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The Racah-Wigner Algebra in Quantum Theory

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The Racah–Wigner Algebra in Quantum Theory

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

GIAN-CARLO ROTA

Foreword

The study of the symmetries of physical systems remains one of the principal contemporary theoretical activities. These symmetries, which basically express the geometric structure of the physical system in question, must be clearly analyzed in order to understand the dynamical behavior of the system. The analysis of rotational symmetry, and the behavior of physical quantities under rotations, is the most common of such problems. Accordingly, every professional physicist must achieve a good working knowledge of the "theory of angular momentum."

In addition, the theory of angular momentum is the prototype of continuous symmetry groups of many types now found useful in the classification of the internal symmetries of elementary particle physics. Much of the intuition and mathematical apparatus developed in the theory of angular momentum can be transferred with little change to such research problems of current interest.

If there is a single essential book in the arsenal of the physicist, it is a good book on the theory of angular momentum. I have worn out several earlier texts on this subject and have spent much time checking signs and Clebsch-Gordan coefficients. Such books are the most borrowed and least often returned. I look forward to a long association with the present fine work.

A good book on the theory of angular momentum needs to be thoroughly reliable yet must develop the material with insight and good taste in order to lay bare the elegant texture of the subject. Originality should not be erected in opposition to current practices and conventions if the text is to be truly useful.

The present text, written by two well-known contributors to the field, satisfies all these criteria and more. Subtleties and scholarly comments are presented clearly yet unobtrusively. Moreover, the footnotes contain fascinating historical material of which I was previously unaware. The two chapters on the "theory of turns" and "boson calculus" are significant new additions to the pedagogical literature on angular momentum. Much of the theory of turns presented here was developed by the authors. By means of this approach the concept of "double group" is made very clear. The development of the boson calculus employs Gel'fand patterns in an essential way, in addition to the more traditional Young tableaux. This section provides an excellent prototype for the analysis of all compact groups.

The representation theory is developed in the complete detail required for physical applications. This exposition of the lore of rotation matrices is especially thorough, including the Euler angle parametrization as well as others of practical value.

The text ends with a long chapter on applications well chosen to illustrate the power of the general techniques. The book concludes with a masterly development of the group theoretical description of the spectra of spherical top molecules. To my mind the recent experimental confirmation of this theory in high resolution laser spectrometry experiments is one of the most spectacular confirmations of quantum theory.

The present text is really a book for physicists. Nevertheless, the theory generates substantial material of interest for mathematicians. Recent research (for example in non-Abelian gauge field theory) has produced topics of common interest to both mathematicians and physicists. Some of the more interesting mathematical outgrowths of the theory of angular momentum are developed in the companion volume currently in press.

PETER A. CARRUTHERS
General Editor, Section on Mathematics of Physics

Preface

"The art of doing mathematics," Hilbert[1] has said, "consists in finding that *special* case which contains all the germs of *generality*." In our view, angular momentum theory plays the role of that "special case," with symmetry—one of the most fruitful themes of modern mathematics and physics—as the "generality." We would only amend Hilbert's phrase to include physics as well as mathematics. In the Preface to the second edition of his famous book *Group Theory and its Applications to the quantum Mechanics of Atomic Spectra*, Wigner[2] records von Laue's view of how remarkable it is that "almost all the rules of [atomic] spectroscopy follow from the symmetry of the problem." The symmetry at issue is *rotational symmetry*, and the spectroscopic rules are those implied by *angular momentum conservation*. In this monograph, we have tried to expand on these themes.

The fact that this monograph is part of an encyclopedia imposes a responsibility that we have tried to take seriously. This responsibility is rather like that of a library. It has been said that a library must satisfy two disparate needs: One should find the book one is looking for, but one should also find books that one had no idea existed. We believe that much the same sort of thing is true of an encyclopedia, and we would be disappointed if the reader did not have both needs met in the present work. To accomplish this objective, we have found it necessary to split our monograph into two volumes, one dealing with the "standard" treatment of angular momentum theory and its applications, the other dealing in depth with the fundamental concepts of the subject and the interrelations of angular momentum theory with other areas of mathematics.

Fulfilling this responsibility further, we have made an effort to address readers who seek *very* detailed answers on *specific* points—hence, we have a large index, and many notes and appendices—as well as readers who seek an overview of the subject, especially a description of its unique and appealing aspects. This accounts for the uneven level of treatment which varies from chapter to chapter, or even within a chapter, quite unlike a

[1]Quoted in M. Kac, "Wiener and Integration in Function Spaces," *Bull. Amer. Math. Soc.* **72** (1966), p. 65. (The italics are in the original; Kac notes that the statement may be apocryphal.)

[2]E. P. Wigner, *Group Theory and Its Applications to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York, 1959, p. v. (We have added in brackets the word "atomic," since this was clearly von Laue's intended meaning.)

textbook with its uniformly increasing levels of difficulty. The variation in the treatment applied particularly to the Remarks. Quite often these Remarks contain material that has not been developed or explained earlier. Such material is intended for the advanced reader, and it can be disregarded by others. We urge the reader to browse and skip, rather than trying, at first, any more systematic approach.

These considerations apply also to the applications. Some applications may be almost too elementary, whereas others are at the level of current research. The field of applications is so broad that we have surely failed to do justice in many cases, but we do hope that the treatment of some applications is successful.

In discussing a particular subject, we have given more detail than is usual in mathematical books, where terseness is considered the cardinal virtue. Here we have followed the precepts of Littlewood[3] who points out that "*two* trivialities omitted can add up to an *impasse*."

Let us acknowledge one idiosyncrasy of our treatment: We have not explicitly used the methods of group theory, per se, but have proceeded algebraically so that the group theory, if it appears at all, appears naturally as the treatment develops. No doubt this method of treatment is an overreaction to the censure—(now disappearing?)—with which many physicists greeted the *Gruppenpest*.[4] In any event, we think that this treatment does make the material more accessible to some readers.

Let us make some brief suggestions as to how to use the first volume, *Angular Momentum in Quantum Physics* (AMQP). Part I: (i) Chapters 2 and 3 and parts of Chapter 6 constitute the standard treatment of angular momentum theory and will suffice for many readers who wish to learn the mechanics of the subject. The methods used are elementary (but by no means imprecise), and the whole treatment flows from the fundamental commutation relations of angular momentum. (ii) Chapters 4 and 5 are recommended to the reader who wishes a general overview of the subject with methods capable of great generalization. Paradoxically, although these two chapters contain much new material, this material also belongs to the very beginnings of the subject—in the multiplication of forms of Clebsch and Gordan, and in the ξ-η calculus of Weyl—all of which are now incorporated under the rubric of the "boson calculus." Part II: The applications given in Chapter 7 are totally independent of one another, and can be understood from the results given in Chapter 3.

The second volume, *Racah–Wigner Algebra in Quantum Theory* (RWA), is also presented in two parts. (The Contents for RWA appears also at the

[3] J. E. Littlewood, *A Mathematician's Miscellany*, Methuen and Co., London, 1953, p. 30. (The italics are in the original.)

[4] B. G. Wybourne, "The Gruppenpest yesterday, today, and tomorrow," *International Journal of Quantum Chemistry*, Symposium No. 7 (1973), pp. 35–43.

beginning of AMQP.) Part I: In Chapters 2, 3, and 4 the algebra of the operators associated with the two basic quantities in angular momentum theory—the Wigner and Racah coefficients—is developed within the framework of the algebra of bounded operators acting in Hilbert space. These chapters are intended to rephrase the concept of a "Wigner operator" (tensor operator) in algebraic terms, using methods from Gel'fand's development of Banach algebras. This approach to angular momentum theory is rather new, and is intended for the reader who wishes to pursue the subject from the viewpoint of mathematics. Part II: The twelve topics developed in Chapter 5 establish diverse interrelations between concepts in angular momentum theory and other areas of mathematics. These topics are independent of one another, but do draw for their development on the material of Chapter 3 of AMQP, and to a lesser extent on Chapters 1–3 of RWA. This material should be of interest to both mathematicians and physicists.

Acknowledgments

This monograph could not have been completed without the extensive help of friends and colleagues. Professor L. P. Horwitz performed the vital chore of a thorough reading and criticism of the entire two volumes; his help is gratefully acknowledged.

Other colleagues who have helped us by reading and criticizing particular chapters, the applications, or topics in *Angular Momentum in Quantum Physics* and in *Racah–Wigner Algebra in Quantum Theory* are: (i) AMQP. D. Giebink, Chapters 2–4; Professors H. Bacry and B. Wolf, Chapter 4; Professor R. Rodenberg and Dr. M. Danos, Chapters 2–6; Professor B. R. Judd, Chapter 7, Section 5; Drs. H. W. Galbraith, C. W. Patterson, and B. J. Krohn, Chapter 7, Section 10; (ii) RWA. Professor L. Michel, Chapters 2–4; Drs. H. Ruch and R. Petry, Topic 2; Dr. M. M. Nieto, Professors M. Reed, N. Mukunda, and H. Bacry, Topic 7; Professor T. Regge, Topic 9; Dr. B. J. Krohn, Topic 10; Dr. C. W. Patterson and Professor J. Paldus, Topic 12. Dr. W. Holman read both volumes to assist us with the indexing and suggested many improvements. Further acknowledgment of help from those not mentioned here is indicated in the relevant chapters.

In a more general way, we are indebted also to Professors H. van Dam, E. Merzbacher, A. Bohm, and Dr. N. Metropolis for discussions and help extending over several years.

This monograph is dedicated to Professor Eugene P. Wigner, whose picture (courtesy of the Niels Bohr Library of the American Institute of Physics) appears as the frontispiece. We not only acknowledge the inspiration of Wigner's research, but also record his personal encouragement and help, and we are honored that he has accepted this dedication. We wish also to acknowledge our great indebtedness to the late Professor Giulio Racah, who encouraged our work at its most critical time, the very beginning. Professor Ugo Fano also helped us greatly in this same period.

Special thanks are due to Professor Gian-Carlo Rota, editor of this Encyclopedia, for encouraging us to write at length on the subject of this monograph.

Most monographs begin as course notes and lectures series. The present monograph is no exception and evolves from such notes and lectures given over the years. We are particularly indebted to Dr. H. William Koch for urging us to write up the lectures on angular momentum theory (based on the concept of the turn) given at the National Bureau of Standards. This was finally achieved many years later, when a much expanded version was presented at Canterbury University (Christchurch, New Zealand, in 1973) and supported by an Erskine Fellowship arranged by Professor Brian Wybourne.

Without the patience and helpful attitude of the editorial and production staffs of the Advanced Book Program of Addison-Wesley, and the free-lance copy editor, Dorothea Thorburn, it would not have been possible to split our original manuscript into the present two volumes.

The entire typing of the original manuscript and several of its revisions were carried through, courageously and without flinching, by Nancy Simon. Her loyalty to this task, extending over several years, is especially appreciated. Thanks are also given to Lena Diehl for typing many of the final revisions. We are also indebted to Graphic Arts and Illustration Services of the Los Alamos Scientific Laboratory for the figures and reproductions.

The writing of a monograph can be traumatic to others besides the authors. We wish to thank our wives, Sarah Biedenharn and Marge Louck, for their unfailing support and encouragement in keeping us at our task.

L. C. BIEDENHARN
J. D. LOUCK

Introduction

1. This book is a sequel to its authors' recently published *Angular momentum in quantum physics: Theory and application*; it treats various advanced topics that could not be covered in the earlier volume without making it inconveniently long. My purpose is to explain the subject matter from a mathematician's point of view, but it would be awkward and difficult to do this without taking into account the contents of both books. Thus, in spite of its tardy appearance, this essay will, in effect, be an introduction to the two-volume work as a whole.

When a physicist speaks of "angular momentum theory," he is alluding to a theory that a mathematician would be more likely to describe as "the theory of rotational invariance." This theory, whatever we call it, is concerned (a) with a technique for exploiting the fact that many physical laws are independent of orientation in space and (b) with the many important consequences of this fact.

The physicists' choice of the words "angular momentum theory" illustrates a tendency that is one of the many factors inhibiting communication between mathematicians and physicists. This is the tendency physicists have to avoid thinking in the abstract and instead to keep a concrete physical problem constantly in mind and use physical terminology whenever possible. From the mathematician's point of view, the physicist is behaving like a beginner who will not take the step from "two oranges and two oranges is four oranges" to "two plus two equals four." The physicist is much less practiced in abstract thinking and is quite properly reluctant to give up an important source of intuition and inspiration.

But what is the connection between rotational invariance and angular momentum that inspires this terminology? It derives from a fundamental

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theorem in mechanics—both classical and quantum—setting up a natural one-to-one correspondence between certain "one-parameter symmetry groups" on the one hand, and "integrals of the motion" on the other. The group $\mathfrak{E}$ of all rigid motions of Euclidean space defines an action of $\mathfrak{E}$ on the phase space Ω of an n-particle system, and for each x in $\mathfrak{E}$ the associated one-to-one map of Ω on Ω is a "symmetry" of the system in an obvious sense. If $s \to \alpha_s$ is any one-parameter subgroup of $\mathfrak{E}$—that is, any continuous homomorphism of the additive group of the real line into $\mathfrak{E}$—then this homomorphism composed with the action of $\mathfrak{E}$ defines an action of the real line on Ω, each map of which is a symmetry. Thus, one has an integral of the motion that is, a function on Ω that is constant in time) for each one-parameter subgroup α of $\mathfrak{E}$. These integrals, which are evidently of special interest, are called momentum integrals. Given a line l in space, let α_s^l denote the rotation about l through an angle of s radians. The integral of the motion corresponding to this one-parameter symmetry group is called the *total angular momentum* about the axis l. Linear momentum is defined similarly, with one-parameter groups of translations. Although the linear and angular momentum integrals were discovered long before anyone thought in terms of groups of symmetries, it is gratifying to have such an elegant a priori reason for their existence.

2. Before proceeding further, it will be useful to recall the basic structure of quantum mechanics in the rigorous form given it by von Neumann. This can be done quite concisely and completely, and a reader unfamiliar with quantum mechanics (at least in this formulation) should not hesitate to make a serious effort to understand it.

In classical mechanics the future of a system of n particles in uniquely determined by the positions and velocities of these particles at any instant of time t. The $6n$-dimensional space Ω of all possible positions and velocities of the particles is called the *phase space* of the system, and its points ω are called the *states* of the system. (For reasons that need not concern us, one actually uses positions and momenta; the momentum of a particle being the mass times its velocity.) For each positive real number t and each $\omega \in \Omega$, let $\alpha_t(\omega)$ denote that point ω' of Ω such that the positions and velocities corresponding to ω' are precisely those that describe the system t time units after it was described by the positions and velocities corresponding to ω. Then in all "reversible" systems (and we consider no others), each α_t is a one-to-one map of Ω on Ω, and setting $\alpha_{-t} = \alpha_t^{-1}$ we obtain a one-parameter group $t \to \alpha_t$ of one-to-one transformations of Ω into itself. Let us call this the *dynamical group* of the system. The parameterized curves $t \to \alpha_t(\omega)$ are the *trajectories* of the system, and we obtain a vector field X^α in Ω by assigning to each point ω the tangent vector to the unique trajectory through ω. This vector field is called the *infinitesimal generator* of the dynamical group α, and via uniqueness theorems for systems of ordinary differential equations it determines α uniquely. The unique determination of α by X^α is of the greatest importance for physics, because in most cases X^α can be written down explicitly, whereas α cannot. Thus, nontrivial mathematical problems remain to be solved after the physical law has been precisely formulated. Real valued functions on Ω—that is, functions of the coordinates and velocities—are called *observables* or *dynamical variables*. If f is an observable and $\omega \in \Omega$, then $f(\omega)$ is said to be the *value of the observable* in *the state defined by* ω. Since ω varies with time, the value of any observable f will also vary with time according to the formula $t \to f(\alpha_t(\omega))$. However, there are certain observables g that are such that $t \to g(\alpha_t(\omega))$ is a constant for all ω. These are called *integrals of the motion*, and they are precisely those functions g on Ω that are constants on the trajectories.

In quantum mechanics the states (points of Ω) are replaced by the one-dimensional subspaces of a separable complex Hilbert space $\mathcal{H}$, and the dynamical group is replaced by a continuous one-parameter group $t \to V_t$ of unitary operators mapping $\mathcal{H}$ onto $\mathcal{H}$. By a celebrated theorem of M. H. Stone (inspired by the needs of quantum mechanics), every one-parameter unitary group $t \to V_t$ may be put uniquely in the form $V_t = e^{iHt}$, where H is a (not necessarily bounded) self-adjoint operator. This operator H is the analog of the vector field X^α in classical mechanics and is what one can write down explicitly. If ϕ is a unit vector in the one-dimensional subspace specifying a state at time 0, then this state will be specified t time units later by the one-dimensional subspace containing $V_t(\phi)$, and the variable vector $t \to V_t(\phi) = \phi_t$ will satisfy the differential equation $d\phi_t/dt = iH(\phi_t)$. This (in abstract form) is the Schrödinger equation—the quantum mechanical substitute for the equations of motion of a classical mechanical system. Just as in classical mechanics, the state of a system at a future time t is uniquely determined by t and the state at time 0.

The key difference between quantum mechanics and classical mechanics lies in the fact that the number $f(\omega)$, which the state defined by ω assigns to the observable f, is replaced in quantum mechanics by a probability distribution. In every quantum mechanical state there will be observables that do not have a well-determined value. If one makes the appropriate measurements, one gets different values, but some occur much more frequently than others, and one does have a well-defined probability measure on the line. Our task now is to explain how to calculate the probability distribution of an observable $\mathcal{O}$ in a state s when we know the self-adjoint operator A defining $\mathcal{O}$ and the one-dimensional subspace L of $\mathcal{H}$ defining s. This will be the quantum mechanical substitute for $f(\omega)$. The task is quite trivial when the operator A has a pure point spectrum—that is, when $\mathcal{H}$ admits an orthonormal basis $\phi_1, \phi_2, \dots$ such that $A(\phi_j) = \lambda_j \phi_j$ for $j = 1, 2, \dots$. Let ψ be any unit vector in L. Then $\psi = \sum_{j=1}^{\infty} c_j \phi_j$, where $c_j = (\psi \cdot \phi_j)$ and

$\sum_{j=1}^{\infty} |c_j|^2=1$. Also, $|c_j|$ is independent of the choice of ψ in L. Setting $\mu_L(E)=\sum_{\lambda_j \in E} |c_j|^2$, we obtain a probability measure on the real line, and this is the probability measure assigned to the observable $\mathcal{O}$ defined by A when the system is in the state s defined by L. Note that the probability that the measurement of $\mathcal{O}$ is not one of the eigenvalues λ_j of the operator A is zero. Of course, self-adjoint operators may have continuous spectra, and then the associated probability measures will not be concentrated in countable sets—not all quantum mechanical observables are "quantized." To compute μ_L when A has a (partially or totally) continuous spectrum, it is necessary to resort to the spectral theorem. We shall not attempt to explain the spectral theory here. Readers who are familiar with the theorem will have no difficulty in adapting the above.

Although it is necessary to diagonalize A in order to compute the probability distribution of the corresponding $\mathcal{O}$ in the various states, the "expected value" of $\mathcal{O}$ can be computed directly from A and ψ. When A has a pure point spectrum so that $\mu_L(E)=\sum_{\lambda_j \in E} |c_j|^2$, it follows at once from the definition that the expected value of $\mathcal{O}$ is $\sum_{j=1}^{\infty} \lambda_j |c_j|^2$. On the other hand, if $\psi=\sum_{j=1}^{\infty} c_j\phi_j$, then

$$A(\psi)=\sum_{j=1}^{\infty} c_j A(\phi_j)=\sum_{j=1}^{\infty} c_j\lambda_j\phi_j,$$

so

$$(A(\psi)\cdot\psi)=\sum_{j=1}^{\infty} c_j\overline{c_s}\lambda_j(\phi_j\cdot\phi_j)=\sum_{j=1}^{\infty} \lambda_j|c_j|^2.$$

Thus, the expected value is just $(A(\psi)\cdot\psi)$. This result can be shown to hold even where A does not have a pure point spectrum.

Finally, let A be the self-adjoint operator defining an observable $\mathcal{O}$. Under what conditions on A shall we say that $\mathcal{O}$ is an "integral of the motion"? In classical mechanics we required that $f(\alpha_t(\omega))$ be independent of t for every ω in Ω. The obvious analog is that the probability distribution defined by A and $V_t(\psi)$ be independent of t for every unit vector ψ. This is equivalent to demanding that the probability distribution defined by $V_tAV_t^{-1}$ and ψ be independent of t for every unit vector ψ: This can be shown to happen if and only if $V_tAV_t^{-1}$ is independent of t—that is, if and only if A commutes with all V_t. Accordingly, an integral of the motion in quantum mechanics is an observable whose corresponding self-adjoint operator A commutes with all V_t. Recall that $V_t = e^{iHt}$ for some self-adjoint operator H. Evidently the observable corresponding to H is an integral of the motion and, moreover, one that plays a special role. It is a constant multiple of the quantum mechanical analog of the energy integral of classical mechanics. Note that the state defined by the unit vector ψ will be stationary—that is, independent of the time—if and only if $V_t(\psi) = e^{i\lambda t}\psi$ for some real λ and all t. On the other hand, it is easy to see that $V_t(\psi) = e^{i\lambda t}\psi$ if and only if $H(\psi) = \lambda\psi$. Thus, the stationary states are precisely the states in which the energy observable has a definite value with probability 1, the possible values being constant multiples of the eigenvalues of H. As will be explained more fully below, this fact is the key to the quantum mechanical explanation of atomic spectra. In particular, it largely reduces the problem of predicting spectral lines to finding the eigenvalues of certain self-adjoint operators.

3. With the abstract structure of quantum mechanics before us, it is possible to explain the correspondence between one-parameter symmetry groups and integrals of the motion alluded to in section 1. By definition, a symmetry of a quantum mechanical system is a pair α, β consisting of a one-to-one mapping α of the states on the states and a one-to-one mapping β of the observables on the observables such that the following two conditions are satisfied:

*For all states s and all observables $\mathcal{O}$, the probability measure in the line assigned to $\beta(\mathcal{O})$ by $\alpha(s)$ is the same as that assigned to $\mathcal{O}$ by s.
**For all states s and all real numbers t, $\tilde{V}_t(\alpha(s)) = \alpha(\tilde{V}_t(s))$, where $\tilde{V}_t$ is the map of states into states defined by the unitary operator V_t.

It is a theorem that any pair α, β that satisfies (*) is defined by an operator U that is either unitary or anti-unitary. If s corresponds to the one-dimensional subspace L, and $\mathcal{O}$ to the self-adjoint operator A, then $\alpha(s)$ corresponds to $U(L)$, and $\beta(\mathcal{O})$ to UAU^{-1}. The operator U is uniquely determined up to multiplication by a complex number of modulus 1. In order that (**) should also be satisfied, it is evidently necessary and sufficient that for each real t we have $UV_tU^{-1} = c(t)V_t$, where $c(t)$ is a complex number of modulus 1. Since the square of an anti-unitary operator is always unitary, an obvious argument shows that only unitary operators occur in one-parameter symmetry groups. A less easy argument allows one to eliminate the constant $c(s_1, s_2)$ in $U_{s_1+s_2} = U_{s_1}U_{s_2}c(s_1, s_2)$ and to show that every one-parameter symmetry group is implemented by a one-parameter unitary group $s \to U_s$. By Stone's theorem, $U_s = e^{iAs}$ for some self-adjoint operator A. The operator A is determined by the symmetry group up to an additive constant. Condition (**) is satisfied if and only if $U_sV_t = c(s,t)V_tU_s$

for all s and t, and unless V has a very special form this can be shown to imply that the U_s and V_t commute. This special form seldom if ever arises in actual physical problems. In other words, the one-parameter symmetry groups are defined by those one-parameter unitary groups $s \to U_s$ such that $U_s V_t = V_t U_s$ for all s and t. Now $U_s = e^{iAs}$, where A is a self-adjoint operator, and one shows easily that $U_s V_t = V_t U_s$ for all s and t if and only if $V_t A = A V_t$ for all t. This last condition, however, is precisely the condition that the observable corresponding to A be an integral of the motion. In other words, the one-parameter symmetry groups are just those of the form $s \to e^{iAs}$, where A varies over the self-adjoint operators corresponding to those observables that are integrals of the motion.

4. The fact that the laws of physics are independent of position and orientation in space implies the existence of certain symmetries for an isolated physical system. Let $\mathfrak{E}$ be the group generated by the translations and rotations in space. Then there will be a symmetry for each member α of $\mathfrak{E}$. Since every member of $\mathfrak{E}$ is the square of another member, these symmetries will be implemented by unitary operators, and there will exist a certain natural map $\alpha \to U_\alpha$ of $\mathfrak{E}$ into the unitary operators of the Hilbert space $\mathcal{H}$ of the system—uniquely determined up to multiplication of each U_α by a complex number c_α such that $|c_\alpha| = 1$. This mapping will be a homomorphism of $\mathfrak{E}$ into the group of symmetries and hence will have the property that $U_{\alpha\beta} = U_\alpha U_\beta \sigma(\alpha, \beta)$. Here, for each α and β, $\sigma(\alpha, \beta)$ is a complex number of modulus 1. As such, it is a so-called "projective unitary representation of $\mathfrak{E}$ with multiplier σ." If we replace each U_α by $c_\alpha U_\alpha$, then σ changes to σ', where $\sigma'(\alpha, \beta) = \sigma(\alpha, \beta) c_{\alpha\beta} / c_\alpha c_\beta$, and it is natural to try to eliminate σ by choosing the c_α properly. This can almost be done, but not quite. There is a multiplier σ_0, taking on only the values ± 1 that cannot be eliminated in this fashion. It can be shown, however, that every other σ can either be eliminated or be changed into σ_0 by suitably choosing the c_α. Actually, the most convenient way to proceed is to replace $\mathfrak{E}$ by its simply connected covering group $\tilde{\mathfrak{E}}$. This has a two-element normal subgroup Z and a homomorphism ψ onto $\mathfrak{E}$ whose kernel is Z. Defining $\tilde{U}_\alpha = U_{\psi(\alpha)}$ for all α in $\tilde{\mathfrak{E}}$, one can always choose the c_α so that $\tilde{U}_{\alpha\beta} = \tilde{U}_\alpha \tilde{U}_\beta$ for all α, β in $\tilde{\mathfrak{E}}$. When this is done, the $\tilde{U}_s$ are uniquely determined. Thus, to every "isolated" physical system one has a canonically associated unitary representation of the group $\tilde{\mathfrak{E}}$. Mild and plausible physical assumptions make it possible to prove that this representation is *continuous* in the sense that for each vector ϕ in $\mathcal{H}$ the mapping $\alpha \to \tilde{U}_\alpha(\phi)$ is a continuous function from $\tilde{\mathfrak{E}}$ to $\mathcal{H}$. Thus, the theorems of the theory of unitary group representations apply.

One can also show that $V_t \tilde{U}_\alpha = \tilde{U}_\alpha V_t$ for all t and α—that is, that the constant that the definition of symmetry permits is actually 1 for all t and α. It follows that V and $\tilde{U}$ can be combined to yield a natural unitary representation of the product group $\mathfrak{S}\times T$, where T is the group of all translations in time.

5. Let us now introduce a rectangular coordinate system in space with origin 0, and let $\tilde{K}$ denote the subgroup of $\tilde{\mathfrak{S}}$ consisting of all elements that leave 0 fixed. Then $\tilde{K}$ is the simply connected covering group of the group K of all rotations about 0, and there is a natural homomorphism of $\tilde{K}$ on K whose kernel is the two-element center of K. The group $\tilde{K}$ is isomorphic to the group $SU(2)$ of all 2×2 unitary matrices of determinant 1, and its center consists of $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. With respect to our rectangular coordinate system, we may distinguish three one-parameter subgroups of $\tilde{K}$; they are the groups of rotations about the x-, y-, and z-axes. Let α_θ^x denote the rotation through θ radians about the x-axis in some fixed sense. Then $\theta\to\alpha_\theta^x$ will be a continuous homomorphism of the additive group of the real line into $\tilde{K}$—that is, a one-parameter subgroup of $\tilde{K}$. In a similar fashion, one defines the one-parameter subgroups $\theta\to\alpha_\theta^y$ and $\theta\to\alpha_\theta^z$. Of course, if $\theta\to\beta_\theta$ is any one-parameter subgroup of $\tilde{K}$ (or, more generally, of $\tilde{\mathfrak{S}}$), then $\theta\to\tilde{U}_{\beta_\theta}$ will define a one-parameter symmetry group of our system, and by the considerations of section 3 there will correspond a well-defined integral of the motion. The integrals of the motion thus defined by the one-parameter subgroups α^x, α^y, and α^z (multiplied by a universal constant) are called the x-, y-, and z-components of the *total angular momentum* of the system. This constant is the same as the one relating the operator H in $V_t=e^{iHt}$ to the operator defining the energy observable. It occurs because in quantum mechanics, unlike classical mechanics, there is a "natural" unit of mass. More precisely, such a unit exists once units have been chosen for time and distance. If one makes use of this unit, the constant turns out to be 1 and can be ignored. However, physicists are conservative and stick to old traditions as much as possible. They still use the arbitrary mass unit of classical physics and need a conversion factor to carry them from the "natural" measures of energy and momentum to the classical ones. This constant is usually denoted by $\hbar$ and is $h/2\pi$, where h is the celebrated constant introduced by Planck in the theory of "black-body radiation," which he formulated in 1900.

Of course, every one-parameter subgroup of $\tilde{K}$ leads to an integral of the motion just as α^x, α^y, and α^z do. We do not get anything essentially new in this way, however. Every such integral is a linear combination with real coefficients of the x-, y-, and z-components of the total angular momentum. Equivalently, it is equal to a constant multiple of the total angular momentum about some axis through 0.

6. A fact about the operators describing the angular momentum observables that is of great significance for the whole theory is that they satisfy

certain simple identities called commutation relations. Specifically,

$$M_xM_y - M_yM_x = i\hbar M_z,$$
$$M_yM_z - M_zM_y = i\hbar M_x,$$
$$M_zM_x - M_xM_z = i\hbar M_y,$$

where M_x, M_y, and M_z denote the self-adjoint operators associated, respectively, with the x-, y-, and z-components of the angular momentum about 0. Note that the second and third commutation relations can be obtained from the first by cyclical permutations of x, y, and z.

From the point of view of pure mathematics, these identities are immediate consequences of the definition of M_x, M_y, and M_z and the application to the rotation group of some of the fundamental ideas of Lie's theory of continuous groups. Consider the group $GL(n, R)$ of all $n \times n$ matrices with real coefficients and determinant different from zero. If A is *any* $n \times n$ real matrix, then e^{At} is defined by the convergent infinite series $1 + At + (A^2t^2/2!) + \ldots$ for all real t, and $t \to e^{At}$ is a continuous one-parameter subgroup of $GL(n, R)$. Conversely, every continuous one-parameter subgroup of $GL(n, R)$ can be obtained from a unique A. This one-to-one correspondence between matrices A and one-parameter subgroups of $GL(n, R)$ permits one to define an "infinitesimal version" of the group $GL(n, R)$ that is easier to analyze than the group itself but reflects many of its most important properties. Moreover, the construction of this "infinitesimal version" is capable of vast generalization and is applicable to any group locally describable by finite sets of real numbers in such a way that the group operations are continuous.

Consider two one-parameter subgroups, $t \to e^{At}$ and $t \to e^{Bt}$. Their product, $t \to e^{At}e^{Bt}$, is not a one-parameter subgroup but becomes more and more like one as t is restricted to smaller and smaller values. Indeed, $e^{At}e^{Bt} = (1 + At + At^2/2! + \ldots)(1 + Bt + Bt^2/2! + \ldots) = 1 + (A+B)t + (t^2/2!)(A^2 + 2A + B^2) + \ldots$, so that $e^{At}e^{Bt}$ is approximated by $e^{(A+B)t}$ for small t. Thus, we may associate a unique "sum" $t \to e^{(A+B)t}$ to each pair of one-parameter subgroups $t \to e^{At}$ and $t \to e^{Bt}$, and under this sum the set of all one-parameter subgroups is itself a group. This group is commutative and becomes a real vector space under a definition of real multiplication that is easily defined for arbitrary one-parameter subgroups of arbitrary groups without any need for infinitesimal considerations. One simply uses the trivial fact that, if $t \to \psi(t)$ is a one-parameter subgroup, then $t \to \psi(\lambda t)$ is also a one-parameter subgroup for every real λ, and defines this to be $\lambda\psi$.

The fact that the one-parameter subgroups of $GL(n, R)$ can be made into an n^2-dimensional real vector space is not very interesting in itself. This vector space tells us nothing but the number of parameters describing the group. The significant fact is that one can define a kind of product that captures much more of the structure of the group. Consider $t \to e^{At}e^{Bt}(e^{At})^{-1}(e^{Bt})^{-1} = e^{At}e^{Bt}e^{-At}e^{-Bt}$. This also is not a one-parameter subgroup, but it becomes more and more like one as t is restricted to smaller and smaller values. Indeed, replacing e^{At} by $1+At+A^2t^2/2!+\ldots$ and $e^{B}t$ by $1+B^t+B^2t^2/2!+\ldots$, one finds that $e^{At}e^{Bt}e^{-At}e^{-Bt}=1+(AB-BA)t+t^2(\ldots)+\ldots$, so that, for small t, $t \to e^{At}e^{Bt}e^{-At}e^{-Bt}$ is approximated by the one-parameter subgroup $t \to e^{(AB-BA)t}$. This one-parameter subgroup uniquely determined by $t \to e^{At}$ and $t \to e^{Bt}$ is called the *commutator product* of these two one-parameter subgroups.

The key idea in Sophus Lie's theory of continuous groups is that for any such group one can convert the one-parameter subgroups into a finite-dimensional vector space with a "commutator product" in a strictly analogous fashion and that the resulting object, the so-called Lie algebra of the group, reflects many of its most important properties. In the particular case of the group $GL(n, R)$, the above considerations show that the Lie algebra of the group is isomorphic to the vector space of all $n \times n$ real matrices, with the commutator product $[A, B]$ being defined by $[A, B]=AB-BA$. It is easy to check that this product obeys the distributive laws

$$[A, B+C]=[A, B]+[A, C],$$
$$[A+B, C]=[A, C]+[B, C],$$

but is neither commutative nor associative. Instead of commutativity one has anticommutativity, $[A, B]=-[B, A]$, and instead of associativity one has the so-called Jacobi identity,

$$[[A, B], C]+[[C, A], B]+[[B, C], A]=0.$$

These properties persist in the general case and in fact characterize Lie algebras.

Now consider the group of rotations in three-dimensional space. It is three-dimensional and has a three-dimensional Lie algebra $\mathfrak{L}$ spanned by the one-parameter subgroups $\alpha_x, \alpha_y, \alpha_z$. Thus, every element of $\mathfrak{L}$ is uniquely of the form $\lambda_1\alpha_x+\lambda_2\alpha_y+\lambda_3\alpha_z$, and by the distributive law the commutator product may be completely described by specifying the nine products of the basis elements. Since $[A, A]=-[A, A]$, $[A, A]=0$ for all A, and it suffices to specify $[\alpha_x, \alpha_y]$, $[\alpha_y, \alpha_z]$, and $[\alpha_z, \alpha_x]$. A computation shows that

$$[\alpha_x, \alpha_y]=\alpha_z, \qquad [\alpha_y, \alpha_z]=\alpha_x, \qquad [\alpha_z, \alpha_x]=\alpha_y,$$

in evident analogy with the commutation relations for the operators defining the angular momentum observables.

Quite generally, if one has a continuous unitary representation $x \to U_x$ of a continuous group G, and $\mathfrak{L}$ is the Lie algebra of G, then each α in $\mathfrak{L}$ defines a one-parameter group of unitary operators $t \rightarrow U_{\alpha(t)}$. By Stone's theorem, one has $U_{\alpha(t)} = e^{iT_\alpha t}$, where T_α is a self-adjoint operator depending on α, and one proves easily that $(iT_\alpha)(iT_\beta) - (iT_\beta)(iT_\alpha) = iT_{[\alpha,\beta]}$ for all α and β in $\mathfrak{L}$. This implies that $T_\alpha T_\beta - T_\beta T_\alpha = (1/i)T_{[\alpha,\beta]}$ for all α and β in $\mathfrak{L}$. The commutation relations for the angular momentum operators result from applying this theorem to the commutation relations defining the Lie algebra of the rotation group.

7. It is not difficult to show that the unitary operators e^{iT}, where T varies over all angular momentum operators about a given 0, coincide exactly with the operators U_x, where $x \rightarrow U_x$ is the associated unitary representation of the rotation group. Thus, in exploiting rotational symmetry, it is often a matter of indifference whether one argues from the theory of group representations or from properties of the angular momentum operators. One arrives at much the same conclusions in either case. Physicists tend to prefer calculations with matrices—especially when these have a direct physical interpretation—to arguments involving the more abstract and conceptually more difficult theory of group representations. They refer to using "algebraic methods" to eliminate group theory, much to the astonishment of mathematicians, for whom group theory is one of the principal branches of algebra. To a physicist, however, "algebra" means computing with symbols, not the abstract conceptual arguments dear to the hearts of mathematicians.

8. The first and one of the most important applications of angular momentum theory to quantum mechanics is to the analysis of atomic spectra. Let $\mathcal{H}$ be the Hilbert space of states for the quantum mechanical system that models an atom consisting of a nucleus surrounded by N electrons. Just as in classical mechanics, one can separate the motion of the center of gravity of the system from motion relative to the center of gravity and replace the problem by one in which N electrons move in a central force field. With this reduction the problem of calculating the frequencies of the spectral lines emitted by the atom becomes that of computing the eigenvalues of that multiple of the dynamical operator that corresponds to the total energy of the system. Indeed, if $E_1 \leqslant E_2 \leqslant \ldots$ are these eigenvalues, then the possible frequencies are included among the numbers $(E_i - E_j)/h$, where h is Planck's constant. This is because a light quantum of energy $E_i - E_j$ is emitted when a "perturbation" causes the atom to shift from a stationary state with energy E_i to a continuous state of energy $E_j < E_i$ (see the last half of the last paragraph in section 2 above) and the frequency ν of the light in a quantum of energy E is such that $E = h\nu$.

In the special case in which there is only one electron (the hydrogen atom) and one neglects the effects of "spin," the operator whose eigenvalues must be found is a relatively simple partial differential operator in three variables. It is the (densely defined) operator in $\mathfrak{L}^2(R^3)$ that takes ψ into

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)-\frac{e^2\psi}{\sqrt{x^2+y^2+z^2}}.$$

Here m and $-e$ are, respectively, the mass and charge of the electron, and h is Planck's constant. For the helium atom (two electrons) the operator is only slightly more complicated. It is (densely) defined in $\mathfrak{L}^2(R^6)$ and takes ψ into

$$-\frac{h^2\psi}{8\pi^2 m}\left(\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial y_1^2}+\frac{\partial^2\psi}{\partial z_1^2}+\frac{\partial^2\psi}{\partial x_2^2}+\frac{\partial^2\psi}{\partial y_2^2}+\frac{\partial^2\psi}{\partial z_2^2}\right)$$
$$-\frac{2e^2\psi}{\sqrt{x_1^2+y_1^2+z_1^2}}-\frac{2e^2\psi}{\sqrt{x_2^2+y_2^2+z_2^2}}$$
$$+\frac{e^2\psi}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}.$$

The reader should now be able to guess the form that the operator takes when there are N electrons. The Hilbert space is $\mathfrak{L}^2(R^{3N})$, and the operator H takes ψ into

$$-\frac{h^2}{8\pi^2 m}\sum_{j=1}^{N}\left(\frac{\partial^2\psi}{\partial x_j^2}+\frac{\partial^2\psi}{\partial y_j^2}+\frac{\partial^2\psi}{\partial z_j^2}\right)$$
$$-Ne^2\psi\sum_{j=1}^{N}\frac{1}{\sqrt{x_j^2+y_j^2+z_j^2}}$$
$$+e^2\psi\sum_{i>j}\frac{1}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}}.$$

When $N=1$, the eigenvalues can be found exactly by relatively easy arguments. They are the numbers $-2\pi^2e^4m/h^2n^2$, where $n=1,2,3,\dots$; m, e, and h are as described above; and the eigenvalue $-2\pi^2e^4m/h^2n^2$ occurs with multiplicity n^2. This formula for the energy levels of the hydrogen atom was announced by Bohr in 1913 a dozen years before the discovery of quantum mechanics. He obtained it as a consequence of the assumption that the electron in a hydrogen atom moves in a circular orbit about the nucleus and that only certain "quantized" orbits occur—those in which the angular momentum is an integer multiple of $h/2\pi$. [In a circular orbit of radius r, the velocity v is a constant and is such that $e^2/r^2=mv^2/r$

(because the attraction of the nucleus for the electron must balance the centrifugal force). Thus, $mv^2=e^2/r$. Now Bohr's quantization hypothesis implies that $mvr=nh/2\pi$, and eliminating v from these two equations yields the formula $r=n^2h^2/4\pi^2e^2m$ for the radii of the allowed orbits. The total energy $mv^2/2-e^2/r=-e^2/2r$ is thus limited to the values $\dfrac{-e^2}{2(n^2h^2/4\pi^2e^2m)}=\dfrac{-2\pi^2e^4m}{h^2n^2}$ as stated.]

When N is 2 or greater, no such simple formula exists, and approximate methods have to be used. Although rotational symmetry can be effectively utilized in discovering the simple formula for the eigenvalues in the $N=1$ case, the arguments are relatively simple and elementary. It is in dealing with more than one electron that one has to make use of the more advanced and interesting parts of angular momentum theory.

9. The approximation method that one uses is based on the fact that, if only the electrons did not repel one another, the eigenvalues of an N-electron atom could be written down at once. Indeed, in that case the final term

$$e^2\psi\sum_{i>j}\frac{1}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}}$$

in the formula for H would be missing, and one checks easily that, if $\psi_1,\psi_2,\ldots,\psi_n$ are any N eigenfunctions of the operator ${}_1H$,

$$\psi\to-\left(\frac{h^2}{8\pi^2m}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)-N\frac{e^2\psi}{\sqrt{x^2+y^2+z^2}},$$

with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_N$, then

$$x_1,y_1,z_1,\ldots,x_N,y_N,z_N\to\psi_1(x_1,y_1,z_1)\psi_2(x_2,y_2,z_2)\ldots\psi_N(x_N,y_N,z_N)$$

is an eigenfunction of the modified H with eigenvalue $\lambda_1+\lambda_2+\cdots+\lambda_N$. Moreover, it is also easy to see that every eigenfunction is a finite linear combination of such, all having the same eigenvalue. The operator H' differs from the operator H in the one-electron case only in that the coefficient e^2 has been replaced by e^2N. Thus, the eigenvalues of H' can be obtained from those of H by replacing e by $e\sqrt{N}$; they are the numbers $-2\pi^2N^2e^4m/h^2n^2$.

Of course, one cannot neglect the mutual repulsion of the electrons, and one obtains a second approximation to the desired eigenvalues by using the following device. Define a family of operators H_ε by replacing the term

$$e^2\psi \sum_{i>j} \frac{1}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}}$$

$$\varepsilon e^2\psi \sum_{i>j} \frac{1}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}},$$

where ε is a real parameter. Then $H_1=H$, and H_0 is an operator whose eigenvalues have just been described and whose eigenfunctions are known. Assume that the eigenvalues and eigenfunctions of H_ε vary analytically with ε. Let λ_j be one of the known eigenvalues of H_0, and suppose that it occurs with multiplicity 1. Let ψ_j be a corresponding eigenfunction (determined up to a multiplicitive constant). Then there exist constants $\lambda_j^1, \lambda_j^2, \ldots$ and functions $\psi_j^1, \psi_j^2, \ldots$ so that, for every sufficiently small ε, $\psi_j+\varepsilon\psi_j^1+\varepsilon^2\psi_j^2+\ldots$ is an eigenfunction for H_ε with eigenvalue $\lambda_j(\varepsilon)=\lambda_j+\varepsilon\lambda_j^1+\varepsilon^2\lambda_j^2+\ldots$. Thus, if the radius of convergence of these paired series exceeds 1, $\lambda(1)=\lambda_j+\lambda_j^1+\lambda_j^2+\ldots$ will be an eigenvalue of H that one thinks of as a "perturbation" of the eigenvalue λ_j of H_0.

When our assumptions are valid, the problem reduces to computing the coefficients $\lambda_j^1, \lambda_j^2, \ldots$ of the power series expansion of $\lambda_j(\varepsilon)$. This turns out to be quite easy—at least in principle. Let us write $H_\varepsilon=H_0+\varepsilon J$, where $J=H-H_0$, and consider the equation

$$\begin{aligned}(H_0+\varepsilon J)&\left(\psi_j+\varepsilon\psi_j^1+\varepsilon^2\psi_j^2+\ldots\right)\\ &=\left(\lambda_j+\varepsilon\lambda_j^1+\varepsilon^2\lambda_j^2+\ldots\right)\left(\psi_j+\varepsilon\psi_j^1+\varepsilon^2\psi_j^2+\ldots\right).\end{aligned}$$

Expanding and equating coefficients of corresponding powers of ε, one obtains equations that may be solved iteratively to obtain explicit expressions for the λ_j^k in terms of the matrix elements of J. These are rather complicated for large k, but for $k=1$ one has the very simple expression

$$\lambda_j^1=\left(J(\psi_j)\cdot\psi_j\right).$$

Correspondingly, one thinks of $\lambda_j+\lambda_j^1$ as a first approximation to the perturbation of the eigenvalue λ_j of H_0.

Unfortunately, the assumption that λ_j has multiplicity 1 is quite unrealistic, and the simple formula $\lambda_j^1=(J(\psi_j)\cdot\psi_j)$ can seldom be used. For reasons that will be explained below, the highly symmetrical character of the operator H_0 not only makes it easy to determine its eigenvalues and eigenfunctions but also forces most of the eigenvalues to have rather high multiplicities. Although it is not difficult to modify the above argument and find an elegant generalization of the formula $\lambda'_j=(J(\psi_j)\cdot\psi_j)$, using this generalization leads to difficult new problems.

The perturbation J is much less symmetrical than the operator H_0, and this causes some of the multiplicities to decrease. Thus, an eigenvalue λ_j of H_0 whose multiplicity is ρ_j may break up into ρ_j different eigenvalues or at least into several eigenvalues of lower multiplicities when H_0 is perturbed by εJ. There will be a number of different functions of λ, all of which reduce to λ_j when $\varepsilon=0$, and one has to find not just λ'_j but perhaps as many as ρ_j different numbers $\lambda'_{j,1}\lambda'_{j,2},\ldots,\lambda'_{j,\rho_j}$. The algorithm for finding these is simple in principle. One introduces an orthonormal basis $\psi_j^1,\psi_j^2,\ldots,\psi_j^{\rho_j}$ in the ρ_j-dimensional vector space of all the eigenfunctions of H_0 of eigenvalue λ_j and calculates the matrix $\|(J(\psi_j^k)\cdot\psi_j^{k'})\|$. The eigenvalues of this matrix are then the $\lambda'_{j.k}$. Since ρ_j can be quite large, diagonalizing this matrix presents practical problems of considerable magnitude. However, J retains some of the symmetry of H_0, and this can be exploited to make extensive simplifications in the diagonalization problem. Working this out has led to a considerable body of theory, and this theory is a major part of the content of angular momentum theory insofar as it applies to atomic spectra. We shall describe it in some detail beginning in section 12 below.

10. Although little if anything has been done in the direction of extending the theory to be described below to simplify the calculation of higher-order terms, there is a way of making the approximation of first-order perturbation theory a bit less crude. Instead of writing $H=H_0+\varepsilon J$, with H_0 and J defined as above, one replaces the term $-Ne^2\psi\sum_{j=1}^{N}\frac{1}{\sqrt{x_j^2+y_j^2+z_j^2}}$ in the definition of H_0 by a term of the form $\sum_{j=1}^{N} g_N\left(\sqrt{x_j^2+y_j^2+z_j^2}\,\right)$, where g_N is chosen in such manner that the resulting new H_0 is a much better approximation to H. We need not discuss here the details of how g_N is chosen. The basic idea is to diminish the attractive force of the nucleus by a force representing the average repulsion of all the other electrons. We shall denote this modification of H_0 by H_0' and define J' as $H-H_0'$.

Of course, once H_0 has been replaced by H_0', our determination of the eigenvalues of H_0 is no longer relevant, and it must be replaced by a determination of the eigenvalues of H_0'. Just as with H_0, this reduces to determining the eigenvalues of an operator ${}_1H_0'$ in three-dimensional space, the eigenvalues of H_0 being sums $\lambda_1+\lambda_2+\cdots+\lambda_N$ of eigenvalues of ${}_1H_0'$. The operator ${}_1H_0'$ in three-dimensional space is that which takes ψ into

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)-g_N\left(\sqrt{x^2+y^2+z^2}\,\right)\psi.$$

Unlike the special case considered above in which $g_N(\sqrt{x^2+y^2+z^2})$ is $Ne^2/\sqrt{x^2+y^2+z^2}$, there is no explicit, exact formula for the eigenvalues of ${}_1H_0'$. However, they can be shown to be eigenvalues of certain *ordinary* second-order differential operators, and good approximate values can be obtained quite easily.

This reduction to ordinary differential operators is worth looking at in some detail, as it is an excellent illustration of the exploitation of rotational invariance and leads directly to an important classification of the eigenvalues of ${}_1H_0'$, which is insensitive to the exact choice of g_N. It can perhaps be understood most easily by looking first at a two-dimensional analog in which the corresponding rotation group is commutative and the application of the theory of group representations reduces to ordinary Fourier analysis.

Consider, then, the differential operator

$$\psi \to -A\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-g\left(\sqrt{x^2+y^2}\right)\psi,$$

where A is a positive constant, and h is a real valued function defined on the nonnegative real axis. Let ψ be an eigenfunction with eigenvalue λ. For each $r>0$, the restriction of ψ to the circle $x=r\cos\theta$, $y=r\sin\theta$, is a function $\theta\to\psi_r(\theta)$, which may be expanded into a Fourier series $\psi_r(\theta)=\sum_{l=-\infty}^{\infty} a_l(r)e^{il\theta}$. In this way the determination of ψ_r is reduced to the determination of the countably many functions of one variable $r\to a_l(r)$, and for each l these satisfy an ordinary differential equation. Indeed, one computes without difficulty that for any $l=0,\pm1,\pm2,\ldots$ and any differentiable f, one has

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(f(r)e^{il\theta}\right)=e^{il\theta}\left(\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{l^2f}{r^2}\right).$$

Thus, our differential operator takes $\sum_{l=-\infty}^{\infty} a_l(r)e^{il\theta}$ into

$$\sum_{l=-\infty}^{\infty}\left(-A\frac{d^2a_l}{dr^2}-\frac{A}{r}\frac{da_l}{dr}+\frac{Al^2}{r^2}a_l-ga_l\right)e^{il\theta},$$

and this will equal $\lambda\sum_{l=-\infty}^{\infty} a_l(r)e^{il\theta}$ if and only if, for each $l=0,\pm1,\pm2,\ldots,$ the function a_l is an eigenfunction with eigenvalue λ for the second-order ordinary differential operator

$$f\to\left(-A\frac{d^2f}{dr^2}-\frac{A}{r}\frac{df}{dr}+\frac{Al^2}{r^2}f-gf\right)=K^l(f).$$

Usually a given λ will be an eigenvalue for only one of the differential operators K^l, and thus it forces the corresponding eigenfunction to have the form $f(r)e^{il\theta}$, where f is the λ eigenfunction for K^l. In any case, eigenfunctions of the form $f(r)e^{il\theta}$, where f is an eigenfunction of K^l, will constitute a basis for the space spanned by all eigenfunctions.

Thus, one sees that there is a natural division of the eigenfunctions and eigenvalues of our partial differential operator into classes, with one class for each integer l, in which the members of the class l are obtained by finding the eigenfunctions and eigenvalues of the ordinary differential operator K^l.

To apply the same method to the partial differential operator of actual concern to us, we need a substitute for the functions $e^{il\theta}$ on the unit circle. For each $l=0,1,2,\ldots,$ let S^l denote the complex vector space of all homogeneous polynomials of the lth degree in three variables that satisfy Laplace's equation $\partial^2P/\partial x^2+\partial^2P/\partial y^2+\partial^2P/\partial z^2=0$. It is almost trivial to see that this vector space is $(2l+1)$-dimensional and is invariant under rotations. Moreover every element may be written uniquely in the form

$$\left(\sqrt{x^2+y^2+z^2}\right)^l P\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

The functions $\left(1/\sqrt{x^2+y^2+z^2}\right)^l P(x,y,z)$, where P is in S^l, thus define a $(2l+1)$-dimensional vector space $\tilde{S}^l$ of functions on the unit sphere called surface harmonics. This vector space can be shown to be irreducible under rotations, and the corresponding representation of the group R of all rotations in three-space is customarily denoted by the symbol D^l. Toward the end of the eighteenth century, two decades before Fourier's famous first memoir on heat conduction was sent to the French Academy, Laplace and Lagrange studied surface harmonics and showed how to write more or less general functions defined on the sphere as infinite linear combinations of surface harmonics. It turns out that surface harmonics of different degrees l are orthogonal functions, and, choosing an arbitrary orthonormal basis is each $\tilde{S}^l$, one obtains functions that in the aggregate form an orthonormal basis in the Hilbert space of all square-summable functions on the unit sphere.

Using expansions in surface harmonics as a substitute for Fourier expansions of functions on the circle, one can easily adapt the arguments given in the two-dimensional case and show that the eigenfunctions of our three-dimensional partial differential operator ${}_1H_0$ are all linear combinations of functions of the special form

$$f\left(\sqrt{x^2+y^2+z^2}\right)P\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right),$$

where P is a surface harmonic of some degree $l=0,1,2,\ldots$, and f is an eigenfunction of an ordinary second-order differential operator $\bar{K}^l$ depending on l (and also on N and g). For a given such l and f, P can be *any* surface harmonic of degree l, and so the corresponding eigenvalue will occur with multiplicity at least equal to $2l+1$.

In understanding the final result, obtained by finding the eigenvalues of the ordinary differential operators $\bar{K}^l$, it is useful to consider first the special case in which $g_N(r)=Ne^2/r$, so that $H_0'=H_0$. One finds in this case that the eigenvalues of $\bar{K}^l$ can be exactly and explicitly determined; they are equal to the numbers $-2\pi^2N^2me^4/h^2(k+l+1)^2$, where $k=0,1,2,3\ldots$, and each occurs with multiplicity 1. Correspondingly, H_0 has eigenvalues depending on the two parameters k and l, and for each l the eigenvalue $-2\pi^2N^2me^4/h^2(k+l+1)^2$ occurs $(2l+1)$ times. The actual value of the eigenvalue depends only on the sum of k and l, so that eigenvalues belonging to different l's can be equal. For a given value of $n=k+l+1$, the possible l values vary from 0 up to $n-1$. Thus, the total multiplicity of the eigenvalue $-N^2me^4/2h^2n^2$ is $1+3+5+\cdots+(n-1)=n^2$, as announced in section 8 for the case $N=1$.

When Ne^2/r is replaced by a function $g_N(r)$ that yields an H_0' giving a better approximation to H than H_0, the eigenvalues $-2\pi^2N^2me^4/h^2(k+l+1)^2$ are perturbed slightly in a manner that varies with l. However, the change is small enough so that each can be unambiguously associated with the particular $-2\pi^2N^2me^4/h^2(k+l+1)^2$ from which it came. The value of $k+l+1=n$ is called the *principal quantum number* and the value of l is called the *azimuthal quantum number* for the eigenvalue in question. We need not concern ourselves with the exact value of the eigenvalue with given quantum numbers n and l. It suffices to know that it may be written in the form $-2\pi^2N^2me^4/h^2(n+\varepsilon(l))^2$, where $|\varepsilon(l)|$ is small and does vary with l and that it occurs with multiplicity $2l+1$.

11. Before returning to the rôle of symmetry in carrying out perturbation theory for an N-electron atom alluded to at the end of section 9, we shall continue the considerations of section 10 by relating the analysis given there to the general theory of unitary group representations. This will make it possible (a) to see the methods used in the two- and three-dimensional cases from a unified point of view, (b) to understand that these methods work because of the rotational symmetry of the operators, and (c) to explain the physical significance of the azimuthal quantum number l.

Let G be a compact group, and let $\alpha\to U_\alpha$ be a continuous unitary representation of G in the separable Hilbert space $\mathcal{H}$. It follows from the compactness of G and the general theory of unitary representations that U is a discrete direct sum of subrepresentations each of which is irreducible and finite-dimensional. This means that $\mathcal{H}$ admits a sequence $\mathcal{H}_1,\mathcal{H}_2,\ldots$ of mutually orthogonal subspaces having the following properties: (a) Each $\mathcal{H}_j$

is *invariant* in the sense that $U_\alpha(\mathcal{H}_j)=\mathcal{H}_j$ for all α, and *irreducible* in the sense that no proper subspace is invariant. (b) Every element ϕ in $\mathcal{H}$ is expressible as the sum of an infinite series $\phi_1+\phi_2+\ldots$, where each $\phi_j\in\mathcal{H}_j$. Although the decomposition is not unique, it is "essentially" unique in the following sense. If $\mathcal{H}_1',\mathcal{H}_2',\ldots$ is a second such decomposition, then there exists a permutation π of the integers such that the subrepresentation of G defined by restricting U to $\mathcal{H}_j$ is for all j "equivalent" to the subrepresentation defined by restricting U to $\mathcal{H}'_{\pi(j)}$. In this connection one says that two representations V and V' in Hilbert spaces $\mathcal{H}(V)$ and $\mathcal{H}(V')$ are *equivalent* if there exists a unitary operator W mapping $\mathcal{H}(V)$ on $\mathcal{H}(V')$ such that $WV_\alpha W^{-1}=V'_\alpha$ for all α in G. In particular, it follows that for each irreducible unitary representation W the number of j for which W is equivalent to U restricted to $\mathcal{H}_j$ is the same for all decompositions. This number is called the *multiplicity* of W in U, and the representation U is said to be *multiplicity-free* if this number is 0 or 1 for all W.

There is another element of uniqueness in the decomposition of U into irreducibles. For each irreducible W whose multiplicity in U is not zero, let $\mathcal{H}_W$ denote the closed subspace spanned by *all* irreducible invariant subspaces that define subrepresentations equivalent to W. Then $\mathcal{H}_W$ is clearly invariant, and $\mathcal{H}_W$ and $\mathcal{H}_{W'}$ are orthogonal whenever W and W' are inequivalent. Evidently $\mathcal{H}$ is *uniquely* a direct sum of invariant subspaces of the form $\mathcal{H}_W$. The corresponding decomposition of U is called the canonical decomposition into primary representations—a primary representation being (by definition) a direct sum of mutually equivalent irreducible representations. Any decomposition into irreducibles is clearly a refinement of the canonical decomposition into primaries. Moreover, it is obvious that the multiplicity-free representations are precisely those in which the decomposition into irreducibles is unique and coincides with the canonical decomposition into primary representations.

At this point it is possible to make a simple, easily proved general statement whose truth is fundamental for the application of group representations to the theory of atomic spectra.

Theorem. Let U be a continuous unitary representation of the compact group G in the Hilbert space $\mathcal{H}$, and let $\mathcal{H}=\mathcal{H}_{W^1}\oplus\mathcal{H}_{W^2}\oplus\ldots$ define the canonical decomposition of U into primary representations. Let T be any self-adjoint operator in $\mathcal{H}$ that lies in the commuting algebra of U in the sense that $TU_\alpha=U_\alpha T$ for all α in G. Then (a) for each W^j, $\mathcal{H}_{W^j}$ is carried into itself by T. (b) If λ is any eigenvalue for the restriction T^j of T to $\mathcal{H}_{W^j}$, and $\mathfrak{M}_\lambda\subseteq\mathcal{H}_{W^j}$ is the corresponding eigenspace, then $\mathfrak{M}_\lambda$ is invariant under all U_α and hence is a direct sum of irreducible U-invariant subspaces in each of which U defines a representation equivalent to W^j. In particular, the dimension of $\mathfrak{M}_\lambda$ is a multiple of the dimension $d(W^j)$ of the space in which W^j acts.

It follows at once from (a) that a partial diagonalization of a self-adjoint operator is provided by the canonical decomposition into primaries of any unitary group representation that commutes with it, and from (b) that this diagonalization is essentially complete whenever the unitary group representation is multiplicity-free. It also follows from (b) that the operator is forced to have multiple eigenvalues whenever the group representation has irreducible constituents which are not one-dimensional irreducible constituents.

In this section we shall be concerned only with the applications of the theorem to the interpretation of the results of section 10. Later we shall exploit it heavily in simplifying the diagonalization of the finite-dimensional matrices that arise in the perturbation theory of many-electron atoms.

Returning to the two- and three-dimensional partial differential operators of section 10, let $M=2$ or 3, and let $\mathcal{H}$ be the Hilbert space of all square-summable complex-valued functions on Euclidean M space E^M. Let G be the group of all rotations α about the origin, and let U be the unitary representation of G such that $U_\alpha(f)(p)=f(\alpha(p))$ for each $p\in E^M$. The operators considered in section 10 are all rotationally invariant in the sense that they commute with all U_α. Thus, the theorem stated above applies, and in particular it provides a natural division of the eigenfunctions and the eigenvalues of these operators into families parameterized by those equivalence classes of irreducible unitary representations of G that actually occur in the decomposition of U.

When $M=2$, G is commutative and is isomorphic to the multiplicative group of all complex numbers of modulus 1. All irreducible unitary representations of a commutative group are one-dimensional and hence are of the form $\alpha\to\chi(\alpha)I$, where I is the identity operator, and χ is a continuous function with $|\chi(\alpha)|=1$ such that $\chi(\alpha_1\alpha_2)=\chi(\alpha_1)\chi(\alpha_2)$. For each integer l, $\chi_l(e^{i\theta})=e^{il\theta}$ is such a function, and it can be shown that there are no others. One checks easily that the decomposition of U into irreducibles contains each χ_l and contains it with multiplicity ∞. Thus, the canonical decomposition of U into primary representations carries with it a direct sum decomposition of $\mathcal{H}$ into subspaces parameterized by l and each invariant under our operator. Our original diagonalization problem is replaced by countably many others. As shown in section 10, these are much simpler, since they involve finding the eigenvalues of *ordinary* differential operators.

When $M=3$, the situation is much the same. The only added complication is that G is noncommutative and has irreducible unitary representations that are more than one-dimensional. As explained in section 10, the natural action of G on the surface harmonics of degree l gives a $(2l+1)$-dimensional example D^l of an irreducible unitary representation of G for each $l=0,1,2,\ldots$. It turns out that there are no others—every irreducible unitary representation of G is equivalent to some D^l. These all occur with multiplicity ∞, and the theorem applies as when $M=2$. Now, however, conclusion (b) comes into play and tells us that the eigenvalues in the primary component corresponding to D^l must have a multiplicity that is a multiple of $2l+1$, the dimension of D^l.

Keeping $M=3$, consider an arbitrary eigenspace $\mathfrak{M}_\lambda$ of ${}_1H_0'$. Unless the function g_N is very special, the restriction of U to the invariant subspace $\mathfrak{M}_\lambda$ will be irreducible and thus equivalent to D^l for some $l=0,1,2\ldots$, where l is the azimuthal quantum number for the eigenvalue λ. Since the subspace $\mathfrak{M}_\lambda$ is U-invariant, it is in particular invariant under all $U_{\alpha(t)}$, where $t\to\alpha(t)$ is an arbitrary one-parameter subgroup of G. Hence it is invariant under all angular momentum operators and consequently under the operator $\Omega_x^2+\Omega_y^2+\Omega_z^2=\Omega$, where Ω_x, Ω_y, and Ω_z give the angular momenta about the x-, y-, and z-axes, respectively. It is natural to think of the corresponding observable as the square of the total angular momentum. It is not surprising to find it to be independent of the orientation of the axes and equivalently that it commutes with all U_α. This and the irreducibility of U in $\mathfrak{M}_\lambda$ imply that Ω restricted to $\mathfrak{M}_\lambda$ is just multiplication by a constant. In other words, in the $(2l+1)$-dimensional vector space of state vectors having a given definite energy value with azimuthal quantum number l, the square of the total angular momentum also has a definite value. A simple computation shows that this value depends only on l and is equal to $(h^2/4\pi^2)l(l+1)$. Thus, the physical significance of the azimuthal quantum number is that it determines the total angular momentum in the associated stationary states. The principal quantum number determines the *approximate* energy.

12. Now let us return to the line of thought begun at the end of section 9 and discuss the manner in which rotational symmetry can be used to simplify the problem of diagonalizing the matrices that arise when one attempts to apply perturbation theory to the N-electron atom. Our first observation is that the relationship between the unitary representation U of the rotation group $G=SO(3)$ and the eigenspaces of ${}_1H_0'$ considered in the last section has an obvious generalization in which $SO(3)$ is replaced by a group containing $SO(3)\times SO(3)\times\cdots\times SO(3)$ (N factors) and ${}_1H_0'$ is replaced by H_0'. Let $\alpha_1,\alpha_2,\ldots,\alpha_N$ be any N members of $SO(3)$. Then there is a unique unitary operator $W^0_{\alpha_1,\alpha_2,\ldots\alpha_N}$, which takes $f_1(x_1,y_1,z_1)$, $f_2(x_2,y_2,z_2),\ldots,f_N(x_N,y_N,z_N)$ into $f_1(\alpha_1(x_1,y_1,z_1))$, $f_2(\alpha_2(x_2,y_2,z_2)),\ldots,f_N(\alpha_N(x_N,y_N,z_N))$ whenever the f_j are square-summable. The mapping $\alpha_1,\alpha_2,\ldots,\alpha_N\to W^0_{\alpha_1,\alpha_2,\ldots\alpha_N}$ is then a continuous unitary representation of $G^0=SO(3)\times SO(3)\times\cdots\times SO(3)$, and it is clear that $W^0_\alpha H_0^1=H_0^1W^0_\alpha$ for all $\alpha=\alpha_1,\alpha_2,\ldots,\alpha_N\in G^0$. In the language of the general theory of group representations, W^0 is the "outer tensor product" $U\times U\times\cdots\times U$ of N copies of U. However, W^0 is not quite the N-electron generalization of U that we need. When $N>1$, there are further symmetries. Indeed, let π be any permutation of the N integers $1,2,3,\ldots,N$. Then there is a unique unitary operator in our Hilbert space that takes

$\phi(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N)$ into $\phi(x_{\pi(1)}, y_{\pi(1)}, z_{\pi(1)}, x_{\pi(2)}, y_{\pi(2)}, z_{\pi(2)}, \ldots, x_{\pi(N)}, y_{\pi(N)}, z_{\pi(N)})$ for all square-summable ϕ. Let us denote it by W'_π. One sees at once that $\pi \to W'_\pi$ is a unitary representation of S_N, the symmetric group on N objects, and that H'_0 also commutes with all W^0_π. The two representations W^0 and W^1 of G^0 and S^N can now be combined into a single representation W of a compact group G containing both G^0 and S_N as subgroups. The group G consists of all pairs α, π, where $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_N \in G_0$ and $\pi \in S_N$. The multiplication law in G is given by $(\alpha_1, \alpha_2, \ldots, \alpha_N, \pi)(\alpha'_1, \alpha'_2, \ldots, \alpha'_N, \pi') = \alpha''_1, \alpha''_2, \ldots, \alpha''_N, \pi\pi'$, where $\alpha''_j = \alpha_j \alpha'_{\pi(j)}$, and G^0 and S^N may be identified with the subgroups of G defined by setting α (respectively π) equal to the identity. The representation W of G is then defined by setting $W_{\alpha,\pi} = W^0_\alpha W'_\pi$.

Evidently H'_0 commutes with all $W_{\alpha,\pi}$, and the representation W of $G = SO(3) \times SO(3) \times \cdots \times SO(3) Ⓢ S_N$ is the analog of the representation U of $SO(3)$, whose relationship to the eigenspaces of ${}_1H'_0$ was analyzed in section 11. Just as in the one-electron case, the commutativity of H'_0 and the $W_{\alpha,\pi}$ implies that each eigenspace of H'_0 is invariant under all $W_{\alpha,\pi}$ and hence defines a subrepresentation of the representation W. Moreover, when there are no "accidental degeneracies"—that is, whenever $\lambda_1 + \lambda_2 + \cdots + \lambda_N \neq \lambda'_1 + \lambda'_2 + \cdots + \lambda'_N$—unless there is a permutation π such that $\lambda'_j = \lambda_{\pi(j)}$ for all j (and certain other conditions hold), then this subrepresentation of W can be shown to be an *irreducible* unitary representation of G. Each eigenspace and eigenvalue of H'_0 has a "label" that is an equivalence class of (often irreducible) unitary representations of the compact group G and plays the role in the N-electron case played by the azimuthal quantum number l in the one-electron case.

Our next observation is that a certain closed subgroup $\tilde{G}$ of G has the property that not only H'_0 but also J' and H all lie in the commuting algebra of the representation $\tilde{W}$ of $\tilde{G}$ obtained by restricting W to $\tilde{G}$. This is the group of all $\alpha_1, \alpha_2, \ldots, \alpha_N, \pi$ for which $\alpha_1 = \alpha_2 = \cdots = \alpha_N$ and is obviously isomorphic to $SO(3) \times S_N$. Consider now an eigenspace $\mathfrak{M}_{\lambda_1 + \cdots + \lambda_N}$ of H'_0, and let P be the projection operator whose range it is. Then P lies in the commuting algebra of W and *a fortiori* in that of $\tilde{W}$. Thus, $PJ'P$ lies in the commuting algebra of $\tilde{W}$. But $PJ'P$ restricted to the range of P is precisely the operator whose matrix we must diagonalize to find the first-order approximations to those eigenvalues of H obtained by perturbing $\lambda_1 + \lambda_2 + \cdots + \lambda_N$. (See the end of section 9.)

The key point may now be formulated as follows. Each eigenspace $\mathfrak{M}$ of H'_0 is the space of a (usually irreducible) unitary representation W of $G = SO(3) \times SO(3) \times \cdots \times SO(3) Ⓢ S_N$, and the operator $J'_{\mathfrak{M}} =$ restriction to $\mathfrak{M}$ of $PJ'P$ (whose eigenvalues must be found in order to determine first-order approximations to the corresponding values of H) is in the commuting algebra of the restriction $\tilde{W}$ of W to a subgroup $\tilde{G}$ of G isomorphic to $SO(3) \times S_N$.

13. Because of the facts summarized at the end of the preceding section, one can apply the theorem of section 11 and use the decomposition of W to facilitate determining the eigenvalues of $J'_{\mathfrak{M}}$. It is time to explain just how this is done and just what one needs to know about the unitary representation theory of $SO(3)\times S_N$ in order to do it. It will be convenient to proceed abstractly and consider an arbitrary self-adjoint operator T, which lies in the commuting algebra of an arbitrary finite-dimensional unitary representation V of an arbitrary separable locally compact group K.

Let $L^1, L^2, \ldots, L^r$ denote the irreducible representations into which V decomposes, let m_j denote the multiplicity with which L^j occurs, and let $\mathfrak{M}_j$ denote the unique invariant subspace in which V is equivalent to L^j repeated m_j times. According to the theorem of section 11, T must take each $\mathfrak{M}_j$ into itself, and the eigenvalues of the restriction T_j of T to $\mathfrak{M}_j$ must have multiplicities that are integer multiples of the dimension d_j of the space of L^j. Let $\lambda_1^j, \lambda_2^j, \ldots, \lambda_{m_j}^j$ be the eigenvalues of T_j, each repeated as many times as its multiplicity divided by d_j. We now make the important observation that for each j one can compute the sum

$$\lambda_1^j+\lambda_2^j+\cdots+\lambda_{m_j}^j$$

as a linear combination of the matrix elements $t_{i,i'}$ of T with respect to any convenient orthonormal basis $\phi_1, \phi_2, \ldots, \phi_d$, the coefficients being determined by the properties of the group representation V. Indeed, let $\theta_1^j, \theta_2^j, \ldots, \theta_{m_j d_j}^j$ be any orthonormal basis for $\mathfrak{M}_j$. Then the trace of T_j is equal on the one hand to

$$d_j\left(\lambda_1^j+\lambda_2^j+\cdots+\lambda_{m_j}^j\right) \tag{$*$}$$

and on the other to

$$\sum_{l=1}^{m_j d_j}\left(T\left(\theta_l^j\right)\cdot\theta_l^j\right). \tag{$**$}$$

Now, expanding θ_l^j in terms of the basis $\phi_1, \phi_2, \ldots, \phi_d$ yields

$$\theta_l^j=\sum_i\left(\theta_l^j\cdot\phi_i\right)\phi_i \tag{$***$}$$

and

$$T\left(\theta_l^j\right)=\sum_{i'}\left(\theta_l^j, \phi_{i'}\right)T\left(\phi_{i'}\right), \tag{$****$}$$

equating $(*)$ and $(**)$ and substituting $(***)$ and $(****)$ into $(**)$, one finds that

$$d_j\left(\lambda_1^j+\lambda_2^j+\cdots+\lambda_{m_j}^j\right)=\sum_{l=1}^{m_j d_j}\left(\sum_{i,i'}\left(T(\phi_{i'})\cdot\phi_i\right)\left(\overline{\theta_l^j\cdot\phi_i}\right)\left(\theta_l^j\cdot\phi_{i'}\right)\right),$$

and this evidently implies that

$$\left(\lambda_1^j+\lambda_2^j+\cdots+\lambda_{m_j}^j\right)=\sum_{i,i'}\left(T(\phi_i)\cdot\phi_{i'}\right)c_{i,i'}^j, \qquad (\dagger)$$

where the coefficients $c_{i,i'}^j$ are computed from the formula

$$c_{i,i'}^j=\frac{1}{d_j}\sum_{l=1}^{m_j d_j}\left(\theta_l^j\cdot\phi_i\right)\left(\theta_l^j\cdot\phi_{i'}\right). \qquad (\dagger\dagger)$$

The two equations (†) and (††) are fundamental. They make it possible to compute the sums $\lambda_1^j+\lambda_2^j+\cdots+\lambda_{m_j}^j$ for the eigenvalues of T directly from the matrix elements of T as soon as one knows the expansion coefficients $(\theta_l^j\cdot\phi_i)$ for the members of the basis $\{\theta_l^j\}$ relative to the basis $\{\phi_i\}$. The determination of these coefficients is a purely group-representational problem that is a substantial part of the entire theory. It does not become definite until one has chosen the bases in question, but it is important to realize that it cannot be trivialized by choosing the ϕ_i to coincide with the θ_l^j. This would beg the question. To be able to compute the matrix elements $(T(\phi_i)\cdot\phi_{i'})$, the ϕ_i must be chosen in a way that takes no cognizance of the $\mathfrak{M}_j$. This "coefficient problem" will be described in some detail below. The rest of the present section will be devoted to the problem of passing from the sums $\lambda_1^j+\cdots+\lambda_{m_j}^j$ to the individual eigenvalues.

Evidently, when V is multiplicity-free—that is, when each m_j is equal to 1—there is no problem. Each sum has just one term, and formula (†) gives the actual eigenvalues. In the general case one notices that, when T is in the commuting algebra of V, so is every power of T, and also that the eigenvalues of T^k are just $\lambda_1^k, \lambda_2^k, \ldots$, where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of T. Knowing the matrix elements of T, one can compute those of $T^2, T^3, \ldots$ by matrix multiplication and so use formula (†) (with T replaced by its various powers) to find explicit expressions for $(\lambda_1^j)^2+\cdots+(\lambda_{m_j})^2$, $(\lambda_1^j)^3+(\lambda_2^j)^3+\cdots+(\lambda_{m_j})^3$, etc. From these one can obtain the eigenvalues $\lambda_1^j, \lambda_2^j, \ldots, \lambda_{m_j}^j$ as the roots of a polynomial equation of degree m_j. Indeed, if $m_j=2$, one will have two unknown eigenvalues a and b but will know both $a+b$ and a^2+b^2. Now a and b are the roots of the quadratic equation $x^2-(a+b)x+ab=0$, and $ab=[(a+b)^2-(a^2+b^2)]/2$. Thus, knowing $a+b$

and a^2+b^2 permits one to find a and b by solving a quadratic equation. Similarly, one can find a cubic equation whose roots are a, b, and c whenever $a+b+c$, $a^2+b^2+c^2$, $a^3+b^3+c^3$, etc., are known for all possible values of m_j.

The point of the reduction is that it is much easier to solve r polynomial equations of degrees $m_1, m_2, \dots, m_r$ than one polynomial equation of degree $m_1d_1+\cdots+m_rd_r$. From another point of view, the polynomial of degree $m_1d_1+\cdots+m_rd_r$, which must be factored into linear factors in order to solve the equation, is partially factored by the method into $P_1^{d_1}P_2^{d_2}\cdots P_r^{d_r}$, where each P_j is a polynomial of degree m_j, and it remains only to factor the P_j.

14. At this point it will be useful to descend from the abstract to the concrete and consider the actual V's that arise in the theory of atomic spectra in the special case in which $N=2$. In that case the general eigenvalue of H_0' will be of the form $\lambda_1+\lambda_2$, where λ_1 and λ_2 are eigenvalues of ${}_1H_0'$ with principal quantum numbers n_1 and n_2 and azimuthal quantum numbers l_1 and l_2. With no "accidental" degeneracies, the corresponding eigenspace will have dimension $2(2l_1+1)(2l_2+1)$, and an orthonormal basis for it will consist of all functions of the form $\phi(x_1, y_1, z_1)\psi(x_2, y_2, z_2)$ and all functions of the form $\psi(x_1, y_1, z_1)\ \phi(x_2, y_2, z_2)$, where ϕ ranges over the $2l_1+1$ members of some orthonormal basis for the λ_1 eigenspace of ${}_1H_0'$, and ψ ranges over the $2l_2+1$ members of some orthonormal basis for the λ_2 eigenspace of ${}_1H_0'$. However, in the special case in which $m_1=m_2$ and $l_1=l_2$, so that the two eigenspaces coincide, the dimension is only $(2l_1+1)(2l_2+2)$-dimensional, since interchanging ϕ and ψ does not lead to a different eigenfunction.

In any case, the eigenspace is the space of an irreducible unitary representation of the group $SO(3)\times SO(3)$ Ⓢ S_2, which is irreducible unless $n_1\neq n_2$ and $l_1=l_2$ hold simultaneously. In the latter event the representation in the eigenspace decomposes into two inequivalent subrepresentations. Fortunately, these two subeigenspaces are easily shown to be carried with themselves by the operator J' and so can be treated as separate eigenspaces —just as though $E_1+E_2\neq E_2+E_1$. Making the appropriate modification in the definition of eigenspace, one may say that in all cases the eigenspace is the space of an *irreducible* unitary representation of the group $SO(3)\times SO(3)$ Ⓢ S_2 and that the coefficient problem is a problem about the restriction of this irreducible representation to the subgroup of all x, y, π, with $x=y$. It is useful to analyze this restriction by looking first at the restriction to the intermediate subgroup $SO(3)\times SO(3)$. It is almost evident that this restriction is irreducible when $l_1=l_2$ and otherwise is a direct sum of the two inequivalent irreducibles $D^{l_1}\times D^{l_2}$ and $D^{l_2}\times D^{l_1}$, whose spaces are interchanged by the unitary map defined by the nontrivial member of S_2. The coefficient problem in either case thus reduces to the corresponding problem in which $SO(3)\times SO(3)Ⓢ S_2$ is replaced by $SO(3)\times SO(3)$ and the subgroup is the set of all x, y with $x=y$.

To revert to the abstract for the moment, let G_1 and G_2 be any two compact groups, let L be any finite-dimensional unitary representation of G_1, and let M be any finite-dimensional unitary representation of G_2. One can then prove that there exists a finite-dimensional unitary representation V of $G_1\times G_2$ that is uniquely determined up to equivalence by the fact that $\mathrm{trace}(V_{x,y})=\mathrm{trace}(L_x)\mathrm{trace}(M_y)$ for all x in G_1 and all y in G_2. It is called the *tensor product* of L and M and is denoted by the symbol $L\times M$. Its dimension is the product of the dimensions of L and M. One proves that $L\times M$ is irreducible if and only if *both* L and M are irreducible and that every irreducible unitary representation of $G_1\times G_2$ is equivalent to one of the form $L\times M$, where L and M are uniquely determined up to equivalence. In this way the problem of finding the irreducible unitary representations of a product group is completely reduced to the corresponding problem for the factors. Evidently there is a step-by-step procedure permitting one to deal in an analogous fashion with the irreducible unitary representations of any finite product of compact groups $G_1\times G_2\times \cdots \times G_m$.

In the special case in which $G_1=G_2=G$, one can identify G with a subgroup of $G\times G$—namely, the diagonal subgroup of all x, y with $x=y$. Given any two irreducible unitary representations L and M of G, one can form $L\times M$ and by restricting to the diagonal form a new unitary representation of G. This representation is sometimes called the *inner tensor product* of L and M, and it is seldom irreducible. When one has found the possible irreducible unitary representations of a compact group G, another important problem that presents itself is that of determining their inner tensor products. For each triple L, M, N of irreducible unitary representations of G, one wants the (possibly zero) multiplicity with which N is contained in the inner tensor product $L\otimes M$.

Returning now to the concrete problem set by the two-electron atom, one sees that the representation V occurring in the coefficient problem is just the inner tensor product $D^l\otimes D^{l'}$ of two irreducible unitary representatives of $SO(3)$. The structure of this representation is easily determined by using an important fact about the restrictions of D^l and $D^{l'}$ to the subgroup A_z of all rotations about the z-axis. The group A_z is commutative and isomorphic to the multiplicative group of all complex numbers of modulus 1—that is, all $e^{i\theta}$, with θ real. The irreducible unitary representations are all one-dimensional and correspond one-to-one to the integers, the representation L^k for $k=0,\pm 1,\pm 2,\ldots$ being $e^{i\theta}\to e^{ik\theta}$. Moreover, it follows easily from the definition of D^l in terms of surface harmonics that D^l restricted to A_z is multiplicity-free and equivalent to the direct sum $L^{-l}\oplus L^{-l+1}\oplus \cdots \oplus L^{l-1}\oplus L^l$. It follows at once from the relevant definitions that $D^l\otimes D^{l'}$ restricted to A_z must be the direct sum of all $L^{k+k'}$, where k and k' are integers and

$-l \leqslant k \leqslant l$, $-l' \leqslant k' \leqslant l'$. Since $L^{l+l'}$ occurs just once in this sum and $l+l'$ is the maximum integer that occurs, it follows that $D^{l+l'}$ occurs once and only once in the decomposition of $D^l \otimes D^{l'}$. Continuing in this vein, one deduces the truth of the celebrated Clebsch–Gordan formula

$$D^l \otimes D^{l'} = D^{|l-l'|} \oplus D^{|l-l'|+1} \oplus D^{|l-l'|+2} \oplus \cdots \oplus D^{l+l'}$$

and in particular that $D^l \otimes D^{l'}$ is multiplicity-free for all l and l'. It follows that the formula of section 13 gives the eigenvalues of J' directly without the necessity of factoring any nonlinear polynomials.

To use the formula of section 13, one must choose an orthonormal basis in the space of $D^l \otimes D^{l'}$ and another in the space of each irreducible constituent. A useful and natural choice is suggested by the fact that each D^k has the property that its restriction to A_z is multiplicity-free and so defines a direct sum decomposition of the space of D^k into one-dimensional A_z-invariant subspaces. Choosing the basis elements to lie in these subspaces determines them up to multiplication by a complex number of modulus 1. The properties of surface harmonics make it possible to write down these basis elements quite explicitly, and from the physicist's point of view they have the added advantage of having a simple physical interpretation. They are eigenvectors of the operator representing the z-component of an angular momentum and hence represent states in which this angular momentum component has a definite value. This natural (modulo a choice of z-axis) basis in the space of each D^l carries with it a corresponding basis in the space of each product $D^l \otimes D^{l'}$ and in the space of each component $D^{|l-l'|+k}$ in the reduction of $D^l \otimes D^{l'}$. The basis $\{\phi_i\}$ of section 13 is thus parameterized by pairs of integers k_1, k_2, where $-l \leqslant k_1 \leqslant l$ and $-l' \leqslant k_2 \leqslant l'$, and the basis $\{\theta_\rho^j\}$ by pairs of integers j, ρ, where $|l-l'| \leqslant j \leqslant l+l'$ and $-j \leqslant \rho \leqslant j$. In the physical interpretation the z-components of angular momentum ~~corresponding to k_1, k_2, and ρ are those of the two electrons individually~~ and of the total system. The coefficients of section 13 for $SO(3) \times SO(3)$ and its diagonal subgroup with the bases chosen as indicated are known as *Clebsch–Gordan coefficients*, and their determination and detailed study is one of the major topics in angular momentum theory. They not only solve the problem in the two-electron case but serve as building blocks in solving the coefficient problem for more than two electrons. They are functions of six integer variables—l, l', j, k_1, k_2, and ρ—where $0 \leqslant l$, $0 \leqslant l'$, and j, k_1, k_2, and ρ are restricted as indicated above.

At this point the better-informed reader will be protesting that in fact the variables are not integers but half-integers. He is correct, but half-integers appear only when the effect of "electron spin" has been taken into account and the group $SO(3)$ has been replaced by the group $SU(2)$ of all 2×2 unitary matrices of determinant 1. The quotient of this group by its two-element center is isomorphic to $SO(3)$, and its irreducible unitary representations include those of $SO(3)$ as well as a supplementary family $D^{1/2}, D^{3/2},\ldots$, which reduce on the center to the negative of the identity and so may be regarded as "double-valued" representations of $SO(3)$. For all l, integral or not, the dimension of D^l is $2l+1$.

The Clebsch–Gordan coefficients (and slight variants thereof) are also known as Wigner coefficients, as Wigner 3-j symbols, and as vector coupling coefficients. The literature on them is extensive, including tables of values, explicit formulas, and formulas for computing them recursively.

15. The case $N=2$, although fundamental, is rather special in that $D^l \otimes D^{l'}$ is multiplicity-free and also in that S_2 is a commutative group with a very trivial representation theory. It is accordingly necessary to take a look at the three-electron case as well as the two-electron case in order to get an adequate introduction to the complexities of the general case.

When $N=3$, the group whose irreducible unitary representations help to label the eigenvalues of H_0' is $SO(3)\times SO(3)\times SO(3)\circledS S_3$. We begin by describing the irreducible unitary representations of this group. Let V be any such representation, and consider the restriction to $SO(3)\times SO(3)\times SO(3)$. This will be a finite direct sum of irreducible unitary representations of $SO(3)\times SO(3)\times SO(3)$—that is, of representations of the form $D^{l_1}\times D^{l_2}\times D^{l_3}$. Moreover, it is easy to prove that, whenever $D^{l_1}\times D^{l_2}\times D^{l_3}$ occurs, then the others that occur are precisely those obtainable by permuting l_1, l_2, and l_3 in all possible ways. The number of inequivalent ones depends, of course, upon how many equalities there are between l_1, l_2, and l_3 and is either 1, 3, or 6. The (unordered) triple l_1, l_2, l_3 "almost" determines the irreducible unitary representation V. Indeed, when l_1, l_2, and l_3 are distinct, it does determine it. On the other hand, when $l_1 \neq l_2 = l_3$, there are two inequivalent V's for each l_1, l_2, l_3, and when $l_1 = l_2 = l_3$, there are three inequivalent V's for each. To analyze these possibilities, consider the restriction of V to S_3, and let $H(V)=\mathfrak{M}_1\oplus\mathfrak{M}_2\oplus\cdots\oplus\mathfrak{M}_f$ be the decomposition of $H(V)$ into orthogonal subspaces corresponding to the distinct irreducibles in the restriction of V to $SO(3)\times SO(3)\times SO(3)$. By the above, $f=1$, 3, or 6. One sees easily that for π in S_3 the V_π permute the $\mathfrak{M}_i$ among themselves so that the decomposition defines a "system of imprimitivity" for V and for its restriction to S_3. Let S be the subgroup of S_3 consisting of all π in S_3 such that $V_\pi(\mathfrak{M}_1)=\mathfrak{M}_1$. It is not hard to show that, when V is restricted to $\mathfrak{M}$ and at the same time to S, one obtains a representation of S that is a multiple of an *irreducible* unitary representation L of S. It is (the equivalence class of) this representation L that one needs to know in addition to the (unordered) triple l_1, l_2, l_3 in order to determine the equivalence class of V. Every irreducible unitary representation of S (up to equivalence) actually occurs. Thus, depending upon whether none, two, or three of the l's are equal, S will be S_1, S_2, or S_3, and there will be correspondingly one, two, or three inequivalent unitary V's for each triple l_1, l_2, l_3.

In order to pass from a given unitary representation L of S back to the corresponding unitary representation V of $SO(3)\times SO(3)\times SO(3)$ Ⓢ S_3, one employs a general construction in group representation theory that is a sort of inverse to restricting a representation to a subgroup. In the special case in which the subgroup has only a finite number of cosets, this construction is quite easy to describe. Let G' be a closed subgroup of a topological group G, and let there be only a finite number of right G' cosets $G'x$ (x in G). Let W be an arbitrary continuous finite-dimensional unitary representation of G'. One constructs a unitary representation U^w of the whole group G as follows. Its space is the vector space F of all functions g from G to the Hilbert space $\mathcal{H}(W)$ in which W acts, which satisfy the equation

$$g(\xi x)=W_\xi g(x) \tag{$*$}$$

for all ξ in G' and all x in G. Note that $(*)$ implies that g is uniquely determined throughout the coset $G'x_0$ as soon as its value at x is determined and that this value can be assigned arbitrarily. It follows that the dimension of F is finite and equal to the number of right G' cosets in G multiplied by the dimension of $\mathcal{H}(W)$. It is obvious that, whenever $(*)$ holds for g, one has also $g(\xi xy)=W_\xi g(xy)$ for all y, and hence that the right translates by y of every g in F is also in F. We define U_y^w by setting $U_y^w(g)(x)=g(xy)$. Evidently $y\to U^w$ is a representation of G whose space is F. It is called the *representation of G induced by W*.

Returning to the problem at hand and given l_1, l_2, l_3 and the appropriate subgroup S of S_3, let L be an arbitrary irreducible unitary representation of S. Recall that the representation $D^{l_1}\times D^{l_2}\times D^{l_3}$ of $SO(3)\times SO(3)\times SO(3)$ has a "natural" extension to an irreducible unitary representation $(D^{l_1}\times D^{l_2}\times D^{l_3})^e$ of $SO(3)\times SO(3)\times SO(3)$ Ⓢ S_3. (Compare the relationship between W^0 and W in section 12.) On the other hand, $\alpha_1, \alpha_2, \alpha_3$, $\pi\to L_\pi$, is an irreducible unitary representation L' of $SO(3)\times SO(3)\times SO(3)$ Ⓢ S. One shows easily that the inner tensor product of L' with the restriction to $SO(3)\times SO(3)\times SO(3)$ Ⓢ S of $(D^{l_1}\times D^{l_2}\times D^{l_3})^e$ is irreducible. Denote it by $A^{l_1,l_2,l_3,L}$. The unitary representation $U^{A^{l_1,l_2,l_3,L}}$ of $SO(3)\times SO(3)\times SO(3)$ Ⓢ S, induced by the representation $A^{l_1,l_2,l_3,L}$ of $SO(3)\times SO(3)\times SO(3)$ Ⓢ S is irreducible and is the desired representation associated with l_1, l_2, l_3, L.

The fact that V can be so induced from an irreducible representation of a subgroup is intimately related to the existence of the system of imprimitivity $\mathcal{M}_1\oplus\mathcal{M}_2\oplus\cdots\oplus\mathcal{M}_f$. Returning to the general group G considered above, let V be any unitary representation of G, and let $\mathcal{H}(V)$, the space of V, be a direct sum of a finite number of orthogonal subspaces $\mathcal{M}_1\oplus\mathcal{M}_2\oplus\cdots\oplus\mathcal{M}_f$ such that $V_x(\mathcal{M}_j)=\mathcal{M}_k$ for each j and x and some k depending on x and j. Suppose also that this system of imprimitivity is "transitive" in the sense that for each j and k there exists x, with $V_x(\mathcal{M}_j)=\mathcal{M}_k$. Let G' be the subgroup of G consisting of all x with $V_x(\mathfrak{M}_1)=\mathfrak{M}_1$. Then the simultaneous restriction of V to G' and $\mathfrak{M}_1$ defines a unitary representation W of G', and it is easy to see that V can be reconstructed from W and the way in which G permutes the $\mathfrak{M}_j$. Actually $G'x \rightarrow V_x(\mathfrak{M}_1)$ sets up a one-to-one correspondence between the right G' cosets and the $\mathfrak{M}_j$, and one can show that V is equivalent to the induced representation U^w.

The determination of the irreducible unitary representations of $SO(3)\times SO(3)\times SO(3)Ⓢ S_3$ just described has a straightforward generalization in which 3 is replaced by an arbitrary positive integer N. To construct the most general irreducible unitary representation (up to equivalence) of $SO(3)\times SO(3)\times \cdots \times SO(3)Ⓢ S_N$, one chooses an N-tuple $l_1, l_2, l_3, \ldots, l_N$ of non-negative integers l and standardizes the arbitrary ordering by requiring $l_1 \leq l_2 \leq l_3 \leq \cdots \leq l_N$. Next one chooses an irreducible unitary representation L of the subgroup S of S_N consisting of all permutations π of the superscripts such that $l_{\pi(j)}=l_j$ for $j=1,2,\ldots,N$ and forms a representation $A^{l_1,l_2,\ldots,l_N,L}$ of $SO(3)\times SO(3)\times \cdots \times SO(3)Ⓢ S$ by direct analogy with what was done above in the case $N=3$. The induced representation $U^{A^{l_1,l_2,\ldots,l_N,L}}$ is then an irreducible unitary representation of $SO(3)\times SO(3) \times \cdots \times SO(3)Ⓢ S_N$. To within equivalence it is the most general possible, and two such are equivalent if and only if the l_j are the same and the representations L of S are equivalent. Note that S is always isomorphic to a group of this form $S_{f_1}\times S_{f_2}\times \cdots \times S_{f_r}$, where S_j is the symmetric group on j objects and $f_1+f_2+\cdots+f_r=N$. Thus, to find all irreducible unitary representations for every S that arises, one need only know the irreducible unitary representations of the symmetry groups S_k for all k.

16. Having described the irreducible unitary representations of $SO(3)\times SO(3)\times SO(3)Ⓢ S_3$, we have paved the way for generalizing the considerations of section 14 to the case of three electrons. Each eigenspace will now be determined by an (unordered) triple of eigenvalues of H_0' with quantum numbers n_1, l_1, n_2, l_2, and n_3, l_3. The dimensions of this eigenspace will be $k(2l_1+1)(2l_2+1)(2l_3+1)$, where $k=1$, 3, or 6 according to whether the number of equalities among the pairs n_i, l_i is 3, 1, or 0. Moreover, if $n_i=n_j$ whenever $l_i=l_j$, this eigenspace will be the space of an irreducible unitary representation of $SO(3)\times SO(3)\times SO(3)Ⓢ S$—namely, that induced by the representation $(D^{l_1}\times D^{l_2}\times D^{l_3})^e$ restricted back to $SO(3)\times SO(3)\times SO(3)Ⓢ S$. Here $S=S_3$ if $l_1=l_2=l_3$, and otherwise is isomorphic to S_2 or S_1. If $l_i=l_j$ and $n_i\neq n_j$ for one or more pairs i, j with $i\neq j$, the situation is more complicated in that the representation in the eigenspace is no longer irreducible. In the extreme case in which $l_1=l_2=l_3=l$ while all n_j are distinct, this representation is that induced by $D^l\times D^l\times D^l$ and is a direct sum of four irreducible representations of $SO(3)\times SO(3)\times SO(3)Ⓢ S_3$, two of which are equivalent. The three inequivalent ones exhaust the irreducible

unitary representations of $SO(3)\times SO(3)\times SO(3)\circledS S_3$ associated with the triple l, l, l and correspond to the three inequivalent irreducible representations of S_3. In all cases the situation can be described by introducing the subgroup S' of all permutations, leaving the pairs $n_1, l_1, n_2, l_2, n_3, l_3$ fixed, as well as the subgroup S leaving l_1, l_2, l_3 fixed. Of course $S'\subseteq S$, and one forms the representation B of S induced by the identity of S'. The reduction of B exactly parallels that of the representation of $SO(3)\times SO(3)\times SO(3)\circledS S_3$ defined by the eigenspace. In particular, it has at most four irreducible constituents and is multiplicity-free except in the extreme case discussed above.

Whatever the actual representation of $SO(3)\times SO(3)\times SO(3)\circledS S_3$ in the eigenspace is, one can collect the irreducible components and obtain the canonical decomposition into primary parts. The corresponding subspaces of the eigenspaces are J'-invariant and so (as in the case $N=2$) may be treated as though they belonged to distinct eigenvalues. Unlike the case $N=2$, the associated unitary representations need not be irreducible. Instead they will be primary—that is, direct sums of *equivalent* irreducibles. However, the multiplicity will always be either 2 or 1 and will be 2 only when $l_1=l_2=l_3$ and n_1, n_2, and n_3 are all distinct. In any case the problem is to carry out the program of section 14 with the irreducible representations of $SO(3)\times SO(3)\circledS S_2$ replaced by primary representations of $SO(3)\times SO(3)\times SO(3)\circledS S_3$. Whether the primary representation is irreducible or not, the major part of the problem concerns the corresponding irreducible representation. Thus, to solve the coefficient problem in most cases and most of the problem in all cases, one must restrict an irreducible unitary representation V of $G=SO(3)\times SO(3)\times SO(3)\circledS S_3$ to the subgroup G_3 of all x, y, z, π with $x=y=z$ and seek two orthonormal bases for $\mathcal{H}(V)$, the space of V, as well as the expansion coefficients of one basis with respect to the other. One of these bases must be defined and be readily computable without the knowledge of how V reduces when restricted. The other must reduce the restriction of V to G_3 in the sense that each basis element must lie in one of the irreducible subspaces of some fixed restriction.

Finding the first basis is easily reduced to finding a basis for the space of the inducing representation $A^{l_1,l_2,l_3,L}$ of $SO(3)\times SO(3)\times SO(3)\circledS S$, since inducing commutes with the taking of direct sums. When l_1, l_2, and l_3 are all distinct, $S=S_1$ and may be ignored. In that case (and more generally whenever L is one-dimensional), it suffices to choose a convenient basis in the space of $D^{l_1}\times D^{l_2}\times D^{l_3}$, and this is carried out by considering the subgroup $A_z\times A_z\times A_z$ of $SO(3)\times SO(3)$ in evident generalization of what was done in section 14 for the case in which $N=2$.

A straightforward analysis shows that, when the induced representation $U^{A^{l_1,l_2,l_3,L}}$ is restricted back to G_3 (and G_3 is identified with $SO(3)\times S_3$), it coincides with the representation of $SO(3)\times S_3$ induced by a certain representation of $SO(3)\times S$. When $S=S_1$ (as it does when l_1, l_2, and l_3 are all distinct), $SO(3)\times S$ is just $SO(3)$, and the inducing representation is just $D^{l_1}\otimes D^{l_2}\otimes D^{l_3}$, the inner tensor product of three irreducible representations D^{l_1}, D^{l_2}, D^{l_3} of $SO(3)$. Since inducing commutes with the taking of direct sums, it suffices in finding the second basis to find an orthonormal basis in the space of $D^{l_1}\times D^{l_2}\times D^{l_3}$ that reduces $D^{l_1}\otimes D^{l_2}\otimes D^{l_3}$. If one attempts to use the method that was used above for $D^{l_1}\times D^{l_2}$, one runs into difficulty because $D^{l_1}\otimes D^{l_2}\otimes D^{l_3}$ is multiplicity-free only in rather special cases. To overcome this difficulty, one effects the reduction in two stages. In the first stage one restricts $D^{l_1}\times D^{l_2}\times D^{l_3}$ to the subgroup of all x, y, z in $SO(3)\times SO(3)\times SO(3)$ with $x=y$, and in the second one restricts further to the subsubgroup of all x, y, z with $x=y=z$. At the first stage the restriction may be identified with $D^{l_1}\otimes D^{l_2}\times D^{l_3}=(D^{|l_1-l_2|}\oplus D^{|l_1-l_2|+1}\oplus D^{|l_1-l_2|+2}\oplus\cdots\oplus D^{l_1+l_2})\times D^{l_3}=(D^{|l_1-l_2|}\times D^{l_3})\oplus(D^{|l_1-l_2|+1}\times D^{l_3})\oplus\cdots\oplus(D^{l_1+l_2}\times D^{l_3})$, which is a multiplicity-free representation of $SO(3)\times SO(3)$ and so defines a unique direct sum decomposition of the space of $D^{L_1}\otimes D^{l_2}\otimes D^{l_3}$. The kth summand in this decomposition is the space of the representation $D^{|l_1-l_2|+k-1}\times D^{l_3}$, and in the second stage this becomes the multiplicity-free representation $D^{|l_1-l_2|+k-1}\otimes D^{l_3}=D^{|l_1-l_2|+k-1+l_3}\oplus D^{|l_1-l_2|+k+l_3}\oplus\cdots\oplus D^{|l_1-l_2|+k-1+l_3}$ of $SO(3)$. Thus, each summand has a uniquely determined direct sum decomposition into spaces of irreducible representations of $SO(3)$. Putting them together, one has a well-defined direct sum decomposition of $D^{l_1}\otimes D^{l_2}\otimes D^{l_3}$ into irreducible components.

One obtains the second basis by restricting each of these to A_z, as in the case in which $N=2$. To designate a member of the second basis, given l_1, l_2, l_3, one chooses first an integer l_4, with $|l_1-l_2|\leq l_4\leq l_1+l_2$, denoting the particular component of $D^{l_1}\otimes D^{l_2}$ that is to be combined with D^{l_3}. Then one chooses an integer l^5, with $|l_4-l_3|\leq l_5\leq l_4+l_3$, denoting the component of $D^{l_4}\otimes D^{l_3}$ whose space is to contain the basis element. Finally one chooses an integer m_5, with $-l_5\leq m_5\leq l_5$, to select a basis element in the space of l_5. ~~The ordered triple~~ $\cancel{l_4, l_5, m_5}$ ~~determines the basis element (once suitable~~ corrections have been made about "phase factors"—that is, arbitrary constants of absolute value 1).

The members of the first basis are indexed by the triples m_1, m_2, m_3, where $-l_1\leq m_1\leq l_1-l_2\leq m_2\leq l_2-l_3\leq m_3\leq l_3$. The coefficients that must be computed in the $N=3$ case (when L is one-dimensional) are thus functions of the nine integer variables l_1, l_2, l_3, m_1, m_2, m_3, l_4, l_5, and m_5, which vary independently subject to the restrictions listed above. To carry out the computation, one introduces a third basis and proceeds in two steps. In the first step, members of the second basis are expressed in terms of those of the third, and in the second step, members of the third basis are expressed in terms of those of the first. The members of the third basis are defined by using the method of the $N=2$ case to introduce a second basis in the space of $D^{l_1}\times D^{l_2}$ and then combining this with the A_z basis for D^{l_3} by the tensor product construction. It is almost obvious that all coefficients involved in expressing members of the third basis in terms of members of the second are Clebsch–Gordan coefficients as defined in section 14 (whenever they are not zero) and that the same is true of the coefficients involved in expressing members of the second basis in terms of the third. It follows immediately that the coefficients that replace the Clebsch–Gordan coefficients when $N=3$ can all be explicitly given as sums of products of pairs of Clebsch–Gordan coefficients.

The nondegenerate cases for any N can be treated by a straightforward (although tedious to describe) generalization of the method just described for $N=3$. The coefficients involved can all be computed from the Clebsch–Gordan coefficients; they are complicated sums of $(N-1)$-fold products of them.

17. The analysis given so far is incomplete in several important respects. First, it has concentrated on the strongly nondegenerate cases in which $l_1, l_2, \ldots, l_N$ are all distinct, whereas in practice the most interesting and most frequently encountered cases are those in which there are coincidences among the l_j as well as among the n_j. Second, nothing has been said about how one computes the matrix elements of J' with respect to the first basis. This can be a formidable problem—especially since there are so many of them—and rotational symmetry can be as useful in simplifying the solution as it is in facilitating the diagonalization of the matrix after it has been found. Finally, no account has yet been taken of two fundamental physical facts: the so-called "spin" of the electron, and the "Pauli exclusion principle."

It is clear from the discussion in sections 14, 15, and 16 that an analysis of the degenerate cases along the lines indicated there demands considerable involvement with the representation theory of the symmetric group S_N. This fact was quite disturbing to most physicists of the late 1920s. They disliked the idea of having to learn to think in terms of an unfamiliar "unphysical" and abstract subject like group theory, and the device of passing to the operators occurring in the corresponding Lie algebra representation was not available. The symmetric group S_N is discrete, and its Lie algebra is consequently trivial. There was thus considerable relief when it was realized that the Pauli exclusion principle excluded so many eigenvectors and eigenvalues from consideration that only relatively easy parts of the representation theory of S_N continued to play a role. Indeed, if the electron did not have a "spin," this representation theory (effectively) would not be involved at all. Finally in 1929, only two or three years after Wigner, Weyl, and von Neumann introduced group-theoretic methods, J. C. Slater published a remarkable and extremely influential paper entitled "The Theory of Complex Spectra," in which he apparently got rid of the "group pest" altogether. In addition, he attacked and partially solved the problem of computing the matrix elements of J' with respect to the first basis. In brief, he outlined a practical procedure for computing first-order perturbations to energy levels that made no explicit use of the representation theory of either $SO(3)$ or S_N and in particular no explicit use of Clebsch–Gordan coefficients. Its only limitations were that it sometimes yielded only finite sums of energy levels and that it required very lengthy (but elementary) computations except when N and the l_j were small.

Before explaining what Slater did in detail, it will be necessary to say a few words about the nature of spin and the Pauli exclusion principle and how they affect our mathematical model. In the discussion up to now it has been assumed that the Hilbert space of states for an N-electron atom is the space $\mathcal{L}^2(E^{3N})$ of all square-summable complex-valued functions on $3N$-dimensional Euclidean space, or equivalently the tensor product $\mathcal{H}\times\mathcal{H}\times\cdots\times\mathcal{H}$ of N copies of the one-electron space $\mathcal{L}^2(E^3)$. The Pauli exclusion principle, in the more general and sophisticated form given it (independently) by Heisenberg and Dirac, changes this assumption by replacing $\mathcal{L}^2(E^{3N})$ by a certain closed subspace invariant under the energy operator H. This subspace is the subspace of all functions f in $\mathcal{L}^2$ that are *antisymmetric* in the sense that

$$f(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N)$$
$$\equiv f(x_2, y_2, z_2, x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$$

and more generally that the analogous identity holds for the interchange of any two pairs $x_i, y_i, z_i; x_j, y_j, z_j$. This statement can be put into a different form, which relates it to the symmetric group S_N by considering the unitary representation W' of S_N introduced in section 12. If one forms the canonical decomposition of W' into primary parts, one obtains an H-invariant subspace for each equivalence class of irreducible representations of S_N, and the subspace of antisymmetric functions is precisely that associated with the so-called "alternating representation" of S_N. The latter is the unique one-dimensional representation other than the identity; it may be described explicitly by the assertion that it is -1 times the identity on every permutation that interchanges two distinct integers while fixing the rest.

The replacement of $\mathcal{H}^N=\mathcal{H}\times\mathcal{H}\times\cdots\times\mathcal{H}$ by its antisymmetric subspace $\mathcal{H}_A^N$ has the effect of eliminating many of the stationary states and energy levels that would be predicted by the more naïve theory. As already explained in earlier sections, each eigenspace of the approximate energy operator is associated with an irreducible unitary representation of $SO(3)\times SO(3)\times\cdots\times SO(3)$ Ⓢ S_N, and one has a different first-order perturbed energy level associated with each irreducible representation of $SO(3)\times S_N$, which occurs in the restriction of the eigenspace representation to the subgroup of all $x_1, x_2, \ldots, x_N$, with $x_1=x_2=\cdots=x_N$. In particular, every perturbed eigenvalue is canonically associated with an irreducible unitary representation of S_N. The replacement of $\mathcal{H}^N$ by $\mathcal{H}_A^N$ retains only those energy levels for which the corresponding representation of S_N is the alternating representation. Investigating what these are, using certain general theorems in the theory of group representations, one finds that the program described in section 16 can be carried out without knowing any of the other irreducible unitary representations of S_N. We shall not give details, as Slater's method exploits the fact to produce a simpler and more elementary approach to the whole question.

As indicated above, the existence of the spin of the electron causes a second correction in the mathematical model from which one deduces the energy levels of an atom. This correction consists in changing the one-electron space by replacing complex-valued square-summable functions on E^3 by square-summable functions, taking values in the space of the two-dimensional irreducible unitary representation $D^{1/2}$ of $SU(2)$. The definition of ${}_1H_0$ given in section 10 continues to make sense when ψ is vector-valued and needs only to be altered by adding the (very small) "spin perturbation" to become the energy operator for the case of a single electron with spin. The natural action of $SO(3)$ on $\mathcal{L}^2(E^3)$ can be "lifted" to an action of $SU(2)$ by identifying $SO(3)$ with the quotient of $SU(2)$ by its two-element center. More generally, one has a natural action of $SU(2)$ on the space of all square-summable functions from E^3 to $\mathcal{H}(D^s)$ for any unitary irreducible representation D^s of $SU(2)$ ($s=0,\frac{1}{2},1,\frac{3}{2},\ldots$). It is defined by $U_\alpha(\psi)(x,y,z) = D_\alpha^s(\psi(\alpha(x,y,z,)))$. Retaining the definition of angular momentum about an axis as the self-adjoint operator associated with the restriction of U to the group of all rotations about that axis, one finds that this operator is a sum of two parts. One of these is what one would get if D_α^s were replaced by the identity operator for all α. The other is what one would get if $D_\alpha^s(\psi(\alpha(x,y,z)))$ were replaced by $D_\alpha^s(\psi(x,y,z))$. One refers to "orbital angular momentum" and to "spin angular momentum." The idea is that the occurrence of a nontrivial representation D^s adds something to the angular ~~momentum observable that would not be there if D^s were trivial. The~~ self-adjoint operators representing what is added obey the commutation rules for ordinary angular momentum and suggest that some other rotation must be taking place. The physicists make the hypothesis that the electron is spinning about an axis through its center. Hence the term "spin angular momentum." It is easy to see from its definition that the spin angular momentum takes on only $2s+1$ different values, and from the way in which it affects atomic spectra one is able to deduce that s for an electron is one-half. The actual sequence of events consisted in noticing anomalies in atomic spectra that could be accounted for by postulating a spinning electron and finally in discovering an appropriate modification in the mathematical model.

Neglecting the spin perturbation (as will be done here), one finds that the change in the Hilbert space of states for a single electron brought about by the introduction of spin carries with it an extension of the symmetry group $SO(3)$. Indeed, the operator ${}_0H_1$ commutes not only with the natural action of $SO(3)$ but also with every operator of the form $f(x, y, z,) \to Tf(x, y, z)$, where T is an arbitrary linear transformation acting in the two-dimensional complex Hilbert space in which f takes its values. In particular, then, there is a natural action of $SU(2)$ on the one-particle space that commutes with ${}_0H_1$ and commutes with the action of $SO(3)$ that takes $f(x, y, z)$ into $f(\alpha(x, y, z))$ for every rotation α. Putting these together, one obtains an action of $SO(3) \times SU(2)$ that commutes with ${}_0H_1$. The net effect of this change on the eigenvalues of ${}_0H_1$ is simply to double the multiplicity of each eigenvalue that occurs. However, as the reader should be able to verify for himself, this doubling has more profound effects on the eigenvalues that remain when one passes from $\mathcal{H}^N$ to $\mathcal{H}^N_A$. Many fewer eigenvalues are excluded.

One can distinguish two quite separate ideas in Slater's method. One of these is applicable independently of any consideration of spin and the Pauli exclusion principle. The other consists quite simply in the observation that one can eliminate the complications produced by the existence of spin by taking it into account before, rather than after, taking account of the Pauli exclusion principle. It will be convenient to explain the first idea in an abstract setting. Let K be a compact group, and let A be a commutative subgroup of K having the property that every irreducible unitary representation of K is multiplicity-free when restricted to A and moreover is determined to within equivalence by this restriction. In practice this group will be $SO(3) \times SU(2)$ or $SO(3)$, depending on whether the spin of the electron is included or not. Let V be a unitary representation of K in which each irreducible constituent occurs with finite multiplicity, and let J be a self-adjoint operator in this space that commutes with all V_x. In practice V will be a subrepresentation of the inner tensor product of as many irreducible unitary representations of K as there are electrons.

It follows from the fundamental theorem stated in section 11 that the canonical reduction of V into primary representations decomposes the space of V into J-invariant subspaces and that in the subspace corresponding to the irreducible representation M of K each eigenvalue has a multiplicity that is divisible by the dimension d_M of M. Let σ_M denote the sum of these eigenvalues, each repeated as many times as the quotient of its multiplicity by d_M. In section 13 we have explained how the σ_M may be computed as linear combinations of the matrix elements of J using certain coefficients independent of J and depending on the structure of V. The determination of these coefficients, as explained in subsequent sections, involved a heavy use of group theory—especially when V was a tensor product of repeated factors. Slater found a conceptually simpler and practically shorter procedure for calculating the sums σ_M, which may be formulated in such a way as to involve *no* explicit use of group theory and requires one to compute only the *diagonal* terms of J. From a group-theoretic point of view, Slater exploits the fact that J is (*a fortiori*) in the commuting algebra of the restriction of V to A and that the basis with respect to which it is convenient to compute the matrix elements of J (the first basis in the terminology of sections 14 and 16) is one for which all operators in this restriction are diagonal. It follows that every basis element lies in a primary component of the restriction of V to A. An immediate consequence of this is that many matrix elements of J can be seen to be zero without calculation—namely, those between basis elements belonging to different primary constituents of this restriction. The utility of this consequence, however, is completely overshadowed by another—namely, the existence of a system of linear equations of the form $\Sigma a_{\chi,M}\sigma_M=\lambda_\chi$, which are sufficient to determine the σ_M's. Here each $a_{\chi,M}$ is zero or one and is independent of J, and λ_χ is a sum of *diagonal* elements of J.

To see this, for each character χ of the commutative subgroup A consider the associated primary component of the restriction of V to A, and let J_χ denote the restriction of J to this J-invariant subspace. The trace of J_χ is computable at once as a sum of certain *diagonal* elements of the matrix of J. On the other hand, the space of J_χ is a direct sum of subspaces lying in distinct primary components of V, and the trace of J_χ is correspondingly a sum with one contribution from each primary component. It is an evident consequence of the definitions concerned that the contribution of the primary component corresponding to the irreducible representation M of K is precisely σ_M. Thus, for each χ one has

$$\Sigma a_{\chi,M}\sigma_M=\operatorname{trace} J_\chi,$$

where $a_{\chi,M}=1$ or 0, according as M restricted to A does or does not contain χ. The fact that the irreducible unitary representations of K are uniquely determined by their restrictions to A permits one to show that these equations, one for each χ, suffice to determine the σ_M uniquely. Note that ~~one does not need to compute *any* Clebsch–Gordan coefficients. For~~ obvious reasons, Slater's method is often referred to as his "diagonal sum method." In practice, the subgroup A of K is the subgroup of $SO(3)$ [or $SO(3)\times SU(2)$] defined by restriction to rotations about the z-axis. Moreover, vectors on which V restricted to A is multiplication by a character χ of A have a physical interpretation. These vectors define states in which the z-component of angular momentum (both spin and orbital) has a definite value. By exploiting this fact, the whole procedure may be described in physical terms, and this is how Slater described it.

As Slater emphasized, determining the σ_M does not give all the individual eigenvalues except when V is multiplicity-free. Each σ_M is a sum of as many (possibly distinct) eigenvalues as the multiplicity with which M occurs in V. On the other hand, there are many important cases where V is multiplicity-free or has only rather low multiplicities, so that the method has considerable practical utility. Not only is V multiplicity-free when $N=2$, but the Pauli exclusion principle brings about a major reduction in multiplicities for higher N whenever the same irreducible unitary representation occurs more than once. Thus, Slater's method is especially effective precisely when the group-theoretic method would involve serious consideration of the symmetric group.

18. As indicated at the beginning of section 17, Slater's paper contains more than a conceptually and practically simple procedure for computing the eigenvalue sums σ_M. This procedure depends on knowing the diagonal elements of the matrix J' with respect to a certain basis, and Slater also addresses the problem of computing these diagonal matrix elements. Although the method described in section 17 works for *any* operator J that commutes with all V_x, the J' that occur in practice are quite special. Exploiting this fact, one can reduce rather drastically the amount of work that has to be done in evaluating the $6N$-dimensional integrals to which the matrix elements in question are equal. First, J' is always a multiplication operator. In addition, the function of the coordinates that one multiplies by is a finite sum of functions each of which depends upon the coordinates of only one or two electrons. Finally, the functions of the coordinates of one electron that occur are all rotation-invariant, and the functions of the coordinates of two electrons that occur all have the same form:

$$x_i, y_i, z_i, x_j, y_j, z_j \to \frac{e^2}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}}.$$

Let us begin by looking at the special case of two electrons, and let us for the moment ignore the effects of spin and the Pauli exclusion principle. The Hilbert space is then $\mathfrak{L}^2(x_1, y_1, z_1, x_2, y_2, z_2)$, where x_1, y_1, z_1 and x_2, y_2, z_2 are the coordinates of the two electrons, and the operator J' is multiplication by the function

$$x_1, y_1, z_1, x_2, y_2, z_2 \to R_2\left(\sqrt{x_1^2+y_1^2+z_1^2}\right) + R_2\left(\sqrt{x_2^2+y_2^2+z_2^2}\right) + \frac{e^2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}},$$

where R_2 is a function of one real variable, which depends on the choice of g_2 in section 10.

Each unperturbed eigenvalue will be associated with an eigenspace in which the two electrons have principal quantum numbers n_1 and n_2 and azimuthal quantum numbers l_1 and l_2. Its dimension will be $(2l_1+1)(2l_2+1)$, and the basis with respect to which we seek the diagonal elements of J' will consist of functions of the form

$$x_1, y_1, z_1, x_2, y_2, z_2 \\ \to R_{l_1,n_1}(r_1)R_{l_2,n_2}(r_2)Y_1\left(\frac{x_1}{r_1}, \frac{y_1}{r_1}, \frac{z_1}{r_1}\right)Y_2\left(\frac{x_2}{r_2}, \frac{y_2}{r_2}, \frac{z_2}{r_2}\right),$$

where Y_1 and Y_2 are surface harmonics of degrees l_1 and l_2, respectively, and R_{l_1,n_1} and R_{l_2,n_2} are eigenfunctions of the ordinary second-order differential operators described in section 10. Of course there will be $2l_1+1$ mutually orthogonal surface harmonics Y_1, and $2l_2+1$ mutually orthogonal surface harmonics Y_2, distinguished by the so-called "magnetic" quantum numbers m_1 and m_2, where $m_1=-l_i, -l_i+1, \ldots, l_i$ for $i=1,2$. The diagonal matrix element for the basis element corresponding to the quantum numbers n_1, l_1, m_1 and n_2, l_2, m_2 will accordingly be a sum of three integrals of which the first quickly reduces to

$$\int_0^\infty R_{l_1,n_1}(r_1)R_2(r_1)r_1^2\,dr_1=I(l_1,n_1),$$

and the second to

$$\int_0^\infty R_{l_2,n_2}(r_2)R_2(r_2)r_2^2\,dr_2=I(l_2,n_2).$$

The third is more interesting and difficult and does not reduce immediately to an integral over $[0,\infty]$ because of the factor $e^2/\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$. Slater deals with it by first expanding this factor as a sum of terms each of which is a product of functions of r_1, r_2 and surface harmonics in one variable.

Although Slater simply writes the expansion down as a well-known application of the classical theory of surface harmonics, it seems worth digressing to point out that the existence of such an expansion follows from the general principles discussed in section 10. In section 10, functions f of x and y were expanded in series of the form $\sum_{l=-\infty}^{\infty}\rho_l(r)e^{il\theta}$, where $x=r\cos\theta$, $y=r\sin\theta$, by Fourier analysis of the restrictions of f to the circles $x^2+y^2=r^2$, and functions f of x, y, z were expanded in series of the form $\sum_{l=0}^{\infty}\rho_l(r)Y_l(x/r, y/r, z/r)$, where $r=\sqrt{x^2+y^2+z^2}$ and each Y_l is a surface harmonic, by applying the theory of expansion in surface harmonics to the restrictions of f to the spheres $x^2+y^2+z^2=r^2$. In a similar spirit, functions f of x_1, y_1, z_1 and x_2, y_2, z_2 may be expanded by considering their restrictions to the four-dimensional subsets $x_1^2+y_1^2+z_1^2=r_1^2$, $x_2^2+y_2^2+z_2^2=r_2^2$, and using the fact that these four-dimensional subsets are acted upon transitively by $SO(3)\times SO(3)$ just as $x^2+y^2=r^2$ is acted upon transitively by the circle group and $x^2+y^2+z^2=r^2$ is acted upon transitively by $SO(3)$. The appropriate analogs of the functions $e^{i\theta}$ in the circle case and the surface harmonics in the $SO(3)$ case are related to the irreducible unitary representations of $SO(3)\times SO(3)$, just as the surface harmonics are related to the irreducible unitary representations of $SO(3)$. Since the irreducible unitary representations of $SO(3)\times SO(3)$ are just the tensor products $D^{k_1}\times D^{k_2}$, where k_1 and k_2 range independently over the nonnegative integers, the required functions are products $Y_1(x_1/r_1, y_1/r_1, z_1/r_1)Y_2(x_2/r_2, y_2/r_2, z_2/r_2)$, where Y_1 and Y_2 are surface harmonics.

The surface harmonics of degree k form a $(2k+1)$-dimensional vector space, which is invariant under rotations and defines the irreducible unitary representation D^k of the rotation group $SO(3)$. Restricted to the group A_z of all rotations about the z-axis, this representation is a direct sum of one-dimensional representations corresponding to the characters $\phi\rightarrow e^{imk}$ for $m=-k,-k+1,\ldots,k$, and one obtains an orthonormal basis Y_m^k for the surface harmonics of degree k by choosing an element of norm 1 in each one-dimensional subspace. It turns out that, if θ and ϕ are the angle variables for spherical coordinates r,θ,ϕ with $x=r\cos\theta\cos\phi$, $y=r\cos\theta\sin\phi$, $z=r\sin\theta$, then Y_m^k has the form $Y^k(\theta,\phi)=P_{|m|}^k(\cos\theta)e^{im\phi}$, where $P_{|m|}^k$ is a polynomial. Thus, one expects to be able to expand a more or less arbitrary function f of $r_1,\theta_1,\phi_1,r_2,\theta_2,\phi_2$ in the form

$$\sum_{\substack{k_1,m_1,k_2,m_2\\ k_1,k_2=0,1,2,\ldots\\ |m_j|\leqslant 2k_j+1\\ m_j \text{ integral}}} u_{m_1,m_2}^{k_1,k_2}(r_1,r_2)P_{|m_1|}^{k_1}(\cos\theta_1)P_{|m_2|}^{k_2}(\cos\theta_2)e^{im_1\phi_1}e^{im_2\phi_2},$$

and to obtain the expansion used by Slater one has only to compute the functions $u_{m_1,m_2}^{k_1,k_2}$ for the special case in which

$$f(x_1,y_1,z_1,x_2,y_2,z_2)\equiv\frac{e^2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}.$$

It follows from the fact that f is invariant under the diagonal subgroup of $SO(3)\times SO(3)$ that $u_{m_1,m_2}^{k_1,k_2}(r_1,r_2)=0$ except when $m_1=-m_2$ and $k_1=k_2=k$, so that the expansion takes the simpler form

$$e^2\sum_{\substack{k,m\\ k=0,1,2,\ldots\\ |m|\leqslant 2k+1\\ m \text{ integral}}} u_m^k(r_1,r_2)P_{|m|}^k(\cos\theta_1)P_{|m|}^k(\cos\theta_2)e^{im(\phi_1-\phi_2)}.$$

For the particular function at hand, one computes that $u_m^k(r_1,r_2)$ is the constant $\dfrac{(k-|m|)!}{(k+|m|)!}$ multiplied by the function $r_1, r_2 \to \dfrac{[\min(r_1, r_2)]^k}{[\max(r_1, r_2)]^{k+1}}$.

As suggested above, the important fact about this expansion is that each term is a product of a function of r_1 and r_2 alone and of surface harmonics. Since each basis element is a product of surface harmonics and functions of r_1 and r_2 alone, one is reduced to evaluating integrals over the two-dimensional space of all pairs r_1, r_2 and integrals over the sphere of certain products of surface harmonics. Such integrals are much easier to evaluate than integrals over a twelve-dimensional Euclidean space. Moreover, one does not have to pay for this simplification by evaluating infinitely many integrals for each diagonal matrix element that comes up. It follows easily from the properties of surface harmonics (angular momentum theory again) that most of the surface harmonic integrals are zero—enough, in fact, so that each infinite series has only a small, finite number of nonzero terms. Slater's final result (in the two-electron case under consideration) is that the diagonal matrix element corresponding to a pair of electrons with quantum numbers n_1, l_1, m_1 and n_2, l_2, m_2 is

$$I(l_1, n_1) + I(l_2, n_2) + \sum_{k=0}^{\infty} a^k(l_1, m_1; l_2, m_2) F^k(n_1, l_1; n_2, l_2),$$

where $I(l, n) = \int_0^\infty R_{l\,n}(r) R_2(r) r^2\, dr$,

$$F^k(n_1, l_1; n_2, l_2) = 4\pi^2 e^2 \int_0^\infty \int_0^\infty \left(R_{l_1, m_1}(r_1)\right)^2 \left(R_{l_2, m_2}(r_2)\right)^2 \times \frac{\min(r_1, r_2)^k}{\max(r_1, r_2)^{k+1}} r_1^2 r_2^2\, dr_1\, dr_2,$$

and $a^k(l_1, m_1; l_2, m_2) = c^k(l_1, m_1) c^k(l_2, m_2)$, where

$$c^k(l, m) = \frac{(2l+1)(l-|m|)!}{(l+|m|)!} \int_0^\pi \left[P_l^{|m|}(\cos\theta)\right]^2 P_k^0(\cos\theta) \frac{\sin\theta}{2}\, d\theta.$$

It is easy to show that $c^k(l, m)$ is zero for all odd k and for $k > 2$, so that the infinite sum is actually finite as asserted.

Before looking further into the properties and implications of this still rather complex formula, we pause to remark that its components have a much more general range of application than our two-electron hypothesis would indicate. Indeed, if one considers three electrons instead of two, a straightforward adaptation of the analysis given above leads to the conclusion that the diagonal term corresponding to a triple of electrons with quantum numbers $n_1, l_1, m_1, n_2, l_2, m_2, n_3, l_3, m_3$ is

$$I(l_1,n_1)+I(l_2,n_2)+I(l_3,n_3)$$
$$+\sum_{k=0}^{\infty} a^k(l_1,m_1;l_2,m_2)F^k(n_1,l_1;n_2,l_2)$$
$$+\sum_{k=0}^{\infty} a^k(l_2,m_2;l_3,m_3)F^k(n_2,l_2;n_3,l_3)$$
$$+\sum_{k=0}^{\infty} a^k(l_3,m_3;l_1,m_1)F^k(n_3,l_3;l_1,n_1),$$

where I, the a^k, and the F^k are defined exactly as in the two-electron case (except for replacing R_2 by R_3). Moreover, the formula for N electrons is the obvious generalization from 3 to N of what has just been written down. Thus, in the general case of an N-electron atom, finding the diagonal elements reduces to evaluating the integrals $I(l,n)$, $a^k(l_1,m_1;l_2,m_2)$, and $F^k(n_1,l_1;n_2,l_2)$ and adding up the results. Things become more complicated when the spin of the electron and the Pauli exclusion principle are introduced, but the extra complication is nothing compared with the labor saved by omitting all excluded terms from consideration. Once again the analysis is straightforward but tedious to write down, and we shall content ourselves with the statement that the final formula is a sum of terms of the form $I(l_i,n_i)$ and

$$\sum_{k=0}^{\infty} a^k(l_i,n_i;l_j,n_j)F^k(n_i,l_i;n_jl_j)$$

just as above, and in addition terms of the form

$$-\sum_{k=0}^{\infty} b^k(l_i,m_i;l_jm_j)G^k(n_i,l_i;n_j,l_j),$$

where

$$G^k(n_1,l_1;n_2,l_2)=4\pi^2e^2\int_0^{\infty} R_{n_1,l_1}(r_1)R_{n_2,l_2}(r_1)$$
$$\times R_{n_1,l_1}(r_2)R_{n_2,l_2}(r_2)\frac{\min(r_1,r_2)^k}{\max(r_1,r_2)^{k+1}}r_1^2r_2^2\,dr_1\,dr_2$$

and

$$b^k(l_1, m_1; l_2, m_2)$$
$$= \frac{(k-|m_1-m_2|)!}{(k+|m_1-m_2|)!} \frac{(2l_1+1)(l_1-|m_1|)!}{(l_1+|m_1|)!} \frac{(2l_2+1)(l_2-|m_2|)!}{(l_2+|m_2|)!}$$
$$\cdot \left[\int_0^{\pi} P_{l_1}^{|m_1|}(\cos\theta) P_{l_2}^{|m_2|}(\cos\theta) P_k^{|m_1-m_2|}(\cos\theta) \frac{\sin\theta}{2} d\theta \right]^2 .$$

In the most general case then, computing the diagonal matrix elements reduces in the end to computing single and double integrals of the form $I(l, n)$, $a^k(l_1, m_1; l_2, m_2)$, $b^k(l_1, m_1; l_2, m_2)$, $F^k(n_k, l_1; n_2, l_2)$, and $G^k(n_1, l_1; n_2, l_2)$. Note that the "angular" integrals a^k and b^k, on the one hand, and the "radial" integrals I, F^k, and G^k, on the other, play quite distinct roles. The radial integrals are independent of the m_j and hence may be regarded as fixed constants during any particular perturbation calculation. In particular, each $I(n_j, l_j)$ that occurs just adds the same constant to all perturbed eigenvalues. The dependence on the m_j comes entirely from the a^k and b^k. On the other hand, it is only the radial integrals that depend on the choice of an "effective potential" (see section 10)—and in the case of the F^k and G^k—on finding the eigenfunctions of a one-dimensional eigenvalue problem. The angular integrals can be computed once and for all independent of such choices and of N. In the 1929 paper under discussion, Slater does not attempt to compute the radial integrals at all but contents himself with expressing perturbed eigenvalues in terms of them. As far as the a^k and b^k are concerned, he asserts that it is possible to compute them by making use of the known theory of associated spherical harmonics and presents the results of such a calculation in the form of two tables (one for the a^k and one for the b^k) in which l_1 and l_2 vary independently over the values 0, 1, and 2. In this range one needs consider only values of k equal to 0, 1, 2, 3, and 4.

Using the diagonal sum method in conjunction with the tables for the a^k and b^k, one can obtain explicit formulas for the σ_M (and hence in many cases for the perturbed eigenvalues) as linear combinations of the radial integrals. Slater concluded his paper by working out a number of examples. Although he made no attempt to evaluate the radial integrals, he observed that one often has many σ_M's expressed in terms of a few radial integrals, and that in addition relationships exist between the F's and G's that make it possible to express some as linear combinations of others. In this way he could deduce relationships between perturbed energy levels without knowing any radial integrals.

19. Slater's method dominated atomic spectroscopy for more than a decade. In 1935 E. V. Condon and G. H. Shortley published their celebrated

classic, "*The Theory of Atomic Spectra*," a comprehensive treatise of over four hundred pages. An exposition of Slater's method, which is clearly the core of the book, occupies most of their Chapters VI and VII. Certain extensions and improvements appear in Chapters VII and VIII. In 1931, for instance, Condon and Shortley published a paper comparing the experimental results with those calculated by Slater's method and, in order to make more extensive calculations, extended Slater's tables of the a^k and b^k so that l_1 and l_2 could take on the value 3 as well as 0, 1, and 2. In the physicists' terminology, they considered electrons in "f states" as well as those in "s, p, and d states." They incorporated these results in Chapter VII. A weakness in the diagonal sum method is that it sometimes only gives sums of energy levels. To separate these sums one must find the eigenvalues of a matrix with as many rows and columns as there are summands, and computing the elements of the matrix is a new problem. A method for doing this was worked out by Ufford and Shortley in 1932; it depends upon slightly earlier work of Johnson and of Gray and Wills. This method, which is presented in Chapter VIII, demands a knowledge both of off-diagonal matrix elements and of Clebsch–Gordan coefficients. Slater's procedure for computing the diagonal matrix elements turns out to generalize easily, and this generalized procedure is presented in Chapter VI.

