(r-r_0)^3] & (r \geq r_0), \\ \dfrac{\pi D}{2\sin(\pi/3)}\left[\dfrac{1}{\Gamma(2/3)} - \dfrac{(\xi/2)^{2/3}}{\Gamma(4/3)}\right] + O[(r-r_0)^3] & (r \leq r_0). \end{cases}$$

Since $\xi^{2/3}/|r - r_0|$ is continuous at $r = r_0$, comparing coefficients of powers of $(r - r_0)$ yields

$$A = B = \frac{\pi}{\sqrt{3}}D.$$

4. Asymptotic formulas for Bessel functions of large order. Langer's formulas. The method explained above can be used to obtain asymptotic formulas for Bessel functions whose order ν is large. Let us transform the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

into the form (21) by the substitution $u(x) = \sqrt{x}y(\nu x)$ (see the Lommel equation (14.4)). Then $u(x)$ satsifies

$$u'' + r(x)u = 0, \quad r(x) = \nu^2 - \frac{\nu^2 - 1/4}{x^2}.$$

Here we may use the reasoning of Part 3, putting $x = e^z, u = e^{z/2}v(z)$. Then we obtain

$$v'' + r_1(z)v = 0, \quad r_1(z) = \nu^2(e^{2z} - 1). \tag{23}$$

Since $\nu \gg 1$, we can apply a semiclassical approximation to (23). In the original variables the approximation for $u(x)$ is

$$u(x) = \left(\frac{\xi}{p}\right)^{1/2} \begin{cases} AI_{1/3}(\xi) + BK_{1/3}(\xi) & (x \leq 1), \\ CH^{(1)}_{1/3}(\xi) + DH^{(2)}_{1/3}(\xi) & (x \geq 1). \end{cases} \tag{24}$$

Here

$$p = p(x) = \nu s/x, \quad s = |1 - x^2|^{1/2},$$

$$\xi = \xi(x) = \left| \int_1^x p(t)dt \right| = \begin{cases} \nu(\tanh^{-1} s - s) & (x \le 1), \\ \nu(s - \tan^{-1} s) & (x \ge 1). \end{cases}$$

For example, put $u(x) = \sqrt{x}H_\nu^{(1)}(\nu x)$. To determine C and D we compare the principal terms of the asymptotic formulas (for $x \to \infty$) for the two sides of (24). Since

$$s(x) = x + O(1/x), \quad \xi(x) = \nu(x - \pi/2) + O(1/x),$$

as $x \to \infty$ for fixed ν, we have

$$\begin{aligned} \sqrt{x}H_\nu^{(1)}(\nu x) &\approx \left(\frac{2}{\pi\nu}\right)^{1/2} \exp[i(\nu x - \pi\nu/2 - \pi/4)] \\ &= \left(\frac{2}{\pi\nu}\right)^{1/2} \left\{ C e^{i[\nu(x-\pi/2)-\pi/6-\pi/4]} + D e^{-i[\nu(x-\pi/2)-\pi/6-\pi/4)]} \right\}, \end{aligned}$$

from which it follows that

$$D = 0, \quad C = e^{i\pi/6}.$$

We determine A and B by the joining conditions for $u(x)$ and $u'(x)$ at $x = 1$, as in the previous example. We find

$$A = -2i, \quad B = (2/\pi)e^{-i\pi/3},$$

i.e. in the semiclassical approximation, for large ν,

$$H_\nu^{(1)}(\nu x) = \begin{cases} 2\sqrt{\dfrac{\tanh^{-1} s}{s} - 1} \left[-iI_{1/3}(\xi) + \dfrac{e^{-i\pi/3}}{\pi} K_{1/3}(\xi) \right] & (x \le 1), \\ \sqrt{1 - \dfrac{\tan^{-1} s}{s}} e^{i\pi/6} H_{1/3}^{(1)}(\xi) & (x \ge 1). \end{cases} \tag{25}$$

If we compare real parts in (25), we obtain the semiclassical approximation for $J_\nu(\nu x)$ for large ν:

$$J_\nu(\nu x) = \begin{cases} \dfrac{1}{\pi}\sqrt{\dfrac{\tanh^{-1} s}{s} - 1}\, K_{1/3}(\xi) & (x \le 1), \\ \dfrac{1}{\sqrt{3}}\sqrt{1 - \dfrac{\tan^{-1} s}{s}}\, [J_{-1/3}(\xi) + J_{1/3}(\xi)] & (x \ge 1). \end{cases} \tag{26}$$

Formulas (25) and (26) are *Langer's formulas* [L2]. More precise estimates show that they provide uniform approximations to the Bessel functions with error $O(\nu^{-4/3})$ [L2]. It is interesting that (26) gives the behavior of $J_\nu(\nu x)$ correctly as $x \to 0$, even though it was derived by using the asymptotic formulas for Bessel functions as $x \to \infty$.

