§7 Qualitative behavior and asymptotic properties of the Jacobi, Laguerre and Hermite polynomials.

1. Qualitative behavior. In studying the qualitative behavior of solutions of a differential equation of the form

$$[k(x)y']' + r(x)y = 0 \tag{1}$$

on an interval where $k(x) > 0$ and $R(x) > 0$, it is convenient to use, not the oscillating function $y(x)$, but the function

$$v(x) = y^2(x) + a(x)[k(x)y'(x)]^2, \tag{2}$$

where the multiplier $a(x)$ is chosen so that we know where $v(x)$ is monotonic. For this purpose we calculate $v'(x)$ by using (1):

$$\begin{aligned} v'(x) &= 2yy' + a'(x)[k(x)y']^2 + 2a(x)k(x)[k(x)y']' \\ &= a'(x)[k(x)y']^2 + 2yy'[1 - a(x)k(x)r(x)]. \end{aligned}$$

If we take $a(x) = 1/(k(x)r(x))$ then

$$v'(x) = \left[\frac{1}{k(x)r(x)}\right]' [k(x)y']^2. \tag{3}$$

Since $[k(x)y']^2 \geq 0$, if we choose $a(x)$ in this way the intervals where $v(x)$ is increasing or decreasing are the same as those for $a(x) = 1/(k(x)r(x))$. Notice that the values of $v(x)$ and $y^2(x)$ are equal at the maxima of $y^2(x)$. This lets us find the intervals where the successive maxima of $|y(x)|$ increase or decrease.

Let us apply this transformation to describe the qualitative behavior of the classical orthogonal polynomials on the interval (a, b) of orthogonality, when $\sigma(x) \geq 0$. In this case the polynomials $y = y_n(x)$ satisfy the differential equation (1), where

$$k(x) = \sigma(x)\rho(x), \ r(x) = \lambda\rho(x), \ \lambda = \lambda_n \quad (n \neq 0). \tag{4}$$

Accordingly, we put

$$v(x) = y^2(x) + \lambda^{-1}\sigma(x)[y'(x)]^2. \tag{5}$$

Using the differential equation for $y(x)$, we find

$$v'(x) = \frac{\sigma'(x) - 2\tau(x)}{\lambda}[y'(x)]^2. \tag{6}$$

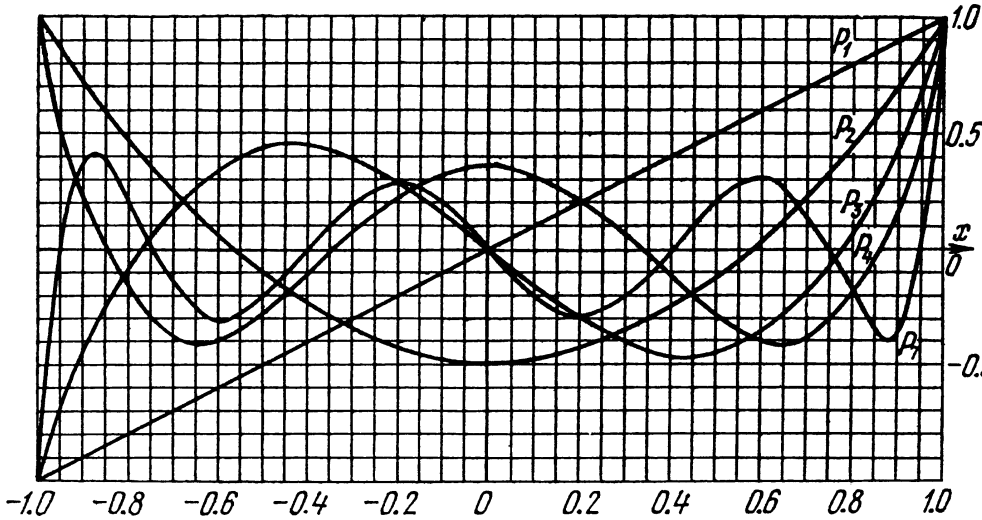

Figure 1.