In resorting to the use of Clebsch–Gordon coefficients, Condon and Shortley appear to be compromising Slater's purity and letting in a little explicit group theory after all. Both in defining these coefficients and in devising an explicit formula for computing their values, Wigner made use of the representation theory of $SO(3)$ and $SU(2)$. However, toward the end of Condon and Shortley's introductory Chapter I, one finds the following statement: "We wish finally to make a few remarks concerning the place of the theory of groups in the study of the quantum mechanics of atomic spectra. The reader will have heard that this mathematical discipline is of great importance for the subject. We manage to get along without it." To ~~understand how the authors reconcile this statement with their later use of~~ Clebsch–Gordan coefficients, one has only to examine their Chapter III, entitled "Angular Momentum." In this chapter they define a triple J_x, J_y, J_z of self-adjoint operators to be an "angular momentum" if the components satisfy the commutation relations

$$J_x J_y - J_y J_z = i\hbar J_z,$$
$$J_y J_z - J_z J_y = i\hbar J_x,$$
$$J_z J_x - J_x J_z = i\hbar J_y.$$

They then proceed to use algebraic manipulations with operators to study the properties of such triples and to prove a number of theorems about them. Later they consider several angular momenta simultaneously, defining two such J^1_x, J^1_y, J^1_z and J^2_x, J^2_y, J^2_z to *commute* with one another if each operator in either triple commutes with each operator in the other. A less obvious relationship between two triples, which plays an important role, consists in satisfying the commutation relations:

$$\begin{array}{lll} J^1_xJ^2_x - J^2_xJ^1_x = 0 & J^1_xJ^2_y - J^2_yJ^1_x = i\hbar J^1_x & J^1_xJ^2_z - J^2_zJ^1_x = -i\hbar J^2_y, \\ J^1_yJ^2_y - J^2_yJ^1_y = 0 & J^1_yJ^2_z - J^2_zJ^1_y = i\hbar J^1_y & J^1_yJ^2_x - J^2_xJ^1_y = -i\hbar J^2_z, \\ J^1_zJ^2_z - J^2_zJ^1_z = 0 & J^1_zJ^2_x - J^2_xJ^1_z = i\hbar J^1_z & J^2_zJ^2_y - J^2_yJ^1_z = -i\hbar J^2_x. \end{array}$$

For reasons that will become clear later, we shall say that J^1_x, J^1_y, J^1_z is a *vector* with respect to J^2_x, J^2_y, J^2_z whenever these nine equations are satisfied. It is easy to see that J^2_x, J^2_y, J^2_z is a vector with respect to J^1_x, J^1_y, J^1_z if and only if J^1_x, J^1_y, J^1_z is a vector with respect to J^2_x, J^2_y, J^2_z, and that J_x, J_y, J_z is a vector with respect to itself if and only if it is an angular momentum. In the latter part of the chapter the methods used in the first part are applied to obtain results about sets of angular momenta whose elements either commute or are vectors with respect to one another. In particular, if J^1_x, J^1_y, J^1_z and J^2_x, J^2_y, J^2_z are two commuting angular momenta, then their sum $J^1_x + J^2_x, J^1_y + J^2_y, J^1_z + J^2_z =$ (def) J_x, J_y, J_z is evidently also an angular momentum, and one sees easily that the two summands are vectors with respect to it. It turns out that general theorems about sets of angular momenta applied to this particular set lead to usable recursion formulas for computing the Clebsch–Gordan coefficients, and the chapter concludes with tables of values for these coefficients computed from these recursion formulas. The authors cite Wigner's group-theoretically derived explicit formula for the Clebsch–Gordan coefficients and declare that it would be difficult to deduce it from their formulas. On the other hand, they claim that it is so complicated that nothing is gained by using it. Many of the arguments and theorems of the chapter are attributed to earlier work of other physicists—especially that in a paper of Güttinger and Pauli published in 1931.

The relationship between the theory of angular momentum as presented in Chapter III of "*The Theory of Atomic Spectra*" and the representation theory of $SO(3)$ and $SU(2)$ has already been explained in principle in sections 6 and 7. Being given an angular momentum as defined above and in Chapter III is precisely the same as being given a representation of the Lie algebra of $SU(2)$ and (by way of the relationship between Lie groups and their Lie algebras) is equivalent to being given a representation of $SU(2)$. If V and W are two different representations of $SU(2)$, and J^V_x, J^V_y, J^V_z and J^W_x, J^W_y, J^W_z are the associated angular momenta, then the two angular momenta commute if and only if $V_\alpha W_\beta = W_\beta V_\alpha$ for all α and β so that $\alpha, \beta \to V_\alpha W_\beta$ defines a representation of $SU(2) \times SU(2)$. The angular momentum corresponding to the inner tensor product $\alpha \to V_\alpha W_\alpha$ is just the sum $J_x^V + J_x^W$, $J_y^V + J_y^W$, $J_z^V + J_z^W$, etc. Translating the results of Chapter III back into the language of group representations using the "dictionary" just described, one finds that Chapter III is little more than a standard development of the unitary representation theory of $SU(2)$, and hence of its quotient $SO(3)$, including a determination of all equivalence classes of irreducible representations, the decomposition of the inner tensor product of any two irreducible representations, and a recursive procedure for computing the associated Clebsch–Gordan coefficients.

Although there is a sense in which Condon and Shortley made heavy use of group theory, they did so in a harmless way. Their readers did not even need to know the definition of a group, much less that of a group representation or character. Everything was done with "purely algebraic" arguments—that is, arguments involving the addition, subtraction, and multiplication of operators and matrices—better still with operators and matrices that could be interpreted as quantum mechanical observables.

20. A major weakness of Slater's method is that it requires very lengthy calculations when N and the l_i are not rather small. One must first compute the a^k and b^k for each relevant quadruple l_1, m_1, l_2, m_2 and then use the diagonal sum procedure to compute the coefficients of the radial integrals F^k and G^k. Even the case $N=2$ is difficult, and by 1942 these coefficients had been computed only for values of l_1 and l_2 not exceeding 4. (In 1938 Shortley and Fried extended the tables in "*The Theory of Atomic Spectra*" to allow l_1 and l_2 to take on the value 4.) It was thus a considerable advance when Guilio Racah published a paper giving a convenient explicit formula for computing each coefficient in the two-electron case as a function of k, l_1, l_2, and M. For example, the coefficient of F^2 is equal to

$$\frac{6\lambda^2+3\lambda-2l_1l_2(l_1+1)(l_2+1)}{(2l_1-1)(2l_1+3)(2l_2-1)(2l_2+3)},$$

where

$$\lambda=\frac{M(M+1)-(l_1^2+l_2^2+l_1+l_2)}{2}.$$

This paper was the first of a now-famous series of four entitled "Theory of Complex Spectra x," where $x=$ I, II, III, and IV, published in the *Physical Review* in 1942, 1942, 1943, and 1949, respectively. These papers make frequent reference to the book of Condon and Shortley cited above. It will be convenient to follow Racah's lead and refer to their book as TAS.

Racah's derivation of his formulas for the coefficients f_k and g_k of Slater's radial integrals F^k and G^k is based on systematic use of the concept of a "vector operator." This is a special case of the concept of an "irreducible tensor operator," which plays an important role in Racah's later papers. To explain this concept it is useful to introduce the notion of a "canonical basis" in the space of an irreducible representation of a group. Let L^1 and L^2 be two equivalent unitary irreducible representations of the same group K. It follows from Schur's lemma that any linear map T from the space $\mathcal{H}(L^1)$ of L^1 to the space $\mathcal{H}(L^2)$ of L^2, such that $TL_x^1T^{-1}=L_x^2$ for all x, is uniquely determined up to a multiplicative constant. Thus, if $\phi_1, \phi_2, \ldots, \phi_r$ is an arbitrary basis for $\mathcal{H}(L^1)$ and $T \neq 0$, then $T(\phi_1)$, $T(\phi_2), \ldots, T(\phi_r)$ is a basis for $\mathcal{H}(L^2)$, which is independent of the choice of T up to multiplying all basis elements by the same constant. If we choose a particular model for each equivalence class of irreducible unitary representations of K and in this model choose a particular basis, we shall have a well-defined basis in the space of each concrete irreducible unitary representation of K modulo multiplying all basis elements by the *same* constant. In the particular case of the group $SO(3)$, there is a model for every irreducible unitary representation D^l based on surface harmonics, and once a z-axis has been chosen there is a canonical way of choosing a basis in $\mathcal{H}(D^l)$ whose elements are carried into multiples of themselves by all D_x^l with x a rotation about the z-axis. If L is a unitary irreducible representation of $SO(3)$ defined in some other way, we shall call a basis for $\mathcal{H}(L)$ *canonical* if it is the image of the surface harmonic basis for some operator setting up an equivalence between L and the appropriate surface harmonic representation. Now let U be an arbitrary unitary representation of $SO(3)$, and let $t=0, 1, 2, \ldots$. An *irreducible tensor operator of rank* t is a $(2t+1)$-tuple $T_{-t}, T_{-t+1}, \ldots, T_{t-1}, T_t$ of linear operators having the following two properties. (1) For each x in $SO(3)$ and each $m=-t,-t+1,\ldots,t$, the operator $U_xT_mU_x^{-1}$ lies in the linear span V of the $2t+1$ operators T_m so that V is the space of a representation $\tilde{U}^V$ of $SO(3)$. (2) This representation is irreducible, and the T_m constitute a canonical basis for it. A *vector operator* is by definition an irreducible tensor operator of rank *one*.

Like many useful mathematical concepts, the notion of an irreducible tensor operator owes its utility to the fact that it provides a connecting link between two rather different lines of argument. On the one hand, one can see from geometric and physical considerations that various operators and sets of operators that arise in concrete problems either are irreducible tensor operators or are constructed out of such operators in some transparent way. (Note in particular that, when U is finite-dimensional, *every* operator is a sum of components of irreducible tensor operators.) On the other hand, basic results in the theory of group representations permit one to draw rather far-reaching conclusions from the fact that a given $(2l+1)$-tuple of operators is an irreducible tensor operator. Indeed, given a decomposition of the unitary representation U of $SO(3)$ as a direct sum of irreducible representations $U=U^1 \oplus U^2 \oplus \ldots$, there is a sense in which one can describe explicitly the most general irreducible tensor operator $T_{-l}, T_{-l+1}, \ldots, T_l$ in the space of U. Let $\mathcal{H}_j$ denote the subspace of $\mathcal{H}(U)$ (the space of U) in which U is U^j, and let U^j be equivalent to D^{l_j}, where l_j and l_k can be equal even when $j \neq k$. For each j, k, and m, let $(T_m)_{j,k}$ be the projection on $\mathcal{H}_k$ of T_m restricted to $\mathcal{H}_j$. Then, for each m, the operator T_m is completely determined by the matrix of operators $(T_m)_{jk}$. Moreover, it follows immediately from the definition of irreducible tensor operator that whether or not a given $(2l+1)$-tuple of operators $T_{-l}, T_{-l+1}, \ldots, T_l$ is such depends only on the individual matrix elements $(T_m)_{j,k}$ and not on any relationship between them. In other words, if one defines $(\tilde{T}_m)_{j,k}$ to be the operator that is zero on $\mathcal{H}_{j'}$ for all $j' \neq j$ and there coincides with $(T_m)_{jk}$, then T_{-l}, $-T_{-l+1}, \ldots, T_l$ is an irreducible tensor operator or $(0,0,0,\ldots,0)$, if and only if for each pair of indices j, k the $(2l+1)$-tuple $(\tilde{T}_{-l})_{j,k}, (\tilde{T}_{-l+1})_{j,k}, \ldots, (\tilde{T}_l)_{j,k}$ is an irreducible tensor operator, or $(0,0,0,\ldots,0)$. This result reduces the question of finding the most general irreducible tensor operator in $\mathcal{H}(U)$ to finding the most general one such that each T_m is zero in the orthogonal complement of some $\mathcal{H}_j$ and takes its values in some $\mathcal{H}_k$ (k and j being independent of m). For each i the irreducible representation U^i is equivalent to some D^{l_i}, and an inspection of the relevant definitions shows that the reduced problem is equivalent to the following. Find the most general set of $2l+1$ elements in the space of the inner tensor product $D^{l_j} \otimes D^{l_k}$ such that this set of elements is a canonical basis for an irreducible subrepresentation equivalent to D^l. But $D^{l_j} \otimes D^{l_k}$ contains D^l just once or not at all, and contains it once if and only if $|l_j - l_k| \leq l \leq l_j + l_k$. (See the Clebsch–Gordan formula in section 14.) Moreover, a given irreducible representation has a canonical basis that is unique up to a multiplicative constant. It follows that the reduced problem has a nontrivial solution if and only if $|l_j - l_k| \leq l \leq l_j + l_k$ and that then this solution is uniquely determined up to a single multiplicative constant. To describe this essentially unique solution explicitly, one has only to introduce a suitable basis in the space of $D^{l_j} \otimes D^{l_k}$ and give the ~~expansion coefficients in terms of this basis of a suitably normalized~~ canonical basis for the subspace defining the unique subrepresentation equivalent to D^l. One chooses the basis in the space of $D^{l_j} \otimes D^{l_k}$ by choosing a suitably normalized canonical basis in the space of each of D^{l_j} and D^{l_k} and taking products. Thus, one has an expansion coefficient for each triple m_1, m_2, m of integers with $|m_1| \leq l_j$, $|m_2| \leq l_k$, and $|m| \leq l$, and it depends only on l_j, l_k, l, m_1, m_2, and m. It is precisely what has been referred to earlier as a Clebsch–Gordan coefficient. The problem of finding explicitly the most general irreducible tensor operator thus reduces to the problem of computing the general Clebsch–Gordan coefficient as a function of the six variables l_j, l_k, l, m_1, m_2, and m.

This result about the form of the most general irreducible tensor operator is known as the Wigner–Eckart theorem. Putting together the components described above, we may reformulate it as follows. Introduce an orthonormal basis in $\mathcal{H}(U)$ by choosing a (suitably normalized) canonical basis ϕ_m^j ($m_1 = -l_j, l_j+1, \ldots, l_j$) in each $\mathcal{H}_j$. Let $T_{-l}, T_{-l+1}, \ldots, T_l$ be a $(2l+1)$-tuple of nonzero operators, and consider the matrix elements $[T_m(\phi_{m_1}^{j_1}) \cdot \phi_{m_2}^{j_2}]$ as functions of the five variables j_1, j_2, m_1, m_2, and m. The $(2l+1)$-tuple will be an irreducible tensor operator of rank l if and only if there exists a function a of j_1 and j_2 alone such that

$$\left[T_m\left(\phi_{m_1}^{j_1}\right) \cdot \phi_{m_2}^{j_2}\right] = a(j_1, j_2) c_{m_1, m_2, m}^{l_{j_1}, l_{j_2}, l},$$

where the second factor is a Clebsch–Gordan coefficient and is taken to be zero whenever D^l is not contained in $D^{l_{j_1}} \otimes D^{l_{j_2}}$. Note that the dependence of the matrix element on m, m_1, m_2 is (up to a multiplicative constant) independent of j_1, j_2 and the particular irreducible tensor operator. It depends only on l_{j_1}, l_{j_2}, and l. The numbers $a(j_1, j_2)$ are called the *reduced matrix elements* and are what distinguishes one irreducible tensor operator from another.

This group representational definition of an irreducible tensor operator (in slightly altered form) and a statement and proof of the Wigner–Eckart theorem appear in Wigner's classic book, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atome*, published in 1931. Wigner's treatment makes reference to a survey article published by C. Eckart in 1930 in *Reviews of Modern Physics*, where the same ideas appear in less explicit and developed form. However, Racah's first article makes no mention of either Wigner or Eckart. Indeed, all use of group theory (as such) is as sedulously avoided in Racah's first three papers as it is in Slater's 1929 paper and in TAS. The notion of vector operator can easily be defined by using the "algebraic" methods of Chapter III of TAS, and, so defined, the notion (without the name) appears there as well as in various places in the physical literature going back at least to 1930. Racah makes particular mention of the 1931 paper of Güttinger and Pauli cited above in section 19. Let U be a unitary representation of $SO(3)$, and let J_x, J_y, J_z be the "angular momentum" defined by the corresponding Lie algebra representation as explained in section 19. A simple calculation shows that a triple of self-adjoint operators T_x, T_y, T_z is a vector operator in the sense defined by Wigner if and only if it is a vector with respect to J_x, J_y, J_z in the sense defined in section 19. A definition equivalent to the latter (but not the name) appears in Section 8 of Chapter III of TAS, and Sections 8 and 9 contain a proof of the Wigner–Eckart theorem in the special case of vector operators.

Racah's use of the vector operator concept in deriving an explicit formula for the Slater coefficients f_k and g_k is based on the following considerations. First, a derivation of the expansion formula for $\dfrac{1}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}$ used by Slater is given in Section 8 of

Chapter VI of TAS. It begins by writing the expression to be expanded in the form $\frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\omega}}$, where $r_j^2=x_j^2+y_j^2+z_j^2$, and ω is the angle between the vectors from 0,0,0 to x_1, y_1, z_1 and x_2, y_2, z_2. For each r_1 and r_2 the resulting function of $\cos\omega$ is then expanded in Legendre polynomials, yielding the infinite series

$$\sum_{k=0}^{\infty}\frac{[\min(r_1,r_2)]^k}{[\max(r_1,r_2)]^{k+1}}P_k(\cos\omega)$$

Whereas Condon and Shortley go on to derive the formula used by Slater by applying the "spherical harmonic addition theorem" to expand $P_k(\cos\omega)$ in surface harmonics on the spheres S_1: $x_1^2+y_1^2+z_1^2=1$ and S_2: $x_2^2+y_2^2+z_2^2=1$, Racah makes the important observation that the desired f_k and g_k can be expressed simply and directly in terms of the multiplication operators on the underlying Hilbert space $\mathfrak{L}^2(x_1, y_1, z_1, x_2, y_2, z_2)$ defined by the functions $x_1, y_1, z_1, x_2, y_2, z_2 \to P_k(\cos\omega)$, where P_k is, as above, the Legendre polynomial of order k. Indeed, for each l_1, l_2, and L, $f_k(l_1, l_2, L)$ is the eigenvalue with label L of the projection of the multiplication operator $P_k(\cos\omega)$ on an $SO(3)\times SO(3)$ invariant irreducible subspace corresponding to the irreducible representation $D^{l_1}\times D^{l_2}$ of $SO(3)\times SO(3)$. Since this operator commutes with the restriction of $D^{l_1}\times D^{l_2}$ to the diagonal ($\cos\omega$ is rotation invariant), it decomposes according to the decomposition of $D^{l_1}\otimes D^{l_2}$ into irreducibles with one (multiple) eigenvalue $f_k(l_1, l_2, L)$ for each D^L that appears. We omit the (rather similar) analysis of the g_k.

To find f_k and g_k by solving the relevant eigenvalue problems, Racah next observes that multiplication by $P_k(\cos\omega)$ is just the polynomial P_k applied to multiplication by $\cos\omega$ and that this simple multiplication operator is (by elementary geometry) a suitably defined "scalar product" of two vector operators. These two vector operators commute with the restrictions to $SO(3)\times e$ and $e\times SO(3)$ of the underlying representation of $SO(3)\times SO(3)$, and this fact (in "algebraic" form) makes it possible to apply a theorem in Chapter III of TAS to find the matrix elements of multiplication by $\cos\omega$ and thence the eigenvalues of multiplication by $P_k(\cos\omega)$ by matrix algebra. Racah establishes his final formula by using a classical recursion relation expressing P_{k+2} in terms of P_{k+1} and P_k.

By using the fact that the contribution of the electron interaction to the potential energy of an atom is a sum of terms each of which involves the coordinates of just two electrons, Racah's formula for the g_k and f_k, when there are just two electrons, can be used to derive similar formulas for problems with three or more electrons. The last two thirds of the paper is devoted to working out several examples. In particular, a general formula is given for the three-electron case in which two of the electrons are in identical p states.

Given the extensive simplification made possible by Racah's ingenious and insightful exploitation of the concept of a vector operator, one must regard "Theory of Complex Spectra I" as a major contribution to the technique of taking advantage of rotational symmetry.

21. The second paper in Racah's series of four is a direct continuation of the first and was submitted less than nine months later. However, it is much more than a tying up of loose ends in the first. It contains a new idea, which led to further extensive simplifications and to the introduction of the important concept now known as a Racah coefficient. The new idea is that one does not have to first work out the matrix coefficients of multiplication by $\cos\omega$ and then compute P_k of this operator. Instead, one can see directly that multiplication by $P_k(\cos\omega)$ is the (suitably defined) "scalar product" of two *irreducible tensor operators of rank k* and apply a suitable generalization of the theorem in Chapter III of TAS used in "Theory of Complex Spectra I" to compute the matrix elements of a scalar product of two vector operators.

Continuing to avoid group theory as such, Racah defines *irreducible tensor operator* in terms of commutation relations, which generalize those used in TAS in defining vector operator. At the same time he introduces the term "vector operator," which, as already mentioned above, does not appear either in TAS or in Racah's first paper. No direct mention is made of the definition of irreducible tensor operator given in Wigner's 1931 book. On the other hand, Racah gives an "algebraic" proof of the Wigner–Eckart theorem and remarks in a footnote: "This relation was already given by Wigner, Chapter 21, Eq. (19), with group theoretical methods." Just as in the vector operator special case, the proof of the scalar product theorem for irreducible tensor operators uses the Wigner–Eckart theorem—but now in full generality. It is in formulating the scalar product theorem that the now-celebrated Racah coefficients make their first appearance in Racah's work. As in the vector operator case, one is interested in the scalar product of two irreducible tensor operators of the same rank k operating in a Hilbert space $\mathcal{H}^1 \times \mathcal{H}^2$, which is a tensor product of two other Hilbert spaces $\mathcal{H}^1$ and $\mathcal{H}^2$. These are spaces of representations V^1 and V^2 of $SO(3)$, and the tensor operators $T_{-k}, T_{-k+1}, \ldots, T_k$; U_{-k}, $U_{-k+1}, \ldots, U_k$ are such that each T_m "operates only on $\mathcal{H}^1$" in the sense that it is of the form $\tilde{T}_m \times I$ for an operator $\tilde{T}_m$ in $\mathcal{H}^1$, and in the same sense each U_m operates only in $\mathcal{H}^2$. If $\mathcal{H}^1$ is a direct sum of V^1-invariant irreducible subspaces $\mathcal{H}^1_\alpha$, and $\mathcal{H}^2$ is a direct sum of V^2-invariant irreducible subspaces $\mathcal{H}^2_\beta$, then the tensor operators $\tilde{T}$ and $\tilde{U}$ will be completely described by reduced matrices $t(\alpha_1, \alpha_2)$ and $u(\beta_1, \beta_2)$, respectively, and their scalar product will be described by a reduced matrix $v(\alpha_1, \beta_1, \alpha_2, \beta_2)$. The scalar product theorem gives a formula for v in terms of the reduced matrices t and u and the rank k of T and U. This formula involves coefficients built out of Clebsch–Gordan coefficients in what seems to be a complicated way. However, there is a simple function W of six variables defined by taking certain sums of products of four Clebsch–Gordan coefficients such that this formula for v in terms of t, u, and k can be expressed quite simply by using values of W instead of Clebsch–Gordan coefficients. The values of W are called *Racah coefficients*.

Both the role they play in the Wigner–Eckart theorem and the fact that the Racah coefficients may be constructed out of them underlines the fundamental importance of the Clebsch–Gordan coefficients. In recognition of this, Racah begins "Theory of Complex Spectra II" with a "purely algebraic" derivation of an explicit formula for them, which he declares to be more symmetrical and useful than the formula Wigner found by using group theory.

Most of the balance of the paper is devoted to showing how its main result may be used to obtain the f_k and g_k for certain configurations for which earlier methods were too tedious. These include any number of equivalent d electrons and three f equivalent electrons.

Although recognition of the fact that $P_k(\cos\omega)$ is a scalar product of tensor operators of rank k facilitated calculating the f_k and g_k for configurations with more than two electrons, the utility of this method was limited by the fact that it could be fully applied only when the electrons were inequivalent or, at most, equivalent in pairs. Racah addressed this problem in "Theory of Complex Spectra III" by beginning a systematic study of those configurations in which all N electrons are equivalent. As in papers I and II, Racah used "purely algebraic" methods, but we shall explain his ideas in group representational terms. The mathematical problem that presents itself is that of finding an *explicit* decomposition into irreducibles of the antisymmetrized N-fold tensor power of an irreducible unitary representation of $SO(3)\times SU(2)$ of the form $D^l\times D^{1/2}$, where $l=0,1,2,\ldots$. The most general irreducible unitary representation of $SO(3)\times SU(2)$ is ~~$D^l\times D^j$, where $l=0,1,2,\ldots$, $j=1/2$, 1, $3/2$, $2,\ldots$. The N-fold antisymm-~~ etrized tensor power of the irreducible representation L of a group G is the subrepresentation of $L\times L\times L\times\cdots\times L$ (N factors) defined by the invariant subspace of $\mathcal{H}(L)\times\mathcal{H}(L)\times\cdots\times\mathcal{H}(L)$ consisting of all antisymmetric elements in $\mathcal{H}(L)\times\mathcal{H}(L)\times\cdots\times\mathcal{H}(L)$. (See the discussion of the Pauli exclusion principle near the beginning of section 17.)

Now the determination of the decomposition of the antisymmetric square of $(D^l\times D^{1/2})$ is quite easy. A routine argument shows that $(D^l\times D^{1/2})\otimes(D^l\times D^{1/2})=(D^l\otimes D^l)\times(D^{1/2}\otimes D^{1/2})$, and the Clebsch–Gordan formula (see section 14) allows one to replace $D^l\otimes D^l$ by $D^0\oplus D^1\oplus\cdots\oplus D^{2l}$ and $D^{1/2}\otimes D^{1/2}$ by $D^0\oplus D^1$. Applying the distributive law, one arrives at the conclusion that $(D^l\times D^{1/2})\otimes(D^l\times D^{1/2})$ is the direct sum $D^0\times D^0\oplus D^0\times D^1\oplus D^1\times D^0\oplus D^1\times D^1\oplus\cdots\oplus D^{2l}\times D^0\oplus D^{2l}\times D^1$, and, since all terms are inequivalent, the decomposition is unique. A simple argument shows that the antisymmetric subrepresentation $D^l \times D^{1/2} Ⓐ D^l \times D^{1/2}$ is obtained by discarding all terms $D^j \times D^k$ with $j+k$ odd so that the decomposition is $D^0 \times D^0 \oplus D^1 \times D^1 \oplus D^2 \times D^0 \oplus \cdots \oplus D^{2l} \times D^0$. For higher powers it is easy to find a decomposition of $(D^l \times D^{1/2}) \otimes (D^l \times D^{1/2}) \cdots (D^l \times D^{1/2})$ into irreducibles by iterating the procedure described above for $(D^l \times D^{1/2}) \otimes (D^l \times D^{1/2})$. The difficulty lies in picking out the antisymmetric subspace. Racah's method is an iterative one. Observing that the antisymmetric Nth power $(D^l \times D^{1/2})^{N,a}$ of $D^l \times D^{1/2}$ occurs as a subrepresentation of the ordinary tensor product $(D^l \times D^{1/2})^{N-1,a} \otimes (D^l \times D^{1/2})$, he concentrates on how to pass from a given decomposition of the first factor to the desired subrepresentation of the product. Note that any decomposition of $(D^l \times D^{1/2})^{N-1,a}$ into (not necessarily inequivalent) irreducible subrepresentations $U^1 \oplus U^2 \oplus \cdots \oplus U^r$ carries with it a canonical decomposition of $(D^l \times D^{1/2})^{N-1,a} \otimes (D^l \times D^{1/2})$ into (not necessarily inequivalent) irreducible subrepresentations labeled by pairs consisting of one of the U^i and an index l, j telling which irreducible representation $D^l \times D^j$ it is equivalent to. This is because $(U^1 \oplus U^2 \oplus \cdots \oplus U^r) \otimes (D^l \times D^{1/2})$ is canonically a direct sum, $[U^1 \otimes (D^l \times D^{1/2})] \oplus [U^2 \otimes (D^l \times D^{1/2})] \oplus \cdots \oplus [U^r \otimes (D^l \times D^{1/2})]$, and each term in this sum is uniquely a direct sum of inequivalent irreducibles. Evidently, a term in the final sum is uniquely specified by its equivalence class and the particular U^i whose product with $D^l \times D^{1/2}$ contains it. One speaks of U^i as its "parent." Given any irreducible subrepresentation L of the fully antisymmetrized Nth power $(D^l \times D^{1/2})^{N-1,a}$, which is equivalent, say, to $D^{l_1} \times D^{j_1}$, it will necessarily be a sub-sub-representation of that subrepresentation of $(D^l \times D^{1/2})^{N-1,a} \otimes (D^l \times D^{1/2})$ consisting of the direct sum of all irreducibles whose second label is l_1, j_1. There will be at most *one* of these having a given U^i as first label. Racah shows how to exploit this fact to describe the subrepresentation L completely by assigning a complex number $C(U^i)$ to each U^i. This set of complex numbers is uniquely determined up to multiplying them all by the same nonzero complex number, and $C(U^i)=0$ when and only when U^i does not occur as a first label. Unlike the canonical subrepresentations of $(D^l \times D^{1/2})^{N-1,a} \otimes (D^l \times D^{1/2})$, the irreducible subrepresentation L of $(D^l \times D^{1/2})^{N,a}$ will generally not be assignable to unique parents. They will have several parents, and the numbers $C(U^i)$ are called *coefficients of fractional parentage*. The main point of Racah's third paper is to define these coefficients, explain their relevance to the problem, and present an iterative method for computing them explicitly. Racah refers to a paper of Goudsmit and Bacher published in the *Physical Review* in 1934, in which what are essentially the squares of his coefficients are defined and used in a somewhat similar way.

The definition of the coefficients of fractional parentage (c.f.p.'s) may be based on a quite simple general principle in the theory of group representations. Let L be an irreducible unitary representation of a group G, and let kL denote the direct sum $L \oplus L \oplus L \oplus \cdots \oplus L$ of L with itself k times. For each k tuple $\lambda_1, \lambda_2, \ldots, \lambda_k$ of complex numbers, let $\mathcal{H}_{\lambda_1,\lambda_2,\cdots\lambda_k}$ denote the subspace of $\mathcal{H}(kL)=\mathcal{H}(L)\oplus\mathcal{H}(L)\oplus\mathcal{H}(L)$ consisting of all vectors of the form $\lambda_1\phi, \lambda_2\phi, \ldots, \lambda_k\phi$. Then, whenever the λ_j are not all zero, it is evident that $\mathcal{H}_{\lambda_1\lambda_2\cdots\lambda_k}$ is a nontrivial invariant subspace of $\mathcal{H}(kL)$, which defines an irreducible subrepresentation of kL. Conversely, it is not hard to show that *every* irreducible subrepresentation of kL has some $\mathcal{H}_{\lambda_1,\lambda_2,\cdots\lambda_k}$ for its space. Since it is clear that $\mathcal{H}_{\lambda_1,\lambda_2,\cdots\lambda_k}=\mathcal{H}_{\mu_1,\mu_2,\cdots\mu_k}$ if and only if $\mu_1=c\lambda_1$, $\mu_2=c\lambda_2, \ldots, \mu_k=c\lambda_k$ for some $c\neq 0$, it follows that the irreducible subrepresentations of kL correspond one-to-one in a natural way to the sequences $\lambda_1, \lambda_2, \ldots, \lambda_k$, provided that we identify $\lambda_1, \lambda_2, \ldots, \lambda_k$ with $c\lambda_1, c\lambda_2, \ldots, c\lambda_k$ whenever $c\neq 0$. Since the U^i, l_1, and j_1 above label equivalent representations, one can parameterize the subrepresentations of $(D^l\times D^{1/2})^{N-1,a}\otimes (D^l\times D^{1/2})$ equivalent to $D^{l_1}\times D^{j_1}$ by such equivalence classes of tuples of complex numbers, and those tuples that come from irreducible subrepresentations of $(D^l\times D^{1/2})^{N,a}$ are just the c.p.f.'s as defined by Racah.

It should now be clear that one can define the subrepresentation of $(D^l\times D^{1/2})^{N-1,a}\otimes D^l\times D^{1/2}$ corresponding to any given irreducible subrepresentation of $(D^l\times D^{1/2})^{N,a}$ by specifying an r tuple of complex numbers (its c.f.p.'s) and that the r tuples specifying the family of all irreducible subrepresentations equivalent to a given one form a vector space whose dimension is equal to the multiplicity with which that irreducible occurs in $(D^l\times D^{1/2})^{N,a}$. To determine the vector space of r tuples associated with any given equivalence class of irreducible unitary representations of $SU(3)\times SU(2)$, Racah writes down a system of homogeneous linear equations in r variables whose solutions constitute this vector space. When $N=3$, the coefficients of these equations are slight modifications of the Racah coefficients introduced in "Theory of Complex Spectra II." For $N>3$, they are products of modified Racah coefficients with coefficients of fractional parentage for the case when N is replaced by $N-1$.

If $(D^l\times D^{1/2})^{N,a}$ were always multiplicity-free, the above procedure would permit the recursive calculation of all coefficients of fractional parentage and an explicit determination of the antisymmetric subspace of the space of $(D^l\times D^{1/2})^N$. Unfortunately, $(D^l\times D^{1/2})^{N,a}$ already fails to be multiplicity-free when $l=2$ and $N=3$, and in order to carry out Racah's procedure it is necessary to find a canonical way of decomposing each primary constituent (see section 11) into irreducibles so that the U^i will be well defined. This turns out to be a nontrivial problem, and Racah presents only the beginnings of a solution in "Theory of Complex Spectra III." For each l and N he defines a self-adjoint operator Q in the space of the representation of $(D^l\times D^{1/2})^{N,a}$, which can be shown to be in the commuting algebra of this representation. It follows that each primary constituent of $(D^l\times D^{1/2})^{N,a}$ acts in a subspace that is invariant under Q and splits into as many subrepresentations as Q has eigenvalues in this subspace. Thus, $(D^l\times D^{1/2})^{N,a}$ decomposes into primary representations (which may belong to the same irreducible), and the different occurrences of this irreducible are distinguished from one another by the eigenspaces of Q to which they belong. It turns out that the eigenvalues of Q all have the form $\frac{1}{4}(n-\nu)(4l+4-n-\nu)$, where ν is a positive integer called the seniority number. If only the subrepresentation associated with each eigenspace of Q (that is, with each seniority number) were multiplicity-free, the introduction of Q and the seniority number would solve the problem. This actually happens when $l=2$, and Racah uses his method to compute all coefficients of fractional parentage for p and d electrons; the results are presented in Tables I, II, III, and IV of Racah's paper. These tables take the form of matrices whose columns are labeled by irreducible subrepresentations of $(D^l\times D^{1/2})^{N-1,a}$ and whose rows are labeled by irreducible subrepresentations of $(D^l\times D^{1/2})^{N,a}$. In comparing these tables with the above remarks, it is important to bear in mind that Racah is not dealing with group representations as such and that he labels the irreducible subspaces of what we consider to be the space of $(D^l\times D^{1/2})^{N,a}$ by spectroscopic labels signifying the orbital and spin angular momenta of the corresponding states. Thus, an irreducible subspace corresponding to the representation $D^{l_1}\times D^j$ would have the label ${}^{j+1}X$ where $X=S$, P, D, F, G, H, or I, according as $l=0$, 1, 2, 3, 4, 5, or 6. The superscript $j+1$ is the dimension of D^j and is the so-called "multiplicity of the term"—that is, the number of levels into which the level in question divides under the influence of the spin orbit perturbation. The seniority number is indicated by a subscript on the superscript. Thus, for any N, ${}^1D$ indicates *the* subrepresentation of $(D^1\times D^{1/2})^{N,a}$ (if any) that is equivalent to $D^2\times D^0$, and ${}^4_3F$ indicates the subrepresentation (if any) of $(D^2\times D^{1/2})^{N,a}$ that is equivalent to $D^3\times D^{3/2}$ and has seniority number 3.

22. Before describing Racah's fourth and final paper on complex spectra—which was separated from the first three by a six-year interval—it seems appropriate to mention a remarkable privately circulated typescript by Wigner. This paper seems to have been written around 1941 and was finally published in 1965. Among other things, it contains an independent definition and discussion of the Racah coefficients. To go back a bit further, Wigner never permitted his belief in group theory to be seriously disturbed by the success of Slater, Dirac, and subsequent writers in avoiding its explicit use in atomic spectroscopy. In addition to writing the book cited earlier on applications of group theory to atomic spectroscopy, he wrote or co-authored a number of pioneering articles applying the theory of group representations to various branches of quantum physics. Among the more important are a paper of 1930 on applications to molecular spectroscopy, a paper (with L. P. Bouckaert and R. Smoluchowski) published in 1936 on applications to the energy band problem in the theory of the solid state; a paper published in 1937 on applications to the structure and classification of atomic nuclei; and a paper published in 1939 determining the "physically

relevant" irreducible unitary representations of the inhomogeneous Lorentz group (Poincaré group) and applying this determination to the classification of relativistic particles. The last of these articles is a landmark paper in pure mathematics as well in physics. Except for the trivial representation, the irreducible unitary representations of the Poincaré group are all *infinite*-dimensional, and this is the first paper in which infinite-dimensional irreducible unitary representations are classified. It was one of the main stimuli in the later development of the representation theory of noncompact semisimple Lie groups.

Wigner's unpublished typescript was meant to be a sequel to a paper published in 1941 in the *American Journal of Mathematics*, and the two together constitute an abstract generalization and an essentially purely mathematical analysis of Clebsch–Gordan (or Wigner) coefficients, Racah coefficients, and related notions. Wigner observed that $SU(3)$ has two properties that play an especially important role in the application of its representation theory to atomic spectra and decided to study compact groups having these properties in the general case. By far the more important of the two is the property that $L\otimes M$ is multiplicity-free whenever both L and M are irreducible. The other is that each conjugacy class is self-inverse (equivalently each irreducible representation is equivalent to its complex conjugate). Any compact group with these two properties is said to be *simply reducible*. It is easy to see that both $SO(3)$ and $SU(2)$ are simply reducible and that any direct product of simply reducible groups is simply reducible—in particular $SO(3)\times SU(2)$. After introducing Clebsch–Gordan coefficients for the general simply reducible group by generalizing the definition for $SO(3)$, Wigner is led to what he calls "6-j symbols" by comparing the two decompositions of $L\otimes M\otimes N$ that one obtains by writing it first as $(L\otimes M)\otimes N$ and then as $L\otimes(M\otimes N)$ and using the (generalized) Clebsch–Gordan series iteratively. Wigner's 6-j symbols are just the Racah coefficients, and it is interesting to compare Wigner's discovery of them with that of Racah. Wigner discovered them while proceeding in a purely mathematical spirit, whereas Racah was led to them (in two different ways) while trying to solve concrete physical problems. On the other hand, Racah's second forced use came precisely because his problem involved comparing $(L\otimes M)\otimes N$ with $L\otimes(M\otimes N)$. Wigner's typescript contains proofs of many of the most important properties of the Clebsch–Gordan and Racah coefficients, all done in the generalized context of simply reducible groups.

23. Racah's fourth paper, "Theory of Complex Spectra IV," is a direct continuation of paper III in spite of the six-year time interval between the two papers. On the other hand, it makes a definite break with the past and with papers I, II, and III in that the author abandons his "purely algebraic"

approach and makes explicit and heavy use of the theory of group representations.

Racah's first observation in paper IV is that the antisymmetric Nth power $(D^l \times D^{1/2})^{N,a}$ of the representation $(D^l \times D^{1/2})$ of $SO(3)\times SU(2)$ may be looked upon as the restriction to $SO(3)\times SU(2)$ of an irreducible representation of a group G containing $SO(3)\times SU(2)$ as a subgroup. He then shows how this fact can be exploited to give a new definition of the seniority number. This new definition suggests a variety of generalizations of the seniority number, and Racah finds such a generalization for the case of f electrons ($l=3$) that leads to a complete reduction of $(D^3 \times D^{1/2})^{N,a}$ into irreducibles with distinct labels.

The essence of Racah's idea can perhaps best be understood by considering it in a slightly more general context. Let us replace the representation $D^l \times D^{1/2}$ of $SO(3)\times SU(2)$ by a finite-dimensional unitary representation L of an arbitrary compact group K. Suppose that the dimension of L is m and that L is "faithful" in the sense that L_x is the identity only when x is the identity element of K. Then the mapping $x \to L_x$ is an isomorphism of K with a closed subgroup of the compact group $U(m)$ of all unitary operators in the m-dimensional space $\mathcal{H}(L)$ of L, and we may identify K with this subgroup $U(m)$. Now let J denote the representation of $U(m)$ that takes each element of $U(m)$ into itself. Then J is an irreducible unitary representation of $U(m)$ whose restriction to K is precisely our given representation L. Thus, the antisymmetric Nth power $L^{N,a}$ of L is just the restriction to K of $J^{N,a}$, the antisymmetric Nth power of J. By the known representation theory of $U(n)$, $J^{N,a}$ is irreducible. Thus, $L^{N,a}$ is realizable as the restriction to K of an irreducible unitary representation of $U(m)$. Now let H be an arbitrary proper closed subgroup of $U(m)$, which contains K properly—$K \subset H \subset U(m)$. We may obtain $L^{N,a}$ by first restricting $J^{N,a}$ to H and then restricting the result to K. However, the restriction of $J^{N,a}$ to H will usually not be primary, and we may consider its canonical decomposition into ~~primary parts $J^{N,a}|H=M^1\oplus M^2\oplus\cdots\oplus M^r$, where the M^j are multiples of~~ inequivalent irreducible representations of H. Now $L^{N,a}$ is clearly a direct sum of the restrictions to K of the representations M^j of H, and the decomposition of these restrictions gives us a decomposition of $L^{N,a}$ into primary components distinguished by *two* labels. One is an equivalence class of irreducible representations of K, and the other is an equivalence class of irreducible representations of H. Ideally one would seek to find an H such that the M^j are irreducible and have multiplicity-free restrictions to K. The resulting decomposition into primaries would then actually be one into irreducibles, and the multiplicity problem would be solved. On the other hand, almost any H will provide *some* splitting of the primary components of $L^{N,a}$, and one can go much further by using several different H's at once as long as the H's contain one another: $U(m) \supset H_1 \supset H_2 \supset \cdots \supset H_t \supset K$. The splittings they provide will be compatible and may be combined.

In comparing the above account of Racah's idea with what he actually did, it is important to realize that he used the full linear group instead of the unitary group $U(n)$ and that he took advantage of the fact that his K was a direct product to make a preliminary reduction replacing K by $SO(3)$ and $D^l \times D^{1/2}$ by D^l. Thus, Racah's m is $2l+1$. He gets the splitting provided by the seniority number by choosing H to be the orthogonal group $SO(2l+1)$. As mentioned in section 22, the seniority number fails to provide a complete splitting when $l=3$. In this special case Racah finds a subgroup of $SO(2l+1)=SO(7)$ that contains $SO(3)$ and that completes the splitting. It is the subgroup leaving invariant a certain antisymmetric trilinear form, and it is isomorphic to G_2, the exceptional simple compact Lie group of lowest order.

Most of the balance of "Theory of Complex Spectra IV" is devoted to finding the coefficients of fractional parentage for the case $l=3$ using the complete splitting provided by $SO(7)$ and G_2. In addition, however, a second general principle is enunciated that simplifies the calculations; we shall only mention it without attempting a full explanation. Using the fact that $(D^l \times D^{1/2})^{N,a}$ is a subrepresentation of $(D^l \times D^{1/2})^{N-1,a} \otimes D^l \times D^{1/2}$ and a simple lemma about restrictions and tensor products of group representations, Racah shows how the coefficients of fractional parentage may be factored as products of functions depending upon smaller numbers of variables.

24. The year 1949 is a significant one in the history of the development of angular momentum theory. First, it is the year in which Racah completed his celebrated series of four papers on angular momentum theory in atomic spectroscopy. Second, it is the year in which that same Racah, a chief advocate and developer of "purely algebraic" methods, reintroduced group theory and did it with a vengeance. Third, it is the year in which Racah's methods and concepts began to find applications to other parts of physics. Finally, it is the year in which L. C. Biedenharn, the senior author of the ~~present volume, completed his doctorate and formally began his scientific~~ career.

One would expect the theory of rotational invariance to have applications to any physical problem in which this kind of symmetry prevailed, and one would expect Racah's methods to play a role as soon as the problem was pursued in sufficient depth. A branch of physics, other than atomic spectroscopy, in which rotational invariance plays an especially significant role is the theory of nuclear structure and more generally the spectroscopy of the excited states of nuclei. A nucleus seems quite different in structure from an atom. It is a "bound state" of roughly equal numbers of two kinds of particles, neutrons and protons, which attract one another with very short-range forces and have roughly the same mass. An atom, on the other hand, is a "bound state" of one massive nucleus and a number of very light, identical electrons. The electrons repel one another and are attracted by the nucleus with long-range forces. Nevertheless, there are important similarities. As Heisenberg pointed out in 1932 (the year the neutron was discovered), one can consistently think of the neutron and the proton as two different states of a single particle differing from one another in somewhat the same way as do two electrons with oppositely oriented spins. Taking this view and calling the single particle a *nucleon*, one finds that a nucleus is a bound state of a set of *identical* nucleons. Thinking of the center of gravity of the system of nucleons as the analog of the nucleus of an atom and a single nucleon as the analog of an electron, one sees that one can apply rotational symmetry in much the same way. There are various a priori difficulties having to do with the strength and nature of the nuclear force, but they seem to cancel one another to a considerable extent. As far as group theory is concerned, one finds a rather close parallelism with atomic spectroscopy. The chief difference is that the nucleon has spin $\frac{1}{2}$ and admits different spin states as well as the neutron and proton states. This (together with the fact that the nuclear force is independent of whether the nucleon is in the proton or neutron state) has the effect of replacing the group $SU(2)$ by the direct product $SU(2)\times SU(2)$, so that the fundamental symmetry group of a single nucleon is $SU(3)\times SU(2)\times SU(2)$ instead of $SU(3)\times SU(2)$, as with a single electron.

Applications of rotational invariance to the theory of the nucleus began in 1937 with independent papers of Wigner and Hund. Wigner (in a paper mentioned above) used group representations and showed that many facts about nuclear spectroscopy were consistent with the hypothesis that the group $SU(2)\times SU(2)$ could be replaced by the larger group $SU(4)$—that is, that nuclear forces were more symmetric than mere "charge independence" would imply. Hund, using Slater's methods, analyzed those nuclei and nuclear states that are the analogs of electron configurations with several equivalent p electrons. Because of the replacement of $SU(2)$ by $SU(2)\times SU(2)$, the Pauli exclusion principle does not exclude so much, and because of this and the complicated nature of nuclear forces, Hund found himself involved in very long calculations. An additional triumph of the formula found by Racah in "Theory of Complex Spectra I" was that it could be adopted to the nuclear case, and it led to Hund's results in a much less tedious way.

Apart from this early observation of Racah, the first application of his methods outside of atomic spectroscopy seems to have been made in an article published in 1949 in the *Proceedings of the Physical Society of London*. The author, J. W. Gardner, was concerned with the correlations between successive γ-ray emissions from atomic nuclei. He found that his formulas could be greatly simplified by using the Racah coefficients. Then, in 1950, H. A. Jahn applied the methods of Racah's paper IV to nuclei and was able to analyze configurations containing nucleons in equivalent d-states. This was the beginning of a series of articles by Jahn himself, Jahn and van

Proceedings of the Royal Society, and carrying applications of Racah's methods to nucleon spectroscopy about as far as Racah had done for atomic spectroscopy. Racah quickly became interested, and in a famous series of lectures on his methods given at the Princeton Institute for Advanced Study in 1951 he emphasized nuclear applications and barely mentioned atomic ones. (These lectures, long unpublished, can now be found in Volume 37 of *Ergebnisse der exakten Naturwissenschaften*," published by Springer Verlag in 1965.)

As mentioned above, Biedenharn, the senior author of this book, was just beginning his scientific career when Racah's paper IV appeared. Its publication coincided with and partly inspired a surge of interest in the theory and applications of Racah's methods. Biedenharn soon became deeply involved, and he is now one of the leading experts on all phases of angular momentum theory, but especially those involving Racah coefficients and Racah's methods generally. In 1951 (in collaboration with Arfken and Rose), he was one of a number who followed Gardner in applying Racah coefficients to γ-ray correlations. The following year he published one of the first tabulations of the Racah coefficients as well as a paper with Blatt on a new application. The new application was to scattering theory—the angular distribution of scattering and reaction cross sections. A third paper, also published in 1952 (with Blatt and Rose), is a general account of the properties of Racah coefficients and certain modifications, which are more convenient in scattering theory. In computing tables it is often more useful to proceed from recursion relations than from explicit formulas for the functions to be computed. In searching for recursion relations for the Racah coefficients, Professor Biedenharn discovered a remarkable new identity between such coefficients. It immediately implies a useful recursion relation and was later found to have an elegant conceptual interpretation. This result was published in the *Journal of Mathematics and Physics* in 1953.

For the next seven years or so, Professor Biedenharn concerned himself ~~almost exclusively with other parts of theoretical physics, but he returned to~~ angular momentum theory in 1961. He has been a prolific and steady contributor ever since. One of the earliest contributions of this second period is "Group-Theoretical Approaches to Nuclear Spectroscopy," a lecture course given in 1962 at the Summer Institute for Theoretical Physics at the University of Colorado in Boulder. However, the main thrust of his later work is in a more purely mathematical direction and very much in the spirit of Wigner's long unpublished typescript on simply reducible groups. (It is thus of some significance that when this paper finally did appear it was in the anthology *Quantum Theory of Angular Momentum*, edited by Biedenharn and van Dam and published in 1965. This anthology also contains two early papers by Biedenharn as well as all four of the papers in Racah's famous series.)

The fact that the groups $SU(3)$ and $SU(2)$ are "simply reducible" in Wigner's sense may be regarded as a lucky accident. We know of no reason why physical space could not have been of higher dimension. Moreover, the analogs of the rotation group in homogeneous Riemannian spaces of higher dimension include all compact semisimple Lie groups (at least to within local isomorphism). There is no difficulty in extending quantum mechanics to higher-dimensional homogeneous spaces and even in deciding what the Schrödinger equation should look like. If the space we live in had been one of these higher-dimensional possibilities, all the problems dealt with in angular momentum theory would have arisen with $SO(3)$ and $SU(2)$ replaced by other compact Lie groups—chosen from a family very few of which are simply reducible. A little reflection makes it clear that one could hope to apply the same kind of method no matter what compact Lie group replaces $SO(3)$—that simple reducibility is a convenience and not a necessity. Its chief function is to trivialize the multiplicity problem in the simple cases and to simplify it in the more complex ones. In other words, it is reasonable to conjecture the existence of a far-reaching generalization of angular momentum theory that would not only extend Wigner's abstract theory by dropping the hypothesis of simple reducibility but go much further and include later developments such as appear in Racah's third and fourth papers and elsewhere. The justification for seeking such a generalization would not lie in our speculative desire to see what the world would be like if space had a different structure. It would be found more in a desire to understand better the mathematics involved in the world we live in by seeing it as a special case of something more general. However, the fact that a more general theory would have a conceivable if speculative application gives us confidence that an interesting and possibly illuminating generalization exists. On a more immediately practical level, a number of other symmetry groups occur both in spectroscopy and in other parts of physics to which a more general Wigner–Racah type analysis might be applied.

In any event, the program of so generalizing angular momentum theory has attracted a number of mathematically minded theoretical physicists including the two authors of this book. The junior author, J. D. Louck, entered the field in the middle 1960's and has since made many contributions, both alone and in collaboration with Biedenharn and others.

Although this program is perhaps the dominant research interest of the two authors, it is not their only interest, and they have treated it quite sparingly in the present work. They have, on the other hand, discussed the rather rich, purely mathematical aspects of classical angular momentum theory in considerable detail. They have used their knowledge of generalizations to lead them to a generalizable discussion but have refrained from presenting the generalizations themselves. It would be tempting to give a more detailed account of what the authors have actually done here, but I have already written a longer introduction than was asked for. It seems time to stop and let the authors speak for themselves.

George W. Mackey

The Racah–Wigner Algebra in Quantum Theory

CHAPTER 1

Introduction

It has been the purpose of the monograph *Angular Momentum in Quantum Physics* (AMQP) to develop comprehensively those aspects of angular momentum theory that are required in carrying out research in modern physics and chemistry. The emphasis there was principally on physical concepts and the symmetry principles that underlie the general applicability of angular momentum theory to a broad area of physical phenomena. Mathematical techniques were introduced within the context of the physical concepts themselves.

Physical applications of angular momentum theory are to a large extent applications of specific properties of Wigner or Racah functions (more generally of $3n$-j coefficients) or of the representation functions themselves. Such applications tend to emphasize the "numerical" aspects of these functions and thus obscure the relationships to other areas of mathematics.

The purpose of the present monograph is to show by specific examples the many interrelations that exist between concepts originating in angular momentum theory and various areas of mathematics. It turns out that this is a substantial task—the diversity of the interrelations is far greater than might be thought.

Even a brief acquaintance with the contents of AMQP will show that the concept that recurs again and again in physical applications is that of an irreducible tensor operator with respect to a group [in our case, $SU(2)$, the quantal rotation group]. Indeed, it is this concept that carries the mathematical apparatus of the physical theory beyond the related but much simpler results of the Lie algebraic theory of the angular momentum itself and the implied representation theory of the group $SU(2)$.

The physical theory of interactions (for rotationally invariant systems) *requires* the introduction of the concepts of irreducible tensor operators and

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the attendant notions of Wigner and Racah coefficients. Although one may take the viewpoint that Wigner coefficients are elements of the matrix that reduces the direct product of two irreducible representations (irreps) of $SU(2)$ into the irreducible constituents of the diagonal $SU(2)$ subgroup, this limited viewpoint misses the equally important aspect of these coefficients as matrix elements of irreducible tensor operators (Wigner–Eckart theorem). Moreover, it is precisely this physical viewpoint as implemented through the Wigner–Eckart theorem that leads naturally to the interpretation of a Racah coefficient as the matrix element of an invariant operator.

It is therefore an important problem to characterize the physical theory of irreducible tensor operators in terms of the mathematics of operator theory. A general theory dealing with the (usually) unbounded operators that occur in physical theory is an extremely difficult undertaking. Fortunately, these difficulties may be circumvented in the characterization of the "angular momentum properties" of a physical system. This fortunate circumstance is due to the existence of the Wigner–Eckart theorem showing that the matrix elements of an irreducible tensor operator factorize into two parts: a physical part, usually unbounded, and a geometric part, a Wigner coefficient, that defines a bounded operator.

This result suggests that one can develop the properties of Wigner and Racah coefficients, and their interrelations, within the framework of bounded operators acting in a separable Hilbert space. The basic problem then becomes one of selecting an appropriate Hilbert space and an appropriate definition of operator actions such that the (schematic) association

$$(\text{Operator}) + (\text{Hilbert space}) \rightarrow \text{coefficient} \tag{1.1}$$

leaves out no essential properties of the coefficients as they arise in physical theory.

In Part I of the present volume (Chapters 2–4), we develop the details of the above program, introducing the concepts of Wigner operators (Chapters 2 and 3) and Racah operators (Chapter 4) as explicit, bounded operators acting on specified separable Hilbert spaces. Wigner operators are bounded operators whose matrix elements are Wigner coefficients. This algebra of Wigner operators also involves Racah coefficients; hence, it is designated Racah–Wigner or RW-algebra. Racah operators are bounded invariant operators whose matrix elements are Racah coefficients. This algebra of invariant operators involves only Racah operators (W-coefficients) and is designated W-algebra. Both RW-algebra and W-algebra are noncommutative, associative algebras.

The significance of these operator realizations lies in the interpretations that are now implied for the many relations between Wigner and Racah coefficients, and, more important, in the structural results that are implied for the coefficients themselves. The nontrivial nature of the interpretive results is nicely illustrated by the invariant operator role that is now assigned to the Racah coefficient in RW-algebra and the fact that the associativity of this algebra implies, and is implied by, the B–E (Biedenharn–Elliott) identity satisfied by the Racah coefficients. There are two principal structural results in RW-algebra and in W-algebra, and these are similar in character. The first result shows that a general element in the algebra is a polynomial form defined on a set of four fundamental operators. The second result establishes the relationship between the null space of a Wigner operator (RW-algebra) or Racah operator (W-algebra) and the zeros of the corresponding coefficient. The result of this analysis is the factorization of a Wigner coefficient or Racah coefficient into a canonical form determined by the characteristic null space zeros.

One, of course, recovers from RW-algebra and W-algebra the many known numerical relations between Wigner and Racah coefficients as indicated schematically by

$$\begin{pmatrix} \text{algebraic relations} \\ \text{between operators} \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{numerical relations} \\ \text{between coefficients} \end{pmatrix}. \tag{1.2}$$

This purely transcriptional aspect of the operator viewpoint is, in itself, not particularly important; as remarked above, it is the interpretation of such relations that affords one new insights and suggests generalizations to other groups.

It should be pointed out that the association (1.1) of operators acting in Hilbert space to Wigner coefficients, for example, does not treat the angular momenta j_1, j_2, j_3 in the 3-j symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ equivalently, since certain angular momenta are to be associated with the "space" and others with the "operator." This is an intrinsic property of the present viewpoint and cannot be avoided. This causes no difficulties, however, since it only implies that alternative, but equivalent, presentations of the theory may be given (by using different identifications of the angular momenta with "operator" versus "space").

Part I of this monograph uses results freely from Chapters 3 and 5 of AMQP, with particular emphasis on the tensor operator concept, the Wigner–Eckart theorem, and boson calculus methods. Our presentation of the theory of angular momentum would be incomplete if we did not relate it to such basic precepts of quantum mechanics as the Wigner theorem on symmetry operators, uncertainty relations for noncommuting observables, and classical limits, or if we failed to note the interrelations with projective geometry, classical functions, and graph theory. These views of the angular momentum functions, as well as others, are developed in Part II of this monograph in a series of twelve Topics.

Each Topic presented in Chapter 5 (which constitutes Part II) may be considered as part of RW- and W-algebra in that it develops some important viewpoint of the Wigner, Racah, and representation functions. Rather than giving a terse summary of the Topics, we introduce each Topic here with a short descriptive remark, which characterizes the subject matter and its relationship to the rest of the monograph.

Topic 1: The Wigner theorem on symmetry operators is basic to the implementation of any physical symmetry in quantum mechanics.

Topic 2: The $SU(2)$ representation functions lend themselves to a natural generalization of the ordinary spherical harmonics as encompassed in the concept of a Hilbert space of *sections*[1] and lead to the monopolar harmonics that describe the states of an electron in the field of a magnetic monopole.

Topic 3: The tensor operator concept leads to an unusual realization of the generators of $SU(2)$ and an underlying Hilbert space of polynomials over a pair of noncommuting variables.

Topic 4: The existence of a Cayley–Hamilton theorem for the operation of commutation of the square of the total angular momentum with a tensor operator is the structure theorem that underlies the explicit construction of operators that shift the angular momentum quantum numbers in physical problems.

Topic 5: Physical theory leads naturally to the consideration of *complex* angular momentum quantum numbers and the classification of the unitary irreps of the noncompact group $SU(1,1)$.

Topic 6: The canonical form of a Wigner coefficient has a natural analytic continuation that gives the value of an important class of integrals over associated Laguerre polynomials that occur in the evaluation of radial integrals for the Coulomb and oscillator problems.

Topic 7: The uncertainty relations are key concepts in the interpretation of quantum mechanics. This Topic develops and interprets the uncertainty relations for angular momentum, relating the definition of canonically conjugate variables for three-space angular momentum to the factorization of the vector Wigner operators into polar form.

Topic 8: The relationship between the Racah function and the complete quadrilateral suggests interpretations of the Racah identity and the B–E identity in terms of theorems in projective geometry.

Topic 9: The refinement of Wigner's classical limits of the 3-j and 6-j symbols to include the rapidly oscillating phase factor (classical region), the connection formulas (transition region), and the exponential decaying factor (nonclassical region) is a classic problem combining angular momentum and JWKB techniques.

[1]These are called cross sections in the mathematical literature on vector bundles.

Topic 10: Only a few occurrences of the accidental zeros of the 3-j and 6-j symbols are presently understood.

Topic 11: The symmetries of the 3-j and 6-j symbols are the "classic" symmetries of the ${}_3F_2$ and ${}_4F_3$ hypergeometric series, respectively.

Topic 12: Each transformation coefficient between binary coupling schemes for n angular momenta may be evaluated by phase and Racah coefficient transformations induced by commutation and association of angular momentum labels. The classification of all transformation coefficients is a problem of constructing a well-defined class of cubic graphs.

CHAPTER 2

Algebraic Structures Associated with Wigner and Racah Operators

1. Introduction and Survey

The application of standard angular momentum techniques to physical problems leads in practice quite often to extensive algebraic manipulations; physicists have designated such calculations as "angular momentum technology" (Danos [1]), or more formally as "Racah algebra" (Sharp [2]), or even pejoratively as "Clebsch–Gordanology." A considerable body of practical results, and methodology, has been accumulated in these applications. To organize this material into a coherent structure is an important problem, which we shall discuss and resolve in this chapter and the succeeding two chapters.

One can distinguish two very different approaches to the problem. It is only to be expected in practical applications involving the angular momenta of many particles that the results should be complicated, since from a formal view one is applying invariant theory to the construction of invariant functions over many variables. Group-theoretically, one is constructing invariants with respect to the diagonal $SU(2)$ subgroup (generated by the total angular momentum) of the n-fold direct product group $SU(2)\times SU(2) \times \cdots \times SU(2)$ (generated by the n kinematically independent angular momenta of n particles).

Such a view of the "Racah–Wigner calculus" is a straightforward generalization of the Racah coefficient (6-j symbol) to the Fano coefficient (9-j symbol),..., leading to the $3n$-j symbols. This generalization has been devel-

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oped primarily by Jucys[1] *et al.* [3] using graphical techniques, and in a variant form by El Baz and Castel [4]. (These results are discussed in Chapter 5, Topic 12.)

In contrast to this "extensive" approach, one may distinguish an "intensive" approach, which seeks to clarify the underlying structure per se. This is the approach of Wigner [5, 6], who categorized the group-theoretic conditions that suffice for defining vector-addition coefficients (Wigner coefficients) for an arbitrary compact group. This more-circumscribed problem (which leads to the concept of a *simply reducible group*) is discussed in Chapter 3, Section 4. It should be recognized that, although these results suffice for a "Racah-Wigner calculus" to exist, they give no characterization of the "calculus" itself. It was to this problem that W. T. Sharp addressed his thesis on Racah algebra (Sharp [2]). Despite considerable progress, this attempt (as Sharp himself states) cannot be considered as wholly successful.

Three algebraic approaches have been developed in connection with compact symmetry groups [such as $SU(2)$] that might be relevant for the purpose of categorizing the Racah–Wigner calculus. The first two are (1) the group algebra of a compact group (Boerner [7]) and, more generally, (2) the universal enveloping algebra of the group generators (Jacobson [8]). *Both these algebraic approaches fail to be sufficiently general for the purposes of treating angular momentum in quantum physics.* This is easily seen from the fact that both approaches exclude spinorial operators [which can enter, for example, in weak interactions (beta decay)]. More generally, both algebras are restricted to algebraic elements (operators) that commute with the Casimir invariant, and hence can only describe physical processes in which $\Delta J = 0$. A third algebraic approach was considered by Sharp as the basis for Racah algebra. This is the "algebra of representations," based on reducing the Kronecker product of irreducible representations. Applications of this structure are given by Sharp [2] and, for finite groups, by Biedenharn *et al.* [9] (see also Derome and Sharp [10]).

As an algebraic structure, this third method has serious drawbacks: The structure lacks the concept of a "null representation" (which would contradict unitarity), and the subtraction of representations is not defined. The resulting algebraic structure is unusual—not that of a ring—and has not been investigated in much detail.[2]

It is our purpose in Chapters 2–4 to develop an algebraic characterization of the Racah–Wigner calculus sufficiently general to satisfy the needs of physics. The present Chapter 2, in particular, defines a Racah–Wigner

[1]The spelling Jucys was preferred by Jucys himself; the Cyrillic transliteration Yutsis is equally frequently used.

[2]The generalized structure found in λ-rings (Knutson [11]) appears to circumvent at least some of these difficulties, but this technically difficult subject is beyond the scope of the present work.

algebra (RW-algebra) precisely, and categorizes the Wigner operators themselves in purely algebraic terms.

To understand the basis of the present approach, let us recall that all quantum mechanical calculations are phrased in the context of Hermitian operators acting in Hilbert space. An approach of this generality has an immediate difficulty: Interesting physics generally involves *unbounded* operators, with all their attendant ills. The first essential task is to limit the subject to more manageable proportions. *This is the fundamental role played by the Wigner–Eckart theorem*, which, as discussed in Chapter 3, AMQP, distinguishes two aspects of the problem: (1) It defines a basis for the set of tensor operators, and (2) it separates the physical aspects of quantum mechanical operators from the geometric (structural) aspects by means of the reduced matrix elements.

This separation of the physical problem into these two distinct aspects is of basic importance, for *all the technical difficulties of operator theory are confined to the physical structure, the invariant reduced matrix elements* (about which little of a general nature can be said anyway). By contrast, the Wigner coefficient aspect can be understood completely.

As an elementary example of what is involved, let us consider the angular momentum generators themselves. The commutation relation, $[J_3, \phi] = -i\hbar$ (ϕ is the azimuthal angle), shows that the operator J_3 necessarily has domain problems (see Chapter 5, Topic 7) and is, moreover, an unbounded operator (as $J_3 \to m$ shows). By contrast, the associated Wigner operator, $J_3(\mathbf{J}^2)^{-\frac{1}{2}}$, is normalized so that it is bounded and thus can be defined on all of Hilbert space. The physical reduced matrix element invariant, $(\mathbf{J}^2)^{\frac{1}{2}}$—although in this case very simple in structure—is responsible for all the technical difficulties.

Once this definitive role of the Wigner–Eckart theorem is understood, the proper algebraic framework becomes clear: The elements of the algebra are the unit tensor operators (Wigner operators), which are *bounded* and inherit from the Hilbert space structure (of quantum physics) the properties of a normed algebra (Banach algebra) with a unity and an involution (Hermitian conjugation). That an RW-algebra should fit into this general framework is not surprising, since this is just the setting (that of a C^*-algebra) long advocated by I. Segal [12] as the appropriate structure for physics, and especially for the axiomatization of quantum physics (von Neumann [13], Mackey [14], Haag and Kastler [15], Kastler [16], Varadarajan[1] [17], Emch [18]).

A framework of the generality of C^*-algebras is, however, more general than is required for the purpose at hand. One recognizes that it suffices for

[1]A recent review by Varadarajan [19] of the book by Piron [20] gives a succinct summary of the mathematical concepts underlying the axiomatic approach to the logical foundations of quantum theory.

the structure of the Wigner coefficient aspect of the problem to study the simplest possible generic realization.

As will be clear from even the briefest glance at AMQP, this simplest realization is none other than that provided by boson operators and the Jordan map or, more explicitly, by the 2×2 matrix boson (summarized for convenience in Section 3). [This realization is familiar to mathematicians as a Weyl algebra over the four variables (bosons) a_i^j $(i, j=1,2)$.]

This realization has the great technical advantage that the Hilbert space of boson polynomials realizes all operators (mapping this space into itself) as rank 1 operators, so that one can go from numerical operator matrix elements (Wigner coefficients in our application) to the operators themselves (Wigner operators or unit tensor operators) *without loss of generality.* (This is discussed in considerable detail in Chapter 5, AMQP.)

It is because of this unusually attractive technical advantage of the boson calculus that we may avoid a framework of the generality of C^*-algebra.

This, then, is the structure that is studied in detail in this chapter. We consider the Hilbert space of polynomials in the elements of the 2×2 matrix boson $A=(a_i^j)$, and develop the Banach star-algebra of operators generated algebraically by the fundamental Wigner operators acting in this space. The concept of a Racah invariant operator arises naturally in this algebra, and the properties of this class of invariant operators lead then to the basic properties of Racah coefficients. (See Ref. [21] for an earlier approach to this algebra.)

The chapter concludes with a formal definition of RW-algebra, and a canonical characterization of the Wigner operators as generators of graded maximal (left) ideals. This characterization is equivalent to that given by the characteristic null space properties of the Wigner operators, and in Chapter 3 a structure theorem is developed to this effect.

It is quite interesting that a purely algebraic characterization of the Racah invariant operators (without the explicit use of Wigner operators as occurs in RW-algebra) is indeed possible. This new algebraic structure, called W-algebra, is developed in Chapter 4.

2. Notational Preliminaries

The choice of a suitable notation is a vexing task, for which there probably can be no "best" decision. In AMQP we have used the more-or-less standard (j, m) notation of physics. To smooth the way for further developments, it is advisable to use a notation designed to generalize to the family of unitary groups $U(n)$, $n=1,2,\ldots$. Accordingly, we shall introduce here a new notation for basis vectors and tensor operators appropriate for $U(n)$—despite the fact that the use of this notation for $SU(2)$ is, admittedly, redundant and often cumbersome. For *basis vectors* in $U(n)$ this notation was introduced by Gel'fand and Tseitlin [22]; the generalization to *operators* was introduced in Ref. [23]. For both basis vectors and operators it is important to realize that the notation expresses by geometric (pattern) constraints the results of known structural theorems for $U(n)$ as discussed below.

Basis vectors. The idea of the notation is to specify a basis vector in the carrier space of an irreducible representation (irrep) of $U(n)$ by a set of integer-valued labels $\{m_{ij}\colon i=1,2,\ldots,j;\, j=1,2,\ldots,n\}$ arranged in a triangular pattern ("Gel'fand pattern"). For $U(2)$ these patterns are of the form[1]

$$\begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix}, \tag{2.1}$$

where the m_{ij} are integers (positive, negative, or zero) that satisfy the betweenness condition

$$m_{12} \geqslant m_{11} \geqslant m_{22}.$$

(Triangular patterns of integers that satisfy the betweenness condition will be said to be *lexical patterns*.)

The basis vector specified by the pattern (2.1) will be denoted by $|(m_{ij})\rangle$, or, equivalently, by $|(m)\rangle$, with (m) denoting implicitly the triangular array of labels (m_{ij}). The fact that the Gel'fand patterns obtained from (2.1) by setting $m_{11}=m_{22}, m_{22}+1,\ldots, m_{12}$, are in one-to-one correspondence with the basis vectors of an irrep of $U(2)$ is a consequence of a fundamental result of Weyl [24, 25] known as the Weyl branching law.[2] This rule, when applied to $U(2)$, asserts that the irrep of $U(2)$ corresponding to the partition $[m_{12}m_{22}]$ reduces on restricting $U(2)$ to $U(1)$ to the direct sum of irreps of $U(1)$ given by

$$\sum_{m_{11}=m_{22}}^{m_{12}} \oplus [m_{11}].$$

It is useful to note, although we shall not make much use of this fact, that the Gel'fand notation is related to the concept of a *standard Weyl tableau*.[3]

[1]Gel'fand patterns of n rows have been discussed in detail in Appendix A to Chapter 5, AMQP.

[2]The Weyl branching law for the reduction of irreps of $U(t)$ into irreps of $U(t-1)$, $t=2,3,\ldots,n$, is the structural result embodied in the general notation for a $U(n)$ Gel'fand pattern.

[3]The relationship between Weyl tableaux, Young tableaux, Gel'fand patterns, and boson state vectors for $U(n)$ is discussed in detail in Appendix A to Chapter 5, AMQP. Results of that generality will not be required in the present discussion.

The *Young frame* corresponding to the pattern (2.1) is given by

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline & & \cdots & & & \cdots & & & \cdots & & \\ \hline & & \cdots & \\ \cline{1-4} \end{array} \quad \begin{array}{l} m_{12} \text{ boxes in row 1} \\ m_{22} \text{ boxes in row 2.} \end{array}$$

Each lexical pattern (2.1) corresponds to the standard Weyl tableau obtained by "filling in" the Young frame with m_{11} 1's followed by $m_{12}-m_{11}$ 2's in row 1 and with m_{22} 2's in row 2:

$$\begin{array}{l} \left|\longleftarrow\quad m_{11} \quad\longrightarrow\right| \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 & 2 \\ \hline 2 & 2 & \cdots & 2 \\ \cline{1-4} \end{array} \end{array} \quad \begin{array}{l} m_{12} \text{ boxes} \\ m_{22} \text{ boxes.} \end{array}$$

An irrep of the unitary group $U(2)$ is irreducible under restriction to $SU(2)$, the unimodular subgroup. Thus, each of the vector spaces spanned by the (orthonormal) basis vectors in the set

$$\left\{ \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle : m_{11} = m_{22}, m_{22}+1, \ldots, m_{12} \right\} \tag{2.2}$$

is the carrier space of an irrep of $SU(2)$. The relationship between the irrep label j and the projection quantum number m and the (m_{ij}) is given by

$$j = (m_{12} - m_{22})/2, \qquad m = m_{11} - (m_{12} + m_{22})/2. \tag{2.3}$$

Thus, without loss of generality, we may take $m_{22}=0$ in using the Gel'fand notation for the basis vectors of irrep spaces of $SU(2)$:

$$|jm\rangle \equiv \left| \begin{pmatrix} 2j & & 0 \\ & j+m & \end{pmatrix} \right\rangle, \qquad \text{or, equivalently,} \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle. \tag{2.4}$$

It is clear that for $SU(2)$ the notation is somewhat awkward; we do, however, gain something, for the range of m values, $j \geqslant m \geqslant -j$, is now clearly implied by the betweenness condition. (The quantum number m is integral or half-integral, depending on j being integral or half-integral; $2j$ and $j+m$ are always integral.)

Tensor operators.[1] In order to adopt a suitable notation for tensor operators, it is necessary first to recall the Wigner–Eckart theorem (which validates the structure we wish to encode notationally) and the definition of

[1] We give in Note 1 a precise mathematical definition of a tensor operator.

a tensor operator, which underlies this theorem. An $SU(2)$ tensor operator $\mathbf{T}^J$ is a *set* of operators $\{T^J_M\colon M=J, J-1,\ldots,-J\}$ such that the operators in this set satisfy the following commutation relations with the generators $\mathbf{J}=(J_1, J_2, J_3)$ of $SU(2)$ (see Chapter 3, Section 14, AMQP, for a more detailed discussion):

$$\left[J_\mu, T^J_M\right]=(2J+1)^{\frac{1}{2}}\sum_{M'} C^{J1J}_{M\mu M'}T^J_{M'}, \tag{2.5}$$

where J_μ ($\mu=+1,0,-1$) are the spherical components of the angular momentum $\mathbf{J}$; that is, $J_{+1}=-(J_1+iJ_2)/\sqrt{2}$, $J_0=J_3$, $J_{-1}=(J_1-iJ_2)/\sqrt{2}$, and $C^{J1J}_{M\mu M'}$ are Wigner coefficients.

The transformation properties of the tensor operator $\mathbf{T}^J$ under finite rotations, which are implied by Eq. (2.5), allow one to associate a Gel'fand pattern to the operator T^J_M itself; that is,

$$T^J_M\to\begin{pmatrix}2J & & 0\\ & J+M & \end{pmatrix}. \tag{2.6}$$

Now recall the Wigner–Eckart theorem, which asserts that, for a generic matrix element of a tensor operator, one has the result

$$\langle(\alpha')j'm'|T^J_M|(\alpha)jm\rangle=\langle(\alpha')j'\|\mathbf{T}^J\|(\alpha)j\rangle C^{jJj'}_{mMm'}. \tag{2.7}$$

In this result $\langle(\alpha')j'\|\mathbf{T}^J\|(\alpha)j\rangle$ is the *reduced matrix element* defined (see Chapter 3, Section 15, AMQP) by the relation

$$\langle(\alpha')j'\|\mathbf{T}^J\|(\alpha)j\rangle\equiv(2j'+1)^{-1}\sum_{mMm'}\langle(\alpha')j'm'|T^J_M|(\alpha)jm\rangle C^{jJj'}_{mMm'}, \tag{2.8}$$

and the Wigner coefficients, $C^{jJj'}_{mMm'}$ are explicitly defined (numerical-valued) coefficients (discussed in detail in Chapter 3, Section 12, AMQP).

It is important to note the explicit appearance of the (α) labels, which serve to denote all remaining observables besides the total angular momentum (see footnote in Chapter 3; p. 32, AMQP).

The Wigner–Eckart theorem is the basis on which all our succeeding results are to be founded, and it is essential to be clear as to the significant features implied by this theorem (even at the risk of being repetitious). We have noted already the separation between the (physical) reduced matrix element structure and the (geometric) Wigner coefficient structure. In addition, it has been noted that it is technically advantageous to normalize the Wigner coefficients so that they correspond to *unit* tensor operators—that is, to *bounded* operators. Accordingly, one introduces a new set of physical operators (unit tensor operators)*defined to be diagonal in the* *variables* (α). That is to say, the reduced matrix elements for these unit tensor operators are defined to be proportional to $\delta_{(\alpha')(\alpha)}$ [see Eq. (2.7)].

We can exploit the separation implied by the Wigner–Eckart theorem to the fullest extent possible if we note that the physical quantum numbers denoted by (α) play a supernumerary role in this new set of physical operators; it is expedient, therefore, simply to discard all these labels, and consider the smallest Hilbert space that can support an angular momentum structure. This smallest Hilbert space is the separable Hilbert space $\mathcal{H}$ whose basis is the set of orthonormal states $\{|jm\rangle: j=0, \frac{1}{2}, 1, \ldots; m=j, j-1, \ldots, -j\}$ constructed from the two boson operators (a_1, a_2) and their conjugates, using the Jordan mapping (discussed in Chapter 5, Section 3, AMQP).

Because of the structural separation implied by the Wigner–Eckart theorem, we are guaranteed that the algebraic structure achieved by this particular realization for the unit tensor operators will in fact be generic.

We can now proceed to complete the notation designed for these unit tensor operators. By definition, these operators will act in the space $\mathcal{H}$ and will carry (as tensor operators) the Gel'fand pattern $\left(\begin{smallmatrix} 2J & & 0 \\ & J+M & \end{smallmatrix}\right)$. Moreover, this set of operators will be defined to have, in the space $\mathcal{H}$, matrix elements that are numerically given by the Wigner coefficients.

Note, however, that for an operator labeled by the pattern $\left(\begin{smallmatrix} 2J & & 0 \\ & J+M & \end{smallmatrix}\right)$, say, this association to a Wigner coefficient is not yet well-defined: *there is a missing operator label*, which we shall now specify to be the shift $\Delta \equiv j_{\text{final}} - j_{\text{initial}}$ $(j_{\text{final}} \equiv j', j_{\text{initial}} \equiv j)$ induced by the action of the tensor operator:

$$\left|\begin{matrix} 2j & & 0 \\ & j+m & \end{matrix}\right\rangle \to \left|\begin{matrix} 2j' & & 0 \\ & j'+m' & \end{matrix}\right\rangle.$$

The situation is summarized, notationally, by denoting a generic operator with the symbol

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle, \tag{2.9}$$

where the labels $\left(\begin{smallmatrix} 2J & & 0 \\ & J+M & \end{smallmatrix}\right)$ in the lower triangle form a Gel'fand pattern [and specify the transformation properties implied by Eq. (2.5)], and the labels $\left(\begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \end{smallmatrix}\right)$ in the upper (inverted) triangle form an *operator pattern*[1] *whose entries satisfy the same relations as the entries in the Gel'fand pattern* (2.1).

[1]It is essential to note that operator patterns (first defined in Ref. [23]) are conceptually very distinct from Gel'fand patterns (for the Gel'fand patterns the "space of labels" is dual to the group space, a concept that does not exist for operator pattern space).

The notation implies (by design) that the unit tensor operator $\left\langle\begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{smallmatrix}\right\rangle$ exists for values $-J\leqslant M\leqslant J$ of the magnetic quantum number label and for values $-J\leqslant\Delta\leqslant J$ of the *shift label.* It is often convenient to suppress the labels (M, Δ) and speak of the set of (unit) tensor operators $\langle 2J\ 0\rangle$.

To complete the definition of the operators $\langle 2J\ 0\rangle$, we specify the action of each operator, on every basis vector in $\mathcal{H}$, by

$$\left\langle\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix}\right\rangle\left|\begin{matrix} 2j & & 0 \\ & j+m & \end{matrix}\right\rangle \equiv C^{jJj+\Delta}_{m,M,m+M}\left|\begin{matrix} 2(j+\Delta) & & 0 \\ & j+\Delta+m+M & \end{matrix}\right\rangle, \tag{2.10}$$

where the coefficient is an $SU(2)$ Wigner coefficient for all triples of angular momentum quantum numbers $(j, J, j+\Delta)$ that satisfy the triangle condition—that is, for

$$2j=J-\Delta, J-\Delta+1,\ldots. \tag{2.11}$$

In order to be fully explicit we note that the Wigner coefficient is defined such that

$$C^{jJj+\Delta}_{m,M,m+M}=0 \tag{2.12}$$

for

$$2j=0,1,\ldots, J-\Delta-1. \tag{2.13}$$

Each operator is accordingly well-defined on all vectors of the basis of $\mathcal{H}$ and, hence, on $\mathcal{H}$ itself. [We shall often refer to the unit tensor operators defined by Eq. (2.10) as *Wigner operators*.]

This concludes the notational preliminaries. (Special Wigner operators of interest in physical applications are considered in Note 2.)

3. Fundamental Wigner Operators

The Hilbert space in which the action of the unit tensor operators, $\langle 2J\ 0\rangle$, is defined will be considered not as an abstract space, but as the specific Hilbert space constructed from the pair of boson operators (a_1, a_2). The basis states are accordingly given by the explicit forms

$$\left|\begin{matrix} 2j & & 0 \\ & j+m & \end{matrix}\right\rangle=[(j+m)!(j-m)!]^{-\frac{1}{2}}a_1^{j+m}a_2^{j-m}\left|\begin{matrix} 0 & & 0 \\ & 0 & \end{matrix}\right\rangle, \tag{2.14}$$

with $\left|\begin{smallmatrix} 0 & & 0 \\ & 0 & \end{smallmatrix}\right\rangle\equiv|0\rangle$ being the "vacuum" ket of no quanta.

The boson operators (a_1, a_2) are, as operators on this space, themselves classifiable as tensor operators. It is easily seen that the pair (a_1, a_2) transform under rotations as tensor operators having $J=\frac{1}{2}$. To see this, we use the Jordan map (Chapter 5, Section 3, AMQP). Accordingly, the realization of the angular momentum operators is given by

$$\begin{aligned} J_+ &= a_1\bar{a}_2, \\ J_- &= a_2\bar{a}_1, \\ J_3 &= (a_1\bar{a}_1 - a_2\bar{a}_2)/2. \end{aligned} \tag{2.15}$$

The boson operators are then found to satisfy the commutation relations:

$$\begin{array}{ll} [J_+, a_1]=0, & [J_+, a_2]=a_1, \\ [J_3, a_1]=\frac{1}{2}a_1, & [J_3, a_2]=-\frac{1}{2}a_2, \\ [J_-, a_1]=a_2, & [J_-, a_2]=0. \end{array} \tag{2.16}$$

Using the defining tensor operator commutation relations, Eq. (2.5), one finds that these results suffice to establish the Gel'fand pattern assignments given by

$$\begin{aligned} a_1 &\leftrightarrow \left(\begin{smallmatrix} 1 & & 0 \\ & 1 & \end{smallmatrix}\right), \\ a_2 &\leftrightarrow \left(\begin{smallmatrix} 1 & & 0 \\ & 0 & \end{smallmatrix}\right). \end{aligned} \tag{2.17}$$

By explicitly evaluating the matrix elements on the basis, Eq. (2.14), one easily finds that

$$a_1\left|\begin{smallmatrix} 2j & & 0 \\ & j+m & \end{smallmatrix}\right\rangle = (j+m+1)^{\frac{1}{2}}\left|\begin{smallmatrix} 2j+1 & & 0 \\ & j+m+1 & \end{smallmatrix}\right\rangle, \tag{2.18}$$

$$a_2\left|\begin{smallmatrix} 2j & & 0 \\ & j+m & \end{smallmatrix}\right\rangle = (j-m+1)^{\frac{1}{2}}\left|\begin{smallmatrix} 2j+1 & & 0 \\ & j+m & \end{smallmatrix}\right\rangle. \tag{2.19}$$

In a similar way, one determines that the Gel'fand pattern assignments for the conjugate boson operators[1] are given by

$$\begin{aligned} -\bar{a}_2 &\leftrightarrow \left(\begin{smallmatrix} 1 & & 0 \\ & 1 & \end{smallmatrix}\right), \\ \bar{a}_1 &\leftrightarrow \left(\begin{smallmatrix} 1 & & 0 \\ & 0 & \end{smallmatrix}\right). \end{aligned} \tag{2.20}$$

[1]Note the minus sign for $\bar{a}_2$. This sign is an important structural property related to the Frobenius–Schur invariant (see Chapter 5, Topic 1, Section 6).

The matrix elements for the conjugate operators are found to be

$$-\bar{a}_2\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle=-(j-m)^{\frac{1}{2}}\left|\begin{matrix}2j-1 & & 0\\ & j+m & \end{matrix}\right\rangle, \tag{2.21}$$

$$\bar{a}_1\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle=(j+m)^{\frac{1}{2}}\left|\begin{matrix}2j-1 & & 0\\ & j+m-1 & \end{matrix}\right\rangle. \tag{2.22}$$

These results show that, to within an invariant operator (the reduced matrix element), we have determined *the fundamental Wigner operator*— that is, the unit tensor operator $\langle 1\ 0\rangle$.

The required invariant operator is determined, by normalizing, to be $(4\mathbf{J}^2+1)^{-\frac{1}{2}}$. Making use of the fact that on the space $\mathcal{H}$ the operator $\mathbf{J}^2$ can be factorized— that is,

$$4\mathbf{J}^2=N(N+2), \tag{2.23}$$

where $N\equiv a_1\bar{a}_1+a_2\bar{a}_2$ and

$$N\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle=(2j)\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle, \tag{2.24}$$

one finds the following boson realization for the operators $\langle 1 0\rangle$:

$$\begin{array}{ccc}
\text{Boson operator realization} & & \text{Fundamental Wigner operator}\\[2ex]
a_1(N+1)^{-\frac{1}{2}} & \leftrightarrow & \left\langle\begin{matrix} & 1 & \\ 1 & & 0\\ & 1 & \end{matrix}\right\rangle\\[2ex]
a_2(N+1)^{-\frac{1}{2}} & \leftrightarrow & \left\langle\begin{matrix} & 1 & \\ 1 & & 0\\ & 0 & \end{matrix}\right\rangle\\[2ex]
-\bar{a}_2(N+1)^{-\frac{1}{2}} & \leftrightarrow & \left\langle\begin{matrix} & 0 & \\ 1 & & 0\\ & 1 & \end{matrix}\right\rangle\\[2ex]
\bar{a}_1(N+1)^{-\frac{1}{2}} & \leftrightarrow & \left\langle\begin{matrix} & 0 & \\ 1 & & 0\\ & 0 & \end{matrix}\right\rangle.
\end{array} \tag{2.25}$$

The results obtained above, Eqs. (2.17)–(2.25), show that the boson operators are necessarily unbounded. This explains why the boson operators themselves are not used to generate the operator algebra, as might otherwise appear reasonable.

Remark. Once having obtained the fundamental Wigner operators $\langle 1\ 0\rangle$, explicitly realized in the space $\mathcal{H}$, it is possible to proceed purely algebraically and use the $\langle 1\ 0\rangle$ to generate a normed operator ring. This is, in fact, the procedure we shall use in Section 7 to categorize the Racah–Wigner algebra. The objective in such an algebraic approach is to achieve a purely algebraic description of the set of all unit tensor operators $\langle 2J\ 0\rangle$ themselves (as is done in Chapter 3).

By contrast, an entirely different algebraic procedure is possible; this approach takes the realization of the set of unit tensor operators $\langle 2J\ 0\rangle$ on $\mathcal{H}$ for granted (since the Wigner coefficients may be considered as known) and seeks to determine further algebraic properties of the resulting structure. This latter approach is the one taken (implicitly) in the physics literature.

We shall use this second approach in the following sections, determining thereby the algebraic properties of certain invariant operators on $\mathcal{H}$, the Racah coefficients.

4. Properties of Unit Tensor Operators

Let us show first that the unit tensor operators defined by Eq. (2.10) do indeed have the required transformation properties in $\mathcal{H}$.

For this purpose, let $U \to T_U$, where $U \in SU(2)$, be the representation of $SU(2)$ induced (using the Jordan mapping) by unitary operators acting on the space $\mathcal{H}$ [see Eqs. (5.42)–(5.44), AMQP]. The action of T_U on the basis vectors of $\mathcal{H}$ is given by

$$T_U \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = \sum_{m'} D^j_{m'm}(U) \left| \begin{matrix} 2j & & 0 \\ & j+m' & \end{matrix} \right\rangle. \tag{2.26}$$

~~Then, since Eq. (2.10) defines an irreducible tensor operator for each Δ, it~~ follows that these unit tensor operators have the transformation property under the action of T_U given by

$$T_U \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle T_{U^{-1}} = \sum_{M'} D^J_{M'M}(U) \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M' & \end{matrix} \right\rangle. \tag{2.27}$$

[See Chapter 3, Section 14, AMQP.]

This equation asserts that, for each fixed angular momentum J and for each fixed shift value Δ, the set of operators

$$\left\{ \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle : -J \leqslant M \leqslant J \right\}$$

transforms irreducibly. (This is why Fano and Racah [26] designated tensor operators by the more explicit terminology "irreducible tensorial sets.")

An alternative verification of Eq. (2.27) may also be given by demonstrating the property

$$\sum_{m'} D^{j+\Delta}_{m+M',\, m'+M}(U)\, C^{jJj+\Delta}_{m',\, M,\, m'+M}\, D^{j}_{m'm}(U^{-1}) = D^{J}_{M'M}(U)\, C^{jJj+\Delta}_{m,\, M',\, m+M'} \tag{2.28}$$

for each $U \in SU(2)$, since Eq. (2.27) is just an operator statement of this matrix element relation. Equation (2.28) is itself a consequence of the following explicit expression of the Clebsch–Gordan series for $SU(2)$ [see Eq. (3.189) of AMQP]:

$$D^{j_1}_{m_1' m_1}(U)\, D^{j_2}_{m_2' m_2}(U) = \sum_{j} C^{j_1 j_2 j}_{m_1',\, m_2',\, m_1'+m_2'}\, C^{j_1 j_2 j}_{m_1,\, m_2,\, m_1+m_2}\, D^{j}_{m_1'+m_2',\, m_1+m_2}(U). \tag{2.29}$$

To obtain Eq. (2.28) from this result, one uses the orthogonality and symmetries of the Wigner coefficients as well as the symmetries of the D-functions.

It is useful also to define the set of the conjugate operators[1]

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger}, \qquad \begin{matrix} \Delta = -J, -J+1, \ldots, J, \\ M = -J, -J+1, \ldots, J, \end{matrix} \tag{2.30}$$

for each $2J = 0, 1, \ldots$ by giving the action of these operators on the basis vectors of $\mathcal{H}$:

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger} \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = C^{j-\Delta Jj}_{m-M,\, M,\, m} \left| \begin{matrix} 2(j-\Delta) & & 0 \\ & j-\Delta+m-M & \end{matrix} \right\rangle, \tag{2.31}$$

where $C^{j-\Delta Jj}_{m-M,\, M,\, m} \equiv 0$ for $2j = 0, 1, \ldots, J+\Delta-1$ and $\Delta > -J$.

[1]The matrix elements of an operator $\mathcal{O}$ and its conjugate $\mathcal{O}^{\dagger}$ are related by $\langle j'm' | \mathcal{O} | jm \rangle = \langle jm | \mathcal{O}^{\dagger} | j'm' \rangle^{*}$. Note, however that the matrix elements of unit tensor operators (Wigner coefficients) are real.

It follows from Eq. (2.27) that the conjugate operators transform as

$$T_U \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger} T_{U^{-1}} = \sum_{M'} D^J_{M'M}(U^*) \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M' & \end{matrix} \right\rangle^{\dagger}, \tag{2.32}$$

so that, for each fixed shift value Δ $(-J \leqslant \Delta \leqslant J)$ in the operator pattern (2.30), there exists a tensor operator having irrep label J, the components of which transform among themselves according to the *complex conjugate representation* of $SU(2)$.

The orthogonality relations for Wigner coefficients may now be given an alternative, but equivalent, formulation in terms of Wigner operators. Thus, the relationship

$$\sum_M \left\langle \begin{matrix} & J+\Delta' & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger} = \delta_{\Delta'\Delta} \mathbf{I}^J_{-\Delta} \tag{2.33}$$

is equivalent to the orthogonality of coefficients given by

$$\sum_M C^{jJj+\Delta'}_{m'-M,M,m'} C^{jJj+\Delta}_{m'-M,M,m'} = \varepsilon_{j,J,j+\Delta} \delta_{\Delta'\Delta}. \tag{2.34}$$

Similarly, the relationship

$$\sum_\Delta \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M' & \end{matrix} \right\rangle^{\dagger} \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle = \delta_{M'M} \tag{2.35}$$

is equivalent to the orthogonality of coefficients given by

$$\sum_\Delta C^{jJj+\Delta}_{m',M',m+M} C^{jJj+\Delta}_{m,M,m+M} = \delta_{m'm} \delta_{M'M}. \tag{2.36}$$

The symbol $\mathbf{I}^J_\Delta$ denotes the *invariant operator* defined explicitly on the basis vectors of $\mathcal{H}$ by

$$\mathbf{I}^J_\Delta \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = \varepsilon_{j,J,j+\Delta} \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle, \qquad \Delta = J, J-1, \ldots, -J. \tag{2.37}$$

We have used in Eqs. (2.34) and (2.37) the symbol $\varepsilon_{j_1 j_2 j}$ to denote the *unit step function* (characteristic function), which is defined to be unity for all triples of angular momenta (j_1, j_2, j) that satisfy the triangle relation, $|j_1 - j_2| \leqslant j \leqslant j_1 + j_2$, and zero for all other triples of angular momenta. [The necessity of including the invariant operator $\mathbf{I}^J_{-\Delta}$ in Eq. (2.33) is discussed below.]

Let us repeat that relations (2.33) and (2.35) serve the same purpose in expressions involving unit tensor operators as do the orthogonality relations for Wigner coefficients in purely coefficient-type relations, and that they are entirely equivalent to these orthogonality relations.

The concept of characteristic null space. In order to understand the necessity for including the invariant operator $\mathbf{I}^J_{-\Delta}$ in the right-hand side of Eq. (2.33), it is helpful to re-examine now the definition of a unit tensor operator, Eq. (2.10), from another point of view.

Let us recognize that, if we wish to study a *specific* Wigner operator (unit tensor operator) on our underlying Hilbert space, we should in our defining equation, Eq. (2.10), consider that J and Δ are specified (fixed) but that the label $2j$ in the basis vector should run over all possible integral values $0, 1, \ldots$. With this view in mind, let us recall the origin of the triangle rule. This rule is derived from the Kronecker (direct) product relation,

$$D^J \otimes D^j = \sum_{\Delta=-J}^{J} \oplus INT(J\otimes j;\ j+\Delta) D^{j+\Delta}, \tag{2.38}$$

which is just the Clebsch–Gordan series written in a form appropriate to the present discussion. The *intertwining number* $INT(J\otimes j;\ j+\Delta)$ denotes the number of occurrences of the irreducible representation $j+\Delta$ in the direct product of irrep J by irrep j. In accordance with our view of associating J and Δ with a fixed operator, we may specify the intertwining number as a function of $2j$:

$$INT(J\otimes j;\ j+\Delta) = \varepsilon_{j,J,j+\Delta} \equiv \begin{cases} 1 & \text{for } 2j \geqslant J-\Delta, \\ 0 & \text{for } 2j < J-\Delta. \end{cases} \tag{2.39}$$

Thus, in this language, the familiar triangle rule becomes the statement that *the intertwining number is a step function, which has value* 0 *at all points* $2j$ *in the set*

$$\{0, 1, \ldots, J-\Delta-1\}$$

and the value 1 *at all points* $2j$ *in the set*

$$\{J-\Delta, J-\Delta+1, \ldots\}.$$

(The first set is the empty set for $\Delta = J$.)

The preceding results imply the following: Let $\mathcal{H}_j$ denote the subspace of $\mathcal{H}$ spanned by the $2j+1$ basis vectors in the set

$$\left\{\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle : m=j,\, j-1,\ldots,-j\right\}. \tag{2.40}$$

Then the characteristic null space of the irreducible Wigner operator (unit tensor operator)

$$\left\langle\begin{matrix} & J+\Delta & \\ 2J & & 0\\ & \cdot & \end{matrix}\right\rangle \equiv \left\{\left\langle\begin{matrix} & J+\Delta & \\ 2J & & 0\\ & J+M & \end{matrix}\right\rangle : J\geqslant M\geqslant -J\right\} \tag{2.41}$$

is the subspace of $\mathcal{H}$ *given by the direct sum*

$$\mathcal{H}_0\oplus\mathcal{H}_{\frac{1}{2}}\oplus\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_{(J-\Delta-1)/2}\equiv\mathcal{N}_{(J-\Delta-1)/2}. \tag{2.42}$$

Remarks. (*a*) The concept of the null space of an operator acting on a Hilbert space is a familiar one in mathematics: *The null space of an operator is the set of vectors annihilated by that operator.* (This set of vectors is also termed the *kernel of the operator.*)

This concept must be modified to accord with the fact that our Hilbert space supports an additional structure: Under the group action, our Hilbert space is split invariantly into equivalence classes of subspaces, each carrying an irrep of $SU(2)$.

The characteristic null space of a unit tensor operator is defined to be an invariant subset of the null space of that operator. Put in simplest terms, if one vector of an irrep space belongs to the characteristic null space, then *all* vectors in that irrep space must so belong. (The restriction to *unit* tensor operators is for convenience to avoid zeros from invariant multiplicative factors.) Observe that the concept of the characteristic null space of a Wigner operator is basis-independent to the extent that it does not depend on how one introduces a basis into the carrier space of a given irrep $[j]\sim[2j\ 0]$.

(*b*) Note that the only Wigner operator of irrep label J that has an *empty characteristic null space* is the one having $\Delta=J$. This property may be attributed to the fact that maximal shift Wigner operators having $\Delta=J$ may be considered as generating the state vectors themselves from the "vacuum" ket:

$$\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle = \left\langle\begin{matrix} & 2j & \\ 2j & & 0\\ & j+m & \end{matrix}\right\rangle\left|\begin{matrix}0 & & 0\\ & 0 & \end{matrix}\right\rangle. \tag{2.43}$$

(c) The necessity for including the invariant operator $\mathbf{I}^{J}_{-\Delta}$ on the right-hand side of Eq. (2.33) can now be seen to be a consequence of the fact that the left-hand side of Eq. (2.33) will annihilate each vector in $\mathcal{H}$ that belongs to the characteristic null space of the operator $\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix} \right\rangle^{\dagger}$. No such null space factor is required in Eq. (2.35), since at least one operator in the sum produces a nonzero result on an arbitrary vector of $\mathcal{H}$ [see Remark (b) above].

(d) The idea of characteristic null space is conceptually simple, but it is nevertheless fundamental. In Chapter 3, Section 3, we demonstrate that transformation properties combined with null space properties determine the unit tensor operators essentially uniquely—that is, to within a purely numerical constant.

(e) The set of Wigner operators that effect nonpositive shifts leave invariant every subspace of $\mathcal{H}$ of the form $\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$ for each $k=0,\frac{1}{2},1,\ldots$. This structure is discussed further in Note 3 to this chapter.

(f) In addition to the "operator orthogonality" relations expressed by Eqs. (2.33) and (2.35), the unit tensor operators also satisfy the "trace orthogonality" relation:

$$\sum_{m} \left\langle \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right| \left\langle \begin{matrix} & J'+\Delta' & \\ 2J' & & 0 \\ & J'+M' & \end{matrix} \right\rangle \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger} \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = \frac{2j+1}{2J+1}\delta_{J'J}\delta_{M'M}\delta_{\Delta'\Delta}\varepsilon_{j,J,j-\Delta}. \tag{2.44}$$

This relation follows easily from the definitions (2.10) and (2.31) together with the orthogonality and symmetry of the Wigner coefficients. [Another orthogonality relation that is similar to Eq. (2.44) may be written in which the order of the Wigner operator and the conjugate operator is reversed.]

5. Coupling of Unit Tensor Operators

In Chapter 3 of AMQP we have noted, and discussed, the fact that the Wigner coefficients could be viewed in two quite distinct ways: From one view these coefficients could be considered as matrix elements of tensor operators (operator aspect), but from another view (considered as numerical coefficients) they defined a coupling of tensor operators to yield a new tensor operator (coupling aspect). One may exploit this coupling aspect to achieve a variety of different algebraic "multiplications" defined on the set of unit tensor operators. One such multiplication produces invariants; another multiplication produces vectors (and is a more general version of the familiar cross product of vectors).

To be explicit, let us consider the coupling of two unit tensor operators, $\langle 2b\ 0\rangle$ and $\langle 2a\ 0\rangle$, using the Wigner coefficients to obtain a new irreducible tensor operator $T^{c}_{\gamma}(\rho,\sigma)$ of irrep label c for *each* shift label $\rho=a, a-1,\ldots,-a$ and $\sigma=b,b-1,\ldots,-b$:[1,2]

$$T^{c}_{\gamma}(\rho,\sigma)=\sum_{\alpha\beta} C^{abc}_{\alpha\beta\gamma}\left\langle \begin{matrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{matrix}\right\rangle \left\langle \begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix}\right\rangle . \tag{2.45}$$

From the discussion in Chapter 3, AMQP, or directly from the commutation properties with the operator $\mathbf{J}$ (in the Jordan mapping), we can verify that the product in Eq. (2.45) defines an irreducible tensor operator transforming as T^{c}_{γ}. But note that the product operator is no longer necessarily a *unit* tensor operator. Consider the Wigner–Eckart theorem applied to $T^{c}_{\gamma}(\rho,\sigma)$:

$$\begin{aligned}\langle j'm'|T^{c}_{\gamma}(\rho,\sigma)|jm\rangle &= \delta_{j',j+\rho+\sigma}\delta_{m',m+\gamma}\langle j+\rho+\sigma\|\mathbf{T}^{c}(\rho,\sigma)\|j\rangle \\ &\times \left\langle \begin{matrix} 2(j+\rho+\sigma) & & 0 \\ & j+\rho+\sigma+m+\gamma & \end{matrix}\right| \left\langle \begin{matrix} & c+\rho+\sigma & \\ 2c & & 0 \\ & c+\gamma & \end{matrix}\right\rangle \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix}\right\rangle .\end{aligned} \tag{2.46}$$

This result shows that the product operator (2.45) differs from the unit tensor operator

$$\left\langle \begin{matrix} & c+\rho+\sigma & \\ 2c & & 0 \\ & c+\gamma & \end{matrix}\right\rangle$$

by a *multiplicative invariant operator*;[3] that is,

$$\sum_{\alpha\beta} C^{abc}_{\alpha\beta\gamma}\left\langle \begin{matrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{matrix}\right\rangle \left\langle \begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix}\right\rangle = \mathbf{W}^{abc}_{\rho,\sigma,\rho+\sigma}\left\langle \begin{matrix} & c+\rho+\sigma & \\ 2c & & 0 \\ & c+\gamma & \end{matrix}\right\rangle , \tag{2.47}$$

[1]We follow the Racah convention of using Roman letters $a,b,c,\ldots$ to denote angular momenta, and Greek letters $\alpha,\beta,\gamma,\ldots$ to denote the corresponding projection (m-type) quantum numbers. We shall also use Greek letters $\rho,\sigma,\tau,\ldots$ to denote Δ-type shift labels.

[2]Observe that this coupling differs from the standard $\times$ coupling given by Eq. (3.233) of AMQP by the factor $(-1)^{a+b-c}$. This slight change is clearly of no general significance, and it has the technical advantage of leading to a whole series of operator identities that contain no phase factors when written as relations between standard Wigner and Racah coefficients.

[3]The placement of $\mathbf{W}^{abc}_{\rho,\sigma,\rho+\sigma}$ to the left or to the right of the operator $\langle 2c\ 0\rangle$ is important. When taking matrix elements of Eq. (2.47), the left invariant operator $\mathbf{W}^{abc}_{\rho,\sigma,\rho+\sigma}$ is evaluated on the final angular momentum label j'; a right invariant operator would be evaluated on the initial angular momentum label j.

where we have denoted the invariant operator by $\mathbf{W}^{abc}_{\rho,\sigma,\rho+\sigma}$ (using a notation designed to be suggestive of the interpretation to be given below).

The invariant operator $\mathbf{W}^{abc}_{\rho,\sigma,\rho+\sigma}$ may be given explicitly in terms of the unit tensor operators themselves by using the conjugate operators defined by Eq. (2.31). Multiplying Eq. (2.47) from the right by

$$\left\langle 2c \begin{matrix} c+\tau \\ \\ c+\gamma \end{matrix} 0 \right\rangle^{\dagger},$$

summing over γ, and using the orthogonality relation (2.33), we obtain

$$\mathbf{W}^{abc}_{\rho\sigma\tau} = \sum_{\alpha\beta\gamma} C^{abc}_{\alpha\beta\gamma} \left\langle 2b \begin{matrix} b+\sigma \\ \\ b+\beta \end{matrix} 0 \right\rangle \left\langle 2a \begin{matrix} a+\rho \\ \\ a+\alpha \end{matrix} 0 \right\rangle \left\langle 2c \begin{matrix} c+\tau \\ \\ c+\gamma \end{matrix} 0 \right\rangle^{\dagger}, \qquad (2.48)$$

where we have defined

$$\mathbf{W}^{abc}_{\rho\sigma\tau} = \varepsilon_{abc}\delta_{\tau,\rho+\sigma}\mathbf{W}^{abc}_{\rho,\sigma,\rho+\sigma}.$$

Notice, however, that the right-hand side of Eq. (2.48) is zero unless the triple (abc) satisfies the triangle condition, and unless $\tau=\rho+\sigma$. Thus, we may take Eq. (2.48) as the definition of $\mathbf{W}^{abc}_{\rho\sigma\tau}$ without the necessity of including the factors ε_{abc} and $\delta_{\tau,\rho+\sigma}$. We then have

$$\mathbf{W}^{abc}_{\rho\sigma\tau} = 0 \text{ unless } (abc) \text{ satisfy the triangle condition and } \tau=\rho+\sigma. \qquad (2.49)$$

A comment on the choice of notation $\mathbf{W}^{abc}_{\rho\sigma\tau}$ for the invariant operator[1] occurring in Eq. (2.47) is appropriate here. This notation has been chosen in ~~analogy to the notation~~ $\cancel{C^{abc}_{\alpha\beta\gamma}}$ ~~for Wigner coefficients for the following~~ reasons: Not only does the property expressed by Eq. (2.49) suggest this notation, but so also does the fact that the ranges of ρ, σ, and τ are, respectively, given by

$$-a \leqslant \rho \leqslant a, \qquad -b \leqslant \sigma \leqslant b, \qquad -c \leqslant \tau \leqslant c; \qquad (2.50)$$

that is, these ranges are precisely the same as those of the projection quantum numbers in a Wigner coefficient. An even more persuasive reason for our choice of notation appears in Chapter 3, Section 18, AMQP, where it was proved that the eigenvalue $W^{abc}_{\rho\sigma\tau}(j)$ [see Eqs. (2.52) and (2.53) below]

[1]The use of the boldface letter **W** is to signify that this object is an operator as opposed to a number.

of $\mathbf{W}^{abc}_{\rho\sigma\tau}$ had the remarkable limit property

$$\lim_{j\to\infty} W^{abc}_{\rho\sigma\tau}(j)=C^{abc}_{\rho\sigma\tau}. \tag{2.51}$$

To establish the significance of the invariant operator $\mathbf{W}^{abc}_{\rho\sigma\tau}$ in relation to the work of Racah, we evaluate its eigenvalue on an arbitrary basis vector in $\mathcal{H}$:

$$\mathbf{W}^{abc}_{\rho\sigma\tau}\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle = W^{abc}_{\rho\sigma\tau}(j)\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle, \tag{2.52}$$

where, using Eqs. (2.48), (2.10), and (2.31), we find for the numerical value of $W^{abc}_{\rho\sigma\tau}(j)$ the explicit equation

$$\begin{aligned} W^{abc}_{\rho\sigma\tau}(j) &= \delta_{\rho+\sigma,\tau}\sum_{\alpha\beta\gamma} C^{abc}_{\alpha\beta\gamma} C^{j-\sigma\, b\, j}_{m-\beta,\beta,m} C^{j-\tau\, a\, j-\sigma}_{m-\gamma,\alpha,m-\beta} C^{j-\tau\, c\, j}_{m-\gamma,\gamma,m} \\ &= \delta_{\rho+\sigma,\tau}\left[(2c+1)(2j-2\sigma+1)\right]^{\frac{1}{2}} W(j-\tau,a,j,b;\, j-\sigma,c). \end{aligned} \tag{2.53}$$

We have denoted by $W(abcd;ef)$ the coefficient introduced by Racah [27] (in his notation) for the sum over four Wigner coefficients, which occurs in Eq. (2.53). The properties of these coefficients were discussed in detail in Chapter 3 of AMQP.

We conclude from Eq. (2.53) that *the eigenvalue of the invariant operator occurring in the coupling law, Eq.* (2.47), *is a Racah coefficient.*

We note again that the definition of the Racah invariant operator $\mathbf{W}$ in Eq. (2.47) depends very much on the placement of the operator $\mathbf{W}$ on the left. Were the operator $\mathbf{W}$ to be placed on the right of the operator $\langle 2c\ 0\rangle$, the $\mathbf{W}$ operator would act on a different basis vector in $\mathcal{H}$, and would have a different numerical value.

Let us turn now to a discussion of some of the general properties of the *Racah invariant operators* $\mathbf{W}^{abc}_{\rho\sigma\tau}$.

Note first that the set of Racah invariant operators, $\{\mathbf{W}^{abc}_{\rho\sigma\tau}\}$, is defined by our considerations only for integer and half-integer triples of nonnegative numbers (abc) that satisfy the triangle condition $|a-b|\leqslant c\leqslant a+b$ and for $\rho=a,a-1,\ldots,-a$; $\sigma=b,b-1,\ldots,-b$; $\tau=c,c-1,\ldots,-c$. It is only for these ranges of indices that we assign (in the present discussion[1]) a significance to the symbols $\mathbf{W}^{abc}_{\rho\sigma\tau}$. Although, by definition, $\mathbf{W}^{abc}_{\rho\sigma\tau}=0$, unless $\tau=\rho+\sigma$, it is useful to retain the redundant third label in ρ, σ, τ, not only for symmetry, but also because it is not always τ that one wishes to eliminate explicitly from the symbol.

[1]It proves useful in Chapter 4 to relax this strict view of the Racah coefficients.

Let us consider first the characteristic null space of the invariant operator $\mathbf{W}^{abc}_{\rho\sigma\tau}$. This is the set of irrep spaces $\mathcal{H}_j \subset \mathcal{H}$ annihilated by the operator, and these spaces are identified by the set of values of j for which we have $W^{abc}_{\rho\sigma\tau}(j)=0$. A value j will determine an $\mathcal{H}_j$ in the null space of $\mathbf{W}$ if any one of the three Wigner operators appearing in the right-hand side of Eq. (2.48) annihilates the state vector resulting from the action on $\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle$ of the Wigner operators lying to its right. Applying this result to $\mathbf{W}^{abc}_{\rho\sigma\tau}$, we easily see then that

$W^{abc}_{\rho\sigma\tau}(j)=0$, unless each of the triples $(j-\tau,\ a,\ j-\tau+\rho)$, $(j-\tau+\rho, b, j)$, $(j-\tau, c, j)$ consists of nonnegative integers/half-integers that satisfy the triangle conditions.

In Racah's notation, the preceding result becomes the following statement:

$W(abcd;ef)=0$, unless the four triples of nonnegative integers/half-integers—(abe), (cde), (acf), (bdf)—satisfy the triangle conditions.

We have included the triangle conditions on the superscript labels of $\mathbf{W}$ in this statement.

The Racah invariant operators also satisfy orthogonality relations that are the operator analogs to the orthogonality relations for Wigner coefficients. (It follows that the operators are normed invariant operators on $\mathcal{H}$.) We summarize now these relations both in operator form and in coefficient form. The proofs of the operator relations [Eqs. (2.54) below] are given in the Appendix, using the present algebraic formulation:

$$\sum_{\rho\sigma} \mathbf{W}^{abc}_{\rho\sigma\tau}\mathbf{W}^{abd}_{\rho\sigma\tau}=\delta_{cd}\mathbf{I}^{c}_{-\tau},$$

$$\sum_{c} \mathbf{W}^{abc}_{\rho,\tau-\rho,\tau}\mathbf{W}^{abc}_{\rho',\tau-\rho',\tau}=\delta_{\rho\rho'}\mathbf{I}^{ab}_{-\rho,\rho-\tau}, \tag{2.54}$$

where $\mathbf{I}^{c}_{\tau}$ and $\mathbf{I}^{ab}_{\rho\sigma}$ are invariant operators [see Eq. (2.37)] whose values on a state $\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle$ are zero or one as given by the characteristic functions

$$I^{c}_{\tau}(j)=\varepsilon_{j,c,j+\tau}, \qquad \tau=c, c-1,\ldots,-c,$$

$$I^{ab}_{\rho\sigma}(j)=\varepsilon_{j+\sigma,a,j+\rho+\sigma}\varepsilon_{j,b,j+\sigma}. \tag{2.55}$$

The occurrence in Eqs. (2.54) of the invariant idempotent operators $\mathbf{I}$ (that is, operators with eigenvalues zero or one) is an expression of the unit normalization of Racah operators acting on those angular momentum states not annihilated by the operator.

In terms of Racah coefficients, Eqs. (2.54) become

$$\sum_{\rho\sigma} W^{abc}_{\rho\sigma\tau}(j)\, W^{abd}_{\rho\sigma\tau}(j) = \delta_{cd}\,\varepsilon_{j,c,j-\tau},$$

$$\sum_{c} W^{abc}_{\rho,\tau-\rho,\tau}(j)\, W^{abc}_{\rho',\tau-\rho',\tau}(j) = \delta_{\rho\rho'}\,\varepsilon_{j-\tau+\rho,a,j-\tau}\,\varepsilon_{j,b,j-\tau+\rho}. \qquad (2.56)$$

Using the transcription,

$$W^{abc}_{\rho,\sigma,\rho+\sigma}(j) = \left[(2c+1)(2j-2\sigma+1)\right]^{\frac{1}{2}} W(j-\rho-\sigma, a, j, b; j-\sigma, c), \qquad (2.57)$$

we may write these relations in standard Racah coefficient form:

$$\sum_{e} (2e+1)(2f+1)\, W(abcd; ef)\, W(abcd; eg) = \delta_{fg}\,\varepsilon_{acf}\,\varepsilon_{bdf},$$

$$\sum_{e} (2e+1)(2f+1)\, W(abcd; fe)\, W(abcd; ge) = \delta_{fg}\,\varepsilon_{cdf}\,\varepsilon_{abf}. \qquad (2.58)$$

Note that we have omitted the factor ε_{abc} from the right-hand side of Eq. (2.56), since by our convention the superscripts in that equation are understood to satisfy the triangle conditions and the equation as written has index balance. It is less explicit in Racah's notation as to which angular momenta satisfy the triangle conditions, and the two triangle factors that we have included explicitly on the right-hand side of Eqs. (2.58) are generally ignored in the physics literature. The two orthogonality relations (2.58) are, in fact, identical statements in consequence of a symmetry of the Racah coefficients. We have displayed both equations and, in particular, the two in Eq. (2.56) to show the similarity (and identity for infinite j) of these orthogonality relations to those for Wigner coefficients [cf. Eqs. (2.34) and (2.36) of the present chapter and also Eqs. (3.178) and (3.179) of AMQP].

We conclude this section by noting three additional forms into which the basic relation, Eq. (2.47), may be cast by straightforward application (hence, the proofs are omitted) of the orthogonality relations for Wigner operators [Eqs. (2.33) and (2.35)] and Racah invariant operators [Eqs. (2.54)]. (Below each operator relation, we have also written the corresponding matrix element expression.)

Coupling law for two unit tensor operators to a unit tensor operator:

$$\sum_{\substack{\alpha\beta \\ \rho\sigma}} \mathbf{W}^{abd}_{\rho\sigma\tau} C^{abc}_{\alpha\beta\gamma} \left\langle \begin{matrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{matrix} \right\rangle \left\langle \begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix} \right\rangle = \delta_{cd} \left\langle \begin{matrix} & c+\tau & \\ 2c & & 0 \\ & c+\gamma & \end{matrix} \right\rangle; \quad (2.59)$$

$$\sum_{\alpha\beta\rho} [(2d+1)(2j+2\rho+1)]^{\frac{1}{2}} W(j,a,j+\tau,b;j+\rho,d) C^{abc}_{\alpha\beta\gamma}$$
$$\times C^{j+\rho\, b\, j+\tau}_{m+\alpha,\beta,m+\gamma} C^{j\,a\,j+\rho}_{m,\alpha,m+\alpha} = \delta_{dc} C^{j\,c\,j+\tau}_{m,\gamma,m+\gamma}. \quad (2.60)$$

Open product law:

$$\left\langle \begin{matrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{matrix} \right\rangle \left\langle \begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix} \right\rangle = \sum_{c} \mathbf{W}^{abc}_{\rho,\sigma,\rho+\sigma} C^{abc}_{\alpha,\beta,\alpha+\beta} \left\langle \begin{matrix} & c+\rho+\sigma & \\ 2c & & 0 \\ & c+\alpha+\beta & \end{matrix} \right\rangle; \quad (2.61)$$

$$C^{j+\rho\, b\, j+\rho+\sigma}_{m+\alpha,\beta,m+\alpha+\beta} C^{j\,a\,j+\rho}_{m,\alpha,m+\alpha}$$
$$= \sum_{c} [(2c+1)(2j+2\rho+1)]^{\frac{1}{2}} W(j,a,j+\rho+\sigma,b;j+\rho,c)$$
$$\times C^{abc}_{\alpha,\beta,\alpha+\beta} C^{j\,c\,j+\rho+\sigma}_{m,\alpha+\beta,m+\alpha+\beta}. \quad (2.62)$$

Operator pattern or Racah invariant coupling law:

$$\sum_{\rho\sigma} \mathbf{W}^{abc}_{\rho\sigma\tau} \left\langle \begin{matrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{matrix} \right\rangle \left\langle \begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix} \right\rangle = C^{abc}_{\alpha,\beta,\alpha+\beta} \left\langle \begin{matrix} & c+\tau & \\ 2c & & 0 \\ & c+\alpha+\beta & \end{matrix} \right\rangle; \quad (2.63)$$

$$\sum_{\rho} [(2c+1)(2j+2\rho+1)]^{\frac{1}{2}} W(j,a,j+\tau,b;j+\rho,c)$$
$$\times C^{j+\rho\, b\, j+\tau}_{m+\alpha,\beta,m+\alpha+\beta} C^{j\,a\,j+\rho}_{m,\alpha,m+\alpha} = C^{abc}_{\alpha,\beta,\alpha+\beta} C^{j\,c\,j+\tau}_{m,\alpha+\beta,m+\alpha+\beta}. \quad (2.64)$$

The matrix element forms of Eqs. (2.47), (2.48), (2.59), (2.61), and (2.63), relating Wigner and Racah coefficients, are very useful for "manipulating" relations between these coefficients that turn up in physical applications. For example, Eq. (2.64) may be viewed as a "recoupling" in which the coupling scheme on the right-hand side ($\mathbf{a}+\mathbf{b}=\mathbf{c}$, $\mathbf{j}+\mathbf{c}=\mathbf{j}+\boldsymbol{\tau}$) is replaced by coefficients effecting the scheme on the left-hand side [$\mathbf{j}+\mathbf{a}=\mathbf{j}+\boldsymbol{\rho}$, $(\mathbf{j}+\boldsymbol{\rho})+\mathbf{b}=\mathbf{j}+\boldsymbol{\tau}$.] These relations have been summarized in standard notation by Eqs. (3.266)–(3.270) in AMQP.

Remark. We have expressed the various product laws for Wigner operators [Eqs. (2.47) and (2.59) are particularly important examples] in the language of physicists using an index notation. This mode of expression, though indeed having the merit of being fully explicit, is—almost certainly for a mathematician!—very close to a "debauch of indices" in Cartan's expressive complaint. As such, the notation itself becomes a barrier to understanding the genuinely interesting content of these equations. The present remark is intended to discuss this content.

Let us re-express Eq. (2.47) in a more symbolic way. The Wigner coefficient $C^{abc}_{\alpha\beta\gamma}$ that appears in that equation simply expresses an algebraic coupling. Let us suppress all the indices (including the coupling coefficient itself) and indicate the product by $\underset{c}{\blacklozenge}$. Equation (2.47) now reads

$$\langle 2a\;0\rangle \underset{c}{\blacklozenge} \langle 2b\;0\rangle = \mathbf{W}^{abc}\langle 2c\;0\rangle . \tag{2.47$'$}$$

The invariant content of this relation is now evident: *The Wigner product* $\underset{c}{\blacklozenge}$ *of two unit tensor operators, here* $\langle 2a\;0\rangle$ *and* $\langle 2b\;0\rangle$, *yields a new unit tensor operator, here* $\langle 2c\;0\rangle$, *normalized by the Racah invariant operator* $\mathbf{W}^{abc}$.

Now let us examine Eq. (2.59) from this structural point of view. Clearly, the Wigner coefficient once again effects the product $\underset{c}{\blacklozenge}$ as before. It is then clear that the Racah operator $\mathbf{W}^{abd}_{\rho\sigma\tau}$ effects a *new type of product involving operator pattern labels.* Let us denote this new multiplication ("Racah product") by $\overset{d}{\blacklozenge}$. As before, just *which* multiplication is meant (that is, the particular Racah product yielding the final angular momentum d) is indicated by the index d.

With this mode of expression, Eq. (2.59) achieves a more evident invariant significance:

$$\langle 2b\;0\rangle \underset{\underset{c}{\blacklozenge}}{\overset{\overset{d}{\blacklozenge}}{}} \langle 2a\;0\rangle = \delta^{d}_{c}\langle 2c\;0\rangle . \tag{2.59$'$}$$

The Racah and Wigner product of two unit tensor operators yields a new unit tensor operator (if the products agree) or the zero operator (if not).

The effect of the Racah product alone is given by Eq. (2.63), which now would read

$$\langle 2b\;0\rangle \overset{d}{\blacklozenge} \langle 2a\;0\rangle = \#\langle 2c\;0\rangle , \tag{2.63$'$}$$

with $\# = C^{abc}$. *That is, the Racah product of two unit tensor operators yields a* *new unit tensor operator multiplied by a numerical factor* (a Wigner coefficient).

To express this even more abstractly, and more generally, one may say that *the Wigner product for unit tensor operators is the coupling of Gel'fand patterns*, and *the Racah product is the coupling of operator patterns*. This mode of expressions then extends to $U(n)$ (once the multiplicity problem is resolved, so that the two products are well-defined).

This sort of global (or "Gibbsian") symbolism is, by design, useful for grasping the structural significance of the various coupling laws, but we have made no systematic use of this elsewhere in the manuscript for two reasons: (a) Much is "swept under the rug" by the symbolism [the three possible placements of the (implied) Racah invariant operator in the Racah product are not equivalent but are buried in the symbolism], and (b) the applications of angular momentum techniques in physics are already surfeited with notations, and to impose still another is embarrassing.

6. Consequences of Associativity

We have defined the operator algebra of the unit tensor operators by embedding this structure in the space of linear operators acting on a (specific) separable Hilbert space $\mathcal{H}$. It follows that the multiplication laws for this structure are *necessarily associative*. This elementary fact has important implications for the Racah invariant operators, as we now demonstrate.

In the previous section, we determined the explicit form of the (open) product of any two unit tensor operators. This result is given by Eq. (2.61), and using this result enables us to examine directly the question of associativity.

Associativity requires that the following identity exist:

$$\left\langle 2c \begin{matrix} c+\tau \\ \\ c+\gamma \end{matrix} 0 \right\rangle \left(\left\langle 2b \begin{matrix} b+\sigma \\ \\ b+\beta \end{matrix} 0 \right\rangle \left\langle 2a \begin{matrix} a+\rho \\ \\ a+\alpha \end{matrix} 0 \right\rangle \right)$$

$$= \left(\left\langle 2c \begin{matrix} c+\tau \\ \\ c+\gamma \end{matrix} 0 \right\rangle \left\langle 2b \begin{matrix} b+\sigma \\ \\ b+\beta \end{matrix} 0 \right\rangle \right) \left\langle 2a \begin{matrix} a+\rho \\ \\ a+\alpha \end{matrix} 0 \right\rangle \qquad (2.65)$$

for arbitrary unit tensor operators. Using the product law, Eq. (2.61), twice to transform first the left-hand side and then the right-hand side of Eq.

(2.65) yields the relation

$$\sum_{de} W^{abd}_{\rho,\sigma,\rho+\sigma}(j'-\tau)C^{abd}_{\alpha,\beta,\alpha+\beta}W^{dce}_{\rho+\sigma,\tau,\rho+\sigma+\tau}(j')C^{dce}_{\alpha+\beta,\gamma,\alpha+\beta+\gamma}$$
$$\times\left\langle 2e \begin{matrix} e+\rho+\sigma+\tau \\ \\ e+\alpha+\beta+\gamma \end{matrix} 0 \right\rangle \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle$$
$$=\sum_{fe} W^{bcf}_{\sigma,\tau,\sigma+\tau}(j')C^{bcf}_{\beta,\gamma,\beta+\gamma}W^{afe}_{\rho,\sigma+\tau,\rho+\sigma+\tau}(j')C^{afe}_{\alpha,\beta+\gamma,\alpha+\beta+\gamma}$$
$$\times\left\langle 2e \begin{matrix} e+\rho+\sigma+\tau \\ \\ e+\alpha+\beta+\gamma \end{matrix} 0 \right\rangle \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle, \qquad (2.66)$$

where $j'\equiv j+\rho+\sigma+\tau$.

Since this result is to be an identity, we conclude that the coefficients (the sum over d and f) must be the same on each side of this expression. We set these two coefficients equal and use Eq. (2.64) in the form

$$C^{bcf}_{\beta,\gamma,\beta+\gamma}C^{afe}_{\alpha,\beta+\gamma,\alpha+\beta+\gamma}$$
$$=\sum_{d}[(2d+1)(2f+1)]^{\frac{1}{2}}W(abec;df)C^{dce}_{\alpha+\beta,\gamma,\alpha+\beta+\gamma}C^{abd}_{\alpha,\beta,\alpha+\beta} \qquad (2.67)$$

to eliminate the Wigner coefficients. The result is the following relation:

$$W^{abd}_{\rho,\sigma,\rho+\sigma}(j-\tau)W^{dce}_{\rho+\sigma,\tau,\rho+\sigma+\tau}(j)$$
$$=\sum_{f}[(2d+1)(2f+1)]^{\frac{1}{2}}W(abec;df)W^{bcf}_{\sigma,\tau,\sigma+\tau}(j)W^{afe}_{\rho,\sigma+\tau,\rho+\sigma+\tau}(j), \qquad (2.68)$$

---

where we have replaced j' in Eq. (2.66) by j. We conclude that associativity implies the identity given by Eq. (2.68). The argument clearly can be reversed so that the identity implies associativity. Thus, we have proved the fundamental proposition: *The algebra of Wigner operators is an associative algebra if and only if the Racah coefficients satisfy the identity expressed by Eq.* (2.68).

When expressed in standard Racah coefficient notation, Eq. (2.68) takes the form

$$W(a'ab'b;c'e)W(a'ed'd;b'c)$$
$$=\sum_{f}(2f+1)W(abcd;ef)W(c'bd'd;b'f)W(a'ad'f;c'c). \qquad (2.69)$$

Equation (2.69) expresses a well-known identity on the Racah coefficients, first derived by considering the coupling of four angular momenta (Biedenharn [28], Elliott [29]), but derived here as a consequence of the associativity law of multiplication for unit tensor operators.[1] This latter approach stems from Racah (unpublished work). The customary designation in the literature for this identity (the Biedenharn–Elliott identity) is due to Fano and Racah [26]. We shall follow this usage by occasionally referring to Eq. (2.69) as the B–E identity.

Using the orthogonality of the Racah coefficients, we may write Eq. (2.69) in a variety of forms, a structure we have exploited in Chapter 3, Section 18, of AMQP.

7. A Characterization of Racah–Wigner Algebra[2]

The developments carried out in the preceding sections have taken for granted the usual Hilbert space and operator structures assumed in quantum physics; building on the results of Chapter 3 of AMQP, we have also taken, as presupposed, full knowledge of the Wigner coefficients, in order to define unit tensor operators and to determine their properties. The essential element being explored was, accordingly, the structure of certain invariant operators, the Racah operators **W**. Once this has been accomplished, we are in a good position to pause and resurvey the whole undertaking, with particular emphasis on elucidating the underlying structure. We wish, in other words, to replace the informal methods of theoretical physics by the more formal, and more searching, methods of mathematics.

It is the purpose of this concluding section of Chapter 2 to state precisely the defining characteristics of Racah–Wigner algebra. These characteristics are to be considered as abstracted from the explicit model of the algebraic system constructed in the previous sections. (For brevity, we shall call this an RW-algebra, even though the definition of this term is not explicit as yet.)

It follows from this explicit model that *an* RW-*algebra is first of all a subalgebra of the Banach star-algebra of bounded operators on the Hilbert space* $\mathcal{H}$*—that is, a* C^**-algebra over the space* $\mathcal{H}$.

This is a matter of checking the definitions. As operators in a Hilbert space (denote generic operators by $x, y, \ldots$), there is a natural involution

[1]This method of proof was suggested to us by our earlier work developing the multiplication properties of symplecton polynomial forms (see Chapter 5, Topic 3). The B–E identity is the analog in RW-algebras of the Jacobi bracket identity in Lie algebras, the Jacobi identity itself being a consequence of the associativity of multiplication of group elements.

[2]We thank Professor David A. Smith for critically reading this section.

denoted by * that satisfies the defining properties (Loomis [30, p. 87]):[1]

$$
\begin{aligned}
x^{**} &= x, \\
(x+y)^* &= x^* + y^*, \qquad\qquad\qquad\qquad\qquad\qquad (2.70) \\
(\lambda x)^* &= \bar{\lambda} x^*, \qquad \text{where } \lambda \in \mathbb{C}, \\
(xy)^* &= y^* x^*.
\end{aligned}
$$

Since we restrict the operators to be *bounded*, there is a natural Banach algebra inherited from the Hilbert space realization. The significant fact here is that the Wigner operators, as unit tensor operators, are indeed bounded and from the Hilbert space structure admit of the usual algebraic operations, with bounded invariant operators playing the role of generalized scalars. For explicitness, we state the definition[2] (Gel'fand *et al.* [31, p. 241]) for the resulting *normed ring* R (Banach algebra):

(a) R is a ring; that is, operations of addition and multiplication satisfying the usual algebraic conditions are defined in R. We also assume that R has a unit element e.

(b) R is a linear vector space with multiplication by complex numbers, where this multiplication is permutable with the operation of multiplication of elements in R.