5. Finding the energy eigenvalues for the Schrödinger equation in the semiclassical approximation. The Bohr-Sommerfeld formula. The solution of the Schrödinger equation

$$-\frac{1}{2}\psi''(x) + U(x)\psi(x) = E\psi(x) \quad (-\infty < x < \infty) \tag{27}$$

describing the motion of a particle in a field with potential $U(x)$ can be found explicitly for only a few special forms of $U(x)$ (E is the total energy of the particle; we use a system of units in which the mass m and Planck's constant $\hbar$ are both 1). This can be done, for example, when (27) can be reduced to a generalized equation of hypergeometric type (see the theorem of §9, part 2). However, it is more useful to have methods for solving (27) approximately for any potential $U(x)$.

Let us find the energy eigenvalues of (27) in the semiclassical approximation. We are to find the values of E for which $E - U(x) < 0$ as $x \to \pm\infty$, and $\psi(x)$ satisfies the normalizing condition

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1. \tag{28}$$

Let

$$E - U(x) \ge 0 \text{ for } x_1 \le x \le x_2$$

(this is the interval of classical motion) and let

$$E - U(x) < 0 \quad \text{for } x < x_1 \quad \text{and} \quad x > x_2,$$

where x_1 and x_2 are simple roots of the equation $E = U(x)$ (called turning points in quantum mechanics).

In solving the problem, we suppose that the integrals

$$\int_{-\infty}^{x_1} p(x)dx, \qquad \int_{x_2}^{\infty} p(x)dx$$

diverge ($p(x) = (2|E - U(x)|)^{1/2}$) and that $\int_{x_1}^{x_2} p(x)dx$ is sufficiently large. In the semiclassical approximation we have, for $-\infty < x \leq x_2$,

$$\psi(x) = \begin{cases} (\xi/p)^{1/2}[A_1 I_{1/3}(\xi) + B_1 K_{1/3}(\xi)] & \text{for } x \leq x_1, \\ (\xi/p)^{1/2}[A_2 J_{-1/3}(\xi) + B_2 J_{1/3}(\xi)] & \text{for } x_1 \leq x \leq x_2, \end{cases}$$
$$\xi = \xi(x) = \left| \int_{x_1}^{x} p(s)ds \right|. \tag{29}$$

We put $A_1 = 0$ so that $\int_{-\infty}^{\infty} |\psi(x)|^2 dx$ will converge. We can express A_2 and B_2 in terms of B_1 by using the joining conditions for $\psi(x)$ and $\psi'(x)$ at $x = x_1$ (see the example in Part 3). This yields $A_2 = B_2 = \pi B_1/\sqrt{3}$. Consequently

$$\psi(x) = A_2(\xi/p)^{1/2}[J_{-1/3}(\xi) + J_{1/3}(\xi)] \tag{30}$$

for $x_1 \leq x \leq x_2$.

We are considering values of x which make $\xi(x)$ sufficiently large. For such values of x we can use the asymptotic formula for $J_{\pm 1/3}(z)$:

$$J_{\pm 1/3}(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z \mp \frac{\pi}{6} - \frac{\pi}{4}\right).$$

We obtain

$$\psi(x) = \frac{c_1}{\sqrt{p(x)}} \cos\left[\int_{x_1}^{x} p(s)ds - \frac{\pi}{4}\right] \tag{31}$$

(c_1, a constant).

In a similar way we can obtain an expression for $\psi(x)$ for $x_1 < x \leq x_2$, by starting from the behavior of the function as $x \to \infty$. If we consider values of x for which $\int_x^{x_2} p(s)ds$ is sufficiently large, we obtain

$$\psi(x) = \frac{c_2}{\sqrt{p(x)}} \cos\left[\int_x^{x_2} p(s)ds - \frac{\pi}{4}\right] \tag{32}$$

(c_2, a constant).