It is clear from this formula that the sign of $v'(x)$ is the same as the sign of the linear polynomial $(1/\lambda)[\sigma'(x) - 2\tau(x)]$. The values of $v(x)$ and $y^2(x)$ agree at the points where $\sigma(x) = 0$, and also at the maxima of $y^2(x)$ at which $y'(x) = 0$. Hence, in an interval where $v'(x) < 0$, the successive values of $|y(x)|$ at such points will decrease, whereas when $v'(x) > 0$ they will increase.

Examples. 1) For the *Jacobi polynomials*,

$$\sigma(x) = 0 \text{ for } x = \pm 1, \; \sigma'(x) - 2\tau(x) = 2[\alpha - \beta + (\alpha + \beta + 1)x]. \tag{7}$$

Let $\alpha + 1/2 > 0$, $\beta + 1/2 > 0$; then $\lambda_n \geq 1$. When

$$-1 < x < \tilde{x} = \frac{\beta - \alpha}{\alpha + \beta + 1}$$

we have $\sigma'(x) - 2\tau(x) < 0$ and $|P_n^{(\alpha,\beta)}(x)| < |P_n^{(\alpha,\beta)}(-1)|$, and the heights of the maxima of $|P_n^{(\alpha,\beta)}(x)|$ decrease as x increases. Similarly, when $\tilde{x} < x < 1$ the heights of the successive maxima of $|P_n^{(\alpha,\beta)}(x)|$ will increase.

Therefore when $\alpha + 1/2 > 0$ and $\beta + 1/2 > 0, -1 < x < 1$, we have

$$|P_n^{(\alpha,\beta)}(x)| < \max\left[|P_n^{(\alpha,\beta)}(-1)|, |P_n^{(\alpha,\beta)}(1)|\right]. \tag{8}$$

In particular, for the Legendre polynomials, we have (see Figure 1)

$$|P_n(x)| < 1 \text{ for } -1 < x < 1. \tag{9}$$

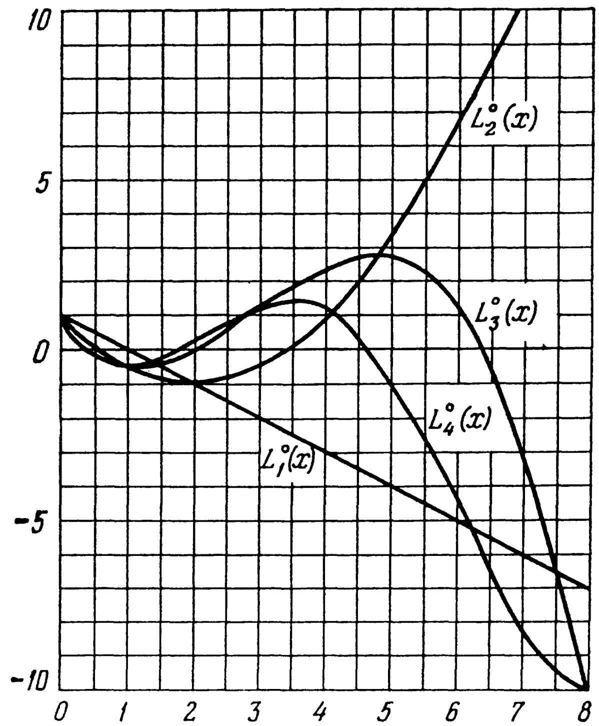

Figure 2.

2) For the *Laguerre polynomials* with $\alpha + 1/2 > 0$ and $0 < x < \tilde{x} = \alpha + 1/2$, we have $|L_n^\alpha(x)| < |L_n^\alpha(0)|$, and the heights of the successive maxima of $|L_n^\alpha(x)|$ decrease. If, however, $x > \tilde{x}$, the heights of successive maxima of $|L_n^\alpha(x)|$ increase (see Figure 2 for $L_n^\alpha(x), \alpha = 0$).

3) For the *Hermite polynomials* $\sigma'(x) - 2\tau(x) = 4x$. Therefore the heights of the successive maxima of $|H_n(x)|$ increase with increasing $|x|$.