(c) A norm is defined in R; that is, every element x is associated with a number $\|x\|$ such that

$$
\begin{aligned}
&\|x+y\| \leq \|x\| + \|y\|, \qquad \|xy\| \leq \|x\|\,\|y\|, \\
&\|x\| \geq 0, \text{ and is equal to zero only for } x=0, \qquad\qquad (2.71) \\
&\|\lambda x\| = |\lambda|\,\|x\|, \qquad \|e\| = 1.
\end{aligned}
$$

(d) The ring is complete; that is, from

$$\lim_{m,n\to\infty} \|x_n - x_m\| = 0 \tag{2.72}$$

there follows the existence of an x such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

[1] In this section only we follow the usual custom in mathematics by denoting an involution by * and complex conjugation of a complex number λ by $\bar{\lambda}$.

[2] Chapter VII of Ref. [31] contains the material relevant to this discussion. That chapter was written by I. M. Gel'fand and N. A. Naimark and is an adaptation to the book of their earlier paper [32].

The norm for the operator x is taken to be $\|x\| \equiv \underset{\psi\in\mathcal{H}}{LUB}\|x\psi\|/\|\psi\|$, where $\|\psi\|^2 = \langle\psi|\psi\rangle$ with $\langle \ | \ \rangle$ denoting the inner product in $\mathcal{H}$.

The feature that distinguishes the Racah–Wigner algebraic structure from the more general C^*-algebra is the existence of the tensor operator condition, Eq. (2.5). *The existence of this condition shows that an* RW-*algebra is a graded algebra, graded by the tensor operator label J.*

To define the grading, we first set up a filtration that differs slightly from the standard procedure given by Jacobson [8]. Let us define the subspace $\mathcal{A}^{(i)}$ to be the collection of all elements of the RW-algebra $\mathcal{A}$ that may be written in the form:[1]

$$x^{(i)} \equiv \sum_{J=0}^{i/2} \sum_{M,\Delta} \left\langle 2J \begin{array}{c} J+\Delta \\ \\ J+M \end{array} 0 \right\rangle I(\Delta, J, M), \qquad i=0,1,2,\ldots, \tag{2.73}$$

where the $I(\Delta, J, M)$ are arbitrary scalars (invariant operators and/or complex numbers.}

Then this defines a filtration, since, by construction, we have the (defining) properties:

$$\begin{aligned} &\mathcal{A}^{(i)} \subseteq \mathcal{A}^{(j)} \qquad \text{if } i \leqslant j, \\ &\cup \mathcal{A}^{(i)} = \mathcal{A} = \text{RW-algebra} \\ &\mathcal{A}^{(i)}\mathcal{A}^{(j)} \subseteq \mathcal{A}^{(i+j)}. \end{aligned} \tag{2.74}$$

To obtain the associated graded algebra, we define the elements $\overline{\mathcal{A}}^{(i)}$, where

$$\overline{\mathcal{A}}^{(i)} \equiv \mathcal{A}^{(i)}/\mathcal{A}^{(i-1)} \tag{2.75}$$

with $\mathcal{A}^{(-1)} \equiv 0$. The graded algebra is then

$$\overline{\mathcal{A}} = \sum_{i=0}^{\infty} \oplus \overline{\mathcal{A}}^{(i)}. \tag{2.76}$$

A multiplication is defined component-wise by

$$\left(a_i + \mathcal{A}^{(i-1)}\right)\left(b_j + \mathcal{A}^{(j-1)}\right) = a_i b_j + \mathcal{A}^{(i+j-1)}, \tag{2.77}$$

where $a_i \in \mathcal{A}^{(i)}$, $b_j \in \mathcal{A}^{(j)}$. This multiplication then extends to all of $\overline{\mathcal{A}}$. It is clear that $\overline{\mathcal{A}}$ is a graded algebra, since it satisfies, by construction, the

[1]For the purpose of this general discussion, it is convenient to write invariant operators to the right of the unit tensor operator $\langle 2J\ 0\rangle$.

property

$$\overline{\mathcal{A}}^{(i)}\overline{\mathcal{A}}^{(j)} \subseteq \overline{\mathcal{A}}^{(i+j)}. \tag{2.78}$$

It is also clear that $\overline{\mathcal{A}}$ is, in fact, simply the RW-algebra from which we started.

The necessity for this rather involved construction is caused by the fact that in the original grading defined by the angular momentum the product of graded elements could have *any* grading between the sum and difference. This required that the grading be defined as above to avoid any complications.

In terms of this grading, we can now identify the Wigner $\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix} \right\rangle$ operators themselves. The Wigner operators are the generators of the elements of $\overline{\mathcal{A}}^{(J)}$ in the algebra having grade $2J$, under algebraic operations by numerical scalars and bounded invariant operators (grade 0).

With these preliminaries accomplished, we can now proceed to a formal definition of an RW-algebra. Definition: *An* RW-*algebra is that subalgebra of the Banach star-algebra of bounded operators on a Hilbert space $\mathcal{H}$, which is graded by the tensor operator condition, and which is algebraically generated by the fundamental Wigner operators $\langle 1\,0 \rangle$ acting on $\mathcal{H}$.*

Remarks. (*a*) This definition is clearly abstracted from our defining model, and is equally clearly not vacuous, since it suffices to obtain this very model. On the other hand, the structure does not appear to be overly specific, but is, in fact, generic, since (as we shall discuss in Chapter 3, Section 4) there does exist at least one other (distinct) RW-algebra [incorporating $SU(3)$ symmetry] which uses a tensor operator condition and generators appropriate to an $SU(3)$ adapted Hilbert space.

(*b*) Note that the Wigner operators appear in an RW-algebra as the *linear* ~~generators of the algebra.~~

(*c*) The Racah operators do not explicitly appear in this characterization of an RW-algebra, and are not distinguished from other bounded invariant (grade 0) operators.

The algebraic characterization of the Racah operators requires the construction of a new algebra, the W-algebra discussed in Chapter 4.

We shall now complete our characterization of the $[SU(2)]$ RW-algebra by constructing a set of graded maximal (left) ideals whose generators are precisely the Wigner operators $\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix} \right\rangle$. This result will then establish a canonical definition of the Wigner operator $\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{smallmatrix} \right\rangle$, whose labels are determined by the grading $2J$, by the transformation properties under commutation with the $SU(2)$ generators $\mathbf{J}$ (M label) and the graded maximal left ideal Δ, the latter information being equivalent to the characteristic null space properties of the operator. (A structure theorem, to be developed in Chapter 3, will then demonstrate that this information uniquely characterizes the operator $\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle$ in such a way that the operator is explicitly constructable.)

The technique to be used in constructing the left ideals has been developed by Gel'fand *et al.* [31] and by Gel'fand and Naimark [32] in their classic papers on noncommutative Banach star-algebras; it involves the construction of a set of positive linear functionals on the RW-algebra.

We recall first the definition of a positive linear functional: *A positive linear functional is a function $f(x)$ that assigns to every $x \in R$ a complex number $f(x)$ such that*

$$f(\lambda x+\mu y)=\lambda f(x)+\mu f(y) \qquad \text{for } \lambda, \mu \in \mathbb{C};$$

$$f(x^*x) \geqslant 0 \qquad \text{for every } x. \tag{2.79}$$

We have defined RW-algebra in terms of bounded operators on the Hilbert space $\mathcal{H}$. It is quite easy to determine the desired positive linear functionals from this structure: The (positive) functional f_j is determined by the (unpolarized) density matrix $\rho_j \equiv (2j+1)^{-1}\Sigma_{m=-j}^{j}|jm\rangle\langle jm|$ (see Chapter 7, Section 7, AMQP). Let x denote an arbitrary element of the RW-algebra. Then the functional f_j associates to the element x the complex number given by

$$x \to f_j(x)=(2j+1)^{-1}\sum_m \langle jm|x|jm\rangle. \tag{2.80}$$

It is clear that this defines a linear functional. To verify that it is positive, one observes that

$$\begin{aligned} f_j(x^*x) &= (2j+1)^{-1}\sum_m \langle jm|x^*x|jm\rangle \\ &= (2j+1)^{-1}\sum_m (\phi_m, \phi_m) \geqslant 0. \end{aligned} \tag{2.81}$$

(Here ϕ_m is the vector $x|jm\rangle \in \mathcal{H}$.)

By means of this functional we can construct a representation of the RW-algebra, in general distinct from the defining (self) representation. We represent the element $x \in$RW-algebra by $x \to A_x$, where A_x is the left translation generated by x. It follows from a theorem of Gel'fand and Naimark [31, Theorem 3, p. 250] that this is a cyclic representation generated by a cyclic vector $\phi_0^{(j)}$, despite the fact that in the original representation $\phi_0^{(j)}$ may not correspond to any vector in $\mathcal{H}$. (This remark is essential for understanding the maximality proof below.)

Each of the functionals f_j determines an inner product in the associated representation. (See Remark below.) This inner product is given by

$$f_j(y^*x) \equiv (y, x)_j, \tag{2.82}$$

where the subscript j is a reminder that this inner product may differ from the original inner product in $\mathcal{H}$.

The importance of the set of functionals $\{f_j\}$ is that they can be used to define left ideals in the RW-algebra. Let us define the set $\mathcal{I}(j)$ to be the set of elements $\{y\}$ whose norm vanishes in the inner product defined by f_j. Thus, the set $\mathcal{I}(j)$ is defined by

$$\mathcal{I}(j) = \{y \in RW: f_j(y^*y) = 0\}. \tag{2.83}$$

Assertion: *The set $\mathcal{I}(j)$ is a left ideal of* RW.

Proof. The Schwartz inequality shows that

$$|f_j(x^*y)|^2 \leqslant f_j(x^*x) f_j(y^*y). \tag{2.84}$$

Now let $y \in \mathcal{I}(j)$ and x be an arbitrary element of the RW-algebra. The condition that xy belongs to $\mathcal{I}(j)$ is that

$$f_j(y^*x^*xy) = 0. \tag{2.85}$$

But by the Schwartz inequality [applied to $(x^*xy)^*$ and y] this is true, since $f(y^*y) = 0$.

We must also verify that, if y_1 and $y_2 \in \mathcal{I}(j)$, then so does the element $y_1 + y_2$. One finds that

$$f_j((y_1+y_2)^*(y_1+y_2)) = f_j(y_1^*y_1) + f_j(y_1^*y_2) + f_j(y_2^*y_1) + f_j(y_2^*y_2) = 0, \tag{2.86}$$

since the first and fourth terms are zero (by hypothesis) and the second and third terms vanish by the Schwartz inequality. ■

Remark. The functional f_j determines an equivalence relation in the RW-algebra. Two elements x and $y \in$ RW are called equivalent, $x \sim y$, if the difference $x - y$ belongs to $\mathcal{I}(j)$. Modulo this equivalence relation, the functional f_j determines an inner product.

This construction has, so far, been essentially a transcription of the general Gel'fand–Raikov procedure, but the method becomes categoric for RW-algebras when we apply the Wigner–Eckart theorem. Recall that the W–E theorem asserts that a general element of the RW-algebra having grade $2J$ and projection M may be written in the form

$$x_{JM}=\sum_{\Delta=-J}^{J}\left\langle\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix}\right\rangle I(\Delta), \tag{2.87}$$

where $I(\Delta)$ is an invariant operator (grade 0).

Now consider the functional f_j applied to $x_{JM}^{*}x_{JM}$ (or, equivalently, x_{JM} applied to the cyclic vector $\phi_0^{(j)}$ of the representation generated by f_j). We find that $\left\langle\begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix}\right\rangle I(\Delta)\rightarrow 0$, if j lies in the characteristic null space of this Wigner operator. Since the characteristic null space is a step function on $\left|\begin{smallmatrix} 2j & & 0 \\ & j+m & \end{smallmatrix}\right\rangle$, having zero for all $j<(J-\Delta)/2$, we see that the ideal $\mathcal{G}(j)$ becomes a condition on the label Δ.

To express this somewhat differently, we see that the existence of the set of ideals allows one to determine, purely algebraically, some information on the *operator shift* label Δ. For the ideal $\mathcal{G}(j)$, we find that all Wigner operators $\left\langle\begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix}\right\rangle$ having $\Delta=-J,-J+1,\ldots,J-2j-1$ $(J\geqslant j)$ are *equivalent to zero* [belong to $\mathcal{G}(j)$]. Note that membership in the ideal $\mathcal{G}(j)$ depends only on the labels J and Δ of the Wigner operators. To avoid the complication of vanishings due to the magnetic quantum number M, it is essential to use an averaging (called incoherent mixing in quantum physics) in the density matrix operator ρ_j (or, expressed equivalently with different words, on the functional f_j).

In order to sharpen this concept of "splitting" the operators by the label Δ, let us introduce an ordering in the set of functionals f_j. Consider two positive functionals, f_a and f_b. The functional f_b will be termed *subordinate* to f_a—denoted by $f_a>f_b$—if, for some constant $\lambda>0$, λf_a-f_b is a positive functional.

Let us demonstrate now that *the set of functionals* $\{f_j\}$ *is simply ordered*: $f_j>f_{j'}$, if $j>j'$.

To do this consider first the set of Wigner operators $\{\left\langle\begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{smallmatrix}\right\rangle : M=J,\ldots,-J\}\equiv\{w_{J\Delta}\}$, and consider the set of associated norms $(w,w)_j$. Using Eqs. (2.10), (2.31), and the orthogonality and symmetry properties of the Wigner coefficients, one finds that

$$f_j(w^{*}w)=(w_{J\Delta},w_{J\Delta})_j=\frac{2j+1+2\Delta}{(2j+1)(2J+1)}\varepsilon_{j,J,j+\Delta}, \tag{2.88}$$

where $\varepsilon_{j,J,j+\Delta}$ is the unit step function defined in Eq. (2.37) (it has the value 1 for $\Delta=J, J-1,\ldots, J-2j$, and the value 0 for $\Delta=J-2j-1,\ldots,-J$ for $J\geqslant j$, and the value 1 for all $\Delta=J, J-1,\ldots,-J$ for $J\leqslant j$.) Choosing $\lambda=(2j'+1)/(2j+1)$ and $j'>j$, we obtain from Eq. (2.88) the following results:

$$(\lambda f_{j'}-f_j)(w^*w)=\begin{cases}\dfrac{2(j'-j)}{(2j+1)(2J+1)}, & \Delta=J,\ldots,J-2j;\\[2ex] \dfrac{2j'+1+2\Delta}{(2j+1)(2J+1)}, & \Delta=J-2j-1,\ldots,J-2j';\\[2ex] 0, & \Delta=J-2j'-1,\ldots,-J,\end{cases}\tag{2.89}$$

where these relations apply in the case where J satisfies $J>j'>j$. If J satisfies $J\leqslant j<j'$, then only the top equation applies with $\Delta=J,\ldots,-J$; if J satisfies $j'\geqslant J>j$, then the top two equations apply with $\Delta=J,\ldots,J-2j$ and $\Delta=J-2j-1,\ldots,-J$, respectively. These results establish that $f_{j'}>f_j$ for $j'>j$, since the invariant factors (elements of grade 0) that enter in the general element of the RW-algebra contribute the *same* numerical factor (positive or zero) in both f_j and $f_{j'}$.

Let us observe next that the associated ideals $\mathcal{I}(j)$ are *maximal ideals.* (They are clearly proper ideals.) To establish that the ideals are maximal, let us assume, on the contrary, that there is an element g of the ideal that is of the form $f_j(g^*g)\neq 0$. But then all elements $x\in$ RW-algebra would belong to the ideal, since we may represent x by the identity

$$x=\left[\frac{xP_\phi g^*}{f_j(g^*g)}\right]g+x\left[e-\frac{P_\phi g^*g}{f_j(g^*g)}\right],\tag{2.90}$$

where P_ϕ is the projection operator onto the subspace of the cyclic vector ϕ of the representation generated by f_j. Since the element belongs to the ideal $\mathcal{I}(j)$—that is,

$$e-\frac{P_\phi g^*g}{f_j(g^*g)}\sim 0,\tag{2.91}$$

we see that the identity expresses x as an element of the left ideal generated by g and $\mathcal{I}(j)$. This is a contradiction, thereby proving the stated maximality.

The fact that the functionals f_j are ordered allows one to determine the Wigner operator label Δ *purely algebraically.* Consider the set of elements of grade $2J$ that do not belong to the ideal $\mathcal{I}(0)$, $j=0$. Such elements necessarily are of the form

$$x=\left\langle \begin{matrix} & 2J & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \times (\text{invariant operator}). \tag{2.92}$$

By normalizing the element x, we can recover the Wigner operator $\left\langle \begin{smallmatrix} & 2J & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix} \right\rangle$, thereby determining the operator having $\Delta=J$. To proceed further, one considers the set of elements of grade $2J$ that do not belong to the ideal $\mathcal{I}(1/2)$ and are orthogonal to the operator $\left\langle \begin{smallmatrix} & 2J & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix} \right\rangle$. This determines, after normalizing, the operator $\langle 2J\ 0\rangle$ having $\Delta=J-1$. By iterating this procedure, the set of Wigner operators $\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{smallmatrix} \right\rangle$ is, in principle, uniquely determined (to within an overall phase), purely algebraically, as asserted. The converse— to construct the Wigner operator from the transformation properties (grading) and the characteristic null space information (the ordered ideals)—is carried out in the next chapter.

8. Notes

1. *Precise definition of a tensor operator.* Following the discussion of Michel [33], we give in this note a precise definition of a tensor operator.

Let $\mathcal{H}$ denote the Hilbert space of state vectors of a given physical system. Assume that we have a symmetry group G acting on $\mathcal{H}$ with the unitary representation $\{U(g): g\in G\}$. Let us denote by $\mathcal{L}(\mathcal{H})=Hom\ (\mathcal{H},\mathcal{H})$ the vector space of linear operators on $\mathcal{H}$. [Thus, both $\mathcal{H}$ and $\mathcal{L}(\mathcal{H})$ are group spaces (G-spaces).] Assume further that we have a finite-dimensional space $\mathfrak{S}$ on which G acts with the representation $\{D(g): g\in G\}$. Then we define: *An $\mathfrak{S}$-tensor operator for G is an element of Hom* $(\mathfrak{S}, \mathcal{L}(\mathcal{H}))^G$, *that is, it is an equivariant linear map T, T: $\mathfrak{S}\to\mathcal{L}(\mathcal{H})$, which satisfies*

$$U(g)T(m)U(g^{-1})=T(D(g)m), \text{ each } m\in\mathfrak{S}, \text{ each } g\in G. \tag{2.93}$$

In this result, $T(m)$ denotes the operator in $\mathcal{L}(\mathcal{H})$ to which the element $m\in\mathfrak{S}$ is mapped by T, that is, $T: m\to T(m)$; the notation $D(g)m$ denotes the element of $\mathfrak{S}$ to which the element $m\in\mathfrak{S}$ is mapped under the action of the group element $g\in G$, that is, $g: m\to D(g)m$.

Next, let A denote the Lie algebra of G and let $\{L(a): a\in A\}$ and $\{iF(a): a\in A\}$ denote, respectively, the representations of A obtained by differentiating the representations $\{U(g): g\in G\}$ and $\{D(g): g\in G\}$. One finds

$$[L(a), L(a')]=L(a\wedge a') \text{ on } \mathfrak{S},$$
$$[F(a), F(a')]=iF(a\wedge a') \text{ on } \mathcal{H},$$

for all elements $a, a' \in A$, where $a \wedge a' \in A$ denotes the (Lie bracket) product of elements a and a'. The equivariance condition (2.93) now takes the form:

$$[F(a), T(m)] = iT(L(a)m), \tag{2.94}$$

where $L(a)m$ is the element of $\mathfrak{S}$ to which $m \in \mathfrak{S}$ is mapped by the operator $L(a)$. [When specialized to $G = SU(2)$ and $\{F(a)\colon a \in su(2)\} = (J_1, J_2, J_3) =$ {generators of $SU(2)$}, the condition (2.94) is precisely the commutation relation given by Eq. (2.5).]

By definition an "$\mathfrak{S}$-tensor operator" is a G-morphism from $\mathfrak{S}$ to $\mathfrak{L}(\mathcal{H})$. If the representation $\{D(g)\colon g \in G\}$ is irreducible, then the corresponding G-morphism is called in physics "an irreducible tensor operator."

Let us remark also that one can define the sum $T_1 + T_2$ of two $\mathfrak{S}$-tensor operators, T_1 and T_2, by

$$T_1 + T_2\colon m \to T_1(m) + T_2(m), \text{ each } m \in \mathfrak{S}.$$

Moreover, if T_1 and T_2 are an $\mathfrak{S}_1$-tensor operator and an $\mathfrak{S}_2$-tensor operator, respectively, then one can also define the sum $T_1 \oplus T_2$ and the product $T_1 \otimes T_2$ by

$$T_1 \oplus T_2\colon m_1 \oplus m_2 \to T_1(m_1) + T_2(m_2), \text{ each } m_1 \oplus m_2 \in \mathfrak{S}_1 \oplus \mathfrak{S}_2,$$

$$T_1 \otimes T_2\colon m_1 \otimes m_2 \to T_1(m_1) T_2(m_2), \text{ each } m_1 \otimes m_2 \in \mathfrak{S}_1 \otimes \mathfrak{S}_2.$$

2. *Special Wigner operators.* The unit tensor operators of the type[1]

$$\left\langle \begin{matrix} & J & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \sim \left\langle \begin{matrix} & 0 & \\ J & & -J \\ & M & \end{matrix} \right\rangle \tag{2.95}$$

map each irrep space $\mathcal{H}_j$ into itself. Accordingly, these Wigner operators form a subalgebra of RW-algebra. Moreover, *each such Wigner operator may be realized as a normalized element in the enveloping algebra of the Lie algebra of $SU(2)$.*

In particular, one has for $M = J$:

$$\left\langle \begin{matrix} & J & \\ 2J & & 0 \\ & 2J & \end{matrix} \right\rangle = (-1)^J \left[\frac{(2J)!}{J!J!} \right]^{\frac{1}{2}} J_+^J \left[\prod_{s=1}^{J} (4\mathbf{J}^2 + 1 - s^2) \right]^{-\frac{1}{2}}. \tag{2.96}$$

The operator (2.95) may now be generated by repeated commutation with

[1]See the Note at the end of Chapter 3 for the definition of equivalent Wigner operators.

J_-:

$$\left\langle \begin{matrix} & J & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle = \left[\frac{(J+M)!}{(2J)!(J-M)!}\right]^{\frac{1}{2}} \left[J_-, \left\langle \begin{matrix} & J & \\ 2J & & 0 \\ & 2J & \end{matrix} \right\rangle \right]_{(J-M)}, \qquad (2.97)$$

where the notation $[A, B]_{(k)}$ denotes k-fold commutation with A; that is, $[A, B]_{(0)} = B$, $[A, B]_{(1)} = [A, B]$, $[A, B]_{(2)} = [A, [A, B]]$, etc.

The Wigner operators (2.97) occur often in physical applications and have been discussed extensively in Note 10 to Chapter 3 in AMQP. We have summarized their properties in Tables 1 and 2 in the Appendix of Tables.

3. *Finite-dimensional matrix representations of special Wigner operators.* The set of Wigner operators for which the shift label Δ is nonpositive can be represented by finite-dimensional matrices; correspondingly, all relations between Wigner operators and Racah invariant operators that entail only "free" (no summation) shift labels may be realized, when restricted to cases of nonpositive shift labels, by finite-dimensional matrices.

This result is a consequence of the fact that we may split the Hilbert space $\mathcal{H}$ into two orthogonal subspaces,

$$\mathcal{H} = \mathcal{V}_k \oplus \mathcal{V}_k^{\perp},$$

where $\mathcal{V}_k$ is defined by

$$\mathcal{V}_k = \mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$$

for each $k = 0, \frac{1}{2}, 1, \ldots$. (The symbol $\mathcal{V}_k^{\perp}$ denotes the orthogonal complement of $\mathcal{V}_k$ in $\mathcal{H}$.) It follows, then, that

$$\left\langle \begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix} \right\rangle : \mathcal{V}_k \to \mathcal{V}_k, \qquad \text{each } a, \text{ each } k, \text{ each } \rho \leqslant 0.$$

Furthermore, it follows from the characteristic null space of a Wigner operator that on the space $\mathcal{V}_k$ we have

$$\left\langle \begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix} \right\rangle : \mathcal{V}_k \to 0, \qquad \text{for } a-\rho \geqslant 2k+1 \text{ and } \rho \leqslant 0.$$

Thus, all but a finite number of Wigner operators having a nonpositive shift label annihilate $\mathcal{V}_k$ (and hence are represented on $\mathcal{V}_k$ by the zero matrix).

One may further split the space $\mathcal{V}_k$ into two subspaces,

$$\mathcal{V}_k = \mathcal{V}_k^{(0)} \oplus \mathcal{V}_k^{(\frac{1}{2})},$$

where $\mathcal{V}_k^{(0)}$ and $\mathcal{V}_k^{(\frac{1}{2})}$ denote, respectively, the sum of subspaces in $\mathcal{V}_k$ corresponding to integral and half-integral values of j. Then each subspace, $\mathcal{V}_k^{(0)}$ and $\mathcal{V}_k^{(\frac{1}{2})}$, is invariant under the action of the Wigner operators having *integral* a and $\rho=0,-1,\ldots,-a$. An example of this latter structure is provided by the geometric action of the dot and cross product of vectors as discussed in AMQP [see Eqs. (6.117)], where the space is $\mathcal{V}_1^{(0)}$.

9. Appendix Orthogonality of the Racah Invariant Operators

In this appendix, we prove the orthogonality relations, Eqs. (2.54), for the Racah invariant operators, using the definition (2.48) and the orthogonality relations, Eqs. (2.33) and (2.35), for the unit tensor operators.

Consider the proof of the first orthogonality relation of Eqs. (2.54). We proceed directly from the definition (2.48), using the fact that a Racah invariant operator is Hermitian:

$$\begin{aligned}
\sum_{\rho\sigma}\left(\mathbf{W}^{abc}_{\rho\sigma\tau}\right)^{\dagger}\mathbf{W}^{abd}_{\rho\sigma\tau} &= \sum_{\rho\sigma}\mathbf{W}^{abc}_{\rho\sigma\tau}\mathbf{W}^{abd}_{\rho\sigma\tau}\\
&= \sum_{\substack{\alpha\beta\gamma\\ \alpha'\beta'\gamma'}} C^{abc}_{\alpha'\beta'\gamma'}C^{abd}_{\alpha\beta\gamma}\left\langle\begin{matrix} & c+\tau & \\ 2c & & 0\\ & c+\gamma' & \end{matrix}\right\rangle\\
&\quad\times\left[\sum_{\rho}\left\langle\begin{matrix} & a+\rho & \\ 2a & & 0\\ & a+\alpha' & \end{matrix}\right\rangle^{\dagger}\left(\sum_{\sigma}\left\langle\begin{matrix} & b+\sigma & \\ 2b & & 0\\ & b+\beta' & \end{matrix}\right\rangle^{\dagger}\left\langle\begin{matrix} & b+\sigma & \\ 2b & & 0\\ & b+\beta & \end{matrix}\right\rangle\right)\right.\\
&\quad\left.\times\left\langle\begin{matrix} & a+\rho & \\ 2a & & 0\\ & a+\alpha & \end{matrix}\right\rangle\right]\left\langle\begin{matrix} & d+\tau & \\ 2d & & 0\\ & d+\gamma & \end{matrix}\right\rangle^{\dagger}\\
&= \sum_{\alpha\beta\gamma} C^{abc}_{\alpha\beta\gamma}C^{abd}_{\alpha\beta\gamma}\left\langle\begin{matrix} & c+\tau & \\ 2c & & 0\\ & c+\gamma & \end{matrix}\right\rangle\left\langle\begin{matrix} & d+\tau & \\ 2d & & 0\\ & d+\gamma & \end{matrix}\right\rangle^{\dagger}\\
&=\delta_{cd}\sum_{\gamma}\left\langle\begin{matrix} & c+\tau & \\ 2c & & 0\\ & c+\gamma & \end{matrix}\right\rangle\left\langle\begin{matrix} & c+\tau & \\ 2c & & 0\\ & c+\gamma & \end{matrix}\right\rangle^{\dagger}=\delta_{cd}\mathbf{I}^{c}_{-\tau}.
\end{aligned}\tag{A.1}$$

To prove the second orthogonality relation, Eq. (2.54), for the Racah invariants, we start with Eq. (2.47). Taking the Hermitian conjugate of this equation, forming the product with the original equation, and summing over

γ and c yields the following result:

$$
\begin{aligned}
&\sum_{c}\mathbf{W}^{abc}_{\rho,\tau-\rho,\tau}\left(\sum_{\gamma}\left\langle\begin{matrix} & c+\tau & \\ 2c & & 0 \\ & c+\gamma & \end{matrix}\right\rangle\left\langle\begin{matrix} & c+\tau & \\ 2c & & 0 \\ & c+\gamma & \end{matrix}\right\rangle^{\dagger}\right)\mathbf{W}^{abc}_{\rho',\tau-\rho',\tau}\\
&\quad=\sum_{c}\mathbf{W}^{abc}_{\tau,\tau-\rho,\tau}\mathbf{I}^{c}_{-\tau}\mathbf{W}^{abc}_{\rho',\tau-\rho',\tau}=\sum_{c}\mathbf{W}^{abc}_{\tau,\tau-\rho,\tau}\mathbf{W}^{abc}_{\rho',\tau-\rho',\tau}\\
&\quad=\sum_{\gamma\alpha\alpha'}\left(\sum_{c}C^{abc}_{\alpha,\gamma-\alpha,\gamma}C^{abc}_{\alpha',\gamma-\alpha',\gamma}\right)\left\langle\begin{matrix} & b+\tau-\rho & \\ 2b & & 0 \\ & b+\gamma-\alpha & \end{matrix}\right\rangle\left\langle\begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{matrix}\right\rangle\\
&\qquad\times\left\langle\begin{matrix} & a+\rho' & \\ 2a & & 0 \\ & a+\alpha' & \end{matrix}\right\rangle^{\dagger}\left\langle\begin{matrix} & b+\tau-\rho' & \\ 2b & & 0 \\ & b+\gamma-\alpha' & \end{matrix}\right\rangle^{\dagger}\\
&\quad=\sum_{\gamma'}\left\langle\begin{matrix} & b+\tau-\rho & \\ 2b & & 0 \\ & b+\gamma' & \end{matrix}\right\rangle\left(\sum_{\gamma}\left\langle\begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\gamma-\gamma' & \end{matrix}\right\rangle\left\langle\begin{matrix} & a+\rho' & \\ 2a & & 0 \\ & a+\gamma-\gamma' & \end{matrix}\right\rangle^{\dagger}\right)\\
&\qquad\times\left\langle\begin{matrix} & b+\tau-\rho' & \\ 2b & & 0 \\ & b+\gamma' & \end{matrix}\right\rangle^{\dagger}\\
&\quad=\delta_{\rho\rho'}\sum_{\gamma'}\left\langle\begin{matrix} & b+\tau-\rho & \\ 2b & & 0 \\ & b+\gamma' & \end{matrix}\right\rangle\mathbf{I}^{a}_{-\rho}\left\langle\begin{matrix} & b+\tau-\rho & \\ 2b & & 0 \\ & b+\gamma' & \end{matrix}\right\rangle^{\dagger}\\
&\quad=\delta_{\rho\rho'}(\mathbf{I}^{a}_{-\rho})'\sum_{\gamma'}\left\langle\begin{matrix} & b+\tau-\rho & \\ 2b & & 0 \\ & b+\gamma' & \end{matrix}\right\rangle\left\langle\begin{matrix} & b+\tau-\rho & \\ 2b & & 0 \\ & b+\gamma' & \end{matrix}\right\rangle^{\dagger}\\
&\quad=\delta_{\rho\rho'}(\mathbf{I}^{a}_{-\rho})'\mathbf{I}^{b}_{\rho-\tau}=\delta_{\rho\rho'}\mathbf{I}^{ab}_{-\rho,\rho-\tau},
\end{aligned}
\tag{A.2}
$$

where

$$(\mathbf{I}^{a}_{-\rho})'\mathbf{I}^{b}_{\rho-\tau}|jm\rangle=I^{a}_{-\rho}(j+\rho-\tau)I^{b}_{\rho-\tau}(j)|jm\rangle.$$

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CHAPTER 3

Null Space Properties and Structure Theorems for RW-Algebra

1. The Pattern Calculus

The term "pattern calculus" is an informal designation given to the technique of evaluating explicitly matrix elements of tensor operators by means of symbolic diagrams and rules (associating diagrams to numerical matrix elements). The utility of such techniques cannot be fully appreciated in a structure as simple as that of angular momentum theory; rather, one must consider the whole family of unitary groups $[U(n)]$—for which the pattern calculus was in fact designed—before it becomes apparent just how great the simplifications achieved by the pattern calculus really are.

The need for such techniques is quickly apparent in $U(n)$ and can already be seen in the introduction of the Gel'fand pattern [see Appendix A to Chapter 5, AMQP]. The great merit of the Gel'fand pattern lies in the replacement of the many complicated inequalities among the state vector labels by "obvious" geometric constraints (betweenness); this replacement, in fact, makes explicit the freedom of independent (but globally constrained) "small motions" of the various labels. The pattern calculus attempts to extend such considerations to the tensor operators in $U(n)$.

One achieves great simplifications by formally extending the $SU(2)$ states of Chapter 2 to $U(2)$ states denoted by a Gel'fand pattern:

$$\left|\begin{matrix} 2j & & 0 \\ & j+m & \end{matrix}\right\rangle \to \left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle \equiv |(m)\rangle. \tag{3.1}$$

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Since the angular momentum $j=(m_{12}-m_{22})/2$ and the projection $m=m_{11}-(m_{12}+m_{22})/2$ are invariant to an arbitrary constant shift of all Gel'fand labels,

$$m_{ij} \rightarrow m_{ij}+k, \tag{3.2}$$

this formal extension to $U(2)$ changes nothing whatsoever in $SU(2)$, and we may simply regard this formal change as defining an equivalence class of patterns for $SU(2)$ (see the Note at the end of this chapter for further discussion).

Consider, then, the $U(2)$ unit tensor operator

$$\left\langle \begin{matrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & M_{11} & \end{matrix} \right\rangle. \tag{3.3}$$

The action (see the Note) of this operator on the generic $U(2)$ basis vector $|(m)\rangle$ is to map this vector to (a scalar multiple of) a new vector $|(m')\rangle$ whose labels are shifted:

$$\begin{aligned}(m')=\begin{pmatrix} m'_{12} & & m'_{22} \\ & m'_{11} & \end{pmatrix}&=\begin{pmatrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 \\ & m_{11}+M_{11} & \end{pmatrix}\\ &\equiv\begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix}+\begin{bmatrix} \Delta_1 & & \Delta_2 \\ & W_1 & \end{bmatrix}.\end{aligned} \tag{3.4}$$

The shift pattern

$$[\Delta]\equiv\begin{bmatrix} \Delta_1 & & \Delta_2 \\ & W_1 & \end{bmatrix},$$

introduced in Eq. (3.4) is defined in terms of the unit tensor operator (3.3) itself by two rules:

(a) The *operator pattern*

$$\begin{pmatrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \end{pmatrix}$$

specifies the shift in the $U(2)$ representation labels m_{12} and m_{22}: $m_{12} \rightarrow m_{12}+\Delta_1$ and $m_{22} \rightarrow m_{22}+\Delta_2$, where

$$\Delta_1=\Gamma_{11}, \qquad \Delta_2=M_{12}+M_{22}-\Gamma_{11}. \tag{3.5}$$

(*b*) The *Gel'fand pattern*

$$\begin{pmatrix} M_{12} & & M_{22} \\ & M_{11} & \end{pmatrix}$$

specifies the shift in the $U(1)$ representation label m_{11}: $m_{11} \to m_{11} + W_1$, where

$$W_1 = M_{11}. \tag{3.6}$$

It should be noted explicitly that the labels appearing in a shift pattern do not, in general, satisfy the betweenness conditions.

The relevant idea of the pattern calculus rules is to associate an algebraic factor with the shift pattern

$$\begin{bmatrix} \Delta_1 & & \Delta_2 \\ & W_1 & \end{bmatrix} \tag{3.7}$$

and the initial basis vector labels

$$\begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \tag{3.8}$$

and then to relate this algebraic factor to the nonvanishing matrix elements of the unit tensor operator (3.3) itself.

The first part of the pattern calculus rules associates a pattern of dots and arrows (called an *arrow pattern*) with the shift pattern (3.7), and is a symbolic diagram representing the shift pattern; the second part of the rules assigns to the arrow pattern an algebraic factor depending on the basis vectors labels (3.8) [this algebraic factor is called the *pattern calculus factor* of the operator (3.3)].

We shall first state the pattern calculus rules for the four *fundamental Wigner operators* $\langle 10 \rangle$, since these rules will imply the more general rules to be stated later.[1]

Pattern Calculus Rules for Fundamental Operators

1. *The arrow pattern*

The arrow pattern is obtained from a shift pattern by representing each position in the shift pattern by a *dot* (node) and then drawing an arrow *from* each dot where a 1 occurs *to* each dot where a 0 occurs.

[1] The general pattern calculus rules are stated fully on pp. 58–59, 65–66.

The association of the fundamental Wigner operators to shift patterns to arrow patterns is thus given by

$$
\begin{array}{ccccc}
\text{Fundamental Wigner operator} & & \text{Shift pattern} & & \text{Arrow pattern} \\
\left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle & \rightarrow & \begin{bmatrix} 1 & & 0 \\ & 1 & \end{bmatrix} & \rightarrow & \begin{matrix} \bullet & \rightarrow & \bullet \\ & \nearrow & \\ & \bullet & \end{matrix} \\
\left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle & \rightarrow & \begin{bmatrix} 1 & & 0 \\ & 0 & \end{bmatrix} & \rightarrow & \begin{matrix} \bullet & \rightarrow & \bullet \\ & \searrow & \\ & \bullet & \end{matrix} \\
\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle & \rightarrow & \begin{bmatrix} 0 & & 1 \\ & 1 & \end{bmatrix} & \rightarrow & \begin{matrix} \bullet & \leftarrow & \bullet \\ & \nwarrow & \\ & \bullet & \end{matrix} \\
\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle & \rightarrow & \begin{bmatrix} 0 & & 1 \\ & 0 & \end{bmatrix} & \rightarrow & \begin{matrix} \bullet & \leftarrow & \bullet \\ & \swarrow & \\ & \bullet & \end{matrix}
\end{array}
\tag{3.9}
$$

2. *The labeled arrow pattern*

A *labeled arrow pattern* is obtained from an arrow pattern by assigning the *partial hooks* $p_{ij}=m_{ij}+j-i$ to the corresponding dots in the arrow pattern, where the dots are labeled

$$
\begin{matrix} 12 & & 22 \\ & 11 & \end{matrix} .
$$

For example, the labeled arrow pattern for

$$
\begin{matrix} \bullet & \rightarrow & \bullet \\ & \nearrow & \\ & \bullet & \end{matrix} \quad \text{is} \quad \begin{matrix} p_{12} & & p_{22} \\ \bullet & \rightarrow & \bullet \\ & \nearrow & \\ & \bullet & \\ & p_{11} & \end{matrix} .
$$

3. *The pattern calculus factor*

An algebraic factor, called the pattern calculus factor, is obtained from a labeled arrow pattern by applying the following three rules:

(*a*) Assign to each arrow the factor

$$
p_{\text{tail}}-p_{\text{head}}+e_{\text{tail}}, \tag{3.10}
$$

where

$$
e_{\text{tail}}=\begin{cases} 0 & \text{if tail is in top row,} \\ 1 & \text{if tail is in bottom row.} \end{cases}
$$

(The symbols p_{tail} and p_{head} denote, respectively, the partial hook assigned to the dot at the tail and head of an arrow.)

(*b*) Write out the algebraic factor:

$$\left[\frac{\text{factor for arrow going between rows}}{\text{factor for arrow going within top row}}\right]^{\frac{1}{2}}$$

This is called the pattern calculus factor for the Wigner operator represented by the arrow diagram.

Using these rules, we find the following pattern calculus factors for the fundamental Wigner operators:

Wigner operator | Labeled arrow pattern | Pattern calculus factor

$$\left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle \qquad \begin{matrix} p_{12} & & p_{22} \\ \bullet & \rightarrow & \bullet \\ & \nearrow & \\ & \bullet & \\ & p_{11} & \end{matrix} \quad \rightarrow \quad \left[\frac{p_{11}-p_{22}+1}{p_{12}-p_{22}}\right]^{\frac{1}{2}},$$

$$\left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle \qquad \begin{matrix} p_{12} & & p_{22} \\ \bullet & \rightarrow & \bullet \\ & \searrow & \\ & \bullet & \\ & p_{11} & \end{matrix} \quad \rightarrow \quad \left[\frac{p_{12}-p_{11}}{p_{12}-p_{22}}\right]^{\frac{1}{2}},$$

$$\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle \qquad \begin{matrix} p_{12} & & p_{22} \\ \bullet & \leftarrow & \bullet \\ & \nwarrow & \\ & \bullet & \\ & p_{11} & \end{matrix} \quad \rightarrow \quad \left[\frac{p_{11}-p_{12}+1}{p_{22}-p_{12}}\right]^{\frac{1}{2}},$$

$$\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle \qquad \begin{matrix} p_{12} & & p_{22} \\ \bullet & \leftarrow & \bullet \\ & \swarrow & \\ & \bullet & \\ & p_{11} & \end{matrix} \quad \rightarrow \quad \left[\frac{p_{22}-p_{11}}{p_{22}-p_{12}}\right]^{\frac{1}{2}}. \tag{3.11}$$

If we put these results in terms of standard (jm)-notation, we find, for example, that

$$\left\langle \begin{matrix} m_{12}+1 & & m_{22} \\ & m_{11} & \end{matrix} \middle| \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle \middle| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle = \left[\frac{p_{12}-p_{11}}{p_{12}-p_{22}}\right]^{\frac{1}{2}} = \left[\frac{j-m+1}{2j+1}\right]^{\frac{1}{2}}.$$

In each case the pattern calculus factor is, to within sign $(\pm)$, *the matrix element* (Wigner coefficient) *of the corresponding fundamental Wigner operator.*

The sign of the Wigner coefficient corresponding to the Wigner operator

$$\left\langle \begin{matrix} & \tau & \\ 1 & & 0 \\ & i & \end{matrix} \right\rangle, \qquad i,\tau=0,1$$

is given by the rule

$$\text{sign}=\text{sign}(\tau-i),$$

where sign $a=+1$ for $a\geqslant 0$ and sign $a=-1$ for $a<0$. We shall generally consider the pattern calculus factor itself to have a + sign, leaving the problem of the sign of a Wigner coefficient as a separate issue.

Remarks. (*a*) At this point, no doubt, one may be inclined to feel that the pattern calculus is an overly complicated and elaborate way to calculate [for $SU(2)$] merely four matrix elements! Let us try to dispel this feeling: *The rules have a natural extension to the general unitary group $U(n)$, the shift pattern*[1] *having now two rows of n and $n-1$ entries, respectively, and this extension yields directly all matrix elements of all elementary Wigner operators* (operators whose irrep labels are 1's and 0's). These rules thus yield the matrix elements (up to phase) of a denumerably infinite set of operators ($n=2,3,4,\ldots$).

(*b*) Probably more important, however, is the fact that one achieves by means of the pattern calculus a fully explicit algebra of operators, for which the product of an arbitrary number of elementary operators is fully defined. In other words, *the pattern calculus can implement for the general unitary group a significant generalization of the RW-algebra of angular momentum theory* (see Ref. [1]).

2. Structural Results for the General [$SU(2)$] Wigner Operator

Wigner operators as forms on the fundamental operators. As a first application of the pattern calculus, we shall give a form for the general Wigner operator as a structure based on the fundamental Wigner operators. This general structure can be specialized to yield either Wigner's or Racah's series defining the general Wigner coefficient.

To begin, let us recall the result obtained in Chapter 5 of AMQP for the rotation matrices expressed abstractly in terms of an arbitrary 2×2 matrix

[1] These shift patterns are those of the so-called $U(n):U(n-1)$ projective operators. The extension of the $U(2)$ pattern calculus rules requires only that one now multiply together all factors associated with arrows going between rows to obtain the numerator, and all factors associated with arrows going within rows to obtain the denominator, of the pattern calculus factor.

V. This result reads

$$D^j_{mm'}(V)=[(j+m)!(j-m)!(j+m')!(j-m')!]^{\frac{1}{2}} \times \sum_{\boxed{\alpha}} \frac{(v_{11})^{\alpha_1^1}(v_{12})^{\alpha_1^2}(v_{21})^{\alpha_2^1}(v_{22})^{\alpha_2^2}}{(\alpha_1^1)!(\alpha_1^2)!(\alpha_2^1)!(\alpha_2^2)!}, \tag{3.12}$$

where the sum is over all nonnegative integers (α_i^j) satisfying the row and column sum constraints indicated by the array

$$\begin{array}{cc|l} \alpha_1^1 & \alpha_1^2 & j+m \\ \alpha_2^1 & \alpha_2^2 & j-m \\ \hline j+m' & j-m' & \end{array}.$$

Thus, the (α_i^j) are to satisfy $\alpha_1^1+\alpha_2^1=j+m'$, $\alpha_1^2+\alpha_2^2=j-m'$, $\alpha_1^1+\alpha_1^2=j+m$, $\alpha_2^1+\alpha_2^2=j-m$.

The defining property of these rotation matrices is that they satisfy the multiplicative identity

$$D^j(U)D^j(V)=D^j(UV). \tag{3.13}$$

We have seen in Chapter 5 of AMQP that there is a connection (umbral transformation) between the Wigner coefficients and the rotation matrices. This connection will now be given in a new form, as an operator relation.

Let us simply replace, in Eq. (3.12), the four complex numbers v_{ij} by the four fundamental Wigner operators:

$$v_{ij} \to \left\langle \begin{matrix} & 2-j & \\ 1 & & 0 \\ & 2-i & \end{matrix} \right\rangle. \tag{3.14}$$

(The particular labels, $2-i$ and $2-j$, are chosen to replace the numerals 1, 2 by the 1, 0 of the pattern notation.) This replacement clearly defines a new operator, explicitly determined by the pattern calculus. We now seek to interpret this operator.

There is, however, a slight problem to be disposed of first. The fundamental operators do not all commute, so that in the substitution $V \to \langle 1\,0 \rangle$, given by Eq. (3.14), one must pay attention to the order of the factors.[1] One easily verifies (directly) that *fundamental operators having the same operator pattern*

[1]Since the underlying operator structure is associative, there are no problems of bracketing any sequence of operators, all bracketings being the same.

commute; that is,

$$\left[\left\langle \begin{smallmatrix} & \tau & \\ 1 & & 0 \\ & i & \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} & \tau & \\ 1 & & 0 \\ & j & \end{smallmatrix} \right\rangle\right]=0, \qquad \text{for } \tau, i, j=0,1. \tag{3.15}$$

We shall choose two particular orderings for the terms in the series, and show that both orderings lead—to within an overall invariant factor—to the same Wigner operator

$$\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{smallmatrix} \right\rangle.$$

These two orderings are[1]

$$\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{smallmatrix} \right\rangle \mathcal{G}_{21}^{(\Delta_1\Delta_2)} = \left[\prod_{i=1}^{2} (\Delta_i)!(W_i)!\right]^{\frac{1}{2}}$$
$$\times \sum_{\boxed{\alpha}} \left\langle \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix} \right\rangle^{\alpha_1^2} \left\langle \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix} \right\rangle^{\alpha_2^2} \left\langle \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix} \right\rangle^{\alpha_1^1} \left\langle \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix} \right\rangle^{\alpha_2^1} \Big/ \prod_{i,j=1}^{2} (\alpha_i^j)!, \tag{3.16}$$

$$\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{smallmatrix} \right\rangle \mathcal{G}_{12}^{(\Delta_1\Delta_2)} = \left[\prod_{i=1}^{2} (\Delta_i)!(W_i)!\right]^{\frac{1}{2}}$$
$$\times \sum_{\boxed{\alpha}} \left\langle \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix} \right\rangle^{\alpha_1^1} \left\langle \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix} \right\rangle^{\alpha_2^1} \left\langle \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix} \right\rangle^{\alpha_1^2} \left\langle \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix} \right\rangle^{\alpha_2^2} \Big/ \prod_{i,j=1}^{2} (\alpha_i^j)!. \tag{3.17}$$

In these equations, we have used the following labels: $[\Delta_1\Delta_2]=[J+\Delta\ J-\Delta]$, $[W_1W_2]=[J+M\ J-M]$; the symbol $\boxed{\alpha}$ designates [see Eq. (3.12)] that the summation is over all nonnegative integers satisfying the fixed row and column sum constraints $\alpha_1^1+\alpha_2^1=\Delta_1$, $\alpha_1^2+\alpha_2^2=\Delta_2$, $\alpha_1^1+\alpha_1^2=W_1$, $\alpha_2^1+\alpha_2^2=W_2$. We indicate this by writing

$$\begin{array}{|cc|l} \alpha_1^1 & \alpha_1^2 & W_1 \\ \alpha_2^1 & \alpha_2^2 & W_2\,. \\ \hline \Delta_1 & \Delta_2 & \end{array} \tag{3.18}$$

[1]Note that, by using the correspondences given by Eq. (2.25), we obtain from Eqs. (3.16) and (3.17) two explicit boson realizations of the general $SU(2)$ Wigner operator acting on the boson space $\mathcal{H}$ defined in Chapter 2.

In order to prove that the operator defined by the right-hand side of each of Eqs. (3.16) and (3.17) is, in fact, a multiple of $\left\langle \begin{smallmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{smallmatrix} \right\rangle$, let us note first that the shift induced by any product of fundamental operators is *additive*. Thus, using the $\boxed{\alpha}$ constraints, we find for the two operators in Eqs. (3.16) and (3.17) precisely the shift pattern

$$\begin{bmatrix} J+\Delta & & J-\Delta \\ & J+M & \end{bmatrix}.$$

This verifies the operator pattern labels for the operator

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle.$$

Next consider the behavior of the fundamental operators under an $SU(2)$ transformation $U=(u_{ij})$. To each U there corresponds a transformation $\mathcal{O}_U$ of the basis states $|(m)\rangle$. [In the boson realization, $\mathcal{O}_U$ is the operator T_U defined by Eq. (2.26); see also Appendix E to Chapter 5, AMQP.] Since the $\langle 1 0 \rangle$ are unit tensor operators, we find

$$U: \left\langle \begin{matrix} & \tau & \\ 1 & & 0 \\ & j & \end{matrix} \right\rangle \to \mathcal{O}_U \left\langle \begin{matrix} & \tau & \\ 1 & & 0 \\ & j & \end{matrix} \right\rangle \mathcal{O}_{U^{-1}} = \sum_i \left\langle \begin{matrix} & \tau & \\ 1 & & 0 \\ & i & \end{matrix} \right\rangle u_{ij}. \tag{3.19}$$

Note that commuting sets of fundamental operators are transformed by Eq. (3.19) into commuting sets.

Thus, under the similarity transformation of fundamental Wigner operators given by the middle part of Eq. (3.19), the right-hand side (denoted RHS) of either Eq. (3.16) or (3.17) undergoes the transformation

$$U: (\mathrm{RHS}) \to \mathcal{O}_U(\mathrm{RHS})\mathcal{O}_{U^{-1}}.$$

On the other hand, the summation in the RHS of Eq. (3.16) or (3.17) is designed so that it transforms under the substitution

$$\left\langle \begin{matrix} & \tau & \\ 1 & & 0 \\ & j & \end{matrix} \right\rangle \to \sum_i \left\langle \begin{matrix} & \tau & \\ 1 & & 0 \\ & i & \end{matrix} \right\rangle u_{ij}$$

in the same way as the representation function (3.12) under the substitution

$$v_{jk} \to \sum_i v_{ik} u_{ij}.$$

Since $D^J(UV)=D^J(U)D^J(V)$, we find that the RHS of Eq. (3.16) or (3.17) undergoes the transformation that is characteristic of a tensor operator T_M^J:

$$U\colon T_M^J \to \mathcal{O}_U T_M^J \mathcal{O}_{U^{-1}} = \sum_{M'} D_{M'M}^J(U) T_{M'}^J.$$

These two properties (the shift and transformation properties) together imply that the right-hand side of Eq. (3.16) or (3.17) can differ from a Wigner operator having the labels of the left-hand side of these equations by at most a multiplicative invariant operator.

The eigenvalues of the invariant operators may be found from a special case—namely, by setting $M=J$. Equations (3.16) and (3.17) then reduce to monomials:

$$\left\langle \begin{array}{ccc} & J+\Delta & \\ 2J & & 0 \\ & 2J & \end{array} \right\rangle \mathcal{I}_{21}^{(\Delta_1\Delta_2)} = \left[\frac{(2J)!}{(J+\Delta)!(J-\Delta)!} \right]^{\frac{1}{2}} \left\langle \begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & 1 & \end{array} \right\rangle^{J-\Delta} \left\langle \begin{array}{ccc} & 1 & \\ 1 & & 0 \\ & 1 & \end{array} \right\rangle^{J+\Delta}, \tag{3.20}$$

$$\left\langle \begin{array}{ccc} & J+\Delta & \\ 2J & & 0 \\ & 2J & \end{array} \right\rangle \mathcal{I}_{12}^{(\Delta_1\Delta_2)} = \left[\frac{(2J)!}{(J+\Delta)!(J-\Delta)!} \right]^{\frac{1}{2}} \left\langle \begin{array}{ccc} & 1 & \\ 1 & & 0 \\ & 1 & \end{array} \right\rangle^{J+\Delta} \left\langle \begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & 1 & \end{array} \right\rangle^{J-\Delta}. \tag{3.21}$$

Operating with Eqs. (3.20) and (3.21) on an arbitrary basis vector, using the pattern calculus rules repeatedly, and using the value of the special Wigner coefficient, which arises on the left-hand side, we obtain[1]

$$\mathcal{I}_{21}^{(\Delta_1\Delta_2)}(m_{12}, m_{22}) = \left[\frac{(m_{12}-m_{22}-\Delta_2+1)_{\Delta_1}}{(m_{12}-m_{22}+1)_{\Delta_1}} \right]^{\frac{1}{2}}, \tag{3.22}$$

$$\mathcal{I}_{12}^{(\Delta_1\Delta_2)}(m_{12}, m_{22}) = \left[\frac{(m_{12}-m_{22}+\Delta_1-\Delta_2+2)_{\Delta_2}}{(m_{12}-m_{22}-\Delta_2+2)_{\Delta_2}} \right]^{\frac{1}{2}}, \tag{3.23}$$

where $\mathcal{I}_{21}^{(\Delta_1\Delta_2)}(m_{12}, m_{22})$ and $\mathcal{I}_{12}^{(\Delta_1\Delta_2)}(m_{12}, m_{22})$ denote the eigenvalues of the corresponding invariant operators on the state $|(m)\rangle$.

These results, together with Eqs. (3.16) and (3.17), may now be used to obtain explicit expressions for the general $SU(2)$ Wigner coefficient. The evaluation of the matrix elements in Eqs. (3.16) and (3.17) is not difficult when the pattern calculus techniques to be developed in the next section are used.[2]

[1]We use the Pochhammer notation $(x)_a$ to denote a rising factorial: $(x)_a = x(x+1)\ldots(x+a-1)$ for $a=1,2,\ldots$; $(x)_0=1$, $(x)_{-a}=0$ for $a=1,2,\ldots$.

[2]The results expressed by Eqs. (3.16)–(3.23) generalize directly to the $\langle p0\ldots0\rangle$ tensor operators in $U(n)$ (see Ref. [2]).

It is interesting to observe that the form (3.17) produces the van der Waerden (or Racah) form[3] of the coefficients, whereas the form (3.16) produces the Wigner form of the coefficients [see Eqs. (3.170) and (3.171) of AMQP].

Further development of the pattern calculus. The general Wigner operator has been expressed above as a form defined on the fundamental operators. We should like to demonstrate now that the pattern calculus when applied to this result leads to significant insights into the structure of this otherwise rather formal result.

Let us consider first a product of any number of fundamental Wigner operators. The operators in this product may be represented by a sequence of arrow patterns by using the correspondences (3.9). Furthermore, the action of this product of Wigner operators on a basis vector $|(m)\rangle$ may itself be represented symbolically as the action of the corresponding sequence of arrow patterns on the basis vector labels (m). In this latter case, we write, for example, the symbolic equation

$$\begin{array}{ccc}\bullet & \to & \bullet\\ & \nearrow & \\ \bullet & & \end{array}\begin{pmatrix} m_{12} & & m_{22}\\ & m_{11} & \end{pmatrix} = \begin{array}{c} p_{12}\quad p_{22}\\ \begin{array}{ccc}\bullet & \to & \bullet\\ & \nearrow & \\ \bullet & & \end{array}\\ p_{11}\end{array}\begin{pmatrix} m_{12}+1 & & m_{22}\\ & m_{11}+1 & \end{pmatrix},$$

where we regard the labeled arrow pattern as "numerical." The action of a second arrow pattern on this relation then gives, for example,

$$\begin{array}{ccc}\bullet & \to & \bullet\\ \searrow & & \\ \bullet & & \end{array}\;\begin{array}{ccc}\bullet & \to & \bullet\\ & \nearrow & \\ \bullet & & \end{array}\begin{pmatrix} m_{12} & & m_{22}\\ & m_{11} & \end{pmatrix}$$

$$= \begin{array}{ccc}\bullet & \to & \bullet\\ \searrow & & \\ \bullet & & \end{array}\;\begin{array}{c} p_{12}\quad p_{22}\\ \begin{array}{ccc}\bullet & \to & \bullet\\ & \nearrow & \\ \bullet & & \end{array}\\ p_{11}\end{array}\begin{pmatrix} m_{12}+1 & & m_{22}\\ & m_{11}+1 & \end{pmatrix}$$

$$= \begin{array}{c} p_{12}+1\quad p_{12}\\ \begin{array}{ccc}\bullet & \to & \bullet\\ \searrow & & \\ \bullet & & \end{array}\\ p_{11}+1\end{array}\;\begin{array}{c} p_{12}\quad p_{22}\\ \begin{array}{ccc}\bullet & \to & \bullet\\ & \nearrow & \\ \bullet & & \end{array}\\ p_{11}\end{array}\begin{pmatrix} m_{12}+2 & & m_{22}\\ & m_{11}+1 & \end{pmatrix}.$$

[3] The van der Waerden form of the coefficients was rediscovered by Racah; see Chapter 3, Section 12, AMQP, for a discussion of the various forms in which the Wigner coefficient may be written.

The matrix element of the product of Wigner operators

$$\left\langle\begin{matrix} & 1 & \\ 1 & & 0\\ & 0 & \end{matrix}\right\rangle\left\langle\begin{matrix} & 1 & \\ 1 & & 0\\ & 1 & \end{matrix}\right\rangle \to \begin{matrix}\bullet & \to & \bullet\\ & \searrow & \\ & \bullet & \end{matrix}\quad\begin{matrix}\bullet & \to & \bullet\\ & \nearrow & \\ & \bullet & \end{matrix}$$

is now obtained by making the mappings (3.11) on the labeled arrow patterns:

$$\begin{matrix} p_{12}+1 & & p_{22}\\ \bullet & \to & \bullet\\ & \searrow & \\ & \bullet & \\ & p_{11}+1 & \end{matrix}\qquad\begin{matrix} p_{12} & & p_{22}\\ \bullet & \to & \bullet\\ & \nearrow & \\ & \bullet & \\ & p_{11} & \end{matrix}\to\left[\frac{(p_{12}-p_{11})(p_{11}-p_{22}+1)}{(p_{12}-p_{22}+1)(p_{12}-p_{22})}\right]^{\frac{1}{2}}.$$

The preceding method implements symbolically the use of the pattern calculus rules for evaluating matrix elements (accounting for phases) of products of fundamental Wigner operators. Although useful for this purpose, it does not alone lead to the desired generalization of the pattern calculus rules, since an arbitrary product of fundamental Wigner operators is *not*, in general, again a Wigner operator.

The exception to this result is seen from Eqs. (3.16) and (3.17) to occur when all the fundamental Wigner operators in a product commute: *Any product of commuting fundamental Wigner operators is again a Wigner operator* (up to a multiplicative invariant). This result, when interpreted in terms of arrow patterns, will lead to the desired generalization of the pattern calculus rules.

The first significant result is as follows: *The two arrow patterns for two commuting fundamental Wigner operators contain no opposing arrows*. (Two arrow patterns are said to contain no opposing arrows if there are no arrows going in opposite directions between corresponding pairs of points. In this case we say that the two Wigner operators have the "no-opposing-arrow" property.)

An example will now indicate the generalization of the pattern calculus rules implied by the no-opposing-arrow property. Consider the following product:

$$\left\langle\begin{matrix} & 1 & \\ 1 & & 0\\ & 1 & \end{matrix}\right\rangle\left\langle\begin{matrix} & 1 & \\ 1 & & 0\\ & 0 & \end{matrix}\right\rangle^{2} \to \begin{matrix}\bullet & \to & \bullet\\ & \nearrow & \\ & \bullet & \end{matrix}\quad\begin{matrix}\bullet & \to & \bullet\\ & \searrow & \\ & \bullet & \end{matrix}\quad\begin{matrix}\bullet & \to & \bullet\\ & \searrow & \\ & \bullet & \end{matrix},$$

$$\begin{matrix} p_{12}+2 & & p_{22}\\ \bullet & \to & \bullet\\ & \nearrow & \\ & \bullet & \\ & p_{11} & \end{matrix}\quad\begin{matrix} p_{12}+1 & & p_{22}\\ \bullet & \to & \bullet\\ & \searrow & \\ & \bullet & \\ & p_{11} & \end{matrix}\quad\begin{matrix} p_{12} & & p_{22}\\ \bullet & \to & \bullet\\ & \searrow & \\ & \bullet & \\ & p_{11} & \end{matrix}$$

$$\to\left[\frac{(p_{12}-p_{11})(p_{12}-p_{11}+1)(p_{11}-p_{22}+1)}{(p_{12}-p_{22})(p_{12}-p_{22}+1)(p_{12}-p_{22}+2)}\right]^{\frac{1}{2}}.$$

This result is just that obtained from the combined shift pattern

$$\begin{bmatrix} 3 & & 0 \\ & 1 & \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \end{bmatrix} + \begin{bmatrix} 1 & & 0 \\ & 0 & \end{bmatrix}\begin{bmatrix} 1 & & 0 \\ & 0 & \end{bmatrix}$$

in the following way: We first form the arrow pattern with multiple arrows given by

$$\begin{array}{ccc} \bullet & \twoheadrightarrow\!\!\!\!\rightarrow & \bullet \\ & \searrow\!\!\!\!\searrow \;\; \nearrow & \\ & \bullet & \end{array} .$$

We then assign the partial hooks $\begin{pmatrix} p_{12} & & p_{22} \\ & p_{11} & \end{pmatrix}$ to the corresponding points and interpret the partial hook differences *factorially*:

$$\begin{array}{ccc} p_{12} & & p_{22} \\ \bullet & \twoheadrightarrow\!\!\!\!\rightarrow & \bullet \\ & \searrow\!\!\!\!\searrow \;\; \nearrow & \\ & \bullet & \\ & p_{11} & \end{array} \rightarrow \left[\frac{(p_{12}-p_{11})_2 (p_{11}-p_{22}+1)}{(p_{12}-p_{22})_3} \right]^{\frac{1}{2}} .$$

This result leads to the following rule for associating an arrow pattern with a general shift pattern, and a pattern calculus factor with a labeled arrow pattern:

1. *The arrow pattern*

   The arrow pattern is obtained from the shift pattern

$$\begin{bmatrix} \Delta_1 & & \Delta_2 \\ & W_1 & \end{bmatrix}$$

   by representing (as before) each position in the shift pattern by a dot and then drawing arrows between pairs of dots. The number of arrows going between each pair of dots is equal to the difference between the numerical shift values corresponding to the respective dots, and the direction of each arrow is from the dot having the larger shift, Δ_{tail}, to the dot having the smaller shift, Δ_{head}.

2. *The labeled arrow pattern*

   The labeled arrow pattern is obtained from the arrow pattern by assigning the partial hooks appearing in the pattern

$$\begin{pmatrix} p_{12} & & p_{22} \\ & p_{11} & \end{pmatrix}$$

   to the corresponding dot.

3. *The pattern calculus factor*

   The pattern calculus factor is obtained from the labeled arrow pattern by assigning the rising factorial to each pair of labeled dots, where A denotes the number of arrows going between the pair of dots. The pattern calculus factor is defined by

$$\left[\left|\frac{\text{product of the two rising factorials for arrows going between rows}}{\text{rising factorial for arrows in top row}}\right|\right]^{\frac{1}{2}}. \tag{3.24}$$

It is to be emphasized that the rule obtained above is a *consequence* of the original pattern calculus rules for fundamental operators and is not an arbitrary new rule.

This generalization of the pattern calculus rules applies to products of fundamental Wigner operators (no opposing arrows), but we still must interpret the relationship of these rules to the evaluation of the matrix elements of the general Wigner operator itself. Equations (3.16) and (3.17) are the key to this interpretation.

Recalling that two fundamental Wigner operators with no opposing arrows commute, we see from Eqs. (3.16) and (3.17) that there are just two general Wigner operators that may be composed of products of fundamental operators having no opposing arrows:

$$\left\langle \begin{matrix} & 2J & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle = \left[\frac{(2J)!}{(J+M)!(J-M)!}\right]^{\frac{1}{2}} \times \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle^{J+M} \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle^{J-M}, \tag{3.25}$$

$$\left\langle \begin{matrix} & 0 & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle = \left[\frac{(2J)!}{(J+M)!(J-M)!}\right]^{\frac{1}{2}} \times \left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle^{J+M} \left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle^{J-M}. \tag{3.26}$$

From these results, we conclude: The pattern calculus rules, as generalized above and applied to the shift patterns,

$$\left[\begin{matrix} 2J & & 0 \\ & J+M & \end{matrix}\right] \quad \text{and} \quad \left[\begin{matrix} 0 & & 2J \\ & J+M & \end{matrix}\right]$$

yield, respectively, the matrix elements of the Wigner operators

$$\left\langle \begin{matrix} & 2J & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \quad \text{and} \quad \left\langle \begin{matrix} & 0 & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle$$

having *extremal* operator patterns,[1] but only up to a sign and the multiplicative factor $[(2J)!/(J+M)!(J-M)!]^{\frac{1}{2}}$. It is the geometric property of no opposing arrows that underlies this result.

We may now use this result for extremal patterns to obtain the significance of the pattern calculus factor for the general Wigner operator, obtaining at the same time the most general form of the pattern calculus rules.

Using the results given by Eqs. (3.25) and (3.26), we can express the forms for $\langle 2J\ 0\rangle$ given in Eqs. (3.16) and (3.17) in terms of two extremal Wigner operators:

$$\left\langle \begin{array}{ccc} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{array} \right\rangle \mathcal{G}_{21}^{(\Delta_1\Delta_2)} = [(W_1)!(W_2)!]^{\frac{1}{2}} \times \sum_{\boxed{\alpha}} \left\langle \begin{array}{ccc} & 0 & \\ \Delta_2 & & 0 \\ & \alpha_1^2 & \end{array}\right\rangle \left\langle \begin{array}{ccc} & \Delta_1 & \\ \Delta_1 & & 0 \\ & \alpha_1^1 & \end{array}\right\rangle \Big/ \left[\prod_{i,j=1}^{2} (\alpha_i^j)!\right]^{\frac{1}{2}}, \tag{3.27}$$

$$\left\langle \begin{array}{ccc} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{array} \right\rangle \mathcal{G}_{12}^{(\Delta_1\Delta_2)} = [(W_1)!(W_2)!]^{\frac{1}{2}} \times \sum_{\boxed{\alpha}} \left\langle \begin{array}{ccc} & \Delta_1 & \\ \Delta_1 & & 0 \\ & \alpha_1^1 & \end{array}\right\rangle \left\langle \begin{array}{ccc} & 0 & \\ \Delta_2 & & 0 \\ & \alpha_1^2 & \end{array}\right\rangle \Big/ \left[\prod_{i,j=1}^{2} (\alpha_i^j)!\right]^{\frac{1}{2}}. \tag{3.28}$$

In these equations, we have $[\Delta_1\Delta_2]=[J+\Delta\ J-\Delta]$, $[W_1W_2]=[J+M J-M]$, and the symbol $\boxed{\alpha}$ designates, as earlier, the constraints indicated by (3.18).

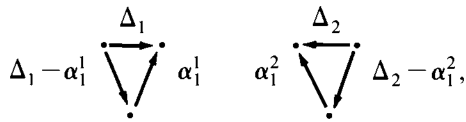
~~Let us turn now to the problem posed by the existence of *opposing arrows*~~ in the evaluation of the matrix element associated with the general operator $\langle 2J\ 0\rangle$, using the representation given by Eq. (3.28) above.

For a generic term in the sum over $\boxed{\alpha}$, we have the operator product associated with the pair of shift patterns [using (3.28)]:

$$\left[\begin{array}{ccc} \Delta_1 & & 0 \\ & \alpha_1^1 & \end{array}\right]\left[\begin{array}{ccc} 0 & & \Delta_2 \\ & \alpha_1^2 & \end{array}\right].$$

[1]An extremal operator pattern is one in which Γ_{11} in $\left(\begin{array}{ccc} & \Gamma_{11} & \\ M_{12} & & M_{22} \end{array}\right)$ reaches a boundary; that is, $\Gamma_{11}=M_{12}$, or $\Gamma_{11}=M_{22}$. This same terminology is applied also to Gel'fand patterns. If either pattern (operator or Gel'fand) in a Wigner operator is extremal, then the Wigner operator is a monomial in the fundamental operators [see Eqs. (3.20)–(3.21)].

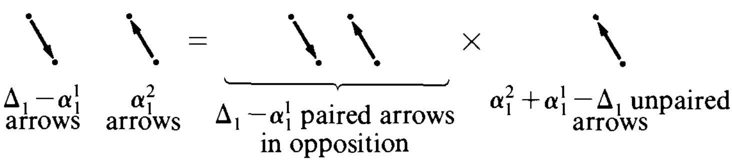
This, in turn, implies the pair of arrow patterns given by



in which the label placed near an arrow designates the number of arrows going between the two points in the indicated direction. Note that the constraints imposed by the $\boxed{\alpha}$ matrix imply that $\Delta_1 \geqslant \alpha_1^1$ and $\Delta_2 \geqslant \alpha_1^2$; this illustrates that the entries in a shift pattern need not obey betweenness.

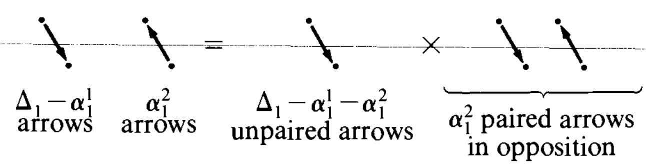
It is the great merit of the pattern calculus that it suggests an approach to the desired answer. First one notes that attention can be confined to the arrows between any pair of dots in the sequence of arrow patterns. This line of reasoning further suggests that one may consider the numerator and denominator patterns separately.

Consider then the numerator pattern, and confine attention to one pair of dots, the pair indexed by (12) and (11). We necessarily have one of two cases: Either $\alpha_1^2 \geqslant \Delta_1-\alpha_1^1$, which implies the patterns



(3.29)

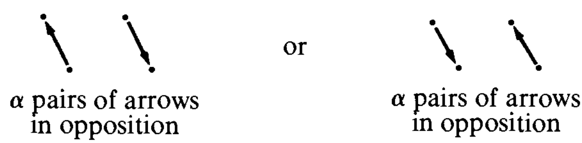
or $\alpha_1^2 < \Delta_1-\alpha_1^1$, which implies the patterns



. (3.30)

In obtaining these symbolic results, we have used the associativity of the pattern calculus to regroup the arrows.

Observe now that each of the composite arrow patterns of the type



effects the same shift,

$$(p_{12}, p_{11}) \to (p_{12}+\alpha, p_{11}+\alpha),$$

on the partial hooks of a generic Gel'fand pattern. Since only differences enter into the algebraic factor associated with an arrow pattern, the presence of the paired arrow patterns in the diagrams (3.29) and (3.30) has no effect whatsoever on the algebraic factor associated with the unpaired arrow patterns. Indeed, using the constraint $\alpha_1^1+\alpha_1^2=W_1=J+M$, we see that the algebraic factors corresponding to the arrow patterns

$W_1-\Delta_1$ arrows

and

Δ_1-W_1 arrows

in the diagrams (3.29) and (3.30), respectively, factor out of the $\boxed{\alpha}$ sum. *These factors are precisely those obtained by applying the pattern calculus rules for multiple arrows directly to the arrow pattern for the* (12) *and* (11) *dots in the shift pattern*

$$\begin{bmatrix} \Delta_1 & & \Delta_2 \\ & W_1 & \end{bmatrix} \textit{ of the operator } \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle .$$

To evaluate the algebraic factor associated with the opposing arrows in diagrams (3.29) and (3.30), we again use the pattern calculus rules. One finds the remarkable result that each of the algebraic factors is a *perfect square* (opposing arrows *always* yield squared factors in the partial hook differences). The square root is thus a product of linear factors. Explicitly, ~~for the two possibilities, one finds, respectively, the following (positive)~~ factors:[1]

$$|(p_{11}-p_{12}+W_1-\Delta_1+1)_{\alpha_2^1}| = (-1)^{\alpha_2^1}(p_{11}-p_{12}+W_1-\Delta_1+1)_{\alpha_2^1},$$

$$|(p_{11}-p_{12}+1)_{\alpha_1^2}| = (-1)^{\alpha_1^2}(p_{11}-p_{12}+1)_{\alpha_1^2}.$$

(We have made use of the constraint $\alpha_2^1=\Delta_1-\alpha_1^1$.)

[1]Careful attention must be paid to the verification that each factor in the rising factorial in these results is negative (or zero) for all values of the initial labels (m) and of the final labels $(m)+[\Delta]$ for which the Gel'fand patterns are lexical. One may, of course, use the relation $(x)_a=(-1)^a(-x-a+1)_a$ to write the rising factorials in various equivalent forms.

The pattern calculus rules, when applied to the arrow patterns for dots (11) and (22),

α_1^1 arrows $\quad \Delta_2 - \alpha_1^2$ arrows

lead to a similar result.

The situation is even simpler for the denominator. The pattern calculus rules, when applied to the arrow patterns for dots (12) and (22),

Δ_1 arrows Δ_2 arrows,

give a factor[1] that is independent of the summation indices, and, hence, this factor appears as an overall multiplicative factor in the matrix element of the operator (3.28). This factor combines with the value of the invariant operator $\mathcal{G}_{12}^{(\Delta_1\Delta_2)}$ given by Eq. (3.23) to yield a term that we denote by D:

$$D_{[\Delta_1\Delta_2]}(p_{12}, p_{22}) = \left[|(p_{12} - p_{22} - \Delta_2)_{\Delta_1}(p_{22} - p_{12} - \Delta_1)_{\Delta_2}| \right]^{\frac{1}{2}}. \tag{3.31}$$

This denominator factor agrees with that given in the denominator of (3.24) only for the extremal cases $\Delta_1 = 2J$, $\Delta_2 = 0$, and $\Delta_1 = 0$, $\Delta_2 = 2J$, in which case the invariant factor $\mathcal{G}_{12}^{(\Delta_1\Delta_2)}$ is unity. We give below the interpretation of D itself directly in terms of the pattern calculus.

Let us summarize.[2] The evaluation of the general Wigner operator as represented by Eq. (3.17), when acting on a generic state vector $|(m)\rangle$, has led us to a result that may be written symbolically in the form

$$\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 \\ & m_{11}+W_1 & \end{matrix} \right| \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle$$
$$= \# \times (\mathrm{PCF}) \times (\text{polynomial}) = \# \times (\mathrm{NPCF}) \times D^{-1} \times (\text{polynomial}). \tag{3.32}$$

We next describe in detail each factor, number (#), pattern calculus factor (PCF), numerator pattern calculus factor (NPCF), denominator (D), and (polynomial) in this result. Before doing so, let us remark that this

[1] The squaring property, typical of the factors associated with opposing arrows going *between* rows, does not occur for opposing arrows going *within* a row.

[2] The results given in the remaining part of this section, in Section 3, and in Appendices A and B have been adapted from Ref. [3].

form, Eq. (3.32), is, in fact, a canonical form for the general Wigner opperator in which the individual factors play decisive structural roles, as discussed in the Remarks on pp. 68–69.

The symbol # denotes a number (independent of the m_{ij}) that could be combined with the polynomial factor, but which has been factored out to simplify the properties of the polynomial described below [see, in particular, Eqs. (3.39), which show that the polynomial has value 1 for extremal patterns]:

$$\# = (-1)^{l_1} \frac{k!(2J-k)!}{[(\Delta_1)!(\Delta_2)!(W_1)!(W_2)!]^{\frac{1}{2}}}, \tag{3.33}$$

where

$$k = \min(\Delta_1, \Delta_2, W_1, W_2), \qquad l_1 = \max(0, W_1 - \Delta_1). \tag{3.34}$$

The factor denoted by PCF is the pattern calculus factor determined by the pattern calculus rules, which we state now in *full generality*, noting that this factor is the ratio of a *numerator pattern calculus factor* (NPCF) and a *denominator pattern calculus factor D*; that is, PCF$=$NPCF$/D$.

The rules refer to the shift pattern

$$\begin{bmatrix} \Delta_1 & & \Delta_2 \\ & W_1 & \end{bmatrix} = \begin{bmatrix} J+\Delta & J-\Delta \\ & J+M \end{bmatrix}$$

of the Wigner operator

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle.$$

The Pattern Calculus Rules for the General Wigner Operator

1. *The arrow pattern and the labeled arrow pattern*

The arrow pattern and the labeled arrow pattern are obtained as previously described by rules (1) and (2) on p. 59.

2. *The numerator pattern calculus factor*

The NPCF is obtained from the labeled arrow pattern as previously described by rule (3) on p. 59, using now only the numerator of that result. Thus,

$$\text{NPCF} = [|\text{product of the two rising factorials for arrows going between rows}|]^{\frac{1}{2}}.$$

3. *The denominator pattern calculus factor*
The complete denominator D is obtained by a variation of the pattern calculus rules as previously given. The shift pattern $[\Delta_1\Delta_2]$ (coming from the operator pattern alone) is first split into a sum of *extremal shift patterns*,

$$[\Delta_1\Delta_2]=[\Delta_1\ 0]+[0\ \Delta_2],$$

and arrow patterns are associated to these shift patterns in the standard way:

$$\underset{\Delta_1 \text{ arrows}}{\bullet\!\!\longrightarrow\!\!\bullet} \qquad \underset{\Delta_2 \text{ arrows}}{\bullet\!\!\longleftarrow\!\!\bullet}.$$

Each of these two arrow patterns is now labeled by the partial hooks (p_{12}, p_{22}) *shifted* by the shift pattern of the other pattern:

$$\underset{\Delta_1 \text{ arrows}}{\overset{p_{12}\qquad\quad p_{22}+\Delta_2}{\bullet\!\!\longrightarrow\!\!\bullet}} \qquad \underset{\Delta_2 \text{ arrows}}{\overset{p_{12}+\Delta_1\qquad\quad p_{22}}{\bullet\!\!\longleftarrow\!\!\bullet}}.$$

The denominator pattern calculus factor is obtained now by applying the factorial rule to multiple arrows in the standard way, thus yielding Eq. (3.31).

Note that this rule for obtaining the denominator pattern calculus factor reduces to the denominator factor in (3.24) whenever the operator pattern is extremal.

The relationship between the pattern calculus rules as applied to the shift pattern of a general Wigner operator is seen now to be the following: The pattern calculus rules give the multiplicative "square-root part" of the coefficient, the remaining part being a polynomial in the differences of the partial hooks (p_{ij}).

For all Wigner operators having extremal operator patterns, the pattern calculus rules give (up to a numerical factor) *the matrix elements of the Wigner operator itself.*

The numerator pattern calculus factor may be written out explicitly with the aid of the step functions u_1, u_2, l_1, and l_2 defined by[1]

$$u_i=\max(0, \Delta_i - W_1), \qquad l_i=\max(0, W_1-\Delta_i), \qquad i=1,2.$$

[1] The nonnegative integer u_i denotes the number of arrows going from dot $(i2)$ to dot (11) in the arrow pattern; similarly, l_i denotes the number of arrows going from dot (11) to dot $(i2)$.

The result is

$$\mathrm{NPCF}=\left[\left|\prod_{i=1}^{2}(p_{11}-p_{i2}-u_i+1)_{u_i}(p_{11}-p_{i2}+1)_{l_i}\right|\right]^{\frac{1}{2}}. \tag{3.35}$$

The polynomial in Eq. (3.32) is explicitly given by

$$\begin{aligned}
&P_k(\Delta_1,\Delta_2,W_1,W_2;p_{11}-p_{12},p_{11}-p_{22})\\
&\quad\equiv(-1)^{l_1+k}\frac{(\Delta_1)!(\Delta_2)!(W_1)!(W_2)!}{k!(2J-k)!}\\
&\qquad\times\sum_{\boxed{\alpha}}(-1)^{\alpha_1^2}\frac{(p_{11}-p_{12}+W_1-\Delta_1+1)_{\sigma_1}(p_{11}-p_{12}+1)_{\sigma_2}}{(\alpha_1^1)!(\alpha_1^2)!}\\
&\qquad\qquad\times\frac{(p_{22}-p_{11}+\Delta_2-W_1)_{\sigma_3}(p_{22}-p_{11})_{\sigma_4}}{(\alpha_2^1)!(\alpha_2^2)!}
\end{aligned} \tag{3.36}$$

in which $k\equiv\min(\Delta_1,\Delta_2,W_1,W_2)$, and the rising factorials are determined from the step functions σ_i, which are

$$\begin{aligned}
\sigma_1&=\begin{cases}\alpha_2^1 & \text{if } \Delta_1\leqslant W_1\\ 0 & \text{if } \Delta_1\geqslant W_1,\end{cases}\\
\sigma_2&=\begin{cases}\alpha_1^2 & \text{if } \Delta_1\geqslant W_1\\ 0 & \text{if } \Delta_1\leqslant W_1,\end{cases}\\
\sigma_3&=\begin{cases}\alpha_1^1 & \text{if } \Delta_2\geqslant W_1\\ 0 & \text{if } \Delta_2\leqslant W_1,\end{cases}\\
\sigma_4&=\begin{cases}\alpha_2^2 & \text{if } \Delta_2\leqslant W_1\\ 0 & \text{if } \Delta_2\geqslant W_1.\end{cases}
\end{aligned} \tag{3.37}$$

The degree of the polynomial P_k in the partial hook differences is given by the integer k, since one sees that k may also be written as $k=\sigma_1+\sigma_2+\sigma_3+\sigma_4$.

The polynomial given by Eq. (3.36) has been stated in a form that resulted directly from the pattern calculus rules for the fundamental Wigner operators. In particular, the curious occurrence of step functions has its origin in these rules.

There are four forms of the polynomial (3.36), depending on the relative magnitudes of Δ_1, Δ_2, W_1, and W_2 and corresponding to the following arrow patterns:

(3.38)

case 1 case 2 case 3 case 4

A single arrow in these diagrams represents zero, one, or more arrows. For determining the properties of the polynomials (3.36), it is convenient to express all four cases in more conventional notation:

$$
\begin{aligned}
P_{W_2}(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2) &= \sum_{k_1+k_2=W_2} \binom{\Delta_1}{k_1}\binom{\Delta_2}{k_2}(z_1+W_1-\Delta_1+1)_{k_1} \\
&\qquad \times (z_2-k_2+1)_{k_2}, \\
P_{\Delta_2}(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2) &= \sum_{k_1+k_2=\Delta_2} \binom{W_1}{k_1}\binom{W_2}{k_2}(z_1+1)_{k_1}(z_2-k_2+1)_{k_2}, \\
P_{\Delta_1}(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2) &= \sum_{k_1+k_2=\Delta_1} \binom{W_2}{k_1}\binom{W_1}{k_2}(z_1+W_1-\Delta_1+1)_{k_1} \\
&\qquad \times (z_2+W_1-\Delta_2-k_2+1)_{k_2}, \qquad (3.39) \\
P_{W_1}(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2) &= \sum_{k_1+k_2=W_1} \binom{\Delta_2}{k_1}\binom{\Delta_1}{k_2}(z_1+1)_{k_1} \\
&\qquad \times (z_2+W_1-\Delta_2-k_2+1)_{k_2}.
\end{aligned}
$$

In stating these results, we have introduced the notations

$$
z_1=p_{11}-p_{12}=-(j-m+1), \qquad z_2=p_{11}-p_{22}=j+m. \qquad (3.40)
$$

[The relations $\Delta_1+\Delta_2=W_1+W_2$ and $(x)_a=(-1)^a(-x-a+1)_a$ have been used in obtaining Eqs. (3.39) from Eq. (3.36).]

Remarks. The results expressed by Eqs. (3.31)–(3.40) no doubt appear quite complicated. Some complication is, of course, to be expected for the general Wigner operator, but it is our claim that the situation is actually the reverse: The form expressed by Eq. (3.32) has a significant and simple structure, and the complication is largely superficial (and notational). To substantiate this claim, let us note the following:

(*a*) The denominator D^2 is a product of linear factors such that $D^2(j)=0$ if and only if the angular momentum j is such that $\mathcal{H}_j$ belongs to the *characteristic null space* of the operator

$$
dim^{-\frac{1}{2}} \left\langle \begin{array}{ccc} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{array} \right\rangle dim^{\frac{1}{2}},
$$

where *dim* denotes the invariant dimension operator defined on each angular momentum space $\mathcal{H}_j$ by *dim*: $\mathcal{H}_j \to \mathcal{H}_j$ and $dim\,|jm\rangle=(2j+1)|jm\rangle$. (It is given on the space of bosons by Eq. (2.24): $dim=N+1$). [The fact that it is the characteristic null space of the operator $dim^{-\frac{1}{2}}\langle\ \rangle dim^{\frac{1}{2}}$, rather than of the Wigner operator $\langle\ \rangle$ itself, that determines the zeros of the denominator function D is due to the particular way in which we normalize a Wigner operator, using Eq. (2.33). This aspect of the denominator function is discussed in detail in the next section.]

(b) The factor NPCF is a product of linear factors such that the matrix elements of the operator vanish *if and only if the lexicality constraints on the final state vector are violated.*

(c) The polynomial is uniquely determined by the remaining characteristic null space zeros.

It is the merit of the pattern calculus that it has implemented these features. The properties (a), (b), and (c) are demonstrated in detail in Section 3 below.

A symmetry relation and generalized Wigner coefficients. Before concluding this section, let us give an important symmetry property of the canonical Wigner coefficients (3.32) and show how this leads to a generalization of these coefficients. Recall that the polynomial (3.36) was obtained by starting from Eq. (3.17). If, instead, we had started from Eq. (3.16), we would have obtained precisely the same two factors, D and NPCF, *but a different polynomial.* Comparing these two polynomials leads to a new symmetry (an alternative proof of this symmetry relation is given in Appendix B):

$$\begin{aligned} P_k(\Delta_1, \Delta_2, W_1, W_2;\, p_{11}-p_{12}, p_{11}-p_{22}) \\ = P_k(\Delta_2, \Delta_1, W_1, W_2;\, p_{11}-p_{22}, p_{11}-p_{12}). \end{aligned} \tag{3.41}$$

In other words, the polynomial part of Eq. (3.32) is invariant under the substitution

$$p_{12} \leftrightarrow p_{22}, \qquad \Delta_1 \leftrightarrow \Delta_2. \tag{3.42}$$

The denominator factor D given by Eq. (3.31) is also invariant under the substitution $p_{12} \leftrightarrow p_{22}$, $\Delta_1 \leftrightarrow \Delta_2$. Indeed, in terms of the notation (3.40), we have

$$\begin{aligned} D_{[\Delta_1\Delta_2]}(p_{12}, p_{22}) &= D'(z_1, z_2) = D''(j) \\ &= \left[|(z_2-z_1-\Delta_2)_{\Delta_1}(z_1-z_2-\Delta_1)_{\Delta_2}|\right]^{\frac{1}{2}} \\ &= \left[|(2j+1-\Delta_2)_{\Delta_1}(-2j-1-\Delta_1)_{\Delta_2}|\right]^{\frac{1}{2}}, \end{aligned} \tag{3.43}$$

since $z_2 - z_1 = 2j+1$. Thus, *D is defined for all complex j.*

Similarly, one finds that NPCF given by Eq. (3.35) is invariant under the substitution $p_{12} \leftrightarrow p_{22}$, $\Delta_1 \leftrightarrow \Delta_2$, and is defined for all complex (m_{ij})—that is, z_1 and z_2.

If one considers now the pattern calculus rules, it is clear that the substitution (3.42) is necessarily a symmetry of the coefficient (3.32), since it does not change in any way the actual numerical factors given by the rules, *except for the sign convention*.

It is useful to note that the substitution $p_{12} \leftrightarrow p_{22}$, $\Delta_1 \leftrightarrow \Delta_2$, implies the following substitutions of the variables $z_1, z_2, \Delta_1, \Delta_2$, and of the angular momentum variables (jm) and $(j'm')$:

$$\begin{aligned}&(z_1, z_2, \Delta_1, \Delta_2) \leftrightarrow (z_2, z_1, \Delta_2, \Delta_1),\\ &(j'm'; jm) \leftrightarrow (-j'-1, m'; -j-1, m).\end{aligned} \tag{3.44}$$

Here (jm) and $(j'm')$ denote, respectively, the angular momentum labels of the initial and shifted Gel'fand patterns, (m) and $(m')=(m)+[\Delta]$.

The substitution $p_{12} \leftrightarrow p_{22}$, $\Delta_1 \leftrightarrow \Delta_2$ does not leave the lexicality conditions (betweenness) of the initial and shifted Gel'fand patterns invariant; accordingly, it is not really a symmetry of the Wigner coefficient, but defines what may be called a *continuation* or *generalization* of the coefficient.

Thus, we may use the right-hand side of Eq. (3.32) to define a *generalized Wigner coefficient* for all complex values (m_{ij}) (with no betweenness). We define

$$\mathcal{C}^{jJj+\Delta}_{m,M,m+M} = \# \times (\text{NPCF}) \times D^{-1} \times (\text{polynomial}). \tag{3.45}$$

In this definition the (m_{ij}) are arbitrary complex numbers, or, equivalently, j and m are arbitrary complex numbers with no relations between them, where we recall that

$$\begin{aligned}&j=(m_{12}-m_{22})/2, \qquad m=m_{11}-(m_{12}+m_{22})/2,\\ &\Delta_1=J+\Delta, \qquad \Delta_2=J-\Delta.\end{aligned} \tag{3.46}$$

The labels J, M, and Δ, however, are still required to assume standard values $J=0,\frac{1}{2},1,\ldots,$ with $M,\Delta=J, J-1,\ldots,-J$. Equation (3.45) defines a generalized Wigner coefficient for all complex (jm), except for those real values of j for which the denominator vanishes, namely, those values of j given by

$$2j+1=-\Delta_1, -\Delta_1+1, \ldots, \Delta_2-\Delta_1-1, \Delta_2-\Delta_1+1, \ldots, \Delta_2-1, \Delta_2. \tag{3.47}$$

By convention the coefficient (3.45) is taken to be zero for values of j given by Eq. (3.47).

The generalized Wigner coefficients defined above then satisfy the symmetry relation

$$(-1)^{l_1}\mathcal{C}^{jJj+\Delta}_{m,M,m+M}=(-1)^{l_2}\mathcal{C}^{-j-1,J,-j-\Delta-1}_{m,M,m+M}, \tag{3.48}$$

where

$$l_1=\max(0,M-\Delta), \qquad l_2=\max(0,M+\Delta).$$

This generalization of the Wigner coefficients has important applications to physical problems; the properties of these coefficients are developed further in Chapter 5, Topic 6.

Remark. The involutory (reflection) symmetry expressed by the substitution $p_{12}\leftrightarrow p_{22}$, $\Delta_1\leftrightarrow\Delta_2$ [or, equivalently, $j+\frac{1}{2}\leftrightarrow-(j+\frac{1}{2})$, $\Delta\leftrightarrow-\Delta$] is the generator of the discrete symmetry group S_2 ($\cong Z_2$). For the general unitary group $U(n)$, the pattern calculus implies the existence of the discrete symmetry group S_n for the (squares of the) Wigner operators. This symmetry is the *operator pattern* analog of the Weyl group of $U(n)$.

Let us note that the use of the partial hooks (p_{ij}), as opposed to the (m_{ij}) themselves, is motivated by the desire to put this symmetry explicitly in evidence.

3. Null Space Aspects of Wigner Coefficients

A motivating example. In order to make clear the essential ideas in the null space concept, it is convenient to discuss first an elementary example, which will lead us directly to the more abstract general formulation. (For clarity we have chosen an example that is independent of the detailed discussion of the previous sections.)

~~Let us suppose that we want to determine explicitly the set of (multipole[1])~~ operators defined by

$$T^J=\left\langle \begin{array}{ccc} & J & \\ 2J & & 0 \\ & J & \end{array}\right\rangle, \qquad J=0,1,\ldots. \tag{3.49}$$

Each of these operators is diagonal on the space[2] $\mathcal{H}_j$:

$$T^J|jm\rangle=C^{jJj}_{m0m}|jm\rangle. \tag{3.50}$$

[1]These tensor operators enter into the description of the ($M=0$) electromagnetic multipole moments of a quantal system (see Chapter 7, Section 6, AMQP).

[2]For this initial discussion, we use the standard ket vector notation for the basis of $\mathcal{H}_j$.

The scalar operator ($J=0$) is the identity operator; the operator for $J=1$ is the $M=0$ component of a vector operator and is therefore (by the Wigner–Eckart theorem) proportional to the angular momentum component J_3 as verified by

$$C_{m0m}^{j1j}=m/\sqrt{j(j+1)}\,. \tag{3.51}$$

How could one determine these operators, including the normalization, using the minimal technical assumptions? To determine the normalization, the answer is to use *multiplet averaging*, a method well-known and frequently applied in the early days of quantum mechanics. Thus, one has

$$\sum_{\text{multiplet}} C_{m0m}^{jJj}C_{m0m}^{jJ'j}=\delta_{JJ'}\times\frac{(\text{dimension of multiplet})}{(\text{dimension of operator})}, \tag{3.52}$$

or, equivalently,

$$\text{trace } T^J T^{J'}=\delta_{JJ'}(2j+1)/(2J+1), \tag{3.53}$$

where the trace of an arbitrary operator $\mathcal{O}\colon \mathcal{H}_j\to\mathcal{H}_j$ is defined by

$$\text{trace}\,\mathcal{O}=\sum_{m=-j}^{j}\langle jm|\mathcal{O}|jm\rangle\,. \tag{3.54}$$

Applying this method to the case $J=1$, using $T^{J=1}=NJ_3$, one finds

$$N^2\sum_{m=-j}^{j}m^2=\sum_{m=-j}^{j}\left(C_{m0m}^{j1j}\right)^2=(2j+1)/3.$$

Since $\sum_{m=-j}^{j}m^2=j(j+1)(2j+1)/3$, one obtains the familiar result, Eq. (3.51).

One now sees that the complete set of diagonal multipole operators (3.49) may be determined in this way. One recognizes that one is simply using the Schmidt orthonormalization process on the ordered set $1, J_3, J_3^2,\ldots$ of operators and using multiplet averaging as the inner product.

To see how null space concepts can fit into this problem, let us consider the quadrupole operator ($J=2$). This operator has the general form

$$T^J=(\text{normalization})\left(a_jJ_3^2+b_jJ_3+c_j\right)$$

so that

$$C_{m0m}^{j2j}=(\text{normalization})\left(a_jm^2+b_jm+c_j\right),$$

where a_j, b_j, and c_j are *polynomials* in $2j+1$ of degrees 0, 1, and 2, respectively.

We now introduce structural information: (a) the knowledge that the matrix element of a quadrupole operator *vanishes* when connecting $j=0$ to $j=0$ or $j=\frac{1}{2}$ to $j=\frac{1}{2}$; that is, the characteristic null space of the diagonal quadrupole operator is $\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}}$; and ($b$) the knowledge that $a_j m^2 + b_j m + c_j$ is invariant under the reflection symmetry $j \to -j-1$. [Note that this transformation leaves the dimension $|2j+1|$ invariant, as well as the eigenvalue $j(j+1)$ of $\mathbf{J}^2$.] This information determines the quadratic form in m up to a multiplicative constant:

$$C^{j2j}_{m0m} = (\text{normalization})\left[3m^2 - j(j+1)\right].$$

Instead of normalizing by multiplet averaging—a tedious method really!—let us apply the characteristic null space idea once again, using also the reflection symmetry:

$$\sum_{m=-j}^{j} \left[3m^2 - j(j+1)\right]^2 = \begin{array}{l}\text{fifth-degree polynomial in } j, \text{ which vanishes for} \\ j=0, \tfrac{1}{2}, -1, -\tfrac{3}{2}, \text{ and contains the dimension of} \\ \text{the multiplet, } 2j+1, \text{ as a factor.}\end{array}$$

This information implies that up to a multiplicative constant the sum is given by

$$(2j+1)(2j)(2j-1)(2j+2)(2j+3),$$

so that

$$C^{j2j}_{m0m} = \#\left[3m^2 - j(j+1)\right] / \left[(2j-1)(2j)(2j+2)(2j+3)\right]^{\frac{1}{2}},$$

where # denotes a numerical constant. Thus, the normalization of the quadrupole operator is reduced to evaluating a numerical special case, and, evaluating the sum (3.52) for $j=1$, we find $\# = \pm 2$.

We conclude from this example the following fact: *Knowledge of the characteristic null space of the diagonal multipole operator*, T^J, *completely determined the explicit functional form of this operator, using only abstract properties of the* $SU(2)$ *structure.*

Remark. The examples we have discussed above have one simplifying feature, which deserves comment. This feature is that for operators of the form T^J given by Eq. (3.49) the numerator pattern calculus factor (NPCF)—using the canonical form, Eq. (3.32), for the general Wigner coefficient—happens to be unity. This simplifies the construction of the polynomial, since it is determined by the characteristic null space zeros alone.

We shall show in the following section that the more general result is also true: A Wigner operator is completely characterized by its null space properties, but in this determination the zeros from the NPCF (lexicality zeros), in addition to the zeros from the characteristic null space, are required.

Detailed investigation of null space. We turn now to a discussion of the null space properties of the general operator as determined by the form given in Eq. (3.32). We have noted earlier (Chapter 2, Section 2) that the characteristic null space of the Wigner operator (see the Note at the end of this chapter)

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{matrix} \right\rangle,$$

where the lower pattern is arbitrary, is the space

$$\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{(J-\Delta-1)/2}$$

for an $SU(2)$ Wigner operator. For a $U(2)$ Wigner operator, the null space is correspondingly given by

$$\sum_{m_{12},\, m_{22}} \oplus \mathcal{H}(m_{12}, m_{22}), \tag{3.55}$$

where the summation is over all irrep labels (m_{12}, m_{22}) such that

$$m_{12} - m_{22} \in \{0, 1, \ldots, J-\Delta-1\}. \tag{3.56}$$

Consider now the canonical form (3.32) of a Wigner operator. Recognizing that the term D is a normalizing factor, we may rewrite an unnormalized Wigner coefficient in the form

$$(\text{NPCF}) \times (\text{polynomial in the two variables } p_{11} - p_{12},\, p_{11} - p_{22}).$$

This form implies that the polynomial part of this expression must vanish for each vector $|(m)\rangle$—or, equivalently, for each point specified by the two variables $z_1 = p_{11} - p_{12}$ and $z_2 = p_{11} - p_{22}$—which belongs to the characteristic null space and *for which the numerator pattern calculus factor is nonvanishing*. Let us give an explicit determination of these points.

We can easily enumerate the set of points $Z = \{(z_1, z_2)\}$ belonging to the characteristic null space of the operator. These points are conveniently tabulated in a triangular array:

$$Z=\{(z_1,z_2)\}=\begin{Bmatrix} (-1,0) & & & \\ (-1,1) & (-2,0) & & \\ (-1,2) & (-2,1) & \ddots & \\ \vdots & & & \\ (-1,\Delta_2-1) & (-2,\Delta_2-2) & \cdots & (-\Delta_2,0) \end{Bmatrix}. \tag{3.57}$$

On the other hand, the set of points for which the numerator pattern calculus factor (NPCF) vanishes may be read off directly from the pattern calculus rules [see Eq. (3.35)]. Denoting this set by Z', one may express it generally by

$$Z'=\left\{\begin{matrix}(z_1,z_2)\text{: either } z_1\geqslant\Delta_1-W_1,\text{ or}\\ z_2\leqslant\Delta_2-W_1-1,\text{ or both}\end{matrix}\right\}. \tag{3.58}$$

Let us remark again that this set of points Z' is also completely characterized by the following property: Z' is the set of points such that the final labels

$$\begin{pmatrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 \\ & m_{11}+W_1 & \end{pmatrix} \tag{3.59}$$

appearing in the Wigner coefficient, Eq. (3.32), *fail* to satisfy the betweenness conditions.

It is now apparent that the polynomial P_k must vanish on the subset of points $Z''\subset Z$ of the characteristic null space given by $Z''=Z-(Z'\cap Z)$:

$$Z''=\{(z_1,z_2):(z_1,z_2)\in Z,\, z_1\leqslant\Delta_1-W_1-1,\, z_2\geqslant\Delta_2-W_1\}. \tag{3.60}$$

A structure theorem based on null space. The purpose of this section is to demonstrate that the null space properties of the generic unit tensor operator $\langle 2J\ 0\rangle$ are categoric; that is, from the null space conditions—together with the symmetry relation, Eq. (3.41)—the explicit generic matrix elements can be deduced. This uniquely determines the operator (after normalization) to within the phase factor, ± 1.

To demonstrate this result we recall the canonical form given by Eq. (3.32):

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \to \#(\text{NPCF})\times D^{-1}\times(\text{polynomial}). \tag{3.61}$$

The first remark is as follows: *Up to a multiplicative factor that is independent of* $z_1=p_{11}-p_{12}$ *and* $z_2=p_{11}-p_{22}$, *the polynomial* $(NPCF)^2$ *is the unique polynomial of smallest degree that is invariant under the substitution given by Eq.* (3.42) *and vanishes on the point set* Z'.

The proof is by direct construction.

The second remark concerns the denominator function, D. We can deduce this function from the requirement that #(NPCF)(polynomial) be normalized. Let us carry out this procedure with the specific intention of determining how the zeros of D^2 are related to the characteristic null space of the operator

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{matrix} \right\rangle . \tag{3.62}$$

How is this Wigner operator to be normalized? This question appears trivial, since surely the normalization is determined from the basic relation, Eq. (2.33). *But observe that this equation does not have the operator* (3.62) *on the right* (that is, acting first) *but rather its conjugate*!

In order to relate the normalization condition (2.33) to the characteristic null space of the operator (3.62), we must rewrite this condition in a form in which the operator (3.62), and not its conjugate, acts first on the generic basis vector $|(m)\rangle$.

[Before proceeding, let us remark that the normalization condition, Eq. (2.33), as written, is due not to perversity, but to the desire to preserve (as far as is practical) standard results in the physics literature.]

Consider now the basic orthonormality relation, Eq. (2.33), setting $\Delta'=\Delta$. If we take matrix elements of the left-hand side of this relation, we obtain

$$\sum_M \left(\langle (m)-[\Delta] | \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger} |(m)\rangle \right)^2 . \tag{3.63}$$

The conjugate operator in this expression is defined by Eq. (2.31) and may be given in the form:

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger} = (-1)^{\Delta-M} dim^{-\frac{1}{2}} \left\langle \begin{matrix} & J-\Delta & \\ 2J & & 0 \\ & J-M & \end{matrix} \right\rangle dim^{\frac{1}{2}}, \tag{3.64}$$

where *dim* is the "dimension operator" defined by

$$dim \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle = (m_{12}-m_{22}+1) \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle . \tag{3.65}$$

[The proof of Eq. (3.64) uses the symmetry relation

$$C^{j-\Delta\, J\, j}_{m-M,\, M,\, m} = (-1)^{\Delta - M}\left(\frac{2j+1}{2j-2\Delta+1}\right)^{\frac{1}{2}} C^{j\, J\, j-\Delta}_{m,\, -M,\, m-M}$$

of the Wigner coefficients.]

With these results we can now put the expression (3.63) in the desired form by replacing Δ by $-\Delta$ and M by $-M$:

$$\sum_M \left(\langle (m)+[\Delta] \| dim^{-\frac{1}{2}} \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle dim^{\frac{1}{2}} | (m)\rangle \right)^2. \tag{3.66}$$

To apply this result, let us recall that the unnormalized Wigner operator can be expressed as a polynomial over boson operators; that is, there exists an explicit polynomial boson operator whose action on basis vectors yields precisely the matrix element #(NPCF)(polynomial). We denote this unnormalized operator as

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & \cdot & \end{matrix} \right\rangle_{\text{unnormalized}} .$$

To define the norm D^2 of this boson operator, we must, in order to agree with convention, use the form given by (3.66):

$$D^2_{[\Delta_1\Delta_2]}(p_{12}, p_{22})$$
$$= \sum_M \left(\langle (m)+[\Delta] \| dim^{-\frac{1}{2}} \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle_{\text{unnormalized}} dim^{\frac{1}{2}} | (m)\rangle \right)^2,$$

or, equivalently,

$$\frac{p_{12}-p_{22}+\Delta_1-\Delta_2}{p_{12}-p_{22}} D^2_{[\Delta_1\Delta_2]}(p_{12}, p_{22})$$
$$= \sum_M \left(\langle (m)+[\Delta] \| \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle_{\text{unnormalized}} | (m)\rangle \right)^2. \tag{3.67}$$

Since the expression on the right-hand side in this result is an $SU(2)$ *invariant* and is the square of a matrix element of a boson operator defined on all basis vectors of $\mathcal{H}$, the right-hand side, and hence the left-hand side, must be a nonnegative number *that is zero on all vectors in the characteristic* *null space of the Wigner operator* (3.62). Where the number is nonzero, we may normalize the Wigner operator, that is,

$$\left\langle 2J \begin{matrix} J+\Delta \\ \\ J+M \end{matrix} 0 \right\rangle = \left\langle 2J \begin{matrix} J+\Delta \\ \\ J+M \end{matrix} 0 \right\rangle_{\text{unnormalized}} \times D^{-1}_{[\Delta_1\Delta_2]}, \tag{3.68}$$

to accord with the (conventional) orthonormality relation (2.33).

This discussion has been rather lengthy, but without this detail it would be very difficult to understand (a) why the characteristic null space of the Wigner operator (3.62) is given by

$$\frac{p_{12}-p_{22}+\Delta_1-\Delta_2}{p_{12}-p_{22}} D^2_{[\Delta_1\Delta_2]}(p_{12}, p_{22})=0;$$

or (b) why the normalization relation is written in the form

$$\sum_M \left\langle 2J \begin{matrix} J+\Delta \\ \\ J+M \end{matrix} 0 \right\rangle \left\langle 2J \begin{matrix} J+\Delta \\ \\ J+M \end{matrix} 0 \right\rangle^{\dagger} = \mathbf{I}^J_{-\Delta},$$

where the characteristic function $\mathbf{I}^J_{-\Delta}$ with values

$$\begin{aligned} I^J_{-\Delta}(m_{12}, m_{22}) &= \varepsilon_{\frac{1}{2}(m_{12}-m_{22}),J,\frac{1}{2}(m_{12}-m_{22})-\Delta} \\ &= \begin{cases} 0 & \text{for } m_{12}-m_{22}=0,1,\ldots, J+\Delta-1 \\ 1 & \text{for } m_{12}-m_{22} \geqslant J+\Delta \end{cases} \end{aligned} \tag{3.69}$$

is essential on the right-hand side (to avoid meaningless division by zero in normalizing the operator $\langle\ \rangle^{\dagger}$). [It is reassuring to note that, for $\Delta=M=0$, the normalization given by Eq. (3.67) agrees with that given by the multiplet averaging method, Eq. (3.52).]

Let us summarize the results obtained above for the denominator function D. We have determined that the function

$$\frac{p_{12}-p_{22}+\Delta_1-\Delta_2}{p_{12}-p_{22}} D^2_{[\Delta_1\Delta_2]}(p_{12}, p_{22}) \tag{3.70}$$

is zero for all points (p_{12}, p_{22}) such that

$$p_{12}-p_{22} \in \{1,2,\ldots,\Delta_2\}. \tag{3.71}$$

Utilizing the invariance under the substitution (3.42), we find that the function (3.70) must also vanish on the points (p_{12}, p_{22}) such that

$$p_{12}-p_{22} \in \{-1,-2,\ldots,-\Delta_1\}. \tag{3.72}$$

Thus, $D^2_{[\Delta_1\Delta_2]}(p_{12}, p_{22})$ itself must vanish on precisely the points given by

$$p_{12}-p_{22}\in\{-\Delta_1, -\Delta_1+1, \ldots, \Delta_2-\Delta_1-1, \Delta_2-\Delta_1+1, \ldots, \Delta_2-1, \Delta_2\}, \tag{3.73}$$

in which only $\Delta_2-\Delta_1$ is missing from the sequence.

It is not difficult to see that, to within a multiplicative factor independent of $(p_{12}-p_{22})$, this categorizes D^2 as the polynomial of least degree in $p_{12}-p_{22}$ with these zeros.

We remark that it is precisely the polynomial in $p_{12}-p_{22}$ with the zeros (3.73) that is constructed from the pattern calculus rules, and from the discussion above, we see that D^2 itself does not define the characteristic null space, as might otherwise be assumed.

Let us turn now to the polynomial term in Eq. (3.32); this term is given explicitly by Eq. (3.36) or, equivalently, by the four equations in (3.39). The expression (3.36) is quite complicated in appearance and would be most unhandy to attempt to deduce from its zeros. The relations given in the form of Eqs. (3.39) are much more amenable and may, in fact, be rewritten in a common form that is somewhat simpler than Eq. (3.36) by using the step functions introduced in Eqs. (3.35):

$$\begin{aligned}&P_k(\Delta_1, \Delta_2, W_1, W_2; z_1, z_2)\\&\quad=\sum_{k_1+k_2=k}\binom{\Delta_1}{k_1+u_1}\binom{\Delta_2}{k_2+u_2}(k_1!)\binom{z_1+\Delta_2-u_2-k_2}{k_1}(k_2!)\binom{z_2-u_2}{k_2}\end{aligned} \tag{3.74}$$

In this expression we have chosen to use (in place of rising factorials) the binomial functions defined for arbitrary x by

$$\binom{x}{a}=x(x-1)\ldots(x-a+1)/a!=(x-a+1)_a/a!. \tag{3.75}$$

We now assert: *Up to a multiplicative factor independent of* z_1 *and* z_2, *the polynomial* $P_k(\Delta_1, \Delta_2, W_1, W_2; z_1, z_2)$ *is the unique polynomial of degree* $k=$ *min*$(\Delta_1, \Delta_2, W_1, W_2)$ *in* z_1 *and* z_2, *which has the symmetry, Eq.* (3.41), *and which vanishes on the point set* Z''.

The proof of this assertion is given in Appendix A.

These results suffice to demonstrate the structure theorem stated at the beginning of this section—that *the operator* $\langle 2J\ 0\rangle$ *is uniquely determined* (*to within* ± 1) *by the null space zeros, the abstract reflection symmetry* (*Eq.* (3.41)) *and the requirement of normalization.*

Remarks. It is known that the Wigner coefficients possess two further classes of zeros. The first class of zeros may be called "symmetry" zeros, since the Wigner coefficient vanishes because of a symmetry. (These symmetries are discussed in detail in Chapter 3, Section 12, AMQP.) For example, we have $C_{000}^{JJJ}=0$ for $J=1,3,\ldots$. Consequently, the vector $|J0\rangle$ is in the null space of the operator $\left\langle \begin{smallmatrix} & J & \\ 2J & & 0 \\ & J & \end{smallmatrix} \right\rangle$ whenever J is an odd integer. The second class of zeros are "accidental" zeros—that is, zeros that occur despite the fact that all triangle conditions (betweenness conditions) are satisfied, and these zeros are not implied by any (known) general symmetry.

The existence of these "extra" zeros points up very clearly the distinction between *structural zeros* [lexicality zeros (or "trivial null space" zeros) and the zeros from the characteristic null space] and all other zeros in the null space of a given unit tensor operator. As shown earlier, the structural zeros are essential in determining the Wigner coefficient itself.

The distinction may be better understood in this way: The *invariant* information on the Wigner operation is contained in the *characteristic null space* such that, if any vector in an irrep space is annihilated, then all vectors in the irrep space are annihilated by the operator. Now, all lexicality constraints are encoded in the NPCF, so that, if a vector in the characteristic null space fails to give a zero of the NPCF, then it *must* be a zero of the polynomial (this is the structural information).

By contrast, the "symmetry" or the "accidental" zeros necessarily require that the polynomial itself vanish (since the NPCF and the denominator do not vanish). The resulting zero occurs for a particular value of j *and* m and is "special" in that other vectors in the irrep space specified by j do *not* vanish. Clearly, these special zeros are not invariant under the group action and, hence, are not of structural significance for the Wigner operator. [It is easily seen that a symmetry zero or an accidental zero (jm) cannot belong to a characteristic set of zeros—that is, a set of zeros $m=-j, -j+1, \ldots, j$—since this would contradict the orthogonality relations.]

Accidental zeros are of intrinsic interest, however; we tabulate and discuss this further in Chapter 5, Topic 10 (along with a similar class of zeros of the Racah coefficients).

4. The Possibility of Defining RW-Algebras for Symmetries Other Than Angular Momentum

The question as to whether or not one can define an RW-algebra for a symmetry other than that of angular momentum reduces to the equivalent question as to the possibility of defining the analog to Wigner coefficients for the irreps of the underlying group structure for the symmetry at issue. Such questions were first considered by E. P. Wigner around 1940, but the results of his far-reaching investigations [which defined the 6-j symbols ("Racah coefficients")] were only partly published at the time[1] (Wigner [4]).

[1] The unpublished second part of this investigation was later published in the collection of fundamental papers on the quantum theory of angular momentum (Biedenharn and van Dam [5]).

In this paper, Wigner singled out two characteristics of a group that are sufficient to guarantee the existence of (what we now call) Wigner coefficients. Let us restrict the symmetry groups under consideration to be compact.[1]

The two characteristics are these:

(a) Every group element g is equivalent to its inverse g^{-1}; expressed in different words, all classes are *ambivalent.*

(b) The Kronecker (or "direct") product of any two irreps contains no irrep more than once. A group possessing this property is said to be *multiplicity-free.*

The investigations of Wigner, and later those of Sharp [6], showed that for every ambivalent multiplicity-free compact group—termed a *simply-reducible* (SR) group by Wigner—the appropriate analog to the Wigner coefficients could be constructed. Using the algebraic approach of the previous chapter, one then obtains the equivalent conclusion: *An RW-algebra can be constructed for every simply-reducible group.*

This result should not be considered as completely satisfactory, however, on two grounds: (a) Many (if not most) of the symmetries important to physics belong to groups that are *not* simply-reducible; (b) the property of being simply-reducible is *sufficient* to guarantee an RW-algebra, but the *necessity* has not been shown.

Mackey [7] showed that the requirement of ambivalence could be weakened considerably. The ambivalence requirement is the same as asserting that the group correspondence

$$R: g \to g^{-1}$$

preserves classes. The correspondence R has the properties that it is (a) an involution: $R^2 = E$ (identity), and (b) an *anti*automorphism: $g_1 \cdot g_2 \to R(g_2) \cdot R(g_1)$, reversing the order of multiplication. Mackey's proposal was to replace the correspondence $g \to g^{-1}$ (for ambivalence) by an arbitrary involutory antiautomorphism, which preserves classes. A group that admits of such an antiautomorphism is called *quasi-ambivalent* (denoted QA).

Sharp [6] investigated the properties of (compact) quasi-ambivalent groups that are multiplicity-free and showed that Wigner coefficients can be satisfactorily defined. The changes are, in fact, quite minimal and are connected only with defining a suitable analog to various ($\pm$) phase factors in the angular momentum case.

The replacement of ambivalence by quasi-ambivalence is eminently satisfactory for physical applications, since most physically important symmetries admit this property. In particular, the unitary groups $U(n)$ are all QA,

[1] In Wigner [4] the more restrictive condition that the group be finite was imposed; the more general compact case was treated in the unpublished manuscript (see footnote p. 80).

but ambivalent only for $n=2$. (A list of QA groups is given in Sharp *et al.* [8].)

The restriction to compactness also appears to be inessential. For the noncompact form of $SU(2)$—the group $SU(1,1)$ or $SL(2,\mathrm{R})$—one can define some of the Wigner coefficients by analytic continuation from the explicit functional forms obtained in $SU(2)$ (see Chapter 5, Topic 5, Section 6). The procedure can be generalized to yield all Wigner coefficients in the double covering of $SU(1,1)$ (Holman and Biedenharn [9]). The lack of compactness for the symmetry group is accordingly not an essential requirement for the existence of an RW-algebra.

The single, most difficult requirement to eliminate has been the multiplicity-free condition. This requirement is the key to the existence of the Wigner–Eckart theorem (cf. Wigner's article in Ref. [5]) and thus to the construction of the associated RW-algebra. The essential difficulty posed by the existence of multiplicity is the lack of a canonical[1] procedure to distinguish among these multiple occurrences (there exist distinct unit tensor operators having the same transformation and shift properties).

The concept of null space as a way to characterize Wigner operators intrinsically (as discussed in this chapter) has considerable promise as a general procedure to resolve the "multiplicity problem." It has been shown in Refs. [11–14] that this concept allows a canonical resolution of the multiplicity problem for the unitary symmetry (which is fundamental to particle physics) $SU(3)$, and hence allows of the construction of an $SU(3)$ RW-algebra. [Partial results have been obtained for general $U(n)$ (see Ref. [15]), and there are good indications that further progress can be achieved.] It is our belief that the concepts of null space and of an RW-algebra offer the appropriate techniques for the general analysis of symmetry in quantum physics, far beyond the original restrictive requirements of simple-reducibility. (Recent surveys appear by Butler [18] and by Kibler and Grenet [19].)

5. Note

Mapping of $U(2)$ Wigner operators onto $SU(2)$ Wigner operators. We have seen in Appendices B and E to Chapter 5, AMQP, that the Wigner coefficients of the group $U(2)$ may be chosen to be numerically equal to the Wigner coefficients of $SU(2)$. This property allows one to establish a mapping of the set of $U(2)$ Wigner operators onto the set of $SU(2)$ Wigner operators.

[1]We use canonical in the sense discussed by Artin [10]; that is, canonical applies to a mathematical construction that is unique in as much as no free choices of objects are used in it, to within equivalence.

A $U(2)$ Wigner operator

$$\left\langle \begin{array}{ccc} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & M_{11} & \end{array} \right\rangle \tag{3.76}$$

is defined by giving its action on the separable Hilbert space $\mathcal{H}'$ defined by

$$\mathcal{H}' = \sum_{m_{12} \geqslant m_{22}} \oplus \mathcal{H}(m_{12}, m_{22}), \tag{3.77}$$

where $\mathcal{H}(m_{12}, m_{22})$ denotes the vector space of dimension $m_{12} - m_{22} + 1$ spanned by the set of orthonormal vectors

$$\left\{ \left| \begin{array}{ccc} m_{12} & & m_{22} \\ & m_{11} & \end{array} \right\rangle : m_{11} = m_{22}, m_{22}+1, \ldots, m_{12} \right\}. \tag{3.78}$$

Explicitly, this action is defined by

$$\left\langle \begin{array}{ccc} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & M_{11} & \end{array} \right\rangle \left| \begin{array}{ccc} m_{12} & & m_{22} \\ & m_{11} & \end{array} \right\rangle = C^{jJj+\Delta}_{m,M,m+M} \left| \begin{array}{cc} m_{12}+\Delta_1 & m_{22}+\Delta_2 \\ \multicolumn{2}{c}{m_{11}+M_{11}} \end{array} \right\rangle, \tag{3.79}$$

where the shift pattern

$$\left[\begin{array}{ccc} \Delta_1 & & \Delta_2 \\ & M_{11} & \end{array} \right] \tag{3.80}$$

is defined in Eqs. (3.5) and (3.6), and the angular momentum labels (jm), (JM), and $(j+\Delta, m+M)$ are defined by

$$\begin{aligned} j &= (m_{12} - m_{22})/2, & m &= m_{11} - (m_{12} + m_{22})/2; \\ J &= (M_{12} - M_{22})/2, & M &= M_{11} - (M_{12} + M_{22})/2,. \\ & & \Delta &= \Gamma_{11} - (M_{12} + M_{22})/2. \end{aligned} \tag{3.81}$$

With each $U(2)$ Wigner operator (3.76) acting on the space $\mathcal{H}'$, we now associate an $SU(2)$ Wigner operator acting on the subspace

$$\mathcal{H} \equiv \sum_{j} \oplus \mathcal{H}(2j, 0) \tag{3.82}$$

by using the rule

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle_{SU(2)} \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle$$

$$\equiv \left\langle \begin{matrix} m_{12}+\Delta_1 & m_{22}+\Delta_2 \\ \multicolumn{2}{c}{m_{11}+M_{11}} \end{matrix} \right| \left\langle \begin{matrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & M_{11} & \end{matrix} \right\rangle \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle$$

$$\times \left| \begin{matrix} 2(j+\Delta) & & 0 \\ & j+\Delta+m+M & \end{matrix} \right\rangle. \tag{3.83}$$

Thus, each $U(2)$ Wigner operator (acting in $\mathcal{H}'$) in the set

$$\left\{ \left\langle \begin{matrix} & \Gamma_{11}+k & \\ M_{12}+k & & M_{22}+k \\ & M_{11}+k & \end{matrix} \right\rangle : k=\ldots,-2,-1,0,1,2,\ldots \right\} \tag{3.84}$$

is mapped to the single $SU(2)$ Wigner operator (acting in $\mathcal{H}$):

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle_{SU(2)} \tag{3.85}$$

In particular, note that

$$\left\langle \begin{matrix} & k & \\ k & & k \\ & k & \end{matrix} \right\rangle \rightarrow \left\langle \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0 & \end{matrix} \right\rangle_{SU(2)} = \text{identity operator} \tag{3.86}$$

for all integral k.

One should distinguish in the notation (as above) between the $U(2)$ Wigner operator

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \qquad \text{with shift pattern} \begin{bmatrix} J+\Delta & J-\Delta \\ \multicolumn{2}{c}{J+M} \end{bmatrix} \tag{3.87}$$

acting in $\mathcal{H}'$ and the $SU(2)$ Wigner operator

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \qquad \text{with shift pattern} \begin{bmatrix} 2\Delta & & 0 \\ & M+\Delta & \end{bmatrix} \tag{3.88}$$

acting in $\mathcal{H}$. It is usually clear from the context which interpretation is being used [$SU(2)$ in Chapter 2; $U(2)$ in Chapter 3], and no notational distinctions are introduced. Note that either interpretation will lead to the same numerical results for the Wigner coefficients themselves. The $U(2)$ interpretation is simpler to implement in terms of the pattern calculus rules.

6. Appendices

A. Determination of P_k from Its Characteristic Zeros

The purpose of this appendix is to prove that the polynomial (3.36) is uniquely determined up to a multiplicative factor that is independent of z_1 and z_2 by the zeros in the point set Z'' [Eq. (3.60)] and the symmetry property (3.41). We give the (constructive) proof only for the case $u_1 = u_2 = 0$, so that $\Delta_1 \leqslant W_1$, $\Delta_2 \leqslant W_2$, and $k = W_2$. [A similar procedure applies to the other three cases given in Eqs. (3.39).] In this case the polynomial in question becomes [see the first of Eqs. (3.39); also, Eq. (3.74)]

$$P_k(\xi_1, \xi_2, W_1, W_2; z_1, z_2) = \sum_{k_1+k_2=k} [\xi_1]_{k_1}[\xi_2]_{k_2} k_1! \binom{z_1+\xi_2-k_2}{k_1}\binom{z_2}{k_2}, \tag{A.1}$$

where $k = W_2$ and $\xi_i = \Delta_i$ (the proof below is valid for arbitrary parameters ξ_i).

Our task is to prove that the polynomial (A.1) is determined up to a multiplicative factor independent of z_1 and z_2 by the zeros of the point set Z'' (specialized to the case at hand) and the symmetry property (3.41).

We shall show here that the polynomial (A.1) is determined up to an arbitrary multiplicative factor in ξ_1 and ξ_2 by the sets of zeros $\{(z_1, z_2)\}$ given by

---

$$T_\alpha = \{(k-\xi_2-\beta, \alpha) : \beta = 1, 2, \ldots, k-\alpha\}, \tag{A.2}$$

where $\alpha = 0, 1, \ldots, \mathrm{k}-1$;

$$S_\alpha = \{(\alpha, k-\xi_1-\beta) : \beta = 1, 2, \ldots, k-\alpha\}, \tag{A.3}$$

where $\alpha = 0, 1, \ldots, k-1$. Under the identifications $\xi_i = \Delta_i$ and $k = W_2$, the sets $T_0, T_1, \ldots, T_{k-1}$ contain precisely the points of Z'' given by Eq. (3.60) [for case 1 in the arrow patterns (3.38)]; the sets $S_0, S_1, \ldots, S_{k-1}$ contain the points obtained from the T_α by the substitution (3.42).

Let $P_k(z_1, z_2)$ denote a polynomial of (total) degree k in z_1 and z_2, which vanishes on the sets T_α, S_α, $\alpha = 0, 1, \ldots, k-1$. The vanishing of $P_k(z_1, z_2)$ on the set of points of T_0 yields the result $P_k(z_1, 0) = \alpha_0(\xi_1, \xi_2)\binom{z_1+\xi_2}{k}$, where

$\alpha_0(\xi_1, \xi_2)$ is arbitrary. Putting $P_k(z_1, z_2) = P_k(z_1, 0) + z_2 Q_{k-1}(z_1, z_2)$, where Q_{k-1} is a polynomial of degree not greater than $k-1$, we obtain $P_k(z_1, 1) = Q_{k-1}(z_1, 1) = 0$ on the set T_1, and therefore $Q_{k-1}(z_1, 1) = \alpha_1(\xi_1, \xi_2)\binom{z_1+\xi_2-1}{k-1}$. Putting $P_k(z_1, z_2) = P_k(z_1, 0) + z_2 Q_{k-1}(z_1, 1) + z_2(z_2 - 1)Q_{k-2}(z_1, z_2)$, where Q_{k-2} is of degree not greater than $k-2$, we obtain $P_k(z_1, 2) = 2Q_{k-2}(z_1, 2) = 0$ on the set T_2; that is, $2Q_{k-2}(z_1, 2) = \alpha_2(\xi_1, \xi_2)\binom{z_1+\xi_2-2}{k-2}$. Continuing this procedure in an obvious manner, we come to the conclusion that the most general polynomial of degree k, which vanishes on the points in the sets $T_0, T_1, \ldots, T_{k-1}$, is

$$P_k(z_1, z_2) = \sum_{s=0}^{k} a_s(\xi_1, \xi_2)\binom{z_1+\xi_2-s}{k-s}\binom{z_2}{s}, \tag{A.4}$$

in which the $a_s(\xi_1, \xi_2)$ are arbitrary.

We now repeat the argument using the points of the sets $S_0, S_1, \ldots, S_{k-1}$ to come to the conclusion that

$$P_k(z_1, z_2) = \sum_{s=0}^{k} b_s(\xi_1, \xi_2)\binom{z_1}{s}\binom{z_2+\xi_1-s}{k-s}, \tag{A.5}$$

in which the $b_s(\xi_1, \xi_2)$ are arbitrary.

The two expressions (A.4) and (A.5) must agree identically in z_1 and z_2. Setting $z_2 = 0$ and equating the expressions gives

$$\sum_{s=0}^{k} b_s(\xi_1, \xi_2)\binom{\xi_1-s}{k-s}\binom{z_1}{s} = a_0(\xi_1, \xi_2)\binom{z_1+\xi_2}{k}. \tag{A.6}$$

We now set $z_1 = 0, 1, 2, \ldots,$ in turn, in Eq. (A.6) to obtain a triangular system of equations, which uniquely yields

$$a_0(\xi_1, \xi_2) = a(\xi_1, \xi_2)\binom{\xi_1}{k}, \tag{A.7}$$

$$b_s(\xi_1, \xi_2) = a(\xi_1, \xi_2)s!(k-s)!\binom{\xi_1}{s}\binom{\xi_2}{k-s}, \tag{A.8}$$

in which $a(\xi_1, \xi_2)$ is arbitrary. Using this result in Eq. (A.5), we obtain

$$P_k(z_1, z_2) = a(\xi_1, \xi_2)\sum_{k_1+k_2=k}(k_1)!(k_2)!\binom{\xi_1}{k_1}\binom{\xi_2}{k_2}\binom{z_1}{k_1}\binom{z_2+\xi_1-k_1}{k_2}. \tag{A.9}$$

We now apply the substitution $\xi_1 \leftrightarrow \xi_2$, $z_1 \leftrightarrow z_2$ to obtain the desired result:

$$P_k(z_1, z_2) = a(\xi_1, \xi_2) P_k(\xi_1, \xi_2, W_1, W_2; z_1, z_2). \tag{A.10}$$

B. Symbolic Forms of the Wigner Coefficients

Symbolic methods for generating the Wigner coefficients have been noted by several authors (Sato and Kaguei [16], Gel'fand *et al.* [17]). The existence of such symbolic forms may be traced to the "discretized rotation matrices" discussed in Chapter 5, Section 8, AMQP, or to the expansions of the types (3.16) and (3.17), although there is no (known) theory of such symbolic techniques.

The polynomial P_k has a simple symbolic interpretation that we find useful to discuss as typical of symbolic methods of this type; this symbolic form exhibits the symmetries of P_k in a particularly nice way.

Consider the following expression, in which the ξ_i and z_i are indeterminates:

$$\{\!\!\{\xi_1 z_1 + \xi_2 z_2 + \xi_1 \xi_2\}\!\!\}^k / k!. \tag{B.1}$$

The $\{\!\!\{\cdots\}\!\!\}$ bracket around the enclosed linear form in z_1 and z_2 symbolizes the following operations: Expand the form by the usual trinomial theorem, collect together the powers of each variable, and map each power ζ^a into $a!\binom{\zeta}{a}$. Thus, the explicit definition of the expression (B.1) is

$$[\xi_1 z_1 + \xi_2 z_2 + \xi_1 \xi_2]^k / k! = \sum_{(k)} \frac{(k_1+k_3)!(k_2+k_3)!}{(k_3)!} \binom{\xi_1}{k_1+k_3} \binom{\xi_2}{k_2+k_3} \binom{z_1}{k_1} \binom{z_2}{k_2}, \tag{B.2}$$

where the summation is over all nonnegative integers $(k) = (k_1 k_2 k_3)$ such that $k_1 + k_2 + k_3 = k$.

The symmetry property

$$\{\!\!\{\xi_1 z_1 + \xi_2 z_2 + \xi_1 \xi_2\}\!\!\}^k = \{\!\!\{\xi_2 z_2 + \xi_1 z_1 + \xi_2 \xi_1\}\!\!\}^k \tag{B.3}$$

under the substitution $\xi_1 \leftrightarrow \xi_2$, $z \leftrightarrow z_2$ is an obvious property of the expansion (B.2).

One of the internal summations occurring in the right-hand side of Eq. (B.2) can be carried out by using the binomial addition theorem. There are two ways of doing this, leading to the following forms:

$$
\begin{aligned}
&\{\xi_1 z_1+\xi_2 z_2+\xi_1\xi_2\}^k/k! \\
&=\sum_{k_1+k_2=k}(k_1)!(k_2)!\binom{\xi_1}{k_1}\binom{\xi_2}{k_2}\binom{z_1+\xi_2-k_2}{k_1}\binom{z_2}{k_2} \\
&=\sum_{k_1+k_2=k}(k_1)!(k_2)!\binom{\xi_1}{k_1}\binom{\xi_2}{k_2}\binom{z_1}{k_1}\binom{z_2+\xi_1-k_1}{k_2}.
\end{aligned}
\tag{B.4}
$$

Notice that, whereas the symmetry relation (B.3) is transparent in the expression (B.2), it is lost in the individual expressions occurring in the right-hand side of Eq. (B.4), but is regained through the equality of the two summation expressions in the right-hand side of Eq. (B.4).

