Let

$$I_n \;=\; n\int_0^1 \frac{x^n}{1+x}\,dx.$$

Evaluate the refined limit

$$L \;=\; \lim_{n\to\infty} n\Big(I_n - \tfrac{1}{2}\Big).$$

1. **Boundary layer and scaling.** For large n, x^n concentrates near $x=1$, so most mass comes from a thin region close to $x=1$.

2. **Substitution $x = e^{-t/n}$.** Then $x^n = e^{-t}$ and $dx = -\frac{1}{n}e^{-t/n}\,dt$. As $x: 0 \to 1$, we have $t: \infty \to 0$. Hence

$$I_n = \int_0^\infty \frac{e^{-t}}{1+e^{-t/n}}\,dt.$$

3. **Expand the slow factor.** Let $\varepsilon = \frac{1}{n}$. For fixed t,

$$e^{-t/n} = 1-\varepsilon t+\frac{\varepsilon^2 t^2}{2}-\frac{\varepsilon^3 t^3}{6}+O(\varepsilon^4), \quad 1+e^{-t/n} = 2-\varepsilon t+\frac{\varepsilon^2 t^2}{2}+O(\varepsilon^3).$$

Invert:

$$\frac{1}{1+e^{-t/n}} = \frac{1}{2}\cdot\frac{1}{1-\frac{\varepsilon t}{2}+\frac{\varepsilon^2 t^2}{4}+O(\varepsilon^3)} = \frac{1}{2}\left(1+\frac{\varepsilon t}{2}\right)+O(\varepsilon^2).$$

4. **Domination (to justify integrating the expansion).** For all n and $t \ge 0$,

$$0 \le \frac{e^{-t}}{1+e^{-t/n}} \le e^{-t},$$

and e^{-t} is integrable on $[0,\infty)$. The remainder is $O(\varepsilon^2)e^{-t}(1+t^2)$, also integrable uniformly in n.

5. **Integrate term by term.**

$$I_n = \int_0^\infty e^{-t}\left(\frac{1}{2}+\frac{t}{4n}\right)dt+O\left(\frac{1}{n^2}\right) = \frac{1}{2}\underbrace{\int_0^\infty e^{-t}dt}_{=1}+\frac{1}{4n}\underbrace{\int_0^\infty te^{-t}dt}_{=1}+O\left(\frac{1}{n^2}\right).$$

Thus

$$I_n = \frac{1}{2}+\frac{1}{4n}+O\left(\frac{1}{n^2}\right).$$

6. **Take the limit.**

$$n\left(I_n - \frac{1}{2}\right) = \frac{1}{4} + O\left(\frac{1}{n}\right) \longrightarrow \boxed{\frac{1}{4}}.$$