2. Asymptotic properties and some inequalities. The preceding inequalities describe the qualitative behavior of $y = y_n(x)$ on the interval of orthogonality. We are now going to obtain some simple quantitative inequalities for the Jacobi and Laguerre polynomials. These describe more precisely how the values of the polynomials depend on n at interior points of (a, b), under the restrictions on the parameters given in part 1. Corresponding inequalities for Hermite polynomials can be obtained by using the connection between the Hermite and Laguerre polynomials, (6.14) and (6.15).*

---

* The estimates to be obtained in part 2 may be derived more easily (but less rigorously) by means of a quasiclassical approximation (see §19, part 2).

We start from the generalized equation of hypergeometric type

$$u'' + \frac{\tilde{\tau}(x)}{\sigma(x)}u' + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}u = 0, \tag{10}$$

which the transformation $u = \phi(x)y$ discussed in §1 transforms into the equation

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0. \tag{11}$$

We recall the connection between the coefficients of (10) and (11):

$$\begin{aligned}\tilde{\tau}(x) &= \tau(x) - 2\pi(x),\ \tilde{\sigma}(x) = \lambda\sigma(x) - q(x),\\ q(x) &= \pi^2(x) + \pi(x)[\tilde{\tau}(x) - \sigma'(x)] + \pi'(x)\sigma(x).\end{aligned}$$

Here $\pi(x)$ is the polynomial, at most of degree 1, in the differential equation

$$\phi'(x)/\phi(x) = \pi(x)/\sigma(x)$$

that determines $\phi(x)$. It is convenient to write (10) in the form

$$\sigma(x)u'' + \tilde{\tau}(x)u' + \left[\lambda - \frac{q(x)}{\sigma(x)}\right]u = 0. \tag{12}$$

In order to estimate $u(x)$ we consider the function

$$w(x) = u^2(x) + \lambda^{-1}\sigma(x)[u'(x)]^2,$$

which is similar to $v(x)$. It is evident that on the interval (a, b) we have, for either Jacobi or Laguerre polynomials,

$$|u(x)| \le \sqrt{w(x)}. \tag{13}$$

By using (12), we obtain

$$w'(x) = \frac{\sigma'(x) - 2\tilde{\tau}(x)}{\lambda}[u'(x)]^2 + \frac{2q(x)}{\lambda\sigma(x)}u(x)u'(x).$$

The simplest expression for $w'(x)$ is obtained when $\pi(x)$ is chosen so that $\sigma'(x) - 2\tilde{\tau}(x) = 0$, which leads to

$$\pi(x) = \frac{1}{4}[2\tau(x) - \sigma'(x)].$$

In this case

$$w'(x) = \frac{2q(x)}{\lambda\sigma(x)}u(x)u'(x). \tag{14}$$

It follows from the evident inequality $2ab \le a^2+b^2$ (a and b any real numbers) that

$$2\left(\frac{\sigma(x)}{\lambda}\right)^{1/2} uu' \le u^2 + \frac{\sigma(x)}{\lambda}[u'(x)]^2 = w(x).$$

Consequently by (14)

$$w'(x) \le \frac{|q(x)|}{\sqrt{\lambda}\sigma^{3/2}(x)} w(x).$$

Hence when $x \ge x_0$ we have

$$w(x) = w(x_0)\exp\left[\int_{x_0}^{x} \frac{w'(s)}{w(s)} ds\right] \le w(x_0)\exp\left[\int_{x_0}^{x} \frac{|q(s)|}{\sqrt{\lambda}\sigma^{3/2}(s)} ds\right] \tag{15}$$

and consequently by (13)

$$|u(x)| \le \sqrt{w(x_0)}\exp\left[\int_{x_0}^{x} \frac{|q(s)|}{2\sqrt{\lambda}\sigma^{3/2}(s)} ds\right]. \tag{16}$$

Let us apply (16) to the Jacobi polynomials. By the symmetry relation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x),$$

it is enough to estimate the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for $-1 < x \le 0$. For these values we have

$$\sigma(x) = 1 - x^2 \ge 1 + x,$$

whence

$$\frac{|q(x)|}{2\sqrt{\lambda}\sigma^{3/2}(x)} \le \frac{A_1}{2\sqrt{\lambda}(1+x)^{3/2}},$$

where $A_1 = \max\limits_{-1\le x\le 0} |q(x)|$. For $-1 < x_0 \le x \le 0$ we then find from (16) that

$$|u(x)| \le \sqrt{w(x_0)}\exp\left[\frac{A_1}{\sqrt{\lambda(1+x_0)}}\right].$$

It is evident that we cannot take $x_0 = -1$ here. Let us determine x_0 from the equation $\sqrt{\lambda(1+x_0)} = C$, where C is a constant independent of n (since

$\lambda = \lambda_n$ does depend on n, the number x_0 also depends on n). Now we have

$$|u(x)| \le A_2 \sqrt{w(x_0)} \tag{17}$$

(A_2 is independent of n).