We have thus proved [see Eq. (A.1)] that

$$
P_{W_2}(\xi_1,\xi_2,W_1,W_2;z_1,z_2)=\{\xi_1 z_1+\xi_2 z_2+\xi_1\xi_2\}^{W_2}/W_2!. \tag{B.5}
$$

Noticing that the general polynomial, Eq. (3.36), may be expressed as

$$
P_k(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2)=\frac{P_{W_2}(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2)}{\prod_i u_i!\binom{z_i}{u_i}} \tag{B.6}
$$

(the denominator polynomial always divides the numerator), we find

$$
P_k(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2)=\frac{\{\Delta_1 z_1+\Delta_2 z_2+\Delta_1\Delta_2\}^{W_2}/W_2!}{\prod_i u_i!\binom{z_i}{u_i}}. \tag{B.7}
$$

The symmetry relation (3.41) follows immediately from expression (B.7).

Remarks. (*a*) Equation (B.4) expresses the equivalence of the Wigner and Racah forms of the Wigner coefficients; Eq. (B.7) also shows that these two forms are different reductions of a common form (B.2), which exhibits *termwise* not only the 72 symmetries of the Wigner coefficient discussed in Appendix C to Chapter 5, AMQP but also the (nonlexical) symmetry (3.48). (*b*) The division of the numerator of (B.7) by the denominator is the discrete analog for Wigner coefficients of the properties of Jacobi polynomials expressed by Eqs. (3.71), Chapter 3, AMQP. (*c*) A uniform generation [one that does not single out the case $k=W_2$; see Eq. (B.7)] of the polynomials P_k by symbolic expansions is given in Appendix D to Chapter

4, where symbolic forms of the Racah coefficients are discussed, the limiting relation (2.51), Chapter 2, being the origin of these relations.

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CHAPTER 4

W-Algebra: An Algebra of Invariant Operators

1. Introduction

In the previous two chapters we have discussed the algebra, called the RW-algebra, generated by algebraic operations on the fundamental Wigner operators [realized as normed linear operators in the specific Hilbert space $\mathcal{H}$ having as a basis the (equivalence classes of) angular momentum ket vectors $\left\{\left|\begin{matrix} 2j & & 0 \\ & j+m & \end{matrix}\right\rangle\right\}$.] The Racah operators appeared in this algebra as merely one type of invariant operator. The existence of several fundamental identities for the Racah invariants, however, clearly suggests that the Racah invariants realize on their own an interesting algebraic structure, but it was not clear at that point in the development how to exploit this suggested structure further.

We shall demonstrate in this chapter an algebraic structure realized by the Racah invariants, denoted as W-algebra, which parallels in a remarkable way the RW-algebra discussed earlier. It will be shown, in fact, that the RW-algebra is a well-defined limit for W-algebra, and thus W-algebra is, in this sense, the more fundamental structure.

In order to develop the concepts of the W-algebra, we must first enlarge the scope of the investigation to include the Wigner coefficients of the direct product group $SU(2)\times SU(2)$. This straightforward generalization leads to the 9-j symbols, and the Racah coefficients reappear as the subset of "Wigner operators" in $SU(2)\times SU(2)$, which are invariant operators in the diagonal subgroup.

These developments are ancillary to our main purpose, and, even though the 9-j symbols play an interesting role in LS-coupling problems physically

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(see Chapter 3, Section 19, and Chapter 7, Sections 5 and 9, of AMQP), we do not pursue this topic further. The real purpose for considering the group $SU(2)\times SU(2)$ is to obtain a well-defined Hilbert space on which the fundamental Racah invariant operators generate a graded Banach star-algebra—called W-algebra—which closely parallels the earlier construction of RW-algebra.

It is remarkable, we feel, that it proves possible to construct a pattern calculus for this new structure. This calculus is a generalization of the previous pattern calculus, but it is based on precisely the same arrow pattern rules. [Even more remarkable is the fact that this generalization was found earlier in connection with $SU(3)$ operator structures (see Refs. [1–3]) and had originally no connection with Racah invariants in any way.]

By means of this pattern calculus realization for W-algebra, we are able to prove a structure theorem for the Racah invariants (in terms of the null space properties) that parallels the earlier structural results for the Wigner operators in the RW-algebra.

The realization achieved here for the W-algebra acts in a Hilbert space whose basis vectors are labeled by *discrete triangles* (triangles whose sides are integral multiples of an elementary unit length). This geometric structure is quite suggestive; for example, by letting one point of the triangle go to infinity, we recover the earlier RW-algebra; a second limit (taking one of the two remaining points to infinity) would recover an algebra based on the Jacobi polynomials. These limit properties are briefly discussed in Section 4.

2. Wigner Operators for the Direct Product Group $SU(2)\times SU(2)$

Let us recall that the group $SU(2)\times SU(2)$ is the set of ordered pairs $\{(U,V)\colon U\in SU(2)$ and $V\in SU(2)\}$ with the multiplication rule $(U,V)(U',V')=(UU',VV')$.

Let

$$(U,V)\to \mathcal{O}_{(U,V)} \tag{4.1}$$

denote a realization of $SU(2)\times SU(2)$ by unitary operators acting in a separable Hilbert space $\mathcal{H}$ having the orthonormal direct product basis

$$\begin{aligned}|j_1m_1;\, j_2m_2\rangle\equiv|j_1m_1\rangle\otimes|j_2m_2\rangle\colon j_i=0,\tfrac{1}{2},1,\ldots;\\ m_i=j_i,\, j_i-1,\ldots,-j_i;\, i=1,2.\end{aligned} \tag{4.2}$$

Alternatively, we may span the same space by the *coupled* vectors

$$|(j_1j_2)jm\rangle\equiv\sum_{m_1m_2} C^{j_1j_2j}_{m_1m_2m}|j_1m_1;\, j_2m_2\rangle, \tag{4.3}$$

where the coefficient denotes the Wigner coefficient of $SU(2)$. The action of $\mathcal{O}_{(U,V)}$ on these two types of basis vectors is given by the following expressions:

$$\mathcal{O}_{(U,V)}|j_1m_1;\ j_2m_2\rangle = \sum_{m_1'm_2'} D^{j_1}_{m_1'm_1}(U)D^{j_2}_{m_2'm_2}(V)|j_1m_1';\ j_2m_2'\rangle, \tag{4.4}$$

$$\mathcal{O}_{(U,V)}|(j_1j_2)jm\rangle = \sum_{j'm'} D^{(j_1j_2)}_{j'm';\,jm}(U,V)|(j_1j_2)j'm'\rangle, \tag{4.5}$$

where the correspondence

$$U \to D^j(U) \tag{4.6}$$

is the standard unitary matrix irreducible representation (irrep) of $SU(2)$. The relation of the elements of the unitary matrix irrep of $SU(2)\times SU(2)$,

$$(U,V) \to D^{(j_1j_2)}(U,V), \tag{4.7}$$

to the equivalent unitary matrix direct product irrep,

$$(U,V) \to D^{j_1}(U)\otimes D^{j_2}(V), \tag{4.8}$$

is given by

$$D^{(j_1j_2)}_{j'm';\,jm}(U,V) = \sum_{\substack{m_1'm_2' \\ m_1m_2}} C^{j_1j_2j'}_{m_1'm_2'm'}C^{j_1j_2j}_{m_1m_2m}D^{j_1}_{m_1'm_1}(U)D^{j_2}_{m_2'm_2}(V). \tag{4.9}$$

Observe that the *diagonal subgroup* representation satisfies the relation

$$D^{(j_1j_2)}_{j'm';\,jm}(U,U) = \delta_{j'j}\varepsilon_{j_1j_2j}D^j_{m'm}(U), \tag{4.10}$$

where

$$\varepsilon_{j_1j_2j} = \begin{cases} 1 & \text{if } j=|j_1-j_2|, |j_1-j_2|+1,\ldots, j_1+j_2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Using Eq. (4.9), the multiplication property of the $SU(2)$ representation functions given by Eq. (2.29), and the definition of Wigner's 9-j symbol [in the form given by Eq. (3.252) of AMQP], we now derive the fundamental result for the $SU(2)\times SU(2)$ representation matrix $D^{(j_1j_2)}(U,V)$:

$$D^{(j_1j_2)}_{j'm';\,jm}(U,V)D^{(k_1k_2)}_{k'\mu';\,k\mu}(U,V) = \sum_{l_1l_2}\sum_{l'\nu'l\nu} C\left[\begin{pmatrix} j_1j_2 \\ j'm' \end{pmatrix}\begin{pmatrix} k_1k_2 \\ k'\mu' \end{pmatrix}\begin{pmatrix} l_1l_2 \\ l'\ \nu' \end{pmatrix}\right]$$
$$\times C\left[\begin{pmatrix} j_1j_2 \\ j\ m \end{pmatrix}\begin{pmatrix} k_1k_2 \\ k\ \mu \end{pmatrix}\begin{pmatrix} l_1l_2 \\ l\ \nu \end{pmatrix}\right]D^{(l_1l_2)}_{l'\nu';\,l\nu}(U,V), \tag{4.12}$$

where the C-coefficient is defined by

$$C\left[\begin{pmatrix} j_1 j_2 \\ j\ m \end{pmatrix}\begin{pmatrix} k_1 k_2 \\ k\ \mu \end{pmatrix}\begin{pmatrix} l_1 l_2 \\ l\ \nu \end{pmatrix}\right] \equiv [(2l_1+1)(2l_2+1)(2j+1)(2k+1)]^{\frac{1}{2}}$$
$$\times \begin{Bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ l_1 & l_2 & l \end{Bmatrix} C^{jkl}_{m\mu\nu}. \tag{4.13}$$

The structure of Eq. (4.12) allows us to identify the coefficients (4.13) as the "Wigner coefficients" of the group $SU(2)\times SU(2)$ (coefficients of the real orthogonal matrix that reduces the direct product of two irreps into a direct sum of irreps).

We have introduced a notation for these $SU(2)\times SU(2)$ Wigner coefficients that is analogous to the $C^{j_1 j_2 j}_{m_1 m_2 m}$ notation for $SU(2)$ Wigner coefficients—superscripts denote irrep labels and subscripts denote subgroup labels. These coefficients satisfy orthogonality relations of the same type as $SU(2)$ coefficients:

Orthogonality of rows:

$$\sum_{jmk\mu} C\left[\begin{pmatrix} j_1 j_2 \\ j\ m \end{pmatrix}\begin{pmatrix} k_1 k_2 \\ k\ \mu \end{pmatrix}\begin{pmatrix} l_1 l_2 \\ l\ \nu \end{pmatrix}\right] C\left[\begin{pmatrix} j_1 j_2 \\ j\ m \end{pmatrix}\begin{pmatrix} k_1 k_2 \\ k\ \mu \end{pmatrix}\begin{pmatrix} l_1' l_2' \\ l'\nu' \end{pmatrix}\right] = \delta_{l_1 l_1'}\delta_{l_2 l_2'}\delta_{ll'}\delta_{\nu\nu'};$$
$$\tag{4.14}$$

Orthogonality of columns:

$$\sum_{l_1 l_2 l\nu} C\left[\begin{pmatrix} j_1 j_2 \\ j\ m \end{pmatrix}\begin{pmatrix} k_1 k_2 \\ k\ \mu \end{pmatrix}\begin{pmatrix} l_1 l_2 \\ l\ \nu \end{pmatrix}\right] C\left[\begin{pmatrix} j_1 j_2 \\ j'm' \end{pmatrix}\begin{pmatrix} k_1 k_2 \\ k'\mu' \end{pmatrix}\begin{pmatrix} l_1 l_2 \\ l\ \nu \end{pmatrix}\right] = \delta_{jj'}\delta_{mm'}\delta_{kk'}\delta_{\mu\mu'}.$$
$$\tag{4.15}$$

These orthogonality relations are proved directly from the definition, Eq. (4.13), and the orthogonality relations for the Wigner coefficients and 9-j coefficients [see Eqs. (3.175)–(3.179) and (3.321) of AMQP]. Similarly, the range of summation is determined from the right-hand side of Eq. (4.13)—standard triangle conditions and projection quantum number ranges are to be observed. Note that the C-coefficient (4.13) is zero unless the condition $\nu = m+\mu$ is satisfied. Correspondingly, the summations in Eqs. (4.14) and (4.15) may be simplified [cf. Eqs. (3.175)–(3.179) in AMQP].

One may now use the orthogonality relations (4.14) and (4.15) to give various equivalent forms of Eq. (4.12) in complete analogy to the results obtained in Chapter 3 of AMQP.

We now come to the definition of a tensor operator. An $SU(2)\times SU(2)$ tensor operator is a set of operators (each operator mapping $\mathcal{H}$ into itself) that is transformed linearly into itself under the similarity action of the operators $\mathcal{O}_{(U,V)}$. The coefficients of this linear transformation form a unitary matrix representation of $SU(2)\times SU(2)$. This representation may then be reduced into a direct sum of irreps.

Correspondingly, each such general tensor operator splits into a family of tensor operators, each tensor operator in the family being itself a set of operators that is transformed into itself under the similarity action of $\mathcal{O}_{(U,V)}$, the matrix of the transformation being now an irrep of $SU(2)\times SU(2)$. In this manner one arrives at the concept of an irreducible tensor operator in $SU(2)\times SU(2)$: An irreducible tensor operator in $SU(2)\times SU(2)$ is a set of operators denoted by

$$\mathbf{T}^{(J_1J_2)}, \tag{4.16}$$

where the labels (J_1J_2) designate that the set of operators $\mathbf{T}^{(J_1J_2)}$ is transformed linearly into itself according to the irrep $D^{(J_1J_2)}$ of $SU(2)\times SU(2)$.

It is customary to label the components of the irreducible tensor (4.16) by the same scheme as that used for the basis vectors of $\mathcal{H}$. Thus, corresponding to the two standard bases (4.2) and (4.3), the irreducible tensor (4.16) is represented by the components

$$\left\{ T^{J_1J_2}_{M_1M_2}: M_i = J_i, J_i-1, \ldots, -J_i \right\} \tag{4.17}$$

and

$$\left\{ {}^{(J_1J_2)}T^J_M : \begin{array}{l} J=|J_1-J_2|,\ldots,J_1+J_2 \\ M=J, J-1,\ldots,-J \end{array} \right\}, \tag{4.18}$$

respectively.

The relationship between these two ways of describing the components of one and the same irreducible tensor (4.16) is given by the orthogonal transformation

$${}^{(J_1J_2)}T^J_M = \sum_{M_1M_2} C^{J_1J_2J}_{M_1M_2M} T^{J_1J_2}_{M_1M_2}. \tag{4.19}$$

The transformation properties of the sets (4.17) and (4.18) are expressed by

$$\begin{aligned} \mathcal{O}_{(U,V)} T^{J_1J_2}_{M_1M_2} \mathcal{O}_{(U^{-1},V^{-1})} &= \sum_{M_1'M_2'} D^{J_1}_{M_1'M_1}(U) D^{J_2}_{M_2'M_2}(V) T^{J_1J_2}_{M_1'M_2'}, \\ \mathcal{O}_{(U,V)} {}^{(J_1J_2)}T^J_M \mathcal{O}_{(U^{-1},V^{-1})} &= \sum_{J'M'} D^{(J_1J_2)}_{J'M';JM}(U,V) {}^{(J_1J_2)}T^{J'}_{M'}. \end{aligned} \tag{4.20}$$

Consider next the specialization of the second of Eqs. (4.20) to the diagonal subgroup. The correspondence

$$U \to (U,U) \to \mathcal{O}_{(U,U)} \tag{4.21}$$

is a unitary operator realization of the diagonal subgroup $SU(2)\subset SU(2)\times SU(2)$. Furthermore, in consequence of relation (4.10), we have the result

$$\mathcal{O}_{(U,U)}{}^{(J_1J_2)}T^J_M \mathcal{O}_{(U^{-1},U^{-1})} = \sum_{M'} D^J_{M'M}(U)\,{}^{(J_1J_2)}T^J_{M'}, \tag{4.22}$$

which expresses the fact that the $SU(2)\times SU(2)$ irreducible tensor operator $\mathbf{T}^{(J_1J_2)}$ with components given by Eq. (4.18) splits into the $SU(2)$ irreducible tensor operators $\mathbf{T}^J$ for $J=|J_1-J_2|,\ldots,J_1+J_2$.

We consider next two sets of unit tensor operators: first, we consider the operators

$$\left\langle \begin{matrix} & J_1+\Delta_1 & \\ 2J_1 & & 0 \\ & J_1+M_1 & \end{matrix} \right\rangle$$

acting in a Hilbert space, $\mathcal{H}_1$, spanned by

$$\{|j_1m_1\rangle : j_1=0,\tfrac{1}{2},1,\ldots;\ m_1=j_1, j_1-1,\ldots,-j_1\};$$

second, we consider the operators

$$\left\langle \begin{matrix} & J_2+\Delta_2 & \\ 2J_2 & & 0 \\ & J_2+M_2 & \end{matrix} \right\rangle$$

acting in a Hilbert space, $\mathcal{H}_2$, spanned by

$$\{|j_2m_2\rangle : j_2=0,\tfrac{1}{2},1,\ldots;\ m_2=j_2, j_2-1,\ldots,-j_2\}.$$

(Thus, we have two copies of abstractly identical Hilbert spaces and Wigner operators.) By definition these operators act in the product space $\mathcal{H}_1\otimes\mathcal{H}_2\equiv\mathcal{H}$ according to the rule

$$(\mathcal{O}_1\otimes\mathcal{O}_2)(f_1\otimes f_2)=(\mathcal{O}_1f_1)\otimes(\mathcal{O}_2f_2)$$

for each $f_1\in\mathcal{H}_1$, $f_2\in\mathcal{H}_2$, and for each operator $\mathcal{O}_1:\mathcal{H}_1\to\mathcal{H}_1$ and each operator $\mathcal{O}_2: \mathcal{H}_2 \to \mathcal{H}_2$. In particular, one has

$$\left\langle \begin{matrix} & J_1+\Delta_1 & \\ 2J_1 & & 0 \\ & J_1+M_1 & \end{matrix} \right\rangle \otimes \left\langle \begin{matrix} & J_2+\Delta_2 & \\ 2J_2 & & 0 \\ & J_2+M_2 & \end{matrix} \right\rangle |j_1m_1\rangle \otimes |j_2m_2\rangle$$

$$= \left(\left\langle \begin{matrix} & J_1+\Delta_1 & \\ 2J_1 & & 0 \\ & J_1+M_1 & \end{matrix} \right\rangle |j_1m_1\rangle \right) \otimes \left(\left\langle \begin{matrix} & J_2+\Delta & \\ 2J_2 & & 0 \\ & J_2+M_2 & \end{matrix} \right\rangle |j_2m_2\rangle \right)$$

$$= C^{j_1J_1j_1+\Delta_1}_{m_1,M_1,m_1+M_1} C^{j_2J_2j_2+\Delta_2}_{m_2,M_2,m_2+M_2} |j_1+\Delta_1, m_1+M_1\rangle \otimes |j_2+\Delta_2, m_2+M_2\rangle. \tag{4.23}$$

If we now make the identification

$$T^{J_1J_2}_{M_1M_2} = \left\langle \begin{matrix} & J_1+\Delta_1 & \\ 2J_1 & & 0 \\ & J_1+M_1 & \end{matrix} \right\rangle \otimes \left\langle \begin{matrix} & J_2+\Delta_2 & \\ 2J_2 & & 0 \\ & J_2+M_2 & \end{matrix} \right\rangle, \tag{4.24}$$

then it is easily verified from the transformation property of unit Wigner operators that $\{T^{J_1J_2}_{M_1M_2}: M_i = J_i, J_i-1, \ldots, -J_i\}$ is an $SU(2)\times SU(2)$ tensor operator. Thus, the operators defined by

$$T^{J_1J_2J}_{\Delta_1\Delta_2M} = \sum_{M_1M_2} C^{J_1J_2J}_{M_1M_2M} \left\langle \begin{matrix} & J_1+\Delta_1 & \\ 2J_1 & & 0 \\ & J_1+M_1 & \end{matrix} \right\rangle \otimes \left\langle \begin{matrix} & J_2+\Delta_2 & \\ 2J_2 & & 0 \\ & J_2+M_2 & \end{matrix} \right\rangle \tag{4.25}$$

for $J=|J_1-J_2|, \ldots, J_1+J_2$; $M=J, \ldots, -J$, constitute the components of an $SU(2)\times SU(2)$ irreducible tensor operator that has been split into an irreducible tensor (the components are $M=J, J-1, \ldots, -J$) with respect to the diagonal subgroup $SU(2)\subset SU(2)\times SU(2)$.

Applying the Wigner–Eckart theorem [where the relevant $SU(2)$ group is the diagonal subgroup of $SU(2)\times SU(2)$] to the tensor operator (4.25), we obtain

$$\langle (j_1+\Delta_1, j_2+\Delta_2)j'm'|T^{J_1J_2J}_{\Delta_1\Delta_2M}|(j_1j_2)jm\rangle$$

$$= \delta_{j',j+\Delta}\delta_{m',m+M}\langle j_1+\Delta_1, j_2+\Delta_2, j+\Delta | \begin{bmatrix} J_1J_2J \\ \Delta_1\Delta_2\Delta \end{bmatrix} | j_1j_2j\rangle$$

$$\times \langle j+\Delta, m+M| \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle |jm\rangle, \tag{4.26}$$

where

$$\langle j_1+\Delta_1, j_2+\Delta_2, j+\Delta \| \begin{bmatrix} J_1 J_2 J \\ \Delta_1 \Delta_2 \Delta \end{bmatrix} \| j_1 j_2 j \rangle$$
$$\equiv \langle j_1+\Delta_1, j_2+\Delta_2, j+\Delta \| \mathbf{T}^{J_1 J_2 J}_{\Delta_1 \Delta_2} \| j_1 j_2 j \rangle \tag{4.27}$$

denotes the *reduced matrix element* of $T^{J_1 J_2 J}_{\Delta_1 \Delta_2 M}$. Notice that we have observed the Condon–Shortley rule of shifting j to $j+\Delta$ in the definition (4.27) (see p. 97 of AMQP for a discussion of this rule).

Using the definition (4.25) of $T^{J_1 J_2 J}_{\Delta_1 \Delta_2 M}$ and the definition

$$\langle j+\Delta, m+M | \left\langle 2J \begin{matrix} J+\Delta \\ \\ J+M \end{matrix} \; 0 \right\rangle | jm \rangle = C^{jJj+\Delta}_{m,M,m+M} \tag{4.28}$$

together with the orthogonality of the Wigner coefficients, we may now evaluate the reduced matrix element (4.27). The result is

$$\langle j_1+\Delta_1, j_2+\Delta_2, j+\Delta \| \begin{bmatrix} J_1 J_2 J \\ \Delta_1 \Delta_2 \Delta \end{bmatrix} \| j_1 j_2 j \rangle$$
$$= \sum_{\substack{M_1 M_2 \\ m_1 m_2 \\ mM}} C^{j_1 j_2 j}_{m_1 m_2 m} C^{J_1 J_2 J}_{M_1 M_2 M} C^{j_1+\Delta_1 j_2+\Delta_2 j+\Delta}_{m_1+M_1, m_2+M_2, m'} C^{j_1 J_1 j_1+\Delta_1}_{m_1, M_1, m_1+M_1} C^{j_2 J_2 j_2+\Delta_2}_{m_2, M_2, m_2+M_2} C^{jJj+\Delta}_{mMm'}$$
$$= [(2j+1)(2J+1)(2j_1+2\Delta_1+1)(2j_2+2\Delta_2+1)]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & j \\ J_1 & J_2 & J \\ j_1+\Delta_1 & j_2+\Delta_2 & j+\Delta \end{Bmatrix}. \tag{4.29}$$

We next combine Eqs. (4.26), (4.28), (4.29), and (4.13) to obtain the following basic result: *The irreducible tensor operator $T^{J_1 J_2 J}_{\Delta_1 \Delta_2 M}$ is an $SU(2)\times SU(2)$ Wigner operator; that is, it has the action on the Hilbert space $\mathcal{H}=\mathcal{H}_1 \otimes \mathcal{H}_2$ given by*

$$T^{J_1 J_2 J}_{\Delta_1 \Delta_2 M} |(j_1 j_2) jm\rangle = \sum_{\Delta=-J}^{J} C\left[\begin{pmatrix} j_1 j_2 \\ j\; m \end{pmatrix} \begin{pmatrix} J_1 J_2 \\ J\; M \end{pmatrix} \begin{pmatrix} j_1+\Delta_1\; j_2+\Delta_2 \\ j+\Delta, m+M \end{pmatrix} \right]$$
$$\times |(j_1+\Delta_1, j_2+\Delta_2) j+\Delta, m+M\rangle, \tag{4.30}$$

where the C-coefficient is an $SU(2)\times SU(2)$ Wigner coefficient.

Let us summarize. Since the direct product of SR groups is itself an SR group (see Chapter 3, Section 4), the developments obtained in this section can be considered as a particular example of the standard procedure for determining unit tensor operators in an SR group [here $SU(2)\times SU(2)$]. One could now proceed to develop the algebra generated by the fundamental operators, following the procedure developed in Chapter 3. We shall defer this development to Notes 1 and 2, since it is peripheral to our main purpose in the present chapter.

From the point of view of physics, there is a special reason for interest in the $SU(2)\times SU(2)$ group; this group has the same Lie algebra as the $SO(4)$ [or $R(4)$] group, which, in turn, is the underlying symmetry group of hydrogen-like atoms. This subject is discussed in Chapter 7, Section 4, of AMQP. (Racah algebra for the group $R(4)$ has been discussed by Ponzano [4].)

3. Construction of the W-Algebra

Invariant operators in the diagonal subgroup. Let us recall some results found in the previous section. We are dealing with a particular Hilbert space whose basis is the set of vectors

$$\left\{|(j_1j_2)jm\rangle : j_1, j_2=0,\tfrac{1}{2},1,\ldots;\ j_1+j_2\geqslant j\geqslant |j_1-j_2|;\ j\geqslant m\geqslant -j\right\}.$$

The unit tensor operators on this space whose actions leave invariant the carrier space of the irreps of the diagonal subgroup are just the invariant $SU(2)\times SU(2)$ Wigner operators, as we see from Eq. (4.30). These invariant Wigner operators are

$$\begin{aligned}T^{JJ0}_{\Delta_1\Delta_2 0}|(j_1j_2)jm\rangle &= [(2j+1)(2j_1+2\Delta_1+1)(2j_2+2\Delta_2+1)]^{\frac{1}{2}}\\ &\quad\times\begin{Bmatrix} j_1 & j_2 & j\\ J & J & 0\\ j_1+\Delta_1 & j_2+\Delta_2 & j\end{Bmatrix}|(j_1+\Delta_1,\, j_2+\Delta_2)jm\rangle\\ &=\left[\frac{(2j_1+2\Delta_1+1)(2j_2+2\Delta_2+1)}{(2J+1)}\right]^{\frac{1}{2}}\\ &\quad\times W(j,j_1,j_2+\Delta_2,J;\,j_2,j_1+\Delta_1)|(j_1+\Delta_1,\,j_2+\Delta_2)jm\rangle,\end{aligned} \tag{4.31}$$

where in obtaining this result we have used the identity

$$\begin{Bmatrix} a & b & c\\ d & d & 0\\ e & f & c\end{Bmatrix}=\frac{(-1)^{a+c+d+f}\begin{Bmatrix} c & a & b\\ d & f & e\end{Bmatrix}}{[(2c+1)(2d+1)]^{\frac{1}{2}}}=\frac{W(cafd;\,be)}{[(2c+1)(2d+1)]^{\frac{1}{2}}}. \tag{4.32}$$

Equation (4.31) is the basic relation that we need to introduce the notion of a *unit Racah operator*. We find, however, that, by modifying this relation slightly, the resulting unit Racah operators will satisfy relationships that are in complete analogy to those of $SU(2)$ unit tensor operators (Wigner operators) as developed in Chapters 2 and 3. [Were we to use Eq. (4.31) directly, unwanted phase factors and dimension factors would appear.] The appropriate definition of a unit Racah operator in terms of the invariant $SU(2)\times SU(2)$ Wigner operator given by Eq. (4.31) is

$$\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}=(2J+1)^{\frac{1}{2}}(1\otimes dim^{-\frac{1}{2}})T^{JJ0}_{\Delta\Delta' 0}(1\otimes dim^{\frac{1}{2}})$$

$$=(-1)^{J+\Delta'}\sum_{M}\left\langle\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix}\right\rangle\otimes\left\langle\begin{matrix} & J-\Delta' & \\ 2J & & 0 \\ & J+M & \end{matrix}\right\rangle^{\dagger}. \tag{4.33}$$

In obtaining this result, we have used Eq. (4.25), the special Wigner coefficient $C^{JJ0}_{M,-M,0}=(-1)^{J-M}/(2J+1)^{\frac{1}{2}}$, and Eq. (3.64) for the conjugate Wigner operator. The form of right-hand side of Eq. (4.33) shows clearly the $SU(2)\times SU(2)$ transformation properties of a unit Racah operator, and, in particular, that it is an invariant under the transformations $\mathcal{O}_{(U,U)}$ corresponding to the diagonal subgroup (see Chapter 3, Section 20, AMQP).

The following two orthogonality relations for unit Racah operators are an immediate consequence of the orthogonality relations (2.33) and (2.35) for Wigner operators:

$$\sum_{\Delta'}\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}\left\{\begin{matrix} & J+\Delta'' & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}^{\dagger}=\delta_{\Delta\Delta''}\mathbf{I}^{J}_{-\Delta}\otimes 1,$$

$$\sum_{\Delta}\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}^{\dagger}\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta'' & \end{matrix}\right\}=\delta_{\Delta'\Delta''}1\otimes\mathbf{I}^{J}_{\Delta'}. \tag{4.34}$$

[It was to obtain these two relations as analogs to Eqs. (2.33) and (2.35) that the phase and dimension factors were introduced into the definition (4.33).]

Combining Eqs. (4.33) and (4.31), we find that a unit Racah operator has the following shift action on an arbitrary basis vector $|(j_1j_2)jm\rangle$ of the tensor product space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$:

$$\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}|(j_1j_2)jm\rangle=[(2j_1+2\Delta+1)(2j_2+1)]^{\frac{1}{2}}$$
$$\times W(j,j_1,j_2+\Delta',J;j_2,j_1+\Delta)|(j_1+\Delta,j_2+\Delta')jm\rangle. \tag{4.35}$$

Observe that in the generic unit Racah operator

$$\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\} \tag{4.36}$$

$2J$ may assume all integer values $0, 1, 2, \ldots,$ whereas for each specified value of $2J$, the values that Δ and Δ' may assume are

$$\Delta, \Delta' = J, J-1, \ldots, -J. \tag{4.37}$$

It is important to note that, although the notation for unit Racah operators is, by design, in one-to-one correspondence with that of unit $SU(2)$ Wigner operators, *neither the upper pattern* $\begin{pmatrix} & J+\Delta & \\ 2J & & 0 \end{pmatrix}$ *nor the lower pattern* $\begin{pmatrix} 2J & & 0 \\ & J+\Delta' & \end{pmatrix}$ *has a subgroup significance*, and each will be called an *operator pattern*. [An extension of a unit Racah operator to "$U(2)$ patterns" is considered in Note 3.]

Unit Racah operators are clearly bounded operators on $\mathcal{H}$, since they are defined by the elements of an orthogonal matrix. Moreover, since the property of being an invariant operator is preserved under sums and products, we may consider the subalgebra (of the Banach star-algebra) generated by the set of all unit Racah operators. This algebra we term "W-algebra."

It is essential to note that in this construction *the quantum number m plays no role*; accordingly, we may replace the individual vectors labeled by different m-values by *an equivalence class of vectors* whose representative, say, is the vector having $m=j$.

Let us now adapt the notation to this situation. The basis vectors of the new Hilbert space (equivalence classes of vectors) will be labeled as the set of ket vectors

$$\{|j_1 j_2 j\rangle \text{ with } j_1, j_2 = 0, \tfrac{1}{2}, \ldots;\ j_1 + j_2 \leqslant j \leqslant |j_1 - j_2|\}. \tag{4.38}$$

The operators we consider in this space will be the operators given in Eq. (4.35). The action of the operators is to produce a new basis vector multiplied by a Racah coefficient as the relevant matrix element (aside from normalizing factors).

Since the basis vectors of the new Hilbert space are specified by three quantum numbers, j_1, j_2, and j, which are constrained to form a triangle of angular momenta, we see that we have an *algebra of invariant operators that act on discrete (integral) triangles in the literal geometric sense.*

A calculus of extended patterns. In order to make the algebra completely explicit, we shall now demonstrate that there exists a pattern calculus for this structure, realized by precisely the same pattern calculus rules as those employed in Chapter 3. The sole difference is that the Gel'fand pattern $(m)=\begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix}$ which occurs in the ket $|(m)\rangle$ in the earlier case, is replaced by an *extended pattern*,

$$\begin{pmatrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{pmatrix}, \tag{4.39}$$

in which *the betweenness conditions are considered operative*; that is, to be explicit, these conditions are

$$m_{12} \geqslant m_{11} \geqslant m_{22} \geqslant m_{21}, \tag{4.40}$$

with all (m_{ij}) being integers.

Before proceeding further, it is useful to verify that these extended patterns do indeed include the vectors $|j_1 j_2 j\rangle$ of the previous section. The vectors map in this way:

$$\left|\begin{matrix} 2j_1 & & 0 & \\ & j_1+j_2-j & & j_1-j_2-j \end{matrix}\right\rangle = |j_1 j_2 j\rangle. \tag{4.41}$$

The m_{ij} in this extended pattern are $m_{11}=2j_1$, $m_{22}=0$, $m_{11}=j_1+j_2-j$, $m_{21}=j_1-j_2-j$, and these are clearly integral for all integral and half-integral j_1, j_2, j that satisfy the triangle condition $j=j_1+j_2, j_1+j_2-1, \ldots, |j_1-j_2|$. Conversely, the betweenness conditions imply

$$2j_1 \geqslant j_1+j_2-j \geqslant 0 \geqslant j_1-j_2-j, \tag{4.42}$$

which, in turn, imply $j=j_1+j_2, j_1+j_2-1, \ldots, |j_1-j_2|$; this recovers the triangle condition for j_1, j_2, j.

The relationship between a general pattern (4.39) and the special pattern[1]

$$\begin{pmatrix} 2j_1 & & 0 & \\ & j_1+j_2-j & & j_1-j_2-j \end{pmatrix} \tag{4.43}$$

is obtained [in analogy to $SU(2)$ versus $U(2)$ patterns—see the Note at the

[1] There is a feature in pattern (4.43) that deserves comment: the specialization of the label j_1 versus j_2. The betweenness conditions, Eq. (4.42), are, in fact, invariant to permutations of j_1, j_2, j. The specific form chosen in Eq. (4.41) is adapted to the notation (4.36) for a unit Racah operator in which the upper pattern specifies the shift in j_1 in the upper row of the basis vector labels, and the lower pattern specifies the shift in j_2 in the lower row of basis vector labels.

end of Chapter 3] by considering all patterns in the set

$$\left\{\begin{pmatrix} m_{12}+k & & m_{22}+k & \\ & m_{11}+k & & m_{21}+k \end{pmatrix} : k=\ldots,-1,0,1,\ldots\right\} \tag{4.44}$$

to be equivalent. This equivalence then determines (choose $k=-m_{22}$) the relations between j_1, j_2, j and the (m_{ij}):

$$\begin{aligned} j_1&=(m_{12}-m_{22})/2,\\ j_2&=(m_{11}-m_{21})/2,\\ j&=(m_{12}+m_{22}-m_{11}-m_{21})/2. \end{aligned} \tag{4.45}$$

We now apply this equivalence relation to the initial and final extended patterns in the following matrix element:

$$\begin{aligned} &\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta_1' & & m_{21}+\Delta_2' \end{matrix} \right| \left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\} \left| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle \\ &= \left\langle \begin{matrix} 2(j_1+\Delta) & & 0 & \\ & j_1+j_2-j+\Delta+\Delta' & & j_1-j_2-j+\Delta-\Delta' \end{matrix} \right| \\ &\qquad \times \left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\} \left| \begin{matrix} 2j_1 & & 0 & \\ & j_1+j_2-j & & j_1-j_2-j \end{matrix} \right\rangle \\ &= \langle j_1+\Delta, j_2+\Delta', j| \left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\} | j_1 j_2 j\rangle \\ &= \left[(2j_1+2\Delta+1)(2j_2+1)\right]^{\frac{1}{2}} W(j, j_1, j_2+\Delta', J; j_2, j_1+\Delta), \end{aligned} \tag{4.46}$$

where we have defined the shift labels Δ_1, Δ_2, Δ_1', Δ_2' by

$$\begin{aligned} &\text{Upper pattern: } \Delta_1=J+\Delta, \quad \Delta_2=J-\Delta,\\ &\text{Lower pattern: } \Delta_1'=J+\Delta', \quad \Delta_2'=J-\Delta'. \end{aligned} \tag{4.47}$$

It is in the sense of this equivalence relation, Eq. (4.44), on patterns that we interpret the following expression for the operator form of the unit Racah operator:

$$\left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\} \left| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle = \# \left| \begin{matrix} m_{12}+J+\Delta & & m_{22}+J-\Delta & \\ & m_{11}+J+\Delta' & & m_{21}+J-\Delta' \end{matrix} \right\rangle, \tag{4.48}$$

where the numerical value, #, is the Racah coefficient defined by

$$\# = [(m_{12}-m_{22}+2\Delta+1)(m_{11}-m_{21}+1)]^{\frac{1}{2}}$$
$$\times W\left(\frac{m_{12}+m_{22}-m_{11}-m_{21}}{2}, \frac{m_{12}-m_{22}}{2}, \frac{m_{11}-m_{21}}{2}+\Delta', J; \frac{m_{11}-m_{21}}{2}, \frac{m_{12}-m_{22}}{2}+\Delta\right). \tag{4.49}$$

Since this result is very complicated in appearance, let us discuss it further in order to bring out the essential simplicity, and even naturalness, concealed under the thicket of indices. The assertion of Eq. (4.48) is that there exists an operator—denoted $\{2J\ \ 0\}$ and carrying the angular momentum label J—which acts on a state vector $|(m)\rangle$, labeled by a triangle whose three sides are $\frac{1}{2}(m_{12}-m_{22}), \frac{1}{2}(m_{11}-m_{21}), \frac{1}{2}(m_{12}+m_{22}-m_{11}-m_{21})$; the action of the operator produces a new vector whose triangle has its sides *shifted* by the lengths $(\Delta, \Delta', 0)$. This is, of course, just the action already found for the unit Racah operators in $SU(2)\times SU(2)$, but the interesting point at issue *is the way in which the extended pattern structure encodes this information in an optimal way.*

To give content to the use of extended patterns, let us now verify that the pattern calculus rules yield precisely the correct values for the four fundamental Racah operators.

Consider first the operator denoted by $\left\{\begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix}\right\}$. The shifts induced by this operator when acting on an extended pattern are

$$\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\} : \begin{pmatrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{pmatrix} \equiv (m) \to (m) + \begin{bmatrix} 1 & & 0 & \\ & 1 & & 0 \end{bmatrix},$$

where the *shift pattern*

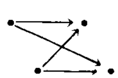
$$[\Delta] \equiv \begin{bmatrix} \Delta_1 & & \Delta_2 & \\ & \Delta_1' & & \Delta_2' \end{bmatrix} = \begin{bmatrix} J+\Delta & & J-\Delta & \\ & J+\Delta' & & J-\Delta' \end{bmatrix} \tag{4.50}$$

for this operator is

$$\begin{bmatrix} 1 & & 0 & \\ & 1 & & 0 \end{bmatrix}.$$

We next construct the arrow pattern in exactly the same way as was discussed in Chapter 3:

Shift pattern Arrow pattern

$$\begin{bmatrix} 1 & & 0 & \\ & 1 & & 0 \end{bmatrix} \to$$ 

We then assign to the dots in this arrow pattern the values of the partial hooks $p_{ij} \equiv m_{ij} + j - i$. The values of these labels (which denote the underlying Hilbert space vector) are given by

$$
\begin{array}{cccc}
p_{12} = m_{12} + 1 & & p_{22} = m_{22} & \\
\bullet & & \bullet & \\
& \bullet & & \bullet \\
& p_{11} = m_{11} & & p_{21} = m_{21} - 1
\end{array}
\tag{4.51}
$$

(Note in particular the -1 in p_{21}.) Thus, we obtain a labeled arrow pattern

$$
\begin{tikzcd}
p_{12} \arrow[r] \arrow[dr] & p_{22} \\
p_{11} \arrow[r] \arrow[ur] & p_{21}
\end{tikzcd}
\tag{4.52}
$$

In this manner, we associate a labeled arrow pattern to each *fundamental Racah operator* (the operators $\{1\,0\}$).

The pattern calculus rules now define an algebraic factor, denoted PCF (pattern calculus factor), for each labeled arrow pattern in the following manner (just as in Chapter 3): To each arrow in the labeled arrow pattern, we assign the factor

$$
p_{\text{tail}} - p_{\text{head}} + e_{\text{tail}}, \qquad e_{\text{tail}} = \begin{cases} 1 & \text{if tail in bottom row,} \\ 0 & \text{if tail in top row.} \end{cases} \tag{4.53}
$$

Then

$$
\mathrm{PCF} = \left[\left| \frac{\text{product of all factors for arrows going between rows}}{\text{product of all factors for arrows going within rows}} \right| \right]^{\frac{1}{2}}. \tag{4.54}
$$

The PCF assigned to the pattern (4.52) is thus found to be

$$
\mathrm{PCF} = \left[\frac{(p_{11} - p_{22} + 1)(p_{12} - p_{21})}{(p_{12} - p_{22})(p_{11} - p_{21} + 1)} \right]^{\frac{1}{2}}.
$$

Using the relation of partial hooks to the m_{ij} given by (4.51) and the map to angular momentum labels j_1, j_2, j given by Eqs. (4.45), we obtain the result

$$
\mathrm{PCF} = \left[\frac{(j_1 + j_2 + j + 2)(j_1 + j_2 + 1 - j)}{(2j_1 + 1)(2j_2 + 2)} \right]^{\frac{1}{2}},
$$

which is precisely the value of the Racah coefficient:

$$\mathrm{PCF}=\left[(2j_1+2)(2j_2+1)\right]^{\frac{1}{2}}W(j,j_1,j_2+\tfrac{1}{2},\tfrac{1}{2};j_2,j_1+\tfrac{1}{2}).$$

We conclude: *The pattern calculus rules have evaluated exactly the matrix element of the fundamental Racah operator* $\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\}$:

$$\mathrm{PCF}=\langle j_1+\tfrac{1}{2},j_2+\tfrac{1}{2},j|\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\}|j_1j_2j\rangle=\langle(m)+[\Delta]|\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\}|(m)\rangle.$$

For completeness we tabulate the results obtained from the pattern calculus for the four fundamental operators $\{10\}$ in Table 4.1, where we note that the sign of the matrix element of the fundamental Racah operator

$$\left\{\begin{matrix} & \rho & \\ 1 & & 0 \\ & \sigma & \end{matrix}\right\}, \qquad \rho,\sigma=0,1 \tag{4.55}$$

is given by (see p. 52)

$$\mathrm{sign}(\rho-\sigma). \tag{4.56}$$

In all four cases the pattern calculus rules evaluate exactly the matrix elements of the corresponding operator as defined by Eq. (4.35) *or* (4.48).

Null space of a unit Racah operator. To determine the characteristic null space of a unit Racah operator, we use the defining relation, Eq. (4.33). We see from this equation that *both* a Wigner operator *and* a conjugate Wigner operator enter into the $\otimes$-product defining the Racah operator. The characteristic null space of each of these operators is known from Eqs. (2.42) and (3.64). It follows that the desired characteristic null space is given by

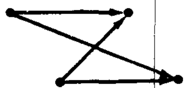
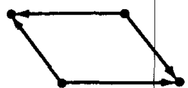
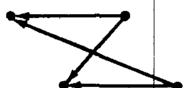
$$\text{Characteristic null space of }\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}=\sum_{j_1j_2}{}' \oplus\mathcal{N}^J_{\Delta\Delta'}(j_1j_2), \tag{4.57}$$

where the space $\mathcal{N}^J_{\Delta\Delta'}(j_1j_2)$ is defined to be

$$\mathcal{N}^J_{\Delta\Delta'}(j_1j_2)\equiv\left[\mathcal{N}_{(J-\Delta-1)/2}\otimes\mathcal{H}_{j_2}\right]\oplus\left[\mathcal{H}_{j_1}\otimes\mathcal{N}_{(J-\Delta'-1)/2}\right]. \tag{4.58}$$

The summation symbol $\sum_{j_1j_2}'$ denotes that the subspace $\mathcal{N}_{(J-\Delta-1)/2}\otimes\mathcal{N}_{(J-\Delta'-1)/2}$ is to be included only once in carrying out the summation $j_1,j_2=0,\tfrac{1}{2},1,\ldots$.

Table 4.1. Matrix Elements of the Fundamental Racah Operators $\{1\ 0\}$

Operator	Arrow Pattern	Matrix Element	Conventional Notation
$\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\}$		$\left[\frac{(p_{11}+1-p_{22})(p_{12}-p_{21})}{(p_{12}-p_{22})(p_{11}+1-p_{21})}\right]^{\frac{1}{2}}$	$[(2j_1+2)(2j_2+1)]^{\frac{1}{2}}W(j\ j_1\ j_2+\frac{1}{2}\ \frac{1}{2};j_2\ j_1+\frac{1}{2})$
$\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix}\right\}$		$\left[\frac{(p_{21}+1-p_{22})(p_{12}-p_{11})}{(p_{12}-p_{22})(p_{21}+1-p_{11})}\right]^{\frac{1}{2}}$	$[(2j_1+2)(2j_2+1)]^{\frac{1}{2}}W(j\ j_1\ j_2-\frac{1}{2}\ \frac{1}{2};j_2 j_1+\frac{1}{2})$
$\left\{\begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\}$		$-\left[\frac{(p_{11}+1-p_{12})(p_{22}-p_{21})}{(p_{22}-p_{12})(p_{11}+1-p_{21})}\right]^{\frac{1}{2}}$	$[(2j_1)(2j_2+1)]^{\frac{1}{2}}W(j\ j_1\ j_2+\frac{1}{2}\ \frac{1}{2};j_2\ j_1-\frac{1}{2})$
$\left\{\begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix}\right\}$		$\left[\frac{(p_{21}-p_{12}+1)(p_{22}-p_{11})}{(p_{22}-p_{12})(p_{21}+1-p_{11})}\right]^{\frac{1}{2}}$	$[(2j_1)(2j_2+1)]^{\frac{1}{2}}W(j\ j_1\ j_2-\frac{1}{2}\ \frac{1}{2};j_2\ j_1-\frac{1}{2})$

Notation

$p_{ij} \equiv m_{ij} + j - i$

$j_1 = (m_{12}-m_{22})/2 = (p_{12}-1-p_{22})/2$

$j_2 = (m_{11}-m_{21})/2 = (p_{11}-p_{21}-1)/2$

The null space of a unit Racah operator is, however, *larger* than the characteristic null space given in Eq. (4.57). The unit Racah operator shifts the labels of the initial vector $|(m)\rangle$ to produce a final vector $|(m)+[\Delta]\rangle$. If the latter pattern

$$(m)+[\Delta]=\begin{pmatrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta_1' & & m_{21}+\Delta_2' \end{pmatrix}$$

is nonlexical, then the initial vector $|(m)\rangle$ must belong to the null space of the operator. Basis vectors belonging to the null space because of lexicality constraints, although possibly infinite in number, can be systematically, and easily, treated by means of the pattern calculus. (This will be developed in Section 4.) For this reason (and, moreover, because such vectors are not of invariant importance), we shall call the set of vectors belonging to the null space because of lexicality constraints the "trivial null space." Zeros of the operator associated with this subset of null space are called "trivial zeros." Vectors in the trivial null space may, or may not, belong to the characteristic null space.

We shall show in Section 4 that the unit Racah operator may be constructed from knowledge of the trivial and characteristic null space (together with symmetry information). This construction is quite analogous to the structural theorem stated in Chapter 3 for Wigner operators.

The lexicality constraints (betweenness conditions) on the initial and final patterns

$$\begin{pmatrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{pmatrix}$$

and

$$\begin{pmatrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta_1' & & m_{21}+\Delta_2' \end{pmatrix}$$

are equivalent to two of the four triangle conditions on the Racah coefficient. Expressed in terms of the (m_{ij}), the triangle determined by the initial vector is

$$\left(\frac{m_{12}+m_{22}-m_{11}-m_{21}}{2}, \frac{m_{12}-m_{22}}{2}, \frac{m_{11}-m_{21}}{2}\right). \qquad (4.59)$$

Similarly, the final pattern determines the triangle

$$\left(\frac{m_{12}+m_{22}-m_{11}-m_{21}}{2}, \frac{m_{12}-m_{22}}{2}+\Delta, \frac{m_{11}-m_{21}}{2}+\Delta'\right). \qquad (4.60)$$

The remaining two triangles associated with the Racah coefficient are determined by the action of the Racah operator:

$$\begin{Bmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{Bmatrix} : \begin{array}{l} j_1=\frac{1}{2}(m_{12}-m_{22})\rightarrow j_1'=\frac{1}{2}(m_{12}-m_{22})+\Delta, \\ j_2=\frac{1}{2}(m_{11}-m_{21})\rightarrow j_2'=\frac{1}{2}(m_{11}-m_{21})+\Delta' \end{array} .$$

Thus, we obtain the two triangles

$$\left(\frac{m_{12}-m_{22}}{2}, J, \frac{m_{12}-m_{22}}{2}+\Delta\right), \tag{4.61}$$

$$\left(\frac{m_{11}-m_{21}}{2}, J, \frac{m_{11}-m_{21}}{2}+\Delta'\right). \tag{4.62}$$

The triangle condition applied to the triple given in (4.61) implies that $m_{12}-m_{22} \geqslant J-\Delta$. Similarly, for the triple (4.62), we find that $m_{11}-m_{21} \geqslant J-\Delta'$. These two conditions are precisely the requirement that the initial vector $|(m)\rangle$ *not* belong to the characteristic null space of the unit Racah operator.

Since the initial basis vectors necessarily have a lexical pattern, the constraints expressed by the triangle condition on the triple (4.59) are not operative. The trivial zeros accordingly all come from the triangle condition on the triple (4.60) or, equivalently, from lexicality constraints on the labels of the final vector.

It is known that there are additional zeros of the Racah coefficient for which all triangle constraints are satisfied; that is, the associated basis vectors do not belong to either the trivial or characteristic null space.[1] It is the merit of the pattern calculus approach that one can understand why these otherwise mysterious "accidental" zeros are not of any structural importance. We discuss this in Remark (a), p. 122, in the context of the ~~structural theorem developed there.~~

The accidental zeros are of intrinsic importance both mathematically and physically; this is discussed in Topic 10 of Chapter 5.

Let us summarize: The null space of the Racah operator

$$\begin{Bmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{Bmatrix} \tag{4.63}$$

consists of three sets of vectors: (a) the trivial null space (based on lexicality constraints); (b) the characteristic null space [given by Eq. (4.57)]; and (c)

[1]Unlike a Wigner coefficient, there are no symmetry vanishings of a Racah coefficient. This is most readily seen from the fact that there are no sign changes in the 6-j symbol under any of the 144 symmetries (see Chapter 3, Section 18, AMQP).

the accidental null space (Topic 10, Chapter 5). The two spaces described in (a) and (b) are *not* necessarily disjoint, whereas the space described in (c) *is* disjoint from that of (a) and (b). The dimension of the trivial null space is either 1 (the zero vector) or denumerably infinite, and this same property holds for the characteristic null space. It is not known whether the accidental null space is finite or not. (Over 1400 of these accidental zeros have been tabulated in Ref. [5], as we discuss in Topic 10, Chapter 5.)

Basis property of the unit Racah operators. The set of $SU(2)\times SU(2)$ Wigner operators [see Eq. (4.30)]

$$T^{J_1 J_2 J}_{\Delta_1\Delta_2 M}:\ \Delta_i = J_i, \ldots, -J_i, \qquad i=1,2, \tag{4.64}$$

constitute a basis for all $SU(2)\times SU(2)$ Wigner operators of the type

$$^{(J_1J_2)}T^J_M = \left[\mathbf{T}^{J_1}(1)\times\mathbf{T}^{J_2}(2)\right]^J_M. \tag{4.65}$$

(See pp. 99–102, AMQP.) Correspondingly, the set of Racah operators

$$\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}: \qquad \Delta, \Delta' = J, \ldots, -J, \tag{4.66}$$

constitute a basis for the $SU(2)\times SU(2)$ Wigner operators that are *invariant* with respect to the diagonal $SU(2)$ subgroup. They are a basis in the sense that one may write

$$\left[\mathbf{T}^J(1)\times\mathbf{T}^J(2)\right]^0_0 = \sum_{\Delta'\Delta} \mathbf{C}^J_{\Delta'\Delta} \left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}, \tag{4.67}$$

where $\mathbf{C}^J_{\Delta'\Delta}$ denotes an $SU(2)\times SU(2)$ invariant operator. Its eigenvalues may be found by taking matrix elements $\langle(\alpha')j_1+\Delta\, j_2+\Delta' j|\cdots|(\alpha)j_1 j_2 j\rangle$, using Eq. (4.46), and comparing the result with Eq. (3.260) of AMQP:

$$C^J_{\Delta'\Delta}(j_1+\Delta, j_2+\Delta') = \left[\frac{(2j_2+2\Delta'+1)}{(2j_2+1)(2J+1)}\right]^{\frac{1}{2}} \langle(\alpha_1')j_1+\Delta\|\mathbf{T}^J(1)\|(\alpha_1)j_1\rangle$$
$$\times\langle(\alpha_2')j_2+\Delta'\|\mathbf{T}^J(2)\|(\alpha_2)j_2\rangle. \tag{4.68}$$

Again using the definition (4.33), expressing unit Racah operators in terms of $SU(2)$ Wigner operators, we also derive the trace orthogonality relation for unit Racah operators directly from that for unit Wigner operators [see Eqs. (2.44) and (3.64)]:

$$\sum_j \langle j_1 j_2 j|(2j+1)\left\{\begin{matrix} & J'+\Delta'' & \\ 2J' & & 0 \\ & J'+\Delta''' & \end{matrix}\right\}\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}^{\dagger}|j_1 j_2 j\rangle$$

$$= \sum_{m_1 m_2} \langle j_1 m_1;\, j_2 m_2|\left\{\begin{matrix} & J'+\Delta'' & \\ 2J' & & 0 \\ & J'+\Delta''' & \end{matrix}\right\}\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}^{\dagger}|j_1 m_1;\, j_2 m_2\rangle$$

$$= \frac{(2j_1+1)(2j_2-2\Delta'+1)}{(2J+1)}\delta_{J'J}\delta_{\Delta'''\Delta'}\delta_{\Delta''\Delta}\varepsilon_{j_1,J,j_1-\Delta}\varepsilon_{j_2,J,j_2-\Delta'}. \tag{4.69}$$

Using this orthogonality relation, one can express the eigenvalues $C^J_{\Delta'\Delta}(j_1, j_2)$ of the invariants $\mathbf{C}^J_{\Delta'\Delta}$ on a state vector $|j_1 j_2 j\rangle$ in the form

$$C^J_{\Delta'\Delta}(j_1, j_2) = \frac{2J+1}{(2j_1+1)(2j_2-2\Delta'+1)}$$

$$\times \sum_j \langle j_1 j_2 j|(2j+1)\left[\mathbf{T}^J(1)\times\mathbf{T}^J(2)\right]^0_0\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}^{\dagger}|j_1 j_2 j\rangle. \tag{4.70}$$

(The eigenvalues (4.68) are obtained from this result by making the shifts $j_1 \to j_1+\Delta,\, j_2 \to j_2+\Delta'$.)

Algebra of unit Racah operators. In this section we present a number of relations between Racah operators. These relations are in complete analogy to relations already given for Wigner operators in Chapter 3. We shall therefore be very brief.

The conjugate to a unit Racah operator is related to another unit Racah operator by

$$\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}^{\dagger} = (-1)^{\Delta-\Delta'}\left(\frac{dim}{dim'}\right)^{-\frac{1}{2}}\left\{\begin{matrix} & J-\Delta & \\ 2J & & 0 \\ & J-\Delta' & \end{matrix}\right\}\left(\frac{dim}{dim'}\right)^{\frac{1}{2}}, \tag{4.71}$$

where the dimension operators dim and dim′ are defined by

$$dim|j_1 j_2 j\rangle = (2j_1+1)|j_1 j_2 j\rangle,$$
$$dim'|j_1 j_2 j\rangle = (2j_2+1)|j_1 j_2 j\rangle. \tag{4.72}$$

Equation (4.71) is a direct consequence of the definition (4.33) and the conjugation property (3.64) of $SU(2)$ Wigner operators. It is equivalent to the symmetry relation

$$W(abcd;ef) = (-1)^{b+c-e-f}W(afed;cb). \tag{4.73}$$

We consider next the coupling laws for unit Racah operators.

It is the B–E identity, Eq. (2.69), that provides the *general product law*. Its interpretation in terms of the unit Racah operators *and* the Racah invariant operators[1] $\mathbf{W}^{abc}_{\alpha\beta\gamma}$ is

$$\begin{Bmatrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{Bmatrix}\begin{Bmatrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{Bmatrix}=\sum_{c}\underline{\mathbf{W}}^{abc}_{\alpha,\beta,\alpha+\beta}\overline{\mathbf{W}}^{abc}_{\rho,\sigma,\rho+\sigma}\begin{Bmatrix} & c+\rho+\sigma & \\ 2c & & 0 \\ & c+\alpha+\beta & \end{Bmatrix}, \tag{4.74}$$

where $\overline{\mathbf{W}}^{abc}_{\rho\sigma\tau}$ and $\underline{\mathbf{W}}^{abc}_{\alpha\beta\gamma}$ denote Racah invariant operators of the type introduced in Chapter 2, Section 5, and here have the following actions on a generic basis vector $|j_1j_2j\rangle$:

$$\begin{aligned}\overline{\mathbf{W}}^{abc}_{\rho\sigma\tau}|j_1j_2j\rangle &= W^{abc}_{\rho\sigma\tau}(j_1)|j_1j_2j\rangle,\\ \underline{\mathbf{W}}^{abc}_{\alpha\beta\gamma}|j_1j_2j\rangle &= W^{abc}_{\alpha\beta\gamma}(j_2)|j_1j_2j\rangle.\end{aligned} \tag{4.75}$$

Thus, when we operate on a generic state with Eq. (4.74), we obtain the B–E relation between Racah coefficients.

Using the orthogonality of the Racah invariant operators [expressed by Eq. (2.54)] and the orthogonality of the Racah operators [expressed by Eq. (4.34)], we obtain from Eq. (4.74) the following three relations:
Lower pattern coupling:

$$\sum_{\alpha\beta}\underline{\mathbf{W}}^{abc}_{\alpha\beta\gamma}\begin{Bmatrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{Bmatrix}\begin{Bmatrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{Bmatrix}=\overline{\mathbf{W}}^{abc}_{\rho,\sigma,\rho+\sigma}\begin{Bmatrix} & c+\rho+\sigma & \\ 2c & & 0 \\ & c+\gamma & \end{Bmatrix}, \tag{4.76}$$

Upper pattern coupling:

$$\sum_{\rho\sigma}\overline{\mathbf{W}}^{abc}_{\rho\sigma\tau}\begin{Bmatrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{Bmatrix}\begin{Bmatrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{Bmatrix}=\underline{\mathbf{W}}^{abc}_{\alpha,\beta,\alpha+\beta}\begin{Bmatrix} & c+\tau & \\ 2c & & 0 \\ & c+\alpha+\beta & \end{Bmatrix}, \tag{4.77}$$

Lower and upper pattern coupling:

$$\sum_{\alpha\beta\rho\sigma}\overline{\mathbf{W}}^{abd}_{\rho\sigma\tau}\underline{\mathbf{W}}^{abc}_{\alpha\beta\gamma}\begin{Bmatrix} & b+\sigma & \\ 2b & & 0 \\ & b+\beta & \end{Bmatrix}\begin{Bmatrix} & a+\rho & \\ 2a & & 0 \\ & a+\alpha & \end{Bmatrix}=\delta_{cd}\begin{Bmatrix} & c+\gamma & \\ 2c & & 0 \\ & c+\gamma & \end{Bmatrix}. \tag{4.78}$$

These operator relations are, of course, fully equivalent to the various forms into which the B–E identity may be cast by using the orthogonality of the Racah coefficients. Equation (4.78), in particular, is the operator analog of Eq. (3.296) of AMQP and correspondingly may be used to determine the general Racah operator from the fundamental (spin-$\frac{1}{2}$) Racah operators.

[1]Recall from our definition, Eq. (4.33), that a unit Racah operator is a *sum* of $SU(2)\times SU(2)$ tensor operators, which is invariant with respect to the diagonal $SU(2)$ subgroup, whereas a Racah invariant operator is now to be interpreted as an $SU(2)\times SU(2)$ invariant.

Remarks. (*a*) The associativity of multiplication of unit Racah operators implies (and is implied by) the B–E identity.

(*b*) There are *no* commuting pairs of fundamental Racah operators [each pair of operators has at least one opposing arrow (see Table 4.1)]. Moreover, there is no group transformation property analogous to Eq. (3.19) for the fundamental Wigner operators. It is, accordingly, more difficult to derive an expression for the general unit Racah operator in terms of the fundamental operators, although such expressions clearly exist, since the iteration of Eq. (4.78) yields such forms. This iteration method is feasible for obtaining the unit Racah operators in which one pattern is extremal (four cases). Carrying out this procedure, we find that these operators are, up to an $SU(2)\times SU(2)$ invariant, monomials in the fundamental Racah operators. Indeed, one sees that the monomial property is characterized by the fact that there are *no opposing arrows going between rows* (see the arrow diagrams in Table 4.1) of the pair of fundamental operators—the numerator pattern calculus factor is independent of the order of the operators. [This characterization of monomials is also valid for $SU(2)$ Wigner operators.] These monomial Racah operators are

$$
\begin{aligned}
\begin{Bmatrix} & J+\Delta & \\ 2J & & 0 \\ & 2J & \end{Bmatrix}\bar{I}_{\Delta_1\Delta_2} &= \left[\frac{(2J)!}{(J+\Delta)!(J-\Delta)!}\right]^{\frac{1}{2}} \\
&\quad \times \begin{Bmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{Bmatrix}^{J+\Delta} \begin{Bmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{Bmatrix}^{J-\Delta}, \\
\begin{Bmatrix} & J+\Delta & \\ 2J & & 0 \\ & 0 & \end{Bmatrix}\bar{I}_{\Delta_1\Delta_2} &= \left[\frac{(2J)!}{(J+\Delta)!(J-\Delta)!}\right]^{\frac{1}{2}} \\
&\quad \times \begin{Bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{Bmatrix}^{J+\Delta} \begin{Bmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{Bmatrix}^{J-\Delta},
\end{aligned}
\tag{4.79}
$$

$$
\begin{aligned}
\begin{Bmatrix} & 2J & \\ 2J & & 0 \\ & J+\Delta' & \end{Bmatrix}\underline{I}_{\Delta'_1\Delta'_2} &= \left[\frac{(2J)!}{(J+\Delta')!(J-\Delta')!}\right]^{\frac{1}{2}} \\
&\quad \times \begin{Bmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{Bmatrix}^{J+\Delta'} \begin{Bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{Bmatrix}^{J-\Delta'}, \\
\begin{Bmatrix} & 0 & \\ 2J & & 0 \\ & J+\Delta' & \end{Bmatrix}\underline{I}_{\Delta'_1\Delta'_2} &= \left[\frac{(2J)!}{(J+\Delta')!(J-\Delta')!}\right]^{\frac{1}{2}} \\
&\quad \times \begin{Bmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{Bmatrix}^{J+\Delta'} \begin{Bmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{Bmatrix}^{J-\Delta'}.
\end{aligned}
$$

The $SU(2)\times SU(2)$ invariants appearing in the left-hand side of these relations have the following values on a generic state $|(m)\rangle$:

$$\bar{I}_{\Delta_1\Delta_2}(p_{12}, p_{22})=\left[(p_{12}-p_{22}+\Delta_1-\Delta_2+1)_{\Delta_2}/(p_{12}-p_{22}-\Delta_2+1)_{\Delta_2}\right]^{\frac{1}{2}},$$
$$\underline{I}_{\Delta'_1\Delta'_2}(p_{11}, p_{21})=\left[(p_{11}-p_{21}+1)_{\Delta'_1}/(p_{11}-p_{21}-\Delta'_2+1)_{\Delta'_1}\right]^{\frac{1}{2}}. \qquad (4.80)$$

The first of these invariants is the same as $\mathcal{G}_{12}^{(\Delta_1\Delta_2)}$ occurring in Eqs. (3.21) and (3.23). The second invariant in Eq. (4.80) is similar to the first one, except that it is defined on the irrep label $2j_2=m_{11}-m_{21}$.

(c) We may apply the pattern calculus rules developed in Chapter 3, Sections 1 and 2, directly to the general shift pattern

$$[\Delta]=\begin{bmatrix}\Delta_1 & & \Delta_2 & \\ & \Delta'_1 & & \Delta'_2\end{bmatrix}$$
$$=\begin{bmatrix}J+\Delta & & J-\Delta & \\ & J+\Delta' & & J-\Delta'\end{bmatrix}=\begin{bmatrix}\Delta_{12} & & \Delta_{22} & \\ & \Delta_{11} & & \Delta_{21}\end{bmatrix} \qquad (4.81)$$

of the unit Racah operator

$$\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}. \qquad (4.82)$$

For clarity, we restate briefly these rules in this somewhat more general context (a preliminary discussion is on p. 103).

The associations are shift pattern $\to$ arrow pattern $\to$ labeled arrow pattern $\to$ PCF. The arrow pattern now contains four dots located at positions labeled

$$\begin{matrix}(12) & & (22) & \\ & (11) & & (21)\end{matrix} \qquad (4.83)$$

with multiple arrows going from dot (ij) to dot $(i'j')$ if $\Delta_{ij}>\Delta_{i'j'}$, in which case the number of arrows is $\Delta_{ij}-\Delta_{i'j'}$. The labeled arrow pattern is obtained from the arrow pattern by assigning the partial hook p_{ij} to point (ij). The rising factorial,

$$(p_{\text{tail}}-p_{\text{head}}+e_{\text{tail}})_A, \qquad (4.84)$$

is then assigned to each pair of points, where A is the number of arrows going between the pair of points. The numerator pattern calculus factor, denoted NPCF, is defined as follows:

$$\text{NPCF} = [|\text{product of all rising factorials for arrows going between rows}|]^{\frac{1}{2}}. \tag{4.85}$$

The denominator pattern calculus factor, denoted DPCF, is defined as follows:

$$\text{DPCF} = [|\text{product of all rising factorials for arrows going within rows}|]^{\frac{1}{2}}. \tag{4.86}$$

The DPCF has, however, a significant generalization (see Ref. [6]), as described earlier in Chapter 3, pp. 65, 66. We now consider two denominator functions, D and D', which are associated, respectively, with the top row $[\Delta_1\ \Delta_2]$ and the bottom row $[\Delta_1'\Delta_2']$ of the shift pattern (4.81). The shift pattern $[\Delta_1\Delta_2]$ is split into the two extremal parts:

$$[\Delta_1\Delta_2] = [\Delta_1\ 0\ ] + [0\ \Delta_2]. \tag{4.87}$$

The labeled arrow patterns associated with each of these shift patterns are

$$\underset{\Delta_1 \text{ arrows}}{\overset{p_{12} \qquad\qquad p_{22}+\Delta_2}{\bullet\!\xrightarrow{\qquad\qquad}\!\bullet}} \qquad \underset{\Delta_2 \text{ arrows}}{\overset{p_{12}+\Delta_1 \qquad\qquad p_{22}}{\bullet\!\xleftarrow{\qquad\qquad}\!\bullet}} \qquad \text{(top row)}. \tag{4.88}$$

One now defines the denominator function D by applying the factorial rule (4.84) ($e_{\text{tail}} = 0$, since the tail is in the top row) to each pattern and forming the product:

$$D_{[\Delta_1\Delta_2]}(p_{12}, p_{22}) = [|(p_{12} - p_{22} - \Delta_2)_{\Delta_1}(p_{22} - p_{12} - \Delta_1)_{\Delta_2}|]^{\frac{1}{2}}. \tag{4.89}$$

This general rule for the denominator function D is a direct consequence of the (extended) pattern calculus rules for the fundamental operators. This result follows from either of the top two equations in (4.79), *just as it did for SU(2) Wigner operators.*

To calculate the denominator D', we must account for the fact that the pattern calculus rules assign a shift $e_{\text{tail}} = +1$ to arrows with tails in the bottom row of the arrow pattern. Thus, we cannot assume a priori that the general rule for determining the denominator function D applies also to D'.

To determine D', we apply the pattern calculus rules for the fundamental operators to either of the last two equations in Eqs. (4.79) and combine the resulting factors with the invariant given by the second of Eqs. (4.80).

The result is as follows: to calculate the denominator function D', we split the shift pattern $[\Delta_1'\Delta_2']$ into two extremal parts,

$$[\Delta_1'\Delta_2']=[\Delta_1'\;0]+[0\;\Delta_2'], \tag{4.90}$$

and associate with each of these two shift patterns the labeled arrow patterns

$$\underset{\Delta_1' \text{ arrows}}{\overset{p_{11}\qquad\quad p_{21}}{\bullet\!\xrightarrow{\qquad\quad}\!\bullet}} \qquad \underset{\Delta_2' \text{ arrows}}{\overset{p_{11}\qquad\quad p_{21}}{\bullet\!\xleftarrow{\qquad\quad}\!\bullet}} \qquad \text{(bottom row)} \tag{4.91}$$

with *no shifts of labels.* Applying the factorial rule ($e_{\text{tail}}=+1$, since the tail is in the bottom row) to each pattern and forming the product yields the desired denominator function:

$$D'_{[\Delta_1'\Delta_2']}(p_{11},p_{21})=\left[|(p_{11}-p_{21}+1)_{\Delta_1'}(p_{21}-p_{11}+1)_{\Delta_2'})|\right]^{\frac{1}{2}}. \tag{4.92}$$

We can now state the pattern calculus factor, PCF, which we obtain from the pattern calculus rules as applied to the shift pattern (4.81). We have

$$\text{PCF}=(\text{NPCF})/DD'. \tag{4.93}$$

Note that, if the upper operator pattern $\begin{pmatrix} & J+\Delta & \\ 2J & & 0\end{pmatrix}$ *and* the lower operator pattern $\begin{pmatrix} 2J & & 0 \\ & J+\Delta' & \end{pmatrix}$ are extremal, then Eq. (4.93) reduces to

$$\text{PCF}=(\text{NPCF})/(\text{DPCF}). \tag{4.94}$$

~~We must, of course, justify the applicability of these pattern calculus rules~~ to the *general unit Racah operator*—that is, demonstrate the relationship of PCF, defined by Eq. (4.93), to the structure of a Racah coefficient. This is carried out in detail in Section 4.

It is essential to note again that the preceding two rules for obtaining denominator functions are consequences of the pattern calculus rules for fundamental operators *and* the special monomial Racah operators (4.79); they have in no way been tailored to the general unit Racah operator.

(d) Despite the fact that D and D' as defined by Eqs. (4.89) and (4.92) have a genuine structural significance in the general unit Racah operator (see Section 4), one is still inclined to ask: Why are there two denominator functions and two general rules? We may answer this question by appealing directly to the relationship between these two denominators in the case of the fundamental Racah operators. We recognize from Table 4.1 that $D_{[10]}$

and $D'_{[10]}$ are related by the dimension operator dim':

$$D'_{[10]} = (dim')^{\frac{1}{2}} D_{[10]} (dim')^{-\frac{1}{2}},$$
$$D'_{[01]} = (dim')^{\frac{1}{2}} D_{[01]} (dim')^{-\frac{1}{2}}. \tag{4.95}$$

For example, the value $D'_{[10]}(p_{11}, p_{21})$ of $D'_{[10]}$ at the point (p_{11}, p_{21}) is given by

$$D'_{[10]}(p_{11}, p_{21}) = \left(\frac{p_{11} - p_{21} + 1}{p_{11} - p_{21}} \right)^{\frac{1}{2}} D_{[10]}(p_{11}, p_{21}),$$

where $D_{[10]}(p_{11}, p_{21})$ is the value of $D_{[10]}$ at the point (p_{11}, p_{21}) [set $p_{12} = p_{11}$ and $p_{22} = p_{21}$ in Eq. (4.89)]. Note that the dimension operator on the left in Eq. (4.95) is to be evaluated on the shifted labels:

$$(p_{11} + \Delta'_1, p_{21} + \Delta'_2) = (p_{11}, p_{21}) + [\Delta'_1 \Delta'_2],$$

that is, on $(p_{12} + 1, p_{22}) = (p_{12}, p_{22}) + [1,0]$ in our example.

It is a *consequence* of relation (4.95) between the fundamental denominator functions that the general denominator function also satisfies this same relation:

$$D'_{[\Delta'_1 \Delta'_2]} = (dim')^{\frac{1}{2}} D_{[\Delta'_1 \Delta'_2]} (dim')^{-\frac{1}{2}}. \tag{4.96}$$

This relationship may, of course, be verified directly from Eqs. (4.89) and (4.92). Thus, despite the superficial distinction in the pattern calculus rules for denominator functions, these two rules serve only to distinguish the two functions by a ratio of final to initial dimension factors:

$$\left(\frac{p_{11} - p_{21} + \Delta'_1 - \Delta'_2}{p_{11} - p_{21}} \right)^{\frac{1}{2}}.$$

It would not serve any useful purpose here to give a detailed proof that Eq. (4.96) is a consequence of the same relation, Eq. (4.95), for the fundamental operators. It is nonetheless significant that this result generalizes to $U(n)$, and, in particular, that the denominator functions given by Eqs. (4.89) and (4.92) (using the associated pattern calculus rules) generalize directly to the $\langle p0\ldots0\rangle$ tensor operators. The structure of the proof is, however, easily appreciated without any detail (see Ref. [6]): When applied to the case at hand, one finds that $(D_{[\Delta'_1 \Delta'_2]})^{-2}$ is the *same polynomial* over ordered products of the fundamental denominator functions $(D_{[10]})^{-1}$ and

$(D_{[01]})^{-1}$ as $(D'_{[\Delta'_1\Delta'_2]})^{-2}$ is over $(D'_{[10]})^{-1}$ and $(D'_{[01]})^{-1}$. Thus, the similarity transformation (4.95) between fundamental denominators extends to the general denominator functions themselves.

Finally, let us remark that the appearance of the dimension factors in Eq. (4.96) is closely related to the normalization of the unit Racah operators, as discussed further in Section 4.

(e) It is, we believe, quite interesting—even surprising—that the curious insertion of $+1$ for arrow pattern factors having their tails in the bottom row is actually successful in yielding correct matrix elements [as was already demonstrated for $SU(2)$ Wigner operators, and below for Racah operators]. Despite the fact that the "e_{tail} rule," which governs this insertion, appears quite bizarre at first, it is the essential element of the calculus that accounts for the "squaring phenomenon" associated with opposing arrows in products of operators. The origin of these rules is clearly buried in the boson calculus, which more and more appears to us as the key to a general calculus of patterns. (In this regard, it is of interest to recall the original use of the concept of "hook" in the Hall–Robinson formula for the symmetric group, and the use of "hooks" in the pattern measure discussed in Appendix A to Chapter 5, of AMQP.)

(f) The betweenness relation for extended patterns, $(m) = \begin{pmatrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{pmatrix}$, is fully consistent with the formal operation $m_{21} \to -\infty$. By means of this formal limit one recovers from the extended pattern a standard Gel'fand pattern:

$$\begin{pmatrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{pmatrix} \underset{m_{21}\to-\infty}{\to} \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix}.$$

This formal operation becomes significant when applied to explicit matrix elements, *since the pattern calculus rules ensure that the limit exists.*

~~If we apply this limit to the fundamental Racah operators, given above,~~ we obtain the following result: *In the limit $m_{21} \to -\infty$ the matrix elements of the fundamental Racah operators limit to the matrix elements of the fundamental Wigner operators.*

If one examines the limit more closely, one finds that the state vectors have the limit

$$\left| \begin{matrix} 2j_1 & & 0 & \\ & j_1+j_2-j & & j_1-j_2-j \end{matrix} \right\rangle \to \left| \begin{matrix} 2j_1 & & 0 \\ & j_1+m & \end{matrix} \right\rangle$$

or $j_2 \to \infty$, $j \to \infty$, $j_2 - j = m$ (fixed). It is in this precise sense that one can assert: Under the limit above, *the W-algebra generated by the Racah operators $\{2J\ 0\}$ limits to the RW-algebra generated by the Wigner operators* $\langle 2J\ 0 \rangle$.

4. Null Space Properties and Structure Theorems for W-Algebra[1]

Structure of the general Racah operator. The pattern calculus was invented as a tool for writing out the matrix elements of certain Wigner operators. Remarkably, these same methods allow one to write out a class of Racah coefficients, as demonstrated in Remark (*b*) in Section 3. Moreover, as we shall see, these rules have a significant role for the general Racah coefficient.

Polynomial forms expressing the general Racah operator in terms of the fundamental Racah operators such that one obtains *directly* the symmetric expression of the Racah coefficients [Eq. (3.292) of AMQP] do not appear in the literature. As noted in Remark (*b*) above, one *can* develop polynomial forms from Eq. (4.78), but they do not yield the Racah coefficient in the desired form (given below). We therefore appeal directly to the symmetric form to obtain an expression for a Racah coefficient that exhibits the pattern calculus features and null space properties in a form analogous to Eq. (3.32) for a Wigner coefficient (for convenience, Racah's form of the coefficient is repeated in Appendix A).

The appropriate notational changes for making the desired transformation of the standard Racah coefficient $W(abcd;ef)$ are given in Eqs. (4.45), (4.47), and (4.49):

$$
\begin{aligned}
a&=(p_{12}+p_{22}-p_{11}-p_{21}-2)/2,\\
b&=(p_{12}-p_{22}-1)/2,\\
c&=(p_{11}+\Delta'_1-p_{21}-\Delta'_2-1)/2,\\
d&=J=(\Delta_1+\Delta_2)/2=(\Delta'_1+\Delta'_2)/2,\\
e&=(p_{11}-p_{21}-1)/2,\\
f&=(p_{12}+\Delta_1-p_{22}-\Delta_2-1)/2.
\end{aligned}
\tag{4.97}
$$

One now substitutes these definitions of a, b, c, d, e, and f into Racah's expression for $W(abcd;ef)$ to obtain the desired result. This calculation is quite tedious, however, and we have given the principal steps in Appendix A.

The canonical form obtained for a Racah coefficient is

$$
\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta'_1 & & m_{21}+\Delta'_2 \end{matrix} \right| \left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\} \left| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle
$$

$$
=\#(\mathrm{NPCF})\times D^{-1}\times D'^{-1}\times P_k(\Delta;p), \tag{4.98}
$$

[1]This section is adapted from Ref. [7].

where (a) NPCF is the numerator pattern calculus factor written out directly from the stated rules; (b) # is a numerical factor independent of the m_{ij}; (c) D and D' are the $SU(2)\times SU(2)$ invariant denominator functions obtained from the pattern calculus rules and given explicitly by Eqs. (4.89) and (4.92); and (d) $P_k(\Delta; p)$ is a polynomial form of degree $k=\min(\Delta_1, \Delta_2, \Delta_1', \Delta_2')$ in the variables p_{ij}, which is discussed in detail in Appendix A.

Aside from the null space features (discussed below) that the form (4.98) exhibits, it is interesting for several additional reasons:

(a) The limit relation

$$\lim_{m_{21}\to-\infty}\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta_1' & & m_{21}+\Delta_2' \end{matrix} \left| \begin{Bmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{Bmatrix} \right| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 \\ & m_{11}+\Delta_1' & \end{matrix} \left| \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\rangle \right| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle \tag{4.99}$$

is verified in a particularly simple way in consequence of the asymptotic properties (see Appendices A and D):

$$\begin{gathered} P_k(\Delta; p) \sim (-m_{21})^k P_k(\Delta_1, \Delta_2, \Delta_1', \Delta_2'; z_1, z_2), \\ (\mathrm{NPCF})_{\text{extended rules}}/D' \sim (\mathrm{NPCF})/(-m_{21})^k, \end{gathered} \tag{4.100}$$

where NPCF on the right refers to the $SU(2)$ rules.

(b) The denominator functions D and D' are, respectively, invariant under the substitutions $\mathcal{P}$ and $\mathcal{P}'$ defined by

$$\begin{gathered} \mathcal{P}: \Delta_1 \leftrightarrow \Delta_2, p_{12} \leftrightarrow p_{22}, \\ \mathcal{P}': \Delta_1' \leftrightarrow \Delta_2', p_{11} \leftrightarrow p_{21}, \end{gathered} \tag{4.101}$$

whereas NPCF is invariant to each of these operations. It is also true that the polynomial $P_k(\Delta; p)$ is invariant under the substitutions (4.101):

$$\begin{aligned} P_k(\Delta_1, \Delta_2, \Delta_1', \Delta_2'; p_{12}, p_{22}, p_{11}, p_{21}) &= P_k(\Delta_2, \Delta_1, \Delta_1', \Delta_2'; p_{22}, p_{12}, p_{11}, p_{21}) \\ &= P_k(\Delta_1, \Delta_2, \Delta_2', \Delta_1'; p_{12}, p_{22}, p_{21}, p_{11}). \end{aligned} \tag{4.102}$$

Thus, one may extend the domain of definition of a Racah coefficient so as to include the symmetry under the substitutions (4.101). [Observe that the first substitution in (4.101) survives the limit (4.99) and implies the corresponding symmetry of the (generalized) Wigner coefficient.]

No simple method of proof of the symmetry (4.102) has been found, and since its validity is essential to the construction of a Racah coefficient from its null space, we have sketched a proof in Appendix B.

Null space properties. We have identified the set of null space vectors of a Racah operator in Section 3. Consider now the characterization by null space of the various factors—NPCF, polynomial, and denominators—appearing in the Racah coefficient (4.98). We assert that the following properties are true:

(*a*) Define the set of points $Z'=\{(m_{12}, m_{22}, m_{11}, m_{21})\}$ to be the set of values of the m_{ij} such that the final pattern

$$\begin{pmatrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta_1' & & m_{21}+\Delta_2' \end{pmatrix} \tag{4.103}$$

violates the betweenness conditions

$$m_{12}+\Delta_1 \geqslant m_{11}+\Delta_1' \geqslant m_{22}+\Delta_2 \geqslant m_{21}+\Delta_2', \tag{4.104}$$

the initial pattern (m) being lexical. The set of vectors $\{|(m)\rangle \in \mathcal{H}\}$ corresponding to the points of Z' constitute a basis of the trivial null space.

We assert: *The pattern calculus factor*, $(NPCF)^2$, *is the unique*[1] *polynomial of smallest degree in the variables*

$$z_{ij} \equiv p_{i1}-p_{j2}, \qquad i, j=1,2 \tag{4.105}$$

that is invariant under the substitutions $\mathcal{P}$ and $\mathcal{P}'$ and that vanishes on the point set Z'.

(*b*) Define the set of points $Z=\{(m_{12}, m_{22}, m_{11}, m_{21})\}$ to be the set of values of the m_{ij} such that one or both of the following conditions obtain:

$$\begin{aligned} &m_{12}-m_{22} \in \{0,1,\ldots, J-\Delta-1\}, \\ &m_{11}-m_{21} \in \{0,1,\ldots, J-\Delta'-1\}. \end{aligned} \tag{4.106}$$

(If $\Delta=J$ in the first set or $\Delta'=J$ in the second set, there is no condition on the corresponding variable.) The set of vectors $\{|(m)\rangle \in \mathcal{H}\}$ corresponding to the points of Z constitute a basis of the characteristic null space given by Eq. (4.57).

Now consider the set $Z-(Z\cap Z') \equiv Z''$. The polynomial $P_k(\Delta; p)$ must then vanish on the points of Z'', since the Racah coefficient must vanish on all points Z associated with the characteristic null space.

[1]In the discussion to follow, we use the word unique somewhat loosely to mean "unique up to a numerical factor" independent of the variables $p_{i1}-p_{j2}$.

We assert: $P_k(\Delta;\, p)$ *is uniquely determined by the requirement that it be the polynomial of smallest degree in the* z_{ij} *that is invariant under the substitutions* $\mathcal{P}$ *and* $\mathcal{P}'$ *and that vanishes on the point set* Z''.

(*c*) Consider the first set of points in Eq. (4.106). These are just the points corresponding to the basis vectors of the factor $\mathfrak{N}_{(J-\Delta-1)/2}$ occurring in the characteristic null space given by Eqs. (4.57) and (4.58). The denominator D is associated with these points. In order to determine the zeros of the denominator function in terms of the associated points of the characteristic null space, we must examine the precise form of the normalizing conditions on the Racah operator, Eq. (4.34). These conditions show that the denominator function D is related to the characteristic null space factor $\mathfrak{N}_{(J-\Delta-1)/2}$ in precisely the same way as is the denominator function of the $SU(2)$ Wigner coefficient. (This relationship is complicated; it is discussed in detail on pp. 76–79.) Accordingly, we find that

$$\frac{p_{12}-p_{22}+\Delta_1-\Delta_2}{p_{12}-p_{22}} D^2_{[\Delta_1\Delta_2]}(p_{12},\, p_{22}) \tag{4.107}$$

must vanish on all points such that

$$p_{12}-p_{22}\in\{1,2,\ldots,\Delta_2\}. \tag{4.108}$$

We assert: $D^2_{[\Delta_1\Delta_2]}(p_{12},\, p_{22})$ *is uniquely determined by the requirement that it be the polynomial of smallest degree in* $p_{12}-p_{22}$ *that is invariant under the substitution* $\mathcal{P}$, *such that the expression* (4.107) *is a polynomial that vanishes on the point set* (4.108)

(*d*) Consider the second set of points in Eq. (4.106). These are just the points corresponding to the basis vectors of the factor $\mathfrak{N}_{(J-\Delta'-1)/2}$ occurring in the characteristic null space given by Eqs. (4.57) and (4.58). The denominator D' is associated with these points. Just as before, to determine the zeros of this denominator function, we must examine the normalizing condition, Eq. (4.34). *We find that* D' *is determined differently than* D *is*: Because the Wigner operator associated with D' enters the Racah operator as the conjugate [see Eq. (4.33)], the points on which $[D'_{[\Delta'_1\Delta'_2]}(p_{11},\, p_{21})]^2$ must vanish are those such that

$$p_{11}-p_{21}\in\{1,2,\ldots,\Delta'_2\}. \tag{4.109}$$

[To be explicit, we note that using the conjugate Wigner operator removes the dimension factors that appear in Eq. (4.107) for D as opposed to D'.]

We assert: $D'^2_{[\Delta'_1\Delta'_2]}(p_{11},\, p_{21})$ *is uniquely determined by the requirement that it be the polynomial of smallest degree in* $p_{11}-p_{21}$ *that is invariant under the substitution* $\mathcal{P}'$, *such that it vanishes on the point set* (4.109).

The proof of each of the italicized statements in (*a*), (*c*), and (*d*) is by direct construction of the polynomials in question and then comparison with the known answers (see Appendix A). The proof of the italicized statement in (*b*) is, however, a nontrivial construction and is sketched in Appendix C.

These results suffice to demonstrate the structure theorem for W-algebra: *The Racah operator*

$$\left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\}$$

is uniquely determined (to within ± 1) *by the zeros associated with both the trivial and characteristic null spaces, by the reflection symmetry* [*Eq.* (4.101)], *and by the requirement of normalization.*

Remarks. (*a*) The structure theorem for the Racah coefficients makes clear the distinction between the trivial null space and the characteristic null space; moreover, this theorem shows in what sense any additional zeros of the Racah coefficient are "accidental," or "isolated." The key point is that, for any additional zeros to constitute structural information, then these zeros—to be invariant under the symmetry structure $SU(2)\times SU(2)$—must occur as characteristic sets of zeros [j varying over all values allowed by the triangle condition on $(j_1 j_2 j)$]. It is easily seen that a given accidental zero *cannot* belong to a characteristic set of zeros. (To do so would contradict the orthonormality relations of the Racah coefficient.)

The structure theorem makes it clear that an accidental zero must be a zero of the polynomial. (This follows from the fact that the NPCF and the denominator do not vanish.)

Very little is known about these isolated zeros of the Racah function; whether or not there are even finitely many such zeros is not known. (Topic 10 of Chapter 5 tabulates some of these zeros.)

(*b*) In view of the emphasis [in (*c*) and (*d*), p. 121] on the distinction in normalization between the two denominator functions, D and D', one might wonder as to how the pattern calculus rules can apply uniformly. It is quite remarkable that the rule—"add unity to factors for arrows beginning on the bottom row"—precisely takes care of the distinction between D and D'. Since this rule was inherited from the original calculus, it was in no sense tailored a priori to the normalization problem.

(*c*) The canonical form given by Eq. (4.98) may be used to define a *continuation* or *generalization*[1] of the Racah coefficient for all complex values of the (m_{ij}) (no betweenness) for which the denominator functions are not zero. Thus, using the notation (2.53) for the Racah coefficient, we

[1]Generalized 6-j coefficients have been introduced by Regge [8] and discussed by Raynal [9], among others.

find

$$\mathcal{W}^{j_1 J j_1+\Delta}_{j_2-j,\Delta',j_2-j+\Delta'}(j_2+\Delta')\equiv(\#)(\mathrm{NPCF})(\mathrm{polynomial})/DD'. \quad (4.110)$$

In this definition the (m_{ij}) may be arbitrary complex numbers, or, equivalently, j_1, j_2, and j may be arbitrary complex numbers with no relations between them, where we recall that

$$\begin{aligned}
&j_1=(m_{12}-m_{22})/2, \qquad j_2=(m_{11}-m_{21})/2,\\
&j=(m_{12}+m_{22}-m_{11}-m_{21})/2,\\
&\Delta_1=J+\Delta, \qquad \Delta_2=J-\Delta,\\
&\Delta_1'=J+\Delta', \qquad \Delta_2'=J-\Delta'.
\end{aligned} \quad (4.111)$$

The labels J, Δ, and Δ' are, however, still required to assume standard values: $J=0,\frac{1}{2},1,\ldots,$ with $\Delta,\Delta'=J,J-1,\ldots,-J$. Equation (4.110) defines a generalized Racah coefficient for all complex (j_1, j_2, j), except for those real values of j_1 and j_2 given by

$$\begin{aligned}
&2j_1+1=-\Delta_1,-\Delta_1+1,\ldots,\Delta_2-\Delta_1-1,\Delta_2-\Delta_1+1,\ldots,\Delta_2-1,\Delta_2,\\
&2j_2+1=-\Delta_1',-\Delta_1'+1,\ldots,-1,1,\ldots,\Delta_2'-1,\Delta_2'.
\end{aligned} \quad (4.112)$$

[By convention, we define the coefficient to be zero for the values of j_1 and j_2 given by Eqs. (4.112).]

These generalized Racah coefficients then satisfy the following symmetry relations in consequence of the symmetry relation (4.102) for the polynomial and a similar symmetry relation for NPCF and the denominator functions:

$$\begin{aligned}
&(-1)^{l_{11}}\mathcal{W}^{j_1 J j_1+\Delta}_{j_2-j,\Delta',j_2-j+\Delta'}(j_2+\Delta')\\
&\quad=(-1)^{l_{12}}\mathcal{W}^{-j_1-1,J,-j_1-1-\Delta}_{j_2-j,\Delta',j_2-j+\Delta'}(j_2+\Delta')\\
&\quad=(-1)^{l_{21}}\mathcal{W}^{j_1 J j_1+\Delta}_{-j_2-1-j,-\Delta',-j_2-1-j-\Delta'}(-j_2-1-\Delta')\\
&\quad=(-1)^{l_{22}}\mathcal{W}^{-j_1-1,J,-j_1-1-\Delta}_{-j_2-1-j,-\Delta',-j_2-1-j-\Delta'}(-j_2-1-\Delta'),
\end{aligned} \quad (4.113)$$

where [see Eq. (A.8) of Appendix A]

$$\begin{aligned}
&l_{11}=\max(0,\Delta'-\Delta),\\
&l_{12}=\max(0,\Delta'+\Delta),\\
&l_{21}=\max(0,-\Delta'-\Delta),\\
&l_{22}=\max(0,-\Delta'+\Delta).
\end{aligned} \quad (4.114)$$

5. The Matrix Boson Realization of Unit Racah Operators

In this section we give explicitly the boson operator realization of the unit Racah operators and the Hilbert space on which they act. The resulting theory of unit Racah operators is an elegant application of boson operator techniques, leading to a new characterization of a unit Racah operator: *A unit Racah operator is a normalized element in the enveloping algebra of the Lie algebra of the group $SU(2)\times SU(1,1)$.* (We shall make this result explicit in this section.)

It is well-known from Schwinger's [10] classic monograph that the coupling of two angular momenta may be developed by using three sets of angular momentum operators: Two sets are ordinary angular momenta $[SU(2)]$, the third set is a hyperbolic angular momentum $[SU(1,1)]$, and the angular momenta from different sets commute. It is the purpose of this section to relate the unit Racah operators to the Lie algebra of these groups, using the boson operator realization of the generators.

The explicit boson realization of the fundamental Racah invariants may be found from the defining relation (4.33) and the maps (2.25) between boson operators and fundamental Wigner operators. Letting (a_1^1, a_2^1) and (a_1^2, a_2^2) denote the boson pairs corresponding to the first and second fundamental Wigner operator factors in Eq. (4.33), we find

$$
\begin{aligned}
\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\} &\leftrightarrow \left(a_1^1 a_2^2 - a_2^1 a_1^2\right) N_1^{-\frac{1}{2}}, \\
\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix}\right\} &\leftrightarrow \left(a_1^1 \bar{a}_1^2 + a_2^1 \bar{a}_2^2\right) N_0^{-\frac{1}{2}}, \\
-\left\{\begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\} &\leftrightarrow \left(\bar{a}_1^1 a_1^2 + \bar{a}_2^1 a_2^2\right) N_1^{-\frac{1}{2}}, \\
\left\{\begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix}\right\} &\leftrightarrow \left(\bar{a}_1^1 \bar{a}_2^2 - \bar{a}_2^1 \bar{a}_1^2\right) N_0^{-\frac{1}{2}},
\end{aligned}
\tag{4.115}
$$

where N_0 and N_1 are the invariant operators defined by

$$
\begin{aligned}
N_0 | j_1 j_2 j\rangle &= (2j_1+1)(2j_2) | j_1 j_2 j\rangle, \\
N_1 | j_1 j_2 j\rangle &= (2j_1+1)(2j_2+2) | j_1 j_2 j\rangle.
\end{aligned}
\tag{4.116}
$$

Observe that this boson realization of the fundamental Racah operators shows explicitly their invariance under the $SU(2)$ diagonal subgroup: The generators of the diagonal subgroup are

$$\begin{aligned}
J_+ &= a_1^1\bar{a}_2^1 + a_1^2\bar{a}_2^2,\\
J_- &= a_2^1\bar{a}_1^1 + a_2^2\bar{a}_1^2,\\
J_3 &= \tfrac{1}{2}\left(a_1^1\bar{a}_1^1 + a_1^2\bar{a}_1^2 - a_2^1\bar{a}_2^1 - a_2^2\bar{a}_2^2\right).
\end{aligned} \tag{4.117}$$

One may verify directly that **J** commutes with the fundamental Racah operators. Alternatively, one may demonstrate the invariance of these Racah operators to the unitary transformation

$$a_i^\alpha \to \sum_j u_{ji} a_j^\alpha, \qquad \alpha = 1,2.$$

Let us now remove the normalizing factors from Eqs. (4.115) and calculate all multiple commutators of the four invariants. Carrying this out, we obtain the following closed angular momentum algebras:
Angular momentum commutation rules:

$$\begin{aligned}
[K_3, K_\pm] &= \pm K_\pm,\\
[K_+, K_-] &= 2K_3.
\end{aligned} \tag{4.118}$$

Hyperbolic angular momentum commutation rules:

$$\begin{aligned}
[H_3, H_\pm] &= \pm H_\pm,\\
[H_+, H_-] &= -2H_3.
\end{aligned} \tag{4.119}$$

Mutual commutivity of **K** and **H**:

$$[K_i, H_j] = 0, \; i, j = 1,2,3. \tag{4.120}$$

The operators $\mathbf{K} = (K_1, K_2, K_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$ with $K_\pm = K_1 \pm iK_2$ and $H_\pm = H_1 \pm iH_2$ are defined by

$$\begin{aligned}
K_+ &= a_1^1\bar{a}_1^2 + a_2^1\bar{a}_2^2,\\
K_- &= a_1^2\bar{a}_1^1 + a_2^2\bar{a}_2^1,\\
K_3 &= \tfrac{1}{2}\left(a_1^1\bar{a}_1^1 + a_2^1\bar{a}_2^1 - a_1^2\bar{a}_1^2 - a_2^2\bar{a}_2^2\right);
\end{aligned} \tag{4.121}$$

$$\begin{aligned}
H_+ &= a_1^1 a_2^2 - a_2^1 a_1^2,\\
H_- &= \bar{a}_1^1\bar{a}_2^2 - \bar{a}_2^1\bar{a}_1^2,\\
H_3 &= \tfrac{1}{2}\left(a_1^1\bar{a}_1^1 + a_2^1\bar{a}_2^1 + a_1^2\bar{a}_1^2 + a_2^2\bar{a}_2^2 + 2\right).
\end{aligned} \tag{4.122}$$

Observe that the generators given by Eqs. (4.121) and (4.122) are all *invariants* with respect to the generators **J** given by Eqs. (4.117). Moreover, the fundamental Racah operators are a subset of the generators (4.121) and (4.122). Explicitly, one has the relations given by

$$\begin{gathered}
\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\} N^{\frac{1}{2}}_{1} \leftrightarrow H_{+}, \\
\left\{\begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix}\right\} N^{\frac{1}{2}}_{0} \leftrightarrow K_{+}, \\
-\left\{\begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix}\right\} N^{\frac{1}{2}}_{1} \leftrightarrow K_{-}, \\
\left\{\begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix}\right\} N^{\frac{1}{2}}_{0} \leftrightarrow H_{-}.
\end{gathered} \tag{4.123}$$

It is a remarkable result that the *three* Casimir operators constructed from **J**, **K**, and **H** are all equal (Schwinger [10]):

$$\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 = K_1^2 + K_2^2 + K_3^2 = H_1^2 + H_2^2 + H_3^2. \tag{4.124}$$

Moreover, we see from the definitions of H_3 and K_3 that these operators are related to the angular momentum operators $\mathbf{J}_1^2$ and $\mathbf{J}_2^2$ by [see Eqs. (2.23) and (2.24)]

$$H_3 + K_3 = \left(4\mathbf{J}_1^2 + 1\right)^{\frac{1}{2}}, \qquad H_3 - K_3 = \left(4\mathbf{J}_2^2 + 1\right)^{\frac{1}{2}}. \tag{4.125}$$

These latter results may now be used to determine the relations between the eigenvalue μ of K_3 and the eigenvalue κ of H_3 and the angular momentum quantum numbers j_1 and j_2:

$$\kappa + \mu = 2j_1 + 1, \qquad \kappa - \mu = 2j_2 + 1. \tag{4.126}$$

In terms of the new quantum numbers μ and κ, the basis vectors (4.41) are denoted by

$$\left|\begin{matrix} \kappa+\mu-1 & & 0 & \\ & \kappa-j-1 & & \mu-j \end{matrix}\right\rangle, \tag{4.127}$$

where the range of the labels μ and κ is determined from the betweenness conditions to be

$$\mu = j, j-1, \ldots, -j; \qquad \kappa = j+1, j+2, \ldots. \tag{4.128}$$

Consider next the explicit construction of the basis vectors (4.127) in terms of bosons, using the boson realization of the basis vectors $|j_1m_1\rangle$ and $|j_2m_2\rangle$ in Eq. (4.3) given by

$$|j_im_i\rangle=\frac{(a_1^i)^{j_i+m_i}(a_2^i)^{j_i-m_i}|0\rangle}{[(j_i+m_i)!(j_i-m_i)!]^{\frac{1}{2}}},\qquad i=1,2. \tag{4.129}$$

This construction was carried out in detail in Chapter 5 of AMQP [see Eq. (5.80)]. In terms of the present notation, the result may be written

$$\begin{aligned}|jm\mu\kappa\rangle&\equiv\left|\left(\frac{\kappa+\mu-1}{2},\frac{\kappa-\mu-1}{2}\right)jm\right\rangle=|(j_1j_2)jm\rangle\\&=\left[\frac{2j+1}{(\kappa-j-1)!(\kappa+j)!}\right]^{\frac{1}{2}}(\det A)^{\kappa-j-1}D^j_{m\mu}(A)|0\rangle,\end{aligned} \tag{4.130}$$

where $D^j_{m\mu}(A)$ are the standard $SU(2)$ representation functions [Eq. (3.12)] in which the 2×2 unitary matrix V is replaced by the 2×2 matrix boson A. [It is convenient in the present discussion to restore quantum number m to the basis vector—the ket vector (4.127) then denotes the equivalence class of vectors (4.130) for different m.]

The actions of the three sets of generators given by Eqs. (4.117), (4.121), and (4.122) on the basis (4.130) are

$$\begin{aligned}&J_\pm|jm\mu\kappa\rangle=[(j\mp m)(j\pm m+1)]^{\frac{1}{2}}|j,m\pm1,\mu,\kappa\rangle,\\&J_3|jm\mu\kappa\rangle=m|jm\mu\kappa\rangle;\end{aligned} \tag{4.131}$$

$$\begin{aligned}&K_\pm|jm\mu\kappa\rangle=[(j\mp\mu)(j\pm\mu+1)]^{\frac{1}{2}}|j,m,\mu\pm1,\kappa\rangle,\\&K_3|jm\mu\kappa\rangle=\mu|jm\mu\kappa\rangle;\end{aligned} \tag{4.132}$$

$$\begin{aligned}&H_+|jm\mu\kappa\rangle=[(\kappa-j)(\kappa+j+1)]^{\frac{1}{2}}|j,m,\mu,\kappa+1\rangle,\\&H_-|jm\mu\kappa\rangle=[(\kappa+j)(\kappa-j-1)]^{\frac{1}{2}}|j,m,\mu,\kappa-1\rangle,\\&H_3|jm\mu\kappa\rangle=\kappa|jm\mu\kappa\rangle.\end{aligned} \tag{4.133}$$

(Note that $H_+=\det A$.)

We may now express the general unit Racah operator (see Appendix A) in terms of the generators **K** and **H** of the Lie algebra of the group[1]

[1] We use the symbol $*$ in place of the usual $\times$ in the direct product of groups to designate that the Casimir invariants are the same.

$SU(2)*SU(1,1)$:

$$\left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}=(-1)^{l_{11}}k!(2J-k)!/\left[(J+\Delta)!(J-\Delta)!(J+\Delta')!(J-\Delta')!\right]^{\frac{1}{2}}$$
$$\times N^J_{\Delta'\Delta}(\mathbf{K},\mathbf{H})P^J_{\Delta'\Delta}\left(\mathbf{J}^2, K_3, H_3\right)\left[\bar{D}^J_{\Delta}(H_3+K_3)\underline{D}^J_{\Delta'}(H_3-K_3)\right]^{-1}, \tag{4.134}$$

where

$$l_{11}=\max(0, \Delta'-\Delta),$$
$$k=\min(J+\Delta, J-\Delta, J+\Delta', J-\Delta').$$

The denominator factors in Eq. (4.134) are operator versions of the denominator functions given by Eqs. (4.89) and (4.92):

$$\bar{D}^J_{\Delta}(H_3+K_3)=\left[(H_3+K_3-J+\Delta)_{J+\Delta}(H_3+K_3+2\Delta+1)_{J-\Delta}\right]^{\frac{1}{2}},$$
$$\underline{D}^J_{\Delta'}(H_3-K_3)=\left[(H_3-K_3+1)_{J+\Delta'}(H_3-K_3-J+\Delta')_{J-\Delta'}\right]^{\frac{1}{2}}. \tag{4.135}$$

The factor $N^J_{\Delta'\Delta}(\mathbf{K},\mathbf{H})$ is an operator version of the numerator pattern calculus factor (NPCF), and $P^J_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3)$ is an operator version of the (polynomial) in the canonical form (4.98) of a Racah coefficient. There are four cases to consider, depending on the relative magnitudes of $J+\Delta$, $J-\Delta$, $J+\Delta'$, and $J-\Delta'$ (see Appendix A).

Case 1. $\Delta'-\Delta\geqslant 0$ and $\Delta'+\Delta\geqslant 0$.

Pattern calculus operator:

$$N^J_{\Delta'\Delta}(\mathbf{K},\mathbf{H})=(K_-)^{\Delta'-\Delta}(H_+)^{\Delta'+\Delta}. \tag{4.136}$$

Polynomial operator:

$$\langle jm\mu\kappa\,|\,P^J_{\Delta'\Delta}\left(\mathbf{J}^2, K_3, H_3\right)|\,jm\mu\kappa\rangle$$
$$=(-1)^{J-\Delta'}\sum_{\substack{k_1+k_2\\=J-\Delta'}}\binom{J+\Delta}{k_1}\binom{J-\Delta}{k_2}(-j-\mu+\Delta'-\Delta)_{k_1}(-j+\mu)_{k_1}$$
$$\times(\kappa-j-k_2)_{k_2}(-\kappa-j-J-\Delta+k_1)_{k_2}. \tag{4.137}$$

Case 2. $\Delta-\Delta'\geqslant 0$ and $\Delta+\Delta'\geqslant 0$.

Pattern calculus operator:

$$N^J_{\Delta'\Delta}(\mathbf{K},\mathbf{H})=(K_+)^{\Delta-\Delta'}(H_+)^{\Delta+\Delta'}. \tag{4.138}$$

Polynomial operator:

$$
\begin{aligned}
&\langle jm\mu\kappa \,|\, P^{J}_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3) \,|\, jm\mu\kappa\rangle \\
&\quad = (-1)^{J-\Delta} \sum_{\substack{k_1+k_2\\ =J-\Delta}} \binom{J+\Delta'}{k_1}\binom{J-\Delta'}{k_2} (-j-\mu)_{k_1}(-j+\mu+\Delta-\Delta')_{k_1} \\
&\qquad \times (\kappa-j-k_2)_{k_2}(-\kappa-j-J-\Delta'+k_1)_{k_2}.
\end{aligned}
\tag{4.139}
$$

Case 3. $\Delta'-\Delta \geqslant 0$ and $-\Delta'-\Delta \geqslant 0$.
Pattern calculus operator:

$$
N^{J}_{\Delta'\Delta}(\mathbf{K},\mathbf{H}) = (K_-)^{\Delta'-\Delta}(H_-)^{-\Delta'-\Delta}.
\tag{4.140}
$$

Polynomial operator:

$$
\begin{aligned}
&\langle jm\mu\kappa \,|\, P^{J}_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3) \,|\, jm\mu\kappa\rangle \\
&\quad = (-1)^{J+\Delta} \sum_{\substack{k_1+k_2\\ =J+\Delta}} \binom{J-\Delta'}{k_1}\binom{J+\Delta'}{k_2} (-j-\mu+\Delta'-\Delta)_{k_1}(-j+\mu)_{k_1} \\
&\qquad \times (\kappa-j-J+\Delta'+k_1)_{k_2}(-\kappa-j-k_2)_{k_2}.
\end{aligned}
\tag{4.141}
$$

Case 4. $\Delta-\Delta' \geqslant 0$ and $-\Delta-\Delta' \geqslant 0$.
Pattern calculus operator:

$$
N^{J}_{\Delta'\Delta}(\mathbf{K},\mathbf{H}) = (K_+)^{\Delta-\Delta'}(H_-)^{-\Delta-\Delta'}
\tag{4.142}
$$

Polynomial operator:

$$
\begin{aligned}
&\langle jm\mu\kappa \,|\, P^{J}_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3) \,|\, jm\mu\kappa\rangle \\
&\quad = (-1)^{J+\Delta'} \sum_{\substack{k_1+k_2\\ =J+\Delta'}} \binom{J-\Delta}{k_1}\binom{J+\Delta}{k_2} (-j-\mu)_{k_1}(-j+\mu+\Delta-\Delta')_{k_1} \\
&\qquad \times (\kappa-j-J+\Delta+k_1)_{k_2}(-\kappa-j-k_2)_{k_2}.
\end{aligned}
\tag{4.143}
$$

Observe that we have defined the polynomial operators $P^{J}_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3)$ by giving their matrix elements. To obtain the operators $P^{J}_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3)$ themselves as polynomials in $\mathbf{J}^2$, K_3, and H_3, one replaces μ by K_3 and κ by H_3 in the right-hand sides of Eqs. (4.137), (4.139), (4.141), and (4.143). This still leaves the angular momentum label j appearing in the polynomials. To eliminate j in favor of $\mathbf{J}^2$, one must rewrite each polynomial in j in terms of a new polynomial in $j(j+1)$, which is then replaced by $\mathbf{J}^2$.

It is not obvious that these polynomials in j can, in fact, be rewritten as polynomials in $j(j+1)$. It may be shown, however, that the polynomials in j given by Eqs. (4.137), (4.139), (4.141), and (4.143) are invariant to the substitution $j\to -j-1$. Since the polynomials are also of even degree in j, an elementary proof shows that they may be written as polynomials in $j(j+1)$. Thus, *the matrix element expressions given above define uniquely the operators* $P^J_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3)$ as polynomials in $\mathbf{J}^2$, K_3, and H_3.

Equation (4.134), expressing a unit Racah operator as a normalized polynomial in the generators $\mathbf{K}$ and $\mathbf{H}$ of $SU(2)*SU(1,1)$, establishes the italicized statement made at the beginning of this section (p. 124).

It is also useful to give the symmetry relations for the operators $P^J_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3)$ that are implied by Eq. (4.102), the conjugation operation (4.71), and the symmetry of the 6-j symbol:

$$\begin{aligned} P^J_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3) &= P^J_{\Delta',-\Delta}(\mathbf{J}^2, -H_3, -K_3)\\ &= P^J_{-\Delta',\Delta}(\mathbf{J}^2, H_3, K_3)\\ &= P^J_{-\Delta',-\Delta}(\mathbf{J}^2, -K_3, -H_3)\\ &= (-1)^{\Delta-\Delta'}P^J_{-\Delta',-\Delta}(\mathbf{J}^2, K_3+\Delta-\Delta', H_3+\Delta+\Delta')\\ &= (-1)^{\Delta-\Delta'}P^J_{\Delta\Delta'}(\mathbf{J}^2, -K_3, H_3). \end{aligned} \tag{4.144}$$

Remarks. (*a*) The explicit form of the polynomial operator $P^J_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3)$ given in terms of $\mathbf{J}^2$, K_3, and H_3 is quite difficult to obtain. Special cases have been tabulated in Table A3 in the Appendix of Tables.

(*b*) The unit Racah operators of the type

$$\left\{\begin{matrix} & J & \\ 2J & & 0 \\ & J & \end{matrix}\right\}$$

are expressible as polynomials in the invariants $\mathbf{J}_1^2$, $\mathbf{J}_2^2$, and $\mathbf{J}_1\cdot\mathbf{J}_2$ (see Table A3). Schwinger [10] has developed various algebraic properties of these special Racah operators. Many of his results are special cases of the general coupling laws given by Eqs. (4.74)–(4.78).

(*c*) The fact that the Racah operator given by Eq. (4.134) splits into four types is not arbitrary. These operators are analogs of the Jacobi polynomials (more precisely of the representation functions $e^{-i\Delta'\alpha}d^J_{\Delta'\Delta}(\beta)e^{-i\Delta\gamma}$), and the Jacobi polynomials also split into four types when written explicitly as polynomials [see Eq. (3.74) in AMQP].

(*d*) Weyl's theorem (Weyl [11], p. 45) on the basic invariants of the unimodular linear group is applicable to bosons. Applied to the bosons $\mathbf{a}^1=(a^1_1, a^1_2)$ and $\mathbf{a}^2=(a^2_1, a^2_2)$, the theorem implies: *Every* $SU(2)$ *polynomial* *invariant is a polynomial in the basic invariants*:

$$\mathbf{a}^{\alpha}\cdot\bar{\mathbf{a}}^{\beta}\equiv\sum_{i=1}^{2}a_i^{\alpha}\bar{a}_i^{\beta},\qquad \alpha,\beta=1,2,$$

$$a_{12}^{12}=\det A,\qquad \bar{a}_{12}^{12}=\det\bar{A}.$$

Thus, the Weyl theorem itself implies that an unnormalized unit Racah operator (in the boson realization) must be expressible in terms of these basic invariants. What is surprising, perhaps, is that the basic invariants themselves define a basis of the Lie algebra of $SU(2)*SU(1,1)$.

6. Notes

1. *Algebra of $SU(2)\times SU(2)$ Wigner operators.* In this chapter, we have developed in detail the algebra of unit Racah operators starting from the concept of an $SU(2)\times SU(2)$ Wigner operator. Since the group $SU(2)\times SU(2)$ is SR, one may also develop RW-algebra for the $SU(2)\times SU(2)$ Wigner operators themselves. These are the operators $T^{J_1J_2J}_{\Delta_1\Delta_2M}$ defined by Eq. (4.25). In this note we summarize several of the relations in this algebra and identify the Racah coefficients for the group $SU(2)\times SU(2)$.

Using the defining relation Eq. (4.25) [or, equivalently, Eq. (4.26)], together with the definition, Eq. (4.13), of the $SU(2)\times SU(2)$ Wigner coefficients, we obtain the open product of two $SU(2)\times SU(2)$ Wigner operators:

$$\begin{aligned}T^{a'b'c'}_{\rho'\sigma'\gamma'}T^{abc}_{\rho\sigma\gamma}=&\sum_{a''b''c''}C\left[\begin{pmatrix}a&b\\c&\gamma\end{pmatrix}\begin{pmatrix}a'&b'\\c'&\gamma'\end{pmatrix}\begin{pmatrix}a''&b''\\c''&\gamma+\gamma'\end{pmatrix}\right]\\&\times\overline{\mathbf{W}}^{a\,a'\,a''}_{\rho,\rho',\rho+\rho'}\underline{\mathbf{W}}^{b\,b'\,b''}_{\sigma,\sigma',\sigma+\sigma'}T^{a''\,b''\,c''}_{\rho+\rho',\sigma+\sigma',\gamma+\gamma'}.\end{aligned}\tag{4.145}$$

Using the orthogonality relations (4.14) for the $SU(2)\times SU(2)$ Wigner coefficients, we may also write Eq. (4.145) as a coupling law for two $SU(2)\times SU(2)$ Wigner operators:

$$\begin{aligned}&\sum_{c\gamma c'\gamma'}C\left[\begin{pmatrix}a&b\\c&\gamma\end{pmatrix}\begin{pmatrix}a'&b'\\c'&\gamma'\end{pmatrix}\begin{pmatrix}a''&b''\\c''&\gamma''\end{pmatrix}\right]T^{a'b'c'}_{\rho'\sigma'\gamma'}T^{abc}_{\rho\sigma\gamma}\\&=\overline{\mathbf{W}}^{a\,a'\,a''}_{\rho,\rho',\rho+\rho'}\underline{\mathbf{W}}^{b\,b'\,b''}_{\sigma,\sigma',\sigma+\sigma'}T^{a''\,b''\,c''}_{\rho+\rho',\sigma+\sigma',\gamma''}.\end{aligned}\tag{4.146}$$

This last result allows us to identify the $SU(2)\times SU(2)$ *Racah invariant operators*:

$$\mathbf{W}\left[\begin{pmatrix}a&b\\\rho&\sigma\end{pmatrix}\begin{pmatrix}a'&b'\\\rho'&\sigma'\end{pmatrix}\begin{pmatrix}a''&b''\\\rho''&\sigma''\end{pmatrix}\right]\equiv\overline{\mathbf{W}}^{aa'a''}_{\rho\rho'\rho''}\underline{\mathbf{W}}^{bb'b''}_{\sigma\sigma'\sigma''}.\tag{4.147}$$

The eigenvalue of this operator on a generic basis vector $|(j_1 j_2)jm\rangle$ is thus a product of two $SU(2)$ Racah coefficients:

$$W\left[\begin{pmatrix} a\, b \\ \rho\, \sigma \end{pmatrix}\begin{pmatrix} a'\, b' \\ \rho'\, \sigma' \end{pmatrix}\begin{pmatrix} a''\, b'' \\ \rho''\, \sigma'' \end{pmatrix}\right]|(j_1 j_2)jm\rangle$$
$$= W^{aa'a''}_{\rho\rho'\rho''}(j_1) W^{bb'b''}_{\sigma\sigma'\sigma''}(j_2)|(j_1 j_2)jm\rangle. \tag{4.148}$$

The $SU(2)\times SU(2)$ Wigner operators and Racah invariant operators satisfy, of course, orthogonality relations that are the analogs of Eqs. (2.33), (2.35), and (2.54):

$$\sum_{c\gamma} T^{abc}_{\rho'\sigma'\gamma}\left(T^{abc}_{\rho\sigma\gamma}\right)^{\dagger} = \delta_{\rho'\rho}\delta_{\sigma'\sigma}\mathbf{I}^{a}_{-\rho}\otimes\mathbf{I}^{b}_{-\sigma},$$
$$\sum_{\rho\sigma}\left(T^{abc'}_{\rho\sigma\gamma'}\right)^{\dagger} T^{abc}_{\rho\sigma\gamma} = \delta_{c'c}\delta_{\gamma'\gamma}; \tag{4.149}$$

$$\sum_{\rho\sigma\rho'\sigma'} \mathbf{W}\left[\begin{pmatrix} a\, b \\ \rho\, \sigma \end{pmatrix}\begin{pmatrix} a'\, b' \\ \rho'\, \sigma' \end{pmatrix}\begin{pmatrix} a''\, b'' \\ \rho''\, \sigma'' \end{pmatrix}\right]\mathbf{W}\left[\begin{pmatrix} a\, b \\ \rho\, \sigma \end{pmatrix}\begin{pmatrix} a'\, b' \\ \rho'\, \sigma' \end{pmatrix}\begin{pmatrix} a'''\, b''' \\ \rho'''\, \sigma''' \end{pmatrix}\right]$$
$$= \delta_{a''a'''}\delta_{b''b'''}\delta_{\rho''\rho'''}\delta_{\sigma''\sigma'''}\mathbf{I}^{a''}_{-\rho''}\otimes\mathbf{I}^{b''}_{-\sigma''},$$
$$\sum_{a''b''\rho''\sigma''} \mathbf{W}\left[\begin{pmatrix} a & b \\ \rho_1 & \sigma_1 \end{pmatrix}\begin{pmatrix} a' & b' \\ \rho_1' & \sigma_1' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ \rho'' & \sigma'' \end{pmatrix}\right]\mathbf{W}\left[\begin{pmatrix} a & b \\ \rho_2 & \sigma_2 \end{pmatrix}\begin{pmatrix} a' & b' \\ \rho_2' & \sigma_2' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ \rho'' & \sigma'' \end{pmatrix}\right]$$
$$= \delta_{\rho_1\rho_2}\delta_{\sigma_1\sigma_2}\delta_{\rho_1'\rho_2'}\delta_{\sigma_1'\sigma_2'}\mathbf{I}^{aa'}_{-\rho_1,-\rho_1'}\otimes\mathbf{I}^{bb'}_{-\rho_2,-\rho_2'}. \tag{4.150}$$

The invariant operators $\mathbf{I}^{a}_{\rho}$, etc., appearing in these results are the characteristic functions defined by Eq. (2.55) when acting on the space $\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}$.

Note also that the $SU(2)\times SU(2)$ Racah invariant (4.147) is zero unless $\rho''=\rho+\rho'$ and $\sigma''=\sigma+\sigma'$, so that the orthogonality relations (4.150) may be simplified by removing the $0=0$ cases.

It is apparent from Eqs. (4.145)–(4.150) that the RW-algebra for the SR group $SU(2)\times SU(2)$ parallels closely that for $SU(2)$ itself. The identity implied by the associativity of this algebra is given in Note 2 below.

2. *Algebra of* $SU(2)\times SU(2)$: $SU(2)$ *projective functions.* The 9-j coefficients occur as the reduced matrix elements in the matrix elements of an $SU(2)\times SU(2)$ Wigner operator [see Eqs. (4.26)–(4.29)]. One can give a (bounded) operator formulation of the properties of the 9-j coefficients themselves by associating an operator with the coefficient in the following manner: We define the operator

$$\begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix} \tag{4.151}$$

by giving its action on each basis vector $|j_1 j_2 j\rangle$ (recall that this notation denotes an equivalence class of basis vectors):

$$\begin{bmatrix} a\ b\ c \\ \rho\ \sigma\ \tau \end{bmatrix} |j_1 j_2 j\rangle = [(2j+1)(2c+1)(2j_1+2\rho+1)(2j_2+2\sigma+1)]^{\frac{1}{2}}$$

$$\times \begin{Bmatrix} j_1 & j_2 & j \\ a & b & c \\ j_1+\rho & j_2+\sigma & j+\tau \end{Bmatrix} |j_1+\rho,\, j_2+\sigma,\, j+\tau\rangle. \quad (4.152)$$

The operators (4.151) are thus defined for all nonnegative integers and half-integers (a, b, c) that satisfy the triangle conditions. The labels (ρ, σ, τ) may range independently over the values given by

$$\begin{aligned} \rho &= a, a-1, \ldots, -a, \\ \sigma &= b, b-1, \ldots, -b, \\ \tau &= c, c-1, \ldots, -c. \end{aligned} \quad (4.153)$$

The operators (4.151) are called $SU(2)\times SU(2): SU(2)$ *unit projective functions*, since they are defined [Eq. (4.152)] in terms of the reduced matrix elements of a unit $SU(2)\times SU(2)$ tensor operator that has been reduced with respect to the diagonal $SU(2)$ subgroup.

The conjugate to the operator (4.151) is denoted by

$$\begin{bmatrix} a\ b\ c \\ \rho\ \sigma\ \tau \end{bmatrix}^{\dagger} \quad (4.154)$$

and is defined by the action

$$\begin{bmatrix} a\ b\ c \\ \rho\ \sigma\ \tau \end{bmatrix}^{\dagger} |j_1 j_2 j\rangle = [(2j-2\tau+1)(2c+1)(2j_1+1)(2j_2+1)]^{\frac{1}{2}}$$

$$\times \begin{Bmatrix} j_1-\rho & j_2-\sigma & j-\tau \\ a & b & c \\ j_1 & j_2 & j \end{Bmatrix} |j_1-\rho,\, j_2-\sigma,\, j-\tau\rangle. \quad (4.155)$$

The properties of $SU(2)\times SU(2): SU(2)$ projective functions have not been worked out to the same extent as those given for $SU(2)$ unit tensor operators in Chapters 2 and 3 and for unit Racah operators in the present chapter. We summarize in this Note several of their basic properties, which may be proved directly from the corresponding properties of the 9-j coefficients:

(*a*) *Null space*. The trivial null space is defined to be the space spanned by the basis vectors in the set $\{|j_1 j_2 j\rangle\}\}$ such that the triple $(j_1+\rho,$ $j_2+\sigma, j+\tau)$ contains a negative entry or contains nonnegative integers or half-integers that do *not* satisfy the triangle condition.

The characteristic null space is defined to be the space spanned by the basis vectors in the set $\{|j_1 j_2 j\rangle\}$ such that at least one of the following conditions holds:

$$\begin{aligned} 2j_1+1 &\in \{1,2,\ldots,a-\rho\},\\ 2j_2+1 &\in \{1,2,\ldots,b-\sigma\},\\ 2j+1 &\in \{1,2,\ldots,c-\tau\}. \end{aligned} \tag{4.156}$$

(Note that the null space vectors enumerated above are all due to violations of triangle conditions in the 9-j symbol. Vectors in the null space of a particular operator because of symmetry vanishings of the 9-j symbol have not been enumerated; little is known about accidental zeros, except that they exist.)

(*b*) *Orthogonality relations.*

$$\begin{aligned} \sum_{c\tau} \begin{bmatrix} a & b & c \\ \rho' & \sigma' & \tau \end{bmatrix} \begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix}^{\dagger} &= \delta_{\rho'\rho}\delta_{\sigma'\sigma}\mathbf{I}^{a}_{-\rho}(1)\mathbf{I}^{b}_{-\sigma}(2),\\ \sum_{\rho\sigma} \begin{bmatrix} a & b & c' \\ \rho & \sigma & \tau' \end{bmatrix}^{\dagger} \begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix} &= \delta_{c'c}\delta_{\tau'\tau}\mathbf{I}^{c}_{\tau}(3), \end{aligned} \tag{4.157}$$

in which $\mathbf{I}^{k}_{\lambda}(i)$ $(k=a,b,c,\ldots,\lambda=\rho,\sigma,\tau,\cdots)$ is the characteristic function defined by

$$\mathbf{I}^{k}_{\lambda}(i)|j_1 j_2 j_3\rangle = \varepsilon_{j_i,k,j_i+\lambda}|j_1 j_2 j_3\rangle \tag{4.158}$$

for $i=1,2,3$ $(j_3 \equiv j)$.

(*c*) *Coupling laws.* In stating the coupling laws it is convenient to introduce the bracket coefficient defined by

$$C\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix} \equiv [(2c+1)(2c'+1)(2a''+1)(2b''+1)]^{\frac{1}{2}} \begin{Bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{Bmatrix}. \tag{4.159}$$

These coefficients then satisfy the orthogonality relation [see Eq. (3.321) of AMQP]:[1]

$$\sum_{hi} C\begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} C\begin{bmatrix} a & b & c' \\ d & e & f' \\ h & i & j \end{bmatrix} = \delta_{cc'}\delta_{ff'}. \tag{4.160}$$

[1]In applying the orthogonality relations in this form, it is to be understood that all entries in the 9-j coefficient are taken to satisfy the required triangle conditions. The more general form of these relations may be obtained by taking matrix elements of Eqs. (4.157).

We now state a number of relations that exist between $SU(2)\times SU(2):SU(2)$ projective functions, 9-j coefficients, and Racah invariant operators:
Open product:

$$\begin{bmatrix} a' & b' & c' \\ \rho' & \sigma' & \tau' \end{bmatrix}\begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix}=\sum_{a''b''c''} C\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix}\mathbf{W}^{aa'a''}_{\rho,\rho',\rho+\rho'}(1)\mathbf{W}^{bb'b''}_{\sigma,\sigma',\sigma+\sigma'}(2)$$
$$\times\mathbf{W}^{cc'c''}_{\tau,\tau',\tau+\tau'}(3)\begin{bmatrix} a'' & b'' & c'' \\ \rho+\rho' & \sigma+\sigma' & \tau+\tau' \end{bmatrix}. \tag{4.161}$$

Single Racah invariant coupling:

$$\sum_{\rho\rho'}\mathbf{W}^{aa'a''}_{\rho\rho'\rho''}(1)\begin{bmatrix} a' & b' & c' \\ \rho' & \sigma' & \tau' \end{bmatrix}\begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix}$$
$$=\sum_{b''c''} C\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix}\mathbf{W}^{bb'b''}_{\sigma,\sigma',\sigma+\sigma''}(2)\mathbf{W}^{cc'c''}_{\tau,\tau',\tau+\tau'}(3)\begin{bmatrix} a'' & b'' & c'' \\ \rho'' & \sigma+\sigma' & \tau+\tau' \end{bmatrix}. \tag{4.162}$$

Double Racah invariant coupling:

$$\sum_{\rho\rho'\sigma\sigma'}\mathbf{W}^{aa'a''}_{\rho\rho'\rho''}(1)\mathbf{W}^{bb'b''}_{\sigma\sigma'\sigma''}(2)\begin{bmatrix} a' & b' & c' \\ \rho' & \sigma' & \tau' \end{bmatrix}\begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix}$$
$$=\sum_{c''} C\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix}\mathbf{W}^{cc'c''}_{\tau,\tau',\tau+\tau'}(3)\begin{bmatrix} a'' & b'' & c'' \\ \rho'' & \sigma'' & \tau+\tau' \end{bmatrix}. \tag{4.163}$$

Triple Racah invariant coupling:

$$\sum_{\rho\rho'\sigma\sigma'\tau\tau'}\mathbf{W}^{aa'a''}_{\rho\rho'\rho''}(1)\mathbf{W}^{bb'b''}_{\sigma\sigma'\sigma''}(2)\mathbf{W}^{cc'c''}_{\tau\tau'\tau''}(3)\begin{bmatrix} a' & b' & c' \\ \rho' & \sigma' & \tau' \end{bmatrix}\begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix}$$
$$=C\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix}\begin{bmatrix} a'' & b'' & c'' \\ \rho'' & \sigma'' & \tau'' \end{bmatrix}. \tag{4.164}$$

Triple Racah invariant coupling and 9-j coefficient coupling:

$$\sum_{cc'}\sum_{\rho\rho'\sigma\sigma'\tau\tau'}\mathbf{W}^{aa'a''}_{\rho\rho'\rho''}(1)\mathbf{W}^{bb'b''}_{\sigma\sigma'\sigma''}(2)\mathbf{W}^{cc'c''}_{\tau\tau'\tau''}(3)\; C\begin{bmatrix} a & b & c \\ a' & b' & c' \\ a''' & b''' & c'' \end{bmatrix}\begin{bmatrix} a' & b' & c' \\ \rho' & \sigma' & \tau' \end{bmatrix}\begin{bmatrix} a & b & c \\ \rho & \sigma & \tau \end{bmatrix}$$
$$=\delta_{a''a'''}\delta_{b''b'''}\begin{bmatrix} a'' & b'' & c'' \\ \rho'' & \sigma'' & \tau'' \end{bmatrix}. \tag{4.165}$$

The Racah invariant operator $\mathbf{W}_{\cdots}^{\cdots}(i)$ appearing in these results is defined by

$$\mathbf{W}_{\rho\sigma\tau}^{abc}(i)|j_1j_2j_3\rangle=W_{\rho\sigma\tau}^{abc}(j_i)|j_1j_2j_3\rangle \tag{4.166}$$

for $i=1,2,3$.

All the above relations, Eqs. (4.161)–(4.165), follow from the first one by using the orthogonality relations for the Racah invariants [see Eqs. (2.54)]. The first relation, Eq. (4.161), may be proved from the product law (4.145) for $SU(2)\times SU(2)$ Wigner operators.

Further relations may be obtained by using the orthogonality relation (4.160) to move the bracket coefficient to the left-hand side in Eqs. (4.161)–(4.164).

The coupling structure of the above equations is very clearly expressed by using the symbolism of Chapter 2, Section 5. Two operator patterns are coupled by a Racah invariant operator to obtain a third operator pattern; for example,

$$\begin{bmatrix} b \\ \bullet \end{bmatrix} \overset{c}{\blacklozenge} \begin{bmatrix} a \\ \bullet \end{bmatrix} = \begin{bmatrix} c \\ \bullet \end{bmatrix};$$

two triangles are coupled by a numerical bracket coefficient to obtain a third triangle; for example,

$$[a'b'\ \bullet\,]\underset{c''}{\blacklozenge}[ab\ \bullet\,]=[\,\bullet\ \ \bullet\ c''].$$

If we take matrix elements of Eqs. (4.161)–(4.165), we recover corresponding relations between 6-j and 9-j symbols. The present operator interpretation of these relations between coefficients illustrates their structural significance in terms of the algebra of projective functions.

For completeness we also give the identity between 9-j and 6-j coefficients that is implied by the associative law for the multiplication of three projective functions. The derivation is a straightforward application of the multiplication rule (4.161), using also the orthogonality of the Racah invariants and the B–E identity, Eq. (2.68), and leads to the relation:

$$\begin{aligned}
&\sum_{xyz}(2x+1)(2y+1)(2z+1)\begin{Bmatrix} a & b & c \\ d & e & f \\ x & y & z \end{Bmatrix}\begin{Bmatrix} x & y & z \\ a' & b' & c' \\ d' & e' & f' \end{Bmatrix} \\
&\qquad\times\begin{Bmatrix} a & d & x \\ a' & d' & x' \end{Bmatrix}\begin{Bmatrix} b & e & y \\ b' & e' & y' \end{Bmatrix}\begin{Bmatrix} c & f & z \\ c' & f' & z' \end{Bmatrix} \\
&\quad=\begin{Bmatrix} a & b & c \\ d' & e' & f' \\ x' & y' & z' \end{Bmatrix}\begin{Bmatrix} x' & y' & z' \\ a' & b' & c' \\ d & e & f \end{Bmatrix}.
\end{aligned} \tag{4.167}$$

3. *Extended definition of unit Racah operators.* In the Note to Chapter 3, we extended the definition of an $SU(2)$ Wigner operator to a $U(2)$ Wigner operator. It is convenient to make the analogous extension of a unit Racah operator, thus including all $U(2)$ operator patterns in the definition.