Let us consider an x for which both $\int_{x_1}^{x} p(s)ds$ and $\int_x^{x_2} p(s)ds$ are large. In this case we have the two expressions (31) and (32) for $\psi(x)$. These expressions and their derivatives coincide at x only if

$$\int_{x_1}^{x} p(s)ds + \int_{x}^{x_2} p(s)ds = \int_{x_1}^{x_2} p(s)ds = \pi\left(n + \frac{1}{2}\right), \tag{33}$$

where $n = 0, 1, 2, \ldots$. It is easy to see that n is the number of zeros of $\psi(x)$ (which can have zeros only for $x_1 < x < x_2$). Moreover, $c_2 = (-1)^n c_1$. Therefore, in the semiclassical approximation, the energy values $E = E_n$ ($n = 0, 1, 2, \ldots$) in the discrete spectrum must be subject to (33). This is the Bohr-Sommerfeld condition in quantum mechanics.

We can also obtain the Bohr-Sommerfeld condition for a particle in a central field $U(r)$. Repeating the preceding reasoning and using the results of Part 3, we obtain the Bohr-Sommerfeld condition for the energy eigenvalues E_{nl} of the discrete spectrum in the form

$$\begin{gathered}\int_{r_1(E)}^{r_2(E)} p(r)dr = \pi\left(n + \frac{1}{2}\right), \\ p(r) = \left\{2\left|E - U(r) - \frac{(l+1/2)^2}{2r^2}\right|\right\}^{1/2}, \quad p(r)|_{r=r_1,r_2} = 0.\end{gathered} \tag{34}$$

Example 1. Find, in the semiclassical approximation, the energy levels of a particle in the field $u(x) = \frac{1}{2}\mu\omega^2 x^2$ (the linear harmonic oscillator).

We obtained the exact solution in §9. If we use the same units as in §9, the Schrödinger equation (27) becomes

$$-\frac{1}{2}\psi'' + \frac{1}{2}x^2\psi = \mathcal{E}\psi \quad (E = \hbar\omega\mathcal{E}).$$

In the present case

$$p(x) = \sqrt{2\mathcal{E} - x^2}, \quad x_1 = -\sqrt{2\mathcal{E}}, \quad x_2 = \sqrt{2\mathcal{E}}.$$

The energy levels are found from the Bohr-Sommerfeld condition

$$\int_{-\sqrt{2\mathcal{E}}}^{\sqrt{2\mathcal{E}}} \sqrt{2\mathcal{E} - x^2}\,dx = \pi\left(n + \frac{1}{2}\right). \tag{35}$$

Since

$$\int \sqrt{\alpha x^2 + \beta}\,dx = \frac{1}{2}x\sqrt{\alpha x^2 + \beta} + \frac{1}{2}\beta \int \frac{dx}{\sqrt{\alpha x^2 + \beta}},$$

we have

$$\int_{x_1}^{x_2} \sqrt{2\mathcal{E} - x^2}\,dx = \mathcal{E} \int_{x_1}^{x_2} \frac{dx}{\sqrt{2\mathcal{E} - x^2}} = \mathcal{E} \arcsin \frac{x}{\sqrt{2\mathcal{E}}}\Bigg|_{x_1}^{x_2} = \pi\mathcal{E},$$

From (35) we find

$$\mathcal{E} = \mathcal{E}_n = n + \frac{1}{2},$$

which agrees with the exact solution even when the condition $\int_{x_1}^{x_2} p(x)dx \gg 1$ is not satisfied.

Example 2. Find, in the semiclassical approximation, the energy levels in the field $U(r) = -Z/r$ (we are using atomic units).

In the present case we put $U(r) = -Z/r$ in (34). After integrating by parts, we find that

$$\begin{aligned}
\int_{r_1}^{r_2} p(r)dr &= rp(r)\Bigg|_{r_1}^{r_2} - \int_{r_1}^{r_2} \frac{r\{(-Z/r^2) + (l+\frac{1}{2})^2/r^3\}}{\{2[E + (Z/r) - \frac{1}{2}(l+\frac{1}{2})^2/r^2]\}^{1/2}}dr \\
&= z\int_{r_1}^{r_2} \frac{dr}{\{2[Er^2 + Zr - \frac{1}{2}(l+\frac{1}{2})^2]\}^{1/2}} \\
&= \int_{x_1}^{x_2} \frac{(l+\frac{1}{2})^2 dx}{\{2[E + Zx - \frac{1}{2}(l+\frac{1}{2})^2 x^2]\}^{1/2}} \\
&= \pi\left(\frac{Z}{\sqrt{-2E}} - l - \frac{1}{2}\right) \qquad (x = 1/r,\ E < 0).
\end{aligned}$$

Substituting this expression for the integral in (34), we obtain

$$E = E_{nl} = -\frac{Z^2}{2(n+l+1)^2},$$

which agrees with the exact value for all n and l.