To estimate $w(x_0)$ we use the connection between $w(x)$ and the Jacobi polynomial $y(x) = P_n^{(\alpha,\beta)}(x)$. We have $u(x) = \phi(x)y(x)$, where $\phi(x)$ is a solution of

$$\frac{\phi'}{\phi} = \frac{\pi(x)}{\sigma(x)}, \quad \pi(x) = \frac{1}{4}[2\tau(x) - \sigma'(x)].$$

Since

$$\frac{\pi(x)}{\sigma(x)} = \frac{1}{2}\frac{\tau(x)}{\sigma(x)} - \frac{1}{4}\frac{\sigma'(x)}{\sigma(x)} = \frac{1}{2}\frac{(\sigma\rho)'}{\sigma\rho} - \frac{1}{4}\frac{\sigma'}{\sigma},$$

we have

$$\phi(x) = \left[\sigma(x)\rho^2(x)\right]^{1/4}, \quad u(x) = \left[\sigma(x)\rho^2(x)\right]^{1/4} y(x),$$

whence

$$\begin{aligned} w(x) &= u^2(x) + \frac{\sigma(x)}{\lambda}[u'(x)]^2 \\ &= \sqrt{\sigma(x)\rho^2(x)}\left[y^2(x) + \left(\frac{2\tau(x) - \sigma'(x)}{4\sqrt{\lambda\sigma(x)}}y + \sqrt{\frac{\sigma(x)}{\lambda}}y'\right)^2\right]. \end{aligned} \tag{18}$$

In estimating $w(x)$ we shall depend on the inequalities already established for $-1 < x < \tilde{x} = (\beta - \alpha)/(\alpha + \beta + 1)$,

$$\begin{gathered} v(x) \le v(-1) = y^2(-1), \\ 0 \le 2\tau(x) - \sigma'(x) < 2\tau(-1) - \sigma'(-1), \end{gathered}$$

and the evident inequalities (see (5))

$$|y(x)| \le \sqrt{v(x)}, \quad \sqrt{\sigma(x)/\lambda}\,|y'(x)| \le \sqrt{v(x)}.$$

If we use these inequalities and the inequality $\sqrt{\lambda\sigma(x_0)} \ge \sqrt{\lambda(1 + x_0)} = c$, we can deduce from (18) that

$$w(x_0) \le A_3 \sqrt{\sigma(x_0)\rho^2(x_0)}\, y^2(-1)$$

for $x_0 < \tilde{x}$, where

$$A_3 = 1 + \left(\frac{2\tau(-1) - \sigma'(-1)}{4C} + 1\right)^2.$$

The condition $x_0 < \tilde{x}$ will be satisfied automatically for $n \geq 1$ if we take $C < \sqrt{1+\tilde{x}}$, since $\lambda = \lambda_n \geq 1$ for $n \geq 1$.

Using the relation

$$\lim_{z\to\infty} \frac{\Gamma(z+a)}{\Gamma(z)z^a} = 1, \quad |\arg z| \leq \pi - \delta \tag{18a}$$

(see Appendix A, formula (26)), formula (5.9), and the equation

$$\sqrt{\lambda(1+x_0)} = C$$

which determines x_0, we can easily see that the numbers

$$\sqrt{n}[\sigma(x_0)\rho^2(x_0)]^{1/4}|y(-1)|$$

are bounded, uniformly in n. Hence it follows from the inequality for $w(x_0)$ and from (17) that when $\alpha + 1/2 > 0$ and $\beta + 1/2 > 0$,

$$(1-x)^{(\alpha/2)+(1/4)}(1+x)^{(\beta/2)+(1/4)}\left|P_n^{(\alpha,\beta)}(x)\right| \leq \frac{A}{\sqrt{n}}, \tag{19}$$

where the number A is independent of n.