A generic Racah operator is now denoted by

$$\left\{\begin{matrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & \Gamma'_{11} & \end{matrix}\right\}, \tag{4.168}$$

where the entries in these patterns are integers that satisfy the betweenness conditions:

$$\begin{aligned} M_{12} &\geqslant \Gamma_{11} \geqslant M_{22}, \\ M_{12} &\geqslant \Gamma'_{11} \geqslant M_{22}. \end{aligned}$$

The action of the operator (4.168) on the basis vector $|(m)\rangle$ is defined by

$$\begin{aligned} &\left\{\begin{matrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & \Gamma'_{11} & \end{matrix}\right\}\left|\begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix}\right\rangle \\ &\quad = \left[(2j_1+2\Delta+1)(2j_2+1)\right]^{\frac{1}{2}} W(j, j_1, j_2+\Delta', J; j_2, j_1+\Delta) \\ &\quad\quad \times \left|\begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta'_1 & & m_{21}+\Delta'_2 \end{matrix}\right\rangle, \end{aligned} \tag{4.169}$$

where

$$\begin{aligned} &J = (M_{12}-M_{22})/2, \\ &\Delta_1 = \Gamma_{11}, \qquad \Delta_2 = M_{12}+M_{22}-\Gamma_{11}, \\ &\Delta'_1 = \Gamma'_{11}, \qquad \Delta'_2 = M_{12}+M_{22}-\Gamma'_{11}. \end{aligned} \tag{4.170}$$

The labels j_1, j_2, j, Δ, and Δ' are defined in terms of the (m_{ij}), J, Δ_1, Δ_2, Δ'_1, and Δ'_2 by Eqs. (4.45) and (4.47).

It now follows from Eqs. (4.46) and (4.48) that we have the following equivalence between unit Racah operators:

$$\left\{\begin{matrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & \Gamma'_{11} & \end{matrix}\right\} \sim \left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\}, \tag{4.171}$$

in which

$$J=(M_{12}-M_{22})/2.$$
$$\Delta=\Gamma_{11}-(M_{12}+M_{22})/2, \qquad \Delta'=\Gamma'_{11}-(M_{12}+M_{22})/2.$$

In particular, it is often convenient to use the equivalence given by

$$\left\{ J \begin{matrix} \Delta \\ \\ \Delta' \end{matrix} -J \right\} \sim \left\{ 2J \begin{matrix} J+\Delta \\ \\ J+\Delta' \end{matrix} 0 \right\}. \tag{4.172}$$

7. Appendices

A. Canonical Form of the Racah Coefficients

The purpose of this appendix is to show that a Racah coefficient can be written in the canonical form

$$\#(\text{NPCF})(\text{polynomial})/DD', \tag{A.1}$$

where (a) NPCF denotes the numerator pattern calculus factor; (b) the polynomial is defined on the m_{ij} as variables; (c) the denominator factors D and D' are, respectively, polynomials in $m_{12}-m_{22}$ and $m_{11}-m_{21}$; and (d) # denotes a numerical factor independent of the m_{ij}.

The method of proof is by direct transformation of the Racah coefficient $W(abcd;ef)$, as given by [see Eq. (3.292) of AMQP]

$$\begin{aligned} W(abcd;ef) &= \Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf) \\ &\times \sum_z \frac{(-1)^{a+b+c+d+z}(z+1)!}{(z-a-b-e)!(z-c-d-e)!(z-a-c-f)!(z-b-d-f)!} \\ &\times \frac{1}{(a+b+c+d-z)!(a+d+e+f-z)!(b+c+e+f-z)!} \\ &= \Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf)w(abcd;ef), \end{aligned} \tag{A.2}$$

to the new variables defined by Eqs. (4.97). We indicate below some of the steps required for bringing Eq. (A.2) to the form (A.1).

The triangle factors take the following forms:

$$\Delta(abe)\Delta(acf)$$

$$=\left[\frac{(p'_{11}-p'_{22})!(p_{12}-p_{11}-1)!(p_{12}-p_{21}-1)!(p_{22}-p_{21}-1)!}{(p_{11}-p_{22})!(p'_{12}-p'_{11}-1)!(p'_{12}-p'_{21}-1)!(p'_{22}-p'_{21}-1)!}\right]^{\frac{1}{2}}$$

$$\times\frac{(p_{11}-p_{22})!(p'_{12}-p'_{11}-1)!(p'_{22}-p'_{21}-1)!}{(p_{12}-p_{21}-1)!},$$

$$(p'_{12}-p'_{22}+1)^{\frac{1}{2}}\Delta(bdf)=\frac{[(\Delta_1)!(\Delta_2)!]^{\frac{1}{2}}}{D_{[\Delta_1\Delta_2]}(p_{12},p_{22})},$$

$$(p_{11}-p_{21}+1)^{\frac{1}{2}}\Delta(cde)=\frac{[(\Delta'_1)!(\Delta'_2)]^{\frac{1}{2}}}{D'_{[\Delta'_1\Delta'_2]}(p_{11},p_{21})}. \tag{A.3}$$

The summation part of $W(abcd;ef)$, denoted $w(abcd;ef)$, when multiplied by the factor that has been separated off in the first of relations (A.3), yields a polynomial:

$$\frac{(p_{11}-p_{22})!(p'_{12}-p'_{11}-1)!(p'_{22}-p'_{21}-1)!}{(p_{12}-p_{21}-1)!}w(abcd;ef)$$

$$=(-1)^{\Delta_1}Q(\Delta;p)$$

$$\equiv(-1)^{\Delta_1}Q(\Delta_1,\Delta_2,\Delta'_1,\Delta'_2;p_{12},p_{22},p_{11},p_{21})$$

$$\equiv(-1)^{\Delta_1}\sum_{k_1+k_2=\Delta'_2}\binom{p'_{11}-p'_{12}+k_1}{k_1}\binom{p_{11}-p_{22}}{k_2}\binom{p_{21}-p_{12}}{\Delta_1-k_1}$$

$$\times\binom{p'_{21}-p'_{22}+\Delta_2-k_2}{\Delta_2-k_2}. \tag{A.4}$$

In these relations we have defined $p_{ij}=m_{ij}+j-i$; $p'_{ij}=m'_{ij}+j-i$, $m'_{12}=m_{12}+\Delta_1$, $m'_{22}=m_{22}+\Delta_2$, $m'_{11}=m_{11}+\Delta'_1$, $m'_{21}=m_{21}+\Delta'_2$; the denominators D and D' are given by

$$D_{[\Delta_1\Delta_2]}(p_{12},p_{22})=\left[|(p_{12}-p_{22}-\Delta_2)_{\Delta_1}(p_{22}-p_{12}-\Delta_1)_{\Delta_2}|\right]^{\frac{1}{2}}, \tag{A.5}$$

$$D'_{[\Delta'_1\Delta'_2]}(p_{11},p_{21})=\left[|(p_{11}-p_{21}+1)_{\Delta'_1}(p_{21}-p_{11}+1)_{\Delta'_2}|\right]^{\frac{1}{2}}. \tag{A.6}$$

Combining Eqs. (A.2)–(A.6), we now obtain the Racah coefficient in the form

$$\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta_1' & & m_{21}+\Delta_2' \end{matrix} \middle| \left\{ 2J \begin{matrix} J+\Delta \\ \\ J+\Delta' \end{matrix} 0 \right\} \middle| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle$$

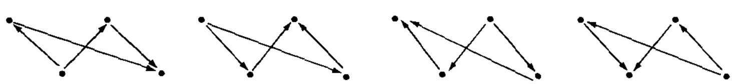
$$= \left[(\Delta_1)!(\Delta_2)!(\Delta_1')!(\Delta_2')! \right]^{\frac{1}{2}} / D_{[\Delta_1\Delta_2]}(p_{12}, p_{22}) D'_{[\Delta_1'\Delta_2']}(p_{11}, p_{21})$$

$$\times \left[\frac{(p_{11}'-p_{22}')!(p_{12}-p_{11}-1)!(p_{12}-p_{21}-1)!(p_{22}-p_{21}-1)!}{(p_{11}-p_{22})!(p_{12}'-p_{11}'-1)!(p_{12}'-p_{21}'-1)!(p_{22}'-p_{21}'-1)!} \right]^{\frac{1}{2}}$$

$$\times (-1)^{\Delta_1} Q(\Delta; p). \quad \text{(A.7)}$$

Equation (A.7) is still not in optimal form, since the polynomial $Q(\Delta; p)$ is not irreducible; this polynomial factorizes into a product of linear factors multiplied by a new polynomial. We wish to remove all such linear factors. The appropriate technique for carrying out this analysis is the pattern calculus.

Depending on the relative magnitudes of the Δ_i and Δ_i', there are four arrow patterns that can be drawn for the arrows going between row 2 (top row) and row 1 (bottom row). The possible cases are illustrated below (a single arrow represents zero, one, or more arrows):



Case 1 Case 2 Case 3 Case 4

It is convenient to introduce the eight nonnegative integers defined by

$$u_{ij} = \max(0, \Delta_j - \Delta_i'), \qquad l_{ij} = \max(0, \Delta_i' - \Delta_j), \qquad i, j = 1, 2. \quad \text{(A.8)}$$

The meaning of the integers u_{ij} and l_{ij} in a given arrow pattern is as follows: u_{ij} is the number of arrows going from dot $(j2)$ in row 2 to dot $(i1)$ in row 1; l_{ij} is the number of arrows going from dot $(i1)$ in row 1 to dot $(j2)$ in row 2.

These numbers have the following properties:

$l_{ij} + u_{ij} = |\Delta_i' - \Delta_j| =$ number of arrows going between dot $(i1)$ in row 1 and dot $(j2)$ in row 2;

$$l_{ij} + \Delta_j = u_{ij} + \Delta_i'; \quad \text{(A.9)}$$

$\sum_{ij} u_{ij} = \sum_{ij} l_{ij}$ (the number of downward-going arrows equals the number of upward-going arrows).

The values of the u_{ij} for the four types of arrow patterns are given explicitly by

$$
\begin{array}{llll}
\text{Case 1: } u_{11}=0, & u_{12}=0, & u_{21}=\Delta_1-\Delta'_2, & u_{22}=\Delta_2-\Delta'_2; \\
\text{Case 2: } u_{11}=\Delta_1-\Delta'_1, & u_{12}=0, & u_{21}=\Delta_1-\Delta'_2, & u_{22}=0; \\
\text{Case 3: } u_{11}=0, & u_{12}=\Delta_2-\Delta'_1, & u_{21}=0, & u_{22}=\Delta_2-\Delta'_2; \\
\text{Case 4: } u_{11}=\Delta_1-\Delta'_1, & u_{12}=\Delta_2-\Delta'_1, & u_{21}=0, & u_{22}=0.
\end{array}
\tag{A.10}
$$

One now finds that the square-root factor in brackets preceding the polynomial Q in Eq. (A.7) may be written as

$$
(\mathrm{NPCF})\times(-1)^{u_{11}+u_{21}+u_{22}} \Big/ \prod_{i,j=1}^{2} (u_{ij})! \binom{p_{i1}-p_{j2}}{u_{ij}}, \tag{A.11}
$$

where

$$
\mathrm{NPCF}=\left[\left|\prod_{i,j=1}^{2} (p_{i1}-p_{j2}-u_{ij}+1)_{u_{ij}}(p_{i1}-p_{j2}+1)_{l_{ij}}\right|\right]^{\frac{1}{2}} \tag{A.12}
$$

denotes the numerator pattern calculus factor. The phase factor $(-1)^{u_{11}+u_{21}+u_{22}}$ is required so that expression (A.11) is nonnegative for all lexical (m_{ij}).

Using this result in Eq. (A.7), we obtain

$$
\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta'_1 & & m_{21}+\Delta'_2 \end{matrix} \middle| \left\{ 2J \begin{matrix} J+\Delta \\ \quad 0 \\ J+\Delta' \end{matrix} \right\} \middle| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle
$$
$$
=\#(\mathrm{NPCF})D^{-1}D'^{-1}P_k(\Delta;p), \tag{A.13}
$$

where

$$
\begin{aligned}
&(a)\ D=D_{[\Delta_1\Delta_2]}(p_{12},p_{22}) \quad \text{and} \quad D'=D'_{[\Delta'_1\Delta'_2]}(p_{11},p_{21});\\
&(b)\ \#\ \text{is given by}\ \frac{(-1)^{l_{11}}k!(2J-k)!}{[(\Delta_1)!(\Delta_2)!(\Delta'_1)!(\Delta'_2)!]^{\frac{1}{2}}};
\end{aligned}
\tag{A.14}
$$

(c) k is given by $\min(\Delta_1,\Delta_2,\Delta'_1,\Delta'_2)$;

(d) $P_k(\Delta;p)$ is a polynomial of degree k in the (p_{ij}) defined as

$$
P_k(\Delta;p)\equiv\frac{(-1)^k(\Delta_1)!(\Delta_2)!(\Delta'_1)!(\Delta'_2)!}{k!(2J-k)!} \times Q(\Delta;p)\Big/\prod_{i,j=1}^{2}(u_{ij})!\binom{p_{i1}-p_{j2}}{u_{ij}}. \tag{A.15}
$$

The numerical factor in Eq. (A.15) has been included in the definition of $P_k(\Delta; p)$ so that

$$P_0(\Delta; p)=1; \tag{A.16}$$

that is, *the polynomial part of Eq.* (A.13) *is unity for any extremal pattern.* This property also simplifies the form of the symbolic expressions developed in Appendix D.

We have yet to demonstrate that $P_k(\Delta; p)$ is a polynomial. To show that the denominator divides the numerator in Eq. (A.15), we examine each of the four cases [corresponding to Eqs. (A.10)] and find the following:
Case 1:

$$\begin{aligned}P_{\Delta_2'}(\Delta; p)=(-1)^{\Delta_2'} \sum_{k_1+k_2=\Delta_2'} \binom{\Delta_1}{k_1}\binom{\Delta_2}{k_2}(p_{11}-p_{12}+\Delta_1'-\Delta_1+1)_{k_1}\\ \times(p_{11}-p_{22}-k_2+1)_{k_2}(p_{21}-p_{12}-\Delta_1+k_1+1)_{k_2}(p_{21}-p_{22}+1)_{k_1};\end{aligned} \tag{A.17}$$

Case 2:

$$\begin{aligned}P_{\Delta_2}(\Delta; p)=(-1)^{\Delta_2} \sum_{k_1+k_2=\Delta_2} \binom{\Delta_1'}{k_1}\binom{\Delta_2'}{k_2}(p_{11}-p_{12}+1)_{k_1}\\ \times(p_{11}-p_{22}-k_2+1)_{k_2}(p_{21}-p_{12}-\Delta_1'+k_1+1)_{k_2}(p_{21}-p_{22}+\Delta_2'-\Delta_2+1)_{k_1};\end{aligned} \tag{A.18}$$

Case 3:

$$\begin{aligned}P_{\Delta_1}(\Delta; p)=(-1)^{\Delta_1} \sum_{k_1+k_2=\Delta_1} \binom{\Delta_2'}{k_1}\binom{\Delta_1'}{k_2}(p_{11}-p_{12}+\Delta_1'-\Delta_1+1)_{k_1}\\ \times(p_{11}-p_{22}-\Delta_2'+k_1+1)_{k_2}(p_{21}-p_{12}-k_2+1)_{k_2}(p_{21}-p_{22}+1)_{k_1};\end{aligned} \tag{A.19}$$

Case 4:

$$\begin{aligned}P_{\Delta_1'}(\Delta; p)=(-1)^{\Delta_1'} \sum_{k_1+k_2=\Delta_1'} \binom{\Delta_2}{k_1}\binom{\Delta_1}{k_2}(p_{11}-p_{12}+1)_{k_1}\\ \times(p_{11}-p_{22}-\Delta_2+k_1+1)_{k_2}(p_{21}-p_{12}-k_2+1)_{k_2}(p_{21}-p_{22}+\Delta_2'-\Delta_2+1)_{k_1}.\end{aligned} \tag{A.20}$$

[Because the pattern calculus rules utilize rising factorials, we have chosen to express these results in terms of the rising factorial notation: $(x)_a = x(x+1)\ldots(x+a-1)$.]

The result expressed by Eq. (A.13), with the polynomials given by Eqs. (A.17)–(A.20), accomplishes our goal of writing a Racah coefficient in a form appropriate to the discussion of null space properties given in Section 4 and Appendices B and C.

Before concluding this Appendix, there are several features of the polynomials (A.17)–(A.20) which we note.

(a) The polynomials (A.17)–(A.20) may all be obtained from (A.17) by interchanging the Δ's and p_{ij} in the following manner:

$$\begin{array}{llll}
\text{Case 2: } \Delta_1 \leftrightarrow \Delta_1', & \Delta_2 \leftrightarrow \Delta_2', & p_{11} \leftrightarrow -p_{22}, & p_{21} \leftrightarrow -p_{12}; \\
\text{Case 3: } \Delta_1 \leftrightarrow \Delta_2', & \Delta_1' \leftrightarrow \Delta_2, & p_{11} \leftrightarrow -p_{12}, & p_{21} \leftrightarrow -p_{22}; \\
\text{Case 4: } \Delta_1' \leftrightarrow \Delta_2', & \Delta_1 \leftrightarrow \Delta_2, & p_{11} \leftrightarrow p_{21}, & p_{12} \leftrightarrow p_{22}.
\end{array} \tag{A.21}$$

(These interchanges of the Δ's are the operations that carry the arrow pattern for Case 1 into the arrow patterns for the respective Cases 2–4.)

(b) Equation (A.13) is in a form that allows an easy proof of the limiting of a Racah coefficient to a Wigner coefficient for $m_{21} \to -\infty$ [see Eqs. (4.99) and (4.100)].

B. Proof of the Reflection Symmetry Property of the P_k

In this Appendix we prove the symmetry property (4.102) of the polynomial P_k. We call this a reflection symmetry, since, in terms of Δ, Δ', $j_1=(m_{12}-m_{22})/2$, and $j_2=(m_{11}-m_{21})/2$, it is expressed as $\Delta \leftrightarrow -\Delta$, $j_1+\frac{1}{2} \leftrightarrow -j_1-\frac{1}{2}$; and $\Delta' \leftrightarrow -\Delta'$, $j_2+\frac{1}{2} \leftrightarrow -j_2-\frac{1}{2}$. Our method of proof is indirect and uses recursion relations for the Racah coefficients.

Consider Eq. (4.77) particularized to the case $a=J-\frac{1}{2}$, $b=\frac{1}{2}$, $c=J$:

$$\sum_{\rho} \overline{\mathbf{W}}^{J-\frac{1}{2}\,\frac{1}{2}\,J}_{\Delta-\rho,\,\rho,\,\Delta} \begin{Bmatrix} & \rho+\frac{1}{2} & \\ 1 & & 0 \\ & \alpha+\frac{1}{2} & \end{Bmatrix} \begin{Bmatrix} & J+\Delta-\rho-\frac{1}{2} & \\ 2J-1 & & 0 \\ & J+\Delta'-\alpha-\frac{1}{2} & \end{Bmatrix}$$
$$= \underline{\mathbf{W}}^{J-\frac{1}{2}\,\frac{1}{2}\,J}_{\Delta'-\alpha,\,\alpha,\,\Delta'} \begin{Bmatrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{Bmatrix}. \tag{B.1}$$

With this equation we take the following steps:

(a) Take matrix elements between states:

$$\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 & \\ & m_{11}+\Delta_1' & & m_{21}+\Delta_2' \end{matrix} \right| \cdots \left| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle;$$

(b) Use Eq. (A.7) to substitute for the generic matrix elements appearing in Eq. (B.1) as a result of (a), and substitute in this result the spin-$\frac{1}{2}$ Racah coefficients.

Corresponding to $\alpha=\frac{1}{2}$, and $\alpha=-\frac{1}{2}$, we obtain (after technically straightforward, but tedious algebra) the following two recursion relations, satisfied by the polynomials $Q(\Delta;\, p)$:[1]

$$\begin{aligned}\Delta_1'(p_{12}-p_{22}+\Delta_1-\Delta_2)&Q(\Delta_1,\Delta_2,\Delta_1',\Delta_2';\,p)\\ &=(p_{12}-p_{22}+\Delta_1)(p_{21}-p_{12}+\Delta_2'-\Delta_1+1)Q(\Delta_1-1,\Delta_2,\Delta_1'-1,\Delta_2';\,p)\\ &\quad+(p_{12}-p_{22}-\Delta_2)(p_{21}-p_{22}+\Delta_2'-\Delta_2+1)Q(\Delta_1,\Delta_2-1,\Delta_1'-1,\Delta_2';\,p),\end{aligned} \tag{B.2}$$

$$\begin{aligned}\Delta_2'(p_{12}-p_{22}+\Delta_1-\Delta_2)&Q(\Delta_1,\Delta_2,\Delta_1',\Delta_2';\,p)\\ &=(p_{12}-p_{22}+\Delta_1)(p_{11}-p_{12}+\Delta_1'-\Delta_1+1)Q(\Delta_1-1,\Delta_2,\Delta_1',\Delta_2'-1;\,p)\\ &\quad+(p_{12}-p_{22}-\Delta_2)(p_{11}-p_{22}+\Delta_1'-\Delta_2+1)Q(\Delta_1,\Delta_2-1,\Delta_1',\Delta_2'-1;\,p).\end{aligned} \tag{B.3}$$

The initial data for the iteration of Eqs. (B.2) and (B.3) is

$$\begin{aligned} Q(0,0,0,0;\,p)&=1, &&\text{all } (p),\\ Q(\Delta;\,p)&=0, &&\text{if any } \Delta\text{-entry is negative.}\end{aligned} \tag{B.4}$$

One then finds: $Q(\Delta;\, p)$ is *uniquely* determined by the recursion relations (B.2) and (B.3) and the initial data (B.4). [Despite the slight difficulty with three-term recursion relations pointed out in Chapter 3, Section 18, AMQP, Eqs. (B.2) and (B.3) for the polynomial part of a Racah coefficient are valid, as explicit calculation shows.]

Let us illustrate how the explicit iteration proceeds: We first derive

$$Q(\Delta_1,\Delta_2,\Delta_1',0;\,p)=\binom{p_{21}-p_{12}}{\Delta_1}\binom{p_{21}-p_{22}}{\Delta_2} \tag{B.5}$$

from Eqs. (B.2) and (B.4). This polynomial now becomes the initial data for the iteration of Eq. (B.3). The iteration of Eq. (B.3) may be carried out, but, unless one is careful to divide out canceling factors at each step of the iteration (a difficult task to carry through), one will not obtain a polynomial form.

[1] It is significant to note here that we made no use of the fact that $Q(\Delta;\, p)$ is a polynomial in deriving Eqs. (B.2)–(B.4), and a more detailed analysis of these relations may be given to show that $Q(\Delta;\, p)$ must be a polynomial (Louck and Biedenharn [7]), thus proving the validity of the structural form (A.7) and, hence, (A.13) without appealing to the known answer.

Our goal in this Appendix is somewhat easier—we wish only to demonstrate two symmetry properties. We give the proof as follows:

(a) The substitution $\Delta_1 \leftrightarrow \Delta_2$, $p_{12} \leftrightarrow p_{22}$ transforms the recursion relations (B.2) and (B.3) into *identical relations* now satisfied by $Q(\Delta_2, \Delta_1, \Delta_1', \Delta_2'; p_{22}, p_{12}, p_{11}, p_{21})$, and the uniqueness of the polynomials satisfying these relations (and the same initial data) proves that

$$Q(\Delta_2, \Delta_1, \Delta_1', \Delta_2'; p_{22}, p_{12}, p_{11}, p_{21}) = Q(\Delta_1, \Delta_2, \Delta_1', \Delta_2'; p_{12}, p_{22}, p_{11}, p_{21}) \tag{B.6}$$

(b) The substitution $\Delta_1' \leftrightarrow \Delta_2'$ and $p_{11} \leftrightarrow p_{21}$ transforms the pair of equations (B.2) and (B.3) into identical equations now satisfied by $Q(\Delta_1, \Delta_2, \Delta_2', \Delta_1'; p_{12}, p_{22}, p_{21}, p_{11})$, hence showing by the argument above that

$$Q(\Delta_1, \Delta_2, \Delta_2', \Delta_1'; p_{12}, p_{22}, p_{21}, p_{11}) = Q(\Delta_1, \Delta_2, \Delta_1', \Delta_2'; p_{12}, p_{22}, p_{11}, p_{21}) \tag{B.7}$$

(c) Using the definition, Eq. (A.15), giving $P_k(\Delta; p)$ in terms of $Q(\Delta; p)$, it follows that P_k also has the symmetries expressed by Eqs. (B.6) and (B.7). (One must exercise considerable care in making this deduction because of the changes in the u_{ij} under permutations of the $\Delta_1, \Delta_2, \Delta_1', \Delta_2'$.)

C. Determination of P_k from Its Zeros

The purpose of this Appendix is to prove, by construction, that *up to a multiplicative factor independent of the variables* $z_{ij} = p_{i1} - p_{j2}$ $(i, j = 1, 2)$ *the polynomial* P_k [*see Eqs.* (A.17)–(A.20)] *is the unique polynomial of smallest degree in the variables* z_{ij} *that possesses the zeros in the point set* Z'' (*see p.* ~~120) *and the reflection symmetry* [*see Eq.* (4.102)].~~

Our proof of this result is quite detailed and lengthy. In our view, however, the fact that one can give the construction of the polynomial part of a Racah coefficient from the minimal degree polynomial possessing a symmetry and the set of zeros implied by the null space is significant not only for angular momentum theory but also for generalizations to $U(n)$. It is therefore essential that one give the proof in some detail.

Zeros of the polynomial P_k. The first task is to give a more explicit and systematic enumeration of the set of zeros of $P_k(\Delta; p)$ that are implied by the null space. This is already a nontrivial task.

From the results of Section 4 and the fact that NPCF vanishes only if the final Gel'fand pattern in (4.103) fails to satisfy the betweenness conditions, we find that the polynomial $P_k(\Delta; p)$ has the set of null space zeros consisting of all points $\{(m_{12}, m_{22}, m_{11}, m_{21})\}$ such that the following conditions are satisfied:

(a) The initial and final Gel'fand patterns in Eq. (4.98) satisfy betweenness; that is,

$$\begin{gathered} m_{12} \geqslant m_{11} \geqslant m_{22} \geqslant m_{21}, \\ m_{12}+\Delta_1 \geqslant m_{11}+\Delta_1' \geqslant m_{22}+\Delta_2 \geqslant m_{21}+\Delta_2'; \end{gathered} \tag{C.1}$$

(b) either

$$m_{12}-m_{22} \in \{0,1,\ldots,\Delta_2-1\}, \tag{C.2}$$

or

$$m_{11}-m_{21} \in \{0,1,\ldots,\Delta_2'-1\}, \tag{C.3}$$

or both. [The point set defined by these requirements clearly has an infinite number of points, since the point $(m_{12}+a, m_{22}+a, m_{11}+a, m_{21}+a)$, $a=$an arbitrary integer, satisfies the conditions if $(m_{12}, m_{22}, m_{11}, m_{21})$ does. For this reason the polynomial $P_k(\Delta; p)$ can depend only on the differences of the m_{ij}.]

Let us first consider that $\Delta_1' \geqslant \Delta_1 \geqslant \Delta_2' \geqslant 1$ and $\Delta_1' \geqslant \Delta_2 \geqslant \Delta_2' \geqslant 1$ (Case 1 of Appendix A). One then verifies that Eqs. (C.1) and (C.2) imply the following relations:

$$\begin{aligned} m_{11}-m_{22}-\Delta_2+1 \leqslant m_{11}-m_{12} &\leqslant -(\Delta_1'-\Delta_1), \\ 0 \leqslant m_{11}-m_{22} &\leqslant \Delta_2'-1, \\ m_{21}-m_{12} &\leqslant -(m_{12}-m_{22}). \end{aligned} \tag{C.4}$$

Conversely, Eqs. (C.4) imply Eqs. (C.1) and (C.2). Similarly, one finds that Eqs. (C.1) and (C.3) imply the following relations, and conversely:

$$\begin{aligned} m_{11}-m_{22}-\Delta_2'+1 \leqslant m_{21}-m_{22} &\leqslant 0, \\ m_{21}-m_{12} &\leqslant -\Delta_2+1, \\ m_{11}-m_{22} &\geqslant 0. \end{aligned} \tag{C.5}$$

(Recall that $\Delta_1+\Delta_2=\Delta_1'+\Delta_2'$.)

We next introduce the variables z_{ij} defined by

$$z_{ij} \equiv p_{i1}-p_{j2}=m_{i1}-m_{j2}+j-i-1, \tag{C.6}$$

and denote by z a point defined by these four variables:

$$z \equiv (z_{11}, z_{12}, z_{21}, z_{22}). \tag{C.7}$$

Note that these variables are not independent, since

$$z_{11}+z_{22}=z_{12}+z_{21}. \tag{C.8}$$

The point sets defined by the conditions (C.4) and (C.5) may be expressed in the notation (C.7) as

$$\left\{\begin{aligned}(a,b,x,x+b-a)\colon & x\leqslant -(p_{12}-p_{22}+1);\\ & b=0,1,\ldots,\Delta_2'-1;\\ & a=-(\Delta_1'-\Delta_1)-1,-(\Delta_1'-\Delta_1)-2,\ldots,-\Delta_2+b\end{aligned}\right\}; \tag{C.9}$$

$$\left\{\begin{aligned}(x+b-d,b,x,d)\colon & x\leqslant -\Delta_2-1;\\ & b=0,1,\ldots,\Delta_2'-1;\\ & d=-1,-2,\ldots,-\Delta_2'+b\end{aligned}\right\}. \tag{C.10}$$

Let us now denote $P_k(\Delta;\, p)$ by the notation $P_k'(\Delta;\, z)$. Then $P_k'(\Delta;\, z)$ must vanish on the point sets (C.9) and (C.10). Observe that the only restriction on x is that it be any negative integer less than some fixed negative integer. Since there are an infinite number of such x, and since $P_k'(\Delta;\, z)$ is of finite degree, it follows that $P_k'(\Delta;\, z)$ must vanish identically in x for each point z in the sets (C.9) and (C.10).

We can infer further sets of zeros of $P_k'(\Delta;\, z)$ by using the symmetry relations, Eq. (4.102), which now takes the forms

$$\begin{aligned}P_k'(\Delta_1,\Delta_2,\Delta_1',\Delta_2';\, z_{11},z_{12},z_{21},z_{22})&=P_k'(\Delta_2,\Delta_1,\Delta_1',\Delta_2';\, z_{12},z_{11},z_{22},z_{21}),\\ P_k'(\Delta_1,\Delta_2,\Delta_1',\Delta_2';\, z_{11},z_{12},z_{21},z_{22})&=P_k'(\Delta_1,\Delta_2,\Delta_2',\Delta_1';\, z_{21},z_{22},z_{11},z_{12}).\end{aligned} \tag{C.11}$$

However, before one can apply these relations, it is necessary to obtain the point sets (C.9) and (C.10) for the general case, since application of the symmetries (C.11) carries one of the cases (Cases 1–4 of Appendix A) to another. We repeat the arguments leading to (C.9) and (C.10) for each of the cases and find that the generic cases can be expressed in terms of the step functions u_{ij} and l_{ij} [see the point sets (C.12) and (C.13) below].

Using each of the symmetries (C.11) on each of the point sets (C.12) and (C.13) below, we obtain the result: *The null space vanishings of a Racah operator and the symmetries* (C.11) *imply that the polynomial* $P_k'(\Delta;\, z)$ $(k\geqslant 1)$ *vanishes on all points in the following sets* (x is an arbitrary integer in these sets):

$$\left\{\begin{array}{l}(a, b, x, x+b-a): b=u_{12}, u_{12}+1, \ldots, \Delta_2-l_{11}-1 \\ \qquad\qquad\qquad a=-l_{11}-1,-l_{11}-2, \ldots,-\Delta_2+b\end{array}\right\} \tag{C.12}$$

$$\left\{\begin{array}{l}(x+b-d, b, x, d): b=u_{12}, u_{12}+1, \ldots, \Delta_2'-l_{22}-1 \\ \qquad\qquad\qquad d=-l_{22}-1,-l_{22}-2, \ldots,-\Delta_2'+b\end{array}\right\} \tag{C.13}$$

$$\left\{\begin{array}{l}(a, b, x+a-b, x): a=u_{11}, u_{11}+1, \ldots, \Delta_1-l_{12}-1 \\ \qquad\qquad\qquad b=-l_{12}-1,-l_{12}-2, \ldots,-\Delta_1+a\end{array}\right\} \tag{C.14}$$

$$\left\{\begin{array}{l}(a, x+a-c, c, x): a=u_{11}, u_{11}+1, \ldots, \Delta_2'-l_{21}-1 \\ \qquad\qquad\qquad c=-l_{21}-1,-l_{21}-2, \ldots,-\Delta_2'+a\end{array}\right\} \tag{C.15}$$

$$\left\{\begin{array}{l}(x, x+d-c, c, d): d=u_{22}, u_{22}+1, \ldots, \Delta_2-l_{21}-1 \\ \qquad\qquad\qquad c=-l_{21}-1,-l_{21}-2, \ldots,-\Delta_2+d\end{array}\right\} \tag{C.16}$$

$$\left\{\begin{array}{l}(x, b, x+d-b, d): d=u_{22}, u_{22}+1, \ldots, \Delta_1'-l_{12}-1 \\ \qquad\qquad\qquad b=-l_{12}-1,-l_{12}-2, \ldots,-\Delta_1'+d\end{array}\right\} \tag{C.17}$$

Construction of P_k from its null space zeros. We shall give the details of the construction only for Case 1 ($\Delta_1' \geqslant \Delta_1 \geqslant \Delta_2'$, $\Delta_1' \geqslant \Delta_2 \geqslant \Delta_2'$), so that $k=\Delta_2'$ throughout. The other three cases then follow by symmetry.

Our procedure follows that already given for the Wigner coefficients in Appendix A of Chapter 3. Consider first the set of zeros of $P_k'(\Delta, z)$ given by point set (C.12):

$$T_b=\{(a, b, x, x+b-a): a=-\Delta_2+k-1,-\Delta_2+k-2, \ldots,-\Delta_2+b\}, \tag{C.18}$$

where $b=0,1, \ldots, k-1$.

Let $P_k'(z)$ denote a polynomial of minimal degree in z_{11} and z_{12} that vanishes on the sets T_b, $b=0,1, \ldots, k-1$. Considering the point sets $T_0, T_1, \ldots, T_{k-1}$, in turn, and following the procedure leading to Eq. (A.4) in Appendix A of Chapter 3 we find that the minimal degree polynomial in z_{11} and z_{12} that vanishes on the points in the sets $T_0, T_1, \ldots, T_{k-1}$ is

$$P_k'(z)=\sum_{s=0}^{k} a_s(\Delta ; z_{21}, z_{22})[z_{11}+\Delta_2-s]_{k-s}[z_{12}]_s, \tag{C.19}$$

where the $a_s(\Delta; z_{21}, z_{22})$ are arbitrary and $[x]_a$ denotes a falling factorial: $[x]_a=x(x-1) \ldots(x-a+1)$.

We repeat this construction on each of the point sets (C.13)–(C.16) with the following results:
Point set (C.14):

$$S_b=\{(b,a,x+b-a,x): a=-\Delta_1+k-1,-\Delta_1+k-2,\dots,-\Delta_1+b\},\qquad \text{(C.20)}$$
$$\text{where } b=0,1,\dots,k-1;$$

Minimal polynomial:

$$P_k'(z)=\sum_{s=0}^{k} b_s(\Delta;z_{21},z_{22})[z_{11}]_s[z_{12}+\Delta_1-s]_{k-s}. \qquad \text{(C.21)}$$

Point set (C.13):

$$T_b'=\{(x+b-d,b,x,d): d=-1,-2,\dots,-k+b\},\qquad \text{(C.22)}$$
$$\text{where } b=0,1,\dots,k-1;$$

Minimal polynomial:

$$P_k'(z)=\sum_{s=0}^{k} c_s(\Delta;z_{11},z_{21})[z_{12}]_s[z_{22}+k-s]_{k-s}. \qquad \text{(C.23)}$$

Point set (C.17):

$$S_b'=\{(x,d,x+b-d+\Delta_2-k,b+\Delta_2-k):$$
$$d=-\Delta_1+k-1,-\Delta_1+k-2,\dots,-\Delta_1+b\},\qquad \text{(C.24)}$$
$$\text{where } b=0,1,\dots,k-1;$$

Minimal polynomial:

$$P_k'(z)=\sum_{s=0}^{k} d_s(\Delta;z_{11},z_{21})[z_{12}+\Delta_1-s]_{k-s}[z_{22}-\Delta_2+k]_s. \qquad \text{(C.25)}$$

Point set (C.15):

$$T_b''=\{(b,x+b-c,c,x): c=-1,-2,\dots,-k+b\},\qquad \text{(C.26)}$$
$$\text{where } b=0,1,\dots,k-1;$$

Minimal polynomial:

$$P_k'(z)=\sum_{s=0}^{k} e_s(\Delta;z_{12},z_{22})[z_{11}]_s[z_{21}+k-s]_{k-s}. \qquad \text{(C.27)}$$

Point set (C.16):

$$S_b'' = \{(x, x+b+\Delta_2-k-c, c, b+\Delta_2-k) : c = -1, -2, \ldots, -k+b\}, \tag{C.28}$$

$$\text{where } b = 0, 1, \ldots, k-1;$$

Minimal polynomial:

$$P_k'(z) = \sum_{s=0}^{k} f_s(\Delta; z_{11}, z_{12})[z_{21}+k-s]_{k-s}[z_{22}-\Delta_2+k]_s. \tag{C.29}$$

The polynomials $a_s, b_s, \ldots, f_s$ appearing in these results are arbitrary, and any multiplicative factors may be considered as absorbed into the $a_s, \ldots, f_s$ so that the six polynomials P_k' defined above must be identically equal.

Consider, then, the consequences of setting

$$\sum_{s=0}^{k} a_s(\Delta; z_{21}, z_{21}-z_{11}+z_{12})[z_{11}+\Delta_2-s]_{k-s}[z_{12}]_s$$
$$\equiv \sum_{s=0}^{k} c_s(\Delta; z_{11}, z_{21})[z_{21}-z_{11}+z_{12}+k-s]_{k-s}[z_{12}]_s, \tag{C.30}$$

where Eq. (C.8) has been used to eliminate the dependent variable. In this relation, we now put $z_{12}=0$ to obtain

$$a_0(\Delta; z_{21}, z_{21}-z_{11})[z_{11}+\Delta_2]_k = c_0(\Delta; z_{11}, z_{21})[z_{21}-z_{11}+k]_k,$$

which must hold identically in z_{11} and z_{21}. Hence, we must have

$$c_0(\Delta; z_{11}, z_{21}) = A_0(\Delta)[z_{11}+\Delta_2]_k,$$
$$a_0(\Delta; z_{21}, z_{21}-z_{11}) = A_0(\Delta)[z_{21}-z_{11}+k]_k, \tag{C.31}$$

where for minimal degree we must take $A_0(\Delta)$ to be independent of z_{11} and z_{21}. Since the second of these relations is valid for arbitrary z_{11}, it follows that

$$a_0(\Delta; z_{21}, z_{22}) = A_0(\Delta)[z_{22}+k]_k, \tag{C.32}$$

in which z_{21} and z_{22} are arbitrary. We next set $z_{12}=1$ in Eq. (C.30) and use Eqs. (C.31) and (C.32) to obtain

$$a_1(\Delta; z_{21}, z_{21}-z_{11}+1)[z_{11}+\Delta_2-1]_{k-1} = c_1(\Delta; z_{11}, z_{21})[z_{21}-z_{11}+k]_{k-1},$$

which must hold identically in z_{11} and z_{21}. Hence, we must have

$$c_1(\Delta; z_{11}, z_{21}) = A_1(\Delta; z_{21})[z_{11}+\Delta_2-1]_{k-1},$$
$$a_1(\Delta; z_{21}, z_{21}-z_{11}+1) = A_1(\Delta; z_{21})[z_{21}-z_{11}+k]_{k-1}, \tag{C.33}$$

where for minimal degree we must take $A_1(\Delta; z_{21})$ to be independent of z_{11}. Again, since the second of these relations is true for arbitrary z_{11}, we conclude

$$a_1(\Delta; z_{21}, z_{22}) = A_1(\Delta; z_{21})[z_{22}+k-1]_{k-1} \tag{C.34}$$

for arbitrary z_{21} and z_{22}. Continuing the argument in an obvious manner, we thus establish the fact that Eq. (C.30) implies

$$a_s(\Delta; z_{21}, z_{22}) = A_s(\Delta; z_{21})[z_{22}+k-s]_{k-s} \tag{C.35}$$
$$c_s(\Delta; z_{11}, z_{21}) = A_s(\Delta; z_{21})[z_{11}+\Delta_2-s]_{k-s}. \tag{C.36}$$

We now repeat the argument leading to Eqs (C.35) and (C.36) for each of the pairs of equations [(C.21), (C.25)], [(C.21), (C.27)], [(C.25), (C.29)], and [(C.27), (C.29)], thus obtaining the following relations:

$$\begin{aligned}
b_s(\Delta; z_{21}, z_{22}) &= B_s(\Delta; z_{21})[z_{22}-\Delta_2+k]_s,\\
d_s(\Delta; z_{11}, z_{21}) &= B_s(\Delta; z_{21})[z_{11}]_s;\\
b_s(\Delta; z_{21}, z_{22}) &= B_s'(\Delta; z_{22})[z_{21}+k-s]_{k-s},\\
e_s(\Delta; z_{12}, z_{22}) &= B_s'(\Delta; z_{22})[z_{12}+\Delta_1-s]_{k-s};\\
d_s(\Delta; z_{11}, z_{21}) &= D_s(\Delta; z_{11})[z_{21}+k-s]_{k-s},\\
f_s(\Delta; z_{11}, z_{12}) &= D_s(\Delta; z_{11})[z_{12}+\Delta_1-s]_{k-s};\\
e_s(\Delta; z_{12}, z_{22}) &= E_s(\Delta; z_{12})[z_{22}-\Delta_2+k]_s,\\
f_s(\Delta; z_{11}, z_{12}) &= E_s(\Delta; z_{12})[z_{11}]_s.
\end{aligned} \tag{C.37}$$

For minimal degree we find that these relations imply

$$\begin{aligned}
b_s(\Delta; z_{21}, z_{22}) &= B_s(\Delta)[z_{21}+k-s]_{k-s}[z_{22}-\Delta_2+k]_s,\\
d_s(\Delta; z_{11}, z_{21}) &= B_s(\Delta)[z_{11}]_s[z_{21}+k-s]_{k-s},\\
e_s(\Delta; z_{12}, z_{22}) &= B_s(\Delta)[z_{12}+\Delta_1-s]_{k-s}[z_{22}-\Delta_2+k]_s,\\
f_s(\Delta; z_{11}, z_{12}) &= B_s(\Delta)[z_{11}]_s[z_{12}+\Delta_1-s]_{k-s},
\end{aligned} \tag{C.38}$$

where $B_s(\Delta) = B_s(\Delta_1, \Delta_2, \Delta_1', \Delta_2')$ is independent of the z_{ij}.

Using the results Eqs. (C.35), (C.36), and (C.38) in the six polynomials $P_k'(z)$ given by Eqs. (C.19)–(C.29), we find one last relation:

$$\begin{aligned} P_k'(z) &= \sum_{s=0}^{k} A_s(\Delta; z_{21})[z_{11}+\Delta_2-s]_{k-s}[z_{12}]_s[z_{22}+k-s]_{k-s} \\ &= \sum_{s=0}^{k} B_s(\Delta)[z_{11}]_s[z_{12}+\Delta_1-s]_{k-s}[z_{21}+k-s]_{k-s}[z_{22}-\Delta_2+k]_s. \end{aligned} \tag{C.39}$$

Since this polynomial is required to be invariant under the substitutions $\Delta_1 \leftrightarrow \Delta_2$, $z_{11} \leftrightarrow z_{12}$, $z_{21} \leftrightarrow z_{22}$, we may effect these interchanges in the right-hand side of Eq. (C.39). We then deduce

$$\begin{aligned} A_s(\Delta_1, \Delta_2, \Delta_1', \Delta_2'; z_{21}) &= B_s(\Delta_2, \Delta_1, \Delta_1', \Delta_2')[z_{21}-\Delta_1+k]_s \\ &= B_s'(\Delta)[z_{21}-\Delta_1+k]_s. \end{aligned} \tag{C.40}$$

Thus, we have found

$$\begin{aligned} P_k'(\Delta; z) &= \sum_{s=0}^{k} B_s'(\Delta)[z_{11}+\Delta_2-s]_{k-s}[z_{12}]_s[z_{21}-\Delta_1+k]_s[z_{22}+k-s]_{k-s} \\ &= \sum_{s=0}^{k} B_s(\Delta)[z_{11}]_s[z_{12}+\Delta_1-s]_{k-s}[z_{21}+k-s]_{k-s}[z_{22}-\Delta_2+k]_s. \end{aligned} \tag{C.41}$$

To obtain the coefficients $B_s'(\Delta)$, we set $z_{11}=z_{21}=0$ and $z_{12}=z_{22}=x$ in Eq. (C.41). The coefficients $B_s'(\Delta)$ must then satisfy the following equation for arbitrary x:

$$\sum_{s=0}^{k} B_s'(\Delta)[\Delta_2-s]_{k-s}[-\Delta_1+k]_s[x+k-s]_k = B_0(\Delta)k![x+\Delta_1]_k.$$

In this result, we set $x=0,1,\ldots,k$, in turn, to obtain a triangular system of equations for the coefficients $B_s'(\Delta)$, $s=0,1,\ldots,k$. The unique solution is given by

$$B_s'(\Delta) = b_0(\Delta)\binom{\Delta_1}{k-s}\binom{\Delta_2}{s}, \tag{C.42}$$

where we have put

$$B_0(\Delta) = b_0(\Delta)\binom{\Delta_2}{k},$$

in which $b_0(\Delta)$ is an arbitrary constant (symmetric in Δ_1 and Δ_2).

Thus, the unique polynomial, which we obtain by requiring it to be (a) zero on the point sets T_b, T'_b, T''_b, S_b, S'_b, S''_b, $(b=0,1,\ldots,k-1)$; (b) minimal in its degree at each step of the construction; and (c) invariant under the exchanges $\Delta_1 \leftrightarrow \Delta_2$, $z_{11} \leftrightarrow z_{12}$, $z_{21} \leftrightarrow z_{22}$, is

$$P'_k(\Delta; z) = b_0(\Delta) \sum_{s=0}^{k} \binom{\Delta_1}{k-s}\binom{\Delta_2}{s}[z_{11}+\Delta_2-s]_{k-s}$$
$$\times [z_{12}]_s [z_{21}-\Delta_1+k]_s [z_{22}+k-s]_{k-s}. \tag{C.43}$$

Making the notational change $[x]_a = (x-a+1)_a$, we obtain the polynomial part of a Racah coefficient [see Eq. (A.17), Appendix A], thus validating the assertion made at the beginning of this Appendix. [The transformation properties between polynomials under the substitutions (A.21) may be used to transform our proof for Case 1 to a proof for the other three cases.]

D. Symbolic Forms of the Racah Coefficients

Symbolic forms for the Racah coefficient have been given in the literature (Sato [12]). We develop here a symbolic expansion of the polynomial part of a Racah coefficient, which is analogous to that given in Appendix B of Chapter 3 for a Wigner coefficient. The essential generalization beyond the rules already given in connection with Wigner coefficients is in handling negative quantities.

Consider the following expression in which ξ, η, x, y, z, and w are indeterminates and k is a nonnegative integer:

$$\phi(\xi)(x)(y) + (\eta)(z)(w) + (\xi)(\eta)(y)\phi^k/k!. \tag{D.1}$$

The $\phi\ldots\phi$ around the enclosed form symbolizes the following operations:
~~(a) Expand the form by the usual trinomial theorem, collect together the~~ powers of each variable, and keep the symbol $\phi\ldots\phi$ and parentheses around each individual factor (to anticipate a further symbolic rule). Thus, by definition, we have

$$\phi(\xi)(x)(y) + (\eta)(z)(w) + (\xi)(\eta)(y)\phi^k/k!$$
$$= \sum_{k_1+k_2+k_3=k} \frac{\phi(\xi)(x)(y)\phi^{k_1}\phi(\eta)(z)(w)\phi^{k_2}\phi(\xi)(\eta)(y)\phi^{k_3}}{k_1!k_2!k_3!}$$
$$= \sum_{k_1+k_2+k_3=k} \frac{\phi(\xi)\phi^{k_1+k_3}\phi(\eta)\phi^{k_2+k_3}\phi(x)\phi^{k_1}\phi(y)\phi^{k_1+k_3}\phi(z)\phi^{k_2}\phi(w)\phi^{k_2}}{k_1!k_2!k_3!}. \tag{D.2}$$

In the second step in the definition of (D.1) we have the following rule: (b) Define the symbol $\{\!\!\{(\zeta)\}\!\!\}^k$ for ζ an indeterminate and k a nonnegative integer by

$$\{\!\!\{(\zeta)\}\!\!\}^k \equiv \zeta(\zeta-1)\ldots(\zeta-k+1)=[\zeta]_k. \tag{D.3}$$

(c) The reason for keeping the parentheses ( ) around ζ is the following: We wish to distinguish $\{\!\!\{(-\zeta)\}\!\!\}^k$ from $\{\!\!\{-(\zeta)\}\!\!\}^k$. The first symbol $\{\!\!\{(-\zeta)\}\!\!\}^k$ is already defined by (D.3). We define the second symbol by

$$\{\!\!\{-(\zeta)\}\!\!\}^k=(-1)^k\{\!\!\{(\zeta)\}\!\!\}^k. \tag{D.4}$$

Using the rule (D.3) in Eq. (D.2), we may now write out the right-hand side in terms of ordinary quantities. It is, however, the expression

$$\{\!\!\{(\xi)(x)(y)-(\eta)(z)(w)+(\xi)(\eta)(y)\}\!\!\}^k/k! \tag{D.5}$$

that is of later interest, and it is this quantity that we now expand.

We now carry out the expansion of (D.5) using the rules (a)–(c) given above:

$$\begin{aligned}
&\{\!\!\{(\xi)(x)(y)-(\eta)(z)(w)+(\xi)(\eta)(y)\}\!\!\}^k/k! \\
&= \sum_{k_1+k_2+k_3=k} \frac{\{\!\!\{(\xi)(x)(y)\}\!\!\}^{k_1}\{\!\!\{-(\eta)(z)(w)\}\!\!\}^{k_2}\{\!\!\{(\xi)(\eta)(y)\}\!\!\}^{k_3}}{k_1!k_2!k_3!} \\
&= \sum_{k_1+k_2+k_3=k} \frac{(-1)^{k_2}\{\!\!\{(\xi)\}\!\!\}^{k_1+k_3}\{\!\!\{(\eta)\}\!\!\}^{k_2+k_3}\{\!\!\{(x)\}\!\!\}^{k_1}\{\!\!\{(y)\}\!\!\}^{k_1+k_3}\{\!\!\{(z)\}\!\!\}^{k_2}\{\!\!\{(w)\}\!\!\}^{k_2}}{k_1!k_2!k_3!} \\
&= \sum_{k_1+k_2+k_3=k} \frac{(-1)^{k_2}[\xi]_{k_1+k_3}[\eta]_{k_2+k_3}[x]_{k_1}[y]_{k_1+k_3}[z]_{k_2}[w]_{k_2}}{k_1!k_2!k_3!} \\
&= \sum_{k_1+k_2=k} \frac{(-1)^{k_2}[\xi]_{k_1}[\eta]_{k_2}[x+\eta-k_2]_{k_1}[y]_{k_1}[z]_{k_2}[w]_{k_2}}{k_1!k_2!}.
\end{aligned} \tag{D.6}$$

The last step in Eq. (D.6) is made by carrying out the internal summation over k_1 in the next to the last term:

$$\begin{aligned}
\sum_{k_1} \frac{[\eta]_{k-k_1}[x]_{k_1}}{(k-k_1-k_2)!k_1!} &= [\eta]_{k_2}\sum_{k_1} \frac{[\eta-k_2]_{k-k_1-k_2}[x]_{k_1}}{(k-k_1-k_2)!k_1!} \\
&= \frac{[\eta]_{k_2}[x+\eta-k_2]_{k-k_2}}{(k-k_2)!}.
\end{aligned}$$

We have then renamed $k-k_2$ to be k_1 in obtaining the final form of (D.6).

It is now straightforward to verify the following result:

$$
\begin{aligned}
P'_{\Delta'_2}(\Delta; z) = \{\!\!|(\Delta_1)(z_{11})(-z_{22}-1) - (\Delta_2)(z_{12})(z_{21}+\Delta'_2-\Delta_1) \\
+(\Delta_1)(\Delta_2)(-z_{22}-1)|\!\!\}^{\Delta'_2}/(\Delta'_2)!. \qquad \text{(D.7)}
\end{aligned}
$$

Symbolic expansions for the other three polynomials [Cases 2–4, Appendix A] are obtained from Eq. (D.7) by performing the operations (A.21).

In the limit of large $-m_{21}$, we have $z_{21} \sim m_{21}$ and $z_{22} \sim m_{21}$. Despite the symbolic meaning of Eq. (D.7), it is correct that

$$
\begin{aligned}
P'_{\Delta'_2}(\Delta; z) &\sim (-m_{21})^{\Delta'_2} \{\!\!|(\Delta_1)(z_{11}) + (\Delta_2)(z_{21}) + (\Delta_1)(\Delta_2)|\!\!\}^{\Delta'_2}/(\Delta'_2)! \\
&= P_{\Delta'_2}(\Delta_1, \Delta_2, \Delta'_1, \Delta'_2; z_{11}, z_{12})(-m_{21})^{\Delta'_2}, \qquad \text{(D.8)}
\end{aligned}
$$

where

$$
\begin{aligned}
&P_{\Delta'_2}(\Delta_1, \Delta_2, \Delta'_1, \Delta'_2; z_{11}, z_{12}) \\
&\quad = \{\!\!|(\Delta_1)(z_{11}) + (\Delta_2)(z_{21}) + (\Delta_1)(\Delta_2)|\!\!\}^{\Delta'_2}/(\Delta'_2)! \qquad \text{(D.9)}
\end{aligned}
$$

is the symbolic expression for the polynomial part of a Wigner coefficient.

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CHAPTER 5

Special Topics

TOPIC 1. FUNDAMENTAL SYMMETRY CONSIDERATIONS

Survey

This Topic considers in detail some fundamental aspects of symmetry in quantum mechanics, with the purpose of placing rotational symmetry and angular momentum in this larger context and thus validating the treatment of rotational symmetry in this monograph. The Topic begins with Wigner's theorem, Section 1, its proof, Section 2, and generalization, Section 3. Applications are made in Section 4 to angular momentum, and in Section 5 to time reversal. In Section 6 the Frobenius–Schur invariant is discussed, and in Section 7 it is applied to the univalence superselection rule for intrinsic angular momentum. Semilinear representations ("corepresentations") are discussed in Section 8. This Topic concludes with a remark, Section 9, on the importance of the Wigner symmetry theorem.

1. The Wigner Theorem on Symmetry Transformations in Quantum Mechanics

In the statistical interpretation of quantum mechanics, the transition probabilities,

$$P(f,g)\equiv|\langle f|g\rangle|^2=P(g,f), \tag{5.1.1}$$

where f, g denote any two normalized vectors in some Hilbert space, are the basic structural elements. Accordingly, if one wishes to interpret a symmetry

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transformation in the widest possible sense—as a redescription, say (choosing the passive interpretation), without change of content[1]—then it is sufficient to require that all probabilities (and hence all physical content) must be invariant. This viewpoint leads to the standard (Weyl [1], Wigner [2]) definition of a symmetry in quantum mechanics, which we formulate in detail below.

Before a precise definition of a symmetry can be given, it is necessary to recall two important details [(*a*) and (*b*) below] concerning quantum mechanical states:

(*a*) The states of a quantum system are to be described in terms of the vectors of a Hilbert space $\mathcal{H}$; it is convenient, because of the probability interpretation, to consider *unit vectors* (vectors of unit norm) as correlated with quantum states. This association of Hilbert space vectors to quantum states is quite natural—especially so because the linearity of Hilbert space is taken to implement the superposition principle so important to quantum (as opposed to classical) physics. Yet it must be constantly borne in mind that this correspondence between unit vectors in Hilbert space and quantum states is not one-to-one, since the two vectors f and $e^{i\lambda}f$ ($\lambda\in\mathbb{R}$), for example, both describe exactly the same state. To eliminate this "freedom-of-phase," the concept of a *unit ray* in a Hilbert space $\mathcal{H}$ is introduced into quantum mechanics.

The concept of a *ray* is that of an equivalence class of vectors in $\mathcal{H}$. Thus, two vectors $f\in\mathcal{H}$ and $g\in\mathcal{H}$ are defined to be equivalent, $f\sim g$, if and only if $f=g\alpha$ for some $\alpha\in\mathbb{C}$, and $|\alpha|=1$.[2] This equivalence relation then (uniquely) partitions $\mathcal{H}$ into subsets of vectors of the type[3]

$$\mathbf{f}=\{f\alpha : f \text{ is a fixed element of } \mathcal{H};\ \alpha\in\mathbb{C} \text{ with } |\alpha|=1\}. \tag{5.1.2}$$

The vector f is a representative of the equivalence class $\mathbf{f}$, as is any vector in $\mathbf{f}$. The set of vectors $\mathbf{f}$ is called a ray in $\mathcal{H}$ with representative f.

Let us denote the inner product of vectors in $\mathcal{H}$ by the notation (f, g), and the norm of f by $\|f\|=(f,f)^{\frac{1}{2}}$. Then we may define the *inner product of two rays* by

$$(\mathbf{f},\mathbf{g})\equiv|(f,g)|, \tag{5.1.3}$$

[1]This is described more fully in Section 4, p. 169.

[2]The more general definition of a ray takes $f\sim g$ if and only if $f=g\alpha$ for some $\alpha\in\mathbb{C}$, and $\alpha\neq0$. In this definition, a ray is a one-dimensional subspace of $\mathcal{H}$, which contains f as a subset of vectors. The definition of a ray given by Eq. (5.1.2) is more convenient for the proof of the Wigner theorem (Bargmann [3]).

[3]Right multiplication by scalars (complex numbers) is used to allow (in later sections) generalization to noncommutative fields.

since $(\mathbf{f},\mathbf{g})$ is clearly independent of the representatives $f \in \mathbf{f}$ and $g \in \mathbf{g}$. [In obtaining this result, we have used the two properties of the inner product in $\mathcal{H}$ given by $(f, g)=(g, f)^*$ and $(f, g\alpha)=(f, g)\alpha$.] Note, then, that the norm of a ray $\mathbf{f}$ is $\|\mathbf{f}\|=\|f\|$.

A *unit ray* is a ray of unit norm; every ray $\mathbf{f}$ $(\mathbf{f}\neq 0)$ may be written uniquely in the form

$$\mathbf{f}=\hat{\mathbf{e}}\rho, \qquad \rho=\|\mathbf{f}\|, \tag{5.1.4}$$

where $\hat{\mathbf{e}}$ is a unit ray. (The notation $\hat{\mathbf{e}}\rho$ denotes that each vector in the set $\hat{\mathbf{e}}$ is to be multiplied by the real positive number ρ.)

Any significant statement about a quantal system is a statement about unit rays.

(*b*) The second detail to be mentioned concerns the assumption that every unit ray corresponds to a state. Such an assumption was explicitly incorporated in the axioms of quantum mechanics in the formulation of von Neumann [4, Axiom VIII]. The existence of superselection rules (Wick *et al.* [5]) in physics shows that this assumption can be invalid. A more general assumption is that the superselection rules determine a decomposition of the Hilbert space into a direct sum of mutually orthogonal subspaces such that every ray in each subspace corresponds to a physical state.

We can now give a precise definition of a symmetry in quantum mechanics (Wigner [2, Chapter 26], Bargmann [3]). Definition:[1] *A symmetry T is a map from the set of unit rays in $\mathcal{H}$ onto the set of unit rays in a second Hilbert space $\mathcal{H}'$ (which may be $\mathcal{H}$) such that the following properties hold*:

(*a*) *T is defined for every unit ray in $\mathcal{H}$.*

(*b*) *Transition probabilities are preserved*, *that is*,

$$\langle T(\hat{\mathbf{e}}), T(\hat{\mathbf{e}}')\rangle=(\hat{\mathbf{e}},\hat{\mathbf{e}}'), \tag{5.1.5}$$

~~*where $\hat{\mathbf{e}},\hat{\mathbf{e}}'$ are unit rays in $\mathcal{H}$, and $T(\hat{\mathbf{e}})$, $T(\hat{\mathbf{e}}')$ are the corresponding unit rays*~~ *in $\mathcal{H}'$.*

(*c*) *The mapping T between unit rays is one-to-one*[2] *and onto.*

A symmetry transformation is thus a one-to-one mapping of unit rays of one Hilbert space onto unit rays of a second Hilbert space; at first glance this definition seems to imply insufficient information to be of much utility for physical purposes. The reason one might feel the information to be too

[1] The alternative $\mathcal{H}\to\mathcal{H}'$ is included to allow the mapping of one coherent subspace onto another coherent subspace; this situation can occur when there are superselection rules. The Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are equipped with inner products denoted by (f, g) and $\langle f', g'\rangle$, respectively; for $\mathcal{H}=\mathcal{H}'$, these inner products are taken to be identical.

[2] Property (*b*) implies that the transformation is one-to-one, since (by the Schwartz inequality) two unit rays coincide if and only if $(\hat{\mathbf{e}},\hat{\mathbf{e}}')=1$. The onto property is a critical assumption in the definition.

minimal is that the superposition principle is fundamental in quantum mechanics, and this principle relates to the addition of *vectors*, not rays or even unit rays. Won't the freedom-of-phase in unit rays lead to discarding superposition?

In the light of such considerations, the Wigner theorem is quite remarkable, for it asserts that every *unit ray* (symmetry) mapping can be replaced by a *vector* mapping, which is either unitary or anti-unitary (the latter occurring only for a symmetry involving time reversal). The Wigner theorem is the fundamental theorem in the analysis of symmetry properties for quantal systems.

MAIN THEOREM (Wigner): *Let T be a symmetry mapping of unit rays in the Hilbert space $\mathcal{H}$ onto the unit rays in the Hilbert space $\mathcal{H}'$. Then there exists an onto mapping $\mathcal{U}$: $\mathcal{H}\to\mathcal{H}'$; that is, $f \xrightarrow{\mathcal{U}} f'=\mathcal{U}(f)\in\mathcal{H}'$ for each $f\in\mathcal{H}$, such that*

$$\mathcal{U}(f+g)=\mathcal{U}(f)+\mathcal{U}(g), \tag{5.1.6}$$

$$\mathcal{U}(f\zeta)=\mathcal{U}(f)\lambda(\zeta), \tag{5.1.7}$$

$$\langle\mathcal{U}(f),\mathcal{U}(g)\rangle=\lambda((f,g)), \tag{5.1.8}$$

where $\lambda(\zeta)$ is defined for each $\zeta\in\mathbb{C}$ to be either

$$\lambda(\zeta)=\zeta \qquad \textit{or} \qquad \lambda(\zeta)=\zeta^*. \tag{5.1.9}$$

2. Proof of the Wigner Theorem[1]

It is useful at this point to introduce some notations. We shall denote by $f, g,\ldots$ vectors in the (coherent) Hilbert space $\mathcal{H}$, and by $e_1, e_2,\ldots$ unit vectors in $\mathcal{H}$ (state vectors). Rays in $\mathcal{H}$ with representative elements $f, g,\ldots$ will be denoted by $\mathbf{f},\mathbf{g},\ldots$, and unit rays in $\mathcal{H}$ with representative elements $e_1, e_2,\ldots$ will be denoted by $\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\ldots$.

Scalars (complex numbers) of modulus 1 will be denoted by $\alpha,\beta,\ldots$ (beginning of the alphabet), whereas positive, real scalars will be denoted by $\rho,\sigma,\ldots$ (end of the alphabet).

The first step in the proof is to extend the mapping T—defined originally on unit rays—to all rays in $\mathcal{H}$, by defining $T(\mathbf{f})$, for each $\mathbf{f}=\hat{\mathbf{e}}\rho$, to be

$$T(\mathbf{f})=T(\hat{\mathbf{e}}\rho)\equiv T(\hat{\mathbf{e}})\rho, \qquad \text{for all } \rho>0, \tag{5.1.10}$$

[1]This proof is modeled after the original proof of Wigner [2, Chapter 26] and includes the additional details supplied in Bargmann's [3] exegesis on the Wigner proof.

and, for completeness, $T(0)=0$. Note that the definition (5.1.10) indeed includes all nonzero rays in $\mathcal{H}$.

For the extended mapping (denoted also by T), it follows that

$$T(\mathbf{f}\sigma)=T(\hat{\mathbf{e}}\rho\sigma)=T(\hat{\mathbf{e}})\rho\sigma=T(\mathbf{f})\sigma, \qquad \text{for all } \rho,\sigma>0; \tag{5.1.11}$$

$$\begin{aligned}\langle T(\mathbf{f}),T(\mathbf{g})\rangle &=\langle T(\hat{\mathbf{e}}_1)\rho, T(\hat{\mathbf{e}}_2)\sigma\rangle=(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2)\rho\sigma\\ &=(\mathbf{f},\mathbf{g}).\end{aligned} \tag{5.1.12}$$

From Eq. (5.1.12) it follows, in particular, that $\langle T(\mathbf{f}),T(\mathbf{f})\rangle=(\mathbf{f},\mathbf{f})$, so that the extended mapping preserves the norm of all rays, including the null ray.

Let us assume that the space $\mathcal{H}$ is at least two-dimensional. Let us fix a particular unit ray $\hat{\mathbf{e}}_1$ and select a representative e_1. The mapping T yields the unit ray $\hat{\mathbf{e}}_1'\equiv T(\hat{\mathbf{e}}_1)$, and we again select a representative e_1'. These (arbitrary) choices constitute the initial step in the construction of the vector mapping $\mathcal{U}$, which, by definition, effects the transformation

$$e_1'=\mathcal{U}(e_1). \tag{5.1.13}$$

From the orthogonal complement to e_1 in $\mathcal{H}$, we now pick a nonzero vector f, and normalize it to define a unit vector e_2. Consider now the vector g defined by

$$g=e_1+f=e_1+e_2(f,e_2)=e_1+e_2x, \tag{5.1.14}$$

where $x\equiv(f,e_2)=(g,e_2)$.

The (extended) mapping T takes the ray $\mathbf{g}$ into the ray $\mathbf{g}'$, the ray $\hat{\mathbf{e}}_1$ into $\hat{\mathbf{e}}_1'$, and the ray $\mathbf{f}$ into $\hat{\mathbf{e}}_2'|x|$. Because the magnitude of every inner product is preserved, it follows that

$$g'\alpha=e_1'\beta+e_2'|x|\gamma, \tag{5.1.15}$$

where α,β,γ are of modulus 1 and g',e_2' are representatives of the rays $\mathbf{g}',\hat{\mathbf{e}}_2'$. Multiplying through by β^{-1}, and renaming the representative of g' ($g'\alpha\beta^{-1}$ is now called g'), we obtain the relation

$$g'=e_1'+e_2'|x|\delta(x), \tag{5.1.16}$$

where $\delta(x)$ is a complex number of unit modulus. Defining $g'=\mathcal{U}(g)$, we have thus extended the vector mapping $\mathcal{U}$ to include linear combinations:

$$\mathcal{U}(e_1+e_2x)=\mathcal{U}(e_1)+e_2'|x|\delta(x). \tag{5.1.17}$$

The essential point in this construction is to recognize that for *every vector* $f=e_2x$ we have obtained a uniquely defined vector $g'=(e_1+f)'$.

Consider now the mapping $\mathcal{U}$ [Eq. (5.1.17)] applied to the two vectors $f_1=e_2x$ and $f_2=e_2y$, where x, y are nonzero complex numbers; that is,

$$\begin{aligned} g_1' &= \mathcal{U}(e_1+e_2x) = \mathcal{U}(e_1)+e_2'|x|\delta(x), \\ g_2' &= \mathcal{U}(e_1+e_2y) = \mathcal{U}(e_1)+e_2'|y|\delta(y). \end{aligned} \tag{5.1.18}$$

The magnitude of the inner products is preserved so that

$$\begin{aligned} |\langle g_1', g_2'\rangle|^2 &= |(g_1, g_2)|^2 = 1 + x^*y + y^*x + |xy|^2 \\ &= 1 + |xy|[\delta^*(x)\delta(y) + \delta^*(y)\delta(x)] + |xy|^2. \end{aligned} \tag{5.1.19}$$

Since $\delta(x)$ and $\delta(y)$ have modulus unity, this relation leads to a quadratic equation,

$$[\delta^*(x)\delta(y)]^2 - \frac{x^*y+y^*x}{|xy|}[\delta^*(x)\delta(y)] + 1 = 0, \tag{5.1.20}$$

with two solutions: either

$$\delta^*(x)\delta(y) = x^*y/|xy|, \tag{5.1.21}$$

or

$$\delta^*(x)\delta(y) = xy^*/|xy|. \tag{5.1.22}$$

These two solutions imply that, for every nonzero x, we have either

$$\delta(x) = \alpha x/|x|, \tag{5.1.23}$$

or

$$\delta(x) = \alpha x^*/|x|, \tag{5.1.24}$$

where α is a fixed complex number of modulus 1. The factor α can be absorbed into the labeling of the representative e_2' of the ray $\hat{e}_2'$, so that the two forms of Eq. (5.1.17) corresponding to the solutions (5.1.23) and (5.1.24) are

$$\begin{aligned} \mathcal{U}(e_1+e_2x) &= \mathcal{U}(e_1)+e_2'x, \\ \mathcal{U}(e_1+e_2x) &= \mathcal{U}(e_1)+e_2'x^*. \end{aligned} \tag{5.1.25}$$

Setting $x=1$ in this result and defining $\mathcal{U}(e_2)$ by

$$\mathcal{U}(e_2)\equiv e_2'=\mathcal{U}(e_1+e_2)-\mathcal{U}(e_1), \tag{5.1.26}$$

we obtain, using Eq. (5.1.9) for λ:

$$\mathcal{U}(e_1+e_2x)=\mathcal{U}(e_1)+\mathcal{U}(e_2)\lambda(x). \tag{5.1.27}$$

If the dimension of $\mathcal{H}$ is greater than two, results of the same form follow for every e_i ($i=2,3,\dots$) in an orthonormal basis of the orthogonal complement to e_1.

The vector mapping $\mathcal{U}$ has thus been defined on all vectors $\mathcal{H}$ of the form e_1, e_1+e_ix, and e_ix. In summary, we have

$$\begin{aligned}\mathcal{U}(e_1)&=e_1',\\ \mathcal{U}(e_1+e_ix)&=e_1'+e_i'\lambda(x),\\ \mathcal{U}(e_ix)&=e_i'\lambda(x),\end{aligned} \tag{5.1.28}$$

where $\lambda(x)$ *may be either* $\lambda(x)=x$ *or* $\lambda(x)=x^*$ *for each value* $i=2,3,\dots$ *of the index* i.

Next we show that the *same* alternative for $\lambda(x)$ occurs for all $i=2,3,\dots$. Consider the vector $h=2^{-\frac{1}{2}}(e_2+e_3)z=ez$, $(e,e)=1$. Since h is orthogonal to e_1, we can repeat the construction (given above, for $f=e_2x$) and arrive at $\mathcal{U}(h)=e'\lambda''(z)$, where $\lambda''(z)$ is one of the two alternatives, z or z^*. Thus, we have

$$\mathcal{U}(h)=e'\lambda''(z)=2^{-\frac{1}{2}}\left[e_2'\lambda(z)+e_3'\lambda'(z)\right] \tag{5.1.29}$$

in which $\lambda(z)$, $\lambda'(z)$, and $\lambda''(z)$ may each assume either of the alternative values, z or z^*. Since $\langle e_2',e_3'\rangle=(e_2,e_3)=0$, we conclude that $\lambda(z)$, $\lambda'(z)$, and $\lambda''(z)$ are all equal. Thus, we have defined the vector mapping $\mathcal{U}$ for all vectors in $\mathcal{H}$ of the form e_1, e_1+f, and f, where f is an arbitrary vector in the orthogonal complement of e_1 in $\mathcal{H}$.

To complete the definition of $\mathcal{U}$, it is necessary only to consider the vector $g=e_1\alpha+f$, where α is a nonzero complex number. To do this, we first consider the vector $h=e_1+f\alpha^{-1}$, so that $g=h\alpha$. Define $\mathcal{U}(g)=\mathcal{U}(h)\lambda(\alpha)$. This is consistent, since

$$\begin{aligned}\mathcal{U}(g)=\mathcal{U}(h)\lambda(\alpha)&=\left[e_1'+\mathcal{U}(f\alpha^{-1})\right]\lambda(\alpha)\\ &=e_1'\lambda(\alpha)+\mathcal{U}(f)=\mathcal{U}(e_1\alpha)+\mathcal{U}(f).\end{aligned} \tag{5.1.30}$$

Thus, the vector mapping $\mathcal{U}$ has been defined on all the vectors in $\mathcal{H}$, and it satisfies all the conditions asserted in the theorem. (We have assumed that the dimension of $\mathcal{H}$ is at least two, but it is clear that for dimensionality one the same result is trivially correct.)

Remarks. (*a*) The two alternatives for the mapping $\mathfrak{U}$ both correspond to an isometry. For the first alternative—that is, for $\lambda(\alpha)=\alpha$—one has a linear isometric mapping (a unitary transformation). For the second alternative, for $\lambda(\alpha)=\alpha^*$, one has an antilinear isometry ("anti-unitary" transformation). (The latter is discussed further in the following sections.)

(*b*) The converse to the Wigner theorem—that every linear or antilinear isometry $\mathfrak{U}$ induces a ray mapping that preserves inner products—can be established directly: From $f'=\mathfrak{U}(f)$, we obtain the ray mapping $\mathbf{f}'=T(\mathbf{f})$, where $\mathbf{f}$ is the equivalence class whose representative is f, and similarly for $\mathbf{f}'$.

(*c*) The linear or antilinear character of the mapping $\mathfrak{U}$ is an intrinsic property of the original ray mapping T. Bargmann [3] has noted that the complex number defined by

$$\Delta(\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3)\equiv(f_1,f_2)(f_2,f_3)(f_3,f_1) \tag{5.1.31}$$

is independent of the representatives chosen for the rays. Defining $\tau(\Delta)=\Delta(T\mathbf{f}_1,T\mathbf{f}_2,T\mathbf{f}_3)$, one sees that $\tau(\Delta)=\lambda(\Delta)$, so that (for dimension$\geqslant 2$) the fact that Δ is not always real can distinguish the linear from the antilinear case.

(*d*) For a given ray mapping T, it is of interest to determine to what extent a vector mapping compatible with T is unique. [The vector mapping $\mathfrak{U}$ is said to be compatible with T if $\mathfrak{U}(f)\in T(\mathbf{f})$ for every $f\in\mathbf{f}$.] Defining the mapping $\mathfrak{U}$ to be *additive* if $\mathfrak{U}(f+g)=\mathfrak{U}(f)+\mathfrak{U}(g)$, Bargmann [3] has shown: *If two additive vector mappings $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are compatible with the same T, and if $\dim\mathcal{H}\geqslant 2$, then $\mathfrak{U}_2(f)=\mathfrak{U}_1(f)\alpha$, $\alpha\in\mathbb{C}$, $|\alpha|=1$, for every vector f in $\mathcal{H}$.*

[This result can be seen already from the proof of the Wigner theorem ~~given above, where $\delta(x)$ differed from $\lambda(x)$ by the factor α. The factor α~~ was removed by redefining the e_i'.]

3. Extension of the Wigner Theorem: Relation to the Fundamental Theorem of Projective Geometry

Uhlhorn [6] has given a very striking generalization of the Wigner theorem in which the requirement that all transition probabilities be preserved is replaced by the much weaker requirement that *only transition probability zero be preserved*! (This is the requirement that $[(\mathbf{f},\mathbf{g})^2=0]\leftrightarrow[\langle\mathbf{f}',\mathbf{g}'\rangle^2=0]$.) It is still necessary to assume (as in the Wigner theorem) that the ray mapping is one-to-one onto, but in addition one must now assume that the dimension of $\mathcal{H}$ is *three* or greater. With these modifications in the hypothesis, the conclusion of the Wigner theorem can be shown to follow.

The essential content in this weakened hypothesis (that only zero probabilities are preserved) is that in place of the equivalence class of quantum states (unit rays) one now considers, in effect, the larger equivalence classes based on rays. In terms of a ray mapping for this more general case, it is to be expected that not only will the phase of the inner product (f, g) change (as before) under the mapping, but the magnitude as well. Only inner products having the value zero will be unchanged by the ray mapping.

When the problem is phrased in this way, one recognizes that the generalization being considered is precisely the construction that characterizes projective geometry, where a "point" in such a geometry is an equivalence class of rays (Artin [7]). Alternatively, and equivalently, from each Hilbert space over a field F, we may form a new object, the corresponding projective space $\overline{\mathcal{H}}$. Its elements are not vectors of $\mathcal{H}$ but subspaces V of $\mathcal{H}$. To each subspace V of $\mathcal{H}$ one assigns a projective dimension, $\dim_p V = (\dim V) - 1$, just one unit smaller than the ordinary dimension. (The zero subspace of $\mathcal{H}$ should be thought of as the "empty element" of $\overline{\mathcal{H}}$.) An incidence relation is introduced in $\overline{\mathcal{H}}$: the inclusion relation $V_1 \subset V_2$ between the subspaces of $\mathcal{H}$.

[To see how this alternative view coincides with the previously mentioned view of rays, consider the usual model of the real projective plane. The points of $\overline{\mathcal{H}}$ are the lines of $\mathcal{H}$ (three-dimensional real space) through the origin. Hence, all nonzero multiples of the same unit vector belong to the same line and correspond to the same point of the projective space.]

The mapping of elements of one projective space onto the elements of another projective space is called a *collineation* if (a) the dimensions of the two spaces are the same; (b) the mapping is one-to-one onto; (c) the incidence relation is preserved.

In order to appreciate that these requirements on a collineation are precisely the same requirements as in Uhlhorn's generalization of the Wigner theorem, it is necessary only to note that condition (c) is equivalent to requiring the ray mapping to preserve zero probabilities. To make this equivalence explicit, let V_1 and V_2 be subspaces of $\mathcal{H}$, and let V_1 be incident with V_2—that is, $V_1 \subset V_2$. To translate this into a statement on inner products, denote by $V_2^{\perp}$ the orthogonal complement of V_2 in $\mathcal{H}$. Then it follows that the incidence relation $(V_1 \subset V_2)$ is *equivalent* to the inner product statement that

$$(V_2^{\perp}, V_1) = 0.$$

Since zero probabilities are to be preserved under the ray mapping, we conclude that $\langle (V_2^{\perp})', V_1' \rangle = 0$, which is the assertion that $V_1' \subset V_2'$; that is, the incidence relation is preserved under the ray mapping. Thus, we have established that a collineation is nothing else than the ray mapping of the generalized Wigner theorem.

We need only one more concept before we can use this "translation guide" to take over the basic results known from projective geometry. This is the concept of a *semilinear mapping* (Artin [7]).

Let V and V' be vector spaces over the field F. *A map* $\mathfrak{U}: V \to V'$ *is called semilinear with respect to the automorphism* μ *of* F *if*

$$\mathfrak{U}(x+y)=\mathfrak{U}(x)+\mathfrak{U}(y), \qquad \mathfrak{U} \text{ is additive,}$$
$$\mathfrak{U}(x\alpha)=\mathfrak{U}(x)\alpha^{\mu}$$

for all vectors $x, y \in V$, *and all* $\alpha \in F$. (Here μ is the automorphism: $\alpha \to \alpha^{\mu}$.)

One recognizes that the semilinear maps are precisely the general concept that is needed for interpreting the Wigner theorem, since the two cases found for this theorem [$\lambda(\alpha)=\alpha$, and $\lambda(\alpha)=\alpha^*$] are just the two (identity and conjugation) continuous automorphisms of the complex numbers.

With these preliminaries completed, we can now cite the fundamental theorem of projective geometry (Artin [7], Baer [8]).

Theorem (Fundamental theorem of projective geometry). Let V and V' be left vector spaces of equal dimension $n \geq 3$ over fields k respectively k', $\bar{V}$ and $\bar{V}'$ the corresponding projective spaces. Let σ be a one-to-one (onto) correspondence of the "points" of $\bar{V}$ and the "points" of $\bar{V}'$ which has the following property: Whenever three distinct "points" L_1, L_2, L_3 (they are lines of V) are collinear: $L_1 \subset L_2 + L_3$, then their images are collinear: $\sigma L_1 \subset \sigma L_2 + \sigma L_3$. Such a map can of course be extended in at most one way to a collineation but we contend more. There exists an isomorphism μ of k onto k' and a semi-linear map λ of V onto V' (with respect to μ) such that the collineation which λ induces on $\bar{V}$ agrees with σ on the points of $\bar{V}$. If λ_1 is another semi-linear map with respect to an isomorphism μ_1 of k onto k' which also induces this collineation, then $\lambda_1(X)=\lambda(\alpha X)$ for some fixed $\alpha \neq 0$ of k and the isomorphism μ_1 is given by $x^{\mu_1}=(\alpha x \alpha^{-1})^{\mu}$. For any $\alpha \neq 0$ the map $\lambda(\alpha X)$ will be semi-linear and induce the same collineation as λ. The isomorphism μ is, therefore, determined by σ up to inner automorphisms of k. (Artin [7, p. 88].)

Remarks. (*a*) We have cited this theorem in the form given by Artin. The proof of the theorem given by Artin is very accessible to physicists, and it begins in a way rather similar to the proof of Wigner's theorem given above.

(*b*) Note the restriction to $\dim V \geq 3$. The two-dimensional case is exceptional and is discussed briefly below in item (*b*) of the Appendix to this section.

(*c*) Note that Bargmann's result [see Remark (*d*) in Section 2] is contained as a special case in the conclusion of this general theorem.

(*d*) Uhlhorn [6] has presented a proof of the generalized Wigner theorem and notes the relation to the fundamental theorem of projective geometry. In an appendix, Uhlhorn discusses critically various formulations and proofs of Wigner's theorem in the literature (Wigner [2, 9], Jauch [10], Hagedorn [11, 12], Lomont and Mendelson [13]). Bargmann [3] notes that Uhlhorn's discussion of the two-dimensional quaternion case is incorrect [see item (*b*) in the Appendix below].

(*e*) There has been much interest in recent years in a modification of the quantum theoretic formalism, which consists in replacing the complex Hilbert space by a quaternionic Hilbert space (Finkelstein *et al.* [14–16]). The general theorem asserts for this case that the semilinear mapping $\mathcal{U}$ compatible with a given ray mapping T is unique to within an inner automorphism of the quaternionic field (there are no outer automorphisms). It follows that the semilinear mapping can be taken to be *linear*, and $\mathcal{U}$ (if taken linear) is determined by T to within an overall sign.

Note in particular that conjugation in the quaternion field is an *anti-automorphism* and is *excluded.*

Appendix to Section 3. (*a*) In our search for the most basic framework in which to incorporate rotational symmetry, we have been led to the Wigner theorem of Section 1 and then to its generalization, the fundamental theorem of projective geometry. This result, aesthetically satisfying in itself, raises several interesting questions, which—although not strictly within the scope of the present monograph—deserve at least some mention along with citation of the relevant literature; hence, this Appendix.

The questions suggested by the results of Section 3 are these: (*i*) What is the relationship between projective geometry and quantum mechanics? (There must surely be *some* relationship, since the Wigner theorem has clearly been incorporated in this wider context.) (*ii*) The theorem of Section 3 is valid for a *general* (associative) field of scalars, which includes, in particular, the real numbers, the complex numbers, and the quaternions. Is there a special role for the complex number field in quantum mechanics? In other words, are there more general types of quantum mechanics hinted at in the generalized Wigner theorem?

These questions lie at the foundation of quantum mechanics; they received serious consideration very soon after the complex Hilbert space formulation of quantum mechanics. Early fundamental work was done by Birkhoff and von Neumann [17], who endowed propositions connected with yes or no experiments (measurement theory) with an algebraic structure. These authors created lattice theory and an associated calculus of propositions that generalized (and of course includes) the measurement theory underlying classical mechanics (Boolean lattices). The propositional calculus was further developed by Jauch [18], Emch [19], Piron [20], and Varadarajan

[21], among others, clarifying in particular both the concept of *compatible* propositions (simultaneously measurable observables), and the relationship of the propositional calculus to projective geometry. The situation as developed in Ref. [18] is a very satisfactory, if not final, synthesis, and may be summarized in this way: (*i*) The propositional calculus is a measurement theoretic structure encompassing both quantal and classical mechanics; (*ii*) every proposition system is a direct union of irreducible proposition systems; (*iii*) every irreducible proposition system is embeddable in a canonical way into a projective geometry; and (*iv*) every projective geometry (dimension three or greater) is algebraically isomorphic with the linear manifolds of a vector space with coefficients from a field. This synthesis nicely disposes of the first question above and shows incidentally that the generalized Wigner theorem is (at least for $n \geqslant 3$) the best possible.

To avoid a misleading impression, let us note that there are two other distinct approaches to the foundations of quantum mechanics: the Jordan algebraic approach (Jordan [22, 23], Emch [19], Gürsey [23a]), which trades the associative, noncommutative matrix algebra of Heisenberg and Dirac (complex Hilbert space) for a nonassociative commutative algebra of observables; and the density matrix approach defined in terms of positivity domains (Koecher [24, 25]) and positive homogeneous cones (Vinberg [26]). The Jordan algebraic approach was the first to show the possibility of an exceptional type of quantum mechanics (which is restricted precisely to the case of projective dimension two, excluded in the fundamental theorem).

What can be said as to the nature of the field? A basic result is that of Birkhoff and von Neumann [17] (see also Jauch (18, p. 130]), who showed that in the propositional calculus the axiom of orthocomplementation, which yields a (physically essential) positive definite metric, also implies that the number field possesses an involutive anti-automorphism. The real, complex, and quaternion fields all have this property, and are suitable candidates for developing three types of quantum mechanics.

Stueckelberg [27] has developed quantum mechanics over the real field; in effect, the proposition system that results is equivalent to that of the standard complex Hilbert space, since a unitary antisymmetric operator J commuting with all observables must be adjoined.

Interest in quaternionic quantum mechanics has long existed (C. N. Yang, for example, has often commented on this possibility). The hope was expressed, at one time, that the extra quaternionic degrees of freedom might be connected with isospin degrees of freedom. A thorough investigation was carried out by Finkelstein *et al.* [14–16]. The requirement that quaternionic quantum mechanics be compatible with special relativity is severely restricting (Emch [28]) and specializes the role of one of the three quaternionic units. The result is, in effect, a variant form of standard complex quantum mechanics.

(*b*) Projective dimension two plays a very special role in these foundational questions. We have already mentioned that it is precisely for projective dimension two that an exceptional quantum mechanics (Refs. [18, 23]) exists. Let us note here a result of Bargmann [3], which shows that *the Wigner theorem does not hold for quaternionic quantum mechanics in two (projective) dimensions.*

4. Implications for Rotational Symmetry

In Chapters 2, 3, and 6 of AMQP, the concept of rotational symmetry was introduced physically, taking as fundamental the postulate that space is isotropic, so that no particular direction is to be preferred in any way. The state of a system was described by a wave function, Φ, as a function of the spatial coordinates, which referred to a fixed (classical) reference frame. Under a rotation, a second observer would ascribe to the state of the system another wave function—call it $\mathcal{O}_R\Phi$—if this second observer were to describe all physical systems and all physical quantities exactly as the first observer would, except that the second observer uses a reference frame that is rotated with respect to the reference frame of the first observer. [Right-handed reference frames $F=(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $F'=(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ of the first and second observers are related by the real, proper, orthogonal transformation $\hat{e}'_i = \sum_j R_{ij}\hat{e}_j$, so that the coordinates $\mathbf{x}=(x_1, x_2, x_3)$ and $\mathbf{x}'=(x'_1, x'_2, x'_3)$ of a point P referred to F and F', respectively, are related by $\mathbf{x}'=R\mathbf{x}$ (column matrix notation), where $R=(R_{ij})$.]

The two aspects of a transformation, alias versus alibi (see Chapter 2, Section 2, AMQP), can be seen to correspond to two observers using frames F and F' and ascribing wave functions Φ and $\mathcal{O}_R\Phi$ to the same state of a physical system versus a single observer who ascribes the wave functions Φ and $\mathcal{O}_R\Phi$, respectively, to the state of a physical system and an identical system whose set of corresponding points $\{P'\}$ is related to the set of points $\{P\}$ of the first system by $\mathbf{x}'=R\mathbf{x}$ for all $\mathbf{x}'\in\{P'\}$ and $\mathbf{x}\in\{P\}$ (in this case all coordinate triples refer to an arbitrary but fixed frame). (See Wigner [2, p. 223] and Houtappel *et al.* [29].)

If the wave function depended only on the Cartesian coordinates (for example, if the system had no intrinsic spin), then we can relate the two wave functions by a point transformation and conclude that

$$(\mathcal{O}_R\Phi)(\mathbf{x}')=\Phi(\mathbf{x}). \tag{5.1.32}$$

This result asserts that for the second observer the transformed function at the transformed point has the same numerical value as the original function at the original point seen by the first observer (alias interpretation).

If, however, the physical system is of a more general type—involving, for example, intrinsic spin, for which a point transformation in three-space is undefined—then one cannot conclude that an equation of the form of Eq. (5.1.32) is necessarily valid. It is essential, therefore, in the general case—even for rotational invariance—to make use of the abstract symmetry analysis of the preceding sections.

It was the conclusion of this analysis (the Wigner fundamental theorem) that there were but two possibilities: either the transformation $\mathcal{O}_R$ is linear unitary, or it is antilinear anti-unitary.

Let us dispose of the second alternative for rotational symmetry. To do so it is necessary to consider the time dependence of the wave functions. It is sufficient for this analysis to consider a Hamiltonian with a purely discrete spectrum. Then a general time-dependent wave function will have the form

$$\psi(t)=\sum_{\kappa} a_{\kappa}\psi_{\kappa}e^{-iE_{\kappa}t/\hbar}, \tag{5.1.33}$$

where the a_κ are complex numbers, and the $\{\psi_\kappa, E_\kappa\}$ are eigenstates and eigenvalues of the Hamiltonian, respectively. If we perform the symmetry transformation at $t=0$ and consider the transformed state at time t, we obtain (using the second alternative in the symmetry theorem) the wave function

$$\psi'(t)=\sum_{\kappa} a_{\kappa}^{*}(\mathcal{O}_R\psi_{\kappa})e^{-iE_{\kappa}t/\hbar}. \tag{5.1.34}$$

If, however, the symmetry transformation is performed at time t, we obtain (using again the second alternative)

$$\psi''(t)=\sum_{\kappa} a_{\kappa}^{*}(\mathcal{O}_R\psi_{\kappa})e^{+iE_{\kappa}t/\hbar}. \tag{5.1.35}$$

~~Clearly these two results differ in an essential way (unless, trivially, all~~ $\cancel{E_\kappa}$ are the same). From a physical viewpoint, rotational invariance is a *time-independent concept*; accordingly, we must conclude that the second alternative is inadmissible and that rotational invariance is necessarily implemented by a linear unitary transformation.

This result does not, however, conclude the matter, since—according to the fundamental theorem—*there is still an overall freedom of phase in implementing the transformation*. Thus, the symmetry operation $\mathcal{O}_R$, although linear unitary, is still undetermined to the extent that one may use equally well $\mathcal{O}'_R=\omega_R\mathcal{O}_R$, $|\omega_R|=1$. Note that the constant ω_R can depend on the particular rotation, although it is the same for all wave functions.

These constants can spoil the representation property, since the product of two rotation operators need then only satisfy the equation

$$\mathcal{O}_R\mathcal{O}_S=\omega_{R.S}\mathcal{O}_{RS}. \tag{5.1.36}$$

The arbitrariness in this multiplication rule is best understood by considering that the rotations R and S are actually transformations belonging to the covering group, the quantal rotation group, $SU(2)$. In this case, with each $U \in SU(2)$, we associate the proper orthogonal matrix R having elements $R_{ij} = \frac{1}{2}\mathrm{tr}(\sigma_i U \sigma_j U^\dagger)$ (see Chapter 2, AMQP); each U then defines the transformation of reference frames given by $\hat{e}_i \to \hat{e}'_i = \sum_j R_{ij}\hat{e}_j$. Note that both U and $-U$ define the same frame rotation.

The arbitrariness in the multiplication rule (5.1.36) may be removed for the quantal rotation group by a continuity method due to Weyl (Wigner [2, Chapter 20]). Clearly, for the identity operator, there is no arbitrary phase, so that $\omega_{\text{identity}} = 1$. The phase ω_U is a continuous function of the paths in group parameter space that go from the point corresponding to the identity rotation to the point corresponding to the rotation U (these paths are curves on the surface of the unit sphere, S^3, in four-space—see Chapter 2, AMQP). Since the parameter space S^3 is simply connected, the phase ω_U is a function only of the point $(\alpha_0, \boldsymbol{\alpha})$ of S^3 corresponding to U, and not of the path. Thus, there is a unique phase for ω_U, and this may be removed by a convention so that there is no arbitrariness at all in the product of two rotation operators:

$$\mathcal{O}_U \mathcal{O}_V = \mathcal{O}_{UV} \tag{5.1.37}$$

for all $U, V \in SU(2)$.

The final result of these considerations is this: *The symmetry transformations of the quantal rotation group are necessarily implemented by linear unitary transformations* $\{\mathcal{O}_U : \mathcal{H} \to \mathcal{H}\}$ *such that for all wave functions* Φ *and* Ψ *in* $\mathcal{H}$ *one has*

$$(\Psi, \Phi) = (\mathcal{O}_U \Psi, \mathcal{O}_U \Phi) \tag{5.1.38}$$

and

$$\mathcal{O}_U(a\Psi + b\Phi) = a\mathcal{O}_U\Psi + b\mathcal{O}_U\Phi, \qquad a, b \in \mathbb{C}. \tag{5.1.39}$$

The implementation of the second equation requires a convention as to an overall phase factor (of modulus 1*) independent both of the rotation and of the wave functions.*

The result obtained here for the quantal rotation group includes the case (5.1.32) of wave functions depending only on Cartesian coordinates, since one may define

$$(\mathcal{O}_U f)(R\mathbf{x}) = f(\mathbf{x}), \tag{5.1.40}$$

where R is the image of U in the homomorphism (given above) of $SU(2)$

onto $SO(3)$. On the Hilbert space of square-integrable functions of coordinates, one then has the identity of rotation operators given by

$$\mathcal{O}_R = \mathcal{O}_U = \mathcal{O}_{-U}. \tag{5.1.41}$$

The transformation (5.1.40) also generalizes to systems having spin-j. Thus, letting $\{\xi_m: m=j, j-1,\ldots, -j\}$ denote an orthonormal basis of spin space, we introduce the functions in the tensor product space of square-integrable functions and spin functions of the form

$$\Psi = \sum_m \psi_m \otimes \xi_m, \tag{5.1.42}$$

where $\Psi(\mathbf{x})$ denotes the vector in spin space given by

$$\Psi(\mathbf{x}) = \sum_m \psi_m(\mathbf{x})\xi_m.$$

The inner product of two functions Ψ and Φ is defined by

$$(\Psi, \Phi) = \sum_m \int \psi_m^*(\mathbf{x})\phi_m(\mathbf{x})\, d\mathbf{x}. \tag{5.1.43}$$

Next, let $\mathcal{U}^{(j)}$ denote the linear unitary rotation operator realization of $U \in SU(2)$ that has the standard action on spin space given by

$$\mathcal{U}^{(j)}\xi_m = \sum_{m'} D^j_{m'm}(U)\xi_{m'}, \tag{5.1.44}$$

where the $D^j(U)$ denote the standard unitary irreps of $SU(2)$ [see Eq. (3.12)].

The desired definition of the transformation $\mathcal{O}_U$, acting in the Hilbert space of functions $\{\Psi\}$, is

$$(\mathcal{O}_U\Psi)(\mathbf{x}) \equiv \left[(1 \otimes \mathcal{U}^{(j)})\Psi\right](R^{-1}\mathbf{x}) = \sum_m \psi_m(R^{-1}\mathbf{x})\left(\mathcal{U}^{(j)}\xi_m\right), \tag{5.1.45}$$

where the 3×3 proper orthogonal matrix $R=(R_{ij})$ and the 2×2 unitary unimodular matrix U are related by $R_{ij} = \frac{1}{2}\operatorname{tr}(\sigma_i U \sigma_j U^\dagger)$. Equivalently, one has the relation

$$\left[(1 \otimes \mathcal{U}^{(j)\dagger})(\mathcal{O}_U\Psi)\right](R\mathbf{x}) = \Psi(\mathbf{x}) \tag{5.1.46}$$

as the generalization of Eq. (5.1.40).

We can now readily understand the arbitrariness in the multiplication rules (5.1.36) for the rotation operators corresponding to $SO(3)$ rotations, using Eq. (5.1.45). Observe that we have the identity between operators given by

$$\mathcal{O}_{-U} = (-1)^{2j}\mathcal{O}_U \tag{5.1.47}$$

on the Hilbert space of functions $\{\Psi\}$. Thus, for integral spin systems, we may always take

$$\mathcal{O}_R = \mathcal{O}_U = \mathcal{O}_{-U}. \tag{5.1.48}$$

Since R is the image of both U and $-U$ in the homomorphism of $SU(2)$ onto $SO(3)$, the multiplication rule (5.1.36) with $\omega_{R,S} = 1$ now follows from Eq. (5.1.37); thus, all phase factors can be removed from Eq. (5.1.36) for systems possessing integral spin. For half-integral spin, however, we find that

$$\mathcal{O}_{-U} = -\mathcal{O}_U, \tag{5.1.49}$$

and there is an essential ± 1 phase in identifying $\mathcal{O}_R$ with the pair of operators $(\mathcal{O}_U, \mathcal{O}_{-U})$ that cannot be removed (for further discussion of this point, see Section 21, Chapter 6, AMQP). The situation represented by the functions $\{\Psi\}$ is general.[1]

It will be readily appreciated that the quantal treatment of rotations as developed in AMQP and in this monograph stems in the final analysis from Wigner's fundamental theorem on symmetry transformations.

Remark. Let us define the product of two functions Ψ and Φ of the type (5.1.42) to be

$$(\Psi\Phi)(\mathbf{x}) = \sum_m \psi_m(\mathbf{x})\phi_m(\mathbf{x})\xi_m. \tag{5.1.50}$$

Then one finds that

$$\mathcal{O}_U(\Psi\Phi) = (\mathcal{O}_U\Psi)(\mathcal{O}_U\Phi). \tag{5.1.51}$$

This result, in turn, implies (under suitable differentiability conditions) the *derivation property* for the total angular momentum (orbital plus spin)

[1]The proper orthogonal group $SO(3)$ is doubly covered by the unitary unimodular group $SU(2)$. It is this double covering that leads to an essential ambiguity of (± 1) phase factor in identifying the irreps of $SO(3)$ with those of $SU(2)$ in the case that j is half-integral. In the strict sense these so-called "two-valued" representations of $SO(3)$ are not representations at all, and we choose, in such cases, to consider the quantal rotation group itself.

operator $\mathbf{J}=\mathbf{L}\otimes 1+1\otimes \mathbf{S}$:

$$\mathbf{J}(\Psi\Phi)=(\mathbf{J}\Psi)\Phi+\Psi(\mathbf{J}\Phi). \tag{5.1.52}$$

(The operator $\mathbf{J}$ is the Lie algebraic generator of the transformations $\mathcal{O}_U$.)

The property expressed by Eq. (5.1.52) is not so fundamental as the linear unitary character of the transformations $\mathcal{O}_R$ and $\mathcal{O}_U$, and, indeed, fails to be satisfied by the $\mathcal{O}_R$ when j is half-integral (Wigner [2, pp. 107, 231]). The derivation property (5.1.52) is often taken for granted in the physics literature, but, in fact, it may be invalid for the basis elements of Lie algebras associated with physical systems (see, for example, Chapter 7, Section 4, AMQP, and Ref. [30]).

5. Time Reversal

The Wigner theorem shows that a symmetry transformation in quantum mechanics is necessarily implemented by either a linear unitary or an antilinear anti-unitary mapping of the Hilbert space; it follows from a slight extension of the argument for the case of rotational symmetry (Section 4) that any symmetry operator[1] leaving the sense of time invariant is necessarily linear unitary. The purpose of the present section is to investigate symmetries in which the sense of time is reversed, and to show that the alternative in the Wigner theorem can be physically important.

The symmetry group of an isolated physical system consists, as we have discussed in Chapter 1 of AMQP, of the transformations of the Poincaré group—that is, space and time translations, rotations, and Lorentz transformations.

It is of interest physically to consider the possibility of adjoining discrete symmetries to the Poincaré group. These symmetries include spatial inversion (parity) and time reversal, but classical physics is no longer a guide, and the physical existence of such symmetries is a matter of experiment[2] (Dirac [31]). For the spectroscopy of such an isolated physical system, we have discussed in Chapter 7 in AMQP the logical (and practical) possibility of choosing that reference frame in which the total linear momentum vanishes. Accordingly, one deals with a finite manifold of states of definite energy—that is, the internal energy of the system under study. Out of the Poincaré group of symmetries, only the rotational subgroup leaves this

[1] We generally use the term "symmetry operator" to apply directly to the linear unitary or antilinear anti-unitary extension of a symmetry operator as defined on p. 159 and acting on rays.

[2] This remark is now commonplace after the discovery of parity violation in 1957. What is remarkable, however, is that this observation was made explicitly by Dirac (1949) in reference to the possible nonexistence of reflection symmetry!

manifold invariant. If we now consider time reversal—that is, the replacement of t by $-t$—then this operation will transform the state ψ into the time-reversed state, denoted by $\mathfrak{T}\psi$, in which all linear momenta are reversed in direction. [A more appropriate name for time reversal would therefore be "motion reversal" (*Bewegungsumkehr*).]

Since time reversal is to be a symmetry, the operation $\mathfrak{T}$ must be either linear or antilinear. Let us show that $\mathfrak{T}$ cannot be linear. Consider the time-dependent state ψ, which is a linear combination of energy eigenstates:

$$\psi(t)=\sum_{\kappa} a_{\kappa}\psi_{\kappa}e^{-iE_{\kappa}t/\hbar}, \qquad a_{\kappa}\in\mathbb{C}. \tag{5.1.53}$$

If $\mathfrak{T}$ is *linear* and we perform a time reversal at $t=0$, then at time t we obtain

$$\psi_1(t)=\sum_{\kappa} a_{\kappa}(\mathfrak{T}\psi_{\kappa})e^{-iE_{\kappa}t/\hbar}. \tag{5.1.54}$$

Physically, however, the state $\psi_1(t)$ must be the same as that obtained by letting time run from $t=0$ to $-t$, and then applying $\mathfrak{T}$.[1]

The latter operation, however, yields the state

$$\psi_2(t)=\sum_{\kappa} a_{\kappa}(\mathfrak{T}\psi_{\kappa})e^{+iE_{\kappa}t/\hbar}, \tag{5.1.55}$$

and clearly ψ_1 and ψ_2 are not the same. Thus, a *linear* time reversal operator is *excluded*, and $\mathfrak{T}$ is therefore antilinear.

The simplest antilinear anti-unitary operator is the operator K_0, which is defined to be the mapping $K_0: \mathcal{H}\to\mathcal{H}$ given by $K_0\Psi=\Psi^*$ for each $\Psi\in\mathcal{H}$. (The notation Ψ^* denotes the wave function whose values are the complex conjugate of the values of Ψ.) The operator K_0 is a symmetry, since it leaves all transition probabilities invariant. The operator K_0 is antilinear and anti-unitary, since $K_0(a\Psi+b\Phi)=a^*(K_0\Psi)+b^*(K_0\Phi)$ and $(K_0\Psi, K_0\Phi)=(\Psi,\Phi)^*$. It is also an involution; that is, $K_0^2=\mathbb{1}=$unit operator in $\mathcal{H}$.

Consider now the product of the time reversal operator with K_0—that is, the operator $\mathfrak{T}K_0$. This operator may be verified to be a symmetry (as is any product of symmetry operators), and since the product of any two antilinear anti-unitary operators is always a linear unitary operator, it follows that $\mathfrak{T}K_0$ is a linear unitary operator $\mathfrak{U}$. Thus, the time reversal operator— in fact, every antilinear anti-unitary operator—may be written in the standard form

$$\mathfrak{T}=\mathfrak{U}K_0 \tag{5.1.56}$$

for some linear unitary operator $\mathfrak{U}$.

[1]This is the same statement as saying that two time reversals leave all velocities unchanged, so that [(time displacement by t)$\times$(time reversal)]^2 is equivalent to the unit operation.

The action of the time reversal operator in the inner product in $\mathcal{H}$ is given by

$$(\mathfrak{T}\Psi,\mathfrak{T}\Phi)=(\mathfrak{U}K_0\Psi,\mathfrak{U}K_0\Phi)=(K_0\Psi,K_0\Phi)$$
$$=(\Psi,\Phi)^*=(\Phi,\Psi). \tag{5.1.57}$$

Recall now that we used above a physical argument that implies that the square of the time reversal operator is the identity operation. Let us note once again that the group to be considered in deriving the consequences of symmetry considerations is not the group of physical transformations, but (as Wigner has emphasized) *the group of the quantum mechanical operators that correspond to the physical transformations*. [This is the reason for our use of the quantum mechanical rotation group $SU(2)$ and not $SO(3)$.] When this argument is applied to the time reversal operator, one sees that $\mathfrak{T}^2$ may be a *multiple* of the identity operator $\mathbb{1}$ and is not necessarily the identity itself—that is,

$$\mathfrak{T}^2=c\mathbb{1}. \tag{5.1.58}$$

where $|c|=1$, since $\mathfrak{T}^2$ is linear unitary.[1] Let us show that $c=\pm 1$.

Proof. Using the standard form of $\mathfrak{T}$ given by Eq. (5.1.56), the unitary property $\mathfrak{U}\mathfrak{U}^\dagger=\mathbb{1}$ of $\mathfrak{U}$, and the involution property $K_0^2=\mathbb{1}$ of K_0, we find: $\mathfrak{T}^2=\mathfrak{U}K_0\mathfrak{U}K_0=c\mathbb{1}=c\mathfrak{U}\mathfrak{U}^\dagger$, which implies $K_0\mathfrak{U}K_0=c\mathfrak{U}^\dagger$ and $\mathfrak{U}=K_0(c\mathfrak{U}^\dagger)K_0=c^*K_0\mathfrak{U}^\dagger K_0$; hence, $c\mathbb{1}=\mathfrak{U}(c\mathfrak{U}^\dagger)=c^*(K_0\mathfrak{U}^\dagger K_0)(K_0\mathfrak{U}K_0)=c^*\mathbb{1}$; that is, $c=c^*$. Thus, the number c is real, and, since it has modulus 1, we obtain the desired result, $c=\pm 1$. ■

We shall show in the next section that the sign in $\mathfrak{T}^2=\pm\mathbb{1}$ is an important physical quantum number that distinguishes the two superselection sectors: fermions versus bosons.

It remains to determine the time reversal operator explicitly for specific physical systems having rotational symmetry. To do so, we first observe that physical observables split into two classes under time reversal: observables such as position and energy, which are invariant, and observables such as linear and angular momenta, which change sign. In the Schrödinger realization, where the coordinate observable x_i is a multiplication operator, we have

$$\mathfrak{T}x_i=x_i\mathfrak{T}. \tag{5.1.59}$$

[1] It is important to note that this result (unlike the convention defining the rotation operators, see p. 171) does not represent a free choice. By the fundamental theorem, the only free choice in an anti-unitary operator is conjugation by a complex constant: $\mathfrak{T}\to a^{-1}\mathfrak{T}a=(a^*/a)\mathfrak{T}$. This causes no change in $\mathfrak{T}^2$, since $(a^*/a)\mathfrak{T}(a^*/a)\mathfrak{T}=\mathfrak{T}^2$.

Similarly, the momentum observable is realized by $p_i = -i\hbar(\partial/\partial x_i)$, which implies

$$\mathfrak{T}p_i = \mathfrak{T}\left(-i\hbar\frac{\partial}{\partial x_i}\right) = +i\hbar\frac{\partial}{\partial x_i}\mathfrak{T} = -p_i\mathfrak{T}. \tag{5.1.60}$$

For orbital angular momentum, defined by $\hbar\mathbf{L}=\mathbf{x}\times\mathbf{p}$, we thus obtain the result

$$\mathfrak{T}\,\mathbf{L} = -\mathbf{L}\mathfrak{T} \qquad \text{or} \qquad \mathfrak{T}\,\mathbf{L}\mathfrak{T}^{-1} = -\mathbf{L}. \tag{5.1.61}$$

Clearly, this same result must hold for the total angular momentum:

$$\mathfrak{T}\,\mathbf{J} = -\mathbf{J}\mathfrak{T}. \tag{5.1.62}$$

It is the implementation of this requirement that determines the form of $\mathfrak{T}$. The only operators available for this purpose (in a nonrelativistic theory) are the spin operators themselves.

Let us now be quite specific and consider a single particle with intrinsic spin-j. The states of such a system are described by functions of the form (5.1.42). We shall, however, now specify that the orthonormal vectors in the set $\{\xi_m : m=j, j-1, \ldots, -j\}$ be *real column matrices* on which the spin operators $(S_1, S_2, S_3)=\mathbf{S}$ have the action given by

$$S_i\xi_m = \sum_{m'} \left(J_i^{(j)}\right)_{m'm}\xi_{m'}, \tag{5.1.63}$$

where the $J_i^{(j)}$ are the standard $(2j+1)\times(2j+1)$ angular momentum matrices (see Chapter 3, Section 4, AMQP).

A general state vector Ψ is now presented as a column matrix of the form

$$\Psi = \sum_m \psi_m\xi_m;\ \text{that is, } \Psi(\mathbf{x}) = \sum_m \psi_m(\mathbf{x})\xi_m \tag{5.1.64}$$

with complex conjugate given by

$$K_0\Psi = \Psi^* = \sum_m \psi_m^*\xi_m;\ \text{that is, } \Psi^*(\mathbf{x}) = \sum_m \psi_m^*(x)\xi_m. \tag{5.1.65}$$

The actions of the orbital angular momentum $\mathbf{L}$ and the intrinsic spin $\mathbf{S}$ on a state Ψ are given by

$$\mathbf{L}\Psi = \sum_m (\mathbf{L}\psi_m)\xi_m,$$

$$\mathbf{S}\Psi = \sum_m \psi_m(\mathbf{S}\xi_m). \tag{5.1.66}$$

(Note that $\mathbf{L}$ and $\mathbf{S}$ commute, as they must.)

We find: For spin-0 particles the time reversal operator may be taken to be the conjugation operator itself; that is,

$$\mathcal{T}=K_0. \tag{5.1.67}$$

For spin-$\frac{1}{2}$ particles the state vectors are of the form

$$\Psi=\psi\xi_{\frac{1}{2}}+\phi\xi_{-\frac{1}{2}} \tag{5.1.68}$$

and

$$K_0\Psi=\Psi^*=\psi^*\xi_{\frac{1}{2}}+\phi^*\xi_{-\frac{1}{2}}. \tag{5.1.69}$$

One now finds that the time reversal operator that satisfies Eq. (5.1.61) with respect to the orbital angular momentum $\mathbf{L}$ and an equation of type (5.1.62) with respect to the spin $\mathbf{S}=\boldsymbol{\sigma}/2$ and the total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{S}$ is

$$\mathcal{T}=-i\sigma_2 K_0, \tag{5.1.70}$$

where σ_2 is the Pauli operator[1] defined in terms of the Pauli matrix $\sigma_2=2J_2^{(\frac{1}{2})}$ by Eq. (5.1.63).

The time reversal operator appropriate to a relativistic spin-$\frac{1}{2}$ particle (for example, the Dirac electron) is the same form as Eq. (5.1.70), except that the Pauli spin operator $i\sigma_2$ is replaced by an operator[2] whose action on the four-component Dirac spinor is given by the 4×4 matrix $i\sigma_2\otimes 1_2$ (matrix direct product of the Pauli matrix $i\sigma_2$ with the 2×2 unit matrix 1_2).

From this explicit realization of the time reversal operator (in both the relativistic and nonrelativistic cases), it follows that

$$\cancel{\mathcal{T}^2=-1 \qquad (\text{spin-}\tfrac{1}{2}\text{ case}),} \tag{5.1.71}$$

whereas for spin-0 one has

$$\mathcal{T}^2=+1 \qquad (\text{spin-0 case}). \tag{5.1.72}$$

[1] The choice of $i\sigma_2$ versus σ_2 is of no consequence and is made solely to have $i\sigma_2$ and K_0 commute.

[2] In terms of Dirac operators, one has $\mathcal{T}=\frac{1}{2}[\gamma_1,\gamma_2]K_0$. The relativistic time reversal operator was given incorrectly in the literature for many years, since the necessity for antilinearity was not recognized. (See, for example, Refs. [32, 33].) It was pointed out in Ref. [34] that Wigner's time reversal operator (explicitly defined to be nonrelativisitic—see Ref. [37a, p. 548]) was, in fact, formally relativistic as well; the Schwinger [35] definition is essentially equivalent to the Wigner definition (Emch [19, p. 152]).

These results are a special case of the general result that $\mathcal{T}^2=(-1)^{2j}1$, where j is the intrinsic spin of the particle (assumed to have nonzero rest mass). This will be shown in the next section.

6. Time Reversal and Rotations; The Frobenius–Schur Invariant

Rotations and time reversal commute as physical operators, but we must examine this result more carefully to see if the corresponding quantal operators commute. Let $\mathcal{O}_U$ denote the quantal operator associated to the rotation U. From the fact that the physical operations commute, one may conclude that

$$\mathcal{O}_U\mathcal{T}=c_U\mathcal{T}\,\mathcal{O}_U, \qquad \text{each } U\in SU(2), \tag{5.1.73}$$

where c_U is a numerical constant of modulus 1 (which can depend on the rotation U). Using the group property (5.1.37) of the rotations, $\mathcal{O}_U\mathcal{O}_V=\mathcal{O}_{UV}$, one finds that Eq. (5.1.73) implies

$$c_Uc_V=c_{UV}; \tag{5.1.74}$$

that is, the constants $\{c_U\}$ form a one-dimensional representation of the quantal rotation group $SU(2)$. There is only one such representation: the identity irrep. Thus, all $\{c_U\}$ are unity; that is

$$\mathcal{O}_U\mathcal{T}=\mathcal{T}\,\mathcal{O}_U, \qquad \text{each } U\in SU(2). \tag{5.1.75}$$

This result is quite important for the representations of the quantal rotation group. Let us apply it, in particular, to the spin-$\frac{1}{2}$ case, where the action of $\mathcal{O}_U$ and $\mathcal{T}$ on an arbitrary state Ψ is given by Eqs. (5.1.44), (5.1.45), and (5.1.64)–(5.1.70). The implication of Eq. (5.1.75) is found to be

$$\left[D^{\frac{1}{2}}(U)\right]^*=U_0^{\dagger}D^{\frac{1}{2}}(U)U_0, \tag{5.1.76}$$

or, equivalently,

$$U^*=U_0^{\dagger}UU_0, \tag{5.1.77}$$

where U_0 is the 2×2 unitary unimodular matrix defined by

$$U_0=-i\sigma_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{5.1.78}$$

This equation asserts that the complex conjugate of the irrep $j=\frac{1}{2}$ is equivalent to the irrep $j=\frac{1}{2}$, and that, moreover, the matrix U_0 effecting the equivalence is *antisymmetric and real.* The fact that the irreps D^j and D^{j*} are equivalent is no surprise, since the (irreducible) characters of the quantal rotation group are real; but the properties of the matrix U_0 are of great interest and will lead us to the Frobenius–Schur invariant.

The generalization of Eq. (5.1.76) to the general irrep D^j is a repetition of the steps leading to that result, using now the definition of Ψ given by Eq. (5.1.64) and the action of $\mathcal{O}_U$ given by Eq. (5.1.45). We write $\mathfrak{T}$ in the standard form $\mathfrak{T}=\mathfrak{U}_0 K_0$, where $\mathfrak{U}_0$ is a linear unitary operator with the action on Ψ given by

$$\mathfrak{U}_0\Psi=\sum_m \psi_m(\mathfrak{U}_0\xi_m),$$
$$\mathfrak{U}_0\xi_m=\sum_{m'} A_{m'm}(\mathfrak{U}_0)\xi_{m'}. \tag{5.1.79}$$

We find that

$$[D^j(U)]^*=A^\dagger(\mathfrak{U}_0)D^j(U)A(\mathfrak{U}_0), \qquad \text{each } U\in SU(2), \tag{5.1.80}$$

where $A(\mathfrak{U}_0)$ is the unitary matrix representation of the unitary operator $\mathfrak{U}_0$ in spin space given by Eq. (5.1.79).

Let us now show that the most general form of $A(\mathfrak{U}_0)$ is

$$A(\mathfrak{U}_0)=cD^j(U_0), \tag{5.1.81}$$

where $|c|=1$, and U_0 is the matrix (5.1.78).

Proof. Observe that the $A(\mathfrak{U}_0)$ given by Eq. (5.1.81) satisfies Eq. (5.1.80) in consequence of the property (5.1.77), the unitary property of $D^j(U_0)$, and the group property of the matrices D^j—that is, $[D^j(U_0)]^\dagger D^j(U)D^j(U_0)=D^j(U_0^\dagger U U_0)=D^j(U^*)=[D^j(U)]^*$, since the D^j are real functions of the u_{ij}. Consider next the solution $D^j(U_0)$ to Eq. (5.1.80) and a second solution $A(\mathfrak{U}_0)$. Using the unitary property of these solutions, we find that the matrix $D^j(U_0)[A(\mathfrak{U}_0)]^{-1}$ commutes with $D^j(U)$ for all $U\in SU(2)$. Schur's lemma now implies that $A(\mathfrak{U}_0)=cD^j(U_0)$, and $|c|=1$ follows from the unitary property of the solutions. ■

The symmetry property under matrix transposition,

$$\tilde{A}(\mathfrak{U}_0)=(-1)^{2j}A(\mathfrak{U}_0), \tag{5.1.82}$$

follows from the properties $\tilde{D}^j(U_0)=D^j(\tilde{U}_0)=D^j(-U_0)=(-1)^{2j}D^j(U_0)$ of the irreps D^j. Correspondingly, we obtain the identity

$$\mathcal{T}^2 = (-1)^{2j}1 \tag{5.1.83}$$

on the space of spin-j wave functions $\{\Psi\}$ given by Eq. (5.1.42).

We generally choose $c = 1$ in Eq. (5.1.81), thus defining $\mathcal{U}_0$ by

$$\mathcal{U}_0 \xi_m = \sum_{m'} D^j_{m'm}(U_0)\xi_{m'}. \tag{5.1.84}$$

A useful alternative form for the matrix $D^j(U_0)$ is found frequently in the literature. Noting that $U_0 = \exp(-i\pi\sigma_2/2)$ and using the general property of the irrep D^j given by

$$D^j(e^{-i\phi\hat{n}\cdot\sigma/2}) = e^{-i\phi\hat{n}\cdot\mathbf{J}^{(j)}}, \tag{5.1.85}$$

we obtain

$$D^j(U_0) = e^{-i\pi J_2^{(j)}}. \tag{5.1.86}$$

In these results $J_2^{(j)}$ denotes the standard $(2j+1)\times(2j+1)$ Hermitian matrix representation of J_2 (see Chapter 3, Section 4, AMQP).

It is important to recognize that the matrix $D^j(U_0)$ is *dependent* on the phase conventions that determined $D^j(U)$—that is, on the conventions of fixing the angular momentum matrices—see Section 3 and Note 1, Chapter 3, AMQP.

The symmetry property of $A(\mathcal{U}_0)$ expressed by Eq. (5.1.82) for arbitrary $|c| = 1$, or, equivalently, by

$$A(\mathcal{U}_0)A^*(\mathcal{U}_0) = \left[D^j(U_0)\right]^2 = (-1)^{2j}1^{(j)}, \tag{5.1.87}$$

where $1^{(j)}$ denotes the unit matrix of dimension $2j+1$, is, however, *independent* of (unitary) basis. (This result is easily proved by replacing the irreps D^j by the irreps $\bar{D}^j = C^\dagger D^j C$ that one obtains by making an arbitrary unitary change of spin basis: $\xi_m \to \sum_{m'} c_{m'm}\xi_{m'}$.) The result expressed by Eq. (5.1.87) is none other than the familiar "rotation by 2π," which distinguishes half-integer from integer angular momenta.

In the present context, we see that the quantum number $(-1)^{2j}$ is determined by time reversal symmetry. Although it is frequently said that time reversal does not lead to any quantum number—since $\mathcal{T}$ is antiunitary—this is not strictly correct, as we now see, since $\mathcal{T}^2$ *is* indeed unitary, and this, in turn, leads to the quantum number, $\mathcal{T}^2 \to (-1)^{2j}$, splitting (massive) particles into two types: integer versus half-integer intrinsic spin.

Let us return to considering the matrix $A(\mathfrak{U}_0)=D^j(U_0)$ [choosing $c=1$ in Eq. (5.1.81)]. It is an easy consequence of the explicit form of $D^j(U)$ given by Eq. (3.12) that

$$D^j_{m'm}(U_0)=(-1)^{j-m}\delta_{m,-m'};\tag{5.1.88}$$

that is, this matrix is the simplest of the 3-j symbols:

$$\begin{pmatrix} j & j & 0 \\ m & m' & 0 \end{pmatrix}=(-1)^{j-m}\delta_{m,-m'}.$$

It follows that *the matrix $D^j(U_0)$ is simply the metric operator*[1] *that couples two angular momentum systems (of the same j) to an invariant.* This aspect of the matrix $D^j(U_0)$ occurs frequently in nuclear physics (see Chapter 7, Section 9, AMQP), where $D^j(U_0)$ relates particle to hole operators).

The fact that the time reversal operator is antilinear, and hence acts on complex numbers, has important implications for the consistency of phase conventions, both for the relative phases in sums of quantal operators, and also for wave functions themselves.

Let us discuss briefly this more general situation in which the angular momentum label j now refers to the total angular momentum of a physical system. We thus consider a physical system whose states have been classified by the total angular momentum quantum numbers (jm), and we denote these states by the standard Dirac ket notation $\{|(\alpha)jm\rangle\}$. Under rotation these states transform according to the standard action of the rotation operator $\mathcal{O}_U$ given by

$$\mathcal{O}_U|(\alpha)jm\rangle=\sum_{m'}D^j_{m'm}(U)|(\alpha)jm'\rangle.\tag{5.1.89}$$

Under time reversal an arbitrary state

$$\Psi=\sum_{(\alpha)jm}c_{(\alpha)jm}|(\alpha)jm\rangle,\qquad c_{(\alpha)jm}\in\mathbb{C}\tag{5.1.90}$$

is transformed by the time reversal operator according to

$$\mathfrak{T}\Psi=\sum_{(\alpha)jmm'}c^*_{(\alpha)jm}A_{m'm}(\mathfrak{T})|(\alpha)jm'\rangle,\tag{5.1.91}$$

where $A(\mathfrak{T})$ is a $(2j+1)\times(2j+1)$ unitary matrix. Using now the relation $\mathcal{O}_U\mathfrak{T}=\mathfrak{T}\mathcal{O}_U$, one is led again to Eq. (5.1.80), satisfied now by the unitary

[1] Wigner [2, p. 293] credits the idea of a metric for the 3-j symbols to Conyers Herring.

matrix $A(\mathcal{T})$. The result $A(\mathcal{T}) = cD^j(U_0)$ follows just as before. We conclude: *The commutivity of the rotation operator* $\mathcal{O}_U$ *and the time reversal operator* $\mathcal{T}$ *implies that the action of* $\mathcal{T}$ *on an arbitrary state* Ψ *of the form given by Eq.* (5.1.90) *is*

$$\mathcal{T}\Psi = \sum_{(\alpha)jm} c^*_{(\alpha)jm} a_{(\alpha)j} (-1)^{j-m} |(\alpha)j, -m\rangle \tag{5.1.92}$$

in which the constants $a_{(\alpha)j} \in \mathbb{C}$ *are of modulus* 1, *but otherwise arbitrary.*

The standard convention is that $a_{(\alpha)j} = 1$ for all (α) and all j. In particular, one then has

$$\mathcal{T}\,|(\alpha)jm\rangle = (-1)^{j-m}|(\alpha)j, -m\rangle. \tag{5.1.93}$$

Since this time reversal property of angular momentum states is to apply to all angular momentum systems, it is essential to verify that this standard phase choice is consistent with the addition of angular momenta—that is, with the phase convention chosen for the Wigner coefficients.

The coupling of two angular momentum systems to states of sharp total angular momentum is given by (suppressing the α-indices)

$$|(j_1 j_2)jm\rangle = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1 m_1\rangle \otimes |j_2 m_2\rangle. \tag{5.1.94}$$

Under time reversal the basis vectors of the Hilbert spaces $\mathcal{H}_1$ (basis $\{|j_1 m_1\rangle\}$) and $\mathcal{H}_2$ (basis $\{|j_2 m_2\rangle\}$) undergo the transformations—using the standard phase—given by

$$\mathcal{T}_1 |j_1 m_1\rangle = (-1)^{j_1 - m_1} |j_1, -m_1\rangle,$$
$$\mathcal{T}_2 |j_2 m_2\rangle = (-1)^{j_2 - m_2} |j_2, -m_2\rangle. \tag{5.1.95}$$

Thus, under time reversal the state vector (5.1.94) for the composite system is transformed to the state vector

$$(\mathcal{T}_1 \otimes \mathcal{T}_2)|(j_1 j_2)jm\rangle = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} \mathcal{T}_1 |j_1 m_1\rangle \otimes \mathcal{T}_2 |j_2 m_2\rangle,$$

since the Wigner coefficients are real. Using Eqs. (5.1.95), the symmetry relation

$$C^{j_1 j_2 j}_{m_1 m_2 m} = (-1)^{j - j_1 - j_2} C^{j_1 j_2 j}_{-m_1, -m_2, -m},$$

and the vanishing of the Wigner coefficient unless $m = m_1 + m_2$, one finds that the phase factors combine to $(-1)^{j-m}$, thus yielding

$$(\mathfrak{T}_1 \otimes \mathfrak{T}_2)|(j_1 j_2) jm\rangle = (-1)^{j-m}|(j_1 j_2) j, -m\rangle. \tag{5.1.96}$$

This result is just that of applying the time reversal operator directly to the state $|(j_1 j_2) jm\rangle$, using the convention (5.1.93).

We conclude:[1] *The standard phase convention under time reversal is consistent with the Wigner coefficient phase conventions for adding angular momenta.*

Do the solid (or, equivalently, the spherical) harmonics satisfy the time reversal phase convention (5.9.93)? (See Chapter 3, Section 10, and Chapter 6, AMQP). Since

$$(-1)^l K_0 \mathcal{Y}_{lm} = (-1)^l \mathcal{Y}^*_{lm} = (-1)^{l-m} \mathcal{Y}_{l,-m}, \tag{5.1.97}$$

there are two alternatives to satisfying the rule (5.1.93): Either one can define the time reversal operator $\mathfrak{T}$ to be such that it has the representation $(-1)^l K_0$ in each orbital angular momentum subspace, or, more simply and satisfying, one can define $\mathfrak{T} = K_0$ on all point functions and rephase the solid harmonics:

$$\mathcal{Y}_l^m \equiv (i)^l \mathcal{Y}_{lm}. \tag{5.1.98}$$

This latter procedure has been advocated in Ref. [37] (see also Wigner [2, p. 345]).

It was mentioned above that time reversal has important implications for relative phases. Since the imaginary unit $i = \sqrt{(-1)}$ changes sign under time reversal, it would be reasonable to shift our point of view and *regard i as an operator in quantum mechanics*, reserving the scalars (numbers) to be *real scalars* (and not complex numbers).[2] Such a viewpoint has been discussed by Dyson [43]. It is not surprising, from this point of view, that the postulate of time reversal invariance can lead to physical implications on relative phases. Assuming time reversal invariance for electromagnetic interactions leads to Lloyd's theorem [44]: *The relative phase of electromagnetic multipole matrix elements is confined to plus or minus* 1. Experimentally the

[1] It was the lack of consistency under addition of angular momenta (stemming from the Wigner [37a] phase choice) that led to a troublesome phase error in the nuclear physics literature (Ref. [36]).

[2] This point of view can be consistently applied to quantum mechanics over any of the Hurwitz division algebras; in essence, one considers quantum mechanics as a *matrix* Hilbert space over the reals only (Stueckelberg [27], Stueckelberg *et al.* [38–40], and Refs. [41] and [42]).

violation of time reversal invariance in electromagnetic transitions is predicated on detecting deviations from Lloyd's result (Jacobsohn and Henley [45]). Similar results obtain in weak interactions, where an application of time reversal invariance leads to real relative phases between the various types of interaction (see Ref. [46]). {Let us remark that historically it was this viewpoint that was instrumental in leading Lüders [47] to the famous TCP theorem linking time reversal, charge conjugation, and parity (see also Pauli [48]). It is outside the scope of the present monograph to attempt even a cursory discussion of this fundamental topic (see Jost [49]).}

Let us conclude by showing the relationship between the time reversal result, $\mathfrak{T}^2 \to (-1)^{2j}$, and the group-theoretically defined Frobenius–Schur invariant. The Frobenius–Schur invariant, I_{FS}, is defined as

$$I_{\mathrm{FS}}(\lambda) \equiv \left[\int_G dg\, \mathrm{tr}\big(D^{\lambda}(g^2)\big)\right] \times (\text{group volume})^{-1}; \qquad (5.1.99)$$

that is, I_{FS} is the mean value of $D^{\lambda}(g^2)$, where D^{λ} is a unitary irrep of the (compact) group G, and g^2 denotes the square of a group element g. The assumption that the irrep is unitary is no restriction, since for a compact group any irrep is equivalent to a unitary irrep (Wigner [2, p. 111]). Using the representation property, we may expand the elements of the matrix $D^{\lambda}(g^2)$:

$$\begin{aligned} D^{\lambda}_{\mu\nu}(g^2) &= \sum_{\sigma} D^{\lambda}_{\mu\sigma}(g) D^{\lambda}_{\sigma\nu}(g) \\ &= \sum_{\sigma} D^{\lambda}_{\mu\sigma}(g) \big[D^{\lambda}_{\nu\sigma}(g^{-1})\big]^{*}. \end{aligned}$$

Consider now the matrices $D^{\lambda *}(g)$. If $D^{\lambda} \equiv \{D^{\lambda}(g) \colon g \in G\}$ is an irrep of G, then so is $D^{\lambda *} \equiv \{[D^{\lambda}(g)]^{*} \colon g \in G\}$. It follows that $D^{\lambda *}$ is equivalent either to D^{λ} or to another irrep of the same dimension. In the latter case, the integral leads to $I_{\mathrm{FS}} = 0$ (using the Peter–Weyl theorem on orthogonality of inequivalent irreps). In the former case, there is a unitary matrix A, which transforms $D^{\lambda *}$ into D^{λ}:

$$\big[D^{\lambda}(g)\big]^{*} = A^{-1} D^{\lambda}(g) A, \qquad \text{each } g \in G. \qquad (5.1.100)$$

Using the orthonormality of the irrep matrices, the integral becomes $I_{\mathrm{FS}} = \mathrm{tr}(AA^{*})/(\text{dimension of irrep})$. We have shown already for the rotation matrices that $AA^{*} = \pm$ (unit matrix), and the same method applies in the general (compact) case. It follows that, when $D^{\lambda *}$ and D^{λ} are equivalent, $I_{\mathrm{FS}} = \pm 1$.