Inequality (19) has been established for $x \geq x_0$. However, it is immediately clear from the behavior of the polynomials $y(x)$ for $-1 \leq x < \tilde{x}$ that (19) remains valid for $x < x_0$, since in this interval

$$[\sigma(x)\rho^2(x)]^{1/4}|y(x)| \leq [\sigma(x_0)\rho^2(x_0)]^{1/4}|y(-1)|,$$

and the numbers $\sqrt{n}\,[\sigma(x_0)\rho^2(x_0)]^{1/4}|y(-1)|$ are bounded, uniformly in n. Consequently (19) holds for $-1 \leq x \leq 0$. From the equation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x),$$

it is easily seen that (19) is also valid for $0 \leq x \leq 1$.

For the Jacobi polynomials, since

$$d_n^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)},$$

we find, by again using (18a), that the numbers nd_n^2 are bounded. Consequently the inequality (19) for the Jacobi polynomials can be rewritten in the form

$$(1-x)^{(\alpha/2)+(1/4)}(1+x)^{(\beta/2)+(1/4)}d_n^{-1}|P_n^{(\alpha,\beta)}(x)| \leq C_1 \tag{20}$$

(the constant C_1 is independent of n).

By the same method as for the Jacobi polynomials, using (15) and putting $\sqrt{\lambda x_0} = C$, we can obtain the following inequality for the Laguerre polynomials $L_n^\alpha(x)$ for $0 \le x \le 1$, $\alpha + 1/2 > 0$:

$$\sqrt{\frac{w(x)}{d_n^2}} \le \frac{C_2}{n^{1/4}}, \tag{21}$$

$$x^{(\alpha/2)+(1/4)} e^{-x/2} d_n^{-1} |L_n^\alpha(x)| \le \left\{ w(x)/d_n^2 \right\}^{1/2} \le C_2/n^{1/4} \tag{22}$$

(the constant C_2 is independent of n).

Inequality (16) is quite poor for the Laguerre polynomials as $x \to +\infty$, since the right-hand side of (15) grows exponentially as $x \to +\infty$, whereas the left-hand side decreases exponentially. The inequality can be improved by using (14) in the following way. Since

$$\sqrt{\lambda^{-1}\sigma(x)}|u'(x)| \le \sqrt{w(x)},$$

we have

$$w'(x) \le \frac{2|q(x)|}{\sqrt{\lambda}\sigma^{3/2}(x)} |u(x)| \sqrt{w(x)},$$

i.e.

$$\frac{d}{dx}\sqrt{w(x)} \le \frac{|q(x)u(x)|}{\sqrt{\lambda}\sigma^{3/2}(x)} = \frac{|q(x)|\sqrt{\rho(x)}|y(x)|}{\sqrt{\lambda}\sigma^{5/4}(x)}. \tag{23}$$

Hence, for $x > 1$,

$$\begin{aligned} \sqrt{w(x)} &= \sqrt{w(1)} + \int_1^x \frac{d}{ds}\left[\sqrt{w(s)}\right] ds \\ &\le \sqrt{w(1)} + \int_1^x \frac{|q(s)|\sqrt{\rho(s)}|y(s)|}{\sqrt{\lambda}\sigma^{5/4}(s)} ds \,. \end{aligned}$$

Applying Schwarz's inequality,* we obtain

$$\sqrt{w(x)} \le \sqrt{w(1)} + \frac{1}{\sqrt{\lambda}} \sqrt{\int_1^x \frac{q^2(s)ds}{\sigma^{5/2}(s)} \int_0^\infty y^2(s)\rho(s)ds},$$

* In Russian, the Cauchy-Bunyakovsky inequality. Although Bunyakovsky's priority is unquestioned, Western readers are likely to recognize the inequality more easily as Schwarz's inequality.–Translator

whence

$$[\sigma(x)\rho^2(x)]^{1/4}\frac{|y(x)|}{d_n} = \frac{|u(x)|}{d_n} \le \sqrt{\frac{w(x)}{d_n^2}}$$
$$\le \sqrt{\frac{w(1)}{d_n^2}} + \frac{1}{\sqrt{\lambda}}\sqrt{\int_1^x \frac{q^2(s)ds}{\sigma^{5/2}(s)}}. \tag{24}$$

By (21), we have

$$\sqrt{\frac{w(1)}{d_n^2}} \le \frac{C_2}{n^{1/4}}. \tag{25}$$

Since $q(x)$ is a quadratic polynomial, and $\sigma(x) = x$, there is a constant C_3, independent of n, such that

$$\sqrt{\int_1^x \frac{q^2(s)ds}{\sigma^{5/2}(s)}} \le C_3 x^{5/4}. \tag{26}$$

Combining (24), (25) and (26), we obtain, for $x > 1$ and $\alpha + 1/2 > 0$,

$$x^{(\alpha/2)+(1/4)}e^{-x/2}\frac{|L_n^\alpha(x)|}{d_n} \le \frac{C_2}{n^{1/4}} + \frac{C_3 x^{5/4}}{n^{1/2}}. \tag{27}$$

It is easily verified that this inequality holds for all $x \ge 0$ (see (22)).

It is interesting to observe that (27) is also valid for $\alpha + 1/2 = 0$, since the Laguerre polynomials have

$$q(x) = \frac{1}{4}x^2 - \left(\alpha + \frac{1}{2}\right)x + \frac{1}{4}\left(\alpha^2 - \frac{1}{4}\right)\Big|_{\alpha=-1/2} = \frac{1}{4}x^2$$

in this case. Consequently the point $x = 0$ is not singular in (23), and in integrating (23) we may take the lower limit at $x = 0$ and immediately obtain (27).

An inequality for Hermite polynomials can be obtained from (27) with $\alpha = \pm 1/2$ by using formulas (6.14) and (6.15):

$$e^{-x^2/2}\frac{|H_n(x)|}{d_n} \le \frac{C_2}{n^{1/4}} + \frac{C_3 x^{5/2}}{n^{1/2}}. \tag{28}$$

Remark. If $x \in [x_1, x_2]$, where $a < x_1 < x_2 < b$, the following simpler inequalities are consequences of (20), (27) and (28):

$$\frac{|P_n^{(\alpha,\beta)}(x)|}{d_n} \leq C_1 \quad \left(\alpha + \frac{1}{2} > 0,\ \beta + \frac{1}{2} > 0\right), \tag{20a}$$

$$|L_n^{\alpha}(x)|/d_n \leq C_2/n^{1/4} \quad \left(\alpha + \frac{1}{2} > 0\right), \tag{27a}$$

$$|H_n(x)|/d_n \leq C_3/n^{1/4} \tag{28a}$$

(the constants C_1, C_2, C_3 evidently depend on x_1, x_2 and the parameters α, β).

We can show that (20a) and (27a) remain valid for arbitrary real values of α and β. Let us show, for example, that (20a) is valid, or (what amounts to the same thing) that

$$\sqrt{n}|P_n^{(\alpha,\beta)}(x)| \leq c, \tag{19a}$$

where c is a constant.

The proof is by induction. We assume that (19a) holds for $P_n^{(\alpha+1,\beta+1)}(x)$ and $P_n^{(\alpha+2,\beta+2)}(x)$. From the differential equation of the Jacobi polynomials and the differentiation formulas (5.6), we have

$$\begin{aligned}
\sqrt{n}P_n^{(\alpha,\beta)}(x) &= -\frac{\sqrt{n}}{\lambda_n}\left[\tau(x)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) + \sigma(x)\frac{d^2}{dx^2}P_n^{(\alpha,\beta)}(x)\right] \\
&= -\frac{\beta - \alpha - (\alpha+\beta+2)x}{2\sqrt{n}}P_{n-1}^{(\alpha+1,\beta+1)}(x) \\
&\qquad - \frac{1-x^2}{4}\left(1 + \frac{\alpha+\beta+2}{n}\right)\sqrt{n}P_{n-2}^{(\alpha+2,\beta+2)}(x).
\end{aligned}$$

Since (19a) holds for $P_{n-1}^{(\alpha+1,\beta+1)}(x)$ and $P_{n-2}^{(\alpha+2,\beta+2)}(x)$, we obtain (19a) for $P_n^{(\alpha,\beta)}(x)$.

Similarly we can prove (27a) for all real values of α.