To summarize: (Frobenius–Schur theorem) *The irreducible representations of a compact group are of three types characterized by the invariant* I_{FS}:

(a) $I_{FS}=+1$. Potentially real irrep with D^* equivalent to D.

(b) $I_{FS}=-1$. Pseudo-real irrep with D^* equivalent to D.

(c) $I_{FS}=0$. Complex irrep with D^* inequivalent to D.

(The meaning of "potentially real" is explained in Note 1.)

For the angular momentum irreps it was determined above that $I_{FS}=(-1)^{2j}$, so that for integer angular momentum the irrep D^j is potentially real, and for half-integer j, D^j is pseudo-real. There are no complex irreps.

Since the addition of angular momenta (reduction of the direct product of irreps) using the Wigner coefficients is compatible with time reversal (and hence with the Frobenius–Schur invariant of the product), it can be easily shown (Wigner [50]) that (a) the product of potentially real irreps, or of pseudo-real irreps, contains only potentially real irreps; and (b) the product of a potentially real and a pseudo-real irrep contains only pseudo-real irreps.

These results follow from the fact that $C^{j_1j_2j_3}_{m_1m_2m_3}=0$ unless $j_1+j_2+j_3=$ integer; hence, $I_{FS}(j_3)=I_{FS}(j_1)I_{FS}(j_2)$. Expressed differently, the Frobenius–Schur classification shows that this invariant for $SU(2)$ obeys modulus 2 arithmetic under the direct product.

Remark. For the quantal rotation group, $SU(2)$, the spinorial irreps (j half-integral) are characterized by the Frobenius–Schur invariant $I_{FS}=-1$. This is a peculiarity of the group $SU(2)$ and is not a general property linking irreps with $I_{FS}=-1$ to spinors, in general. The spinorial property is a consequence of the two-to-one covering of the group $SO(n)$ by Spin(n), whereas the Frobenius–Schur invariant relates to the pseudo-reality of the (complex) representation. The quantal rotation group is thus a misleading example for physicists.

7. The Univalence Superselection Rule for Angular Momentum

The first attempt to axiomatize quantum mechanics was that of von Neumann [4] in his classic monograph. One of von Neumann's axioms (Axiom VII) was quite particular and postulated the (partially ordered) set of all questions in quantum mechanics to be isomorphic to the (partially ordered) set of all subspaces of a separable, infinite-dimensional Hilbert space. This axiom was retained in the Birkhoff–von Neumann [17] lattice theoretic formulation of quantum mechanics.

The first questioning of the necessity of this axiom was by Wigner[1] in connection with possible limitations on the concept of parity (see Wick *et al.*

[1]E. P. Wigner, address at the International Conference on Nuclear Physics and Elementary Particles, September 1951 (University of Chicago).

[5]). By means of an explicit example the invalidity of von Neumann's Axiom VII was demonstrated by showing that the Hilbert space of quantum mechanics was actually split by a "superselection rule" into two noncombining parts. We shall present Wigner's example as discussed in Ref. [5], since this example is based on fundamental aspects of angular momentum symmetry and is accordingly at the heart of both nonrelativistic and relativistic quantum mechanics.

Let us begin by recalling the concept of a "selection rule." It is standard practice to say that a *selection rule* operates between subspaces of a Hilbert space if the state vectors of each subspace are orthogonal to all state vectors of the other subspaces and remain orthogonal as long as the system is isolated and undisturbed. (For example, in a closed system the subspaces of different total angular momentum J are, and remain, mutually orthogonal; this is the content of the familiar selection rule on angular momentum magnitude.) By contrast, a *superselection rule* is said to exist if not only are there no spontaneous transitions between the various subspaces (that is, a selection rule exists), but, in addition, no conceivable (observable) operator can have nonzero matrix elements between the split subspaces.

To demonstrate the existence of at least one superselection rule, let us assume the validity of angular momentum symmetry for some generic quantum system, and divide the Hilbert space $\mathcal{H}$ for this system into two orthogonal parts: The subspace $\mathcal{A}$ is to contain all states $(f_a, g_a, \ldots)$ of *integer* angular momentum, the subspace $\mathcal{B}$ is to contain all states $(f_b, g_b, \ldots)$ of *half-integer* angular momentum, and $\mathcal{H}=\mathcal{A}\oplus\mathcal{B}$. We wish to examine now the meaning of the assumption that the relative phase between state vectors $f_a \in \mathcal{A}$ and $f_b \in \mathcal{B}$ is measurable. At the very least this assumption means that the states described by the vectors Ψ_1 and Ψ_2 defined by

$$\begin{aligned} \Psi_1 &= 2^{-\frac{1}{2}}(f_a + f_b), \\ \Psi_2 &= 2^{-\frac{1}{2}}(f_a - f_b) \end{aligned} \tag{5.1.101}$$

must be distinguishable by some measurement.

Let us now consider the time reversal operation applied to these two vectors. The operation of time reversal will yield

$$\mathcal{T}\Psi_1 = 2^{-\frac{1}{2}}(\omega_a \mathcal{U}_a K_0 f_a + \omega_b \mathcal{U}_b K_0 f_b), \tag{5.1.102}$$

where the phase factors ω_a and ω_b ($|\omega_a| = |\omega_b| = 1$) are indeterminate, but have a *fixed relative phase* ω_a/ω_b. If we again apply the time reversal operation, we find

$$\mathcal{T}^2\Psi_1 = 2^{-\frac{1}{2}}[\omega_a' \mathcal{U}_a K_0(\omega_a \mathcal{U}_a K_0 f_a) + \omega_b' \mathcal{U}_b K_0(\omega_b \mathcal{U}_b K_0 f_b), \tag{5.1.103}$$

where—from the fundamental theorem—ω_a' and ω_b' are indeterminate phases with a fixed relative phase ω_a'/ω_b', *which must be the same as* ω_a/ω_b. Using the relations $(\mathfrak{U}_a K_0)[(\mathfrak{U}_a K_0)f_a] = +f_a$ and $(\mathfrak{U}_b K_0)[\mathfrak{U}_b K_0 f_b] = -f_b$, since $\mathfrak{T}^2 \to (-1)^{2j}$ (as discussed in Section 6), we find

$$\begin{aligned}\mathfrak{T}^2\Psi_1 &= 2^{-\frac{1}{2}}(\omega_a'\omega_a^* f_a - \omega_b'\omega_b^* f_b)\\ &= 2^{-\frac{1}{2}}\omega_b'\omega_b^*(f_a - f_b).\end{aligned} \tag{5.1.104}$$

If time reversal is a physical symmetry, however, then the operation $\mathfrak{T}^2$ applied to Ψ_1 must yield the *same* state Ψ_1 to within an overall (undetermined) phase. Clearly this is not the case, since

$$\mathfrak{T}^2\Psi_1 = 2^{-\frac{1}{2}}\omega_b'\omega_b^*(f_a - f_b) \neq \omega\Psi_1. \tag{5.1.105}$$

It follows that, if time reversal symmetry is valid, then Ψ_1 and Ψ_2 are in fact *indistinguishable.* This contradiction establishes the desired result.

To summarize: *The concept of a relative phase between state vectors from distinct superselection subspaces is operationally meaningless. Distinct superselection subspaces do exist, as shown by the example of angular momentum subspaces, which are split by the unitary operator* $\mathfrak{T}^2 \to (-1)^{2j}$ *into integer and half-integer intrinsic spin superselection subspaces.*

Remarks. (*a*) Since time reversal invariance has been called into question by the Fitch–Cronin [51] experiment—although the issue is not settled—it is important to note that *the use of time reversal in the demonstration is not essential and may be replaced by rotational operations, using rotations by* 2π. The reason (Wick *et al.* [5]) for using time reversal is that the arbitrary phase ω in $\mathfrak{T} = \omega\mathfrak{U}K_0$ drops out of $\mathfrak{T}^2$, whereas the arbitrary phase involved in defining $(\mathfrak{R}(2\pi))^2$ does not drop out and is eliminated in rotations by convention (as noted earlier). This makes the argument using time reversal technically neater, but less firmly based physically.

(*b*) The fact that relative phases may, in some cases, be undefinable in principle causes a very profound change in the fundamental concepts of quantum mechanics. The existence of a superselection rule denies the unrestricted applicability of the superposition principle and limits the operation of the superposition principle to a single superselection subspace (called a *coherent* subspace, or a superselection sector).

The propositional calculus, which includes both classical mechanics and quantum mechanics in a single framework—as discussed, for example, in the monographs by Jauch [18] and Piron [20]—is well adapted to including superselection rules. This is clear from the fact that classical mechanics is distinguished from quantum mechanics precisely in having *all* states as one-dimensional superselection sectors (that is, no superposition principle at all).

(c) The superselection rule that is operative between the subspaces having integer versus half-integer angular momentum is called the *univalence superselection rule*.

The nonobservability of the spinor field operator ψ follows from the univalence superselection rule, since the Hermitian operators $(\psi+\psi^\dagger)$ and $i(\psi-\psi^\dagger)$, if observable, would have to connect superselection sectors of different univalence.

It is generally believed[1] (Wick *et al.* [5]) that various *charge superselection* rules are operative (electric charge, baryonic charge,...). The triality quantum number (Ref. [52]), applied to the $SU(3)$ symmetry known as $SU(3)$ color (Greenberg [52a], Gell-Mann [53]), is a superselection principle that is believed to be exact.

(d) The mathematical problem of determining the superselection rules that are operative for abstract Lie groups was first discussed by Bargmann [54]. The problem is now a standard one in the cohomology of groups (Mackey [55]).

(e) In order to appreciate the profound effects of superselection rules on symmetry considerations, let us discuss here a very instructive example recently given by Lévy-Leblond [56]. This example concerns the Galilei group, the relativity group of nonrelativistic quantum mechanics. From the analysis of Inönü and Wigner [57], it is known that, in quantum mechanics, one necessarily must use projective representations—and hence superselection sectors—each coherent subspace of the Galilei group being defined by the value of the inertial mass of the system.

Landé in his discussion of the foundations of quantum mechanics [58] criticized the de Broglie relationship between momentum and wavelength, $p=h/\lambda$, as being incompatible with Galilean relativity. Landé argued that under a pure Galilean boost (with velocity v) the momentum changes—that is, $p\to p'=p+mv$ (where m is the inertial mass)—while the wavelength does not change. (As Landé put it: "A snapshot of ocean waves taken from a lighthouse displays the same wavelength as one taken from an airplane.")

There is a rather subtle "pseudo-paradox" implicit in Landé's argument, which was resolved by Lévy-Leblond.

The Wigner approach to symmetry, as discussed in Section 1, shows that under a symmetry operation the absolute value of matrix elements is preserved. Denoting the wave function by $\psi(\mathbf{x}, t)$, one sees that, under a Galilean symmetry operation g, the transformed wave function can only be

[1] The validity of charge superselection rules was questioned by Aharonov and Susskind [55a] and by Mirman [55b]; see Wick *et al.* [55c] for a reply.

required to behave as

$$\psi'(\mathbf{x}', t') = \omega_g \psi(\mathbf{x}, t), \qquad |\omega_g| = 1.$$

The content of the fundamental theorem is that there is always an undetermined phase factor ω_g (which, by the theorem, factors out of linear combinations, but which may depend on the operation g).

Whether or not the *relative* phase factor associated with *pairs* of elements of the group can be well-defined is a property of the group. [This, from Remark (d) above, is the problem investigated in Ref. [54].] The conclusion is that there exists a phase function, defined for all group operations, and unique to within an overall phase for all vectors of a single (coherent) superselection sector.

From the analysis in Refs. [54], [57], and [59] of the Galilei group, the transformation of the wave function corresponding to the Galilean boost g defined by

$$g\colon x_1 \to x_1' = x_1 + vt, \qquad x_2 \to x_2' = x_2, \qquad x_3 \to x_3' = x_3,$$
$$t \to t' = t$$

takes the form

$$\mathcal{O}_g\colon \psi(\mathbf{x}, t) \to \psi'(\mathbf{x}', t') = e^{if(x_1, t)} \psi(\mathbf{x}, t),$$

where

$$f(x_1, t) = \frac{m}{h}\left(vx_1 + \frac{v^2 t}{2}\right),$$

in which v = velocity of the boost g. [In this result a common overall constant (but undetermined) phase has been removed.]

If one takes for the wave function a traveling wave along the x_1-direction

$$\psi(\mathbf{x}, t) = \exp\left[2\pi i((x_1/\lambda) - \nu t)\right],$$

then the above transformation implies a resultant wave function of the same functional form *but with changed wavelength and frequency*:

$$(1/\lambda') = (1/\lambda) + (mv/h),$$
$$\nu' = \nu + (v/\lambda) + (mv^2/2h) \text{ (nonrelativistic quantal Doppler effect!).}$$

These two results agree precisely with the de Broglie relation (for the changed momentum and wavelength) *and with the Planck relation*: $E = h\nu$. (To verify this latter relation, it is expedient to use the invariance of the internal energy $[E-(mv^2)/2]$ under Galilean boosts.)

It is especially noteworthy to observe how nicely the Galilean phase factor, e^{if}, comes in (*Deus ex machina*, according to Lévy-Leblond) to put matters right.

This cautionary example shows very clearly how subtle (and nonintuitive) the machinery of symmetry transformations can really become! It was indeed very fortunate for the development of quantum mechanics that for the Poincaré group (aside from the occurrence of the two univalence superselection sectors) all phase factors could effectively be discarded. For Poincaré transformations, none of these intricacies of phase are required to validate the Planck and de Broglie relations directly.

8. The Transformation of Eigenfunctions under Anti-unitary Operators: Corepresentations

The Wigner theorem, as we have seen, becomes in a more general context the fundamental theorem of projective geometry. Correspondingly, as Artin [7] remarks, the associated group of collineations leads to a more general type of realization: the *semilinear maps* of the vector space V into the vector space V'. To the collineation λ one assigns the mapping

$$\lambda: V \rightarrow V', \tag{5.1.106}$$

with the vectors $X, Y, \ldots$ obeying the rules

$$\lambda(X+Y)=\lambda(X)+\lambda(Y) \qquad \text{(additive mapping)} \tag{5.1.107}$$

and

$$\lambda(X\alpha)=\lambda(X)\alpha^{\lambda}, \tag{5.1.108}$$

where α belongs to a field F and α^{λ} is the automorphism of F assigned to the collineation λ. For finite groups the structure of such semilinear mappings was determined by Nakayama and Shoda [60], and by Weyl [61]. For compact groups—in particular, the quantal rotation group including time reversals—the semilinear mappings (restricted to the complex field) were developed by Wigner [2]. Wigner called these semilinear mappings (over $\mathbb{C}$) *corepresentations*, the "co" signifying the inclusion of complex conjugation as an operation. We shall sketch Wigner's analysis here, referring, however, to Ref. [2] for proofs.

Let us consider that we have a symmetry group—for example, the quantal rotation group $SU(2)$—to which we adjoin the symmetry operation of time reversal. Denote the resulting group by G. The quantum mechanical states belong to a Hilbert space $\mathcal{H}$ (over $\mathbb{C}$) which we may take[1] to be of finite dimension n. An element g of G is represented in $\mathcal{H}$ by an operator $\mathcal{O}(g)$, which is either linear unitary or antilinear anti-unitary. Physically the antilinear anti-unitary operators $\mathcal{O}(g)$ will correspond to symmetry operations involving time reversal. Let us use the convention to denote by u those elements of G for which $\mathcal{O}(u)$ is linear unitary; similarly, a denotes an element of G for which $\mathcal{O}(a)$ is antilinear anti-unitary. (The elements $\{u\}$ are easily seen to be an invariant subgroup of G of index 2, called the "unitary subgroup," and denoted G_0.)

The transformations on $\mathcal{H}$ induced by the operators $\{\mathcal{O}(g)\}$ necessarily can be represented by matrices over complex numbers (Artin [7]), but these matrices will *not* form a representation of G. The action of $\mathcal{O}(g)$ on an orthonormal basis $\{f_\mu : \mu = 1,2,\ldots,n\}$ in $\mathcal{H}$ will have the form

$$\mathcal{O}(g)f_\mu = \sum_{\lambda=1}^{n} D_{\lambda\mu}(g)f_\lambda, \tag{5.1.109}$$

where the matrix $D(g)$, each $g\in G$, is unitary. This is clear for the $D(u)$, and, since each $a\in G$, may be written as $a = ua'$ with $\mathcal{O}(a') = \mathfrak{T}$, for the $D(a)$ the unitary property follows from

$$\begin{aligned}(\mathcal{O}(a)f_\mu, \mathcal{O}(a)f_\lambda) &= (\mathcal{O}(u)\mathfrak{T}f_\mu, \mathcal{O}(u)\mathfrak{T}f_\lambda)\\ &= (K_0 f_\mu, K_0 f_\lambda) = (f_\lambda, f_\mu) = \delta_{\lambda\mu}.\end{aligned} \tag{5.1.110}$$

Introducing the transformation (5.1.109) into this equation yields

$$\sum_{\lambda'} D^*_{\lambda'\mu}(a)D_{\lambda'\lambda}(a) = \delta_{\lambda\mu}, \tag{5.1.111}$$

---

so that $D^\dagger(a) = D^{-1}(a)$, as asserted.

Since the anti-unitary operators $\mathcal{O}(a)$ induce complex conjugation, the product laws for the matrices $D(g)$ take the forms

$$\begin{aligned} D(u_1)D(u_2) &= D(u_1u_2),\\ D(u)D(a) &= D(ua),\\ D(a)D^*(u) &= D(au),\\ D(a_1)D^*(a_2) &= D(a_1a_2).\end{aligned} \tag{5.1.112}$$

[1] This involves no loss of generality, since the associated collineation group G is compact.

It is important to note the complex conjugation in the last two equations. In particular, this shows that the matrix $D(a^{-1})$ associated with the inverse of an anti-unitary element is *not* the matrix inverse, but rather one has $D^*(a^{-1})=D^{-1}(a)$.

A set of matrices $D\equiv\{D(g): g\in G\}$ satisfying Eqs. (5.1.112) is called a *corepresentation of the symmetry group G*. Equivalence *of two corepresentations, denoted $D'\sim D$, takes the form*

$$D'(u)=A^{-1}D(u)A,$$

$$D'(a)=A^{-1}D(a)A^*, \tag{5.1.113}$$

with A being a fixed unitary matrix. A corepresentation will be termed *irreducible* if it cannot be brought into reduced form (that is, blocks along the diagonal) by an equivalence transformation of the type (5.1.113).

The subgroup of unitary operators $\{\mathcal{O}(u)\}$ define an ordinary unitary representation $\{D(u)\}$ according to Eqs. (5.1.112). The unitary part of an irreducible corepresentation is, however, not necessarily itself irreducible.

Wigner [2] has shown that there exist precisely three types of irreducible corepresentations. To describe these types it is convenient to single out a particular element a_0 in the set of elements $\{a\}$ corresponding to anti-unitary operators $\mathcal{O}(a)$. Each a can then be written as $a=ua_0$ for some u. The corepresentation is independent of this choice of a_0.

The existence of the three types of corepresentations is based on a trichotomy very similar to that discussed earlier for the Frobenius–Schur invariant. Let $\Delta(u)$ be an irreducible unitary representation of the subgroup $G_0\equiv\{u\}$. Noting that the element $a_0ua_0^{-1}$ belongs to G_0, consider the representation of G_0 defined by $\{\Delta^*(a_0ua_0^{-1}): u\in G_0\}$. If $\Delta^*(a_0ua_0^{-1})$ is equivalent to $\Delta(u)$, then there is a unitary matrix (call it β) that effects the equivalence. There are now two cases: $\beta\beta^*=\pm\Delta(a_0^2)$, corresponding to type ~~I versus type II co-irreps for $+$ versus $-$, respectively. [Note that this result~~ depends on $\Delta(a_0^2)$ as well as the plus or minus sign.] If $\Delta^*(a_0ua_0^{-1})$ is inequivalent to $\Delta(u)$, then one has a type III co-irrep.

The irreducible corepresentations are given explicitly by the following types of matrices:

Type I:

$$D(u)=\Delta(u),$$

where $\Delta(u)$ is an irreducible unitary representation of the subgroup $\{u\}$;

$$D(a)=\Delta(aa_0^{-1})\beta,$$

where β is a fixed unitary matrix satisfying the relation

$$\beta\beta^* = \Delta(a_0^2).$$

Type II:

$$D(u) = \begin{pmatrix} \Delta(u) & 0 \\ 0 & \Delta(u) \end{pmatrix},$$

where $\Delta(u)$ is an irreducible unitary representation of the subgroup $\{u\}$;

$$D(a) = \begin{pmatrix} 0 & \Delta(aa_0^{-1})\beta \\ -\Delta(aa_0^{-1})\beta & 0 \end{pmatrix},$$

where β is a fixed unitary matrix satisfying the relation

$$\beta\beta^* = -\Delta(a_0^2).$$

Type III:

$$D(u) = \begin{pmatrix} \Delta(u) & 0 \\ 0 & \Delta^*(a_0^{-1}ua_0) \end{pmatrix},$$

$$D(a) = \begin{pmatrix} 0 & \Delta(aa_0) \\ \Delta^*(a_0^{-1}a) & 0 \end{pmatrix}.$$

Remarks. (a) To every irrep of the subgroup $\{u\}$ there corresponds precisely one irreducible corepresentation.

(b) It follows from Remark (a) that the anti-unitary operators do not lead to new quantum numbers beyond those provided by the unitary subgroup.[1] This does not imply that the anti-unitary operators are without consequences! Two kinds of results may occur: (i) the anti-unitary operators may imply the vanishing of matrix elements. For example, time reversal invariance can imply the vanishing of certain electric dipole moments (Landau [63]).[2] (ii) The anti-unitary operators may imply the existence of

[1]As pointed out by Wigner [2, p. 344], this result is not in conflict with the theory of types for elementary particles (Barut and Wightman [62]).

[2]Time reversal symmetry also plays a role in the properties of magnetic crystals (Cracknell [63a, p. 173ff.], Lax [63b]).

degeneracies. A famous example of this is Kramer's degeneracy, implied by time reversal invariance. For a quantal system with no spatial symmetry there is only one representation: $\Delta(u)=1$. Hence, there is but one corepresentation, which is of type I if the system has integer spin ($\mathfrak{T}^2=+1$) and of type II if the system has half-integer spin ($\mathfrak{T}^2=-1$). In the latter case, the corepresentation is two-dimensional and hence twofold degenerate.

This latter case applies to Kramer's original example of an electron in an unsymmetric crystalline environment. One can show this degeneracy directly: Since $\mathfrak{T}^2=-1$, one has

$$(\Psi,\mathfrak{T}\Psi)=(\mathfrak{T}\Psi,\mathfrak{T}(\mathfrak{T}\Psi))^*=(\mathfrak{T}(\mathfrak{T}\Psi),\mathfrak{T}\Psi)=-(\Psi,\mathfrak{T}\Psi).$$

Thus, Ψ and $\mathfrak{T}\Psi$ are orthogonal, and, assuming time reversal invariance, degenerate.

(c) Despite the great similarity between the threefold choice of the Frobenius–Schur invariant and the threefold choice defining types I, II, and III, these *two threefold choices are distinct and independent*. (This is discussed below.)

(d) Let us specialize these general results to the quantal rotation group, extended by adjoining the operation of time reversal. Thus, letting u now denote an element of $SU(2)$ and a_0 the physical operation of time reversal, we find, since a_0 commutes with each u (see p. 179), that

$$\Delta^*(a_0 u a_0^{-1})=\Delta^*(u)\sim\Delta(u).$$

The matrix β that effects the equivalence—taking for Δ one of the standard rotation group irreps D^j—is either symmetric ($j=$integer) or antisymmetric ($j=$half-integer), but the corepresentation type also depends on $\Delta(a_0^2)=(-)^{2j}1^{(j)}$. It follows that *only type I corepresentations can occur for the quantal rotation group extended by time reversal.* [Note that this is independent of the Frobenius–Schur invariant $(-1)^{2j}$ for the representation $\Delta=D^j$ of the rotation subgroup.]

(e) Corepresentations may be considered as ordinary, real, orthogonal irreps of an extended group that acts in a real Hilbert space. This is discussed in Note 2.

9. Concluding Remark

The Wigner theorem is one of the fundamental results in the application of symmetry techniques in quantum physics; it goes far beyond the applications to rotational symmetry and angular momentum that are our concerns in this monograph and in AMQP. Just how deep this theorem lies in quantum physics can be seen from Simon's [64] didactic essay *deriving* quantum dynamics (the Schrödinger equation) from the group of automorphisms of the axiomatic structure, thereby reversing the more familiar train of argument. The first step in this chain of arguments is none other than the Wigner theorem.

10. Notes

1. *Potentially real irreducible representations.* An irrep is termed *potentially real* if it has the Frobenius–Schur invariant $+1$. In such a case, the irrep D is unitarily equivalent to the conjugate irrep D^*, and, moreover, the unitary matrix transforming D to D^* is symmetric. We shall show that the irrep D may be transformed to real form.

To begin, let us note an important (although easily proved) fact: *A unitary symmetric matrix possesses eigenvectors that may be assumed real.* Thus, we may write the matrix that transforms D to D^* (call it A) in the form

$$A = r^{-1}\Delta^2 r,$$

where r is real and orthogonal, and Δ is diagonal and unitary. (We have chosen to write the diagonal matrix as the square of a diagonal matrix Δ, which is always possible.) With this result, the equation $D = A^{-1}D^*A$ (which exists by the hypothesis that D is potentially real) may now be put in the form

$$\Delta r D r^{-1}\Delta^{-1} = (\Delta r D r^{-1}\Delta^{-1})^*.$$

This is the desired result, which asserts that the irrep D may be written in an explicitly real form.

Remark. The form A given above for a unitary symmetric matrix is important in the (physical) theory of the scattering matrix ("S-matrix"). The scattering matrix is *unitary* because of the conservation of probability; assuming time reversal invariance, it is also *symmetric*. Thus, the form given for A holds. The consequences of this are discussed in Wigner [65] and in Ref. [66].

2. *Corepresentations as real, orthogonal irreps in a real Hilbert space.* In many ways the use of a Hilbert space over the complex numbers is something of a hindrance in discussing anti-unitary operators. Once the anti-unitary operators are admitted, it is logically clearer—and much simpler—to abandon complex numbers and base the analysis entirely on Hilbert space $\mathcal{H}(\mathbb{R})$ over the real numbers. This is easily accomplished by

regarding the complex number α as a 2×2 matrix over $\mathbb{R}$:

$$\alpha \to \begin{pmatrix} \operatorname{Re}\alpha & -\operatorname{Im}\alpha \\ \operatorname{Im}\alpha & \operatorname{Re}\alpha \end{pmatrix}$$

The imaginary unit i thus maps into $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and is realized as a linear operator i acting on $\mathcal{H}(\mathbb{R})$. The adjunction of the anti-unitary operators can all be expressed in terms of a *linear* operator j, which acts on $\mathcal{H}(\mathbb{R})$ and realizes the complex conjugation operator $K_0 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This point of view has been advocated in Refs. [34] and [43]. Dyson [43] makes a very good case (in his elegant paper on the threefold way) that such a procedure is fundamentally to be preferred.

Let us consider an irreducible corepresentation from this point of view. To the $n\times n$ matrix $D(u)$ we associate the $2n\times 2n$ matrix $M(u)$:

$$M(u) \equiv \begin{pmatrix} \operatorname{Re} D(u) & -\operatorname{Im} D(u) \\ \operatorname{Im} D(u) & \operatorname{Re} D(u) \end{pmatrix}.$$

To the $n\times n$ matrix $D(a)$ we associate the $2n\times 2n$ matrix $M(a)$:

$$M(a) \equiv \begin{pmatrix} \operatorname{Re} D(a) & -\operatorname{Im} D(a) \\ \operatorname{Im} D(a) & \operatorname{Re} D(a) \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix},$$

where 1_n denotes the $n\times n$ unit matrix. Because of the fact that the operators i and j anticommute, their matrix realizations given by

$$i \to \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \quad \text{and} \quad j \to \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$$

also anticommute, and the matrix multiplication of the $\{M(g)\}$ automatically induces the multiplication rules of Eqs. (5.1.112) on the corepresentation matrices. Put in different words, *the correspondence $g \to M(g)$ is a true representation of the group G realized over a $2n$-dimensional real Hilbert space.*

It is useful to mention that the matrices $\{M(g)\}$ are all real and orthogonal. In particular, the unusual inverse, necessitated by Eqs. (5.1.112), for the matrices of the elements $\{a\}$ of the corepresentation,

$$D(a^{-1}) = [D^*(a)]^{-1},$$

now appear in the usual form

$$M(a^{-1})=M^{-1}(a)=\tilde{M}(a).$$

This property is, in fact, valid for all elements $g\in G$.

The analog to the Frobenius–Schur invariant can be constructed for representations over the real field. Dyson [43] has proved that *for a real, orthogonal irreducible representation $M(g)$ of a compact group G the trace $M(g^2)$ averaged over the group G is an invariant characteristic of the representation.*

We may apply this theorem to two special cases: (a) Consider a unitary representation, and apply Dyson's result to the corresponding real representation; one reproduces the Frobenius–Schur invariant. (b) Consider a real representation $M(g)$ corresponding to an irreducible corepresentation of the group G, and adjoin the matrix operator $i\rightarrow\begin{pmatrix}0 & -1_n\\ 1_n & 0\end{pmatrix}$ as a group element. Thus, the representation consists of $\{M(g)\}$ and $\{iM(g)\}$, so that the group represented by these matrices is $\Gamma\equiv G\circledcirc C_2$, an extension of G. In the average over the group Γ, the elements $[M(u)]^2=M(u^2)$ and $[iM(u)]^2$ *cancel*, whereas the elements $[M(a)]^2=M(a^2)$ and $[iM(a)]^2$ *add*. (This is the reason for extending G to Γ.) It follows that the invariant—call it W—becomes $\mathrm{tr}(M(a^2))$ *averaged over the anti-unitary elements $\{a\}$ only.* Since a^2 lies in the unitary subgroup, we can apply the result of (a) above (equivalent to Wigner's threefold choice) to find

$$W=\begin{array}{l}+1 \text{ for Wigner type I corepresentation,}\\ -1 \text{ for Wigner type II corepresentation,}\\ \;\;\,0 \text{ for Wigner type III corepresentation.}\end{array}$$

~~This elegant result was shown by~~ Dyson [43], who accredited the idea to Bargmann.

Let us repeat that, given an irreducible corepresentation, the Frobenius–Schur invariant of the unitary part and the Wigner invariant (W above) for the corepresentation itself are independent. (This independence can be restricted by the particular group and the particular representation.) Realizations of all nine possibilities have been constructed in Ref. [43].

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TOPIC 2. MONOPOLAR HARMONICS

1. Introduction and Survey

In this section we shall discuss a topic— the theory of spherical harmonics adapted to the description of a charged particle moving in a monopolar magnetic field—that at first glance appears to be highly specialized and possibly only of esoteric interest. We shall try to show that, on the contrary, this topic leads one quickly into very beautiful connections between physics and mathematics, which are of current research interest (see Note 1) and as yet not fully explored.

The problem of a point (electric) charge moving in the magnetic field generated by a fixed magnetic monopole was first considered by Poincaré [1] in 1896, using classical Newtonian mechanics. But the real import of the problem became clear only with the astonishing paper of Dirac [2] in 1931, who showed that the *existence* of magnetic charge would imply the *quantization of electric charge*. Dirac's procedure, and his reasoning, in this path-breaking paper was of a qualitatively different type than is customary in physics, and was topological in nature. In effect, Dirac intuitively used arguments that, as C. N. Yang expressed it, can now be seen to underlie the famous Gauss–Bonnet–Allendoefer–Chern–Weil theorem in present-day mathematics.

Since it exceeds our competence to attempt to develop this subject from such a fundamental standpoint, we shall develop instead the necessary concepts (in Section 2) from the physical model of Poincaré, using quantum mechanics. We shall prove thereby Dirac's quantization by direct calculation using a procedure first given by Fierz [3] and by Tamm [4].[1]

In Section 3 we discuss the unusual nature of the solutions found in Section 2, which ultimately imply the necessity *for an enlarged framework for quantum physics*. This framework, as discussed by Yang and Wu [5–8], is that of a modern mathematical construct—fiber bundle theory (Drechsler and Mayer [9]). In this framework a quantum mechanical wave function is now generalized to be a *section* defined on a fiber bundle.

It is our aim to demonstrate quite explicitly, and in detail, how angular momentum theory is related to these phenomena, and how this theory provides a defining example of the general structure.

2. The Defining Physical Problem

Let us consider, in the framework of nonrelativistic classical physics, the problem of a point mass m_0 having electric charge e and moving in the field of a fixed (stationary) magnetic monopole having magnetic charge g. The magnetic field is given by

$$\mathbf{B}=g(\hat{e}_r/r^2). \tag{5.2.1}$$

[1]Jackiw [4a] has shown that the magnetic monopole possesses [in addition to $O(3)$ symmetry] a dynamical symmetry group $O(2,1)$ whose Casimir invariant is determined by the monopole strength.

In this result the magnetic field $\mathbf{B}=(B_1, B_2, B_3)$ is referred to a right-handed Cartesian reference frame[1] $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ with the magnetic point charge at the origin. The coordinates of a point $\mathbf{x}$ are denoted by $\mathbf{x}=(x_1, x_2, x_3)$, and r is the distance $(\mathbf{x}\cdot\mathbf{x})^{\frac{1}{2}}$ from the origin to the point $\mathbf{x}$. We use the notation $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ to denote the right-handed frame that is defined at each point $\mathbf{x}$ such that $\mathbf{x}\neq(0,0,\lambda)$ by

$$\begin{aligned}
\hat{e}_r &= \mathbf{x}/r = (x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3)/r, \\
\hat{e}_\theta &= \left[x_1x_3\hat{e}_1 + x_2x_3\hat{e}_2 - (r^2 - x_3^2)\hat{e}_3\right]/r(r^2 - x_3^2)^{\frac{1}{2}}, \qquad (5.2.2) \\
\hat{e}_\phi &= (-x_2\hat{e}_1 + x_1\hat{e}_2)/(r^2 - x_3^2)^{\frac{1}{2}},
\end{aligned}$$

where the spherical polar coordinates (r, θ, ϕ) define the point $\mathbf{x}=(x_1, x_2, x_3)=(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. (Note that the divergence of $\mathbf{B}$ is no longer zero, and hence Maxwell's equations have been generalized.)

The introduction of a vector potential describing the field $\mathbf{B}$ is necessary if one seeks a Hamiltonian formulation of the problem, and such a formulation is, in turn, essential if one wishes to go over to a quantum description. But, at the classical level there is no necessity for introducing potentials, and one may work directly with the Newtonian equations of motion, and the Lorentz force law for the charge e:

$$m_0\ddot{\mathbf{x}} = (eg/c)(\dot{\mathbf{x}}\times\mathbf{x})r^{-3}, \qquad (5.2.3)$$

where the dot over a vector signifies differentiation with respect to time t.

One can immediately obtain four integrals of the motion:

The energy integral:

$$E = \tfrac{1}{2}m_0\dot{\mathbf{x}}^2. \qquad (5.2.4)$$

~~The (three) angular momentum integrals:~~

$$\mathbf{J} = m_0\mathbf{x}\times\dot{\mathbf{x}} - (eg/c)\hat{e}_r. \qquad (5.2.5)$$

[The energy integral is obtained by taking the dot product of $\dot{\mathbf{x}}$ with Eq. (5.2.3). To verify that $\mathbf{J}$ is an integral of the motion, we differentiate $\mathbf{J}$ with respect to t and use the equation of motion (5.2.3), thus obtaining $d\mathbf{J}/dt=\mathbf{0}$. Note that, for $eg>0$, *the radial angular momentum* $-(eg/c)\hat{e}_r$ *is radially inward.*]

These results establish that we do indeed have the four integrals claimed, but they do not yet justify that the integral $\mathbf{J}$ is the angular momentum. To

[1]We use the vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ in the realization $\hat{e}_1=(1,0,0)$, $\hat{e}_2=(0,1,0)$, $\hat{e}_3=(0,0,1)$ throughout this Topic.

verify this one must prove that the Poisson bracket relations, for angular momentum, are valid for $\mathbf{J}$. This is best accomplished in a Hamiltonian formulation, a task we shall postpone for the time being.

Intuitively $\mathbf{J}_{\text{orb}} = m_0 \mathbf{x} \times \dot{\mathbf{x}}$ is simply the orbital angular momentum of the particle. But what then is the term $-(eg/c)\hat{e}_r$? Remarkably, one can show by direct integration (over the vector $\mathbf{E} \times \mathbf{H}$) that this is the angular momentum of the *total* electromagnetic field (produced by both charges):

$$\mathbf{J}_{\text{field}} = \int_{\substack{\text{all}\\ \text{space}}} d\mathbf{x}\left[\mathbf{x} \times (\mathbf{E} \times \mathbf{H})\right] = -(eg/c)\hat{e}_r. \tag{5.2.6}$$

(What is remarkable is that this angular momentum is independent of the distance of the particle from the origin.) This verifies that $\mathbf{J}$ has, intuitively, the meaning of a *total* angular momentum, but, as we shall show, the implied separation into $\mathbf{J}_{\text{orb}}$ and $\mathbf{J}_{\text{field}}$ is *not* valid.

The classical motion carried out by the particle is a motion, at constant speed, on the surface of a circular cone having apex angle 2θ such that

$$\cos\theta = eg/c\|\mathbf{J}\|.$$

If one develops this conical surface by rolling the cone on a plane, the motion in the plane is along a straight line at constant speed.

Hamiltonian formulation. To formulate the problem in terms of Hamiltonian mechanics requires that one first determine the vector potential $\mathbf{A}$ such that

$$\nabla \times \mathbf{A} = \mathbf{B} \tag{5.2.7}$$

gives the field $\mathbf{B}$ in Eq. (5.2.1). One next defines the Lagrangian L by[1]

$$L = \tfrac{1}{2} m_0 \dot{\mathbf{x}}^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}, \tag{5.2.8}$$

and finally the Hamiltonian H by

$$H = \mathbf{p} \cdot \dot{\mathbf{x}} - L. \tag{5.2.9}$$

In this result the linear momentum $\mathbf{p}$ conjugate to $\mathbf{x}$ is defined by

$$\mathbf{p} = \partial L/\partial \dot{\mathbf{x}}, \tag{5.2.10}$$

that is, $p_i = \partial L/\partial \dot{x}_i$.

[1]The notation $\mathbf{a}^2$ denotes the dot product $\mathbf{a} \cdot \mathbf{a}$ of a vector with itself.

One easily verifies that a vector potential $\mathbf{A}$ that yields $\mathbf{B}=\nabla\times\mathbf{A}$ is given by

$$\mathbf{A}=\frac{g}{r(r+x_3)}(-x_2,x_1,0). \tag{5.2.11}$$

Using this result in the Lagrangian (5.2.8), we find for the canonical linear momentum

$$\mathbf{p}=m_0\dot{\mathbf{x}}+\frac{e}{c}\mathbf{A}. \tag{5.2.12}$$

This result, when introduced into Eqs. (5.2.5) and (5.2.9) to eliminate $m_0\dot{\mathbf{x}}$, gives the angular momentum $\mathbf{J}$ and the Hamiltonian H as

$$\mathbf{J}=\mathbf{x}\times\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)-\frac{eg}{c}\hat{e}_r=\mathbf{L}-\frac{eg}{c}\mathbf{K}, \tag{5.2.13}$$

$$H=\frac{1}{2m_0}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2, \tag{5.2.14}$$

where we have defined[1]

$$\mathbf{L}=\mathbf{x}\times\mathbf{p}, \tag{5.2.15}$$

$$\mathbf{K}=\frac{1}{g}(\mathbf{x}\times\mathbf{A})+\hat{e}_r=\frac{\mathbf{x}+r\hat{e}_3}{r+x_3}. \tag{5.2.16}$$

The Hamiltonian formulation is form-invariant under the replacement of the Poisson brackets by $(i\hbar)^{-1}$ (commutator). Hence, this verifies that these same constants of the motion also occur in the quantum mechanical problem.

Let us now verify that the quantal operator $\mathbf{J}$ in Eq. (5.2.13) satisfies the angular momentum commutation rules. To accomplish this let us first express $\mathbf{J}$ in units of $\hbar$ (that is, replace $\mathbf{J}$ by $\mathbf{J}/\hbar$) and rewrite Eq. (5.2.13) as

$$\mathbf{J}=\mathbf{L}-\mu\mathbf{K}, \tag{5.2.17}$$

where $\hbar\mathbf{L}=\mathbf{x}\times\mathbf{p}$, and μ is defined by

$$\mu=eg/\hbar c. \tag{5.2.18}$$

[1]We have introduced $\mathbf{A}$ explicitly into Eq. (5.2.13) and will do so for the Hamiltonian H below.

The key to a systematic verification of the commutation relations for **J** is the set of relations between **K** and **L** given by[1]

$$\begin{aligned}\mathbf{K}\cdot\mathbf{K}&=2r/(r+x_3),\\ \mathbf{K}\times\mathbf{L}+\mathbf{L}\times\mathbf{K}&=i\mathbf{K},\\ \mathbf{L}\cdot\mathbf{K}=\mathbf{K}\cdot\mathbf{L}&=\left[r/(r+x_3)\right]L_3.\end{aligned}\tag{5.2.19}$$

[The first of these relations follows immediately from the definition of **K**, as does $\mathbf{K}\cdot\mathbf{L}=(r/r+x_3)L_3$, since $\mathbf{x}\cdot\mathbf{L}=0$. The remaining relation is verified by elementary applications of the definitions of **K** and **L** and the canonical commutation relations $[x_i, p_j]=i\delta_{ij}\hbar$.]

Using the first two of Eqs. (5.2.19), we find

$$\mathbf{J}\times\mathbf{J}=\mathbf{L}\times\mathbf{L}-\mu(\mathbf{L}\times\mathbf{K}+\mathbf{K}\times\mathbf{L})=i\mathbf{L}-i\mu\mathbf{K}=i\mathbf{J},\tag{5.2.20}$$

thus verifying that **J** satisfies angular momentum commutation relations.

Using the first and last of Eqs. (5.2.19), we obtain the following expression for the square of the angular momentum **J**:

$$\mathbf{J}^2=\left[\mathbf{x}\times\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\right]^2+\mu^2=\mathbf{L}^2-\frac{2\mu r}{r+x_3}(L_3-\mu).\tag{5.2.21}$$

An alternative form of the Hamiltonian H is obtained by using the following identities:

$$\begin{aligned}\frac{e}{c}\mathbf{p}\cdot\mathbf{A}=\frac{e}{c}\mathbf{A}\cdot\mathbf{p}&=\frac{\mu}{r(r+x_3)}L_3,\\ \hbar^{-2}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2&=-\nabla^2-\frac{2\mu}{r(r+x_3)}L_3+\frac{\mu^2(r-x_3)}{r^2(r+x_3)},\\ -\nabla^2&=-r^{-2}(\mathbf{x}\cdot\nabla)(\mathbf{x}\cdot\nabla+1)+r^{-2}\mathbf{L}^2.\end{aligned}\tag{5.2.22}$$

Thus, we find

$$\begin{aligned}H&=\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2/2m_0\\ &=-\frac{\hbar^2}{2m_0r^2}\left[(\mathbf{x}\cdot\nabla)(\mathbf{x}\cdot\nabla+1)-(\mathbf{J}^2-\mu^2)\right].\end{aligned}\tag{5.2.23}$$

[1]It might be thought that **K** is a vector operator (see Chapter 3, Section 15, AMQP), but this is not the case, since, for example, $[L_1, K_1]\neq 0$.

The Schrödinger equation corresponding to this Hamiltonian then takes the following form in spherical polar coordinates:

$$(H-E)\psi=\left[-\frac{1}{2m_0r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\mathbf{J}^2-\mu^2}{2m_0r^2}-\frac{E}{\hbar^2}\right]\psi=0. \quad (5.2.24)$$

Let us assume[1] that the operator $\mathbf{J}^2$ has standard eigenvalues of the form $j(j+1)$, that is, $\mathbf{J}^2\psi=j(j+1)\psi$. Then the radial equation corresponding to Eq. (5.2.24) is solved by the regular Bessel function (Tamm [4]):

$$\psi_{\text{radial part}}=(kr)^{-\frac{1}{2}}J_\lambda(kr), \quad (5.2.25)$$

where

$$\lambda=\left[j(j+1)-\mu^2+\tfrac{1}{4}\right]^{\frac{1}{2}},$$
$$k=(2m_0E/\hbar)^{\frac{1}{2}}. \quad (5.2.26)$$

For completeness and later use, let us also give the vector potential $\mathbf{A}$, the vector $\mathbf{K}$, and the angular momentum operator $\mathbf{J}$ in terms of the spherical basis (5.2.2):

$$\mathbf{A}=\frac{g(1-\cos\theta)}{r\sin\theta}\hat{e}_\phi,$$
$$\mathbf{K}=-\frac{(1-\cos\theta)}{\sin\theta}\hat{e}_\theta+\hat{e}_r, \quad (5.2.27)$$
$$\mathbf{J}=\mathbf{L}+\mu\frac{(1-\cos\theta)}{\sin\theta}\hat{e}_\theta-\mu\hat{e}_r.$$

The expressions for J_3 and $J_\pm$ in terms of spherical coordinates are accordingly given by [see Chapter 3, Eqs. (3.106), AMQP]

$$J_3=L_3-\mu,$$
$$J_\pm=L_\pm-\mu\frac{(1-\cos\theta)}{\sin\theta}e^{\pm i\phi}; \quad (5.2.28)$$
$$L_\pm=e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta}-\cot\theta L_3\right),$$
$$L_3=-i\frac{\partial}{\partial\phi}. \quad (5.2.29)$$

[1]Since we have not yet given an explicit inner product such that $\mathbf{J}$ is a Hermitian operator, we cannot assert the existence of standard angular momentum multiplets (see Chapter 3, Section 3, AMQP). Indeed, as we show in Section 3, the appropriate definition of the angular momentum $\mathbf{J}$ and an inner product such that $\mathbf{J}$ is a Hermitian operator constitute the principal steps in solving the magnetic monopole problem.

One may, of course, verify the commutation relations (5.2.20) directly, using these explicit spherical coordinate forms.

The angular eigenfunctions. The angular eigenfunctions that diagonalize $\mathbf{J}^2$ and J_3, and that correspond to the separation of variables in Eq. (5.2.24), will be discussed systematically in Section 3 and in Note 2 to this Topic. It will be shown in detail there that these angular functions are closely related to the symmetric top eigenfunctions (see Chapter 7, Section 10, AMQP) or, equivalently, to the $SU(2)$ irrep functions $D^j_{m'm}(\alpha\beta\gamma)$, which are discussed extensively in Chapter 3, Section 8, AMQP. Both the angles α and γ must, however, be identified with $\pm\phi$, and θ with β. The purpose of the present discussion is to present an intuitive argument as to why this is physically reasonable.

We have already seen that (classically) the angular momentum of the electromagnetic field is $-eg/c \equiv -\mu\hbar$ and directed (for $eg>0$) inwardly along the radial direction $\hat{e}_r$. Let us now regard the system of two charges (g and e) as a symmetric top with the symmetry axis along the line joining g to e (that is, along $\mathbf{x}$) and the fixed point of the top being the monopole g at the origin. Then it is clear that for rotations around the symmetry axis of the top we get the (field) angular momentum $(-\mu\hbar)$; this angular momentum corresponds to the body-referred angular momentum $\mathcal{P}_3 \to -\mu$, of a symmetric top (see Chapter 3, Section 8, AMQP). Dirac's theorem, that μ=integer or half-integer, is hence readily understood (but, of course, not proved) through the quantization of angular momentum. [Note that, if there were no magnetic charge ($g=0$), we could still consider the motion as formally that of a "symmetric top." Because the moment of inertia for rotations around the symmetry axis is zero, there is no z-component of the body-referred angular momentum. Thus, $\mathcal{P}_3 \to 0$, and we recover the familiar result that the angular eigenfunctions are $D^{l*}_{m,0}(\phi\theta 0) \propto Y_{lm}(\theta\phi)$.]

This argument thus implies, correctly, that the wave function depends on θ as $d^j_{m,\ \mu}(\theta)$. For the usual symmetric top, rotations around the symmetry axis represent a separate degree of freedom parametrized by the kinematically independent coordinate $\chi=\gamma$. By contrast, in the present problem there is no such freedom, and rotations can involve only the *two* angles θ and ϕ. Thus, the angular momentum corresponding to $\mathcal{P}_3$ is not kinematically independent of the angular momentum conjugate to ϕ. *It follows that there can be no valid separation*[1] *of the angular momentum* $\mathbf{J}$ *into two separate angular momenta—"orbital" and "field."*

This unusual circumstance is related to the fact that the vector potential $\mathbf{A}$ is not rotationally invariant about the origin (position of the magnetic monopole). After a rotation of the coordinates, *a gauge transformation is*

[1] It is interesting to recall that for the photon there is similarly no valid separation of the total angular momentum into "orbital" and "spin" parts (see Chapter 7, Section 6, AMQP).

required to restore the vector potential to its original form. (Such a gauge transformation must exist, since the field **B** is rotationally invariant.)

Fierz's [3] proof of the Dirac quantization condition consisted in showing that without this condition no physically acceptable solutions to the Schrödinger equation existed (that is, solutions that are nonsingular at infinity).

Remarks and criticisms. So far we have followed, more or less uncritically, the physical concepts and ideas generally used in discussing this motion. As we shall now see, this conceptual basis is inadequate.

Let us first remark that the *classical* motion presents no problems; the problem is well-defined for the space $[\mathbb{R}^3 - (0)]$, and, upon choice of a fixed (real) value for eg/c, the motion takes place on a cone minus a single point, the vertex. The set of all possible motions is, moreover, spherically symmetric in the classical sense that, given any one motion, there is another motion obtained from it by any coordinate rotation about the origin. This agrees with one's intuition that the interaction is spherically symmetric, since the magnetic field has this property.

The real problem occurs only when we go over to the quantum mechanical version and introduce the machinery of vector potentials, canonical momenta, and a Hamiltonian. One cannot define these quantities explicitly without introducing a preferred coordinate frame and spurious singularities.

To see this point most clearly, note that the vector potential in Eq. (5.2.11) *is singular on the half-line* $\theta = \pi$. It follows that the required canonical constructs (used above) are not acceptable in all of physical space, despite the evident physical fact that the motion is itself quite unobjectionable. We shall not attempt to review here the extensive literature (in physics) concerning attempts to circumvent the difficulties engendered by this half-line of singularities ("Dirac string"). It should be clear that, if such singularities are explicitly included in the formulation, then one has introduced (infinitely many) extraneous variables, which will necessarily complicate the analysis almost irretrievably.

The resolution of this difficulty is—once found!—simple and elegant. (*One* possible resolution is to note that no magnetic monopole has yet been found, and hence to deny the existence of any meaningful physical problem.)

One first notes that a similar difficulty arises in discussing the coordinatization of a spherical surface: There exist no (analytic) coordinates that are free of singularities over the entire surface. One is forced to use the expedient of defining *coordinate patches* (singularity-free regions) and insisting on piecing together the various patches smoothly on the regions of overlap (defining an *atlas*).

This is the key idea introduced by Yang and Wu [5–8], which will, when carried out fully, eliminate all explicit mention of singularities from the canonical constructs.

To implement this concept we need to define only two regions R_a and R_b, each hemispherical (for fixed r). Using spherical coordinates r, θ, and ϕ with the monopole at the origin, we choose R_a and R_b to be the sets of points $\{P\}$ in $\mathbb{R}^3$ defined by

$$R_a=\left\{P\in\mathbb{R}^3: 0\leqslant\theta<\frac{\pi}{2}+\delta,\, 0<r,\, 0\leqslant\phi<2\pi\right\},$$

$$R_b=\left\{P\in\mathbb{R}^3: \frac{\pi}{2}-\delta<\theta\leqslant\pi,\, 0<r,\, 0\leqslant\phi<2\pi\right\}, \tag{5.2.30}$$

$$R_{ab}=R_a\cap R_b=\left\{P\in\mathbb{R}^3: \frac{\pi}{2}-\delta<\theta<\frac{\pi}{2}+\delta,\, 0<r,\, 0\leqslant\phi<2\pi\right\},$$

where we choose δ such that $0<\delta\leqslant\pi/2$.

The vector potential is now defined to be

$$(A_r)_a=(A_\theta)_a=0, \qquad (A_\phi)_a=\frac{g(1-\cos\theta)}{r\sin\theta}, \qquad \text{each } P\in R_a,$$

$$(A_r)_b=(A_\theta)_b=0, \qquad (A_\phi)_b=\frac{-g(1+\cos\theta)}{r\sin\theta}, \qquad \text{each } P\in R_b. \tag{5.2.31}$$

This satisfactorily eliminates the singularity from each region, but in the overlap (intersection of R_a and R_b) region the two vector potentials do *not* match up. This motivates the second key idea: The vector potential in physics is actually an *equivalence class* of vector potentials, whose members differ by gauge transformations. Hence, one defines: *In the overlap region the two vector potentials must differ at most by a gauge transformation.*

For the case at hand, one finds in the overlap region ($\theta=\pi/2$) that

$$\boxed{\mathbf{A}_a-\mathbf{A}_b=2g\nabla\phi, \qquad \text{each } P\in R_{ab},} \tag{5.2.32}$$

where we have used

$$\nabla\phi=(r\sin\theta)^{-1}\hat{e}_\phi \tag{5.2.33}$$

in obtaining this result. Hence, the vector potentials do indeed differ by a gauge transformation:

$$\mathbf{A}_a-\mathbf{A}_b=\nabla f, \qquad f=2g\phi. \tag{5.2.34}$$

To complete the concept, we must now consider the wave function, ψ. This, too, is defined piecewise in each of the separate regions: *In any overlap region it is required that the two wave functions differ only by a phase factor* (a complex factor of modulus 1). This requirement is tailored to the electromagnetic case where the gauge group is the compact group $U(1)$. More generally, the two wave functions differ at most by a transformation of the gauge group (Drechsler and Mayer [9], Simms and Woodhouse [10]). Thus, we have

$$\psi_b(\mathbf{x})=S_{ba}(\mathbf{x})\psi_a(\mathbf{x}), \qquad \text{each } \mathbf{x}\in R_{ab}, \tag{5.2.35}$$

where $|S_{ba}(\mathbf{x})|=1$.

To ensure the proper smoothness in the overlap region, we now require that the velocity operator $\frac{1}{m_0}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)$ yield the same value when averaged over the overlap region R_{ab}; that is,

$$\int_{R_{ab}}\psi_a^*(\mathbf{x})\left(\mathbf{p}-\frac{e}{c}\mathbf{A}_a\right)\psi_a(\mathbf{x})\,d\mathbf{x}=\int_{R_{ab}}\psi_b^*(\mathbf{x})\left(\mathbf{p}-\frac{e}{c}\mathbf{A}_b\right)\psi_b(\mathbf{x})\,d\mathbf{x}. \tag{5.2.36}$$

Using relation (5.2.35) and the derivation property

$$\mathbf{p}(S_{ba}\psi_a)=(\mathbf{p}S_{ba})\psi_a+S_{ba}(\mathbf{p}\psi_a),$$

we obtain

$$\int_{R_{ab}}\psi_a^*(\mathbf{x})\left[S_{ba}^*(\mathbf{x})(\mathbf{p}S_{ba}(\mathbf{x}))+\frac{e}{c}(\mathbf{A}_a-\mathbf{A}_b)\right]\psi_a(\mathbf{x})\,d\mathbf{x}=\mathbf{0}. \tag{5.2.37}$$

This condition must hold for all choices of overlap region R_{ab} and all functions ψ_a defined in R_a (hence, on R_{ab}). The requirement (5.2.37) can be satisfied if we define the gauge transformation of the vector potential to be

---

$$\frac{e}{c}(\mathbf{A}_a-\mathbf{A}_b)=i\hbar S_{ba}^*(\nabla S_{ba}). \tag{5.2.38}$$

Combining this result with Eq. (5.2.34), we find $2\mu\nabla\phi=iS_{ba}^*(\nabla S_{ba})$; that is,

$$\partial S_{ba}/\partial\phi=-2i\mu S_{ba}. \tag{5.2.39}$$

Hence,

$$S_{ba}=e^{-2i\mu\phi}, \qquad \phi\in R_{ab} \tag{5.2.40}$$

is the simplest choice for S_{ba}.

In order that the wave function be single-valued, we see that the phase factor integrated around a closed loop in the overlap region must give unity, and, hence,

$$eg/\hbar c \equiv \mu = \tfrac{1}{2}(\text{integer}). \tag{5.2.41}$$

This is Dirac's quantization condition, which now appears in the form of a constructibility condition. Indeed, Yang and Wu [5–8] have proved that, if this condition fails, then it is not possible to make the present construction with *any* number of distinct regions (coordinate patches).

The construction we have given here is a standard one in mathematics and has by now a standard nomenclature. What we have called a *wave function* is called in the mathematical literature a *section*. The factor S_{ba} is called a *transition function*, and the vector potential **A** is a *connection on a fiber bundle*.

Let us complete this subsection by giving the form that the angular momentum operators assume for the vector potential in the region R_b given by Eq. (5.2.31). We find

$$\mathbf{J}' = \mathbf{L} - \mu\mathbf{K}', \tag{5.2.42}$$

$$\begin{aligned}\mathbf{K}' &= \frac{1}{g}(\mathbf{x}\times\mathbf{A}_b) + \hat{e}_r = \frac{\mathbf{x} - r\hat{e}_3}{r - x_3} \\ &= \frac{(1+\cos\theta)}{\sin\theta}\hat{e}_\theta + \hat{e}_r;\end{aligned} \tag{5.2.43}$$

$$\begin{aligned}J'_{\pm} &= L_{\pm} - \frac{\mu(1+\cos\theta)}{\sin\theta} e^{\pm i\phi}, \\ J'_3 &= L_3 + \mu.\end{aligned} \tag{5.2.44}$$

~~Note that the operators $\mathbf{J}'$ also satisfy the standard commutation relation~~

$$\mathbf{J}' \times \mathbf{J}' = i\mathbf{J}'. \tag{5.2.45}$$

The angular momentum operators **J** and **J**′ defined, respectively, by Eqs. (5.2.17) and (5.2.42) are the angular momentum operators for regions R_a and R_b:

$$\mathbf{J}_a = \mathbf{J}, \qquad \mathbf{J}_b = \mathbf{J}'. \tag{5.2.46}$$

This follows from the fact that the functions $\mathbf{J}_a\psi_a$ and $\mathbf{J}_b\psi_b$ are well-defined in the regions R_a and R_b, respectively, and, moreover, are correctly related in the overlap region:

$$\mathbf{J}_b\psi_b(\mathbf{x}) = e^{-2i\mu\phi}\mathbf{J}_a\psi_a(\mathbf{x}), \qquad \text{each } \mathbf{x} \in R_{ab}. \tag{5.2.47}$$

An alternative derivation of relation (5.2.40) may be given by the following argument: Since $\mathbf{J}_a\psi_a=\psi_a'$ and $\mathbf{J}_b\psi_b=\psi_b'$ are well-defined functions in the regions R_a and R_b, respectively, we have that $\psi_b=S_{ba}\psi_a$ (in R_{ab}) implies $\psi_b'=S_{ba}\psi_a'$ (in R_{ab}); that is, $(S_{ba}^*\mathbf{J}_bS_{ba})\psi_a=\mathbf{J}_a\psi_a$ for arbitrary functions ψ_a defined in R_a. Thus, we must have

$$S_{ba}^*\mathbf{J}_bS_{ba}=\mathbf{J}_a, \qquad \text{or, equivalently,} \qquad \mathbf{J}_bS_{ba}=S_{ba}\mathbf{J}_a. \quad (5.2.48)$$

Applying this result, in turn, to each component of the angular momentum, and using the derivation property of these operators, we find that S_{ba} must satisfy the following differential equations:

$$\partial S_{ba}/\partial\phi=-2i\mu S_{ba}, \qquad \partial S_{ba}/\partial\theta=0. \quad (5.2.49)$$

Thus, the transition function S_{ba} is *uniquely* determined to be

$$S_{ba}=e^{-2i\mu\phi}, \quad (5.2.50)$$

since an arbitrary numerical factor of modulus 1 can always be absorbed into the wave function ψ_a itself.

3. Reformulation of the Problem

In the analysis of the physical problem presented by a monopolar magnetic field, we have arrived at the desirability of generalizing the concept of a wave function (probability amplitude) in quantum physics to the concept of a *section* (generalizing, that is, to a union of wave functions defined on coordinate patches with joining conditions of the type discussed above). To proceed further we must now specify in detail how the Hilbert space structure of quantum mechanics (which is an additional structure to be imposed on the mathematics of fiber bundles) is to be realized.

We define the scalar product of two *sections* ψ and ϕ to be

$$(\psi,\phi)=(\phi,\psi)^*=\int d\mathbf{x}\psi^*(\mathbf{x})\phi(\mathbf{x}). \quad (5.2.51)$$

(The integral is required to exist with the usual restrictions on an inner product.)

The important fact is that the integrand is well-defined, since in the overlap region one has

$$\psi_b^*\phi_b=\psi_a^*\phi_a. \quad (5.2.52)$$

This result follows from $S_{ab}^*S_{ab}=1$ and the fact that the transition function S_{ab} is the same for all wave functions.

An *operator* is now defined as a mapping of the space of sections into itself. Thus, if $\mathcal{O}$ is to be an operator, one must require that, for any section ψ, the result of the operation, $\mathcal{O}\psi$, is again a section. A Hermitian operator is defined as usual.

It follows from this definition that the coordinate "operator" $\mathbf{x}$ is indeed an operator, since $\mathbf{x}\psi$ is a section; moreover, $\mathbf{x}$ is a Hermitian operator. However, the momentum $\mathbf{p}$ is *not* an operator! (This will cause difficulties in using Fourier analysis, and it is an unresolved technical difficulty in using sections at present.)

The velocity $\dot{\mathbf{x}}=[\mathbf{p}-(e/c)\mathbf{A}]/m_0$ is a valid Hermitian operator when acting on sections, and, in fact, this was used in the defining construction of Section 2.

We have also shown that the angular momentum $\mathbf{J}$ ($\mathbf{J}=\mathbf{J}_a$ in R_a, and $\mathbf{J}=\mathbf{J}_b$ in R_b) is a valid operator on sections, and the Hermitian property may be proved directly. This result may also be verified by using $\mathbf{J}$ directly in the form

$$\mathbf{J}=\mathbf{x}\times\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)-\mu\hat{e}_r. \tag{5.2.53}$$

The Hermitian property of $\mathbf{J}$ acting on sections shows that both $\mathbf{x}$ and $\mathbf{p}-(e/c)\mathbf{A}$ are separately Hermitian (as was already concluded above).

That $\mathbf{J}$ defined by Eq. (5.2.53) is *the* angular momentum is a consequence of the fact (shown in Section 2) that this operator satisfies standard commutation relations (and is Hermitian) when acting on sections.

These results have a surprising implication: The "operator" $\mathbf{L}=\mathbf{x}\times\mathbf{p}$ (using the canonical constructs $\mathbf{x}$ and $\mathbf{p}$) is *not* a valid operator and is *not* the orbital angular momentum!

This rather surprising result has an immediate application in that it answers an old problem for angular momentum in the presence of magnetic ~~fields: What is the proper orbital angular momentum operator: $\mathbf{x}\times\mathbf{p}$ or~~ $\mathbf{x}\times\boldsymbol{\pi}$? [Here $\boldsymbol{\pi}=m_0\mathbf{v}$ is the kinetic momentum $m_0\mathbf{v}=\mathbf{p}-(e/c)\mathbf{A}$, as distinguished from the canonical momentum $\mathbf{p}$.]

To answer this question, let us first recall that in quantum mechanics it is the canonical momentum operator $\mathbf{p}$ that generates displacements, and hence $\mathbf{p}$ is given by $-i\hbar\nabla$ in the Schrödinger realization.

Consider now the situation where the vector potential vanishes (so that $\mathbf{p}$ and $m_0\mathbf{v}$ are the same). Let us recreate the electromagnetic vector potential $\mathbf{A}$ by quickly increasing its value from 0 to $\mathbf{A}$. As we turn on the vector potential, a charged particle necessarily feels a force $e\mathbf{E}=-(e/c)(\partial\mathbf{A}/\partial t)$, and the kinetic momentum $m\mathbf{v}$ changes. Note, however, that the canonical momentum $\mathbf{p}=m_0\mathbf{v}+(e/c)\mathbf{A}$ *remains unchanged*. It follows that it is the canonical momentum $\mathbf{p}$ that must be realized by the invariant association with $-i\hbar\nabla$.

This argument implies that the orbital angular momentum $\mathbf{L}$ must, as the generator of rotations, similarly be realized as $\mathbf{x}\times\mathbf{p}$.

By contrast we see that in the presence of a magnetic monopole we are *forced to realize* $\mathbf{L}$ by $\mathbf{L}=\mathbf{x}\times\boldsymbol{\pi}$, since only in this way do we get a Hermitian operator on sections. This result, which appears rather puzzling, reflects the fact that monopolar fields, being connected with topology and quantized, cannot be "turned on" smoothly, so that the physical argument above (that singled out the canonical momentum) cannot, in fact, be implemented.

It is important to realize that neither this Hilbert space of sections nor the operators possess *any singularities whatever*, by virtue of the construction, which relegates a possible singularity, say, in $\mathbf{A}_b$ to a space outside the region of definition.

Spherical harmonics for sections (monopolar harmonics). We shall study under this heading the problem of developing eigenfunctions for the angular momentum operators $\mathbf{J}$ when acting in a Hilbert space of sections. Since the operator $\mathbf{x}^2$ commutes with $\mathbf{J}$, we can factorize any eigensection into a product of functions defined on each local section, and then recognize that the radial function part can be factored out everywhere. Thus, it is valid simplification to consider purely angular momentum space sections.

Since we are to develop angular "eigensections" in a Hilbert space using operators satisfying angular momentum commutation relations, it is clear that the techniques developed in Chapter 3, Section 3, AMQP, must essentially solve the problem with only relatively minor modifications. Let us recall briefly the technique leading to the quantization of the positive semidefinite operator $\mathbf{J}^2$ and to finite multiplets.

Accordingly, let us consider the invariant operator $\mathbf{J}^2$ defined by

$$\mathbf{J}^2=\left[\mathbf{x}\times\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\right]^2+\mu^2=J_1^2+J_2^2+J_3^2, \tag{5.2.54}$$

where $\mathbf{A}$ is defined by Eqs. (5.2.31). Note that each of the two operators ($\mathbf{J}^2=\mathbf{J}_a^2$ for $\mathbf{x}\in R_a$ and $\mathbf{J}^2=\mathbf{J}_b^2$ for $\mathbf{x}\in R_b$) defined in this equation is positive semidefinite.

Let us now bring both $\mathbf{J}^2$ and J_3 to diagonal form by defining the *sectional ket vectors*:

$$\begin{aligned}\mathbf{J}^2|J'^2,J_3'\rangle&=J'^2|J'^2,J_3'\rangle,\\ J_3|J'^2,J_3'\rangle&=J_3'|J'^2,J_3'\rangle.\end{aligned} \tag{5.2.55}$$

The properties of the raising and lowering operators J_+ and J_- in generating ladders of sectional ket vectors go through exactly as before, and we obtain the standard result (see Chapter 3, Section 3, AMQP):

$$J_\pm|J'^2,J_3'\rangle=\left[J'^2-J_3'(J_3'+1)\right]^{\frac{1}{2}}|J'^2,J_3'\rangle. \tag{5.2.56}$$

Just as before, the ladders must terminate to avoid the contradiction of having $J'^2 - J_3'(J_3' \pm 1) < 0$. This proves that the quantum numbers J'^2 and J_3' are, as before, restricted to the values given by $J'^2 = j(j+1)$ and $J_3' = m$ ($m = -j, -j+1, \ldots, j$) with (j, m) both integer or both half-integer.

The requirement that $\mathbf{J}^2 - \mu^2$ be nonnegative now forces the relation

$$j(j+1) - \mu^2 \geqslant 0. \tag{5.2.57}$$

To examine the question as to the existence of half-integer j, let us consider now the explicit realization for the operator J_3 acting on a *sectional eigenket*. Let $|jm\rangle_a$ and $|jm\rangle_b$ denote sectional eigenkets defined, respectively, on regions R_a and R_b. Then we find

$$\begin{aligned} J_{a3}\langle\theta\phi|jm\rangle_a &= \left(-i\frac{\partial}{\partial\phi} - \mu\right)\langle\theta\phi|jm\rangle_a = m\langle\theta\phi|jm\rangle_a, \qquad \theta,\phi\in R_a, \\ J_{b3}\langle\theta\phi|jm\rangle_b &= \left(-i\frac{\partial}{\partial\phi} + \mu\right)\langle\theta\phi|jm\rangle_b = m\langle\theta\phi|jm\rangle_b, \qquad \theta,\phi\in R_b. \end{aligned} \tag{5.2.58}$$

It follows that the sectional eigenkets have the forms given by

$$\langle\theta\phi|jm\rangle = \begin{cases} \langle\theta\phi|jm\rangle_a = \exp[i(m+\mu)\phi]\, f^a_{jm}(\theta), & \theta,\phi\in R_a, \\ \langle\theta\phi|jm\rangle_b = \exp[i(m-\mu)\phi]\, f^b_{jm}(\theta), & \theta,\phi\in R_b. \end{cases} \tag{5.2.59}$$

Note that

$$\langle\theta\phi|jm\rangle_a = S_{ab}\langle\theta\phi|jm\rangle_b, \tag{5.2.60}$$

where $S_{ab} = \exp(2i\mu\phi)$, as required.

Single-valuedness of these expressions[1] (in all regions) requires that

$$m \pm \mu = \text{integral}. \tag{5.2.61}$$

This result implies the Dirac quantization condition—a general consequence, recall, of the existence of sections—namely, that $2\mu = \text{integer}$.

Thus, if $\mu = \text{half-integer}$, we have the interesting result that m, and hence j, are both *half-integral*. In other words, *we have an explicit realization of spinorial (half-integer) unitary representations realized in ordinary* (θ, ϕ) *space.*

[1] In Note 2 we give an alternative proof, which does not invoke single-valuedness, that $2\mu = \text{integer}$.

This does not contradict the many proofs that half-integer spherical harmonics (defined on θ,ϕ) are impossible, since these proofs assume that the harmonics are to be *functions* (as opposed to *sections*). Note, in particular, that one such proof (Rorschach [11]), based on using $\mathbf{x}$, $\mathbf{p}$, and $\mathbf{L}=\mathbf{x}\times\mathbf{p}$ as operators,[1] fails precisely because $\mathbf{p}$, and hence $\mathbf{x}\times\mathbf{p}$, do *not* exist (as Hermitian operators) in a (nontrivial) space of sections. (We shall discuss the spinorial nature of these solutions further in Section 4 below.)

Let us now complete the development of the monopolar harmonics, $\langle\theta\phi|jm\rangle$. We have determined the ϕ-dependence explicitly in Eq. (5.2.59). From this we can recognize that the sectional behavior is carried entirely by the ϕ-dependence, so that the θ-dependence factors out[2] as a function; that is, $f^a_{jm}(\theta)\times=f^b_{jm}(\theta)$. Thus, we find

$$\langle\theta\phi|jm\mu\rangle=f^j_{m\mu}(\theta)\times\begin{cases}e^{i(m+\mu)\phi} & \text{in } R_a,\\ e^{i(m-\mu)\phi} & \text{in } R_b.\end{cases}\tag{5.2.62}$$

We have introduced μ into the notation for sectional eigenkets to indicate explicitly their dependence on this label. We shall now show that the $f^j_{m\mu}$ that occurs in this result is precisely the function $d^j_{m,-\mu}$ that occurs in the $SU(2)$ representation functions $D^j_{m\mu}(\alpha\beta\gamma)=e^{-im\alpha}d^j_{m\mu}(\beta)e^{-im\gamma}$.

In order to carry out this construction,[3] we first observe using the validity of the standard action of $J_\pm$ on sectional eigenkets, that

$$J_\pm\langle\theta\phi|jm\mu\rangle=[(j\mp m)(j\pm m+1)]^{\frac{1}{2}}\langle\theta\phi|j,m\pm1,\mu\rangle.\tag{5.2.63}$$

Introducing now the explicit forms for the operators $J_\pm=J_{a\pm}$ and $J_\pm=J_{b\pm}$ in the regions R_a and R_b [Eqs. (5.2.28) and (5.2.44), respectively], together with the corresponding sectional eigenkets from Eq. (5.2.62), we find that the following differential-difference equation is satisfied by $f^j_{m\mu}(\theta)$:

$$\left(\pm\frac{\partial}{\partial\theta}-m\cot\theta-\frac{\mu}{\sin\theta}\right)f^j_{m\mu}(\theta)=[(j\mp m)(j\pm m+1)]^{\frac{1}{2}}f^j_{m\pm1,\mu}(\theta).\tag{5.2.64}$$

This is precisely the equation that is satisfied by $d^j_{m,-\mu}(\theta)$, a result that is proved directly from Eqs. (3.102) and (3.104) in AMQP. Since Eq. (5.2.64)

[1]The critical review by van Winter [12] is the best reference for a detailed survey of the literature on this subject, but it omits Ref. [11].

[2]This result becomes explicit when one determines the differential equation for the θ part of the function [see Eq. (5.2.64)].

[3]An alternative method is to use the method of highest weights described in Eqs. (3.17) and (3.18), AMQP.

suffices to determine $f^j_{m\mu}(\theta)$ uniquely, we conclude that

$$f^j_{m\mu}(\theta)=d^j_{m,-\mu}(\theta) \tag{5.2.65}$$

(up to an arbitrary normalization constant).

The final result for the monopolar harmonics is

$$\langle\theta\phi|jm\mu\rangle=\begin{cases} e^{i(m+\mu)\phi}d^j_{m,-\mu}(\theta) & \text{in } R_a, \\ e^{i(m-\mu)\phi}d^j_{m,-\mu}(\theta) & \text{in } R_b. \end{cases} \tag{5.2.66}$$

Equivalently, this result may be expressed as

$$\langle\theta\phi|jm\mu\rangle=\begin{cases} D^{j*}_{m,-\mu}(\phi,\theta,-\phi) & \text{in } R_a, \\ D^{j*}_{m,-\mu}(\phi\theta\phi) & \text{in } R_b. \end{cases} \tag{5.2.67}$$

Remark. In Note 2 we show how the $SU(2)$ representation functions $D^j_{m\mu}(\alpha\beta\gamma)$ may be used *directly* to obtain the sectional eigenkets (5.2.67), including the angular momentum operators $\mathbf{J}$ ($\mathbf{J}=\mathbf{J}_a$ in R_a, and $\mathbf{J}=\mathbf{J}_b$ in R_b). This procedure generalizes that used for the ordinary spherical harmonics in Chapter 3, Section 8, AMQP, and illustrates once again unexpected applications of the $SU(2)$ representation functions themselves. Although this specialization of the $SU(2)$ representation functions yields all the *differential operator actions on specific functions* that have been obtained in this Topic, the key element—the concept of a section so essential to equipping the space with an inner product such that $\mathbf{J}$ is Hermitian—is still required to complete the theory. We have included this alternative derivation, since it is suggestive of possible generalization to other groups.

Completeness of the sectional harmonics. The monopolar harmonics for fixed μ form a complete orthonormal set of *sections* over the sphere. To see that this assertion is valid, it suffices to note that in each region separately the functions $e^{im\phi}$ form a complete set over the circle $0\leqslant\phi<2\pi$. Thus, the index m is determined by Fourier decomposition over ϕ, and the index μ is already fixed.

From the results in Chapter 3, Section 9, AMQP, we know that, for fixed m and μ, the functions $d^j_{m\mu}(\theta)$ are a complete orthonormal set over $0\leqslant\theta\leqslant\pi$. The assertion therefore follows.

It is not difficult to extend to monopolar harmonics various angular momentum analyses (product theorem,...). A recent paper (Wu and Yang [13]) discusses some of these developments, but space is lacking for a detailed treatment here (see also Note 2).

4. Further Discussion

We have stated that the monopolar harmonics for $\mu=\frac{1}{2}$ provide an explicit realization of a *spinor* in (θ,ϕ) space. This statement is correct formally, simply from the fact that the angular momentum has the value $j=\frac{1}{2}$. But the physical meaning is not so obviously correct.

What is a spinor? The fundamental definition is that a spinor is an object that undergoes a sign change under a coordinate rotation by 2π about any axis.

Hence, we must examine the behavior of sections (monopolar harmonics) under rotations by 2π. Taking the general result for sections given in Eq. (5.2.66), we see that for a rotation by 2π around the x_3-axis (taking ϕ to go from ϕ to $\phi+2\pi$) the initial sectional eigenket (at ϕ) is multiplied by the respective factors

$$e^{i(m+\mu)2\pi} \qquad \text{and} \qquad e^{i(m-\mu)2\pi}. \tag{5.2.68}$$

Both phase factors—using the result that $m\pm\mu=$integer—*are unity for all* m,μ *(including* $\mu=0$*)!*

How can this object properly be called a spinor?

To answer this question, note that the method actually used to define the phase is equivalent to using the rotation generator $L_3=-i(\partial/\partial\phi)$, and *not* the total rotation generator J_3. The corresponding finite rotation operators are given by

$$\mathcal{U}_\phi(L_3)=e^{-i\phi L_3} \qquad \text{and} \qquad \mathcal{U}_\phi(J_3)=e^{-i\phi J_3}. \tag{5.2.69}$$

One now finds that the actions of $\mathcal{U}_{2\pi}(L_3)$ and $\mathcal{U}_{2\pi}(J_3)$ on sections are given by

$$\mathcal{U}_{2\pi}(L_3)\to 1 \qquad \text{for } \mu \text{ integer or half-integer,} \tag{5.2.70}$$

$$\mathcal{U}_{2\pi}(J_3)\to\begin{cases} 1 & \text{for } \mu \text{ integer,}\\ -1 & \text{for } \mu \text{ half-integer.}\end{cases} \tag{5.2.71}$$

Thus, with the operator $\mathcal{U}_{2\pi}(J_3)$, we get the desired spinorial property for $j=$half-integer.

This result is not as satisfying as it might appear, for we have simply shifted the problem to another aspect: Why is the operator $\mathcal{U}_{2\pi}(J_3)$, and not $\mathcal{U}_{2\pi}(L_3)$, to be chosen to define the spinorial properties?

Note that the operator $\mathcal{U}_{2\pi}(L_3)$ is well-defined in both regions R_a and R_b and is an acceptable unitary operator on sections. Hence, there is no a priori reason to exclude this as the defining operator for spinorial versus non-spinorial properties.

We have found two reasons to prefer the operator $\mathcal{U}_{2\pi}(J_3)$ over $\mathcal{U}_{2\pi}(L_3)$. The first is to insist that these operators belong to a continuous family of finite rotation operators defined on sections—namely, to

$$\{\mathcal{U}(\psi,\hat{n})=e^{-i\psi\hat{n}\cdot\mathbf{J}}\} \qquad \text{and} \qquad \{\mathcal{U}'(\psi,\hat{n})=e^{-i\psi\hat{n}\cdot\mathbf{L}}\}.$$

However, $\mathcal{U}'(\psi,\hat{n})$ is not, in general, an operator on sections, and this reasoning eliminates **L** as a generator of rotations. The second reason is more compelling: The operator **J** commutes with the Hamiltonian H of the monopole, whereas **L** does not (although L_3 does). Thus, if we insist that H is rotationally invariant, we are forced to the conclusion that **J**, and not **L**, is the generator of rotations.

Concluding remarks. (*a*) The generalization of the concept of quantum mechanical wave functions to include wave functions as sections is structurally a very deep and important generalization, whose significance is as yet not fully worked out. There has been in the fifty-year history of quantum mechanics only one other significant generalization of quantum mechanics, that of Jordan [14] (1931) in defining finite quantum mechanical spaces, where the role of wave functions was subsumed under the concept of a density matrix, and wave functions, as such, were *undefinable*, in general.

(*b*) Let us indicate that there are aspects to the use of sections that have still some puzzling features. Consider the canonical momentum **p**. This operator, as was remarked earlier, is *not* an acceptable operator on sections. Thus, the fundamental (Heisenberg) relation

$$[p_i, x_j] = -i\hbar\delta_{ij}\mathbf{1} \tag{5.2.72}$$

is no longer acceptable on sections.

One must replace the "operator" **p** by the Hermitian sectional operator

$$\boldsymbol{\pi} \equiv \mathbf{p} - (e/c)\mathbf{A}. \tag{5.2.73}$$

Accordingly, one has now the fundamental relation

$$[\pi_i, x_j] = -i\hbar\delta_{ij}\mathbf{1}, \tag{5.2.74}$$

which is a valid operator structure to be applied on sections. It follows that it is the pair of operators $\boldsymbol{\pi}$ and **x** that satisfy the Heisenberg relations and are to be considered as conjugate observables.

The operator $\boldsymbol{\pi}$, however, has unusual commutation properties with itself, since it satisfies the relation[1]

$$[\pi_i, \pi_j] = (ie\hbar/c)e_{ijk}B_k, \tag{5.2.75}$$

[1] Note that Eq. (5.2.76) shows that the Jacobi relation for commutators is violated if the point $r=0$ is not removed from the space accessible to the motion.

in general, and for the monopole field, in particular,

$$\hbar^{-1}\boldsymbol{\pi}\times\hbar^{-1}\boldsymbol{\pi}=i\mu\hat{e}_r/r^2. \tag{5.2.76}$$

One has thereby *lost the commutativity* of the Cartesian components of the operator conjugate to the position operator **x**. This property is well-known for the operator $\boldsymbol{\pi}$, but it could be considered merely a curiosity as long as the canonical momentum **p** is available (trivial section). For nontrivial sections this option is not available, and one is faced with a conceptual difficulty to clarify. The full significance of these interpretational difficulties for sections is as yet unknown.

5. Notes

1. *Renewed interest in monopoles.* Interest in the magnetic monopole problem has been greatly increased by the discovery of 't Hooft [15] and Polyakov [16] that nonabelian gauge field theory possesses physically acceptable solutions ("solitons") with magnetic monopole structure.

It has been claimed by Jackiw and Rebbi [17]—in discussing a theory with isospin as the nonabelian gauge field—that in the field of a monopole in such a nonabelian gauge theory *isospin degrees of freedom are converted into spin angular momentum degrees of freedom*. To quote from Jackiw [18, p. 698]: "In a quantum theory spin has been created from isospin." This astonishing claim is incorrect, as was demonstrated by Troost and Vinciarelli [19].

2. *More detailed investigation of the relationship of monopolar harmonics to the SU(2) representation functions.* In Chapter 3, Section 8, AMQP, we developed in detail the properties of the $SU(2)$ irrep functions $D^j_{m'm}(\alpha\beta\gamma)$ in the Euler angle parametrization. In particular, we obtained the spherical harmonics $Y_{lm'}(\theta\phi)$ by setting $m=0$ in these irrep functions (see Chapter 3, Section 10, AMQP):

$$Y_{lm}(\beta\alpha)=\left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}D^{l*}_{m,0}(\alpha\beta\gamma). \tag{5.2.77}$$

Moreover, this procedure for obtaining the spherical harmonics allowed us to delete the derivative term $\partial/\partial\gamma$ from the angular momentum operator $\boldsymbol{\mathcal{J}}$ [see Eq. (5.2.80) below], thus giving the orbital angular momentum operators themselves.

The purpose of this Note is to show how the monopolar harmonics may be obtained by a generalization of this procedure used for the ordinary spherical harmonics.

For convenience we repeat several of the relations proved in Chapter 3, Section 8, AMQP:

$$\mathcal{J}_{\pm} D^{j*}_{m'm}(\alpha\beta\gamma) = [(j \mp m')(j \pm m' + 1)]^{\frac{1}{2}} D^{j*}_{m' \pm 1, m}(\alpha\beta\gamma),$$
$$\mathcal{J}_3 D^{j*}_{m'm}(\alpha\beta\gamma) = m' D^{j*}_{m'm}(\alpha\beta\gamma); \tag{5.2.78}$$

$$(\mathcal{P}_1 - i\mathcal{P}_2) D^{j*}_{m'm}(\alpha\beta\gamma) = [(j-m)(j+m+1)]^{\frac{1}{2}} D^{j*}_{m', m+1}(\alpha\beta\gamma),$$
$$(\mathcal{P}_1 + i\mathcal{P}_2) D^{j*}_{m'm}(\alpha\beta\gamma) = [(j+m)(j-m+1)]^{\frac{1}{2}} D^{j*}_{m', m-1}(\alpha\beta\gamma),$$
$$\mathcal{P}_3 D^{j*}_{m'm}(\alpha\beta\gamma) = m D^{j*}_{m'm}(\alpha\beta\gamma). \tag{5.2.79}$$

The operators appearing in these relations have the following explicit forms:

$$\mathcal{J}_+ = \mathcal{J}_1 + i\mathcal{J}_2 = e^{i\alpha}\left(i\cot\beta \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} - \frac{i}{\sin\beta}\frac{\partial}{\partial\gamma} \right),$$
$$\mathcal{J}_- = \mathcal{J}_1 - i\mathcal{J}_2 = e^{-i\alpha}\left(i\cot\beta \frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta} - \frac{i}{\sin\beta}\frac{\partial}{\partial\gamma} \right),$$
$$\mathcal{J}_3 = -i\frac{\partial}{\partial\alpha}; \tag{5.2.80}$$

$$\mathcal{P}_1 + i\mathcal{P}_2 = e^{-i\gamma}\left(-i\cot\beta \frac{\partial}{\partial\gamma} + \frac{\partial}{\partial\beta} + \frac{i}{\sin\beta}\frac{\partial}{\partial\alpha} \right),$$
$$\mathcal{P}_1 - i\mathcal{P}_2 = e^{i\gamma}\left(-i\cot\beta \frac{\partial}{\partial\gamma} - \frac{\partial}{\partial\beta} + \frac{i}{\sin\beta}\frac{\partial}{\partial\alpha} \right),$$
$$\mathcal{P}_3 = -i\frac{\partial}{\partial\gamma}. \tag{5.2.81}$$

The representation functions $D^{j*}_{m'm}$ are thus simultaneous eigenfunctions of the three commuting operators $\mathcal{J}^2$, $\mathcal{J}_3$, and $\mathcal{P}_3$:

$$\mathcal{J}^2 D^{j*}_{m'm}(\alpha\beta\gamma) = j(j+1) D^{j*}_{m'm}(\alpha\beta\gamma),$$
$$\mathcal{J}_3 D^{j*}_{m'm}(\alpha\beta\gamma) = m' D^{j*}_{m'm}(\alpha\beta\gamma), \tag{5.2.82}$$
$$\mathcal{P}_3 D^{j*}_{m'm}(\alpha\beta\gamma) = m D^{j*}_{m'm}(\alpha\beta\gamma),$$

where

$$\mathcal{P}_1^2 + \mathcal{P}_2^2 + \mathcal{P}_3^2 = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$$
$$= \mathcal{J}^2 = -\csc^2\beta\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\gamma^2} - 2\cos\beta\frac{\partial^2}{\partial\alpha\,\partial\gamma} \right) - \frac{\partial^2}{\partial\beta^2} - \cot\beta\frac{\partial}{\partial\beta}. \tag{5.2.83}$$

Consider now the following identity:

$$\begin{aligned}[\mathcal{J}_+ e^{im\alpha} f(\beta) e^{-i\mu\gamma}]_{\gamma=-\alpha} &= e^{i\alpha}\left(-m\cot\beta + \frac{\partial}{\partial\beta} - \frac{\mu}{\sin\beta}\right) e^{i(m+\mu)\alpha} f(\beta) \\ &= e^{i\alpha}\left[i\cot\beta\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} - \frac{\mu(1-\cos\beta)}{\sin\beta}\right] e^{i(m+\mu)\alpha} f(\beta) \\ &= J_+ e^{i(m+\mu)\alpha} f(\beta), \end{aligned} \tag{5.2.84}$$

where $f(\beta)$ is an arbitrary differentiable function, and J_+ is the operator defined by Eq. (5.2.28). Repeating this procedure for each of the operators in Eqs. (5.2.78) and (5.2.79), we find

$$\begin{aligned}[\mathcal{J}_\pm D^{j*}_{m,-\mu}(\alpha\beta\gamma)]_{\gamma=-\alpha} &= J_\pm D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha), \\ [\mathcal{J}_3 D^{j*}_{m,-\mu}(\alpha\beta\gamma)]_{\gamma=-\alpha} &= J_3 D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha); \end{aligned} \tag{5.2.85}$$

$$\begin{aligned}[(\mathcal{P}_1 \pm i\mathcal{P}_2) D^{j*}_{m,-\mu}(\alpha\beta\gamma)]_{\gamma=-\alpha} &= M_\pm D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha), \\ [\mathcal{P}_3 D^{j*}_{m,-\mu}(\alpha\beta\gamma)]_{\gamma=-\alpha} &= M_3 D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha); \end{aligned} \tag{5.2.86}$$

where the operators $J_\pm$, J_3, and $M_\pm$, M_3 have the explicit definitions given by

$$\begin{aligned} J_\pm &= L_\pm - \frac{\mu(1-\cos\beta)}{\sin\beta} e^{\pm i\alpha}, \\ J_3 &= L_3 - \mu; \end{aligned} \tag{5.2.87}$$

$$\begin{aligned} M_\pm &= L_\pm - \frac{m(1-\cos\beta)}{\sin\beta} e^{\pm i\alpha}, \\ M_3 &= L_3 - m. \end{aligned} \tag{5.2.88}$$

In these results, $L_\pm$ and L_3 are the operators defined by Eqs. (5.2.29), where $\phi = \alpha$ and $\theta = \beta$.

Using now the right-hand sides of Eqs. (5.2.78) and (5.2.79), we obtain the following actions of the angular momentum operators **J** and **M** on the special representations functions having $\gamma = -\alpha$:

$$\begin{aligned} J_\pm D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha) &= [(j\mp m)(j\pm m+1)]^{\frac{1}{2}} D^{j*}_{m\pm1,-\mu}(\alpha,\beta,-\alpha), \\ J_3 D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha) &= m D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha); \end{aligned} \tag{5.2.89}$$

$$\begin{aligned} M_\pm D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha) &= [(j\mp\mu)(j\pm\mu+1)]^{\frac{1}{2}} D^{j*}_{m,-\mu\mp1}(\alpha,\beta,-\alpha), \\ M_3 D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha) &= \mu D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha). \end{aligned} \tag{5.2.90}$$

We next repeat the entire procedure, but evaluate at $\gamma=\alpha$. The results are

$$\begin{aligned} J'_{\pm}D^{j*}_{m,-\mu}(\alpha\beta\alpha)&=\left[(j\mp m)(j\pm m+1)\right]^{\frac{1}{2}}D^{j*}_{m\pm 1,-\mu}(\alpha\beta\alpha), \\ J'_3D^{j*}_{m,-\mu}(\alpha\beta\alpha)&=mD^{j*}_{m,-\mu}(\alpha\beta\alpha); \end{aligned} \tag{5.2.91}$$

$$\begin{aligned} M''_{\pm}D^{j*}_{m,-\mu}(\alpha\beta\alpha)&=\left[(j\mp\mu)(j\pm\mu+1)\right]^{\frac{1}{2}}D^{j*}_{m,-\mu\mp 1}(\alpha\beta\alpha), \\ M''_3D^{j*}_{m,-\mu}(\alpha\beta\alpha)&=\mu D^{j*}_{m,-\mu}(\alpha\beta\alpha); \end{aligned} \tag{5.2.92}$$

where the operators $J'_{\pm}$, J'_3 and $M''_{\pm}$, M''_3 have the explicit definitions given by

$$\begin{aligned} J'_{\pm}&=L_{\pm}-\frac{\mu(1+\cos\beta)}{\sin\beta}e^{\pm i\alpha}, \\ J'_3&=L_3+\mu; \end{aligned} \tag{5.2.93}$$

$$\begin{aligned} M''_{\pm}&=-L_{\mp}-\frac{m(1+\cos\beta)}{\sin\beta}e^{\mp i\alpha}, \\ M''_3&=-L_3+m. \end{aligned} \tag{5.2.94}$$

(The purpose of the notation $\mathbf{M}''$, as opposed to $\mathbf{M}'$, will become clear below.)

All the equations derived above [Eqs. (5.2.87)–(5.2.94)] are expressions for explicit differential operators and their actions on explicit functions. Accordingly, these relations are valid for all values of α and β for which the differential operators are defined (the functions are defined for all values of α and β, since the dependence on β is that of a polynomial in $\cos(\beta/2)$ and $\sin(\beta/2)$—see Chapter 3, Section 6, AMPQ). *These results are independent of any Hilbert space concepts.* Indeed, one loses the original Hilbert space property of the representation functions with its accompanying Hermitian ~~property for the generators $\mathcal{J}$ and $\mathcal{P}$ in making the identification $\gamma=\pm\alpha$.~~

It is remarkable that the Hilbert space and the Hermitian properties of the operators **J** *and* **M** *may be restored through the concept of sections, but each operator acts in its own Hilbert space of sections.*

Let us next interpret this assertion.

Note that the operators **J** and **J**$'$ defined by Eqs. (5.2.87) and (5.2.93) are identical (set $\phi=\alpha$, and $\theta=\beta$) with those given by Eqs. (5.2.28) and (5.2.44), respectively. Moreover, the actions of these operators expressed by Eqs. (5.2.89) and (5.2.91) are precisely those for the operators given by Eqs. (5.2.63) and (5.2.46) on the sectional eigenkets $\langle\theta\phi|jm\mu\rangle$ in Eq. (5.2.67). Furthermore, the transition function relation (5.2.47) is correctly satisfied.

Thus, when the concept of section is invoked together with its attendant Hilbert space, we recover from Eqs. (5.2.89) and (5.2.91) the desired extension of the standard formulation of the angular momentum multiplets to *sectional multiplets* as developed in Sections 2 and 3.

It is important here to recall that μ is fixed (specified) in the definition of section given in Sections 2 and 3. Accordingly, these sections are more appropriately designated "right μ-sections," since they carry the fixed label μ either implicitly or explicitly, and this label is located to the right in the notation $|jm\mu\rangle$ for the basis states of the Hilbert space of right μ-sections. (A more complete notation would include the label μ in the notation for the operators $\mathbf{J}$ and $\mathbf{J}'$ [see Eqs. (5.2.87) and (5.2.93)] as a reminder that the operator actions expressed by Eqs. (5.2.89) and (5.2.91) apply to fixed μ.)

Let us next consider the operator actions expressed by Eqs. (5.2.90) and (5.2.92). [Recall that these relations originated from the angular momentum multiplets (5.2.79) for the body-referred components of the angular momentum.] We wish to interpret these equations in terms of sections, but Eqs. (5.2.92) are clearly in the wrong form! This situation is easily corrected, however, by complex conjugating Eqs. (5.2.92). The results are

$$\begin{aligned} M'_{\pm} D^{j}_{m,-\mu}(\alpha\beta\alpha) &= [(j\mp\mu)(j\pm\mu+1)]^{\frac{1}{2}} D^{j}_{m,-\mu\mp1}(\alpha\beta\alpha), \\ M'_3 D^{j}_{m,-\mu}(\alpha\beta\alpha) &= \mu D^{j}_{m,-\mu}(\alpha\beta\alpha), \end{aligned} \tag{5.2.95}$$

where

$$\begin{aligned} M'_{\pm} &= M''^{*}_{\pm} = L_{\pm} - \frac{m(1+\cos\beta)}{\sin\beta} e^{\pm i\alpha}, \\ M'_3 &= M''^{*}_3 = L_3 + m. \end{aligned} \tag{5.2.96}$$

It will now be recognized that we have obtained from the relations for body-referred angular momentum operators *precisely* the relations obtained earlier from the space-fixed angular momentum operators, *but with m and μ interchanged*. [Compare Eqs. (5.2.87) with (5.2.88), (5.2.89) with (5.2.90), (5.2.93) with (5.2.96), and (5.2.91) with (5.2.95).] The proof of this result ~~uses the symmetry of the representation functions given by~~

$$\begin{aligned} D^{j*}_{\mu,-m}(\alpha,\beta,-\alpha) &= D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha), \\ D^{j*}_{\mu,-m}(\alpha\beta\alpha) &= D^{j}_{m,-\mu}(\alpha\beta\alpha). \end{aligned} \tag{5.2.97}$$

These results lead us to the concept of a left m-section: A left m-section and the corresponding inner product is defined exactly as for a right μ-section, except that m replaces μ in the definition of a section (hence, m is fixed) and in the definitions of the angular momentum operators that act in m-sections. The term "left" in the designation "left m-section" signifies that the label m to the left in the notation $|jm\mu)$ [see Eq. (5.2.99)] for the basis states of the Hilbert space of left m-sections is to be kept fixed. Note that the Hilbert space of right μ-sections and the Hilbert space of left m-sections are distinct spaces.

The angular momentum operators for left m-sections are defined by

$$\mathbf{M}=\begin{cases}\mathbf{M}=\mathbf{M}_a & \text{in } R_a,\\ \mathbf{M}'=\mathbf{M}_b & \text{in } R_b.\end{cases} \tag{5.2.98}$$

The m-sectional eigenkets, which we denote by $|jm\mu)$ to distinguish them from μ-sectional eigenkets, are given explicitly by

$$(\beta\alpha|jm\mu)=\begin{cases}D^{j*}_{m,-\mu}(\alpha,\beta,-\alpha) & \text{in } R_a,\\ D^{j}_{m,-\mu}(\alpha\beta\alpha) & \text{in } R_b.\end{cases} \tag{5.2.99}$$

(The label μ is introduced explicitly into m-sections through $M_3 \to \mu$.)

The requirements

$$\phi_b(\mathbf{x})=S'_{ba}(\mathbf{x})\phi_a(\mathbf{x}), \qquad \mathbf{x}\in R_{ab} \tag{5.2.100}$$

and

$$\mathbf{M}_b\phi_b(\mathbf{x})=S'_{ba}(\mathbf{x})\mathbf{M}_a\phi_a(\mathbf{x}), \qquad \mathbf{x}\in R_{ab} \tag{5.2.101}$$

uniquely determine the transition function $S'_{ba}(\mathbf{x})$:

$$S'_{ba}(\mathbf{x})=e^{-2im\alpha}, \qquad \mathbf{x}\in R_{ab}. \tag{5.2.102}$$

Note that the m-sectional eigenkets satisfy this condition.

What good are left m-sections? To answer this question let us recall that in the Hilbert space of right μ-sections we were able to extend the standard angular multiplet construction for functions to μ-sections. This result then implied the quantization of (j, m) in the usual way. *In this theory μ appears as a real parameter in the transition function*; *the quantization of μ is obtained by invoking single-valuedness*. But since the Hilbert space of left m-sections is abstractly identical to that of right μ-sections, we may also extend the standard multiplet construction to the Hilbert space of left m-sections: *The quantization of μ (recall $M_3 \to \mu$) to the values $j, j-1, \ldots, -j$ now appears as a consequence of the existence of the Hilbert space of left m-sections—that is, as a consequence of the Hermitian property of the angular momentum* $\mathbf{M}$ *acting in this Hilbert space. Single-valuedness need not be invoked.*

Remarks. (*a*) The operator $\mathbf{J}$ is an operator in the Hilbert space $\mathcal{H}^{\mu}$ of right μ-sections with basis $\{|jm\mu\rangle: j=0,\frac{1}{2},1,\ldots; m=j, j-1,\ldots,-j\}$, but it is not an operator in the Hilbert space $\mathcal{H}^{m}$ of left m-sections $(\mu\neq m)$ with basis $\{|jm\mu): j=0,\frac{1}{2},1,\ldots; \mu=j, j-1,\ldots,-j\}$. Similarly, $\mathbf{M}$ is an operator in $\mathcal{H}^{m}$, but not in $\mathcal{H}^{\mu}(\mu\neq m)$.[1] The product of two operators J_i and M_j is, in general, not defined (in particular, the commutator is not defined).

[1]For $\mu=m$, we have not only the identity of operators $\mathbf{J}=\mathbf{M}$, but also [using the symmetry relation (5.2.97)] the identity of spaces $\mathcal{H}^{m}_j=\mathcal{H}^{m}_j$ for each multiplet space $j\geqslant|m|$ [see Remark (c)].

(b) The operator $\mathbf{M}$ is obtained from $\mathbf{J}$ by replacing μ by m. The relations for $\mathbf{J}$ given in Sections 2 and 3 become relations for $\mathbf{M}$ upon making this replacement. Note, in particular, that $\mathbf{M}^2 \neq \mathbf{J}^2$ [see Remark (a)].

(c) Suppose that we are given a Hilbert space $\mathcal{H}^\mu$ of right μ-sections that splits into a direct sum of right μ-sectional angular momentum multiplets; that is,

$$\mathcal{H}^\mu = \mathcal{H}^\mu_{j_0} + \mathcal{H}^\mu_{j_0+1} \oplus \mathcal{H}^\mu_{j_0+2} \oplus \cdots$$

where $j_0 = |\mu|$, and $\mathcal{H}^\mu_j$ is the space of dimension $2j+1$ characterized by $\mathbf{J}^2 \to j(j+1)$. We may then select basis vectors from this space that have a fixed m-value and j-values given by $j = k_0, k_0+1, \ldots,$ where $k_0 = \max(j_0, |m|)$. Using the angular momentum $\mathbf{M}$, we may now generate a Hilbert space of left m-sections. We may then select basis vectors from this new space that have a fixed μ'-value and j-values given by $j = j_0', j_0'+1, \ldots,$ where $j_0' = \max(|\mu'|, j_0, |m|)$. Using the angular momentum $\mathbf{J}$ (appropriate to μ'), we may now generate a Hilbert space of right μ'-sections of the form

$$\mathcal{H}^{\mu'} = \mathcal{H}^{\mu'}_{j_0'} \oplus \mathcal{H}^{\mu'}_{j_0'+1} \oplus \cdots.$$

In particular, starting from $\mathcal{H}^0$ and $\mathcal{H}^{\frac{1}{2}}$, we may generate the Hilbert spaces of right μ-sections for all $\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots.$

(d) The significance for monopoles of the coexistence of Hilbert spaces of right μ-sections and left m-sections is not known.

6. Addendum. Minimal Coupling and Complex Line Bundles

The discussion of the magnetic monopole problem in the preceding sections has been based largely on the work of Wu and Yang [7, 8]. In point ~~of fact, however, the important observation that the proper mathematical~~ tool with which to treat the monopole problem is the theory of *complex line bundles* was made prior to Wu and Yang's work, by Sniatycki [20] and by Greub and Petry [21]. We shall discuss here the treatment of Greub and Petry, both for its intrinsic interest (Ref. [21] is quite elegant) as well as to show how their mathematical (global and coordinate-free) view accords fully with the physicist's more detailed, coordinate-based treatment.

The key idea is to recognize *the geometric meaning of minimal coupling* for incorporating the electromagnetic interaction into quantum mechanics. For this the proper language is that of differential geometry, and the new interpretation (used also by Wu and Yang) generalizes the concept of a wave function to that of a *section in a complex line bundle*; the momentum operator $\pi_\mu = p_\mu - (e/c)A_\mu$ is generalized to a linear connection in the line bundle whose curvature form is the electromagnetic field. These ideas have already been discussed, at least in part, in the present Topic, but the Greub and Petry treatment has the great merit that it is completely *coordinate-free*; accordingly, we feel it is useful to begin anew, and discuss this approach ab initio.

Complex line bundles.[1] A *complex line bundle* is a collection $L=(E,\pi,M)$, where E and M are smooth manifolds and $\pi: E\to M$ is a smooth map satisfying the following conditions:

(i) For every $x\in M$, the set $F_x=\pi^{-1}(x)$ (the *fiber* over x) is a one-dimensional complex vector space.

(ii) There exists a covering of M by open sets U_α and a family of diffeomorphisms $\Psi_\alpha: U_\alpha\times\mathbb{C}\xrightarrow{\cong}\pi^{-1}(U_\alpha)$, which restrict for every $x\in U_\alpha$ to a (complex) linear isomorphism

$$\Psi_{\alpha x}:\mathbb{C}\xrightarrow{\cong}F_x.$$

Every such family (Ψ_α, U_α) is called a *trivialization* of the complex line bundle L.

The simplest example of a complex line bundle over a manifold M is the product $M\times\mathbb{C}$, where π: $M\times\mathbb{C}\to M$ is the obvious projection. This is called the *trivial bundle*.

A *cross section* in a complex line bundle is a smooth map σ: $M\to E$ such that $\pi\circ\sigma(x)=x$; that is, σ carries every point $x\in M$ into a vector of its fiber. (Therefore, one can define addition of cross sections and multiplication of cross sections with functions on M.)

A *Hermitian metric* is a map $\langle\ ,\ \rangle_L$ that assigns to every pair of cross sections σ_1,σ_2 a function $\langle\sigma_1,\sigma_2\rangle_L$ that is linear in σ_2, and antilinear in σ_1, with[2] $\langle\sigma_1,\sigma_2\rangle_L=\overline{\langle\sigma_2,\sigma_1\rangle}_L$, and such that

$$\langle\phi\sigma_1,\sigma_2\rangle_L=\bar{\phi}\langle\sigma_1,\sigma_2\rangle_L$$

for arbitrary functions ϕ on M. The metric $\langle\ ,\ \rangle_L$ is called *positive definite* if $\langle\sigma,\sigma\rangle_L\geqslant 0$, and $\langle\sigma,\sigma\rangle_L=0$ implies that $\sigma=0$.

A *linear connection* in a complex line bundle L is an operator ∇ that assigns to every pair σ, X (σ a cross section, X a vector field on M), another cross section, $\nabla_X\sigma$, called the *covariant derivative of σ with respect to X*, which is linear in X and σ and satisfies for arbitrary functions ϕ the conditions

$$\nabla_{\phi X}\sigma=\phi\nabla_X\sigma,$$
$$\nabla_X\phi\sigma=\phi\nabla_X\sigma+X(\phi)\sigma.$$

[1]The definitions and theorems given in this section are taken from Ref. [21].

[2]The bar denotes complex conjugation in this Addendum.

In the second result $X(\phi)$ denotes the derivative of ϕ in the direction of X.

Let ∇ be a linear connection in L, let X and Y be arbitrary vector fields on M with Lie product $[X, Y]$, and let σ be an arbitrary cross section. One can show that there is a unique two-form R, called the curvature form, on M such that

$$R(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma.$$

The proof follows directly from the fact that the right-hand side of this equation is function-linear and skew-symmetric in X and Y. *The Bianchi identity states that this two-form is closed; that is* $dR = 0$, as can be verified from the definition of R.

Thus, R represents a de Rham cohomology class $[R]$ of M. Using Cech cohomology theory, one can show that the class $[R]/2\pi i$ is *integral.* Conversely, we have the following theorem:

Existence theorem (see Refs. [22, 23]). Let M be a manifold, and let Φ be a closed two-form on M such that $\Phi/2\pi i$ represents an integral class. Then there exists a line bundle L over M and a linear connection ∇ in L such that the corresponding curvature form coincides with Φ. Moreover, if M is simply connected, then L and ∇ are uniquely determined up to strong bundle isomorphisms. Finally, if $\Phi/2\pi i$ is real-valued, then L admits a positive definite Hermitian metric $\langle\ ,\ \rangle_L$ such that

$$X(\langle \sigma_1, \sigma_2 \rangle_L) = \langle \nabla_X \sigma_1, \sigma_2 \rangle_L + \langle \sigma_1, \nabla_X \sigma_2 \rangle_L$$

for arbitrary cross sections σ_1, σ_2 and vector fields X. This metric is determined up to a positive constant.

Minimal coupling. Let us now see how the usual quantum mechanical concepts fit into this more general framework. The usual wave function is now to be regarded as a section[1] in the trivial bundle, $M \times \mathbb{C}$, with the ~~projection $\Pi: M\times\mathbb{C} \to M$. Denoting this section by $\Psi: M \to M\times\mathbb{C}$, we have~~ that $\Psi(x) = (x, \Psi(x))$ for $x \in M$, where $\Psi(x) \in \mathbb{C}$ is the usual wave function.

To determine the linear connection in this trivial bundle, we recall that minimal coupling involved the substitution

$$p_\mu \to \pi_\mu = p_\mu - \frac{e}{c} A_\mu,$$

where A_μ is the vector potential for the electromagnetic field.

The most general form for the covariant derivative (linear connection) in a trivial bundle is

$$\nabla_X \Psi(x) = (x, \varphi(x)),$$

[1] We often use the term "section" in place of "cross section."

with

$$\varphi(x)=(X\varphi)(x)+\omega(X)\varphi(x).$$

Comparing this with the substitution from minimal coupling, we see that (a) the term $X\varphi$ is given by $(i/\hbar)p_\mu\varphi=\nabla_\mu\varphi$, and (b) the function $\omega(X)$—that is, the one-form ω evaluated on the vector field X—is given by the vector potential $-ieA_\mu/\hbar c$.

It follows that the one-form ω is given by

$$\omega=-\sum_\mu \frac{ie}{\hbar c}A_\mu\, dx_\mu,$$

and that the curvature two-form R is given by

$$R=d\omega\equiv-\frac{ie}{2\hbar c}\sum_{\mu\nu}F_{\mu\nu}\,dx_\mu\, dx_\nu=-\frac{ie}{\hbar c}F,$$

where $F_{\mu\nu}$ is the electromagnetic field $(\mathbf{E},\mathbf{B})$ expressed in tensor form. Note that this curvature is exact—that is, $R=d\omega$, since it is determined by a (global) vector potential.

The Bianchi identity $(dR=0)$ now implies the homogeneous set of Maxwell's equations for the fields $F_{\mu\nu}$.

The probability interpretation of quantum mechanics accords with the Hermitian metric on the sections of these (trivial) bundles.

We conclude that the usual quantum mechanical formulation of minimal coupling is correctly subsumed (as the trivial bundle) under the more ~~general concept of a complex line bundle. The uniqueness theorem asserts~~ that the trivial bundle, with the linear connection stated above, is the only complex line bundle with this particular electromagnetic field (curvature), provided that the region in question is simply connected. (The vector potential is, of course, determined only up to a gauge transformation.)

The new interpretation then extends this description to the case where the complex line bundle is nontrivial and for which the vector potential is no longer globally definable. The concept of a linear connection (covariant derivative) is still meaningful, although no longer determined explicitly by a global vector potential, but only indirectly through the curvature condition. The coordinate-free formulation of minimal coupling, generalizing wave functions to sections in a complex line bundle, obviates the difficulty of nonphysical singularities ("Dirac string") in earlier formulations of the magnetic monopole problem.

The magnetic monopole field is described by the two-form $F=g\Omega$, where g is the pole strength and Ω is the two-form:

$$\Omega=r^{-3}e_{ijk}x_i\,dx_j\,dx_k/2.$$

In order to have a line bundle with the curvature $R=ieF/\hbar c$, the existence theorem implies the condition

$$\int_{S^2} eF/2\pi\hbar c\in\mathbb{Z}\equiv\text{set of integers},$$

where S^2 is a sphere in $\mathbb{R}^3$ centered at the origin. Introducing $F=g\Omega$ and integrating over a surface surrounding the origin, one finds the following result: *Line bundles with the desired curvature exist if and only if the pole strength obeys the Dirac quantization relation* $eg/\hbar c=\mu$, $2\mu\in\mathbb{Z}$.

Conversely, the existence theorem asserts that, for each integer $2\mu\in\mathbb{Z}$, there exists a unique line bundle with the curvature $R=-i\mu\Omega$. Let us denote these line bundles as L_μ.

Explicit construction. The discussion so far has been relatively abstract, and rather in the nature of a vocabulary exercise, in our effort to correlate mathematical and physical concepts. By using angular momentum techniques to implement these coordinate-free concepts, we shall now see explicitly the power, and practicality, of bundle concepts.

The basic idea is to use quaternions to construct a fibering over the space $\mathbb{R}^3-(0)$ (physical three-space, with the origin—the position of the magnetic pole—deleted). We denote the four-dimensional vector space of quaternions by Q, the inner product in Q by $\langle\ ,\ \rangle$, and an individual quaternion by q. Thus, in terms of an orthonormal basis of Q given by e_0, e_1, e_2, e_3, we have (see Chapter 4, Sections 2 and 3, AMQP)

$$q=q_0e_0+\sum_{i=1}^{3}q_ie_i,$$

where the basis quaternions satisfy the multiplication rules

$$e_ie_j=-e_je_i=e_k,\qquad i,j,k\text{ cyclic in }1,2,3,$$
$$e_i^2=-e_0^2=-e_0,\qquad e_0e_i=e_ie_0,\qquad i=1,2,3.$$

(The basis element e_0 is the multiplicative identity element.) The quaternion conjugate to q is denoted by $\bar{q}$ and is defined in terms of q by

$$\bar{q}=q_0e_0-\sum_{i=1}^{3}q_ie_i.$$

The subspace of quaternions orthogonal to e_0 (the space of pure quaternions) is identified, as a vector space, to be the physical space $\mathbb{R}^3$. Thus, an element of $\mathbb{R}^3$ has the form $x = x_1e_1 + x_2e_2 + x_3e_3$.

Consider the set of nonzero quaternions $Q-(0)$ and the map[1] Π from $Q-(0)$ to the $\mathbb{R}^3-(0)$ defined by $\Pi(q) = qe_3\bar{q}$. To verify that this map has the desired properties, note that $\Pi(q) \neq 0$ whenever $q \neq 0$, and that $e_3 \to qe_3\bar{q}$ is a rotation in $\mathbb{R}^3$. Alternatively, $\langle e_0, \Pi(q)\rangle = 0$, so that $\Pi(q)$ belongs to the space of pure quaternions.

To verify that this map defines a fibration, note that $q \to q\varepsilon$, with $\varepsilon(\phi) = e_0 \cos\phi + e_3 \sin\phi$, $\phi \in [0, 2\pi]$, leaves $\Pi(q)$ unchanged. Thus, the fiber is the $U(1)$ group (circle group S^1) with the action in $Q-(0)$ being right multiplication.

Let us determine now the generator of this circle group and, accordingly, the vector field to which this generator corresponds. Consider the path $\alpha: [0,1] \to Q$ that corresponds to the action $q \to q\varepsilon$. The derivative of this path evaluated at $\phi = 0$ defines the vector field V evaluated at the point q; that is, $V_q = qe_3$. Since $\Pi(q)$ is invariant to this action, the vector field V is called *vertical*; vector fields orthogonal to V are called *horizontal* and denoted by H. Thus, $\langle H_q, V_q\rangle = 0$, $q \in Q-(0)$.

Note that in describing vector fields we have made use of the fact that any vector field on $Q-(0)$ may be represented as a quaternion-valued function. Now to every vector field on $\mathbb{R}^3-(0)$ there corresponds a unique horizontal vector field on $Q-(0)$. To determine this correspondence explicitly, we make use of the map Π, differentiating the relation

$$x = \sum_{i=1}^{3} x_i e_i = \Pi(q) = qe_3\bar{q}.$$

~~Corresponding to the vector field X_i on $\mathbb{R}^3-(0)$ along the direction e_i, we~~ obtain the horizontal vector field H_i on $Q-(0)$ given explicitly by

$$H_i = -X_i qe_3/2q\bar{q}, \qquad q \in Q-(0). \qquad (*)$$

[1] More generally we have that $qe_j\bar{q} = (q\bar{q})\sum_{i=1}^{3} R_{ij}e_i$, where $R = (R_{ij})$ is a proper orthogonal matrix with elements given explicitly by Eq. (3.396), AMQP (set $x_\mu = q_\mu$). Moreover, this result is just the quaternionic realization of the homomorphism between the groups $SU(2)$ and $SO(3)$ obtained from the isomorphism between unimodular quaternions and unimodular 2×2 unitary matrices given by $q/(q\bar{q}) \to U$, $\bar{q}/(q\bar{q}) \to U^\dagger$, $e_j \to -i\sigma_j$ $(j = 1,2,3)$ (see AMQP, Chapter 2, Section 5, and Note 3; Chapter 4, Section 3). Thus, $qe_3\bar{q}$ is column 3 of the matrix $(q\bar{q})R$: $x_1 = (2q\bar{q})(q_1q_3 + q_0q_2)$, $x_2 = (2q\bar{q})(q_2q_3 - q_0q_1)$, $x_3 = (q\bar{q})(q_0^2 - q_1^2 - q_2^2 + q_3^2$[see Eq. (2.22), AMQP].

More generally (see Helgason [22]) one can define the horizontal lift H of the vector field X by the derivative $d\Pi$ of the map Π, using

$$(d\Pi)_q H_q = X_{\Pi(q)}.$$

This results in Eq. $(*)$, if we use the quaternion-valued functions to represent vector fields.

The next step is to calculate the covariant derivative and the curvature two-form for this geometric structure. Conceptually there is no difficulty, since the covariant derivative is simply the ordinary derivative orthogonally projected onto the tangent space— that is, the horizontal components of the ordinary derivative. To calculate the curvature two-form, however, one must be aware of the distinction between the Lie bracket for vector fields and the commutator for quaternions. It is simplest, therefore, to revert to the language of derivative operators for vector fields on $Q-(0)$. Denoting the quaternion $q = q_0 e_0 + \sum_i q_i e_i$ by the four vector $\mathbf{q} = (q_0, q_1, q_2, q_3)$, we find for the vector fields H_i $(i = 1,2,3)$ from Eq. $(*)$ that

$$H_1 = (2q\bar{q})^{-1}\left(-q_2\frac{\partial}{\partial q_0} + q_3\frac{\partial}{\partial q_1} - q_0\frac{\partial}{\partial q_2} + q_1\frac{\partial}{\partial q_3}\right),$$

$$H_2 = (2q\bar{q})^{-1}\left(q_1\frac{\partial}{\partial q_0} + q_0\frac{\partial}{\partial q_1} + q_3\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_3}\right),$$

$$H_3 = (2q\bar{q})^{-1}\left(q_0\frac{\partial}{\partial q_0} - q_1\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial q_2} + q_3\frac{\partial}{\partial q_3}\right).$$

The vertical vector field V corresponds to the operator

$$V = (q\bar{q})^{-1}\left(q_3\frac{\partial}{\partial q_0} + q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2} + q_0\frac{\partial}{\partial q_3}\right).$$

It is straightforward now to calculate the Lie bracket of the vector fields H_i, using the explicit operator realizations above. We summarize some of the details of this calculation, since the generators of the right translations, $\mathcal{K}_i$, introduced in Chapter 3, Note 4, AMQP, enter into these commutators.

If we define the differential operators Λ_i by

$$\Lambda_i = (2q\bar{q})H_i,$$

then we find the commutators of the Λ_i to be

$$[\Lambda_i, \Lambda_j] = 4ie_{ijk}\mathcal{K}_k,$$

where the $\mathcal{K}_k$ $(k=1,2,3)$ are defined by [see Eq. (3.389), AMQP]

$$2i\mathcal{K}_k \equiv \left(q_i \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial q_i} \right) - \left(q_0 \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial q_0} \right)$$

with i, j, k cyclic in $1,2,3$. These operators then satisfy the standard angular momentum commutation relations

$$[\mathcal{K}_i, \mathcal{K}_j] = ie_{ijk}\mathcal{K}_k.$$

We next use the commutators

$$[\Lambda_i, q\bar{q}] = 2x_i/q\bar{q}, \qquad [\Lambda_i, (q\bar{q})^{-1}] = -2x_i/(q\bar{q})^3,$$

where the coordinates (x_1, x_2, x_3) in $\mathbb{R}^3 - (0)$ are determined from the map $\Pi(q) = qe_3\bar{q} = \sum_i x_i e_i$, to obtain

$$[H_i, H_j] = 2(q\bar{q})^{-4}[x_j\Lambda_i - x_i\Lambda_j + 2i(q\bar{q})^2\mathcal{K}_k]$$

with i, j, k cyclic in $1,2,3$. Substitution of the Λ_i and $\mathcal{K}_i$ into this result yields the desired commutator

$$[H_i, H_j] = e_{ijk}(x_k/2r^3)V,$$

where we note that $r = \left(\sum_i x_i^2 \right)^{\frac{1}{2}} = q\bar{q}$, and

$$V = (q\bar{q})^{-1} 2i\mathcal{K}_3.$$

Corresponding to this operator commutator, one has the Lie bracket for the vector fields:

$$[H_i, H_j]_q = e_{ijk}(x_k/2r^3)V_q.$$

Recognizing that $e_{ijk}(x_k/2r^3)$ is just the two-form Ω evaluated at the vector fields $X_i = \partial/\partial x_i$ and $Y_j = \partial/\partial x_j$, one can express this result in a coordinate-free way (see Ref. [21]):

$$[H_X, H_Y]_q = \tfrac{1}{2}\Omega(X, Y)_{\Pi(q)} V_q. \qquad (**)$$

The equivariance condition. The construction sketched above (technically called the construction of the principal bundle) does not, by itself, suffice for a coordinate-free description of the monopole problem. To accomplish this latter goal, it is necessary to do two things: (a) Construct complex line bundles associated to the principal bundle, and (b) implement the equivariance condition below (see Ref. [23], p. 406).

The result of (a) is, in fact, the desired solution to the monopole problem, since sections in these associated complex line bundles are precisely the generalized wave functions that were to be obtained.

The result of (b) is, however, of even more practical importance, since (as will be shown) *the equivariance condition allows one to transfer the complex line bundle construction from* $\mathbb{R}^3-(0)$ *to the more amenable problem of constructing equivariant functions over* $Q-(0)$. One has more "freedom" in this larger space, and calculations are technically simpler.

The first step in (a) is to represent the circle group S^1 of the fibers by the representations[1] Φ_μ (characters) given by $\varepsilon(\phi)\to e^{2i\mu\phi}$. Next one constructs the product space $\mathcal{Q}=(Q-(0))\times\mathbb{C}$ and defines an action of S^1 in $\mathcal{Q}$ by

$$\varepsilon(\phi):\mathcal{Q}=(q,z)\to\big(q\varepsilon(\phi),ze^{-2i\mu\phi}\big),\qquad q\in Q-(0),\, z\in\mathbb{C}.$$

This action defines an equivalence relation in $\mathcal{Q}$ given by $(q,z)\sim(q\varepsilon(\phi),ze^{-2i\mu\phi})$. The quotient manifold defined by this equivalence relation will be denoted by $\mathcal{Q}_\mu\equiv(Q-(0))\times{}_\mu\mathbb{C}$ and the corresponding projection by

$$\kappa:\mathcal{Q}=(Q-(0))\times\mathbb{C}\to(Q-(0))\times{}_\mu\mathbb{C}.$$

The manifold $\mathcal{Q}_\mu$ is the desired line bundle over $\mathbb{R}^3-(0)$, and the projection ρ onto the base-space $\mathbb{R}^3-(0)$ is defined by the commutative mapping diagram:

$$\begin{array}{ccc}(Q-(0))\times\mathbb{C} & \xrightarrow{\kappa} & (Q-(0))\times{}_\mu\mathbb{C}\\ \downarrow{\scriptstyle\Pi_1} & & \downarrow{\scriptstyle\rho}\\ Q-(0) & \xrightarrow{\Pi} & \mathbb{R}^3-(0).\end{array}$$

(Here Π_1 is projection onto the first factor, with Π denoting the map defined earlier.)

The complex line bundle $L_\mu=\big((Q-(0))\times{}_\mu\mathbb{C},\rho,\mathbb{R}^3-(0)\big)$ is called *the complex line bundle associated to the principal bundle* $(Q-(0),\Pi,\mathbb{R}^3-(0))$ by the representation Φ_μ. This completes task (a).

Turning to (b), we now define an equivariant function on $Q-(0)$. *A complex-valued function f on Q is called equivariant with respect to the*

[1]The occurrence of the factor 2 in these representations results from the conventional use of half-integers for spinorial representations.

representation Φ_μ *if* f *satisfies the condition*

$$f(q\varepsilon(\phi))=e^{-2i\mu\phi}f(q), \qquad q\in Q-(0), \varepsilon(\phi)\in S^1.$$

It will be recognized at once that this equivariance condition is just the usual symmetry condition expressed by $(Tf)(Tx)=f(x)$.

The importance of the equivariance condition is that there exists an isomorphism between the equivariant functions on $Q-(0)$ and the associated complex line bundles. To see this, consider a cross section σ in the associated bundle L_μ. We can construct a function $f_\sigma\colon Q-(0)\to\mathbb{C}$ by the assignment

$$f_\sigma(q)=\kappa_q^{-1}(\sigma(\Pi(q))),$$

where κ_q denotes the map κ restricted to the point q. Note that the map κ so restricted defines an isomorphism on the fibers; that is,

$$\kappa_q\colon\mathbb{C}\overset{\cong}{\to}\rho^{-1}(\Pi(q)), \qquad q\in Q-(0).$$

The function f_σ is equivariant, since (by the equivalence relation) $\kappa_{q\varepsilon}=e^{-2i\mu\phi}\kappa_q$.

Conversely, every equivariant function defines a unique cross section of the bundle L_μ. Thus, as discussed in detail in Ref. [23, p. 406], *there exists an isomorphism* $\mathcal{K}$ *from the cross sections in* L_μ *to the equivariant functions in* $Q-(0)$.

By means of the isomorphism $\mathcal{K}$, the construction of a covariant derivative in the associated line bundles L_μ becomes conceptually easy. Letting X denote a vector field on $\mathbb{R}^3-(0)$, we can obtain the horizontal vector field H_X on $Q-(0)$. Then, if σ denotes a section in L_μ, we have the linear connection

$$\nabla_X\sigma=\mathcal{K}^{-1}H_X(\mathcal{K}\sigma).$$

To compute the curvature two-form, we let X and Y be constant vector fields on $\mathbb{R}^3-(0)$. Then, since $\nabla_{[X,Y]}=0$, we have

$$R(X,Y)\sigma=\nabla_X\nabla_Y\sigma-\nabla_Y\nabla_X\sigma.$$

Using the isomorphism $\mathcal{K}$, we may express these relations in terms of equivariant functions, $f=\mathcal{K}\sigma$, instead of sections. One finds

$$R(X,Y)\sigma=\mathcal{K}^{-1}[H_X,H_Y](f).$$

Using Eq. (* *) for this Lie bracket, and noting that the equivariance condition implies $\mathcal{K}_3 f = -2i\mu f$, we obtain

$$R(X,Y)\sigma = -i\mu\Omega(X,Y)\sigma.$$

This, in turn, implies (since σ was arbitrary) that the curvature two-form R satisfies the relation $R = -i\mu\Omega$ as required for the physical interpretation of the curvature as the electromagnetic field of a (stationary) monopole of pole strength $g = \mu\hbar c/e$.

To complete the quantum mechanics, let us note that if σ_1 and σ_2 are any two cross sections in L_μ, then the complex-valued functions on $Q-(0)$ given by $f_i = \mathcal{K}^{-1}\sigma_i$ $(i=1,2)$ have a Hermitian metric defined by

$$\bar{f}_1(q)f_2(q) = \langle\sigma_1,\sigma_2\rangle_L(\Pi(q)),$$

which, because of the equivariance condition, *is independent of the choice of* q in the fiber lying over $\Pi(q)$ in $\mathbb{R}^3-(0)$.

Application to the quantum mechanical monopole problem. The concepts and techniques that have been developed so far will, no doubt, seem rather complicated to a physicist; we hope, however, that the specific application to be discussed below will clarify these ideas so that they will appear more natural. For the monopole problem (as discussed earlier in this topic) we have the time-independent Schrödinger equation:

$$E\Psi = (2m_0)^{-1}\sum_{i=1}^{3}\pi_i^2\Psi. \qquad (* * *)$$

Here π_i denotes the kinetic momentum, which (for exact fields) is given by $\pi_i = p_i - (e/c)A_i$. In the monopole case the operator π_i represents the covariant derivative in the direction of the vector field $\partial/\partial x_i$. Recall that Ψ is no longer a function (wave function), but is now to be regarded as a section in a complex line bundle L_μ.

The isomorphism $\mathcal{K}$ between the cross sections in L_μ and the equivariant functions on Q makes it possible now to effect a great simplification in solving the Schrödinger equation. If we apply the isomorphism $\mathcal{K}$ to both sides of Eq. (* * *) and denote $\mathcal{K}\Psi$ by φ_μ, we obtain

$$E\varphi_\mu = -\frac{\hbar^2}{2m_0}\sum_{i=1}^{3}H_i^2\varphi_\mu, \qquad (* * * *)$$

where H_i is now an ordinary derivative in the direction of the horizontal lift.

The differential operators H_i have been explicitly calculated earlier. The great advantage of this step is that we are dealing now with ordinary complex-valued functions, which have, as we shall see, been extensively studied in Chapter 3, AMQP (see, in particular, Note 4). The function φ_μ is equivariant, so that $\varphi_\mu(q\varepsilon)=e^{-2i\mu\phi}\varphi_\mu(q)$, where $\varepsilon(\phi)=e_0\cos\phi+e_3\sin\phi$. The quantal probability density is $\bar{\varphi}_\mu\varphi_\mu$, so that (as mentioned earlier) for equivariant functions this is a probability for physical space, depending only on $x=\Pi(q)=\Pi(q\varepsilon)$.

To solve the Schrödinger equation for φ_μ, we separate variables, writing φ_μ as a product of a radial and angular function,

$$\varphi_\mu(q)=R(r)f_\mu(\hat{q}),$$

where $r=q\bar{q}$ is the radius in physical space, and f_μ is an equivariant function over the unimodular quaternion $\hat{q}=q/\|q\|$. From Chapter 4, Sections 2 and 3, AMQP, we recognize that the unimodular quaternions are, as a manifold, the group manifold of $SU(2)$. Moreover, as a carrier space, functions defined on the unimodular quaternions carry representations of the direct product group $SU(2)\times SU(2)$, realized by the left and right translations (Chapter 3, Note 4, AMQP).

Thus, with reference to previous discussions, we may immediately choose the $SU(2)$ representation functions $D^{j*}_{m,-\mu}(\mathbf{q})$ as a basis for the equivariant functions f_μ. (This result is apparent from the explicit differential operator realization of $\sum_i H_i^2$ obtained below.) The equivariance condition is realized by right translations in the 3-direction (generator $\mathcal{K}_3$). The physical angular momentum $\mathcal{J}=(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3)$ is realized by the generators of the left translations (see Chapter 3, Note 4, AMQP).

To complete the solution of the Schrödinger equation for φ_μ, we calculate ~~the explicit operator $\sum_i H_i^2$. Using the differential operator realizations of~~ the H_i and of $V=2i(q\bar{q})^{-1}\mathcal{K}_3=2i\mathcal{K}_3(q\bar{q})^{-1}$, we find that

$$\sum_{i=1}^{3} H_i^2=\tfrac{1}{4}\rho^{-2}\nabla_4^2-\tfrac{1}{4}V^2=\tfrac{1}{4}\rho^{-2}\nabla_4^2+\rho^{-4}\mathcal{K}_3^2,$$

where $\rho=(q\bar{q})^{\frac{1}{2}}=r^{\frac{1}{2}}$ denotes the radial distance in four-space, and ∇_4^2 denotes the four-space Laplacian

$$\nabla_4^2=\sum_{\nu=0}^{3}\frac{\partial^2}{\partial q_\nu^2}.$$

The Laplacian ∇_4^2 is, however, related to the total angular momentum operator[1] $\mathcal{J}^2$ by

$$\mathcal{J}^2=\mathcal{K}^2=-\tfrac{1}{4}\rho^2\nabla_4^2+\mathcal{J}_0^2+\mathcal{J}_0,$$

where $2\mathcal{J}_0$ is the Euler operator given by

$$2\mathcal{J}_0=\sum_{\nu=0}^{3}q_\nu\frac{\partial}{\partial q_\nu}.$$

Using this result in the expression for $\sum_i H_i^2$, we now obtain

$$\sum_{i=1}^{3}H_i^2=\rho^{-4}\left(\mathcal{J}_0^2+\mathcal{J}_0-\mathcal{J}^2+\mathcal{K}_3^2\right).$$

It is apparent from this result and the eigenfunction properties

$$\mathcal{J}^2D^{j*}_{m,-\mu}(\mathbf{q})=j(j+1)D^{j*}_{m,-\mu}(\mathbf{q}),\qquad \mathcal{J}_0D^{j*}_{m,-\mu}(\mathbf{q})=jD^{j*}_{m,-\mu}(\mathbf{q}),$$
$$\mathcal{K}_3D^{j*}_{m,-\mu}(\mathbf{q})=\mu D^{j*}_{m,-\mu}(\mathbf{q})$$

that the Schrödinger equation (* * * *) will be satisfied by product functions of the form

$$\varphi_\mu(\mathbf{q})=F(\rho)D^{j*}_{m,-\mu}(\mathbf{q})=\rho^{2j}F(\rho)D^{j*}_{m,-\mu}(\hat{q})=R(r)D^{j*}_{m,-\mu}\,(\hat{q}),$$

where $r=\rho^2$ and $R(r)=r^jF(r^{\frac{1}{2}})$. The action of $\mathcal{J}_0$ on $\varphi_\mu(\mathbf{q})$ is the same as that of rd/dr, so that we may write

$$\sum_{i=0}^{3}H_i^2=r^{-2}\left[\left(r\frac{d}{dr}\right)^2+r\frac{d}{dr}-\left(\mathcal{J}^2-\mathcal{K}_3^2\right)\right].$$

What is remarkable about this result is that the radial and angular operator $\sum_i H_i^2$—an operator in the four-dimensional space $Q-(0)$—becomes *precisely* (with no extraneous numerical or algebraic factors) the

[1] The generators $\mathcal{J}=(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3)$ of left translations and the generators $\mathcal{K}=(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3)$ of right translations are related by $-\mathcal{K}_j=\sum_i R_{ij}\mathcal{J}_j=\sum_i\mathcal{J}_jR_{ij}$, where (R_{ij}) is the orthogonal matrix discussed in the footnote on p. 232. This result, the property $[\mathcal{J}_i,\mathcal{K}_j]=0$, as well as the explicit actions of these generators on the $SU(2)$ representation functions have been discussed in detail in Chapter 3, Note 4, AMQP.

radial and angular operators of the *three-dimensional* Laplacian, modified only by the angular momentum term $\mathcal{K}_3^2/r^2 \to \mu^2/r^2$ of the electromagnetic field.

Making the replacements $\mathcal{J}^2 \to j(j+1)$ and $\mathcal{K}_3^2 \to \mu^2$, we obtain the radial equation

$$\left[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{j(j+1)-\mu^2}{r^2}-\frac{2m_0E}{\hbar^2}\right]R(r)=0.$$

This is just the radial equation for the spherical Bessel function found previously [Eq. (5.2.24)]. The angular eigenfunctions are the $SU(2)$ representation functions discussed above.

We have thus found the complete set of equivariant eigenfunctions on the space $Q-(0)$. In summary, these functions are

$$\varphi_\mu(E,j,m;q)=(k\rho^2)^{-\frac{1}{2}}J_\lambda(k\rho^2)D^{j*}_{m,-\mu}(\hat{q}), \qquad k=(2m_0E/\hbar^2)^{\frac{1}{2}}, \qquad E>0,$$

where $\lambda=\frac{1}{2}[(2j+1)^2-4\mu^2]^{\frac{1}{2}}$, and for specified μ the values that j and m may assume are $j=|\mu|,|\mu|+1,|\mu|+2,\ldots;\ m=j,\ j-1,\ldots,-j$.

Remark. It is of interest to note that in this formulation the electromagnetic field angular momentum μ and the total angular momentum projection m appear simply as the components $\mathcal{K}_3\to\mu$ and $\mathcal{J}_3\to m$ of the right and left translations in quaternion space. Accordingly, the two angular quantum numbers clearly play a symmetric role. In the formulation based on eigensections, the symmetry between these quantum numbers is apparent in the final answer but otherwise unexplained (as noted by Wu and Yang). The quaternionic formulation has the merit of making this symmetry apparent.

Relation to the previous formulation. We shall now show the complete equivalence of this abstract formulation to the previous (physically oriented) treatment of the magnetic monopole problem. To do so, we must introduce coordinates into this bundle formulation. [A coordinatization of a bundle is also described as a local trivialization (see discussion, p. 228)]. For the case at hand, we can introduce coordinates in two steps: We first coordinatize the principal bundle, and then we apply this to the associated bundles L_μ.

To coordinatize the principal bundle we introduce (just as in the earlier discussion) two sections. Let $x=\sum_{i=1}^{3}x_ie_i$ denote a point in physical three-space. Then the (upper) section σ_+ defines a point in Q by

$$\sigma_+(x)=\frac{(\|x\|e_3+x)\bar{e}_3}{[2(\|x\|+x_3)]^{\frac{1}{2}}}=[2(r+x_3)]^{-\frac{1}{2}}[(r+x_3)e_0-x_2e_1+x_1e_2].$$

This mapping $\sigma_+ : x \to q$ is well-defined for all points x in $\mathbb{R}^3$ for which $\|x\| + x_3 \neq 0$; that is, for all x *except those on the line* $\theta = \pi$ ("Dirac string"), where θ is the polar angle of x. This is the set of points in $\mathbb{R}^3$ denoted by R_a (for $\delta = \pi/2$) in Eq. (5.2.30). Similarly, the (lower) section σ_- defines a point in Q by

$$\sigma_-(x) = \sigma_+(-x)e_2 = [2(r - x_3)]^{-\frac{1}{2}}[x_1 e_0 + (r - x_3)e_2 + x_2 e_3].$$

This map is defined for all points in $\mathbb{R}^3$ for which $r - x_3 \neq 0$—that is, for all points x except those on the line $\theta = 0$. This is the set of points in $\mathbb{R}^3$ denoted by R_b (for $\delta = \pi/2$) in Eq. (5.2.30). Clearly, $R_a \cup R_b = \mathbb{R}^3 - (0)$, as required. Moreover, in the overlap region $R_a \cap R_b$, we have the transition relation

$$\sigma_-(x) = \sigma_+(x)(x_1 e_0 + x_2 e_3)/(x_1^2 + x_2^2)^{\frac{1}{2}},$$

this result being verified directly from the definitions of $\sigma_\pm(x)$. It follows from this result that the two sections $\sigma_+(x)$ and $\sigma_-(x)$ differ by a rotation in the "vertical direction" (generated by $\mathcal{K}_3$); stated otherwise, they differ by a gauge transformation.

Using the map $\Pi(q) = q e_3 \bar{q}$ from $Q - (0)$ to $\mathbb{R}^3 - (0)$, we find that

$$\Pi(\sigma_\pm(x)) = x.$$

Thus, for each $x \in \mathbb{R}^3$ that belongs to the domain of definition of σ_+ and σ_-, the quaternions $\sigma_+(x)$ and $\sigma_-(x)$ are mapped to x by Π. In particular, if we parametrize $x \in \mathbb{R}^3$ in terms of spherical coordinates

$$x = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta),$$

~~then the corresponding quaternions $\sigma_\pm$ are given by~~

$$\sigma_+(x) = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} e_0 - \sin\frac{\theta}{2} \sin\phi\, e_1 + \sin\frac{\theta}{2} \cos\phi\, e_2\right),$$

$$\sigma_-(x) = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} \cos\phi\, e_0 + \sin\frac{\theta}{2} e_2 + \cos\frac{\theta}{2} \sin\phi\, e_3\right).$$

In order to relate these results to the Euler angle parametrization of the $SU(2)$ representation functions, we need to parametrize a general quaternion $q \in Q$ in terms of Euler angles. This parametrization is obtained by using the isomorphism, $SU(2) \to \hat{Q}$, between the set of 2×2 unimodular unitary matrices and the set of 2×2 unimodular quaternions, $\hat{Q}$. Thus, using

$$U(\alpha\beta\gamma) = e^{-i\alpha\sigma_3/2} e^{-i\beta\sigma_2/2} e^{-i\gamma\sigma_3/2}$$

and the map $\sigma_0 \to e_0$, $\sigma_j \to ie_j$ ($j=1,2,3$), we find [see Eqs. (2.27), (2.40), and (3.133) in AMQP]

$$\begin{aligned}
U(\alpha\beta\gamma) &\to q(\alpha\beta\gamma)/r^{\frac{1}{2}} \\
&= \left(\cos\frac{\alpha}{2}e_0+\sin\frac{\alpha}{2}e_3\right)\left(\cos\frac{\beta}{2}e_0+\sin\frac{\beta}{2}e_2\right)\left(\cos\frac{\gamma}{2}e_0+\sin\frac{\gamma}{2}e_3\right) \\
&= \cos\frac{\beta}{2}\cos\frac{1}{2}(\gamma+\alpha)e_0+\sin\frac{\beta}{2}\sin\frac{1}{2}(\gamma-\alpha)e_1 \\
&\quad +\sin\frac{\beta}{2}\cos\frac{1}{2}(\gamma-\alpha)e_2+\cos\frac{\beta}{2}\sin\frac{1}{2}(\gamma+\alpha)e_3.
\end{aligned}$$

Recalling that the polar angles $(\theta\phi)$ coincide with the Euler angles $(\beta\alpha)$, we see that the quaternions (sections) $\sigma_{\pm}(x)$ are parametrized in terms of Euler angles by

$$\begin{aligned}
\sigma_{+}(x)&=q(\phi,\theta,-\phi), \\
\sigma_{-}(x)&=q(\phi,\theta,\phi).
\end{aligned}$$

To complete the coordinatization, we consider the equivariant functions, $R(r)D^{j*}_{m,-\mu}(\hat{q})$, which constitute the solutions to the problem. Since the radial functions are identical to those found earlier, only the angular functions need to be considered here.

It is clear that the angular eigensections calculated earlier are just the composition of the two maps $\sigma_{\pm}: \hat{x}\to\hat{q}$ and $D^{j*}_{m,-\mu}: \hat{q}\to\mathbb{C}$—that is, $D^{j*}_{m,-\mu}\circ\sigma_{\pm}$. Written explicitly, the angular eigensections are $D^{j*}_{m,-\mu}(\sigma_{\pm}(\hat{x}))$. Using $\sigma_{\pm}(x)=q(\phi,\theta,\mp\phi)$ and the fact that the domains of definition of σ_{+} and σ_{-} are R_a and R_b, respectively, we see that the eigenkets $D^{j*}_{m,-\mu}(\sigma_{\pm}(\hat{x}))$ are exactly those of Eqs. (5.2.67).

~~The conceptual advantage of using coordinate-free methods is evident,~~ but it is also clear that coordinate-free methods have calculational advantages as well, as, for example, in the use of equivariant functions over Q, which allowed a uniform treatment *without any joining conditions.*

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Topic 3. A Minimal Realization Of Angular Momentum States: The Symplecton

1. Preliminary Remarks

In order to contrast the differences between the standard (Jordan–Schwinger) boson operator realization of angular momentum (operators and state vectors) and the realization to be developed in this section,[1,2] let us

[1] This section is based on the work of Ref. [1].

[2] There is still another "nonstandard" boson realization. This is discussed in Note 1.

begin with a few observations on the nature of the standard realization (developed in detail in Chapter 5 of AMQP).

(1) The Jordan–Schwinger realization [Eq. (5.3.3) below], defined in terms of a single boson $\mathbf{a}=(a_1, a_2)$ having two (abstractly equivalent) states, is a "primitive" realization of the generators of $SU(2)$, in the sense that $U(2)$, as opposed to $SU(2)$, is impossible. (This is clearly so, since the necessary antisymmetric structures vanish identically.)

(2) The state vectors in this realization, however, are clearly not conjugation symmetric, since the boson rule $\bar{a}_i|0\rangle=0$ constitutes a distinguished role for the conjugate boson operators. [This remark may seem paradoxical, since the group $SU(2)$ itself has conjugation as an inner automorphism. The resolution lies in the observation that to achieve this symmetry one requires, in addition to conjugation, an interchange of bra and ket vectors.]

(3) The Jordan–Schwinger realization is not, however, minimal in the sense that the largest algebra one can construct, which is quadratic in the boson operators and their conjugates, is not the algebra of $SU(2)$, but that of $Sp(4)$ (using the compact real form), the symplectic group in four dimensions.

(4) The existence of the Wigner–Eckart theorem can be traced to the existence of an isomorphism between operators and state vectors under transformation by the generators J_i $(i=1,2,3)$.

To see how this comes about, let us be fully explicit about the transformation rules. For operators, $\mathcal{O}$ [defined, say, as polynomials over $(a_i, \bar{a}_i)$ with complex (numerical) scalars], the transformation rule is the familiar commutation operation:

$$J_i(\mathcal{O})\equiv[J_i,\mathcal{O}]. \tag{5.3.1}$$

For state vectors, $\nu=|\nu\rangle$, one uses a different rule:

$$J_i(\nu)\equiv J_i|\nu\rangle. \tag{5.3.2}$$

The generators for (infinitesimal) rotations are taken, as in Chapter 2, to be the realization

$$J_+\to a_1\bar{a}_2, \qquad J_-\to a_2\bar{a}_1, \qquad J_3\to\tfrac{1}{2}(a_1\bar{a}_1-a_2\bar{a}_2). \tag{5.3.3}$$

One notes that these generators all have a destruction operator on the right; moreover, every vector ν may be written as $|\nu\rangle=A_\nu|0\rangle$, *where A_ν is a polynomial over the a_i only*. It follows that the transformation law for vectors is formally identical to that for operators; that is,

$$J_i(\nu)\equiv J_i|\nu\rangle=J_iA_\nu|0\rangle=[J_i, A_\nu]|0\rangle. \tag{5.3.4}$$

The operator–state vector isomorphism is evident.

We have emphasized these properties of the Jordan–Schwinger realization only to make it evident that *these properties depend in an essential way on the validity of the boson postulate* $\bar{a}_i|0\rangle \equiv 0$.

2. The Symplecton Realization of $SU(2)$

We wish to demonstrate that there exists another realization of the generators of $SU(2)$ that is minimal and primitive in the sense used above.

Consider a single boson operator, a, and its conjugate $\bar{a}$, obeying the commutation relation, $[\bar{a}, a] = 1$, all other commutators being zero. We may realize the generators of $SU(2)$ by

$$J_+ \to -\tfrac{1}{2}a^2, \qquad J_- \to \tfrac{1}{2}\bar{a}^2, \qquad J_3 \to \tfrac{1}{4}(a\bar{a}+\bar{a}a). \tag{5.3.5}$$

It is easily verified[1] that this realization satisfies the desired commutation relations:

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3. \tag{5.3.6}$$

Since we wish to impose a formal symmetry between a and $\bar{a}$ (this is the meaning denoted by the term "symplecton"), we cannot define the usual vacuum ket. Instead we define a *formal ket*, $|\ \rangle$, and seek to interpret *both* $a|\ \rangle$ and $\bar{a}|\ \rangle$ as nonvanishing vectors.

Operators in this structure will be defined as polynomials over $(a, \bar{a})$ with complex numbers as scalars. State vectors will be defined as operators multiplied on the right by the basic formal ket; that is,

$$\nu \equiv |\nu\rangle \equiv \mathcal{O}_\nu|\ \rangle, \tag{5.3.7}$$

where ν is a vector, and $\mathcal{O}_\nu$ is the operator associated with the vector ν. This ~~establishes the desired operator–state vector isomorphism.~~ ~~*The action of the*~~ *generators* **J** *is defined as commutation on the relevant operator.*

The crucial question now concerns the definition of an inner product in order to convert this structure into a Hilbert space. But first let us attempt to motivate our definition.

Consider the vector $a|\ \rangle$. It is reasonable to associate with this the bra vector $\langle\ |\bar{a}$, where $\langle\ |$ denotes the basic formal bra; this leads to the numeric, $\langle\ |\bar{a}a|\ \rangle$, which we must now interpret. Writing $\bar{a}a$ as the sum

$$\bar{a}a = \tfrac{1}{2}(\bar{a}a + a\bar{a}) + \tfrac{1}{2}[\bar{a}, a], \tag{5.3.8}$$

we note that the first term is just $2J_3$; it is reasonable to associate this first

[1] The minus sign (for J_+) is essential. This choice of phases will be discussed below.

term with angular momentum $j=1$, $m=0$. The second term, $\frac{1}{2}[\bar{a}, a]$, gives $\frac{1}{2}$ and is the scalar ($j=m=0$) part of $\bar{a}a$. Accordingly, we interpret the operation $\langle\ |(\ldots)|\ \rangle$ to mean: *Take only the $j=0$ part of the expression* $(\ldots)$. This rule thus assigns $\frac{1}{2}$ to the vector $a|\ \rangle$.

Proceeding next to the vector $\bar{a}|\ \rangle$, one finds that the $j=0$ part of $a\bar{a}$ is $-\frac{1}{2}$, so that the rule above associates $-\frac{1}{2}$ to the vector $\bar{a}|\ \rangle$. This result motivates the second rule: *The adjoint operation, denoted adj, on the pair of bosons is defined by*

$$adj\colon (a, \bar{a}) \to (\bar{a}, -a). \tag{5.3.9}$$

Accordingly, the vector that is adjoint (or dual) to the vector $\bar{a}|\ \rangle$ is $\langle\ |(-a)$, and the $j=0$ part of $-a\bar{a}$ is $\frac{1}{2}$, thus associating $\frac{1}{2}$ to the vector $\bar{a}|\ \rangle$.

In order to arrive at the general definition of an inner product, we first define the adjoint operation, *adj*, on an arbitrary polynomial:

$$adj\colon P(a, \bar{a}) \to P^*(\bar{a}, -a) \equiv P^{\mathrm{adj}}(a, \bar{a}), \tag{5.3.10}$$

where P^* is the polynomial obtained from P by complex conjugating the numerical coefficients. [For example, if $P(a, \bar{a}) = \alpha a\bar{a}^2 + \beta a^2\bar{a}$, $\alpha, \beta \in \mathbb{C}$, then $P^*(a, \bar{a}) = \alpha^* a\bar{a}^2 + \beta^* a^2\bar{a}$.] The vector that is dual to $P(a, \bar{a})|\ \rangle$ is then $\langle\ |P^{\mathrm{adj}}(a, \bar{a})$.

Let us now give the definition of the inner product of two vectors:

$$|P\rangle = P(a, \bar{a})|\ \rangle \qquad \text{and} \qquad |P'\rangle = P'(a, \bar{a})|\ \rangle, \tag{5.3.11}$$

where P and P' are arbitrary polynomials in the bosons a and $\bar{a}$. The inner product of the vectors $|P\rangle$ and $|P'\rangle$ is denoted by $\langle P|P'\rangle$. By definition, the inner product is the mapping from the set of pairs of polynomials, ~~$\{(P, P')\}$, to the set of complex numbers $\mathbb{C}$ given by $(P, P') \to \langle P|P'\rangle \to \mathbb{C}$,~~ where

$$\begin{aligned}\langle P|P'\rangle &= \langle\ |P^{\mathrm{adj}}(a, \bar{a})P'(a, \bar{a})|\ \rangle \\ &\equiv \text{the } j=0 \text{ part of the polynomial } P^{\mathrm{adj}}(a, \bar{a})P'(a, \bar{a}).\end{aligned} \tag{5.3.12}$$

This definition of inner product is the one suggested by our example above, but, to be fully explicit, we must demonstrate how this result is to be implemented, and, furthermore, that $\langle P|P'\rangle$ does indeed possess the properties required of an inner product.

To carry out this step, we first classify the set of all polynomials $P(a, \bar{a})$ by their properties with respect to the generators $\mathbf{J}$, so that the operation defined on the right-hand side of Eq. (5.3.12) is explicit. The necessary results for doing this are given in Appendices A, B, and C, where the following three properties are proved:

(1) The space of all polynomials in a and $\bar{a}$ contains *characteristic eigenpolynomials* $\mathcal{P}_j^m$ ($j=0,\frac{1}{2},1,\ldots,m=j,j-1,\ldots,-j$) that satisfy standard relations under commutation with $\mathbf{J}$:

$$\left[J_{\pm},\mathcal{P}_j^m\right]=\left[(j\mp m)(j\pm m+1)\right]^{\frac{1}{2}}\mathcal{P}_j^{m\pm 1},$$
$$\left[J_3,\mathcal{P}_j^m\right]=m\mathcal{P}_j^m. \tag{5.3.13}$$

(2) The characteristic polynomials in the set $\{\mathcal{P}_j^m\}$ satisfy the multiplication law:

$$\mathcal{P}_a^{\alpha}\mathcal{P}_b^{\beta}=\sum_{c=|a-b|}^{a+b}\langle c|a|b\rangle C_{\beta,\alpha,\alpha+\beta}^{b\,a\,c}\mathcal{P}_c^{\alpha+\beta}, \tag{5.3.14}$$

where

$$\langle c|a|b\rangle=(2c+1)^{-\frac{1}{2}}\nabla(abc), \tag{5.3.15}$$

$$\nabla(abc)\equiv\left[\frac{(a+b+c+1)!}{(a+b-c)!(a-b+c)!(-a+b+c)!}\right]^{\frac{1}{2}}. \tag{5.3.16}$$

(3) The characteristic polynomials in the set $\{\mathcal{P}_j^m: j=0,\frac{1}{2},1,\ldots; m=j, j-1,\ldots,-j\}$ are a basis of the space of all polynomials in a and $\bar{a}$:

$$P=\sum_j\sum_{m=-j}^{j}\alpha_{jm}\mathcal{P}_j^m,\qquad \alpha_{jm}\in\mathbb{C}. \tag{5.3.17}$$

It is important to remark that the results (1)–(3) are proved without using the existence of an inner product. [Further properties of these polynomials and the symplecton realization of $SU(2)$ are discussed in Notes 2 and 3.]

The expansion property (5.3.17) implies that the complex number defined on the right-hand side of Eq. (5.3.12) exists and is unique.

It is useful to express the complex number $\langle P|P'\rangle$ in an alternative form. If we express P in the form (5.3.17) and similarly express P', then we find

$$\langle P|P'\rangle=\sum_{jm}\alpha_{jm}^{*}\alpha'_{jm}. \tag{5.3.18}$$

[The proof of this result follows from the product law (5.3.14) and the property of the *adjoint characteristic eigenpolynomial* given by

$$\left(\mathcal{P}_j^m\right)^{\text{adj}} = (-1)^{j-m}\mathcal{P}_j^{-m}. \tag{5.3.19}$$

This property of the eigenpolynomials may itself be proved by applying the *adj* operation to Eqs. (5.3.13), using the properties of the generators given by

$$J_+^{\text{adj}} = -J_-, \qquad J_-^{\text{adj}} = -J_+, \qquad J_3^{\text{adj}} = -J_3. \tag{5.3.20}$$

Since the eigenpolynomials that satisfy Eqs. (5.3.13) are unique up to overall normalization, one finds that relation (5.3.19) must be correct up to a multiplicative constant depending only on j. Since $(\mathcal{P}_j^j)^{\text{adj}} = \mathcal{P}_j^{-j}$, the constant is unity.]

Using Eq. (5.3.18) for the complex number $\langle P|P'\rangle$, one now may verify the usual properties of an inner product.

The basis vector $|jm\rangle$ is defined by

$$|jm\rangle = \mathcal{P}_j^m(a,\bar{a})|\;\rangle. \tag{5.3.21}$$

The action of the generators $\mathbf{J}$ on this basis is then given by Eqs. (5.3.4) as

$$\mathbf{J}|jm\rangle = \left[\mathbf{J}, \mathcal{P}_j^m\right]|\;\rangle. \tag{5.3.22}$$

Using Eqs. (5.3.13), one now finds that $\mathbf{J}$ has the standard action on the basis $|jm\rangle$. Moreover, one may prove that $\mathbf{J}$ is a Hermitian operator on the Hilbert space of all polynomial vectors $\{|P\rangle\}$, when equipped with the inner product (5.3.12). Correspondingly, the set of vectors

$$\left\{|jm\rangle : j=0,\tfrac{1}{2},\ldots; m=j, j-1,\ldots,-j\right\} \tag{5.3.23}$$

is an orthonormal basis of this space.

Some examples of the basis vectors $|jm\rangle$ may be useful at this point:[1]

$$\begin{array}{llll}
j=0, & m=0 & |0,0\rangle = |\;\rangle, & \langle 0,0| = \langle\;|; \\
j=\tfrac{1}{2}, & m=\tfrac{1}{2}, & |\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{2}\,a|\;\rangle, & \langle\tfrac{1}{2},\tfrac{1}{2}| = \langle\;|\bar{a}\sqrt{2}, \\
 & m=-\tfrac{1}{2}, & |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{2}\,\bar{a}|\;\rangle & \langle\tfrac{1}{2},-\tfrac{1}{2}| = \langle\;|a(-\sqrt{2}); \\
j=1, & m=1, & |1,1\rangle = \sqrt{2}\,a^2|\;\rangle, & \\
 & m=0, & |1,0\rangle = (a\bar{a}+\bar{a}a)|\;\rangle, & \\
 & m=-1, & |1,-1\rangle = \sqrt{2}\,\bar{a}^2|\;\rangle. &
\end{array} \tag{5.3.24}$$

[1]Note the curious fact that the normalization for both a and a^2 is $\sqrt{2}$.

This last set of eigenvectors, for $j=1$, can be compared with the generators, Eq. (5.3.5); differences in both phase and normalization will be noted, the latter being unimportant. The phase differences are, however, rather more subtle and correspond to the necessary phase changes between operators and vectors that are characteristic of the standard (Condon–Shortley–Wigner) phase convention for the matrix realization of the Wigner operators. It is a necessary (and troublesome) consequence of our use of the standard phase convention that there is now a possible distinction introduced between the two concepts "operators viewed as operators" and "operators viewed as creating state vectors." The $j=1$ case just mentioned is a clear example where this possibility is realized. *We avoid this possible confusion by restricting the product law to apply to operators phased to accord with their role in creating state vectors* [see Eq. (5.3.21)]. That this is a consistent convention can be proved by direct computation.

It should be obvious now that this "symplecton calculus" makes essential use of polynomials that are completely symmetrized in the variables a and $\bar{a}$.

Remarks. (*a*) The product law (5.3.14) as applied to the definition of the inner product verifies the conclusions drawn earlier. In particular (to repeat), the $j=0$ part of the product of two eigenpolynomials vanishes unless $a=b$ and $\alpha=-\beta$. Moreover, there is an associated phase, $(-1)^{a-\alpha}$, verifying the rule for the adjoint operation. (These results are consequences of the properties of the Wigner coefficient.) The explicit definitions of the various numerical factors verify that the actual magnitude (when $a=b, \alpha=-\beta$, and using the adjoint operation) is $+1$.

(*b*) The existence of this product relation demonstrates explicitly that we have constructed a minimal primitive realization of the group $SU(2)$, which is inherently conjugation symmetric.

(*c*) The preceding method of introducing an inner product makes crucial use of the existence of the set of eigenpolynomials $\{\mathcal{P}_j^m: j=0, \frac{1}{2}, 1, \ldots; m=j, j-1, \ldots, -j\}$, the multiplication rule (5.3.14), and the expansion theorem (5.3.17). All this may be avoided by introducing an inner product directly in terms of two arbitrary polynomials P and P'. We have not followed this (basis-independent) procedure at the outset because it appears quite ad hoc. For completeness, however, we state this general definition of the inner product in terms of the coefficients $c_{k\mu}$ appearing in the form P given by Eq. (C.2) in Appendix C:

$$\langle P|P'\rangle = \sum_{kk'\mu} (-1)^{k'-k}(k'+k)!\,c_{k\mu}^{*}c'_{k'\mu}/2^{k'+k}. \qquad (5.3.25)$$

This result may be proved by using the coefficients of Eq. (C.5) in Eq. (5.3.18) and carrying out the summation over j.

(d) An advantage in introducing the inner product (5.3.25) at the outset would be that the splitting of the space of all polynomials in a and $\bar{a}$ into *characteristic multiplets* $\{\mathcal{P}_j^m\colon j=0,\frac{1}{2},1,\ldots;\, m=j,\, j-1,\ldots,-j\}$ is then implied by the standard multiplet construction (see Chapter 3, Section 3 and Note 9, AMQP). One need then only prove that each $\mathcal{P}_j^j$ $(j=0,\frac{1}{2},1,\ldots)$ is unique to obtain the entire multiplet structure of the space—see Appendix A.

(e) The *triangle function*, $\nabla(abc)$, which occurs in the product law, has gratifyingly simple properties: It is a function defined symmetrically on three "lengths" or "sides"—a, b, c—which (from the properties of the factorial function) vanishes unless the *triangle conditions* (that the sum of any two sides equals or exceeds the third side) are fulfilled. The present realization of angular momentum yields the triangle rule of vector addition in a particularly graphic way!

(f) The triangle function has been introduced earlier in angular momentum, but on the grounds of convenience (and, of course, symmetry) in dealing with certain frequently occurring normalizations. Actually, as the proposition shows, the original definition was "upside down" ($1/\Delta$ corresponded to what we here call ∇).

(g) Let us note that the product law has, as an easy consequence, the fact that the triangle function obeys the following transformation law (see Appendix B):

$$\nabla(acf)\nabla(bdf)=(2f+1)\sum_e \nabla(abe)\nabla(cde)W(abcd;ef). \tag{5.3.26}$$

That the Racah function appears in yet another role is interesting, but that it should appear as a sort of "tetrahedral" function "coupling" the triangles by pairs is rather appealing.

(h) Consider the square of the operator J_i—that is, the operator

$$\mathbf{J}^2 \equiv \tfrac{1}{2}(J_+J_- + J_-J_+) + (J_3)^2 \equiv \sum_{i=1}^{3} (J_i)^2. \tag{5.3.27}$$

One readily sees that $\mathbf{J}^2$, far from being $j(j+1)$, is simply the number $-\frac{3}{16}$. But, of course, there is no problem for the present realization: The operator $\mathbf{J}^2$ is properly to be interpreted as the double commutator:

$$\mathbf{J}^2(\nu) \equiv \sum_i J_i(J_i(\nu)) = \sum_i [J_i,[J_i,\mathcal{O}_\nu]]|\,\rangle. \tag{5.3.28}$$

(i) It is clear also that one can extend this structure by adjoining additional symplectons. That is, one considers a symplecton having n "internal" states $a_1, a_2, \ldots, a_n$ and their conjugates $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$. Just as the adjunction of a boson with n "internal" states suffices to realize $SU(n)$, imposing the boson rule $\bar{a}_i|0\rangle=0$, so does an n-state symplecton suffice to realize the structure $Sp(2n)$. [Recall that $SU(2)$ and $Sp(2)$ are isomorphic groups.] It is the particular merit of the product law (5.3.14) that it enables one to reduce all questions in $Sp(2n)$ to mere calculation, just as the Jordan–Schwinger realization similarly reduced $SU(n)$ questions to direct computation. (The symplecton calculus has been studied further by Mukunda [2].)

3. Notes

1. *Nonstandard realization of the generators of* $SU(2)$. Holstein and Primakoff [3] have given a realization of the angular momentum commutation relations in the form

$$\begin{aligned} &J_+\rightarrow(2S-N)^{\frac{1}{2}}\bar{a}, \qquad J_-\rightarrow a(2S-N)^{\frac{1}{2}},\\ &J_3\rightarrow S-N, \end{aligned}$$

where N is the number operator, $N\equiv a\bar{a}$, and $2S$ is a nonnegative integer. (It is easily verified from the boson commutation relation $[\bar{a},a]=\mathbf{1}$ that the angular momentum commutation relations are satisfied.)

Since this realization is achieved with only a single boson operator (and its conjugate), one might—at first glance—consider this realization to be an alternative to the symplecton realization. This would not be correct, however, since the Holstein–Primakoff realization is *confined*, by construction, *to a single irrep* $j=S$.

In actual applications the Holstein–Primakoff realization is the first step in an expansion for large S, and as such, has been the prototypical example for constructing model Hamiltonians, using boson operators. (A recent ~~example is the discussion of the Iachello–Arima nuclear model by Klein~~ and Vallières [4]).

In the large S limit, this realization takes the form:

$$(2S)^{-\frac{1}{2}}J_+\rightarrow\bar{a}, \quad (2S)^{-\frac{1}{2}}J_-\rightarrow a, \qquad S^{-1}J_3\rightarrow\mathbf{1}.$$

One recognizes from these results that the commutation relations for the limiting operators are those of the Heisenberg group: $[\bar{a},a]=\mathbf{1}$, $[\bar{a},\mathbf{1}]=[a,\mathbf{1}]=0$. In other words, the Holstein–Primakoff realization is tailored to effect the group contraction: $SU(2)\rightarrow$ Heisenberg group. [Expressed in this way, one sees that this construction can be generalized: Okubo [5], for example, has given a direct generalization for totally symmetric irreps, $[m_{1n}0\cdots0]$ in $SU(n)$, taking m_{1n} large. More generally, the pattern calculus allows one to effect the sequence of limits $m_{nn}\rightarrow-\infty$, then $m_{n-1,n}\rightarrow-\infty,\ldots$. The extended patterns of Chapter 4 arise in just this way from $SU(3)$; a further limit is discussed on p. 117.]

2. *Further properties of the eigenpolynomials.* The eigenpolynomials $\mathcal{P}_j^m$ satisfy several additional properties that are useful. These relations include (see Ref. [1]) the following:

Generating function. The polynomials $\mathcal{P}_j^m$ may be generated by expanding the form

$$(\zeta a+\eta\bar{a})^{2j}=[(2j)!/2^j]\sum_m \Phi_{jm}(\zeta,\eta)\mathcal{P}_j^m(a,\bar{a}), \tag{5.3.29}$$

where Φ_{jm} are the familiar angular momentum eigenvectors defined on the components (ζ,η) of an arbitrary spinor:

$$\Phi_{jm}(\zeta,\eta)=\zeta^{j+m}\eta^{j-m}/[(j+m)!(j-m)!]^{\frac{1}{2}}. \tag{5.3.30}$$

Transformation property. For each $U\in SU(2)$ given by

$$U=\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \tag{5.3.31}$$

we have the following transformation property of the eigenpolynomials $\mathcal{P}_j^m$:

$$\mathcal{P}_j^m(a',\bar{a}')=\sum_{m'} D^j_{m'm}(U)\mathcal{P}_j^{m'}(a,\bar{a}), \tag{5.3.32}$$

where

$$a'=u_{11}a+u_{21}\bar{a}, \qquad \bar{a}'=u_{12}a+u_{22}\bar{a}. \tag{5.3.33}$$

Observe that this transformation leaves the commutation relations invariant:

$$[\bar{a}',a']=[\bar{a},a]=1. \tag{5.3.34}$$

3. *The symplecton algebra as a semisimple graded Lie algebra.* The symplecton realization is a prototype (Ref. [6]) for a large class of semisimple graded Lie algebras (GLA) (see Corwin *et al.* [7]).

Let us recall the formal definition of a GLA. One has a graded vector space, $L\equiv \bigoplus_k L_k$; that is, L is a vector space, whose elements are finite direct sums of components lying in the vector spaces L_k, and the index k (taken to be an integer here) belongs to a finite abelian group. Then L is a graded Lie algebra if we have a bilinear map (Corwin *et al.* [7], Scheunert

[8]) $[L, L] \to L$ such that

$$[L_k, L_l] \subset L_{k+l}, \tag{5.3.35}$$

$$[x, y] = (-1)^{kl+1}[y, x], \tag{5.3.36}$$

$$[x,[y,z]] = [[x,y],z] + (-1)^{kl}[y,[x,z]], \tag{5.3.37}$$

where $x \in L_k$, $y \in L_l$. It is clear from Eq. (5.3.36) that both commutators and anticommutators will enter the general case.

It is not difficult to check now the defining properties (5.3.35)–(5.3.37), above, and verify that the symplecton structure is indeed a GLA. Denoting a general element by $x \equiv \Sigma x_{a\alpha}\mathcal{P}_a^\alpha$, $x_{a\alpha} \in \mathbb{C}$, we see that the $\{x\}$ have the algebraic structure of a ring equipped with a norm and an involution, the *adj* operation *adj*: $x \to x^{\text{adj}} = \sum_{a\alpha} x^*_{a\alpha}(-1)^{a-\alpha}\mathcal{P}_a^{-\alpha}$. (Note, however, that we do *not* have a Banach algebra.) To carry out the verification, one first splits the eigenpolynomials into the two classes, integer and half-integer; that is, we grade the space $L \equiv \{x\}$ by the two-element group Z_2:

$$L \equiv L_0 \oplus L_1, \qquad L_0 = \{x = \sum_{a\alpha} x_{a\alpha}\mathcal{P}_a^\alpha : a \in \text{integers}\},$$

$$L_1 = \{x = \sum_{a\alpha} x_{a\alpha}\mathcal{P}_a^\alpha : a \in \text{half-integers}\}. \tag{5.3.38}$$

The product $[x, y]$ is then defined to be

$$[x, y] \equiv xy - (-1)^{kl}yx, \qquad x \in L_k, \qquad y \in L_l, \tag{5.3.39}$$

for $k, l = 0$ or 1, where xy denotes the symplecton product obtained from the ~~product given by Eq. (5.3.14) for the basis elements. It is clear that all~~ requirements for a GLA are verified.

It is probably not too surprising that angular momentum theory is rich enough in structure to define a GLA, since after all the fermion–boson split is closely related to angular momentum. In fact, the use of a single boson in this construction is nothing else than the use of the Heisenberg group to define a (two-dimensional) symplectic space $Sp(2)$. If we consider n bosons, then the group-theoretic properties of the Wigner coefficients guarantee that the symplecton structure generalizes to $Sp(2n)$. This is the conclusion of Pais and Rittenberg [9], who show that a general class of semisimple GLA are found in this way.

Physical applications of GLA—to define "supersymmetries" and "supergauges"—were given by Wess and Zumino [10] and by Volkov and Soroka [11].

4. Appendices

A. Definition Of The Characteristic Eigenpolynomials $\mathcal{P}_j^m$

In this Appendix we determine the characteristic eigenpolynomials $\mathcal{P}_j^m$ up to normalization directly *without using an inner product*. [This is essential to the procedure used in Section 2, since the definition of inner product given by Eq. (5.3.12) makes crucial use of the properties of these polynomials.]

One finds that the unique polynomial solution to the two equations

$$[J_+, \mathcal{P}]=0, \qquad [J_3, \mathcal{P}]=j\mathcal{P}, \qquad j \text{ integral or half-integral} \tag{A.1}$$

is

$$\mathcal{P}_j^j(a,\bar{a})=\alpha_j a^{2j}, \qquad \alpha_j \in \mathbf{C} \tag{A.2}$$

for $j=0,\frac{1}{2},1,\ldots$. We now define $\mathcal{P}_j^m$ by

$$\mathcal{P}_j^m(a,\bar{a})=\left[\frac{(j+m)!}{(2j)!(j-m)!}\right]^{\frac{1}{2}}\left[J_-, \mathcal{P}_j^j(a,\bar{a})\right]_{(j-m)}, \tag{A.3}$$

where $m=j,\ j-1,\ldots$.

Carrying out the $(j-m)$ commutations with $J_-=\bar{a}^2/2$ yields the explicit result

$$\mathcal{P}_j^m(a,\bar{a})=\frac{\alpha_j}{2^{j-m}}\binom{2j}{j+m}^{\frac{1}{2}}\sum_s \frac{\bar{a}^{j-m-s}a^{j+m}\bar{a}^s}{(j-m-s)!s!}. \tag{A.4}$$

~~Since $\mathcal{P}_j^{-j}(a,\bar{a})=\alpha_j\bar{a}^{2j}$ and $[J_-, \mathcal{P}_j^{-j}]=0$, it follows directly from the~~ definition (A.3) and the commutator properties (5.3.6) of the generators that the $\mathcal{P}_j^m$ satisfy Eqs. (5.3.13). Conversely, $\mathcal{P}_j^m$ given by Eq. (A.4) is the unique solution to Eqs. (5.3.13).

The complex number α_j is arbitrary in Eq. (A.4). These polynomials with arbitrary α_j will satisfy a multiplication rule (5.3.14) in which $\langle c|a|b\rangle$ is replaced by

$$\frac{\alpha_a\alpha_b}{\alpha_c}\langle c|a|b\rangle\left[\frac{(2a)!(2b)!2^{2c}}{2^{2a}2^{2b}(2c)!}\right]^{\frac{1}{2}}.$$

In order to assign the number α_j, we anticipate here the normalization rule given by Eq. (5.3.12) and the multiplication rule (5.3.14): The $j=0$ part of

$(\mathcal{P}_j^m)^{\text{adj}}\mathcal{P}_j^m = (-1)^{j-m}\mathcal{P}_j^{-m}\mathcal{P}_j^m$ is given by

$$\frac{(\alpha_j)^2}{\alpha_0}\langle 0|j|j\rangle \frac{(2j)!}{2^{2j}}(-1)^{j-m}C^{jj0}_{m,-m,0} = \frac{(\alpha_j)^2}{\alpha_0}\,\frac{(2j)!}{2^{2j}}\,.$$

We choose

$$\alpha_j = \left[2^{2j}/(2j)!\right]^{\frac{1}{2}} \tag{A.5}$$

so that the polynomial $\mathcal{P}_j^m$ will be normalized to unity in the sense of Eq. (5.3.12).

B. Multiplication Law For The Eigenpolynomials $\mathcal{P}_j^m$

The most economical proof of the product law uses induction. It is crucial to the proof to establish the important special case

$$\mathcal{P}_{\frac{1}{2}}^{\alpha}\mathcal{P}_b^{\beta} = (2b+1)^{\frac{1}{2}}\sum_c C^{b\frac{1}{2}c}_{\beta,\alpha,\beta+\alpha}\mathcal{P}_c^{\alpha+\beta}. \tag{B.1}$$

To establish this result, we first invert it:

$$(2b+1)^{\frac{1}{2}}\mathcal{P}_c^{\gamma} = \sum_\beta C^{b\frac{1}{2}c}_{\beta,\gamma-\beta,\gamma}\mathcal{P}_{\frac{1}{2}}^{\gamma-\beta}\mathcal{P}_b^{\beta}. \tag{B.2}$$

We next form the commutator of Eq. (B.2) with J_-, using the general property

$$\left[J_-, \mathcal{P}_d^{\delta}\right] = \left[(d+\delta)(d-\delta+1)\right]^{\frac{1}{2}}\mathcal{P}_d^{\delta-1}. \tag{B.3}$$

(This property follows from the definition of the $\mathcal{P}_j^m$ given in Appendix A.) The resulting equation is then reduced to the form of Eq. (B.2), with γ now replaced by $\gamma-1$, by using the explicit spin-$\frac{1}{2}$ Wigner coefficients. Thus, if Eq. (B.2) is correct for γ, it is correct for $\gamma-1$. Its general validity then follows by induction (on γ) upon demonstrating it to be correct for $\gamma=c$ (there are two cases, $b=c-\frac{1}{2}$ and $b=c+\frac{1}{2}$):

$$\sqrt{2c}\,\mathcal{P}_c^c = \mathcal{P}_{\frac{1}{2}}^{\frac{1}{2}}\,\mathcal{P}_{c-\frac{1}{2}}^{c-\frac{1}{2}}, \tag{B.4}$$

$$(2c+2)\mathcal{P}_c^c = -\mathcal{P}_{\frac{1}{2}}^{\frac{1}{2}}\,\mathcal{P}_{c+\frac{1}{2}}^{c-\frac{1}{2}} + (2c+1)^{\frac{1}{2}}\mathcal{P}_{\frac{1}{2}}^{-\frac{1}{2}}\mathcal{P}_{c+\frac{1}{2}}^{c+\frac{1}{2}}. \tag{B.5}$$

Equation (B.4) is trivially verified, while Eq. (B.5) is seen to be the relation

$$(2c+1)(2c+2)a^{2c} = -a[\bar{a}^2, a^{2c+1}] + 2(2c+1)\bar{a}a^{2c+1},$$

which is also observed to be correct. Thus, Eqs. (B.4) and (B.5) are correct, and the general relation, Eq. (B.2), and hence Eq. (B.1), is proved.

Next, let us prove the general results, Eqs. (5.3.14)–(5.3.16). Assume that these results have been proved for general b and all indices $0, \frac{1}{2}, 1, \ldots, a$. We must prove that the results also hold for $a+\frac{1}{2}$. We start with Eq. (B.2) in the form

$$(2a+1)^{\frac{1}{2}} \mathcal{P}_{a+\frac{1}{2}}^{\varepsilon} = \sum_{\alpha} C_{\alpha, \varepsilon-\alpha, \varepsilon}^{a \frac{1}{2} a+\frac{1}{2}} \mathcal{P}_{\frac{1}{2}}^{\varepsilon-\alpha} \mathcal{P}_{a}^{\alpha}. \tag{B.6}$$

This result is now multiplied from the right by $\mathcal{P}_b^{\beta}$, and the product $\mathcal{P}_{\frac{1}{2}}^{\varepsilon-\alpha}(\mathcal{P}_a^{\alpha}\mathcal{P}_b^{\beta})$ appearing in the right-hand side is expanded as follows: First, we expand $\mathcal{P}_a^{\alpha}\mathcal{P}_b^{\beta}$, using the assumed validity of Eq. (5.3.14) for general b and all indices $0, \frac{1}{2}, \ldots, a$; second, we use the general validity of Eq. (B.1) to expand the product of $\mathcal{P}_{\frac{1}{2}}^{\varepsilon-\alpha}$ and the $\mathcal{P}_c^{\alpha+\beta}$ occurring in the product $\mathcal{P}_a^{\alpha}\mathcal{P}_b^{\beta}$. The result is

$$\begin{aligned}(2a+1)^{\frac{1}{2}} \mathcal{P}_{a+\frac{1}{2}}^{\varepsilon} \mathcal{P}_b^{\beta} &= \sum_d \mathcal{P}_d^{\beta+\varepsilon} \sum_c \langle c|a|b\rangle (2c+1)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{\alpha} C_{\beta, \alpha, \beta+\alpha}^{b\, a\, c} C_{\alpha+\beta, \varepsilon-\alpha, \beta+\varepsilon}^{c \frac{1}{2} d} C_{\alpha, \varepsilon-\alpha, \varepsilon}^{a \frac{1}{2} a+\frac{1}{2}} \right). \end{aligned} \tag{B.7}$$

The sum over α is just

$$[(2c+1)(2a+2)]^{\frac{1}{2}} W(b, a, d, \tfrac{1}{2}; c, a+\tfrac{1}{2}) C_{\beta, \varepsilon, \beta+\varepsilon}^{b\, a+\frac{1}{2} d}.$$

Furthermore, the identity

$$\langle d|a+\tfrac{1}{2}|b\rangle = \sum_c \langle c|a|b\rangle (2c+1) \left(\frac{2a+2}{2a+1} \right)^{\frac{1}{2}} W(b, a, d, \tfrac{1}{2}; c, a+\tfrac{1}{2}) \tag{B.8}$$

follows directly from the explicit algebraic forms of the W-coefficients. Equation (B.7) thus reduces to

$$\mathcal{P}_{a+\frac{1}{2}}^{\varepsilon} \mathcal{P}_b^{\beta} = \sum_d \langle d|a+\tfrac{1}{2}|b\rangle C_{\beta, \varepsilon, \beta+\varepsilon}^{b\, a+\frac{1}{2} d} \mathcal{P}_d^{\beta+\varepsilon}, \tag{B.9}$$

which is just the result, Eq. (5.3.14), for a replaced by $a+\frac{1}{2}$. The induction loop has closed, and the general validity of the product law is proved.

The product law can be inverted:

$$\langle c|a|b\rangle \mathcal{P}_c^{\gamma} = \sum_{\beta} C_{\beta, \gamma-\beta, \gamma}^{b\, a\, c} \mathcal{P}_a^{\gamma-\beta} \mathcal{P}_b^{\beta}. \tag{B.10}$$

In this form, the result appears as a standard coupling of two irreducible tensor operators to form a third. It might seem, therefore, that the result is not particular to the symplecton calculus, since the derivation of the product law, at first glance, seems to entail only general properties of tensor operators. The crucial point, however, where the properties of the symplecton entered the derivation, was in establishing Eq. (B.5). Indeed, for $c=0$, Eq. (B.5) reduces to $[\bar{a}, a]=1$. Conversely, consider any set of polynomials for which the product law is valid with $P_0^0=1$. Then the product law implies $[\bar{a}, a]=1$, where $a\equiv P_{\frac{1}{2}}^{\frac{1}{2}}/\sqrt{2}$ and $\bar{a}\equiv P_{\frac{1}{2}}^{-\frac{1}{2}}/\sqrt{2}$. *The product law is a unique property of the symplecton.*

The general transformation law for the triangle function, Eq. (5.3.26), is a consequence of the fact that the product of symplecton eigenpolynomials is associative:

$$\left(\mathcal{P}_a^{\alpha}\mathcal{P}_b^{\beta}\right)\mathcal{P}_c^{\gamma}=\mathcal{P}_a^{\alpha}\left(\mathcal{P}_b^{\beta}\mathcal{P}_c^{\gamma}\right). \tag{B.11}$$

One uses the product law in this result, and the standard relations of products of Wigner coefficients to Racah coefficients, to establish the general result, Eq. (5.3.26). [Note that Eq. (B.8) is a special case.]

C. Basis Property Of The Eigenpolynomials $\mathcal{P}_j^m$

The set of polynomials

$$\left\{\mathcal{P}_j^m: m=j, j-1,\ldots, -j;\ j=0,\tfrac{1}{2},1,\ldots\right\} \tag{C.1}$$

is a basis for all polynomials in a and $\bar{a}$.

Proof. We first observe that each polynomial $P(a,\bar{a})$ of degree 2λ may be written in the form[1]

$$P(a,\bar{a})=\sum_{k=0}^{\lambda}\sum_{m=-k}^{k} c_{km}a^{k+m}\bar{a}^{k-m} \tag{C.2}$$

by using the property $\bar{a}a=a\bar{a}+1$. Thus, we find

$$P=\sum_{km}2^{-k}c_{km}\left[(k+m)!(k-m)!\right]^{\frac{1}{2}}\mathcal{P}_{\frac{k+m}{2}}^{\frac{k+m}{2}}\mathcal{P}_{\frac{k-m}{2}}^{-\frac{k-m}{2}}. \tag{C.3}$$

Using the product law given by Eq. (5.3.14) and the explicit value of the Wigner coefficient for the special product in Eq. (C.3), we obtain the desired

[1]The summation is over $k=0,\frac{1}{2},1,\ldots,\lambda$.

result:

$$P=\sum_{jm}\alpha_{jm}\mathscr{P}_j^m, \tag{C.4}$$

where

$$\begin{aligned}\alpha_{jm}&=\left[(j+m)!(j-m)!\right]^{-\frac{1}{2}}\\&\quad\times\sum_{k=j}^{\lambda}\left[(-1)^{k-j}(k+m)!(k-m)!/2^k(k-j)!\right]c_{km}.\ \blacksquare\end{aligned} \tag{C.5}$$

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Topic 4. Algebraic Aspects Of Physical Tensor Operators

In Chapter 2 we defined a Wigner operator in terms of its action on a specific Hilbert space $\mathcal{H}$. Indeed, in this presentation of a Wigner operator, each operator is itself realized as a sum of monomials defined on the boson operators a_1, a_2, and their conjugates. [These explicit polynomial forms may be obtained by combining the mappings (2.25) with Eqs. (3.16) and (3.17) of Chapter 3.]

Although this procedure is quite satisfactory for the study of the RW-algebra itself, it does not address directly (except through the Wigner–Eckart theorem) the tensor operator algebra that occurs in many physical problems.

Thus, in physical problems possessing rotational symmetry, we often encounter physical tensor operators that satisfy the defining relation, Eq. (2.5), where **J** is the total angular momentum of the physical system. It is often useful in the analysis of such problems to construct directly operators that shift the state labels $\{(\alpha), j, m\}$ to a new set $\{(\alpha'), j', m'\}$. The purpose of this Topic is to give a uniform treatment[1] of such "shift operators" within the framework of the tensor operator concept and to relate these shift operators to Wigner operators defined on physical state vector space.

We shall use an algebraic approach to the problem that generalizes the method given in detail in Chapter 6, Sections 11 and 12, AMQP, for vector operators.

We assume throughout this discussion that (*a*) **T** is a given irreducible tensor operator with angular momentum label k (components T_μ) with respect to an angular momentum **J** of a physical system possessing states $\{|(\alpha)jm\rangle\}$ of sharp angular momentum; and (*b*) the conjugate tensor operator $\mathbf{T}^\dagger$ exists and satisfies the (defining) relation

$$\langle(\alpha')j'm'|T|(\alpha)jm\rangle = \langle(\alpha)jm|T^\dagger|(\alpha')j'm'\rangle^*. \qquad (5.4.1)$$

We now give in summary form (without attempting to be rigorous) the generalization to an arbitrary irreducible tensor of the algebraic properties of vector operators developed in Chapter 6, AMQP.

Vector space. We define the vector space $\mathfrak{T}_k$ to be the space spanned by the set of all tensor operators generated by repeated commutation of $\mathbf{J}^2$ with **T**—that is, $[\mathbf{J}^2,\mathbf{T}]$, $[\mathbf{J}^2,[\mathbf{J}^2,\mathbf{T}]],\ldots$. (Note that the tensor operators thus generated all have angular momentum label k.) The scalars of this vector space include not only the complex numbers but also invariants constructed from **J** and **T**. The scalar product of $\mathbf{S}'\in\mathfrak{T}_k$ and $\mathbf{S}\in\mathfrak{T}_k$ is the invariant operator defined by[2]

$$(\mathbf{S}',\mathbf{S}) \equiv \sum_\mu S'_\mu S^\dagger_\mu = \mathbf{S}'\cdot\mathbf{S}^\dagger. \qquad (5.4.2)$$

Operator actions on $\mathfrak{T}_k$. We introduce the operator actions of commutation discussed at length in Chapter 3, Note 9, AMQP. For the reasons already discussed in connection with vector operators, it is convenient, however, to redefine the action as taking place to the left. Thus, the linear

[1]This treatment is adapted from notes (unpublished) of one of the authors (JDL) written in the early 1960's. Hughes [1], using a somewhat different approach, has addressed the same problem.

[2]The definition of scalar product has been given in this way to obtain agreement later with the convention for Wigner operators [see Eq. (2.33)].

operators C_i and Ω are defined by

$$\mathbf{S}C_i \equiv [\mathbf{S}, J_i], \qquad i=1,2,3,$$
$$\mathbf{S}\Omega = [\mathbf{S}, \mathbf{J}^2] \tag{5.4.3}$$

for each $\mathbf{S} \in \mathfrak{T}_k$. Observe that the *definition of an irreducible tensor* [see Eq. (2.5)] assures $\mathbf{S}C_i \in \mathfrak{T}_k$. The *irreducibility* of each $\mathbf{S} \in \mathfrak{T}_k$ implies and is implied by the relations

$$\begin{aligned} \mathbf{S}\mathbf{C}^2 &= k(k+1)\mathbf{S}, \\ S_\mu C_3 &= -\mu S_\mu, \\ S_\mu C_\pm &= -\left[(k \mp \mu)(k \pm \mu + 1)\right]^{\frac{1}{2}} S_{\mu \pm 1}, \end{aligned} \tag{5.4.4}$$

where $\mathbf{C}^2 \equiv C_1^2 + C_2^2 + C_3^2$, and $C_\pm = C_1 \pm iC_2$. [The C_i as well as Ω are Hermitian operators with respect to the scalar product (5.4.2).]

Using the definition of Ω, we also find the following result for the action of Ω on an element $\mathbf{S} \in \mathfrak{T}_k$:

$$\begin{aligned} \mathbf{S}\Omega &= [\mathbf{S}, \mathbf{J}^2] = \sum_i \left([\mathbf{S}, J_i] J_i + J_i [\mathbf{S}, J_i]\right) \\ &= 2\sum_i [\mathbf{S}, J_i] J_i - \sum_i [[S_i, J_i], J_i] \\ &= \mathbf{S}\left(2\sum_i C_i J_i - \mathbf{C}^2\right); \end{aligned}$$

that is,

$$\Omega = 2\mathbf{C}\cdot\mathbf{J} - \mathbf{C}^2, \tag{5.4.5}$$

where

$$\mathbf{C}\cdot\mathbf{J} \equiv \sum_i C_i J_i. \tag{5.4.6}$$

The action of $C_i J_i$ on $\mathbf{S}$ is given by $\mathbf{S}(C_i J_i) = (\mathbf{S}C_i) J_i$, where we follow the convention of writing angular momentum operators J_i to the right.

Eigenvalue problem. The Hermitian operators C^2, C_3, and Ω mutually commute on the space $\mathfrak{T}_k$ (Chapter 3, Note 9, AMQP). Accordingly, they may be simultaneously diagonalized. We seek the elements of $\mathfrak{T}_k$ such that

$$\mathbf{S}\Omega = \mathbf{S}\omega, \tag{5.4.7}$$

where ω is an invariant operator.

Equation (5.4.5) is the key relation for solving the eigenvalue problem: We introduce the operators L_i defined by

$$L_i = J_i - C_i \tag{5.4.8}$$

with an action on $\mathbf{S}$ given by $\mathbf{S}L_i = \mathbf{S}J_i - \mathbf{S}C_i = J_i\mathbf{S}$. Since the J_i and the C_j commute and obey angular momentum commutation rules, the L_i likewise obey angular momentum commutation rules. We thus find

$$\Omega = \mathbf{J}^2 - \mathbf{L}^2. \tag{5.4.9}$$

This relation may now be used to solve the eigenvalue problem, since it follows that

$$(\mathbf{S}\Omega)|(\alpha)jm\rangle = \mathbf{S}[j(j+1) - (j+\delta)(j+\delta+1)]|(\alpha)jm\rangle, \tag{5.4.10}$$

where (from the addition of angular momenta) for given j and k we must have

$$\delta \in \{k, k-1, \ldots, -k\}. \tag{5.4.11}$$

Since k is to be regarded as fixed and j is generic (and may therefore be larger than k), all values of δ in the set (5.4.11) must be admitted.

We conclude: *The only possible solutions to the eigenvalue problem* (5.4.7) *are those given by*

$$\mathbf{S}^{(\delta)}\Omega = \mathbf{S}^{(\delta)}\omega_\delta, \qquad \delta = k, k-1, \ldots, -k, \tag{5.4.12}$$

---

where ω_δ is the invariant operator defined by

$$\omega_\delta = -\delta(\delta + dim) \tag{5.4.13}$$

in which dim is the dimension operator:

$$dim|(\alpha)jm\rangle = (4\mathbf{J}^2+1)^{\frac{1}{2}}|(\alpha)jm\rangle = (2j+1)|(\alpha)jm\rangle. \tag{5.4.14}$$

The eigenvectors $\mathbf{S}^{(\delta)}$ in Eq. (5.4.12) satisfy

$$(\mathbf{S}^{(\lambda)} \cdot \mathbf{S}^{(\delta)\dagger})(dim + \lambda - \delta) = 0 \qquad \text{for } \lambda \neq \delta. \tag{5.4.15}$$

Proof. We have

$$\begin{aligned}(\mathbf{S}^{(\lambda)}\cdot\mathbf{S}^{(\delta)\dagger})\Omega=0&=(\mathbf{S}^{(\lambda)}\Omega)\cdot\mathbf{S}^{(\delta)\dagger}+\mathbf{S}^{\lambda}\cdot(\mathbf{S}^{(\delta)\dagger}\Omega)\\&=\left(\mathbf{S}^{(\lambda)}\omega_{\lambda}\right)\cdot\mathbf{S}^{(\delta)\dagger}-\mathbf{S}^{(\lambda)}\cdot\left(\omega_{\delta}\mathbf{S}^{(\delta)\dagger}\right)\\&=(\mathbf{S}^{(\lambda)}\cdot\mathbf{S}^{(\delta)\dagger})(\delta-\lambda)(\lambda-\delta+dim).\end{aligned}$$ ■

Cayley–Hamilton theorem. The operator Ω obeys a Cayley–Hamilton type of identity on $\mathfrak{T}_k$:

$$\prod_{\delta=-k}^{k}(\Omega-\omega_{\delta})=0. \tag{5.4.16}$$

We prove this result by showing that $\mathfrak{T}_k$ is, in fact, the space spanned by the set of tensors

$$\mathbf{T},\mathbf{T}\Omega,\ldots,\mathbf{T}\Omega^{2k}; \tag{5.4.17}$$

hence,

$$\dim\mathfrak{T}_k\leqslant 2k+1. \tag{5.4.18}$$

Proof. We must show that $\mathbf{T}\Omega^{2k+1}$ is dependent on the $2k+1$ tensor operators (5.4.17). An induction on n may be used to prove the identity

$$\sum_{s=0}^{n}(T_k\Omega^s)P_s^{(n)}(\mathbf{J})=(T_kC_-^n)J_+^n$$

for each $n=0,1,\ldots,$ where the $P_s^{(n)}$ are polynomials in $\mathbf{J}^2$ and J_3. In particular, the $P_s^{(2k+1)}$ are independent of J_3, hence, are invariants. (The induction is carried out by operating on this equation from the right with Ω.) One now sets $n=2k+1$, noting that $T_kC_-^{2k+1}=0$ to obtain the desired result. ■

Construction of eigenvectors. We introduce the projection operators P_λ ($\lambda=k,k-1,\ldots,-k$) defined by

$$P_\lambda=\prod_{\substack{\delta=-k\\ \delta\neq\lambda}}^{k}\frac{\Omega-\omega_\delta}{\omega_\lambda-\omega_\delta}. \tag{5.4.19}$$

These (Hermitian) projection operators satisfy the usual relations:

$$
\begin{aligned}
P_\lambda P_{\lambda'} &= \delta_{\lambda\lambda'} P_\lambda, \\
1 &= \sum_\lambda P_\lambda, \\
\Omega &= \sum_\lambda P_\lambda \omega_\lambda.
\end{aligned} \tag{5.4.20}
$$

It follows from the Cayley–Hamilton theorem [Eq. (5.4.16)] that the tensors defined by

$$
\mathbf{S}^{(\delta)} \equiv \mathbf{T} P_\delta, \qquad \delta = k, k-1, \ldots, -k \tag{5.4.21}
$$

are solutions to the eigenvalue problem, Eq. (5.4.12). Moreover, it follows from the properties (5.4.20) of the projection operators P_λ that each $\mathbf{S} \in \mathfrak{T}_k$ is a linear combination of the $\mathbf{S}^{(\delta)}$; that is, the $2k+1$ tensor operators (5.4.21) span $\mathfrak{T}_k$.

Relation to the Wigner–Eckart theorem. Using the resolution of the identity operator given in Eqs. (5.4.20), we have that

$$
\mathbf{T} = \sum_\delta \mathbf{S}^{(\delta)}. \tag{5.4.22}
$$

The nonvanishing matrix elements of $\mathbf{T}$ are thus found to be

$$
\begin{aligned}
\langle (\alpha') j+\delta, m+\mu | T_\mu | (\alpha) jm \rangle &= \langle (\alpha') j+\delta, m+\mu | S_\mu^{(\delta)} | (\alpha) jm \rangle \\
&= \langle (\alpha') j+\delta \| \mathbf{T} \| (\alpha) j \rangle \, C^{j\,k\,j+\delta}_{m,\mu,m+\mu}.
\end{aligned} \tag{5.4.23}
$$

This result shows that up to a multiplicative invariant operator the physical ~~shift operator $\mathbf{S}^{(\delta)}$ (constructed from $\mathbf{T}$ by multiple commutations with $\mathbf{J}^2$)~~ realizes the action of a Wigner operator on physical state space.

The Wigner operators that are realized will depend on the particular physical tensor operator $\mathbf{T}$ that is chosen (recall that we have proved only that $\dim \mathfrak{T}_k \leqslant 2k+1$). For example, choosing $\mathbf{T}$ to be the tensor operator of rank k constructed from the components of $\mathbf{J}$ [see Eqs. (3.441) and (3.442) of AMQP] yields a space $\mathfrak{T}_k$ of dimension 1.

Null space aspects of physical tensor operators. When carrying out the operator actions in the construction of the solutions to the eigenvalue problem, Eq. (5.4.12), the invariant operators ω_δ occur as scalars and possess no further intrinsic properties insofar as one is concerned only with properties of the space $\mathfrak{T}_k$. These scalars, however, do possess structure with respect to the underlying Hilbert space $\{|(\alpha) jm\rangle\}$. The properties of the (invariant) denominators

$$\prod_{\substack{\lambda=-k\\ \lambda\neq\delta}}^{k} (\omega_\delta-\omega_\lambda)=\prod_{\substack{\lambda=-k\\ \lambda\neq\delta}}^{k} (\lambda-\delta)(dim+\delta+\lambda) \qquad (5.4.24)$$

now become important. One sees, in fact, that the action of $\mathbf{T}$, as given by Eq. (5.4.22), is not defined on all states $\{|(\alpha)jm\rangle\}$ because of zeros that develop in the invariant denominators in Eq. (5.4.19), when evaluated for certain values of j. This result was to be expected, however, in view of the existence of the null space of a Wigner operator. Quite remarkably, the denominator D of the Wigner operator

$$\left\langle \begin{array}{ccc} & k+\delta & \\ 2k & & 0 \\ & \bullet & \end{array} \right\rangle \qquad (5.4.25)$$

is exactly [see Eqs. (3.31) and (3.43), Chapter 3]

$$D=\left[\prod_{\substack{\lambda=-k\\ \lambda\neq\delta}}^{k} (dim+\delta+\lambda)\right]^{\frac{1}{2}}. \qquad (5.4.26)$$

This result shows that we should define

$$\mathbf{S}^{(\delta)}|(\alpha)jm\rangle\equiv 0 \qquad (5.4.27)$$

for all j such that $D(j)=0$. With this definition the action of $\mathbf{T}$ is defined on all states $j=0,\frac{1}{2},1,\ldots$.

~~*Remarks*. (*a*) The two shift operators corresponding to a physical *spinor*~~ *operator* $\mathbf{T}$ are

$$\mathbf{S}^{(+\frac{1}{2})}=\left[\mathbf{T}(dim+\tfrac{1}{2})+4\mathbf{Q}\right](2\,dim)^{-1},$$

$$\mathbf{S}^{(-\frac{1}{2})}=\left[\mathbf{T}(dim-\tfrac{1}{2})-4\mathbf{Q}\right](2\,dim)^{-1}, \qquad (5.4.28)$$

where $\mathbf{Q}$ is the spinor operator with components $(Q_{+\frac{1}{2}},Q_{-\frac{1}{2}})$ given by

$$Q_{+\frac{1}{2}}=T_{+\frac{1}{2}}J_3+T_{-\frac{1}{2}}J_+,$$

$$Q_{-\frac{1}{2}}=T_{+\frac{1}{2}}J_--T_{-\frac{1}{2}}J_3. \qquad (5.4.29)$$

(*b*) The construction given in this Topic of operators that shift the angular momentum quantum numbers (j, m) is useful for analyzing the irreps (in an angular momentum basis) of groups that contain $SU(2)$ as a subgroup (see, for example, Ref. [2]). It is also the algebraic structure that underlies the theory of vector spherical harmonics (developed in detail in Chapter 6, Section 12, AMQP; see also Rose [3] and Edmonds [4]).

(*c*) The fact that the denominator function D^2 of the Wigner operator (5.4.25) is [up to an (arbitrary) numerical factor] just the invariant function

$$\prod_{\substack{\lambda \\ \lambda \neq \delta}} (\omega_\lambda - \omega_\delta) \tag{5.4.30}$$

defined on the eigenvalues of the commutation operator Ω is another important structural feature of a Wigner coefficient, thus relating these denominators to classical invariant theory.

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TOPIC 5. COMPLEX ANGULAR MOMENTA, REGGE TRAJECTORIES, AND REGGE POLES

1. Physical Motivation

The problem of treating the scattering of waves by obstacles in the path of the wave train is common to the many branches of physics where wave phenomena occur—for example, in acoustics, in electromagnetic theory, and most prominently in quantum mechanics. If the scattering system has spherical symmetry, the scattering problem may always be treated by series expansions over the rotation matrices (see Chapter 7, Section 6, AMQP); for scalar waves one obtains, as a special case, a series over the Legendre functions. In such a series development, the discrete summation variable has the physical significance of an angular momentum magnitude.

For the scattering of waves in a domain where the wavelength is very small compared with a typical size of the scatterer, the series approach (although valid) becomes more or less infeasible, owing to the large number of significant angular momenta involved. In such a case, the heuristic technique of letting the angular momentum variable in the series assume *complex values*, and approximating the sum by an integral, has proved to be not only of practical importance but, as we shall indicate below, of theoretical significance as well. This aspect of complex angular momenta, for the asymptotic treatment of short wavelength phenomena, was developed primarily by Sommerfeld [1] and by Fock [2] for electromagnetic problems.[1] These methods have become useful in heavy ion scattering in nuclear physics (McVoy [4]).

We shall discuss the essential ideas of this approach in Section 2 by summarizing the work of Sommerfeld [1], Regge [5], and De Alfaro and Regge [6].

The strongest impetus toward the development of complex angular momentum techniques has come, however, from a very different field—namely, the field of particle (high-energy) physics, with very different motivations and concepts. Not until we have discussed relativistic scattering theory from the viewpoint of Mandelstam (see, for example, Omnès and Froissart [7]) and of Chew [8] (the subject of Section 4) will the nature of complex angular momenta become clearer.[2] Here it will develop that complex angular momenta do indeed have a valid group-theoretic (symmetry) origin, and what we began as a heuristic device has in fact a deeper significance.

2. The Regge Treatment of Nonrelativistic Potential Scattering

Let us consider the quantum mechanical problem of the scattering of spinless particles by a spherically symmetric potential. This is a classic problem in quantum mechanics; of the many treatments, that of De Alfaro and Regge [6] is especially rigorous. Our purpose here, though, is essentially motivational and descriptive.

The Schrödinger equation for this system has the form

$$\left[-\frac{\hbar^2}{2m}\nabla^2+V(r)\right]\psi(\mathbf{x})=E\psi(\mathbf{x}). \tag{5.5.1}$$

This equation has the integral equation solution

$$\psi(\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}-\frac{2m}{\hbar^2}\int\frac{e^{ikR}}{4\pi R}V(r')\psi(\mathbf{x}')\,d^3\mathbf{x}', \qquad R\equiv\|\mathbf{x}-\mathbf{x}'\|, \tag{5.5.2}$$

[1] There is a very large literature, which may be traced from the citations in the paper of Nussenzveig [3].

[2] Textbooks on this subject include those of Frautschi [9], Newton [10], Squires [11], and Collins [12]. See also Frautschi *et al.* [13].

appropriate to scattering boundary conditions (incident plane wave along the direction $\mathbf{k}$ with *outgoing* scattered waves—note that $\hbar^2\mathbf{k}^2=2mE$).

In the standard way, one introduces a decomposition of Green's function, $e^{ikR}/4\pi R$, in terms of spherical waves (see Chapter 7, Section 6, AMQP):

$$\frac{e^{ikR}}{R}=ik\sum_{l=0}^{\infty}(2l+1)j_l(kr_<)h_l^{\text{out}}(kr_>)P_l(\cos\theta), \tag{5.5.3}$$

where $\cos\theta=\hat{x}\cdot\hat{x}'$, with $\hat{x}\equiv\mathbf{x}/r$ and $\hat{x}'\equiv\mathbf{x}'/r$. From this result, one obtains the scattering amplitude $f(k,\theta)$:

$$\psi(\mathbf{x})\sim e^{i\mathbf{k}\cdot\mathbf{x}}+f(k,\theta)\frac{e^{ikr}}{r}, \tag{5.5.4}$$

which, in turn, determines the differential cross section $d\sigma/d\Omega$ given by

$$\frac{d\sigma}{d\Omega}=|f(k,\theta)|^2. \tag{5.5.5}$$

Using these results, one finds for the amplitude $f(k,\theta)$ the expression

$$f(k,\theta)=k^{-1}\sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta), \tag{5.5.6}$$

where $a_l(k)\equiv e^{i\delta_l}\sin\delta_l$, and the phase shift δ_l is given by

$$-\sin\delta_l=(2mk/\hbar^2)\int_0^{\infty}j_l(kr')\phi_l(kr')V(r')r'^2\,dr'. \tag{5.5.7}$$

Here the $\phi_l(kr)$ are the exact regular eigenfunctions, which satisfy the radial ~~Schrödinger equation for angular momentum l, and are normalized to be~~ phase-shifted regular spherical Bessel functions at infinity.

The essence of this (standard) development is to obtain a Legendre series, Eq. (5.5.6), for the scattering amplitude, expressed in terms of a denumerably infinite set of (energy-dependent) parameters, the phase shifts $\delta_l(k)$ defined in terms of the potential $V(r)$ by Eq. (5.5.7).

The representation, Eq. (5.5.6), for the scattering amplitude is adapted to *low-energy* scattering, since, physically, at low energies the centrifugal potential prevents all but the very lowest angular momenta from entering significantly.

The problem is to develop a representation adapted to the *high-energy* limit. To accomplish this, we formally consider the terms in the series as functions of a complex variable l and use (as it is now called) the Watson–Sommerfeld (Watson [14]) transformation to write an integral representa-

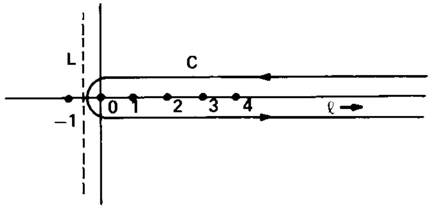

Figure 5.1. The Contours of C and L

tion for the scattering amplitude as a contour integral in the complex l-plane. (See Notes 1 and 2.) For this interpretation to be meaningful, we replace the Legendre polynomial by the hypergeometric function,

$$(-1)^l P_l(\cos\theta)=P_l(-\cos\theta)\equiv {}_2F_1\left(-l,l+1,1;\frac{1+\cos\theta}{2}\right), \quad (5.5.8)$$

which is well-defined for complex l. The scattering amplitude then takes the form

$$f(k,\theta)=(2\pi ik)^{-1}\int_C dl\frac{(2l+1)a_l(k)P_l(-\cos\theta)}{\sin(\pi l)}, \quad (5.5.9)$$

where the contour C is as shown in Fig. 5.1.

It is easily seen that, by construction, the poles from $(\sin\pi l)^{-1}$ reproduce, by the Cauchy formula, the original series.[1]

At this point it is essential to use the analytic results shown by Regge to be valid for a large class of nonrelativistic potentials (potentials that are superpositions of Yukawa forms, $e^{-\mu r}/r$). The elastic scattering amplitude, $a_l(k)$, for a fixed energy (fixed k), if regarded as a function of l, may be analytically continued, Regge showed, into the complex l-plane for $\mathrm{Re}\, l>-\frac{1}{2}$. Moreover, the scattering amplitude is meromorphic, with the poles confined to the upper half-plane ($\mathrm{Im}\, l>0$).

On the basis of these results, the contour C may be deformed into the contour L (see Fig. 5.1), thus picking up pole contributions, so that the scattering amplitude $f(k,\theta)$ now takes the form

$$f(k,\theta)=(2\pi ik)^{-1}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl(2l+1)a_l(k)\frac{P_l(-\cos\theta)}{\sin\pi l}$$
$$+\sum_i \frac{\beta_i(k)}{\sin\pi\alpha_i(k)}P_{\alpha_i(k)}(-\cos\theta). \quad (5.5.10)$$

[1] One might be concerned as to the uniqueness of this representation. Subject to physically reasonable requirements on the potential, the uniqueness is guaranteed by Carlson's theorem (see Ref. [15]). Note that the minus sign in the argument of $P_l(-\cos\theta)$ is compensated by the alternating sign of the derivative of $\sin\pi l$ at odd and even l-values.

In this result the $\{\alpha_i\}$ are the positions of the poles in the complex l-plane, and the $\{\beta_i\}$ are the residues.

The interesting feature of Eq. (5.5.10) is the occurrence of the pole contributions, called "Regge poles," coming from the singularities of $a_l(k)$ that are crossed in deforming the contour. The position of the ith pole is at the complex l-value denoted by α_i. It is important to note that the position of a given pole is *not* fixed, but varies with the energy, $\alpha_i = \alpha_i(k)$. The residue at the ith pole is β_i, which is also a function of energy. The path in the complex l-plane of a given pole as a function of energy is called a "Regge trajectory."

For large energy (short wavelengths) the lowest pole (the pole closest to the real axis) gives the dominant contribution, and the contribution of the background integral (as Regge showed) becomes negligible.

To summarize: The high-energy limit of potential scattering is dominated by Regge poles, which are the singularities in the scattering phase functions $a_l(k)$ for complex l-values. This is the way in which the term "complex angular momenta" was originally introduced.

The physical significance of a Regge pole—as a scattering resonance—is discussed in Section 3.

An alternative physical interpretation of the scattering represented by a Regge pole can be given in terms of the diffraction process for short wavelengths. From this viewpoint a given Regge pole contribution is related to a "creep wave" (Franz and Deppermann [16], Franz [17]), which channels radiation along a region of rapidly varying index of refraction. For sound waves, Schlieren photographs exist (Neubauer [18]), showing graphically the existence of creep waves on a cylindrical surface. This phenomenon has been analyzed in terms of Regge poles, as we shall now sketch.

Let us consider a Regge pole at the complex angular momentum $\alpha = \alpha_1 + i\alpha_2$, and assume that both α_1 and α_2 are large and positive. The angular dependence of the contribution from the pole is given by $P_\alpha(-\cos\theta)$. Using the asymptotic form of P_α, valid for large α, one finds (see Ref. [1], p. 288)

$$\frac{P_\alpha(-\cos\theta)}{\sin\pi\alpha} \sim \left[\frac{(2i)}{(\pi\alpha\sin\theta)}\right]^{\frac{1}{2}} e^{i(\alpha+\frac{1}{2})\theta}, \qquad \theta \gtrsim 0. \qquad (5.5.11)$$

The contribution, $e^{i(\alpha+\frac{1}{2})\theta}$, from the Regge pole is interpreted as a creep wave in θ, decreasing exponentially as θ increases—that is, moving into the diffraction region.[1]

Exactly this sort of exponential dependence on the scattering angle is seen in nuclear reactions taking place at the surface of strongly absorbing nuclei (McVoy [4]).

[1]Creep waves in the opposite sense, for decreasing θ, exist for the opposite side of the diffraction region.

Creep waves can be considered as an interesting special case in a generalization of the Fermat principle of geometric optics. Keller [19] has shown that a ray falling tangentially on the surface of an object, running along a geodesic of the surface, and emerging tangentially to move along a straight line to the image point is a (relative) minimal path joining source and image point.

3. An Illustrative Example

It will be helpful in understanding these ideas to consider a simple, explicitly solvable example for potential scattering. For this we take the Coulomb potential, even though the long range of this potential causes special features not found in the general (exponentially falling off, Yukawa type) potential.[1]

The scattering function (S matrix) for the lth partial wave is easily found to be

$$S_l \equiv e^{2i\delta} = \frac{\Gamma(l+1-i\eta)}{\Gamma(l+1+i\eta)}, \tag{5.5.12}$$

where the Sommerfeld number, η (a function of energy), is given by

$$\eta(E) = \frac{Z_1 Z_2 e^2 m}{\hbar^2 k} = \left(\frac{Z_1 Z_2 e^2}{\hbar}\right)\left(\frac{m}{2}\right)^{\frac{1}{2}} E^{-\frac{1}{2}}. \tag{5.5.13}$$

The analytic extension to complex l-values is immediate from the properties of the Γ function. Let us denote complex l by $\alpha(k)$, as is customary. Since the function $\Gamma^{-1}(z)$ is an entire function, and, moreover, $\Gamma(z)$ has only poles, the only singularities are simple poles at the negative integers: $\alpha+1-i\eta=-N$.

Hence, for each integer $N=0,1,2,\ldots$ we find a *Regge trajectory*:

$$\alpha_N(k) = -N-1+i\eta(E); \tag{5.5.14}$$

that is, the path of the Nth singularity is given as an explicit function of energy.

To interpret the meaning of this trajectory, let us consider the contribution to the scattering amplitude as given by the singular terms in Eq. (5.5.10),—that is, the Regge pole terms. A single trajectory contributes the term $(\beta_N/\sin\pi\alpha_N)P_{\alpha_N}(-\cos\theta)$, where the residue β_N is found from Eq.

[1]Considerations based on a square-well potential, although explicitly solvable, must be viewed with caution, since the typical length (from the sharp edge) is zero, and hence some aspects of the limit are singular.

(5.5.12) to be $\beta_N = (-1)^N/N!\Gamma(2\alpha_N + 2 + N)$. (Note that this is an entire function.)

To interpret this contribution to the scattering amplitude $f(k,\theta)$, we expand this term as a Legendre series, using the formula

$$\int_0^{\pi} P_l(\cos\theta) P_\alpha(-\cos\theta)\sin\theta\, d\theta = \frac{2}{\pi}\frac{\sin(\pi\alpha)}{(\alpha-l)(\alpha+l+1)}. \tag{5.5.15}$$

Thus, the Nth trajectory contributes to the scattering amplitude a term corresponding to angular momentum l with the magnitude

$$\frac{2\beta_N}{\pi}\frac{1}{(\alpha_N - l)(\alpha_N + l + 1)}. \tag{5.5.16}$$

Consider the situation where $\operatorname{Re}\alpha_N = l$ at the energy level E_l. Expanding the trajectory function near this energy, we find

$$\left(\frac{2\beta_N(E_l)}{\pi[\alpha_N(E_l)+l+1]}\right)\frac{1}{(E-E_l)[d(\operatorname{Re}\alpha)/dE] + i\operatorname{Im}\alpha_N(E_l)}. \tag{5.6.17}$$

This result shows that the contribution to the scattering amplitude has the Breit–Wigner form, and hence constitutes a resonance of width $\Gamma = \dfrac{\operatorname{Im}\alpha_N}{d(\operatorname{Re}\alpha)/dE}$. If $\operatorname{Im}\alpha_N$ vanishes, we have a bound state.[1]

Regge gave qualitative arguments that $d(\operatorname{Re}\alpha)/dE$ was *positive* for sharp resonances, and hence (since $\operatorname{Im}\alpha$ is positive) the width Γ is positive, as required physically.

It is this elegant result that provides the interpretation of the singularities (Regge poles): *Each trajectory function*, $\alpha(E)$, *corresponds to a physical resonance or bound state whenever* $\operatorname{Re}\alpha$ *is an integer* (including 0).

Turning the problem around, we may now consider the set of bound states and resonances arising from a given potential and view *the trajectory functions as interpolating functions uniting the resonances and bound states into families*. The set of trajectories is therefore an alternative, and equivalent, formulation of the information contained in a potential. The remarkable nature of this insight lies in its suggestiveness for applications in high-energy physics.

[1]One finds that for the Coulomb case there are *only* bound states, but this is a peculiarity of special choice of potential.

The Coulomb problem is solvable for the relativistic (Dirac) electron problem; a corresponding discussion (in closed form) for a *relativistic* problem (with spin) is thus possible, but unnecessary for our purposes here.

4. The Mandelstam–Chew Viewpoint

The formulation of Regge poles and trajectories given above is, despite the elegance and novelty of the concepts, not essential for potential scattering problems; a direct treatment would lead to the same physics. The critical importance of these ideas lies instead in the realm of high-energy physics where there are, as yet, very few valid general theoretical structures, or procedures, available. Quantum field theory—the best available theoretical structure—is quite limited, and its applicability to the problem of strong interactions is unreliable. It was Mandelstam and Chew who saw in the Regge trajectories the key concepts for a possible alternative approach, which we now sketch.

We begin with a discussion of the kinematic preliminaries. Let the mass, spin, and four-momentum of the particles in the two-particle[1] scattering process be denoted by m_i, s_i, and $\mathbf{k}_i$, with $i=1,2,3,4$. All the momenta are taken to be directed *into* the scattering diagram (Fig. 5.2), and conservation of four-momentum implies $\sum_{i=1}^{4}\mathbf{k}_i=\mathbf{0}$.

There are three invariants defined by the four-momenta, denoted, as is customary, by s, t, and u:

$$\begin{aligned} s&=(\mathbf{k}_1+\mathbf{k}_2)^2=(\mathbf{k}_3+\mathbf{k}_4)^2,\\ t&=(\mathbf{k}_1+\mathbf{k}_3)^2=(\mathbf{k}_2+\mathbf{k}_4)^2,\\ u&=(\mathbf{k}_1+\mathbf{k}_4)^2=(\mathbf{k}_2+\mathbf{k}_3)^2. \end{aligned} \tag{5.5.18}$$

Each of the momenta, $\mathbf{k}_i$, has a fixed length given by $\mathbf{k}_i^2=m_i^2$. The three invariants (s, t, u) are not independent but satisfy the constraint $s+t+u=\sum_{i=1}^{4} m_i^2$.

There are three physical processes associated with the scattering diagram of Fig. 5.2:

$$\begin{aligned} &1+2\to3+4, &&\text{“}s\text{-channel,”}\\ &1+\bar{3}\to\bar{2}+4, &&\text{“}t\text{-channel,”}\\ &1+\bar{4}\to\bar{2}+3, &&\text{“}u\text{-channel.”} \end{aligned}$$

[1] We assume that the energy is such that only two-body final states are possible.

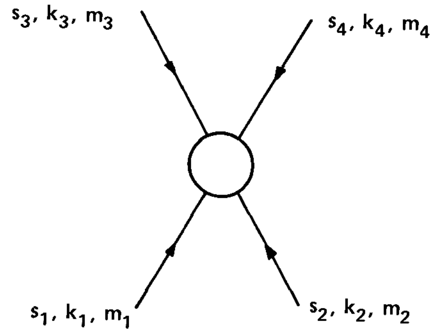

Figure 5.2. Two-Body Scattering Diagram

In the s-channel ("direct channel"), particles 1 and 2 with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ are *incoming*, and particles 3 and 4 with momenta $-\mathbf{k}_3$ and $-\mathbf{k}_4$ are *outgoing*. In this channel, the invariants s and t have the following significance: The quantity s is the energy in the Lorentz frame for which the total three-momentum vanishes (center-of-momentum), and t is linearly related to the scattering angle, $\cos\theta$, in this same frame.

The t-channel ("crossed channel") reaction describes the physical process in which particle 1 and *antiparticle* $\bar{3}$ with momenta $\mathbf{k}_1$ and $\mathbf{k}_3$ are incoming, with antiparticle $\bar{2}$ and particle 4 (momenta $-\mathbf{k}_2$ and $-\mathbf{k}_4$) outgoing. The invariants (t,u) now play the role of energy and angle variables in the center-of-momentum frame.

An analogous interpretation holds for the remaining u-channel.

The constructive principle in the Mandelstam–Chew approach is the *assumption* that the scattering function (the S matrix, which is a unitary Poincaré invariant function) is a *meromorphic function* of its arguments, possibly with cuts, and that *crossing symmetry* obtains. Thus, *the scattering matrices in the s, t, and u channels are to be analytic continuations of one and the same function.* These postulated properties are not in conflict with any basic principles, and have been verified in special cases, but they are as yet generally unproved from any more basic structure.

When the postulate of crossing symmetry is combined with the concept of a Regge pole, a remarkable result can be obtained.[1] To see this, consider a single Regge pole in the direct channel. One has the relation

$$f_{\text{Regge}}(s,\cos\theta)=\frac{\beta(s)}{\sin\pi\alpha(s)}P_{\alpha(s)}(-\cos\theta), \tag{5.5.19}$$

where we use s to denote the center-of-momentum energy in this direct channel.

[1]A different application of crossing symmetry is discussed in Note 3.

Using crossing symmetry, we may view this same contribution, f_{Regge}, in the crossed channel, the t-channel, for which the energy is the invariant t that is linearly related to the variable $\cos\theta$. Thus, for large t (large energy) we obtain

$$f_{\text{Regge}} \sim (\text{function of } s)\, t^{\alpha(s)}. \tag{5.5.20}$$

In other words, we have shown that the *low-energy resonances in the direct channel* determine the Regge poles which in the *crossed channel govern the high-energy scattering.* This remarkable interrelation of previously separated regimes is a result that could not be obtained experimentally, and was not obtained theoretically until Mandelstam and Chew saw the implications of Regge's work. This application of Regge poles has been found experimentally and theoretically to be significant. Although details have had to be modified, the concept itself has survived, playing a key role, for example, in the later dual resonance models (Susskind [20]).

Let us now indicate how these ideas relate to angular momentum theory. In the reaction $1+2\rightarrow3+4$ there are two invariant variables. Using the freedom to choose any Lorentz frame, we may choose these two invariants in the s-channel as the center-of-momentum energy, s, and the associated scattering angle, $\cos\theta$.

The corresponding stability group in the s-channel is the group of rotations, $SU(2)$. *It thus follows from general principles that the unitary scattering matrix may be expanded in angular momentum functions defined by the $SU(2)$ little group.* This is the group-theoretic basis for a relativistic generalization of the Legendre expansion used in Section 2, which in the more general case (with intrinsic spins) becomes a rotation matrix expansion in the helicity basis (see Chapter 7, Section 8e, AMQP).

~~Consider now the same reaction, but looked at in the t-channel. The~~ relevant Lorentz frame is now the "brick-wall" frame in which the total energy is zero, and the total momentum is along, say, the z-direction. Thus, in the t-channel *the stability group is $SU(1,1)$*. This is the group of "complex angular momentum," and, at long last, we see the group-theoretic basis[1] for the Regge poles.[2]

[1] This interpretation of the "unphysical" Poincaré irreps (having spacelike momentum) is due to Joos [21], to Domokos and Surányi [22], and to Toller [22a]. A direct attempt to give a meaning to complex angular momenta, per se, has been reported by Beltrametti and Luzzato [23], but some of the group properties are lost.

[2] The Poincaré group has, for non-null momentum, three little groups, two of which [$SU(2)$ (partial wave analysis), $SU(1,1)$ (Regge poles)] have been discussed above. The third little group, $E(2)$, for lightlike momenta, also has a physical meaning (as pointed out by Patera *et al.* [23a]): it corresponds to the *eikonal* expansion.

It is worth noting explicitly that the physically forbidden (one particle) Poincaré irreps thus have a valid physical application as states of a *composite* system viewed in the crossed channel.

Can one now assert that the scattering matrix may be expanded in the spherical functions of the $SU(1,1)$ group? *No*, because there is no guarantee that the scattering matrix is square-integrable [unlike the s-channel case, where the unitarity relation guarantees the possibility of expanding in $SU(2)$ spherical functions]. The singular terms that prevent the $SU(1,1)$ expansion are just the Regge poles (Boyce [24], Iverson [25]).

To clarify this result, let us determine the irreps of $SU(1,1)$.

5. The Irreducible Representations of $SU(1,1)$[1,2]

The algebraic method we used in Chapter 3, AMQP, to discuss the irreps of $SU(2)$ generalizes directly to the noncompact group $SU(1,1)$. The $SU(1,1)$ group is the group of nonsingular transformations in a two-dimensional complex plane $\mathbb{C}^2$, which leaves the following Hermitian form invariant:

$$\bar{\psi}\psi=\psi^\dagger\sigma_3\psi=\psi_1^*\psi_1-\psi_2^*\psi_2. \tag{5.5.21}$$

This group has been discussed extensively in a paper of Bargmann [27] as incidental to his discussion of the Lorentz group.

From the definition of the group, it follows that it consists of 2×2 complex matrices $\{S\}$ satisfying the relation

$$\sigma_3 S^\dagger \sigma_3 = S^{-1}. \tag{5.5.22}$$

The matrix S thus has the general form

$$S=\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad |\alpha|^2-|\beta|^2=1. \tag{5.5.23}$$

The group $U(1)$ of diagonal matrices of the form

$$U(\alpha)=\begin{pmatrix} \exp(i\alpha/2) & 0 \\ 0 & \exp(-i\alpha/2) \end{pmatrix}, \qquad 0\leqslant\alpha<4\pi \tag{5.5.24}$$

[1] The symmetry group $SU(1,1)$ enters physics in surprisingly many ways. Besides the application discussed in this Topic, this group enters in the analysis of radial integrals (Topic 6), in the symmetry of the motion of an electron in a constant magnetic field, and as a subgroup of the symmetry of the continuum (nonrelativistic) Coulomb problem and the (nonrelativistic) magnetic monopole problem (Topic 2).

[2] The section is adapted from the work presented in Ref. [26].

is a maximal compact subgroup of the double covering of $SU(1,1)$. The corresponding Lie algebra $su(1,1)$ is the set of matrices given by

$$su(1,1)=\{A: \sigma_3 A^\dagger \sigma_3 = A\} \tag{5.5.25}$$

and has the explicit basis

$$k_1 = \frac{i}{2}\sigma_1, \qquad k_2 = \frac{i}{2}\sigma_2, \qquad k_3 = \tfrac{1}{2}\sigma_3. \tag{5.5.26}$$

It should be noted that the matrices k_i are the generators of the two-dimensional nonunitary representation defining $SU(1,1)$.

In a unitary representation of $SU(1,1)$, the corresponding basis elements are self-adjoint operators (K_1, K_2, K_3) acting in a Hilbert space $\mathcal{H}$ and satisfying the commutation relations:

$$\begin{aligned}
[K_1, K_2] &= -iK_3,\\
[K_3, K_1] &= iK_2,\\
[K_2, K_3] &= iK_1.
\end{aligned} \tag{5.5.27}$$

Hence, the invariant operator $\mathbf{K}^2 = -K_1^2 - K_2^2 + K_3^2$ is self-adjoint, but *indefinite*.

Let us now consider a unitary representation of $SU(1,1)$ in a Hilbert space $\mathcal{H}$. The space $\mathcal{H}$ is necessarily infinite dimensional because all *faithful* unitary irreducible representations of noncompact Lie groups are infinite dimensional. We assume that the unitary representation $\{S \to \mathcal{U}(S): S \in SU(1,1)\}$ is continuous in the sense that the inner product $(\Psi, \mathcal{U}(S)\Phi)$ for $\Psi, \Phi \in \mathcal{H}$ is continuous in S.

This representation decomposes with respect to the $U(1)$ subgroup into a direct sum of one-dimensional unitary representations. Correspondingly, the Hilbert space $\mathcal{H}$ splits into a direct sum of one-dimensional spaces $\mathcal{H}_{[m]}$:

$$\mathcal{H} = \sum_m \oplus \mathcal{H}_{[m]}. \tag{5.5.28}$$

The action of the unitary operator $\mathcal{U}(\alpha)$ corresponding to $U(\alpha)$ [see Eq. (5.5.24)] on a vector $\psi_m \in \mathcal{H}_{[m]}$ is given by

$$\mathcal{U}(\alpha)\psi_m = e^{im\alpha}\psi_m. \tag{5.5.29}$$

Since the values of α given by

$$\alpha = 0 (\operatorname{mod} 4\pi) \tag{5.5.30}$$

correspond to the identity element in the group, it follows that m is integer or half-integer (double covering).

It is essential to recognize that this quantization of m does not follow from the Lie algebra, but only from a global property. This reflects the fact that, unlike $SU(2)$, which is its own universal covering group, $SU(1,1)$ is not simply connected. As Bargmann [27] shows, the group manifold (parameter space) is homeomorphic to the direct product of the circle with a two-dimensional Euclidean space. The group manifold of the universal covering group of $SU(1,1)$ is then the three-dimensional Euclidean space and covers $SU(1,1)$ infinitely many times.

If $\mathcal{H}$ is an irrep space carrying a unitary representation of $SU(1,1)$, then in the decomposition of $\mathcal{H}$ given by Eq. (5.5.28), each m appears *at most once.*[1]

Let us next consider the Lie algebra $su(1,1)$. In terms of the raising and lowering operators $K_{\pm}=K_1 \pm iK_2$, the commutation relations (5.5.27) are expressed by

$$[K_3, K_{\pm}] = \pm K_{\pm}, \qquad [K_+, K_-] = -2K_3. \tag{5.5.31}$$

The operator K_3 is the generator of the one-parameter subgroup $U(1)$ [see Eq. (5.5.24)], and its action on an arbitrary vector $\psi \in \mathcal{H}$ is realized by the differential operator $-i\partial/\partial\alpha$. Thus, we find

$$K_3 \psi_m = m \psi_m. \tag{5.5.32}$$

Using next the commutation relations (5.5.31), we obtain

$$\begin{aligned} K_3 K_- \psi_m &= (m-1) K_- \psi_m, \\ K_3 K_+ \psi_m &= (m+1) K_+ \psi_m. \end{aligned} \tag{5.5.33}$$

~~These relations show that $K_-\psi_m$ and $K_+\psi_m$ are eigenvectors of K_3 with~~ eigenvalues $m-1$ and $m+1$, respectively.

Consider then the squares of the norms of the eigenvectors $K_-\psi_m$ and $K_+\psi_m$:

$$\begin{aligned} \|K_-\psi_m\|^2 &= (\psi_m, K_+K_-\psi_m) \geqslant 0, \\ \|K_+\psi_m\|^2 &= (\psi_m, K_-K_+\psi_m) \geqslant 0. \end{aligned} \tag{5.5.34}$$

Since

$$\mathbf{K}^2 = K_3(K_3+1) - K_-K_+ = K_3(K_3-1) - K_+K_-, \tag{5.5.35}$$

[1] This assertion is a consequence of Theorem 1 of Godement [28].

we conclude from Eqs. (5.5.32) and (5.5.34) that

$$
\begin{aligned}
&m(m-1)-K'^2 \geqslant 0,\\
&m(m+1)-K'^2 \geqslant 0,
\end{aligned}
\tag{5.5.36}
$$

where K'^2 denotes the (real) eigenvalue of $\mathbf{K}^2$ on the vector ψ_m (the operators $\mathbf{K}^2$ and K_3 are mutually commuting and Hermitian[1] on $\mathcal{H}$ so that they may be simultaneously diagonalized):

$$
K^2\psi_m = K'^2\psi_m. \tag{5.5.37}
$$

[In the compact case $SU(2)$, the quantities on the left-hand sides of Eqs. (5.5.36) must be negative or zero.]

For sufficiently large $|m|$, the positivity conditions of Eq. (5.5.36) are both satisfied. Assume that among the allowed m there exists one positive m and its corresponding state ψ_m (negative m's are treated analogously). The infinite chain of states

$$
\psi_m, \psi_{m+1}, \ldots \tag{5.5.38}
$$

obtained by applying K_+ repeatedly on ψ_m all exist and are different from zero, since

$$
\begin{aligned}
\|K_+\psi_m\|^2 &= (\psi_m, K_-K_+\psi_m)\\
&= (\psi_m, (2K_3 + K_+K_-)\psi_m)\\
&= 2m\|\psi_m\|^2 + \|K_-\psi_m\|^2 > 0.
\end{aligned}
\tag{5.5.39}
$$

If the lowering operator K_- is applied repeatedly to ψ_m, one of the following alternatives occurs: (i) The chain never terminates; or (ii) the chain terminates.

Consider case (i): The positivity conditions [Eq. (5.5.36)] in terms of eigenvalues for all m imply

$$
K'^2 < 0, \quad \text{if } m \text{ is integral}, \tag{5.5.40}
$$

$$
K'^2 < -\tfrac{1}{4}, \quad \text{if } m \text{ is half-integral}. \tag{5.5.41}
$$

In case (ii), let ψ_{m_0} be the last nonvanishing vector in the descending chain; that is,

$$
K_-\psi_{m_0} = 0. \tag{5.5.42}
$$

[1] One must be careful to distinguish *Hermitian* operators from *self-adjoint* operators when there can be domain problems (see Note 2, p. 350). It is customary in the physics literature to use these terms somewhat loosely and essentially interchangeably.

Then, we find

$$K_+K_-\psi_{m_0}=\left[-K'^2+K_3(K_3-1)\right]\psi_{m_0}=0; \tag{5.5.43}$$

that is, the eigenvalue of $\mathbf{K}^2$ on the vector ψ_{m_0} is

$$K'^2=m_0(m_0-1). \tag{5.5.44}$$

Equation (5.5.39) applied to ψ_{m_0} shows that m_0 must be positive and nonzero.

The case $m_0=0$ occurs only in the identity representation for which

$$K_+\psi_0=K_-\psi_0=0 \tag{5.5.45}$$

and $K'^2=0$. [Note that there are *three* representations corresponding to $K'^2=0$ ($a=0$) (see Fig. 5.3).]

The representations obtained under cases (i) and (ii) are unitary (by construction) and irreducible (again by construction, since every vector of these representations is cyclic[1]). It can be shown that the representations so constructed constitute all the irreducible representations of $SU(1,1)$. The representations with m integral are representations of $SO(2,1)$ [the Lorentz group with signature $(+,+,-)$]; the half-integral m-values correspond to projective representations of $SO(2,1)$ (see Note 4).

It is useful to recapitulate these results in a notation that will prove convenient for more general cases.

The first step is to factorize the Casimir invariant in terms of two symmetric variables—that is,

$$K'^2=-\tfrac{1}{4}ab. \tag{5.5.46}$$

In terms of these variables, the operator K_-K_+ takes on the symmetric ~~form on ψ_m given by~~

$$K_-K_+\psi_m=\left(m-\frac{a}{2}\right)\left(m-\frac{b}{2}\right)\psi_m, \tag{5.5.47}$$

where we require

$$a+b=-2. \tag{5.5.48}$$

Similarly, one obtains

$$K_+K_-\psi_m=\left(m-\frac{a'}{2}\right)\left(m-\frac{b'}{2}\right)\psi_m, \tag{5.5.49}$$

[1]A vector is cyclic if upon applying the enveloping algebra one generates the whole space of the representation.

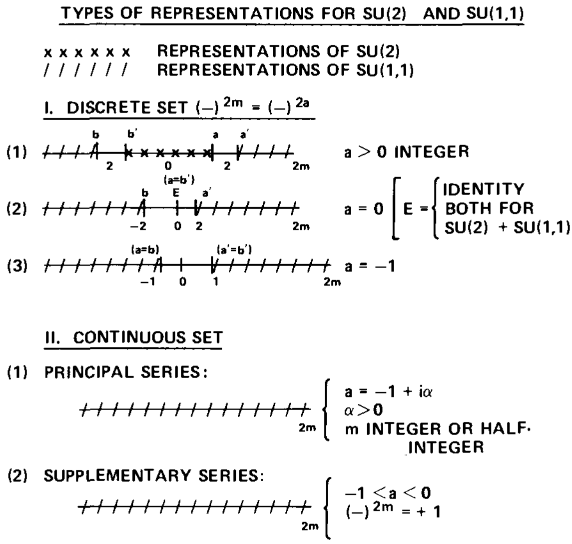

Figure 5.3.

where

$$a'+b'=2 \qquad \text{and} \qquad a'b'=ab. \tag{5.5.50}$$

We may identify the variables (a', b') with (a, b) in four ways, the most useful being

$$\begin{aligned} a'&=a+2,\\ b'&=b+2. \end{aligned} \tag{5.5.51}$$

The purpose of this notation is to make the boundaries for the raising and lowering operators clear. Thus, (a, b) represent boundaries for the raising operator K_+, and (a', b') for the lowering operator K_-.

The different representations of both $SU(2)$ and $SU(1,1)$ may now be plotted as in Fig. 5.3.

It is useful for understanding the procedure by which these irreps were generated to make explicit the following three points:

(i) The boundary points $(a,b)/(a',b')$ separate positive and negative regions for K_-K_+/K_+K_-.

(ii) Motion toward the left terminates, if at all, at a' or b'; toward the right at a or b.

(iii) The motion is discrete; *one may jump over unallowed regions.*

The critical condition that separates the cases is the coincidence of two of the four variables (a, b, a', b').

6. Concluding Remarks

If we now reconsider the formal results of the Watson–Sommerfeld transform—given in Eq. (5.5.10)—in the light of the irreps of $SU(1,1)$, the structure of the Regge pole analysis becomes clearer. One sees that the background integral is nothing but the expansion of the scattering amplitude in terms of the *principal series* of $SU(1,1)$. (See summary in Fig. 5.3.)

The content of the Regge approach is to assert that, although the full amplitude is not square-integrable over the $SU(1,1)$ parameters, the remainder after removing the Regge poles is square-integrable.

Thus, as has been emphasized by Iverson [25] and O'Raifeartaigh [29], the Watson–Sommerfeld transform can be considered in two distinct ways: *Physically* the Regge poles are the important terms (for short wavelengths), and the background integral is a minor correction. *Group-theoretically* the background term is the interesting term in an expansion in spherical functions of the group, and the pole terms are incidental to providing convergence.

To explain why only the principal series occurs in Eq. (5.5.10), it is useful to note a result (due to Bargmann [27]) to the effect that a square-integrable function can be expanded in terms of the principal series and the discrete series for $l > -\frac{1}{2}$. The absence of the discrete series for Eq. (5.5.10) has been explained in Refs. [24] and [25]: these series do occur for scattering in the general case having nonvanishing helicities (intrinsic spins).

Accordingly, we conclude: The introduction of (unphysical) complex angular momenta, although originally a technical device for obtaining asymptotic results, nevertheless turns out to have a valid interpretation in terms of the symmetry of the noncompact group $SU(1,1)$.

7. Notes

1. *Splitting of the amplitude $f(k, \theta)$.* Although (for simplicity) we chose not to do so in our discussion, it is important in applications in particle physics to split the amplitude $f(k, \theta)$ into odd and even "signature" parts $f^{\pm}(k, \theta)$, since these two parts are independent quantities.

The origin of this splitting comes from the possibility of introducing exchange potentials (nonrelativistically) or (relativistically) from crossed channel terms.

It has been emphasized by Frautschi [9] that this splitting has been wrongly confused with parity, and, in fact, the signature splitting is valid even if parity is violated.

2. *Integral representation of the Legendre series.* The transformation of the Legendre series into the integral representation of Eq. (5.5.9) is due to Watson [14]. Sommerfeld's [1] contribution was to relate the short-wave limit directly to a new form of Green's function, thereby providing a new technique for the treatment of the exterior boundary value problem for scattering. The Green's function found by Sommerfeld has many technical advantages; among other properties, the change in form [for $r_1 > r_2$ or $r_1 < r_2$; see Eq. (5.5.3)] is now avoided.

3. *Crossing matrices as Racah coefficients.* Crossing symmetry can be used in a significant way to implement the symmetry (invariance) properties of the scattering matrix. Since the scattering matrix is a rotational invariant, the s-channel amplitude $M_s(J_s)$ (see Fig. 5.2) can be expanded in terms of the rotation matrices d^{J_s} to yield the angular momentum common to the entrance channel $1+2$ and the exit channel $3+4$. The t-channel amplitude similarly can be expanded in terms of d^{J_t}, where J_t is the total angular momentum in channel $1+\bar{3}$ and in channel $\bar{2}+4$. *Crossing symmetry implies that the terms in the two expansions,* $M_s(J_s)$ and $M_t(J_t)$, *are related by a "crossing matrix," which, in simple cases, is a Racah coefficient* (Dyson [30]). (This is intuitively clear from the fact that the various angular momenta have been recoupled.)

For strong interactions, the scattering matrix is invariant under isospin rotations, the group $SU(2)$, and also under unitary symmetry, $SU(3)$. The corresponding "crossing matrices" are the recoupling (Racah) coefficients of the particular invariance group. (A careful discussion, with proper attention to phases, is given in Carruthers [31].)

The concepts underlying crossing symmetry stem from the earliest applications of (relativistic) perturbation theory; the *Fierz transformation* of beta decay theory may be considered as an example involving the symmetric group S_4.

4. *Wigner coefficients of* $SU(1,1)$. The problem of determining the Wigner coefficients of $SU(1,1)$ has a fairly extensive literature, which can be traced from Refs. [33]–[34]. Alternatively, the problem may be considered as the analytic continuation of the known series for the $SU(2)$ Wigner coefficients. Particularly interesting results have been obtained by Verde [35]. Applications of the principal series coefficients occur in the evaluation of radial integrals (see Topic 6).

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Topic 6. Radial Integrals and the Lie Algebra of $SU(1, 1)$

1. Introduction

The Hamiltonian of an isolated physical system is a rotational invariant. It is but one of many rotational invariants that may be constructed from the position, linear momentum, and spin vectors of the particles constituting the system. Thus, the determination of the properties of the Hamiltonian itself may be subsumed under the more general investigation of properties of rotational invariants.

One sometimes finds in the set of all rotational invariants not only the Hamiltonian of a particular physical system, but also several other rotational invariants that together with the Hamiltonian constitute a basis of a Lie algebra. In such favorable cases, the representation theory of Lie algebras may be used to determine properties of the states of the system going beyond those implied by the angular momentum itself.

In this Topic we consider the Coulomb problem and the n-dimensional isotropic harmonic oscillator problem from the viewpoint of rotational invariants. For these problems, one finds that the Hamiltonian is an element of the Lie algebra of the noncompact group $SU(1, 1)$. (This statement must be qualified for the Coulomb problem—see the Remarks at the end of Section 8.) Moreover, one may, by an appropriate change of variables, transform the physical realization of the Lie algebra to a standard form involving first- and second-order differential operators acting on associated Laguerre functions. Thus, although the realizations of the Lie algebra in terms of physical rotational invariants for the Coulomb and oscillator problems are quite different in appearance, it is the same set of basic properties of the associated Laguerre functions that underlies the mathematics.

The principal new result obtained by considering the Coulomb and oscillator problems from the viewpoint of the Lie algebra of $SU(1,1)$ is the explicit evaluation of a class of integrals of radial wave functions in terms of the Wigner coefficients of the noncompact group $SU(1,1)$. As we shall show in this Topic, these Wigner coefficients are the generalized coefficients defined by Eq. (3.45) of Chapter 3.

The relationship between radial integrals for the Coulomb and oscillator problems and the Wigner coefficients of $SU(1,1)$ was first pointed out by Armstrong [1–3], and has been developed further by other authors (Cunningham [4], Moshinsky *et al.* [5–7], Miller [8]).

The subject of radial integrals may be approached from either a physical or a mathematical viewpoint. In the physical approach one would start with the Hamiltonian for the system in question and seek those invariant operators that define the $SU(1,1)$ algebra. One would thus deal separately with each physical problem. Alternatively, one may emphasize the properties of the special functions at the outset (in our case the associated Laguerre functions), thus illustrating still another occurrence of Wigner coefficients in a classic subject. We shall follow this latter course, since the several physical problems considered are simple transformations of the general results for associated Laguerre functions, as noted in detail in Section 8.

We have summarized in an Appendix a number of well-known properties of the associated Laguerre polynomials and functions without giving proofs, since the results are available from standard references, or are easily derived. We make frequent use of these results in the developments below.

2. Associated Laguerre Functions and the Lie Algebra of the Group $SU(1,1)$

We define the functions Φ_{jm} for each real value of j such that $2j+1>-1$ and for each $m=j+1, j+2,\ldots$ in terms of the associated Laguerre functions [see Eq. (A-11) in the Appendix] by

$$\begin{aligned}\Phi_{jm}(x)&\equiv\mathfrak{L}^{2j+1}_{m-j-1}(x)\\&=\left[\frac{(m-j-1)!}{\Gamma(m+j+1)}\right]^{\frac{1}{2}}x^{(2j+1)/2}e^{-x/2}L^{2j+1}_{m-j-1}(x),\end{aligned}\qquad(5.6.1)$$

where $L_k^{\alpha}(x)$ denotes an associated Laguerre polynomial [see Eq. (A.1)], and $\Gamma(z)$ denotes the gamma function.

We next introduce the differential operators H_i $(i=1,2,3)$ defined by

$$\begin{aligned} H_1 &= x\left(\Lambda_{2j+1}^2 - \tfrac{1}{4}\right), \\ H_2 &= -i\left(x\frac{d}{dx} + \frac{1}{2}\right), \\ H_3 &= x\left(\Lambda_{2j+1}^2 + \tfrac{1}{4}\right), \end{aligned} \tag{5.6.2}$$

where

$$\Lambda_{2j+1}^2 = -\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + \frac{\left(j+\frac{1}{2}\right)^2}{x^2}. \tag{5.6.3}$$

These operators satisfy the hyperbolic commutation relations

$$\begin{aligned} [H_3, H_{\pm}] &= \pm H_{\pm}, \\ [H_+, H_-] &= -2H_3, \end{aligned} \tag{5.6.4}$$

where $H_{\pm} = H_1 \pm iH_2$. Moreover, the action of these operators on the functions Φ_{jm} is given by [see Eqs. (A.13)–(A.17)]

$$\begin{aligned} H_+\Phi_{jm} &= \left[(m-j)(m+j+1)\right]^{\frac{1}{2}}\Phi_{j,m+1}, \\ H_-\Phi_{jm} &= \left[(m-j-1)(m+j)\right]^{\frac{1}{2}}\Phi_{j,m-1}, \\ H_3\Phi_{jm} &= m\Phi_{jm}; \end{aligned} \tag{5.6.5}$$

$$\begin{aligned} \mathbf{H}^2\Phi_{jm} &= \left[H_3(H_3-1) - H_+H_-\right]\Phi_{jm} = j(j+1)\Phi_{jm}, \\ H_-\Phi_{j,j+1} &= 0. \end{aligned} \tag{5.6.6}$$

The functions Φ_{jm} may be generated from the function $\Phi_{j,j+1}$ by repeated application of the raising operator H_+:

$$\Phi_{jm} = \left[\binom{2j+1}{m-j-1}\right]^{\frac{1}{2}} H_+^{m-j-1}\Phi_{j,j+1}. \tag{5.6.7}$$

The set of functions

$$\{\Phi_{jm}\colon m=j+1, j+2, \dots\} \tag{5.6.8}$$

is an orthonormal set with inner product defined by

$$(\Phi_{jm}, \Phi_{jm'}) \equiv \int_0^\infty dx\, \Phi_{jm}(x)\Phi_{jm'}(x) = \delta_{mm'}. \tag{5.6.9}$$

In general, the functions Φ_{jm} and $\Phi_{j'm'}$ are not orthogonal for $j \neq j'$ using the inner product (5.6.9) [see, however, Eq. (5.6.15) below]. Note also that the operators H_i are not Hermitian with respect to the inner product (5.6.9); furthermore, they do not possess the derivation property—for example, $H_1(fg) \neq f(H_1 g) + (H_1 f)g$.

In the following sections, we use the Lie algebraic properties of the associated Laguerre functions summarized above to evaluate integrals of the form

$$\left(\Phi_{j'm'}, x^J \Phi_{jm}\right), \qquad J \text{ real}. \tag{5.6.10}$$

It is interesting to note that Schrödinger [9] calculated the integral (5.6.10) [see Eq. (A.18)] in one of his first papers on wave mechanics for the purpose of evaluating radial integrals of the hydrogen atom wave functions. [Actually, Schrödinger's derivation, using generating functions, is applicable to nonnegative integral values of the parameters α', α, p; the extension of Eq. (A.18) to arbitrary complex α', α, and $\operatorname{Re} p > -1$ is a consequence of using the more general definition (A.1) of the associated Laguerre polynomials.]

Before turning to the derivation of the integral (5.6.10), it is useful to interpret Schrödinger's result, Eq. (A.18), in terms of generalized Wigner coefficients.

3. Integrals Involving Associated Laguerre Functions

Using the integral (A.18) and the definition (5.6.7) of the functions Φ_{jm} and setting $p = J + j + j' + 1$, we obtain

$$\left(\Phi_{j'm'}, x^J \Phi_{jm}\right) = N(j'm'; jm)\Gamma(j'+j+J+2) \times \sum_s \binom{J-j+j'}{m-j-1-s}\binom{J+j-j'}{m'-j'-1-s}\binom{j'+j+J+1+s}{s}, \tag{5.6.11}$$

$$N(j'm'; jm) = (-1)^{m'-j'-m+j}\left[\frac{(m'-j'-1)!(m-j-1)!}{\Gamma(j'+m'+1)\Gamma(j+m+1)}\right]^{\frac{1}{2}}, \tag{5.6.12}$$

where we henceforth take j', m', j, m, and J to be real parameters satisfying the conditions

$$\begin{aligned} 2j+1&>-1, \qquad & m&=j+1, j+2,\ldots, \\ 2j'+1&>-1, \qquad & m'&=j'+1, j'+2,\ldots, \\ j'+j+J+2&>0. \end{aligned} \tag{5.6.13}$$

[Note that in the integral (5.6.11) there are, in general, no relations between parameters except those expressed by Eqs. (5.6.13).]

In general, the functions $\Phi_{j'm'}$ and Φ_{jm} ($j'\neq j$) are not orthogonal, as already noted. Equation (5.6.11) gives the general overlap integral as

$$(\Phi_{j'm'}, \Phi_{jm}) = N(j'm'; jm)\Gamma(j'+j+2) \times \sum_s \binom{j'-j}{m-j-1-s}\binom{j-j'}{m'-j'-1-s}\binom{j+j'+1+s}{s}. \tag{5.6.14}$$

Observe, however, from Eq. (5.6.11) that we have the following important orthogonality relation in j:

$$(\Phi_{j'm}, x^{-1}\Phi_{jm}) = \delta_{j'j}/(2j+1) \tag{5.6.15}$$

for all j', j such that $j'+j\geqslant 0$. We also obtain the following special overlap integral from Eq. (5.6.14):

$$(\Phi_{j'm}, \Phi_{jm}) = (-1)^{j'-j}\left[\frac{(m-j')_{j'-j}}{(j+m+1)_{j'-j}}\right]^{\frac{1}{2}} \tag{5.6.16}$$

for $j'>j$.

For restricted domains of definition of the parameters (5.6.13), the value of the integral (5.6.11) is closely related to the generalized Wigner coefficients discussed in Chapter 3, Section 2. Thus, if in Eq. (5.6.11) we put $j'=j+\Delta$, $m'=m+M$, and consider $J=0,\frac{1}{2},1,\ldots$ with the *restriction* that $\Delta=J, J-1,\ldots,-J$ and $M=J, J-1,\ldots,-J$, then we obtain the remarkable result:[1]

$$(\Phi_{j+\Delta,m+M}, x^J\Phi_{jm}) = (-1)^{W_1-\Delta_1}(\text{NPCF})^{\frac{1}{2}}P_k(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2), \tag{5.6.17}$$

[1]For $J=0,\frac{1}{2},1,\ldots$ and $\Delta\in\{J,J-1,\ldots,-J\}$, the integral (5.6.11) vanishes unless also $M\in\{J,J-1,\ldots,-J\}$ [see Eq. (5.6.29)]. For $\Delta\notin\{J,J-1,\ldots,-J\}$ we must appeal directly to Eq. (5.6.11) for the value of the integral.

where we have utilized the notations of Chapter 3:

$$\begin{aligned}&\Delta_1=J+\Delta, \qquad \Delta_2=J-\Delta, \qquad W_1=J+M, \qquad W_2=J-M,\\ &z_1=m-j-1, \qquad z_2=j+m,\\ &k=\min(\Delta_1,\Delta_2,W_1,W_2).\end{aligned} \tag{5.6.18}$$

[In obtaining Eq. (5.6.17) from Eq. (5.6.11), we have used the exchange symmetry (3.41) of the polynomials P_k.]

The polynomial P_k in Eq. (5.6.17) is given in Chapter 3 by Eqs. (3.39) (four cases), and the pattern calculus factor, NPCF, is evaluated by applying the pattern calculus rules (see Chapter 3) to the labeled arrow pattern for the shift pattern

$$\begin{bmatrix}\Delta_1 & & \Delta_2\\ & W_1 & \end{bmatrix}, \tag{5.6.19}$$

using $p_{12}=2j+1$, $p_{22}=0$, $p_{11}=j+m$.

Using Eq. (3.45), we obtain the expression for the integral (5.6.17) in terms of generalized Wigner coefficients:

$$\left(\Phi_{j+\Delta,\,m+M},\, x^J\Phi_{jm}\right)=\binom{2J}{J+M}^{\frac{1}{2}}(j+\Delta\,\|\,J\,\|\,j)\,\overline{C}^{\,jJj+\Delta}_{m,\,M,\,m+M}, \tag{5.6.20}$$

where we have defined[1]

$$(j+\Delta\,\|\,J\,\|\,j)\equiv\left[[2j+2\Delta]_{J+\Delta}(2j+2\Delta+2)_{J-\Delta}\Big/\binom{2J}{J+\Delta}\right]^{\frac{1}{2}}, \tag{5.6.21}$$

$$\overline{C}^{\,jJj+\Delta}_{m,\,M,\,m+M}=(-1)^{u_1}C^{\,jJj+\Delta}_{m,\,M,\,m+M}, \qquad u_1=\max(0,\Delta-M). \tag{5.6.22}$$

In these results we have $J=0,\frac{1}{2},1,\ldots,$ and the values that M and Δ may assume are $J, J-1,\ldots,-J$. [An alternative derivation of Eq. (5.6.20) is sketched below.]

[1] Let us recall that the notations $(x)_a$ and $[x]_a$ denote rising and falling factorials, respectively: $(x)_a\equiv x(x+1)\cdots(x+a-1)$ and $[x]_a\equiv x(x-1)\cdots(x-a+1)$, $a=1,2,\ldots$.

Another result of interest for the radial integral problem is obtained by replacing J by $-J-1$ and setting $j'=j+\Delta$, $m'=m+M$, in Eq. (5.6.11):

$$\left(\Phi_{j+\Delta,m+M}, x^{-J-1}\Phi_{jm}\right)=\left[\frac{(m-j-1+M-\Delta)!}{(m-j-1)!\Gamma(j+m+1)\Gamma(j+m+1+\Delta+M)}\right]^{\frac{1}{2}}$$
$$\times\sum_t \binom{J-\Delta+t}{t}\binom{J+M+t}{J+\Delta}[m-j-1]_t$$
$$\times\Gamma(j+m-J+\Delta-t), \tag{5.6.23}$$

where the summation is over $t=m-j-1, m-j-2,\ldots,\ u_1=\max(0,\Delta-M)$. For the cases of subsequent interest, we also take

$$\begin{aligned} &J=0,\tfrac{1}{2},1,\ldots; \qquad M=J,J-1,\ldots,-J,\\ &\Delta=\ldots,-J-1,-J,-J+1,\ldots,J,J+1,\ldots. \end{aligned} \tag{5.6.24}$$

In addition to these restrictions, one must choose $m=j+1, j+2,\ldots$ and $m+M=j+\Delta+1, j+\Delta+2,\ldots$ in order that the functions Φ_{jm} and $\Phi_{j+\Delta,m+M}$ are defined and also $2j-J+\Delta\geqslant 0$ in order that the integral exist.

Selecting J and Δ from the values given by (5.6.24) and choosing $2j\geqslant J-\Delta$, we find from Eq. (5.6.23) that the following integrals vanish:

$$\left(\Phi_{j+\Delta,m+M}, x^{-J-1}\Phi_{jm}\right)=0, \qquad \text{unless } \Delta=J,J-1,\ldots,-J. \tag{5.6.25}$$

[This result was discovered (in the context of the hydrogen atom radial integrals) in 1962 by Pasternack and Sternheimer [10].]

Now restricting Δ and M to the values $J, J-1,\ldots,-J$, we may introduce the parameters (5.6.18) into the right-hand side of Eq. (5.6.23) and obtain the following relations:

$$\begin{aligned} &\left(\Phi_{j+\Delta,m+M}, x^{-J-1}\Phi_{jm}\right)\\ &=(\mathrm{NPCF})^{\frac{1}{2}}\sum_{s=u_1}^{z_1}\binom{W_1+s}{\Delta_1}\binom{\Delta_2+s}{s}\frac{(z_1-s+1)_{s-u_1}}{(z_2-\Delta_2-s)_{W_1+u_2+s+1}}\\ &=\frac{(W_1)!(W_2)!}{(\Delta_1)!(\Delta_2)!}\,\frac{(\mathrm{NPCF})^{\frac{1}{2}}P_k(\Delta_1,\Delta_2,W_1,W_2;z_1,z_2)}{(z_2-z_1-\Delta_2)_{\Delta_1+\Delta_2+1}}\\ &=(-1)^{M-\Delta}\left[\binom{2J}{J+\Delta}\Big/\binom{2J}{J+M}\right]\frac{\left(\Phi_{j+\Delta,m+M}, x^{J}\Phi_{jm}\right)}{(2j+1-J+\Delta)_{2J+1}}, \end{aligned} \tag{5.6.26}$$

where $u_i = \max(0, \Delta_i - W_1)$. Thus, when the integral on the left exists, it has a simple relation to the integrals of x^J. (We restrict $J=0, \frac{1}{2}, 1, \ldots$ and $\Delta, M=J, J-1, \ldots, -J$ in this result.) The first identity on the right-hand side of Eq. (5.6.26) is quite difficult to prove directly, and we establish this result by proving below [Eq. (5.6.31)] the identity of the integral on the left side to the last expression on the right side of Eq. (5.6.26). A proof of this relation, using the generating function for associated Laguerre polynomials, may also be found in Ref. [11], Part III.

4. Iteration Method for Deriving the Integrals

The preceding results, Eqs. (5.6.11)–(5.6.26), have been obtained from Schrödinger's integral, Eq. (A.18). An alternative procedure, which shows directly, at each iteration, the occurrence of the Wigner coefficients in these integrals, is to write Eqs. (A.12) in the (jm)-notation and then to iterate the relations, which read [see Eq. (5.6.42) below]:

$$\begin{aligned} x^{\frac{1}{2}}\Phi_{jm} &= -(m-j)^{\frac{1}{2}}\Phi_{j-\frac{1}{2}, m+\frac{1}{2}} + (m+j)^{\frac{1}{2}}\Phi_{j-\frac{1}{2}, m-\frac{1}{2}} \\ &= (m+j+1)^{\frac{1}{2}}\Phi_{j+\frac{1}{2}, m+\frac{1}{2}} - (m-j-1)^{\frac{1}{2}}\Phi_{j+\frac{1}{2}, m-\frac{1}{2}} \\ &= (2j+1)^{\frac{1}{2}} \sum_{\mu} \overline{C}^{j\frac{1}{2}j-\frac{1}{2}}_{m,\mu,m+\mu}\Phi_{j-\frac{1}{2}, m+\mu} \\ &= (2j+1)^{\frac{1}{2}} \sum_{\mu} \overline{C}^{j\frac{1}{2}j+\frac{1}{2}}_{m,\mu,m+\mu}\Phi_{j+\frac{1}{2}, m+\mu}. \end{aligned} \tag{5.6.27}$$

These equations express the fact that on the space $\mathcal{H}_j'$ spanned by the set of orthonormal vectors $\{\Phi_{jm}: m=j+1, j+2, \ldots\}$ we have an identity between fundamental Wigner operators:

$$\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle + \left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle - \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle.$$

This circumstance could not occur in $SU(2)$ because of the perpendicularity of the spaces $\mathcal{H}_{j-\frac{1}{2}}$ and $\mathcal{H}_{j+\frac{1}{2}}$ (which itself followed from the Hermitian property of $\mathbf{J}^2$). The failure of the Hermitian property for $\mathbf{H}^2$ in the present case allows this unusual result to occur.

Equations (5.6.27) may be iterated to obtain the following relation:

$$x^J\Phi_{jm} = (j+\Delta \| J \| j) \sum_{M=-J}^{J} \binom{2J}{J+M}^{\frac{1}{2}} \overline{C}^{jJj+\Delta}_{m,M,m+M}\Phi_{j+\Delta, m+M}. \tag{5.6.28}$$

The choice of $\Delta=J, J-1,\ldots,-J$ is *arbitrary* in this result, and the various choices yield *relations* between the functions in the set $\{\Phi_{j+\Delta,m+M}: \Delta, M=J,\ldots,-J\}$. Using the orthogonality relation, Eq. (5.6.9), we now obtain the result, Eq. (5.6.20), as well as

$$\left(\Phi_{j+\Delta,m'}, x^J\Phi_{jm}\right)=0 \tag{5.6.29}$$

for $m'\notin\{m+M: M=J, J-1,\ldots,-J\}$.

A similar iteration procedure may be used to obtain the expansion of $x^{-J}\Phi_{jm}$. One first inverts the pair of equations, (5.6.27), thus obtaining two forms for $x^{-\frac{1}{2}}\Phi_{jm}$, one a linear combination of $\Phi_{j+\frac{1}{2},m+\frac{1}{2}}$ and $\Phi_{j-\frac{1}{2},m+\frac{1}{2}}$, the other a linear combination of $\Phi_{j+\frac{1}{2},m-\frac{1}{2}}$ and $\Phi_{j-\frac{1}{2},m-\frac{1}{2}}$. Iteration of these relations then yields

$$x^{-J}\Phi_{jm}=\binom{2J}{J+M}^{-\frac{1}{2}}\sum_{\Delta=-J}^{J}\frac{(-1)^{\Delta-M}}{(J+\Delta\|J\|j)}\bar{C}^{jJj+\Delta}_{m,M,m+M}\Phi_{j+\Delta,m+M}, \tag{5.6.30}$$

where $M=J, J-1,\ldots,-J$ may be chosen arbitrarily $(J=0,\frac{1}{2},1,\ldots)$.

We now use the orthogonality relation (5.6.15) to obtain from Eq. (5.6.30) the result

$$\begin{aligned}&\left(\Phi_{j+\Delta,m+M}, x^{-J-1}\Phi_{jm}\right)\\ &\qquad=(-1)^{\Delta-M}\binom{2J}{J+M}^{-\frac{1}{2}}\frac{\bar{C}^{jJj+\Delta}_{m,M,m+M}}{(2j+2\Delta+1)(j+\Delta\|J\|j)}.\end{aligned} \tag{5.6.31}$$

This result is seen to agree with the last expression in Eq. (5.6.26) upon noting that

$$\binom{2J}{J+\Delta}(j+\Delta\|J\|j)^2=\frac{(2j+1-J+\Delta)_{2J+1}}{(2j+2\Delta+1)}. \tag{5.6.32}$$

Equation (5.6.30) and the orthogonality relation (5.6.15) also imply[1]

$$\left(\Phi_{j',m+M}, x^{-J-1}\Phi_{jm}\right)=0$$

[1]For $M\notin\{J, J-1,\ldots,-J\}$ we must appeal directly to Eq. (A.18) for the value of the integral $(\phi_{j',m+M}, x^{-J-1}\phi_{jm})$ for $J=0,\frac{1}{2},1,\ldots$.

for $j' \notin \{J+\Delta\colon \Delta = J, J-1, \ldots, -J\}$. This result is a re-expression of Eq. (5.6.25).

5. Further Properties of Generalized Wigner Coefficients

The set of generalized Wigner coefficients

$$\left\{ \begin{matrix} \overline{C}^{jJj+\Delta}_{m,M,m+M}\colon \Delta, M=J, J-1, \ldots, -J \\ J=0, \tfrac{1}{2}, 1, \ldots \end{matrix} \right\}, \tag{5.6.33}$$

where $m=j+1, j+2, \ldots,$ with j an arbitrary real number satisfying $2j \geqslant J-\Delta$, enter significantly into the properties of the associated Laguerre polynomials as shown by the preceding results. We therefore list several additional properties of these coefficients:

Orthogonality relations:

$$\sum_{M=-J}^{J} (-1)^{J+M} \overline{C}^{jJj+\Delta}_{m-M,M,m} \overline{C}^{jJj+\Delta'}_{m-M,M,m} = (-1)^{J+\Delta} \delta_{\Delta\Delta'}, \tag{5.6.34}$$

$$\sum_{\Delta=-J}^{J} (-1)^{J+\Delta} \overline{C}^{jJj+\Delta}_{m-M,M,m} \overline{C}^{jJj+\Delta}_{m-M',M',m} = (-1)^{J+M} \delta_{MM'}, \tag{5.6.35}$$

where

$$\overline{C}^{jJj+\Delta}_{m,M,m+M} \equiv (-1)^{u_1} \mathcal{C}^{jJj+\Delta}_{m,M,m+M}, \qquad u_1 = \max(0, \Delta - M). \tag{5.6.36}$$

Recursion relations:

$$\begin{aligned} &[(J-M)(J+M+1)]^{\frac{1}{2}} \overline{C}^{jJj+\Delta}_{m-M,M+1,m+1} \\ &\quad = [(j+m+\Delta+1)(m-j-\Delta)]^{\frac{1}{2}} \overline{C}^{jJj+\Delta}_{m-M,M,m} \\ &\qquad - [(j+m-M+1)(m-j-M)]^{\frac{1}{2}} \overline{C}^{jJj+\Delta}_{m-M+1,M,m+1}, \end{aligned} \tag{5.6.37}$$

$$\begin{aligned} &-[(J+M)(J-M+1)]^{\frac{1}{2}} \overline{C}^{jJj+\Delta}_{m-M,M-1,m-1} \\ &\quad = [(j+m+\Delta)(m-j-\Delta-1)]^{\frac{1}{2}} \overline{C}^{jJj+\Delta}_{m-M,M,m} \\ &\qquad - [(j+m-M)(m-j-M-1)]^{\frac{1}{2}} \overline{C}^{jJj+\Delta}_{m-M-1,M,m-1}, \end{aligned} \tag{5.6.38}$$

where $J=0,\frac{1}{2},\ldots$ and $\Delta, M=J, J-1,\ldots,-J.$

$$(J+M)^{\frac{1}{2}}\bar{C}^{jJj+\Delta}_{m-M,M,m}$$
$$=\left[\frac{(J+\Delta)(2j+J+\Delta+1)(j+\Delta+m)}{(2j+2\Delta+1)(2j+2\Delta)}\right]^{\frac{1}{2}}\bar{C}^{jJ-\frac{1}{2}j+\Delta-\frac{1}{2}}_{m-M,M-\frac{1}{2},m-\frac{1}{2}}\cdot$$
$$-\left[\frac{(J-\Delta)(2j-J+\Delta+1)(m-j-\Delta-1)}{(2j+2\Delta+1)(2j+2\Delta+2)}\right]^{\frac{1}{2}}\bar{C}^{jJ-\frac{1}{2}j+\Delta+\frac{1}{2}}_{m-M,M-\frac{1}{2},m-\frac{1}{2}}\cdot \tag{5.6.39}$$

The last relation is valid for $J=\frac{1}{2},1,\ldots;\ \Delta=J, J-1,\ldots,-J;\ M=J, J-1,\ldots,-J+1$. The extremal coefficient corresponding to $M=-J$ is given by

$$\bar{C}^{jJj+\Delta}_{m+J,-J,m}=(-1)^{J+\Delta}\left[\frac{(m-j-\Delta)_{J+\Delta}(j+m+\Delta+1)_{J-\Delta}}{(j+\Delta\|J\|j)}\right]^{\frac{1}{2}}. \tag{5.6.40}$$

Symmetry relation:

$$\bar{C}^{jJj+\Delta}_{m,M,m+M}=\left[\frac{2j+2\Delta+1}{2j+1}\right]^{\frac{1}{2}}\bar{C}^{j+\Delta Jj}_{m+M,-M,m}. \tag{5.6.41}$$

Table of $J=\frac{1}{2}$ coefficients:

$$\bar{C}^{j\frac{1}{2}j+\Delta}_{m-M,M,m}$$

$\Delta\backslash M$	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	$\left[\frac{j+m+\frac{1}{2}}{2j+1}\right]^{\frac{1}{2}}$	$-\left[\frac{m-j-\frac{1}{2}}{2j+1}\right]^{\frac{1}{2}}$
$-\frac{1}{2}$	$-\left[\frac{m-j-\frac{1}{2}}{2j+1}\right]^{\frac{1}{2}}$	$\left[\frac{j+m+\frac{1}{2}}{2j+1}\right]^{\frac{1}{2}}$

(5.6.42)

Remarks. (*a*) The generalized Wigner coefficients were defined for all $2j\neq -J-\Delta, -J-\Delta+1,\ldots,-2\Delta-1,-2\Delta+1,\ldots,J+\Delta$ [see Eq. (3.47), Chapter 3]. Observe from these results that for $m=j+1, j+2,\ldots$ the coefficients (5.6.33) may be written out directly from the tables of standard Wigner coefficients by reversing the sign of all factors under the square root (this is the NPCF) of the form $j-m+a$—that is, by replacing each such factor by $m-j-a$.

(b) It follows from Remark (a) that the coefficients (5.6.33) satisfy orthogonality relations similar to those for the standard coefficients [see Eqs. (2.34) and (2.36)], the required modification being to include a phase factor to account for the reversal of signs carried out in (a). The phase factor in Eq. (5.6.34) may be found by determining the number of common factors of the form $j-m+a$ that occur in the NPCF for each of the Wigner operators in the product [see Eq. (2.33)]:

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \left\langle \begin{matrix} & J+\Delta' & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^{\dagger}.$$

The number of such common factors is the number ρ of opposing arrows going between point 1 of the top row and point 1 of the bottom row in the arrow patterns for the shift patterns

$$\begin{bmatrix} J+\Delta & & J-\Delta \\ & J+M & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} J-\Delta' & & J+\Delta' \\ & J-M & \end{bmatrix}.$$

By direct verification of all cases (six in all), one may show that $\rho=u_1+u_1'+(J+M)-(J+\Delta)-\max(0,\Delta'-\Delta)$. Inserting the factor $(-1)^{\rho}$ in the standard orthogonality relation, Eq. (2.34), then gives the result, Eq. (5.6.34). The second orthogonality relation, Eq. (5.6.35), is implied by the first one.

(c) Recursion relation (5.6.39) may be iterated to obtain all coefficients except those corresponding to $M=-J$. The proof of Eq. (5.6.39) is obtained by making the reversal of signs described in (a) above in a recursion relation for ordinary Wigner coefficients given in Ref. [12]. Relations (5.6.37) and (5.6.38) are similarly obtained.

(d) Symmetry relation (5.6.41) is proved directly from the explicit form of the generalized coefficients.

(e) The orthogonality relation (5.6.35) together with the symmetry relation (5.6.41) may be used to derive Eq. (5.6.30) directly from Eq. (5.6.28).

(f) The properties of the set of generalized Wigner coefficients given above are incomplete. In particular, the analogs of Eqs. (2.60), (2.62), and (2.64), involving Racah coefficients, have not been developed.

6. Interpretation of Integrals in Terms of the Lie Algebra of $SU(1,1)$

The lack of unique expansions for x^J and x^{-J} in terms of unit tensor operators [Δ is arbitrary in Eq. (5.6.28) and M is arbitrary in Eq. (5.6.30)] precludes any consistent interpretation of powers of x in terms of tensor operators in the usual sense. Nonetheless, it is possible to utilize the Lie algebra (5.6.4) and tensor operator theory to derive the special case $\Delta=0$ of

Eq. (5.6.28) (hence, J is integral). This special result is a consequence of the relation (see Ref. [6])

$$x=2(H_3-H_1) \tag{5.6.43}$$

[see Eqs. (5.6.2)]. It follows then from the Lie algebra (5.6.4) that the commutators

$$\left[H_{\pm}, x^k\right] \text{ and } \left[H_3, x^k\right] \tag{5.6.44}$$

will be (finite) polynomial forms defined on the components (H_+, H_3, H_{-1}) of $\mathbf{H}$.

Noting next that the operators (J_1, J_2, J_3) defined by

$$J_1=iH_1, \qquad J_2=iH_2, \qquad J_3=H_3 \tag{5.6.45}$$

satisfy standard angular momentum commutation relations, it follows that it must be possible to express x^k as a linear combination of the tensor operators

$$\mathcal{T}_\mu^{\,k}(\mathbf{J}), \qquad \mu=k, k-1,\ldots,-k \tag{5.6.46}$$

introduced in Note 9, Chapter 3, AMQP, and tabulated in Table A1 of the Appendix of Tables. [Observe that the Hermitian conjugate relation $J_+^\dagger=J_-$ (invalid now) is inessential in the definition of the operators (5.6.46)]. We thus seek the coefficients in the expansion

$$(2i)^k(J_1-iJ_3)^k=\sum_\mu A_\mu^k \mathcal{T}_\mu^{\,k}(\mathbf{J}). \tag{5.6.47}$$

Using Eq. (3.45) of Chapter 3 in AMQP to express J_3 as the rotation of J_2,

$$J_3=\mathfrak{U}\left(\frac{\pi}{2},\hat{e}_1\right)J_2\,\mathfrak{U}\left(\frac{\pi}{2},-\hat{e}_1\right),$$

we may write the left-hand side of Eq. (5.6.47) as (see Table A1)

$$(2i)^k\left[\frac{k!k!}{(2k)!}\right]^{\frac{1}{2}}\mathfrak{U}\left(\frac{\pi}{2},\hat{e}_1\right)\mathcal{T}_{-k}^{\,k}\,\mathfrak{U}\left(\frac{\pi}{2},-\hat{e}_1\right),$$

thus yielding

$$A_\mu^k = (2i)^k \left[\frac{k!k!}{(2k)!} \right]^{\frac{1}{2}} D_{\mu,-k}^k\left(\frac{\pi}{2}, \hat{e}_1 \right)$$

$$= (-i)^\mu \left[\binom{2k}{k+\mu} \Big/ \binom{2k}{k} \right]^{\frac{1}{2}} \tag{5.6.48}$$

[see Eq. (3.79), Chapter 3, AMQP].

Restoring the operators **H** in Eq. (5.6.47), we now find the desired expansion:

$$x^k = \sum_\mu (-1)^u \left[\binom{2k}{k+\mu} \Big/ \binom{2k}{k} \right]^{\frac{1}{2}} \mathcal{T}_\mu^k(\mathbf{H}), \tag{5.6.49}$$

where $\mathcal{T}_\mu^k(\mathbf{H})$ is obtained from $\mathcal{T}_\mu^k(\mathbf{J})$ by replacing J_+ by H_+, J_- by H_-, and J_3 by H_3 (see Table A1) *without sign changes*, and $u = \max(0, -\mu)$. Taking matrix elements of Eq. (5.6.49) and accounting for the reduced matrix element

$$\langle j \| \mathcal{T}^k(\mathbf{H}) \| j \rangle = \left[[2j]_k (2j+2)_k \right]^{\frac{1}{2}}, \tag{5.6.50}$$

we obtain Eq. (5.6.28) for the special case $\Delta = 0$.

We have thus shown that Eq. (5.6.28) is implied by the Lie algebra of $SU(1,1)$ in a special case, but we have found no corresponding interpretation for the general results given by Eqs. (5.6.28) and (5.6.30).

7. Interpretation of Integrals in Terms of Tensor Operators

Armstrong [1,2] has derived results equivalent to Eqs. (5.6.20) and (5.6.31) for the special case of integral values of j, m, and J. We describe here briefly how this method works (extending the technique to half-integers as well), since it illustrates nicely the concept of tensor operators for a noncompact $[SU(1,1)]$ group.

Using a slight variation of the technique of Miller [8] and Armstrong [1], we introduce the functions $\Psi_{jm}(x, \theta)$ of two variables (x, θ) with $0 \leqslant x < \infty$ and $0 \leqslant \theta < 4\pi$ defined by

$$\Psi_{jm}(x,\theta) = \sqrt{\frac{2j+1}{4\pi}}\, e^{-im\theta} \Phi_{jm}(x) \tag{5.6.51}$$

for $j = 0, \frac{1}{2}, 1, \ldots$ and $m = j+1, j+2, \ldots$.

The motivation for this definition is easily understood when one examines Eqs. (5.6.5) and (5.6.15). In consequence of Eqs. (5.6.5), the action of H_3 on $\Psi_{jm}(x,\theta)$ is $i\partial/\partial\theta$; in consequence of Eq. (5.6.15) and the exponential factor in the definition (5.6.51), one finds that the set of functions $\{\Psi_{jm}: j=0,\frac{1}{2},1,\dots;\, m=j+1, j+2,\dots\}$ is orthogonal with inner product defined by

$$\begin{aligned}(\Psi_{j'm'}|\Psi_{jm}) &\equiv \int_0^{4\pi} d\theta \int_0^{\infty} x^{-1}\,dx\, \Psi^*_{j'm'}(x,\theta)\Psi_{jm}(x,\theta)\\ &= \delta_{m'm}\delta_{j'j}.\end{aligned} \tag{5.6.52}$$

More generally, we have

$$(\Psi_{j'm'}|x^k e^{-i\mu\theta}|\Psi_{jm}) = \delta_{m',m+\mu}[(2j'+1)(2j+1)]^{\frac{1}{2}}(\Phi_{j'm+\mu}, x^{k-1}\Phi_{jm}) \tag{5.6.53}$$

for arbitrary k (for which the integrals exist) and $\mu=\dots,-1,-\frac{1}{2},0,\frac{1}{2},1,\dots$.

Using the fact that the action of H_3 on $\Psi(x,\theta)$ is given by $i\partial/\partial\theta$, we also find from Eqs. (5.6.2) and (5.6.51) that the operators

$$\begin{aligned}K_\pm &= e^{\mp i\theta}\left(i\frac{\partial}{\partial\theta} \pm x\frac{\partial}{\partial x} - \frac{x}{2} \pm \frac{1}{2}\right),\\ K_3 &= i\frac{\partial}{\partial\theta},\end{aligned} \tag{5.6.54}$$

satisfy the hyperbolic commutation relations (5.6.4) and have the same action on the functions $\{\Psi_{jm}\}$ as do $H_\pm$, H_3 on the Φ_{jm} [see Eqs. (5.6.5)].

Let us next rewrite Eqs. (5.6.20) and (5.6.31) in terms of the inner product (5.6.52). Defining T^J_M $(J=0,\frac{1}{2},1,\dots;\, M=J,J-1,\dots,-J)$ by

$$T^J_M = \binom{2J}{J+M}^{-\frac{1}{2}} x^{J+1} e^{-iM\theta}, \tag{5.6.55}$$

we obtain from Eqs. (5.6.20) and (5.6.53) the result

$$(\Psi_{j+\Delta,m'}|T^J_M|\Psi_{jm}) = \delta_{m',m+M}(j+\Delta\|\mathbf{T}^J\|j)\bar{C}^{jJj+\Delta}_{m,M,m+M}, \tag{5.6.56}$$

where

$$(j+\Delta\|\mathbf{T}^J\|j) \equiv [(2j+2\Delta+1)(2j+1)]^{\frac{1}{2}}(j+\Delta\|J\|j). \tag{5.6.57}$$

Similarly, defining S^J_M ($J=0,\frac{1}{2},1,\ldots;\ M=J, J-1,\ldots,-J$) by

$$S^J_M=(-1)^{J+M}\binom{2J}{J+M}^{\frac{1}{2}}x^{-J}e^{-iM\theta}, \tag{5.6.58}$$

we obtain from Eqs. (5.6.31) and (5.6.53) the result

$$(\Psi_{j+\Delta,m'}|S^J_M|\Psi_{jm})=\delta_{m',m+M}(j+\Delta\|\mathbf{S}^J\|j)\bar{C}^{jJj+\Delta}_{m,M,m+M}, \tag{5.6.59}$$

where

$$(j+\Delta\|\mathbf{S}^J\|j)\equiv(-1)^{J+\Delta}/(2j+2\Delta+1)(j+\Delta\|J\|j). \tag{5.6.60}$$

Equations (5.6.56) and (5.6.59) express the fact that T^J_M and S^J_M are tensor operators with respect to the group $SU(1,1)$ (generated by the Lie algebra with basis $K_\pm$, K_3). Indeed, one verifies directly the commutation relations

$$\begin{aligned}
[K_+,T^J_M]&=[(J-M)(J+M+1)]^{\frac{1}{2}}T^J_{M+1},\\
[K_-,T^J_M]&=-[(J+M)(J-M+1)]^{\frac{1}{2}}T^J_{M-1},\\
[K_3,T^J_M]&=MT^J_M,
\end{aligned} \tag{5.6.61}$$

with an identical set of relations when T^J_M is replaced by S^J_M.

If we take matrix elements [basis $\{\Psi_{jm}\}$] of Eqs. (5.6.61) and utilize Eq. (5.6.56), we are led to the recursion relations (5.6.37) and (5.6.38) for the generalized Wigner coefficients. Armstrong [1–3] used this type of procedure (iterating the recursion relations) to establish the Wigner–Eckart theorem for the tensors T^J_M and S^J_M defined on the space spanned by the $\{\Psi_{jm}\}$, thus validating Eqs. (5.6.56) and (5.6.59).

Remark. Chacón *et al.* [7] have discussed (in the context of several physical problems) the close analogy that exists between the transformation from the Schrödinger representation of quantum mechanics to the Heisenberg representation and the transformation of operators

$$K_i=e^{-iH_3\theta}H_ie^{iH_3\theta}, \tag{5.6.62}$$

which results from the transformation of functions

$$\Psi_{jm}(x,\theta)=\sqrt{\frac{2j+1}{4\pi}}\,e^{-iH_3\theta}\Phi_{jm}(x). \tag{5.6.63}$$

8. Radial Integrals for the Harmonic Oscillator and the Coulomb Potentials

The radial functions for the harmonic oscillator and the Coulomb potentials are related to the functions $\{\Phi_{jm}\}$ by a transformation of the form

$$\Phi_{jm}^{(\lambda)}(x)=x^{-\lambda/2}\Phi_{jm}(x), \tag{5.6.64}$$

where λ is an integer or half-integer. If we carry out the transformation of the operators $H_{\pm}, H_3$ given by $H_i^{(\lambda)}\equiv x^{-\lambda/2}H_i x^{\lambda/2}$, then the operators $H_{\pm}^{(\lambda)}, H_3^{(\lambda)}$ also satisfy the hyperbolic commutation relations (5.6.4), and the action of each of these operators on the $\{\Phi_{jm}^{(\lambda)}\}$ is still given by Eqs. (5.6.5). The operators $H_i^{(\lambda)}$ are given in terms of the H_i by

$$\begin{aligned} H_1^{(\lambda)}&=H_1-\lambda\frac{d}{dx}-\frac{\lambda^2}{4x}, \\ H_2^{(\lambda)}&=H_2-i\frac{\lambda}{2}, \\ H_3^{(\lambda)}&=H_3-\lambda\frac{d}{dx}-\frac{\lambda^2}{4x}. \end{aligned} \tag{5.6.65}$$

The group $SU(1,1)$ thus enters into both the harmonic oscillator and Coulomb radial problems, and the Wigner coefficients $\bar{C}^{jJj+\Delta}_{m,M,m+M}$ will occur in the radial integrals. The details of this relationship will depend on the value of λ, the identification of x with the radial distance for the specific problem, and the normalization of the functions given by Eq. (5.6.64). We summarize these details in this section.

Harmonic oscillator. The harmonic oscillator in N-dimensional Euclidean space has the Hamiltonian

$$H=\tfrac{1}{2}(\mathbf{p}^2+\mathbf{x}^2), \tag{5.6.66}$$

where $\mathbf{x}=(x_1, x_2,\ldots, x_N)$ and $\mathbf{p}=(p_1, p_2,\ldots, p_N)$, $p_k=-i\partial/\partial x_k$; that is $\mathbf{p}=-i\nabla$. (We have transformed the physical Hamiltonian to dimensionless coordinates and set $\hbar=1$.) Introducing the generators of rotations (in Euclidean N-space) in the (i, j)-plane given by

$$L_{ij}=-L_{ji}=x_i p_j-x_j p_i, \qquad i<j=1,2,\ldots,N, \tag{5.6.67}$$

and the (Casimir) operator, $\boldsymbol{\Omega}^2\equiv\sum_{i<j=1}^{N}(L_{ij})^2$, one finds that $\boldsymbol{\Omega}^2$ is the rotationally invariant function of $\mathbf{x}^2, \nabla^2$, and $\mathbf{x}\cdot\nabla$ given by

$$\boldsymbol{\Omega}^2=-\mathbf{x}^2\nabla^2+\mathbf{x}\cdot\nabla(\mathbf{x}\cdot\nabla+N-2). \tag{5.6.68}$$

The eigenfunctions of the (positive semidefinite) Hermitian operator $\mathbf{\Omega}^2$ of physical significance for the oscillator are homogeneous harmonic polynomials $\{P_l\}$ of degree l $(l=0,1,2,\ldots)$:

$$\nabla^2 P_l = 0, \qquad (\mathbf{x}\cdot\nabla)P_l = lP_l. \tag{5.6.69}$$

Using this result in Eq. (5.6.68), we find

$$\mathbf{\Omega}^2 P_l = l(l+N-2)P_l. \tag{5.6.70}$$

Putting

$$\psi_l(\mathbf{x}) = R(r)P_l\left(\frac{\mathbf{x}}{r}\right) = r^{-l}R(r)P_l(\mathbf{x}), \tag{5.6.71}$$

and noting that $\mathbf{\Omega}^2$ commutes with any differentiable function of $r=(x_1^2+\cdots+x_N^2)^{\frac{1}{2}}$, whereas $\mathbf{x}\cdot\nabla$ commutes with any differentiable function of $\mathbf{x}/r$ and has the action $r\,dR(r)/dr$ on $R(r)$, we again use Eq. (5.6.68) to obtain the action of ∇^2 on $\psi_l(\mathbf{x})$:

$$\nabla^2\psi_l(\mathbf{x}) = \left[\frac{d^2R(r)}{dr^2} + \frac{N-1}{r}\frac{dR(r)}{dr}\right]P_l\left(\frac{\mathbf{x}}{r}\right) - \frac{l(l+N-2)}{r^2}\psi_l(\mathbf{x}). \tag{5.6.72}$$

On the other hand, using the boson operator representation (see Chapter 5 of AMQP), one finds that the Hamiltonian (5.6.66) takes the form

$$H = \left(\sum_{i=1}^{N} a_i\bar{a}_i\right) + \frac{N}{2} \tag{5.6.73}$$

and has the normalized eigenstates given by

$$H\prod_{i=1}^{N}\frac{(a_i)^{\alpha_i}}{[(\alpha_i)!]^{\frac{1}{2}}}|0\rangle = \left(n+\frac{N}{2}\right)\prod_{i=1}^{N}\frac{(a_i)^{\alpha_i}}{[(\alpha_i)!]^{\frac{1}{2}}}|0\rangle, \tag{5.6.74}$$

where the α_i $(i=1,2,\ldots,N)$ are nonnegative integers such that $\sum_i \alpha_i = n$ $(n=0,1,2\ldots)$.

Using Eq. (5.6.72) and the eigenvalue $[n+(N/2)]$ of H, we now obtain the radial differential equation for the N-dimensional harmonic oscillator:

$$\left[-\frac{d^2}{dr^2} - \frac{(N-1)}{r}\frac{d}{dr} + \frac{l(l+N-2)}{r^2} + r^2\right]R(r) = (2n+N)R(r). \tag{5.6.75}$$

Making the change of variable $x=r^2$ and writing $G(x)=R(\sqrt{x})$, we find that this equation is transformed to

$$H_3^{(\lambda)}G(x)=mG(x), \tag{5.6.76}$$

where $H_3^{(\lambda)}$ is obtained from Eqs. (5.6.65), (5.6.2), and (5.6.3) by putting

$$\begin{gathered} \alpha=l-1+(N/2), \qquad \lambda=(N-2)/2, \qquad m=(2n+N)/4, \\ j=(\alpha-1)/2=(2l+N-4)/4, \\ m-j-1=(n-l)/2. \end{gathered} \tag{5.6.77}$$

For each $l=0,1,2,\ldots$, the allowed values of the principal quantum number n are $n=l$, $l+2$, $l+4,\ldots$, so that for each j of the form $(2l+N-4)/4$ the allowed values of m are $j+1, j+2,\ldots$.

We thus find that the normalized radial eigenfunctions are given by

$$\mathfrak{R}_{nl}(r)=\sqrt{2}\left[x^{-(N-2)/4}\Phi_{jm}(x)\right]_{x=r^2}, \tag{5.6.78}$$

where j and m are expressed in terms of n, l, and N by Eqs. (5.6.77) and the nomalization is such that

$$\int_0^\infty r^{N-1}dr|\mathfrak{R}_{nl}(r)|^2=(\Phi_{jm},\Phi_{jm})=1. \tag{5.6.79}$$

[A phase factor in Eq. (5.6.78) has been chosen such that the operators (5.6.65) with $\lambda=(N-2)/2$ and $\alpha=l-1+(N/2)$ have the standard action (5.6.5) on the functions $x^{-(N-2)/4}\Phi_{jm}(x)$.]

Introducing the notation

$$(n'l'|r^k|nl)\equiv\int_0^\infty r^{N-1}dr\,\mathfrak{R}_{n'l'}(r)r^k\mathfrak{R}_{nl}(r), \tag{5.6.80}$$

we then find the following important relation for the radial integrals of the N-dimensional oscillator:

$$(n'l'|r^k|nl)=(\Phi_{j'm'},x^{k/2}\Phi_{jm}), \tag{5.6.81}$$

for all k for which the integrals exist. In particular, the results expressed by Eqs. (5.6.20), (5.6.25), (5.6.26), (5.6.29), and (5.6.31) are of special importance for the oscillator problem.[1]

[1] The results expressed by Eqs. (5.6.81) and (5.6.11) are, of course, more general than these particular cases, which involve the generalized Wigner coefficients.

Remarks. (*a*) The three physical rotationally invariant operators [invariant under $O(3)$ as well as $O(n)$ transformations] that define the $SU(1,1)$ algebra for the oscillator are

$$\begin{aligned} H_1 &= (\mathbf{p}^2 - \mathbf{x}^2)/4, \\ H_2 &= (\mathbf{x}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{x})/4, \\ H_3 &= (\mathbf{p}^2 + \mathbf{x}^2)/4. \end{aligned} \tag{5.6.82}$$

[The operators $H_i^{(\lambda)}$ $[\lambda=(N-2)/2]$ of Eqs. (5.6.65) are just the "radial part" of these operators transformed by $x=r^2$.] One can, of course, investigate directly, using Lie algebraic methods, the properties of the operators (5.6.82) on the space of oscillator wave functions. This has been carried out in Ref. [6].

(*b*) Many explicit values of radial integrals for harmonic oscillators have been given by Shaffer [13] and by Shaffer and Krohn [14].

(*c*) A detailed analysis of the N-dimensional oscillator wave functions has been carried out in Refs. [11] and [13].

Hydrogen atom. The radial functions for the Coulomb problem are given by (see Chapter 7, Section 4, AMQP)

$$R_{nl}(r) = (-1)^{n-l-1}\frac{2}{n^2}\left[x^{-\frac{1}{2}}\Phi_{ln}(x)\right]_{x=2r/n}, \tag{5.6.83}$$

where $l=0,1,2,\ldots$ and $n=l+1, l+2,\ldots$. [The operators $H_i^{(1)}$ given by Eqs. (5.6.2) and (5.6.65) $(j=l, \lambda=1)$ will then have the standard action (5.6.5) $(m=n)$ on the functions $x^{-\frac{1}{2}}\Phi_{ln}(x)$.] These functions are normalized according to

$$\int_0^\infty r^2\,dr[R_{nl}(r)]^2 = 1. \tag{5.6.84}$$

Introducing the new variable $\xi=(n+n')r/nn'$, one may obtain the following expression for the radial integrals of the Coulomb problem:

$$\begin{aligned} &\int_0^\infty r^{k+2}\,dr R_{n'l'}(r)R_{nl}(r) \\ &= (-1)^{n'-l'-n+l}\frac{2(n'n)^{k+\frac{1}{2}}}{(n'+n)^{k+2}}\int_0^\infty \xi^{k+1}\,d\xi\,\Phi_{l'n'}\left(\frac{2n\xi}{n'+n}\right)\Phi_{ln}\left(\frac{2n'\xi}{n'+n}\right) \\ &= (-1)^{n'-l'-n+l}\frac{2(n'n)^{k+\frac{1}{2}}}{(n'+n)^{k+2}}\sum_{m'=l'+1}^{n'}\sum_{m=l+1}^{n} A_{m',m}^{n'l';nl}(\Phi_{l'm'}, x^{k+1}\Phi_{lm}), \end{aligned} \tag{5.6.85}$$

where

$$A_{m',m}^{n'l';nl} \equiv \frac{(-1)^{n'-m'}(2n')^{m}(2n)^{m'}(n-n')^{n'+n-m'-m}}{(4n'n)^{\frac{1}{2}}(n'+n)^{n'+n-1}}$$

$$\times\left[\binom{n'+l'}{n'-m'}\binom{n'-l'-1}{m'-l'-1}\binom{n+l}{n-m}\binom{n-l-1}{m-l-1}\right]^{\frac{1}{2}}. \quad (5.6.86)$$

The general result, Eq. (5.6.85), has been obtained by using the scaling (dilatation) operation (A.10).

In the case $n'=n$, one has the much simpler result:

$$\int_0^{\infty} r^{k+2}\,dr\,R_{nl'}(r)R_{nl}(r)=(-1)^{l'-l}\frac{n^{k-1}}{2^{k+1}}\left(\Phi_{l'n}, x^{k+1}\Phi_{ln}\right), \quad (5.6.87)$$

where, again, cases of special interest are those corresponding to Eqs. (5.6.20), (5.6.25) (the Pasternack–Sternheimer result), (5.6.26), (5.6.29), and (5.6.31).

Remarks. (*a*) The occurrence of $2r/n$ as the "intrinsic" dimensionless variable in the hydrogen atom problem complicates the Lie algebraic approach to the Coulomb radial integral problem. Although one may attempt to remedy this difficulty by introducing a scaling operator, this procedure is of no help (Armstrong [2]; Cunningham [4]) in understanding the general result (5.6.85) from a group-theoretic approach. See, however, the treatment by Moshinsky *et al.* [6] for the case $l'=l$.

(*b*) The radial hydrogen atom functions $\{R_{nl}: n=l+1, l+2, \ldots\}$ are orthonormal, a result that is not explicit in the form (5.6.85). Thus, although Eq. (5.6.85) is fully explicit, it is not the optimal way of expressing these integrals.

APPENDIX: SUMMARY OF PROPERTIES OF ASSOCIATED LAGUERRE POLYNOMIALS AND FUNCTIONS

Definition of associated Laguerre polynomials:

$$L_k^{\alpha}(x)=\sum_{s=0}^{k}\frac{[\alpha+k]_{k-s}(-x)^s}{(k-s)!s!}, \quad (A.1)$$

α arbitrary, k nonnegative integer.

Functional relations:

$$xL_k^{\alpha}(x)=(2k+\alpha+1)L_k^{\alpha}(x)-(k+\alpha)L_{k-1}^{\alpha}(x)-(k+1)L_{k+1}^{\alpha}(x), \quad (A.2)$$

$$L_k^{\alpha-1}(x)=L_k^\alpha(x)-L_{k-1}^\alpha(x), \tag{A.3}$$

$$xL_k^{\alpha+1}(x)=(k+\alpha+1)L_k^\alpha(x)-(k+1)L_{k+1}^\alpha(x), \tag{A.4}$$

$$\frac{d}{dx}L_k^\alpha(x)=-L_{k-1}^{\alpha+1}(x), \tag{A.5}$$

$$x\frac{d}{dx}L_k^\alpha(x)=kL_k^\alpha(x)-(k+\alpha)L_{k-1}^\alpha(x). \tag{A.6}$$

Generation by exponential operators (King [15]):

$$L_k^\alpha(x)=\frac{(-1)^k}{k!}e^{-\Omega_\alpha}x^k, \tag{A.7}$$

$$\Omega_\alpha\equiv x\frac{d^2}{dx^2}+(\alpha+1)\frac{d}{dx}. \tag{A.8}$$

Orthogonality:

$$\int_0^\infty dx\,x^\alpha e^{-x}L_k^\alpha(x)L_{k'}^\alpha(x)=\delta_{kk'}\frac{\Gamma(k+\alpha+1)}{k!}, \tag{A.9}$$

[Re $\alpha>-1$].

Scaling property:

$$L_k^\alpha(\lambda x)=\sum_{s=0}^{k}\frac{[\alpha+k]_{k-s}}{(k-s)!}\lambda^s(1-\lambda)^{k-s}L_s^\alpha(x). \tag{A.10}$$

Definition of associated Laguerre functions:

$$\mathfrak{L}_k^\alpha(x)=\left[\frac{k!}{\Gamma(\alpha+k+1)}\right]^{\frac{1}{2}}x^{\alpha/2}e^{-x/2}L_k^\alpha(x), \tag{A.11}$$

[α real; $\alpha>-1$, $k=0,1,\ldots$].

Functional relations:

$$\begin{aligned}x^{\frac{1}{2}}\mathfrak{L}_k^\alpha(x)&=(k+\alpha)^{\frac{1}{2}}\mathfrak{L}_k^{\alpha-1}(x)-(k+1)^{\frac{1}{2}}\mathfrak{L}_{k+1}^{\alpha-1}(x)\\&=(k+\alpha+1)^{\frac{1}{2}}\mathfrak{L}_k^{\alpha+1}(x)-k^{\frac{1}{2}}\mathfrak{L}_{k-1}^{\alpha+1}(x),\end{aligned} \tag{A.12}$$

$$\begin{aligned}x\mathfrak{L}_k^\alpha(x)&=(2k+\alpha+1)\mathfrak{L}_k^\alpha(x)-[k(k+\alpha)]^{\frac{1}{2}}\mathfrak{L}_{k-1}^\alpha(x)\\&\quad-[(k+1)(k+\alpha+1)]^{\frac{1}{2}}\mathfrak{L}_{k+1}^\alpha(x),\end{aligned} \tag{A.13}$$

$$\left(2x\frac{d}{dx}+1\right)\mathfrak{L}_k^\alpha(x)=[(k+1)(k+\alpha+1)]^{\frac{1}{2}}\mathfrak{L}_{k+1}^\alpha(x)-[k(k+\alpha)]^{\frac{1}{2}}\mathfrak{L}_{k-1}^\alpha(x). \tag{A.14}$$

Orthogonality:

$$\int_0^\infty dx\, \mathfrak{L}_k^\alpha(x)\mathfrak{L}_{k'}^\alpha(x) = \delta_{kk'}. \tag{A.15}$$

Differential equation:

$$\left(-\Lambda_\alpha^2 + \frac{2k+\alpha+1}{2x} - \frac{1}{4}\right)\mathfrak{L}_k^\alpha(x) = 0, \tag{A.16}$$

$$\Lambda_\alpha^2 \equiv -\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + \frac{\alpha^2}{4x^2}. \tag{A.17}$$

Integrals:

$$\int_0^\infty x^p e^{-x} L_k^\alpha(x) L_{k'}^{\alpha'}(x)\, dx$$

$$= \Gamma(p+1)\sum_s (-1)^{k+k'+s}\binom{p-\alpha}{k-s}\binom{p-\alpha'}{k'-s}\binom{-p-1}{s}, \tag{A.18}$$

[Re $p > -1$] (Schrödinger [9]).

$$\int_0^\infty dx\, e^{-bx} x^\alpha L_k^\alpha(\lambda x) L_{k'}^\alpha(\mu x)$$

$$= \frac{\Gamma(k+k'+\alpha+1)}{k!k'!}\, \frac{(b-\lambda)^k (b-\mu)^{k'}}{b^{k+k'+\alpha+1}}$$

$$\times {}_2F_1\left(-k, -k'; -k-k'-\alpha; \frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)}\right), \tag{A.19}$$

[Re $\alpha > -1$, Re $b > 0$] (Erdélyi *et al.* [16]).

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TOPIC 7. UNCERTAINTY RELATIONS FOR ANGULAR MOMENTUM[1]

1. The Role of the Uncertainty Relations

Heisenberg's [1] discovery of the uncertainty relations for position and momentum measurements in quantum mechanics played a fundamental role in clarifying the physical content of quantum mechanics (see Refs. [2] and [3]). Using the terminology devised by Dirac, one says that quantum mechanics considers two types of observables: commuting classical-type observables ("c-numbers") and quantal-type observables ("q-numbers"), the latter obeying noncommutative, but associative, algebraic rules. The basic quantal observables for a particle—position **x** and momentum **p**—are postulated in quantum mechanics to satisfy the Heisenberg[2] commutation

[1]We thank Professor Michael Reed and Dr. Michael Nieto for discussions on the content of this Topic.

[2]Alfred Landé has mentioned that in point of fact it was Born, not Heisenberg, who first wrote out these commutation rules in their final form (Heisenberg having found the diagonal elements); but the concept (if not the result) is indisputably Heisenberg's. Hence, we stick to the common usage. The historical context for the uncertainty relation is discussed in the eloquent Heisenberg memorial lecture by Mehra [5a].

rule (Heisenberg [4], Born *et al.* [5]):

$$[p_i, x_j] = -i\hbar\delta_{ij}\mathbb{1}. \tag{5.7.1}$$

The existence of a nonvanishing commutator between two observables implies a *restriction* on the possibility of preparing quantal states in which the two observables simultaneously take definite values. The Heisenberg uncertainty relation for position and momentum measurements,

$$\Delta p_i \Delta x_i \geqslant \tfrac{1}{2}\hbar, \tag{5.7.2}$$

gives a precise statement of this restriction, which asserts an inequality relating the two dispersions, Δp_i and Δx_i, that occur for measurements on a physical system in the state $|\psi\rangle$:

$$\begin{aligned}(\Delta p_i)^2 &\equiv \langle\psi|(p_i - \langle p_i\rangle)^2|\psi\rangle,\\ (\Delta x_i)^2 &\equiv \langle\psi|(x_i - \langle x_i\rangle)^2|\psi\rangle,\end{aligned} \tag{5.7.3}$$

where the expectation value, $\langle A\rangle$, of an observable A for a system in state $|\psi\rangle$ is defined by $\langle A\rangle = \langle\psi|A|\psi\rangle$.[1]

From the Heisenberg uncertainty relation, Eq. (5.7.2), it is clear that a position measurement of great accuracy, having small dispersion $\Delta x \sim \varepsilon$ ($\Delta x = 0$ is both physically and mathematically excluded), necessarily requires a very large uncertainty $\Delta p \sim \hbar/\varepsilon$ for a simultaneous momentum measurement. It follows that in quantum mechanics—in sharp contrast to classical mechanics—there can be no meaning to such a concept as "the path of a particle." Bohr's Principle of Complementarity, formulated at the Solvay Conference[2] of 1927, was an attempt to capture the qualitative implications of the uncertainty relation. This principle asserts that atomic phenomena cannot be described with the completeness demanded by classical mechanics; some of the elements in a classical description (particle versus wave nature—that is, position versus momentum aspects) are actually mutually exclusive, but these complementary elements are all essential in the description of the phenomena. This principle is a basic tenet of the Copenhagen interpretation of quantum mechanics, which is the standard viewpoint of essentially all physicists (see Jammer [6]).

[1] The general definition of the dispersion ΔA of an observable A measured in the state $|\psi\rangle$ is $(\Delta A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle$, where $\langle A\rangle = \langle\psi|A|\psi\rangle$. Note then that, in general, $\langle A\rangle$ is a real number that depends on ψ, and ΔA is a nonnegative real number that depends on ψ. It is customary in the physics literature to suppress this (functional) dependence on the state. However, note that in Eq. (5.7.2) the right-hand side (the minimum value $\hbar/2$) *is* independent of the state $|\psi\rangle$.

[2] This conference marked the beginning of the Einstein–Bohr controversy over quantum mechanics (Robertson [3, pp. 143ff.]).

The uncertainty relations are accordingly one of the essential elements in interpreting and understanding the physical content of quantum mechanics (see Note 1). The purpose of the present Topic is to discuss in a precise mathematical way the uncertainty relations for angular momentum observables. This is clearly of intrinsic interest, but it is a task of particular importance, since the literature contains many confusions and errors. To do this properly it is useful to review first the situation for position–momentum, to which we now turn.

2. Résumé of the Position–Momentum Uncertainty Relation

The essential fact to recognize in any attempt to discuss the uncertainty relation [Eq. (5.7.2)] with precision is that the operators (observables) entering Eqs. (5.7.1) and (5.7.2) are *unbounded*: Both $p=p_i$ and $x=x_i$ have the real line as spectrum. It is a consequence of the Toeplitz–Hellinger theorem[1] that unbounded operators can be defined, at best, only on a dense subset of Hilbert space. This unfortunate fact of life is the source of much difficulty (and suffering) in quantum physics and a chief stock-in-trade of mathematical physics (see Notes 2 and 3).

There is an elegant way to avoid some, but not all, of these difficulties in the case at hand. This is to use Weyl's idea of replacing the unbounded operators p and x by the unitary (and, hence, bounded) operators defined by

$$\begin{aligned} U&=\exp(i\lambda p/\hbar),\\ V&=\exp(i\mu x/\hbar). \end{aligned} \tag{5.7.4}$$

One thereby obtains *the Heisenberg commutation relation in the Weyl form* [8, p. 273]:

$$UV=e^{(i\lambda\mu/\hbar)}VU. \tag{5.7.5}$$

Operators in the Weyl form do not in general suffice for the purposes of physics (U and V, for example, are technically not observables). To justify the (standard) physicist's manipulations to follow, we shall assume that there exists a (dense) domain invariant under p and x, and in the domains of both.

To establish the uncertainty relation, Eq. (5.7.2), let us fix an arbitrary (normalized) Hilbert space vector $|\psi\rangle$ and consider the operators P and Q

[1] The Toeplitz–Hellinger theorem (Reed and Simon [7]) asserts that a symmetric operator [an operator A that satisfies $(A\psi,\psi)=(\psi,A\psi)$] defined on *all* vectors ψ in a Hilbert space is necessarily bounded. The term "Hermitian" is used interchangeably with "symmetric."

defined by[1]

$$P \equiv p - \langle \psi | p | \psi \rangle,$$
$$Q \equiv x - \langle \psi | x | \psi \rangle. \tag{5.7.6}$$

Applying the Schwartz inequality to the vectors $P|\psi\rangle$ and $Q|\psi\rangle$, one finds

$$\langle P\psi | P\psi \rangle \langle Q\psi | Q\psi \rangle \geqslant |\langle P\psi | Q\psi \rangle|^2. \tag{5.7.7}$$

To evaluate the right-hand side in this relation, we note that

$$\begin{aligned} |\langle P\psi | Q\psi \rangle|^2 &= |\langle \psi | PQ | \psi \rangle|^2 \\ &= \tfrac{1}{4} |\langle \psi | [P, Q] | \psi \rangle + \langle \psi | PQ + QP | \psi \rangle|^2 \\ &\geqslant \tfrac{1}{4} |\langle \psi | [P, Q] | \psi \rangle|^2 \geqslant \tfrac{1}{4} \hbar^2, \end{aligned} \tag{5.7.8}$$

since $[P, Q] = -i\hbar\mathbf{1}$.

This result establishes the uncertainty relation[2] of Eq. (5.7.2), since the dispersion $(\Delta x)^2$ is given by $(\Delta x)^2 = \langle Q\psi | Q\psi \rangle$, and, similarly, $(\Delta p)^2 = \langle P\psi | P\psi \rangle$.

It is interesting to pose the question: What is the class of states for which the uncertainty relation actually achieves the minimum? These are the *minimum uncertainty states* that were defined[3] by Schrödinger [10] and that play a major role in quantum optics (Glauber [11], Klauder and Sudarshan [12], Louisell [13]). There are two conditions to be fulfilled: (a) For the Schwartz inequality [Eq. (5.7.7)] to be an *equality* requires that the vector $P|\psi\rangle$ be a multiple of the vector $Q|\psi\rangle$; that is,

$$P|\psi\rangle = \lambda Q |\psi\rangle, \qquad \lambda \in \mathbb{C}; \tag{5.7.9}$$

and (b) for Eq. (5.7.8) to be an equality requires that

$$\langle \psi | (PQ + QP) | \psi \rangle = 0. \tag{5.7.10}$$

Consider Eq. (5.7.10) first. Using $PQ = QP - i\hbar\mathbf{1}$ and then $QP = PQ + i\hbar\mathbf{1}$,

[1] Observe that these operators are functionals of ψ (see footnote 1, p. 308).

[2] This conclusion is inescapable for p and x Hermitian operators defined on a dense subset of a Hilbert space with the assumed domain properties. (The proof we have given is essentially that of von Neumann [9, pp. 230ff.]. It follows that an eigenvector of p or x cannot be a vector in Hilbert space, since such an eigenvector would satisfy $P|\psi\rangle = 0$ or $Q|\psi\rangle = 0$, thus contradicting Eq. (5.7.8). The conclusion is: The commutation relation (5.7.1) is a valid operator identity only when applied to vectors in a common dense invariant domain of Hilbert space; it is clearly invalid when acting on eigenvectors of p or x. (This is discussed further in Section 3.)

[3] Von Neumann [9, p. 237] attributes these states to Heisenberg.

we find that a minimum uncertainty state must satisfy the two conditions expressed by

$$\langle\psi|PQ|\psi\rangle=-i\hbar/2 \qquad \text{and} \qquad \langle\psi|QP|\psi\rangle=i\hbar/2. \quad (5.7.11)$$

Using next the requirement (5.7.9), we find the relations[1]

$$(\Delta p)^2=-i\hbar\lambda/2, \qquad (\Delta x)^2=i\hbar/2\lambda. \quad (5.7.12)$$

Note that we recover from these two relations the minimum uncertainty relation

$$\Delta p\,\Delta x=\hbar/2, \quad (5.7.13)$$

as well as the relation

$$\Delta p/\Delta x=-i\lambda. \quad (5.7.14)$$

Conversely, these two relations imply Eqs. (5.7.12).

Using Eq. (5.7.14) in Eq. (5.7.9), we now obtain the following equation that must be satisfied by a minimum uncertainty state $|\psi\rangle$:

$$\left[(\Delta p)^{-1}P-i(\Delta x)^{-1}Q\right]|\psi\rangle=0. \quad (5.7.15)$$

Using the Schrödinger realization of the operators p and x [$p\to -i\hbar(\partial/\partial x)$, and x is the multiplication operator by x], we can integrate the relation (5.7.15) to obtain the following generic form for every position–momentum minimum uncertainty state, $\psi(x)=\langle x|\psi\rangle$:

$$\boxed{\psi(x)=\left[\pi\hbar(\Delta x/\Delta p)\right]^{-\frac{1}{4}}\exp\left[-\left(\frac{(x-\langle x\rangle)^2}{2\hbar(\Delta x/\Delta p)}+\frac{i}{\hbar}\langle p\rangle x\right)\right].} \quad (5.7.16)$$

At first glance this result is a curious one, since $\langle p\rangle$, $\langle x\rangle$, and $\Delta x/\Delta p$ are properly to be thought of as functionals of ψ itself. On the other hand, the relationship is self-reproducing in the sense that one may set $\langle p\rangle=\hbar b(-\infty<b<\infty)$, $\langle x\rangle=a$, $(-\infty<a<\infty)$, and $\hbar(\Delta x/\Delta p)=\mu$ $(0<\mu<\infty)$ in the right-hand side of Eq. (5.7.16), and then calculate by direct integration the values $\langle p\rangle=b\hbar$, $\langle x\rangle=a$, and $\hbar\Delta x/\Delta p=\mu$, using $\psi(x)=\psi(a,b,\mu;x)$. This result shows that, in fact, $|\psi\rangle$ *is to be interpreted as a three-parameter*

[1] Note then that we must have $\lambda=ig(\psi)$, where $0<g(\psi)<\infty$ for all states ψ in the Hilbert space of states of a physical system (see footnote 1, p. 308).

family of states

$$\psi(a,b,\mu;x)=(\pi\mu)^{-\frac{1}{4}}\exp\left[-\left(\frac{(x-a)^2}{2\mu}+ibx\right)\right], \qquad (5.7.17)$$

where the real parameters a, b, and μ may assume any values such that $-\infty<a<\infty$, $-\infty<b<\infty$, and $0<\mu<\infty$.

Since the minimum uncertainty state (5.7.17) is generic—that is, it refers to no particular (one-dimensional) physical system—it is valid for every such physical system. Each member of this family is a minimum uncertainty state for any physical system and may be realized for that system by suitable (position–momentum) measurements.

The family of minimum uncertainty states (5.7.17) is in one-to-one correspondence with the ground states (lowest energy states) of a *family of harmonic oscillators.* [This statement is to be contrasted with the less carefully phrased one often found in the literature, which (misleadingly) asserts that the states (5.7.17) are "harmonic oscillator states."]

To see more clearly the relationship of the minimum uncertainty states (5.7.17) to the harmonic oscillator, we operate on Eq. (5.7.15) from the left with $(2m)^{-1}[\Delta p)^{-1}P+i(\Delta x)^{-1}Q]$. The result is

$$\left[\frac{1}{2m}(p-\hbar b)^2+\frac{\hbar^2}{2m\mu^2}(x-a)^2\right]|\psi(a,b,\mu)\rangle=\frac{\hbar^2}{2m\mu}|\psi(a,b,\mu)\rangle. \qquad (5.7.18)$$

The operator in the left-hand side of this equation is the Hamiltonian for a particle of mass m in a harmonic oscillator potential centered at $x=a$ and ~~observed from a reference frame moving in the x-direction with momentum~~ $\hbar b$. Moreover, the frequency of the (classical) motion is $\omega=\hbar/m\mu$, so that the energy of the oscillator [right-hand side of Eq. (5.7.18)] is $\hbar\omega/2$. Thus, $|\psi(a,b,\mu)\rangle$ is the ground state of a harmonic oscillator with fixed physical characteristics (as determined by the mass m and the frequency $\omega=\hbar/m\mu$), which has equilibrium position $x=a$ and is observed in a moving reference frame having momentum $\hbar b$.

It is important to note that *each* member of the family of minimum uncertainty states $\{|\psi(a,b,\mu'\rangle:0<\mu'<\infty\}$ is a minimum uncertainty state of the oscillator described above with mass m and frequency $\omega=\hbar/m\mu'$; only *one* member $(\mu'=\mu)$ of the family coincides with the ground state of the oscillator with physical parameters (m,μ). (Coherent states for a *particular* oscillator are a subset of the position–momentum minimum uncertainty states.)

We refer to the literature for further discussion of minimum uncertainty states and coherent states for other physical systems (Nieto and Simmons [14], Perelomov [15]; a recent review is by Santhanam [15a]).

3. Uncertainty Relations for Angular Momentum[1] in Two Dimensions

It is customary (in textbook treatments) to discuss the uncertainty relations for angular momentum by considering in effect only the special case of the subgroup of rotations generated by the orbital rotation operator L_3. Such a restriction limits the discussion essentially to (two-dimensional) planar rotations, a very special case indeed, which (as will be shown below) grossly distorts the actual three-dimensional situation. Since the treatment of rotations in two dimensions is, however, important in its own right, and, moreover, illustrates rather clearly certain typical *technical* difficulties, it is helpful to discuss this case first.

We begin therefore with the orbital rotation operator L_3 defined by

$$L_3 = -i\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right), \tag{5.7.19}$$

and seek to determine the position variable that is conjugate to L_3. Let us define the angle ϕ by

$$\phi = \tan^{-1}(x_2/x_1); \tag{5.7.20}$$

that is, we take ϕ to be the angle of rotation in the $(x_1 - x_2)$-plane measuring the position $\mathbf{x} = x_1\hat{e}_1 + x_2\hat{e}_2$ of a particle initially at the point $(r, 0)$, where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$.

One can verify—at this stage heuristically—that L_3 and ϕ apparently do satisfy the desired canonical commutation rule

$$[L_3, \phi] = -i\mathbf{1}, \tag{5.7.21}$$

since L_3 has the Schrödinger realization

$$L_3 = -i(\partial/\partial\phi). \tag{5.7.22}$$

[1]Angular momentum states which minimize the uncertainty relation $\Delta J_1 \Delta J_2 \geqslant |\langle J_3\rangle|/2$ have also been called "minimum uncertainty states" and discussed in the literature (see Bacry [15b], and references cited there). We are concerned here, however, with states minimizing the uncertainty for an operator and its conjugate observable, as in the prototype Heisenberg relation.

This conclusion agrees intuitively with one's physical understanding of planar rotations, and it agrees with the Dirac quantization prescription whereby the Poisson bracket (PB) is mapped into the quantum mechanical commutator:[1]

$$[L_3,\phi]_{PB}\to(i\hbar)^{-1}[L_3,\phi].$$

The problem one faces in a precise treatment is to *postulate* Eq. (5.7.21) as valid, and then *to determine the exact sense in which the operators L_3 and ϕ—especially the appropriate space on which they act—are to be interpreted.*

That this is not simply a matter of overly fastidious mathematical taste can be seen from the following "fallacy": Taking matrix elements of Eq. (5.7.21) between eigenstates of L_3, one obtains the result $(m'-m)\langle m'|\phi|m\rangle=\delta_{m'm}$, which is absurd, since it implies (for $m=m'$) that $0=1$. This "fallacy" has been known since the beginning of quantum mechanics (Jordan [16]) and is continually being rediscovered (for example, see Perlman and Troup [17]). The resolution of this situation is *not* to deny the commutation rule (as in Ref. [17]), but, as emphasized above, to find the precise conditions for its validity (Kraus [18]).

We begin by observing that L_3 is unbounded, and hence defined only on a dense set of Hilbert space (see Note 2). Next we observe that the most suitable function space for the present problem is the space $AC[0,2\pi)$ of absolutely continuous square-integrable functions on the interval $[0,2\pi)$ (see Note 4). For such a space the fundamental theorem of the calculus is valid, and for each function f in the space there exists a derivative f' almost everywhere. The operator L_3 is then defined as the operator acting on functions $f\in AC[0,2\pi)$ given by

$$L_3\colon f(\phi)\to-i\frac{df(\phi)}{d\phi},\qquad f\in AC[0,2\pi),\qquad f(0)=f(2\pi)=0. \tag{5.7.23}$$

Here the domain of definition for the operator L_3 (denoted $\mathfrak{D}$) is the set of functions defined by $\mathfrak{D}=\{f\in AC[0,2\pi)\colon f(0)=f(2\pi)=0\}$. The domain $\mathfrak{D}$ is dense in $AC[0,2\pi)$, and therefore Eq. (5.7.23) constitutes an adequate definition for L_3.

If we interpret the operator ϕ as the multiplication operator in $AC[0,2\pi)$ defined by

$$\phi\colon f(\phi)\to\phi f(\phi), \tag{5.7.24}$$

[1]Recall that L_3 in Eq. (5.7.19) is in units of $\hbar$, so that these results are dimensionally in accord.

then ϕ is a bounded operator, defined everywhere. In particular, the domain of the product of the two operators, ϕL_3 and $L_3\phi$, coincides with the domain $\mathcal{D}$. *Thus, it follows that the commutator given by Eq.* (5.7.21) *is well-defined on the domain* $\mathcal{D}$, *which is dense in* $AC[0,2\pi)$.

This is quite reassuring, but it is far from the end of the story! There are several problems yet to be discussed:

(a) The operator L_3, although indeed well-defined on $\mathcal{D}$ and Hermitian on $\mathcal{D}$, is *not* self-adjoint. This is important physically, since the spectral theorem (which validates quantum mechanical applications) applies *only* to self-adjoint operators (Reed and Simon [7], Weyl [8], von Neumann [9]).

(b) The domain $\mathcal{D}$ is not physically satisfactory, since *the boundary condition* $[\phi(0)=\phi(2\pi)=0]$ *spoils the rotational symmetry of the physical problem.*

(c) The operator ϕ is also physically unsatisfactory, since it fails to be cyclic at the boundary.

Let us consider the problem of self-adjointness, item (a) above. The operator L_3 is sufficiently simple that one can determine the adjoint[1] L_3^* quite directly without using the complicated analysis developed by von Neumann for the general case.[2] Consider the defining relation for the adjoint operator L_3^*. For $f\in\mathcal{D}(L_3)$ and $g\in\mathcal{D}(L_3^*)$, we have (using an integration by parts)

$$(L_3 f, g)-(f, L_3^* g)=\bar{f}(2\pi)g(2\pi)-\bar{f}(0)g(0)\equiv 0.$$

Since $f\in\mathcal{D}(L_3)$, it is required that $f(0)=f(2\pi)=0$, and hence the right-hand side vanishes *with no conditions on the function* g [aside from $g\in AC(0,2\pi)$]. It is clear that the domain of L_3^* is larger than (and includes) that of L_3; in symbols, $L_3\subset L_3^*$, as required by the general analysis.

In order to construct a self-adjoint extension of L_3, it is evident that one must weaken the boundary conditions on the admissible functions in such a way that the same conditions apply to both $\mathcal{D}(L_3)$ and $\mathcal{D}(L_3^*)$. There is a continuous family of such self-adjoint extensions, which we denote by $L_3^{(\alpha)}$. Then $L_3^{(\alpha)}$ is the operator $-i(d/d\phi)$ acting on the elements ψ of the domain $\mathcal{D}_\alpha$ defined by

$$\mathcal{D}_\alpha=\{\psi\colon \psi\in AC[0,2\pi), \psi(2\pi)=\alpha\psi(0), \alpha\in\mathbf{C}, |\alpha|=1\}. \quad (5.7.25)$$

[1]In this section only (and in related Note 2, p. 346), we follow the custom in mathematics and use the asterisk to denote the adjoint of an operator and the bar to denote complex conjugation.

[2]This discussion is taken from Reed and Simon [7, pp. 141ff]. This same example also appears in von Neumann [9].

Expressed in words: The self-adjoint operator $L_3^{(\alpha)}$ has the domain $\mathfrak{D}_\alpha$, which is defined to be the space of absolutely continuous functions on the interval $[0,2\pi)$ satisfying the boundary condition $\psi(2\pi)=\alpha\psi(0)$, where α is a complex number of modulus 1. We note that $L_3 \subset L_3^{(\alpha)} = L_3^{(\alpha)*} \subset L_3^*$, as required by the general theory.

This result is satisfactory in that it validates the commutation relation, Eq. (5.7.21), not only for a *self-adjoint* operator but also on a *larger* space. The freedom to choose α is also satisfying, since this choice is a matter of physics, and it allows one to choose $\alpha=1$ as *the physical space of cyclic functions*, $f(\phi)=f(\phi+2\pi)$, *as required by rotational symmetry.* [Let us note that the other choices of α correspond each to a group contained in the covering group of the circle (the circle group having $\alpha=1$); this freedom results from the fact that the circle group is infinitely covered.]

This nicely disposes of items (*a*) and (*b*) above, but it *worsens* the problem presented by item (*c*), since the operator ϕ no longer leaves the domain $\mathfrak{D}_\alpha$ invariant! Let us now turn to this problem, the task of finding a satisfactory definition of ϕ. It is not particularly helpful (although it is possible) to define (Nieto [19], Susskind and Glogower [20]) ϕ as a multiplication operator everywhere modulo 2π. Let us rather make use of the suggestion of Jordan [16] and consider, instead of ϕ, the operator $e^{i\phi}$. This disposes at once of the problem, since this operator is everywhere invariant under $\phi \to \phi+2\pi$, and moreover leaves the domains $\mathfrak{D}_\alpha$ invariant. The commutation relation then reads

$$\left[L_3, e^{i\phi}\right] = e^{i\phi}, \tag{5.7.26}$$

where we now understand the operator L_3 to be the self-adjoint operator denoted earlier by $L_3^{(\alpha=1)}$. This commutation relation is valid on the domain $\mathfrak{D}_{\alpha=1}$.

We remark that this resolution of item (*c*) is not without objection, since technically the operator $e^{i\phi}$ is not a physical observable. We may, and shall, replace $e^{i\phi}$ by its physically observable components $\sin\phi$ and $\cos\phi$ where necessary, and regard this final difficulty more as an inconvenience than a genuine flaw.

Before developing the uncertainty relations that follow from Eq. (5.7.26), let us note explicitly the resolution of the "fallacy" with which we began. *The flaw lies in taking matrix elements of Eq.* (5.7.21) *using the eigenfunctions of* $L_3 = L_3^{(\alpha=1)}$, *since the operator* ϕ *takes the eigenfunctions out of the space.* Equivalently, the operator identity expressed by Eq. (5.7.21) is invalid when applied to an eigenfunction of L_3. This resolution of the problem is rather analogous to the resolution of a similar "fallacy" for the Heisenberg commutation relation itself. This latter fallacy results from taking matrix elements between "eigenvectors" of the position operator. The flaw here is that the "eigenvectors" are *improper*, and not vectors in the Hilbert space. Thus, in both cases, the resolution of the "fallacy" is to deny the validity of the commutation relation when acting on eigenvectors.

Let us turn now to the uncertainty relations. It is convenient to consider first the operators $\sin\phi$ and $\cos\phi$ defined by

$$\begin{aligned}\sin\phi &\equiv (e^{i\phi}-e^{-i\phi})/2i,\\ \cos\phi &\equiv (e^{i\phi}+e^{-i\phi})/2,\end{aligned} \tag{5.7.27}$$

for which the commutation relations are

$$\begin{aligned}[L_3,\sin\phi] &= -i\cos\phi,\\ [L_3,\cos\phi] &= i\sin\phi.\end{aligned} \tag{5.7.28}$$

Consider the first commutator in Eqs. (5.7.28). Let us define the operators L and X by

$$\begin{aligned}L &\equiv L_3-\langle L_3\rangle,\\ X &\equiv \sin\phi-\langle\sin\phi\rangle.\end{aligned} \tag{5.7.29}$$

Using $[L, X]=-i\cos\phi$, we may now repeat the procedure given by Eqs. (5.7.7)–(5.7.14). We find that the minimum uncertainty relation

$$\Delta L_3\Delta(\sin\phi)=\tfrac{1}{2}|\langle\cos\phi\rangle| \tag{5.7.30}$$

will hold if there exists a (minimum uncertainty) state $|\psi\rangle$, which satisfies the following conditions:[1]

$$\begin{aligned}L|\psi\rangle &= \lambda X|\psi\rangle,\\ (\Delta L_3)^2 &= -i\lambda\langle\cos\phi\rangle/2,\\ (\Delta(\sin\phi))^2 &= i\langle\cos\phi\rangle/2\lambda.\end{aligned} \tag{5.7.31}$$

Using $L_3=-i\partial/\partial\phi$, we may integrate the first equation in (5.7.31) to obtain the following relation that the minimum uncertainty state $\psi(\phi)=\langle\phi|\psi\rangle$ must satisfy:

$$\psi(\phi)=N^{-\frac{1}{2}}\exp i[\langle L_3\rangle\phi-\lambda\cos\phi-\lambda\langle\sin\phi\rangle\phi], \tag{5.7.32}$$

where N is a normalization factor.

[1]One must be particularly cautious with the (standard) notation used here. For example, $\langle\cos\phi\rangle$ is *not* a function of ϕ, but rather a complex number whose value depends on the state $|\psi\rangle$—namely, $\langle\cos\phi\rangle\equiv\langle\psi|\cos\phi|\psi\rangle=\alpha(\psi)=$ complex number.

The requirement that $\psi(\phi+2\pi)=\psi(\phi)$ forces the relation

$$\exp i2\pi[\langle L_3\rangle-\lambda\langle\sin\phi\rangle]=1;$$

that is,

$$\langle L_3\rangle-\lambda\langle\sin\phi\rangle=m, \tag{5.7.33}$$

where $m=0,\pm 1,\pm 2,\ldots$. Thus, we find a denumerable infinity of minimum uncertainty states of the form

$$\psi_m(\phi)=N^{-\frac{1}{2}}e^{i(m\phi-\lambda\cos\phi)}, \qquad m=0,\pm 1,\pm 2,\ldots, \tag{5.7.34}$$

where we note from Eqs. (5.7.31) that λ is a pure imaginary number of the form $\lambda=i\mu$ with $\mu>0$ (see footnote, p. 311).

Let us digress a moment to give several integrals that are useful for interpreting the results given by Eqs. (5.7.30)–(5.7.34):

$$\begin{aligned} 2\pi I_0(2\mu)&=\int_0^{2\pi}d\phi e^{2\mu\cos\phi}, \\ 2\pi I_1(2\mu)&=\int_0^{2\pi}d\phi e^{2\mu\cos\phi}\cos\phi=2\mu\int_0^{2\pi}d\phi e^{2\mu\cos\phi}\sin^2\phi, \\ 0&=\int_0^{2\pi}d\phi e^{2\mu\cos\phi}\sin\phi. \end{aligned} \tag{5.7.35}$$

In these results, I_n is a modified Bessel function (Watson [21]), and μ is an arbitrary real parameter.

We next introduce $\lambda=i\mu$ explicitly into Eq. (5.7.34) and define

$$\psi_{m\mu}(\phi)=N^{-\frac{1}{2}}e^{im\phi+\mu\cos\phi} \tag{5.7.36}$$

for all $m=0,\pm 1,\pm 2,\ldots$ and for all positive values of μ.

The results given by Eqs. (5.7.35) now allow us to give explicitly the normalization of the states (5.7.36) and the expectation values of $\cos\phi$, $\sin^2\phi$, $\sin\phi$, and L_3 for these states:

$$\begin{aligned} N&=2\pi I_0(2\mu), \\ \langle\cos\phi\rangle&=2\mu\langle\sin^2\phi\rangle=I_1(2\mu)/I_0(2\mu), \\ \langle\sin\phi\rangle&=0, \\ \langle L_3\rangle&=m. \end{aligned} \tag{5.7.37}$$

We conclude: The states in the set $\{|\psi_{m\mu}\rangle: m=0,\pm 1,\pm 2,\ldots;\mu>0\}$ are minimum uncertainty states for the observables L_3 and $\sin\phi$. The dispersions of these observables in the state $|\psi_{m\mu}\rangle$ are given by

$$(\Delta L_3)^2 = \mu\langle\cos\phi\rangle/2,$$
$$(\Delta(\sin\phi))^2 = \langle\sin^2\phi\rangle = \langle\cos\phi\rangle/2\mu \tag{5.7.38}$$

and satisfy the minimum uncertainty relations

$$\Delta L_3 \Delta(\sin\phi) = \tfrac{1}{2}|\langle\cos\phi\rangle| \tag{5.7.39}$$

associated with the commutation relation

$$[L_3, \sin\phi] = -i\cos\phi. \tag{5.7.40}$$

These dispersions also satisfy the relation

$$\Delta L_3/\Delta(\sin\phi) = \mu. \tag{5.7.41}$$

The explicit dependence of ΔL_3 and $\Delta(\sin\phi)$ on μ is determined by Eqs. (5.7.38) and

$$\langle\cos\phi\rangle = I_1(2\mu)/I_0(2\mu). \tag{5.7.42}$$

A similar result to that stated above also holds for the observables L_3 and $\cos\phi$. [Interchange $\cos\phi$ and $\sin\phi$ in all of Eqs. (5.7.36)–(5.7.42), replacing also i by $-i$ in Eq. (5.7.40)].

Remarks. (*a*) The only parameter in the uncertainty relation (5.7.39) is μ. ~~For large μ one may show, using asymptotic properties of the modified~~ Bessel functions, that $\langle\cos\phi\rangle \sim 1 - 1/4\mu$. In this case we see from Eqs. (5.7.38) that ΔL_3 is large and $\Delta(\sin\phi)$ is small. Since $\langle\sin\phi\rangle = 0$ and $\langle\cos\phi\rangle \cong 1$, the uncertainty $\Delta\phi$ in ϕ is small, and we recover the usual uncertainty relation result that ΔL_3 is large when $\Delta\phi$ is small.

(*b*) For the opposite extreme—μ small—we find that $\langle\cos\phi\rangle \sim \mu$, $(\Delta L_3) \sim \mu/\sqrt{2}$, and $\Delta(\sin\phi) \sim 1/\sqrt{2}$. Thus, in the limit $\mu \to 0$, the uncertainty relation (5.7.39) becomes trivial (both sides zero), thereby escaping the dilemma ($\Delta\phi$ undefined) of the incorrect result based naively on Eq. (5.7.21).

Appendix to Section 3. *Quantum mechanics of discrete rotations.* Weyl, in his discussion of quantum kinematics (Ref. [8, pp. 272–276)], recognized that the Heisenberg commutation relations were a particular instance of a much more basic view:[1]

> We have thus found a very natural interpretation of quantum kinematics as described by the commutation rules. *The kinematical structure of a physical system is expressed by an irreducible Abelian group of unitary ray rotations in system space. The real elements of the algebra of this group are the physical quantities of the system; the representation of the abstract group by rotations of system space associates with each such quantity a definite Hermitian form which "represents" it.* If the group is *continuous* this procedure automatically leads to *Heisenberg's* formulation; in particular, we have seen how the *pairs of canonical variables* then result from the requirement of irreducibility, whence the number of parameters in such an irreducible Abelian group must be *even*."

Weyl recognized further that "our general principle allows for the possibility that the Abelian rotation group is entirely discontinuous, or that it may even be a finite group."

In our opinion, the full import of this insight has yet to be obtained. As an illustration, we shall consider the example of discrete rotations (that is, a finite group) in some detail.

Let U and V represent a pair of canonical elements corresponding to the finite group analog to Eq. (5.7.5). Thus, U and V are unitary transformations of a finite-dimensional Hilbert space that satisfy the relation

$$UV = \varepsilon VU,$$

where $\varepsilon = \exp(2\pi i/N)$ is an Nth root of unity for some integer N. Thus, one has

$$U^l V^k = e^{2\pi i k l/N} V^k U^l.$$

(In the applications to angular momentum to be discussed in Section 4, the integer N will be taken to be equal to $2j+1$.)

In his discussion of such discrete Weyl systems, Schwinger[2] [22] found it ~~advantageous to introduce the (bra-vector) orthonormal basis~~

$$\{\langle a^k| : k = 1, 2, \ldots, N\},$$

where

$$\langle a^k|V = \langle a^{k+1}|, \qquad k = 1, 2, \ldots, N,$$

with $\langle a^{N+1}| = \langle a^1|$. Thus, two bra-vectors, $|a^{k'}\rangle$ and $|a^k\rangle$, are equal if $k' \equiv k (\mathrm{mod}\, N)$.

[1] The italics in the quotation are in the original text.

[2] See also the references in [22] to Schwinger's papers in the *Proceedings of the National Academy of Sciences*. We follow Schwinger's notation and presentation of the discrete Weyl systems.

Repetition of the action of the unitary operator V defines a sequence of linearly independent unitary operators, $V, V^2, V^3 \dots$ with the actions given by

$$\langle a^k | V^n = \langle a^{k+n}|, \qquad n=1,2,\dots,$$

until one arrives at

$$\langle a^k | V^N = \langle a^{k+N} | = \langle a^k |, \qquad \text{each } k=1,2,\dots, N.$$

Thus, V^N is the unit operator,

$$V^N = \mathbf{1}.$$

This result shows that the eigenvalues of the operator V are the N distinct complex roots of unity:

$$\{v^k \colon v=e^{2\pi i/N} \text{ and } k=0,\dots, N-1\}.$$

We may now factor the equation $V^N - \mathbf{1} = 0$ in the form

$$[(V/v)-\mathbf{1}] \sum_{k=0}^{N-1} (V/v)^k = 0.$$

This result, in turn, shows that the expression for the projection operator $P(v^k)$ for the eigenspace corresponding to the eigenvalue v^k is

$$P(v^k) = N^{-1} \sum_{l=0}^{N-1} e^{-2\pi i k l/N} V^l.$$

~~Letting $|v^k\rangle$ denote the eigenvector of V corresponding to the eigenvalue~~ v^k, we may also express the projection operator $P(v^k)$ as

$$P(v^k) = |v^k\rangle\langle v^k|.$$

Applying the operator $P(v^k)$ to the basis $\{\langle a^n|\}$, we thus find

$$\begin{aligned}\langle a^n | P(v^k) &= \langle a^n | v^k\rangle\langle v^k|,\\ &= N^{-1} \sum_{l=0}^{N-1} e^{-2\pi i k l/N} \langle a^n | V^l,\\ &= N^{-1} \sum_{l=0}^{N-1} e^{-2\pi i k l/N} \langle a^{n+l}|.\end{aligned}$$

These results determine the transformation coefficients between the two bases, which, with a particular phase choice, read

$$\langle a^k | v^l \rangle = N^{-\frac{1}{2}} e^{2\pi i k l/N},$$
$$\langle v^l | a^k \rangle = N^{-\frac{1}{2}} e^{-2\pi i k l/N}.$$

This is equivalent to expressing the eigen-bras of V in terms of the original basis $\{|\langle a^l|\}$ as given by

$$\langle v^k| = N^{-\frac{1}{2}} \sum_{l=0}^{N-1} \langle a^l| e^{-2\pi i k l/N}.$$

The operator U—in the canonical pair U, V—has the effect of cyclically permuting the basis $\{\langle v^k|\}$, in a way similar to the action of V on the basis $\{\langle a^l|\}$. In particular, one has

$$\langle v^k| U = \langle v^{k-1}|,$$

which, in turn, implies that $U^N = \mathbf{1}$, in accord with

$$U^l V^k = e^{2\pi i k l/N} V^k U^l.$$

The eigenvalues of the operator U are the Nth roots of unity, just as for the operator V. In fact, the vectors in basis $\{\langle a^k|\}$ can be seen to be precisely the eigenvectors of U, so that

$$\langle a^k| = \langle u^k|.$$

The complementary pair of operators, U and V, are the generators of a complete orthonormal operator basis for the set of all operators mapping the space spanned by $\{\langle a^n|\}$ into itself. This operator basis may be taken to be the set of N^2 operators given by

$$X(m,n) = N^{-\frac{1}{2}} U^m V^n, \qquad m, n = 0, 1, \ldots, N-1.$$

The orthonormality and completeness (which may be shown by Schur's lemma) are expressed by the relations

$$\langle X^\dagger(m,n) | X(m',n') \rangle = \delta_{mm'} \delta_{nn'}$$

for $m, m', n, n' = 0, \ldots, N-1$.

Remarks. (*a*) Schwinger has shown that a kind of ergodic theorem is valid for this operator system, in which an average over spectral translations is equated to an average over states.

(*b*) The importance of these discrete Weyl systems lies in the fact that the pair of operators U, V generates a complete operator basis (for any N), and the two operators are maximally incompatible, as expressed by the aspect of complementarity.

(*c*) In the limit where $N \to \infty$, one recovers the Weyl representation theorem for operators of the Hilbert–Schmidt class. Thus, one finds that

$$A = \sum_{m,n} a(m,n) X(m,n),$$

$$\operatorname{tr}(A^{\dagger}A) = \sum_{m,n} |a(m,n)|^2$$

becomes Weyl's result in the limit:

$$A = \iint_{-\infty}^{\infty} d\sigma\, d\tau\, a(\sigma,\tau) e^{i(\sigma q - \tau p)},$$

$$\operatorname{tr}(A^{\dagger}A) = \iint_{-\infty}^{\infty} d\sigma\, d\tau |a(\sigma,\tau)|^2,$$

where p, q denote a canonical pair of momentum and position operators.

(*d*) One can extend this construction to products of Weyl systems defined over the prime numbers and to the infinite prime (thereby obtaining Schwinger's "special canonical group," which is an alternative approach to Dirac's delta function). There would seem to be many interesting generalizations of the Weyl system to, say, idele and adele groups and the like.[1]

Let us conclude by noting that minimum uncertainty states do exist in discrete Weyl systems, although, for reasons of brevity, we shall not give these states here. (It should also be noted that this basic structural insight of Weyl has been extensively developed for continuous systems. A guide to this literature may be found in Grossmann [24a] and Wolf [24b].)

4. Uncertainty Relations for Angular Momentum in Three-Dimensional Space

Introductory remarks. The first question that must be settled before the desired uncertainty relations can be obtained is this: What is the canonical set of variables for the discussion of angular momentum? We have, of

[1]Segal [23] discusses some of these possibilities briefly; Mackey [24, pp. 52–53] also mentions such possible generalizations.

course, no problems as to the angular momentum operators themselves, **J**, and their commutation rules; the question concerns only the conjugate angle operators. There is a standard way to find the desired answer. One considers the corresponding classical problem, specialized to orbital angular momentum, resolves the problem within the algebraic framework of Poisson brackets, and then (following Dirac) maps these answers into the corresponding commutators, *preserving all algebraic relations to the extent possible*. This is the route we shall follow.[1]

Classical discussion using canonical transformations. The question as to what constitutes a canonical set of variables has a complete answer in the Hamilton–Jacobi theory. A set of variables $\{p_i, q_i\}$ will be termed canonical if all the variables in the set satisfy the Poisson bracket relations:

$$\begin{aligned}
&[p_i, p_j]_{PB}=0,\\
&[q_i, q_j]_{PB}=0,\\
&[q_i, p_j]_{PB}=\delta_{ij},
\end{aligned} \tag{5.7.43}$$

where the Poisson bracket is defined by

$$[A, B]_{PB}\equiv\sum_{i=1}^{3}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right). \tag{5.7.44}$$

Here the variables $\{p_i, q_i\}$ are, by postulate, a set of canonical variables (the fundamental set).

The importance of the Hamilton–Jacobi theory of canonical transformations lies in the fact that the Poisson brackets are canonical invariants. The quantal statement of this result is more immediate and is simply that commutators are invariant to unitary transformations (see, for example, Goldstein [26] and Van Vleck [27]).

To apply the standard transformation theory, it suffices to consider a particular Hamiltonian having the angular momentum as a constant of the motion and determine the desired canonical variables. For this purpose, consider the motion of a mass point constrained to move on a spherical surface of radius a. The Hamiltonian is simply

$$H=\mathbf{p}^2/2m, \tag{5.7.45}$$

in which $\mathbf{p}\cdot\hat{e}_r=0$ and $r=a$, or, equivalently, H is expressed in terms of the

[1]This discussion is adapted from the work of Ref. [25].

orbital angular momentum $\mathbf{L}$ as

$$H=\mathbf{L}^2/2ma^2. \tag{5.7.46}$$

Introducing spherical polar coordinates—that is, the canonically conjugate pairs $\{(\theta, p_\theta), (\phi, p_\phi)\}$—the Hamiltonian (5.7.46) takes the form

$$H=\left(\frac{1}{2ma^2}\right)\left(p_\theta^2+\frac{p_\phi^2}{\sin^2\theta}\right). \tag{5.7.47}$$

To determine the canonical transformation $S(\mathbf{q},\mathbf{P})$ to action–angle variables $\{(\alpha_1, P_1),(\alpha_2, P_2)\}$, one uses the Hamilton–Jacobi equation

$$H(\mathbf{q},\partial S/\partial\mathbf{q})=E, \qquad \mathbf{q}=(\theta,\phi), \tag{5.7.48}$$

and assumes S separable. In the standard way, one finds

$$S(\mathbf{q},\mathbf{P})=\frac{J_\phi}{2\pi}\phi+\frac{1}{2\pi}\int_{\theta_0}^{\theta}\left[\left(|J_\phi|+J_\theta\right)^2-\frac{J_\phi^2}{\sin^2\theta}\right]^{\frac{1}{2}}d\theta. \tag{5.8.49}$$

The momenta $\mathbf{P}=(P_1, P_2)=(J_\theta, J_\phi)$ are the action variables, and the $(\alpha_1,\alpha_2)=(\alpha_\theta,\alpha_\phi)=(\partial S/\partial J_\theta, \partial S/\partial J_\phi)$ are the angle variables. (We have followed the customary notation.) A more familiar form results if we note from Eqs. (5.7.46) and (7.5.49) that

$$\begin{aligned}|J_\phi|+J_\theta&=(\mathbf{L}^2)^{\frac{1}{2}},\\ J_\phi&=L_3.\end{aligned} \tag{5.7.50}$$

~~Using $\alpha_i=\partial S/\partial P_i$, one finds the canonically conjugate coordinates.~~

We summarize the results of this analysis: *The canonical set of variables for orbital angular momentum* (in classical mechanics) *are the conjugate pairs given by*

$$\begin{aligned}&(\mathbf{L}^2)^{\frac{1}{2}} \qquad \text{and} \qquad \psi=-\sin^{-1}\left[\cos\theta/\left(1-L_3^2/\mathbf{L}^2\right)^{\frac{1}{2}}\right],\\ &L_3 \qquad \text{and} \qquad \chi=\phi+\sin^{-1}\left[L_3\cot\theta/\left(\mathbf{L}^2-L_3^2\right)^{\frac{1}{2}}\right]\\ &\qquad\qquad\qquad\quad =-\sin^{-1}\left[L_1/\left(\mathbf{L}^2-L_3^2\right)^{\frac{1}{2}}\right].\end{aligned} \tag{5.7.51}$$

These variables necessarily obey the definition of canonical variables and satisfy the Poisson bracket relations of Eq. (5.7.43). That this is indeed correct may be verified directly (although the algebra is somewhat involved).

The angle variables found above are rather complicated in appearance, and it is desirable to interpret them further. To do this, consider the moving frame $(\hat{f}_1, \hat{f}_2, \hat{f}_3)$, whose instantaneous orientation with respect to an inertial frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is given by the Euler angles $(\alpha\beta\gamma)$ as illustrated in Fig. 5.4 (see Chapter 2, Section 6, AMQP, for a detailed discussion of Euler angles). We now take the angular momentum vector $\mathbf{L}$ and the position vector $\mathbf{x}$ to be along the moving frame axes as given by

$$\hat{L} = \mathbf{L}/(\mathbf{L}^2)^{\frac{1}{2}} = \hat{f}_3, \qquad \hat{e}_r = \mathbf{x}/r = -\hat{f}_2. \tag{5.7.52}$$

(See Fig. 5.4.)

Using the explicit direction cosine matrix given by Eq. (2.37), Chapter 2, AMQP (or the spherical geometry from the figure), we may now interpret the significance of the Euler angles $(\alpha\beta\gamma)$ in terms of the angle variables χ and ψ that are conjugate to L_3 and $(\mathbf{L}^2)^{\frac{1}{2}}$:

The angle β is determined by the constants of the motion

$$\cos\beta = L_3/(\mathbf{L}^2)^{\frac{1}{2}};$$

the angle χ is $\alpha - \pi/2$; and the angle ψ is γ.

Proof. The result stated for β is obvious. Since $\sin\chi = -L_1/(\mathbf{L}^2 - L_3^2)^{\frac{1}{2}} = -(\hat{f}_3 \cdot \hat{e}_1)/\sin\beta = -\cos\alpha$, we obtain the result stated for χ. Finally, since $\sin\psi = -\cos\theta/\cos\beta$ and $(\mathbf{x}/r)\cdot\hat{e}_3 = \cos\theta = -\hat{f}_2 \cdot \hat{e}_3 = -\sin\gamma\sin\beta$, we find also the last result, $\psi = \gamma$. ■

Remarks. (*a*) The canonical angle χ conjugate to L_3 is just the precession angle of the orbital plane, or, apart from a constant the azimuthal angle α of the angular momentum $\mathbf{L}$. The canonical angle ψ conjugate to $(\mathbf{L}^2)^{\frac{1}{2}}$ is the

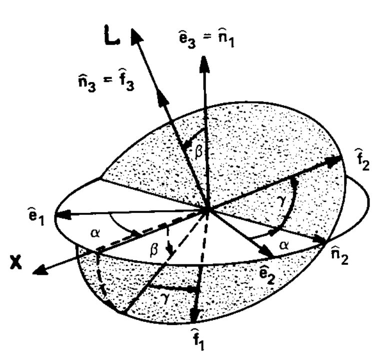

Figure 5.4.

angle $\psi=\gamma$ swept out by the particle in the orbital plane, measured from the line of nodes (the axis $\hat{n}_2$ in Fig. 5.4).

(b) It should be emphasized that the planar angle ϕ associated with the particle position vector $\mathbf{x}/r=(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ is clearly *not* the angle conjugate to the momentum L_3, and the contrary assertion (sometimes found) is clearly a confusion based on the two-dimensional problem. [Indeed, if we restrict $\mathbf{L}\to L_3$ (by going from three to two dimensions), then it is $(\mathbf{L}^2)^{\frac{1}{2}}$ and its conjugate ψ that go into the planar canonical variables.]

In the three-dimensional problem the momentum L_3 is conjugate to the azimuthal angle of the *angular momentum vector* $\mathbf{L}$ itself, not that of the particle vector $\mathbf{x}$.

Heuristic quantal treatment. Let us now consider the quantal transcription of these results. The Dirac formulation of quantization postulates a correspondence between quantal operators (denoted $A, B,\ldots$) and classical variables (denoted also $A, B,\ldots$) such that the quantal commutators $[A, B]$ correspond to the Poisson bracket; that is,

$$(i\hbar)^{-1}[A, B]\leftrightarrow[A, B]_{PB}.$$

Note that there is no guarantee in this correspondence that the algebraic relations within the Poisson bracket structure will necessarily be preserved in the commutator structure; moreover, the form of the quantal operators need not be a literal replacement of the corresponding classical variables. In other words, the Poisson bracket results are to be considered only as strong hints as to the desired quantal formulation. [A good example of the sort of behavior that can occur will be seen below, where not only is there a generalization to operator-valued spherical harmonics with noncommuting angle operators (so that the variables change form), but also the commutator structure itself differs from the Poisson bracket structure.]

To avoid the necessity of working with inverse trigonometric functions, and the problem of cyclic angle variables, we shall introduce the exponentials of the angle variables $Q_1=\psi+\pi/2=\gamma+\pi/2$ and $Q_2=\alpha=\chi+\pi/2$. It is desired then to generalize the classical results given by Eqs. (5.7.51), which we rewrite now as

$$\begin{aligned}
&(1-L_3^2/\mathbf{L}^2)^{\frac{1}{2}}\cos Q_1=\hat{e}_3\cdot\hat{e}_r,\\
&(1-L_3^2/\mathbf{L}^2)^{\frac{1}{2}}\sin Q_1=\hat{e}_3\cdot(\hat{e}_r\times\hat{L})=\left[(\mathbf{L}^2)^{\frac{1}{2}}, \hat{e}_3\cdot\hat{e}_r\right]_{PB},\\
&(\mathbf{L}^2-L_3^2)^{\frac{1}{2}}\cos Q_2=L_1,\\
&(\mathbf{L}^2-L_3^2)^{\frac{1}{2}}\sin Q_2=L_2.
\end{aligned}\tag{5.7.53}$$

Of these four equations, clearly only two are independent.

We take then for the definition of the canonical angles Q_1 and Q_2 the general forms

$$e^{iQ_1}f_1 + e^{-iQ_1}g_1 = \hat{e}_3 \cdot \hat{A} = \hat{A}_3,$$
$$e^{iQ_2}f_2 + e^{-iQ_2}g_2 = L_1, \tag{5.7.54}$$

where we will require that $\hat{A}$ be a (normalized) vector operator with respect to $\mathbf{L}$—that is, $[L_i, \hat{A}_j] = ie_{ijk}\hat{A}_k$.

We seek to determine the operators f_i, g_i, $\hat{A}$, from the properties implied by the canonical equations

$$[P_i, Q_j] = -i\delta_{ij},$$
$$[P_i, P_j] = 0, \tag{5.7.55}$$
$$[Q_i, Q_j] = 0.$$

Note that $P_1 \to (\mathbf{L}^2)^{\frac{1}{2}}$, $P_2 \to L_3$; that is, the P_i only correspond to the classical results and are not necessarily of the same form. Note, moreover, that we are *assuming* the existence of observable (self-adjoint) operators P_1, P_2, Q_1, and Q_2, as well as canonical commutation relations, so that the analysis so far is heuristic ("physical").

The quantal operators L_i are taken to satisfy the relations $[L_i, L_j] = ie_{ijk}L_k$; this corresponds exactly to the Poisson bracket result, and, at the risk of some confusion, we do not distinguish notationally the quantal and classical angular momentum operators.

The operators f_i and g_i are necessarily functions only of P_1 and P_2 in order to satisfy Eqs. (5.7.55) (we take these functions to be analytic—that is, ~~representable by power series). It is easily shown from Eqs. (5.7.55) that~~ $P_i e^{\pm iQ_i} = e^{\pm iQ_i}(P_i \pm 1)$, and from this result that for an analytic function $h(P_1, P_2)$ one has the relations

$$h(P_1, P_2)e^{\pm iQ_1} = e^{\pm iQ_1}h(P_1 \pm 1, P_2),$$
$$h(P_1, P_2)e^{\pm iQ_2} = e^{\pm iQ_2}h(P_1, P_2 \pm 1). \tag{5.7.56}$$

Using this result and the (assumed) Hermitian property of L_1 and $\hat{A}_3$, we find from Eqs. (5.7.54) that the f_i, g_i must satisfy the relations

$$g_1^*(P_1+1, P_2) = f_1(P_1, P_2) \qquad \text{and} \qquad g_2^*(P_1, P_2+1) = f_2(P_1, P_2). \tag{5.7.57}$$

Consider now the following double commutator:

$$\left[e^{iQ_2}f_2+e^{-iQ_2}g_2,\left[P_2,e^{iQ_2}f_2+e^{-iQ_2}g_2\right]\right]=\left[L_1,\left[P_2,L_1\right]\right]. \quad (5.7.58)$$

The left-hand side may be evaluated by using Eqs. (5.7.55) and (5.7.56), with the result

$$f_2(P_1,P_2)g_2(P_1,P_2+1)-f_2(P_1,P_2-1)g_2(P_1,P_2)=\left[L_1,\left[P_2,L_1\right]\right]/2.$$

If we identify P_2 as the angular momentum operator L_3, we thus obtain the finite-difference equation:

$$f_2(P_1,L_3)g_2(P_1,L_3+1)-f_2(P_1,L_3-1)g_2(P_1,L_3)=-(1/2)L_3.$$

The unique polynomial solution to this difference equation is

$$f_2(P_1,L_3-1)g_2(P_1,L_3)=\left[k(P_1)-L_3(L_3-1)\right]/4,$$

where the function $k(P_1)$ does not depend explicitly on L_3.

Using $P_2=L_3$ and the form of L_1 given by Eq. (5.7.54), we obtain

$$L_2=-i\left[L_3,L_1\right]=-i\left(e^{iQ_2}f_2-e^{-iQ_2}g_2\right). \quad (5.7.59)$$

This result when combined with L_1 as given by Eq. (5.7.54) and the properties (5.7.56) now yields the relation

$$\begin{aligned}(1/2)\left(L_1^2+L_2^2\right)&=f_2(P_1,L_3-1)g_2(P_1,L_3)+f_2(P_1,L_3)g_2(P_1,L_3+1)\\&=\left[k(P_1)-L_3^2\right]/2.\end{aligned}$$

~~Since $L_1^2+L_2^2=\mathbf{L}^2-L_3^2$, we conclude that $k(P_1)=\mathbf{L}^2$ and, using Eq. (5.7.57),~~ $g_2^*(P_1,L_3+1)=f_2(P_1,L_3)$, that

$$\begin{aligned}f_2(P_1,L_3)&=(1/2)\left[\mathbf{L}^2-L_3(L_3+1)\right]^{\frac{1}{2}},\\ g_2(P_1,L_3)&=(1/2)\left[\mathbf{L}^2-L_3(L_3-1)\right]^{\frac{1}{2}}.\end{aligned} \quad (5.7.60)$$

The arbitrary phase involved in determining f_2 is absorbed into the definition of the angle Q_2.

The canonical angle Q_2 has thus been determined and is given by the defining equation:

$$e^{iQ_2}\left[\mathbf{L}^2-L_3(L_3+1)\right]^{\frac{1}{2}}+e^{-iQ_2}\left[\mathbf{L}^2-L_3(L_3-1)\right]^{\frac{1}{2}}=2L_1. \quad (5.7.61)$$

By forming the commutator of this relation with L_3, we obtain a similar result for L_2:

$$e^{iQ_2}[\mathbf{L}^2 - L_3(L_3+1)]^{\frac{1}{2}} - e^{-iQ_2}[\mathbf{L}^2 - L_3(L_3-1)]^{\frac{1}{2}} = 2iL_2. \quad (5.7.62)$$

These results may be expressed in a more suggestive form by writing them in terms of the ladder operators $L_{\pm} = L_1 \pm iL_2$:

$$\begin{aligned} L_+ &= e^{iQ_2}[\mathbf{L}^2 - L_3(L_3+1)]^{\frac{1}{2}}, \\ L_3 &= L_3, \\ L_- &= e^{-iQ_2}[\mathbf{L}^2 - L_3(L_3-1)]^{\frac{1}{2}}. \end{aligned} \quad (5.7.63)$$

We shall interpret the result expressed by these equations after we have determined the operators Q_1 and P_1, noting here, however, that the right-hand side defines precisely the vector Wigner operator having $\Delta = 0$.

Let us turn next to the determination of P_1 and Q_1, continuing to work heuristically. Although f_1 and $\hat{A}$ are undetermined in the first of Eqs. (5.7.54) [recall that $g_1(P_1, P_2) = f_1^*(P_1 - 1, P_2)$], the requirement that $\mathbf{A} = (\mathbf{A}^2)^{\frac{1}{2}}\hat{A}$ be a vector operator essentially determines f_1, as we now show.

The algebraic theory of vector operators was developed in detail in Chapter 6, Section 12, AMQP, and in a more general framework in Topic 4 of the present chapter.

An intrinsic property of any vector operator (using the notation of Topic 4) is given by

$$\mathbf{A} \prod_{\delta=-1}^{1} (\Omega - \omega_\delta) = \mathbf{0}. \quad (5.7.64)$$

In this result Ω is the operation of commutation defined by $\mathbf{A}\Omega = [\mathbf{A}, \mathbf{L}^2]$, and the ω_δ are the (operator) eigenvalues given by $\omega_\delta = -\delta(dim + \delta)$, where *dim* is the dimension operator $dim = (4\mathbf{L}^2 + 1)^{\frac{1}{2}}$.

Consider now any Hermitian vector operator with the A_0-component given by

$$A_0 = e^{iQ_1} f_0(P_1, L_3) + e^{-iQ_1} g_0(P_1, L_3).$$

We seek to determine $\mathbf{A}$ itself. Using $\mathbf{L}^2 = k(P_1)$ and the properties (5.7.56), we find

$$A_0(\Omega - \omega_\delta) = e^{iQ_1} f_0'(P_1, L_3) + e^{-iQ_1} g_0'(P_1, L_3),$$

where

$$f_0'(P_1, L_3) = [k(P_1+1) - k(P_1) - \omega_\delta]\, f_1(P_1, L_3),$$
$$g_0'(P_1, L_3) = [k(P_1-1) - k(P_1) - \omega_\delta]\, g_1(P_1, L_3).$$

Thus, applying Eq. (5.7.64), we find, since Eq. (5.7.64) holds for an arbitrary vector operator, that $k(P_1)$ must satisfy the two equations

$$\prod_\delta [k(P_1 \pm 1) - k(P_1) - \omega_\delta] = 0.$$

These two equations determine that

$$k(P_1) = \mathbf{L}^2 = P_1(P_1+1). \tag{5.7.65}$$

Consider next the operator **A** itself. Since **A** is to be a vector operator, its spherical components

$$\begin{aligned} A_{+1} &= -(A_1 + iA_2)/\sqrt{2}, \\ A_0 &= A_3, \\ A_{-1} &= (A_1 - iA_2)/\sqrt{2} \end{aligned} \tag{5.7.66}$$

may be generated from A_0:

$$\begin{aligned} [L_+, A_0] &= \sqrt{2}\, A_{+1}, \\ [L_-, A_0] &= \sqrt{2}\, A_{-1}. \end{aligned} \tag{5.7.67}$$

It is clear from these relations, the explicit forms of $L_\pm$ given by Eqs. (5.7.63), and the multiplication properties (5.7.56) that the form of $A_\mu(\mu = +1, 0, -1)$ is

$$A_\mu = e^{i(Q_1 + \mu Q_2)} f_\mu + e^{-i(Q_1 - \mu Q_2)} g_\mu. \tag{5.7.68}$$

Using the known form (5.7.63) for L_+ together with $\mathbf{L}^2 = P_1(P_1+1)$, the multiplication rules (5.7.56), and $[L_+, A_{+1}] = 0$, one may now derive recursion relations that determine $f_{+1}(P_1, L_3)$ and $g_{+1}(P_1, L_3)$ up to arbitrary operators depending only on P_1. Using these results, in turn, in Eqs. (5.7.67), one then obtains recursion relations which determine f_0 and g_0, and f_{-1} and g_{-1}. We phase all operators such that the standard definition, $[L_i, A_j] = ie_{ijk} A_k$, of a vector operator (with respect to **L**) holds, and we also apply the Hermitian condition $g_0^*(P_1, L_3) = f_0(P_1 - 1, L_3)$ [Eq. (5.7.57)]. The results of this analysis are the following:

$$\begin{aligned} f_{+1} &= a(P_1)[(P_1+L_3+1)(P_1+L_3+2)/2]^{\frac{1}{2}}, \\ f_0 &= a(P_1)[(P_1-L_3+1)(P_1+L_3+1)]^{\frac{1}{2}}, \\ f_{-1} &= a(P_1)[(P_1-L_3+1)(P_1-L_3+2)/2]^{\frac{1}{2}}; \end{aligned} \tag{5.7.69}$$

$$\begin{aligned} g_{+1} &= -a^*(P_1-1)[(P_1-L_3)(P_1-L_3-1)/2]^{\frac{1}{2}}, \\ g_0 &= a^*(P_1-1)[(P_1-L_3)(P_1+L_3)]^{\frac{1}{2}}, \\ g_{-1} &= -a^*(P_1-1)[(P_1+L_3)(P_1+L_3-1)/2]^{\frac{1}{2}}. \end{aligned} \tag{5.7.70}$$

In these results $a(P_1)$ is an arbitrary function of the invariant P_1, and $f_\mu = f_\mu(P_1, L_3)$, $g_\mu = g_\mu(P_1, L_3)$. Moreover, the Hermitian property of $\mathbf{A}$—that is, $A_\mu^\dagger = (-1)^\mu A_{-\mu}$—follows from

$$g_\mu^*(P_1, L_3) = (-1)^\mu f_{-\mu}(P_1 - 1, L_3 + \mu). \tag{5.7.71}$$

We conclude: *Every Hermitian vector operator* $\mathbf{A}$ *with respect to* $\mathbf{L}$ *with the* A_0*-component defined by*

$$A_0 = e^{iQ_1} f_0(P_1, L_3) + e^{-iQ_1} g_0(P_1, L_3) \tag{5.7.72}$$

has the form

$$A_\mu = e^{i(Q_1+\mu Q_2)} f_\mu + e^{-i(Q_1-\mu Q_2)} g_\mu, \tag{5.7.73}$$

where the f_μ *and* g_μ *are given by Eqs.* (5.7.69) *and* (5.7.70). [This result is, of course, subject to the canonical commutation rules (5.7.55) and the Hermiticity of the P_i and Q_i, where we recall also that $P_2 = L_3$ and $\mathbf{L}^2 = P_1(P_1+1)$.]

To put Eq. (5.7.73) in a more familiar form, it is convenient to introduce the standard basis $\{|(\alpha)lm\rangle\}$ for the orbital angular momentum $\mathbf{L}$:

$$L_\mu |(\alpha)lm\rangle = [l(l+1)]^{\frac{1}{2}} C^{l1l}_{m,\mu,m+\mu} |(\alpha)l, m+\mu\rangle. \tag{5.7.74}$$

We then have

$$P_1 |(\alpha)lm\rangle = l |(\alpha)lm\rangle. \tag{5.7.75}$$

Let us now interpret the results given by Eqs. (5.7.63) and (5.7.68)–(5.7.70), first using the Wigner–Eckart theorem for vector operators and the definition of unit tensor operator to write out the nonvanishing matrix elements of the operators L_μ and A_μ. We obtain the following results (the operators H_μ^Δ are defined below):

$$\begin{aligned}
\langle(\alpha')l, m+\mu|L_\mu|(\alpha)lm\rangle &= \delta_{(\alpha')(\alpha)}[l(l+1)]^{\frac{1}{2}}\langle l, m+\mu|e^{i\mu Q_2}H_\mu^{(0)}(P_1, L_3)|lm\rangle \\
&= \delta_{(\alpha')(\alpha)}[l(l+1)]^{\frac{1}{2}}\langle l, m+\mu|\left\langle \begin{matrix} & 1 & \\ 2 & & 0 \\ & 1+\mu & \end{matrix}\right\rangle|lm\rangle \\
&= \delta_{(\alpha')(\alpha)}[l(l+1)]^{\frac{1}{2}}C^{l1l}_{m,\mu,m+\mu};
\end{aligned} \tag{5.7.76}$$

$$\begin{aligned}
\langle(\alpha')l+\Delta, m+\mu|A_\mu|(\alpha)lm\rangle &= \langle(\alpha')l+\Delta\|\mathbf{A}\|(\alpha)l\rangle \\
&\quad\times\langle l+\Delta, m+\mu|e^{i(\Delta Q_1+\mu Q_2)}H_\mu^{(\Delta)}(P_1, L_3)|lm\rangle \\
&= \langle(\alpha')l+\Delta\|\mathbf{A}\|(\alpha)l\rangle \\
&\quad\times\langle l+\Delta, m+\mu|\left\langle \begin{matrix} & 1+\Delta & \\ 2 & & 0 \\ & 1+\mu & \end{matrix}\right\rangle|lm\rangle \\
&= \langle(\alpha')l+\Delta\|\mathbf{A}\|(\alpha)l\rangle C^{l1l+\Delta}_{m,\mu,m+\mu},
\end{aligned} \tag{5.7.77}$$

where the reduced matrix elements are given by

$$\begin{aligned}
\langle(\alpha')l+1\|\mathbf{A}\|(\alpha)l\rangle &= \langle(\alpha')l|a(P_1)|(\alpha)l\rangle, \\
\langle(\alpha')l\|\mathbf{A}\|(\alpha)l\rangle &= 0, \\
\langle(\alpha')l-1\|\mathbf{A}\|(\alpha)l\rangle &= -\langle(\alpha')l-1|a(P_1)|(\alpha)l-1\rangle^*.
\end{aligned} \tag{5.7.78}$$

One may take $\Delta=+1,0,-1$ in Eq. (5.7.77), but observe that one obtains 0 for the $\Delta=0$ matrix element. This is a consequence of $\mathbf{A}\cdot\mathbf{L}=0$, a property that follows directly from Eqs. (5.7.63) and (5.7.73).

Up to phases and factors depending on l—that is, P_1—the operators in Eqs. (5.7.76) and (5.7.77) denoted by $H_\mu^{(\Delta)}(P_1, L_3)$ for $\Delta=+1,0,-1$ are, respectively, the square-root factors in the expressions for f_μ, L_μ, and g_μ. More precisely, one has

$$H_\mu^{(\Delta)}(P_1, L_3)|lm\rangle = C^{l1l+\Delta}_{m,\mu,m+\mu}|lm\rangle. \tag{5.7.79}$$

Observe from the properties

$$\begin{aligned}
L_3e^{i(\Delta Q_1+\mu Q_2)} &= e^{i(\Delta Q_1+\mu Q_2)}(L_3+\mu), \\
P_1e^{i(\Delta Q_1+\mu Q_2)} &= e^{i(\Delta Q_1+\mu Q_2)}(P_1+\Delta)
\end{aligned} \tag{5.7.80}$$

that it is the exponentiated angle operator that effects the shifts $m\to m+\mu$, $l\to l+\Delta$ in Eqs. (5.7.76) and (5.7.77).

Equations (5.7.76) and (5.7.77) provide us with the desired interpretation of the angle operators Q_1 and Q_2. We have that

$$\left\langle \begin{matrix} & 1+\Delta & \\ 2 & & 0 \\ & 1+\mu & \end{matrix} \right\rangle = e^{i(\Delta Q_1+\mu Q_2)} H_\mu^{(\Delta)}(P_1, P_2), \tag{5.7.81}$$

where

$$2P_1+1=(4\mathbf{L}^2+1)^{\frac{1}{2}}, \qquad P_2=L_3. \tag{5.7.82}$$

We conclude: *The angle operators Q_1 and Q_2 are to be defined in terms of the factorization of the vector Wigner operators into polar form.*[1]

Observe that *this significant result* (obtained by heuristic arguments) *is independent of which vector operator* $\mathbf{A}$ *we employ.* It is an abstract result relating to the concept of a Wigner operator itself. Choosing different vector operators $\mathbf{A}$ simply leads to different physical realizations of the shift operators (proportional to the vector Wigner operators) $\mathbf{A}^{(\delta)}$ constructed explicitly in Chapter 6, Section 12, AMQP. It is, however, the case that there is only one general class of vector operators with respect to $\mathbf{L}$ that satisfies $\mathbf{A}\cdot\mathbf{L}=0$—namely, $\mathbf{A}=(a\mathbf{x}+b\mathbf{p})$, where a and b are invariant operators. For definiteness in the definition of the Q_i, one often chooses $\mathbf{A}=\mathbf{x}/r$. This then leads to the explicit representation of Wigner operators given by the shift operators $R^{(\delta)}$ in Chapter 6, Section 13, AMQP, and to the theory of vector spherical harmonics.

Using the vector spherical harmonics, one may now introduce explicitly a Hilbert space and determine directly whether or not there exist operators P_1, P_2, Q_1, Q_2 that have the properties that have been assumed in our heuristic approach.

An alternative approach, which we develop in the next section, is to utilize the fact (mentioned above) that Q_1 and Q_2 are determined by the polar factorization (5.7.81) of the $\langle 2\ 0\rangle$ Wigner operators, and we may therefore use any convenient realization of these operators acting in a Hilbert space. The boson realization discussed in Chapter 2 is ideally suited to this purpose.

Remark. From Eqs. (5.7.53) in the classical case and (5.7.81) in the quantum case it follows that the respective algebras of the canonically conjugate variables $\{L_3, (\mathbf{L}^2)^{\frac{1}{2}}, Q_1, Q_2\}$ and $\{P_1, P_2, e^{iQ_1}, e^{iQ_2}\}$ are *larger* than the enveloping algebra of the generators of the (classical or quantal)

[1] Polar forms for operators—that is, $\mathcal{O}=UH$ (U a partial isometry and H Hermitian positive definite) are the operator analogs of the complex number decomposition $z=re^{i\phi}$ (Reed and Simon [7, p. 197]).

rotation group. The problem of finding the conjugate angle operators to the rotation operators **L**, and therefore of establishing relevant uncertainty relations, cannot be solved within the Lie algebra of the rotation group. (This point is not made clear in many papers; see, for example, Delbourgo [28].)

Precise treatment using the boson operator realization. We have succeeded in the above subsection in identifying the problem of constructing orbital angular momentum operators and their conjugate angle operators with the problem of writing a vector Wigner operator in polar form. To investigate the existence of this polar form (hence, of the conjugate angle operators), we may use any convenient realization of the Wigner operators acting in an explicit Hilbert space. The boson realization for angular momentum—which has proved so helpful in AMQP and in Chapters 2 and 4 of the present volume—not only allows one to extend the theory to half-integer angular momentum, but also allows a precise formulation of the problem.

Let us recall that the boson operator realization [Eqs. (2.15), Chapter 2] of the angular momentum $\mathbf{J}$ is given by

$$\begin{aligned} J_+ &\equiv J_1 + iJ_2 = a_1\bar{a}_2, \\ J_3 &\equiv (a_1\bar{a}_1 - a_2\bar{a}_2)/2, \\ J_- &\equiv J_1 - iJ_2 = a_2\bar{a}_1. \end{aligned} \tag{5.7.83}$$

A significant feature of this realization is that the operator $N/2 = (a_1\bar{a}_1 + a_2\bar{a}_2)/2$ is a linear operator (commuting with $\mathbf{J}$, above) whose eigenvalues are the angular momenta $j = 0, \frac{1}{2}, 1, \ldots$ [see Eq. (2.24)]. Thus, this operator is to be identified as the canonical momentum P_1, in full accord with the properties established for this operator in the heuristic analysis [Eq. (5.7.75)]. Let us identify then the canonical operators P_1 and P_2 as the operators

$$\begin{aligned} P_1 &= (N_1 + N_2)/2, \\ P_2 &= (N_1 - N_2)/2, \end{aligned} \tag{5.7.84}$$

where N_1 and N_2 are the boson number operators

$$\begin{aligned} N_1 &= a_1\bar{a}_1, \\ N_2 &= a_2\bar{a}_2. \end{aligned} \tag{5.7.85}$$

What are the conjugate angle operators? The desired operators are precisely the operators found in the heuristic analysis above, transcribed to boson operator language. Thus, for a uniform derivation for e^{iQ_2} and e^{iQ_1}, we need to transcribe the Wigner operator expressions[1]

$$\left\langle \begin{matrix} & 1 & \\ 2 & & 0 \\ & 2 & \end{matrix} \right\rangle = e^{iQ_2} H_{+1}^{(0)},$$
$$\left\langle \begin{matrix} & 2 & \\ 2 & & 0 \\ & 1 & \end{matrix} \right\rangle = e^{iQ_1} H_0^{(+1)} \tag{5.7.86}$$

into boson relationships. We find, using Eq. (5.7.79) and the explicit definition of the Wigner coefficients and the operators N, N_1, and N_2, that

$$H_{+1}^{(0)} = -\left[\frac{2N_2(N_1+1)}{N(N+2)}\right]^{\frac{1}{2}},$$
$$H_0^{(+1)} = \left[\frac{2(N_1+1)(N_2+1)}{(N+1)(N+2)}\right]^{\frac{1}{2}}. \tag{5.7.87}$$

On the other hand, using the relationship (2.25) between fundamental Wigner operators and bosons, and Eqs. (3.20) and (3.25) for the relevant vector Wigner operators, we find

$$\left\langle \begin{matrix} & 1 & \\ 2 & & 0 \\ & 2 & \end{matrix} \right\rangle = \sqrt{2} \left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle \left[\frac{N+1}{N}\right]^{\frac{1}{2}}$$
$$= -\sqrt{2}\, a_1 \bar{a}_2 [N(N+2)]^{-\frac{1}{2}}, \tag{5.7.88}$$
$$\left\langle \begin{matrix} & 2 & \\ 2 & & 0 \\ & 1 & \end{matrix} \right\rangle = \sqrt{2} \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle \left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle$$
$$= \sqrt{2}\, a_1 a_2 [(N+1)(N+2)]^{-\frac{1}{2}}.$$

Comparing these results with those obtained by combining Eqs. (5.7.86) and (5.7.87), we obtain the desired realization of e^{iQ_2} and e^{iQ_1} in terms of boson operators:

$$e^{iQ_2} = a_1 \bar{a}_2 [(N_1+1)N_2]^{-\frac{1}{2}},$$
$$e^{iQ_1} = a_1 a_2 [(N_1+1)(N_2+1)]^{-\frac{1}{2}}. \tag{5.7.89}$$

[1]One must pay particular attention to the null space of these Wigner operators in validating the results below.

A similar procedure, using again Eq. (5.7.81), leads to

$$e^{-iQ_2}=\bar{a}_1 a_2[N_1(N_2+1)]^{-\frac{1}{2}},$$
$$e^{-iQ_1}=\bar{a}_1 \bar{a}_2(N_1 N_2)^{-\frac{1}{2}}. \tag{5.7.90}$$

It is important to emphasize that these boson operators for the angle operators $e^{\pm iQ_1}$ and $e^{\pm iQ_2}$ are not arbitrary, or ad hoc, but have been determined by the analysis leading to Eq. (5.7.81).

The angle operators $e^{\pm iQ_1}$ and $e^{\pm iQ_2}$ may be put in a more elegant form in terms of the shift operators[1] S_i and their conjugates $\bar{S}_i$ defined by

$$S_i=a_i(N_i+1)^{-\frac{1}{2}}=(N_i)^{-\frac{1}{2}}a_i,$$
$$\bar{S}_i=\bar{a}_i(N_i)^{-\frac{1}{2}}=(N_i+1)^{-\frac{1}{2}}\bar{a}_i, \qquad i=1,2. \tag{5.7.91}$$

In terms of the basis states $|n_1 n_2)$ defined by

$$|n_1 n_2)=\frac{a_1^{n_1}a_2^{n_2}}{[(n_1)!(n_2)!]^{\frac{1}{2}}}|0\,0), \qquad n_1, n_2=0,1,2,\ldots, \tag{5.7.92}$$

one then has the following actions for the S_i and $\bar{S}_i$:

$$S_1|n_1 n_2)=|n_1+1, n_2), \qquad \bar{S}_1|n_1 n_2)=|n_1-1, n_2),$$
$$S_2|n_1 n_2)=|n_1, n_2+1), \qquad \bar{S}_2|n_1 n_2)=|n_1, n_2-1), \tag{5.7.93}$$

where we note that

$$\bar{S}_1|0, n_2)=\bar{S}_2|n_1, 0)=0, \qquad n_1, n_2=0,1,2,\ldots. \tag{5.7.94}$$

Let us also note that the angular momentum states $|jm\rangle$ are given in terms of the notation (5.7.92) by

$$|jm\rangle=|j+m, j-m), \tag{5.7.95}$$

so that $|0\,0)=|0\,0\rangle$.

The angle operators $e^{\pm iQ_1}$ and $e^{\pm iQ_2}$ are now given in terms of the shift operators S_i and $\bar{S}_i$ by

$$e^{iQ_1}=S_1 S_2, \qquad e^{-iQ_1}=\bar{S}_1\bar{S}_2,$$
$$e^{iQ_2}=S_1\bar{S}_2, \qquad e^{-iQ_2}=\bar{S}_1 S_2. \tag{5.7.96}$$

[1]It is interesting to note that these shift operators were explicitly defined by Dirac [29, p. 125] in the first edition of his book, but were omitted in all subsequent editions.

It is important to note that these angle operators inherit the null space of the associated Wigner operator; that is,

$$e^{iQ_1}|jj\rangle = e^{-iQ_1}|j,-j\rangle = e^{iQ_2}|jj\rangle = e^{-iQ_2}|j,-j\rangle = 0 \quad (5.7.97)$$

for all $j=0,\frac{1}{2},1,\ldots$.

The shift operators satisfy very simple algebraic properties that we now summarize for the purpose of working out the multiplication properties of the P_i and the angle operators:

$$\begin{aligned}
&\bar{S}_1 S_1 = \bar{S}_2 S_2 = \mathbf{1}, \qquad S_1\bar{S}_1 = \mathbf{1}-\Lambda_1, \qquad S_2\bar{S}_2 = \mathbf{1}-\Lambda_2,\\
&[S_1, S_2] = [S_1, \bar{S}_2] = [\bar{S}_1, S_2] = [\bar{S}_1, \bar{S}_2] = 0, \qquad (5.7.98)\\
&[P_1, S_1] = S_1/2, \quad [P_1, S_2] = S_2/2,\\
&[P_1, \bar{S}_1] = -\bar{S}_1/2, \quad [P_1, \bar{S}_2] = -\bar{S}_2/2,\\
&[P_2, S_1] = S_1/2, \quad [P_2, S_2] = -S_2/2,\\
&[P_2, \bar{S}_1] = -\bar{S}_1/2, \quad [P_2, \bar{S}_2] = \bar{S}_2/2.
\end{aligned}$$

In these results Λ_1 and Λ_2 are the (Dirac) projection operators defined by

$$\Lambda_1 = \sum_j |j,-j\rangle\langle j,-j|, \qquad \Lambda_2 = \sum_j |jj\rangle\langle jj|, \qquad (5.7.99)$$

where the summations are to be carried out over all $j=0,\frac{1}{2},1,\ldots$. These projection operators satisfy the rules

$$\begin{aligned}
&(\Lambda_1)^2 = \Lambda_1, \qquad (\Lambda_2)^2 = \Lambda_2,\\
&\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = |0\ 0\rangle\langle 0\ 0|. \qquad (5.7.100)
\end{aligned}$$

The occurrence of the projection operators Λ_1 and Λ_2 in the products $S_1\bar{S}_1$ and $S_2\bar{S}_2$ is a direct consequence of the null spaces of the $\langle 2\ 0\rangle$ Wigner operators used in identifying the angle operators and cannot be avoided.

From these relations, it is now straightforward to show that

$$\left[P_i, e^{\pm iQ_j}\right] = \pm\delta_{ij} e^{\pm iQ_j}. \qquad (5.7.101)$$

Since also $[P_1, P_2]=0$, we have thus validated all the required commutation relations except those for the angle operators themselves.

Using Eqs. (5.7.96) and (5.7.98), we find, for example, that

$$[e^{iQ_1}, e^{iQ_2}] = [S_1 S_2, S_1 \bar{S}_2] = S_1^2 [S_2, \bar{S}_2] = -S_1^2 \Lambda_2.$$

Since $S_1^2 |j\, j\rangle = |j+1,\, j+1\rangle$, we thus obtain

$$[e^{iQ_1}, e^{iQ_2}] = -\sum_j |j+1,\, j+1\rangle\langle j\, j|. \tag{5.7.102}$$

Observe then that e^{iQ_1} *and* e^{iQ_2} *fail to commute because of the existence of the null space of the Wigner operator* $\left\langle \begin{smallmatrix} & 1 & \\ 2 & & 0 \\ & \cdot & \end{smallmatrix} \right\rangle$.

A second difficulty is that *angle operators* e^{iQ_1} *and* e^{iQ_2} *are not unitary and, in consequence, there exist no Hermitian angle operators* Q_1 *and* Q_2. The root cause of this behavior is again the existence of the null spaces of the $\langle 2\ 0\rangle$ Wigner operators. The multiplication properties of the angle operators may be determined from Eqs. (5.7.98) to be

$$\begin{aligned}
&e^{-iQ_1} e^{iQ_1} = 1, \\
&e^{iQ_1} e^{-iQ_1} = (1-\Lambda_1)(1-\Lambda_2) = 1 - \Lambda_1 - \Lambda_2 + |0\ 0\rangle\langle 0\ 0|, \\
&e^{-iQ_2} e^{iQ_2} = (1-\Lambda_2), \qquad e^{iQ_2} e^{-iQ_2} = 1 - \Lambda_1, \\
&e^{iQ_1} e^{iQ_2} = S_1^2 (1-\Lambda_1), \qquad e^{iQ_2} e^{iQ_1} = S_1^2.
\end{aligned} \tag{5.7.103}$$

Let us conclude this subsection by giving the forms the sine and cosine operators take in terms of the nonnormal shift operators S_i and their conjugates (the shift operators are called nonnormal, since $S_i \bar{S}_i \neq \bar{S}_i S_i$):

$$\begin{aligned}
\cos Q_1 &= (S_1 S_2 + \bar{S}_1 \bar{S}_2)/2, \qquad \sin Q_1 = (S_1 S_2 - \bar{S}_1 \bar{S}_2)/2i, \\
\cos Q_2 &= (S_1 \bar{S}_2 + \bar{S}_1 S_2)/2, \qquad \sin Q_2 = (S_1 \bar{S}_2 - \bar{S}_1 S_2)/2i.
\end{aligned} \tag{5.7.104}$$

These operators then satisfy the commutation relations with P_1 and P_2 given by

$$\begin{aligned}
&[P_1, \sin Q_1] = -i \cos Q_1, \qquad [P_1, \cos Q_1] = i \sin Q_1, \\
&[P_2, \sin Q_2] = -i \cos Q_2, \qquad [P_2, \cos Q_2] = i \sin Q_2,
\end{aligned} \tag{5.7.105}$$

where we recall, however, that $[\sin Q_i, \cos Q_i] \neq 0$.

Remarks. (*a*) The commutation relation (5.7.102) *is* acceptable in the classical limit (Poisson bracket), since the right-hand side is of negligible size in the limit (N versus N^2 in weight).

(*b*) Although the end result of this (more precise) analysis is the conclusion that there do not exist observable (self-adjoint) angle operators obeying canonical commutation relations for the quantal rotation group $SU(2)$, this conclusion is not as disastrous as it may appear. The nonnormal angle operators e^{iQ_1} and e^{iQ_2} do indeed exist and—using the observable (self-adjoint) sine and cosine operators—do lead to physically important uncertainty relations, as shown in the next subsection.

(*c*) The fundamental reason for the failure of an observable angle operator Q_1 to exist is the fact that the spectrum of the conjugate operator $2P_1$ (the magnitude $2j$ of the angular momentum) is the set of *nonnegative* integers.[1]

Since the spectrum of the operator $P_2(=J_3)$ is discrete and finite, one would expect (see Appendix to Section 3) that there should exist a *discrete* conjugate angle observable. This is, in fact, possible. The resulting discrete structures have the operator realization given by

$$U=e^{iQ_2},$$
$$V=\exp\left(\frac{2\pi iP_2}{2P_1+1}\right).$$

These operators then satisfy the (discrete) Weyl commutation relation:

$$UV=e^{2\pi i/(2P_1+1)}VU.$$

The disadvantage of this operator structure (versus e^{iQ_2} and P_2) is that V does not commute with e^{iQ_1}.

(*d*) The use of nonnormal operators is (as yet) unusual in physics (Lévy-Leblond [30]). It is worth remarking that the analysis of shift operators (such as S_1 and S_2) has received a definitive analysis under the concept of invariant subspaces (Helson [31]). It is particularly worth noting that the eigenspaces of the (downward) shift operators $\bar{S}_i$ have been fully classified in terms of inner functions, Blaschke functions, and singular functions (Helson [31]), a point of importance for physical applications.

(*e*) We can now appreciate the reason for asserting that the use of two-dimensional rotations (for the analysis of uncertainty relations) is a gross distortion of the actual situation. The confusion as to the meaning of the angle ϕ (discussed earlier) is minor compared with the qualitative distinction over the lack of an observable conjugate angle operator. Similarly, the topology of the circle (possessing an infinite covering group) is

[1] This situation is the analog in angular momentum theory to the lack of an observable time operator in quantum mechanics (which results from the fact that the energy spectrum has a lower bound).

qualitatively different from the compactness of the quantal rotation covering group.

(*f*) Since the two-boson structure—having as conjugate pairs of operators N_i and S_i, $i=1,2$—satisfies a set of six canonical commutation relations in the Jordan form,

$$[N_i, S_j]=\delta_{ij}S_j, \qquad [N_1, N_2]=[S_1, S_2]=0, \tag{5.7.106}$$

one might wonder as to why the innocuous-appearing transformation

$$\begin{aligned} P_1&=(N_1+N_2)/2, & P_2&=(N_1-N_2)/2, \\ e^{iQ_1}&=S_1S_2, & e^{iQ_2}&=S_1\bar{S}_2, \end{aligned} \tag{5.7.107}$$

should fail to be canonical. The reason, as might be suspected, lies in the fact that the shift operators S_i are nonnormal in consequence of the existence of the null spaces of the $\langle 2\ 0\rangle$ Wigner operators.

The uncertainty relations for angular momentum. The determination of the uncertainty relations for angular momentum is relatively straightforward (at least in principle) once the proper set of variables—the $\{P_i, Q_i\}$ of the previous section—has been determined. We shall develop the uncertainty relations by constructing explicit minimum uncertainty states (where possible).

Let us first consider minimum uncertainty states in the total angular momentum magnitude P_1 and the conjugate angle Q_1. We have seen that the angle operator Q_1, as well as the (properly cyclic) exponential operator e^{iQ_1}, is not an observable; either of the Hermitian operators $\sin Q_1$ or $\cos Q_1$ is, however, a suitable candidate for an observable involving the conjugate angle Q_1. Note that these two operators do not commute, since they satisfy the commutation rule

$$[\sin Q_1, \cos Q_1]=i(\Lambda_1+\Lambda_2-|0\ 0\rangle\langle 0\ 0|)/2. \tag{5.7.108}$$

(The right-hand side is a projection operator onto the "boundary states" $m=\pm j$ and $j=0,\frac{1}{2},1,\ldots$.)

Let us consider the observable operators $\sin Q_1$ and $\cos Q_1$, which, from Eq. (5.7.105), satisfy the following commutation relations with P_1 and $P_2=J_3$:

$$\begin{aligned} [P_1, \sin Q_1]&=-i\cos Q_1, & [J_3, \sin Q_1]&=0, \\ [P_1, \cos Q_1]&=i\sin Q_1, & [J_3, \cos Q_1]&=0. \end{aligned} \tag{5.7.109}$$

We may use either of the commutation relations of P_1 with $\sin Q_1$ or $\cos Q_1$ to determine minimum uncertainty states. We shall consider here only the lower pair of commutators in (5.7.109).

The action of each of the operators P_l, J_3, $\cos Q_l$, and $\sin Q_l$ on the angular momentum states $|jm\rangle$ is well-defined and, indeed, given explicitly by

$$\begin{aligned} P_l|jm\rangle &= j|jm\rangle, \qquad J_3|jm\rangle = m|jm\rangle, \\ \cos Q_l|jm\rangle &= \tfrac{1}{2}|j+1,m\rangle + \tfrac{1}{2}|j-1,m\rangle, \\ \sin Q_l|jm\rangle &= \frac{1}{2i}|j+1,m\rangle - \frac{1}{2i}|j-1,m\rangle. \end{aligned} \tag{5.7.110}$$

The construction of minimum uncertainty states given in Sections 2 and 3 is therefore applicable. We thus find that a minimum uncertainty state $|\psi\rangle$ for the observables P_l and $\cos Q_l$ must satisfy the relation

$$(P_l + i\gamma\cos Q_l)|\psi\rangle = \lambda|\psi\rangle, \tag{5.7.111}$$

where

$$\begin{aligned} \lambda &= \langle P_l\rangle + i\gamma\langle\cos Q_l\rangle, \\ \gamma &= \Delta P_l/\Delta(\cos Q_l), \\ \Delta P_l\Delta(\cos Q_l) &= \tfrac{1}{2}|\langle\sin Q_l\rangle|. \end{aligned} \tag{5.7.112}$$

In order to determine the minimum uncertainty states $|\psi\rangle$, we expand $|\psi\rangle$ in terms of the angular momentum basis functions $\{|jm\rangle: j=|m|,|m|+1,\ldots\}$ having sharp eigenvalue m of J_3 and use Eqs. (5.7.110) to determine a recursion relation for the expansion coefficients. For simplicity, we choose $\langle\cos Q_l\rangle=0$ and set $\langle P_l\rangle=|m|+k_0$. (Using this form for $\langle P_l\rangle$, one finds that both k_0 and the coefficients in the expansion are independent of m— a result that we anticipate in determining the expansion coefficients below.) We thus put

$$|\psi_m\rangle = \sum_k a_k|k+|m|,m\rangle, \tag{5.7.113}$$

where the summation is over $k=0,1,2,\ldots$, and apply condition (5.7.111) to obtain the following recursion relation for the expansion coefficients:

$$(i\gamma/2)(a_{k-1}+a_{k+1}) = (k_0-k)a_k \tag{5.7.114}$$

with $a_{-1}=0$.

This recursion relation is of the type satisfied by the modified Bessel functions (Watson [21, p. 79], and we find the explicit result for the expansion coefficients in Eq. (5.7.113) (Carruthers and Nieto [32]):

$$a_k = (-i)^k I_{k-k_0}(\gamma). \tag{5.7.115}$$

Let us recall that the modified Bessel function $I_\nu(x)$ is defined for all values of the parameter ν such that $\mathrm{Re}(\nu+\frac{1}{2})>0$ by the integral representation

$$I_\nu(x) = \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi d\theta e^{\pm x\cos\theta} \sin^{2\nu}\theta \tag{5.7.116}$$

and has the ascending series expansion given by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{\nu+2n}}{n!\Gamma(n+\nu+1)}. \tag{5.7.117}$$

Using the integral representation (5.7.116), one may show that

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} \leqslant \frac{\frac{1}{2}x}{\nu+\frac{1}{2}},$$

so that for a large value of the index ν compared to the value of x—that is, $\nu>\frac{1}{2}x$—the ratio of two consecutive terms in the expansion (5.7.117) is strictly less than 1, and by the Weierstrass criterion the series converges for all finite values of the argument.

Since we require that $a_{-1}=0$, one finds that $I_{-1-k_0}(\gamma)=0$; that is, the expectation value k_0 must satisfy

$$2h+1>k_0>2h, \qquad h=0,1,\ldots. \tag{5.7.118}$$

We thus obtain the class of normalized minimum uncertainty states for the observables P_1 and $\cos Q_1$:

$$|\psi_{m\gamma}\rangle = N_{m\gamma}^{-\frac{1}{2}} \sum_k (-i)^k I_{k-k_0}(\gamma)|k+|m|,m\rangle, \tag{5.7.119}$$

where $k_0 \in \{0,1,2,\ldots\}$ is required to satisfy Eq. (5.7.118), and

$$N_{m\gamma} = \sum_k \left[I_{k-k_0}(\gamma)\right]^2,$$

$$\gamma = \Delta P_1/\Delta(\cos Q_1). \tag{5.7.120}$$

Each state in the family $\{|\psi_{m\gamma}\rangle : m=0, \pm\frac{1}{2}, \pm 1, \ldots; 0<\gamma<\infty\}$ then gives the minimum uncertainty in the dispersions ΔP_1 and $\Delta(\cos Q_1)$:

$$\Delta P_1 \Delta(\cos Q_1) = \tfrac{1}{2}\langle \sin Q_1 \rangle . \tag{5.7.121}$$

In order to give the proper interpretation to the minimum uncertainty states given by Eq. (5.7.119), one requires the eigenvectors and eigenvalues of the operator $\cos Q_1$. This is easily accomplished by assuming an expansion of the form (5.7.113) and using the middle equation in (5.7.110) to derive a recursion relation for the coefficients. Carrying out this procedure leads to the following results:

$$\begin{aligned} \cos Q_1 |m, \cos q_1\rangle &= \cos q_1 |m, \cos q_1\rangle , \\ |m, \cos q_1\rangle &= \sqrt{\frac{2}{\pi}} \sum_k \sin[(k+1)q_1] |k+|m|, m\rangle . \end{aligned} \tag{5.7.122}$$

The parameter q_1, which specifies the eigenvalue $\cos q_1$ of the operator $\cos Q_1$, may assume any value in the interval $0 \leqslant q_1 \leqslant \pi$, and to each such eigenvalue there corresponds a single eigenvector $|m, \cos q_1\rangle$.

The eigenstates given by Eqs. (5.7.122) are not normalizable and do not properly belong to Hilbert space. One has, instead, the Dirac delta function normalization and completeness relations given by

$$\begin{aligned} \langle m', \cos q_1' | m, \cos q_1\rangle &= \delta_{m'm}\delta(q_1' - q_1), \\ \sum_m \int_0^{\pi} dq_1 |m, \cos q_1\rangle\langle m, \cos q_1| &= \mathbf{1} . \end{aligned} \tag{5.7.123}$$

For completeness, let us summarize the analogous results for the operator $\sin Q_1$:

$$\begin{aligned} \sin Q_1 |m, \sin q_1\rangle &= \sin q_1 |m, \sin q_1\rangle , \\ |m, \sin q_1\rangle &= \sqrt{\frac{2}{\pi}} \sum_k (i)^k \cos\left[(k+1)q_1 + \frac{k\pi}{2}\right] |k+|m|, m\rangle , \end{aligned} \tag{5.7.124}$$

where there is a one-to-one correspondence between eigenvectors and eigenvalues for each value of the parameter q_1 in the interval $(-\pi/2) \leqslant q_1 \leqslant (\pi/2)$. Again, these eigenvectors satisfy the normalization and completeness relations given by

$$\begin{aligned} \langle m', \sin q_1' | m, \sin q_1\rangle &= \delta_{m'm}\delta(q_1' - q_1), \\ \sum_m \int_{-\pi/2}^{\pi/2} dq_1 |m, \sin q_1\rangle\langle m, \sin q_1| &= \mathbf{1} . \end{aligned} \tag{5.7.125}$$

The physical interpretation of these results, Eqs. (5.7.119)–(5.7.123), is much simpler than might appear from our (necessarily complicated) determination of the states themselves. If there is no dispersion in the total angular momentum (that is, j is sharp) then $\langle \sin Q_1 \rangle$ is 0, and the uncertainty relation (5.7.121) becomes the uninformative $0=0$. The dispersion in $\cos Q_1$ can easily be calculated directly. One finds that for states not on the boundary ($m \neq \pm j$) the expectation value $\langle \cos Q_1 \rangle$ is 0, and $\langle \cos^2 Q_1 \rangle = \frac{1}{2}$. (For states on the boundary, $\langle \cos Q_1 \rangle = 0$, but $\langle \cos^2 Q_1 \rangle = \frac{1}{4}$. This peculiar behavior reflects the fact that a genuine angle observable does not exist, but the "angle" is nonetheless as random as possible.)

For states sufficiently far from the boundary, the expectation values $\langle \cos^m Q_1 \sin^n Q_1 \rangle$ are precisely the expected result[1] *for a randomly distributed angle*. ("Sufficiently far" means that m and n are not large enough for the angle operators to shift the state to the boundary.)

Let us next consider states that have a sharp value for the observable operator $\cos Q_1$. For such (nonnormalizable) states the dispersion $\Delta(\cos Q_1)$ vanishes; the expectation value $\langle \cos q_1 | \sin Q_1 | \cos q_1 \rangle$ has a finite value, and hence from the uncertainty relation, Eq. (5.7.121), the dispersion in the angular momentum, ΔP_1, *becomes unlimitedly large.*

This behavior is exactly as would be expected on physical grounds. A maximally sharp "angle" observable implies a maximally unsharp j-value.

The existence of *normalizable* states $|\psi\rangle$ realizing the minimum uncertainty relation (5.7.121) guarantees that states having the "most classical possible behavior" do exist for $\{P_1, Q_1\}$ and go smoothly into classically observable states in the classical limit. These states are, however, technically difficult to work with and have not been investigated in much detail.

In the discussion above, the variables $\{P_2, Q_2\}$ have been essentially ignored, or taken to have convenient values for the analysis. [For example, $P_2 = J_3$ was taken to be sharp in the minimum uncertainty states of Eq. (5.7.119).]

Let us now consider the variables $\{P_2, Q_2\}$. As a first possibility—which is of practical importance—let us take P_1 to be sharp (and, hence, the "angle" Q_1 is as random as possible). Note that P_1 commutes with both P_2 and Q_2, so that sharp P_1 presumably does not affect these variables. However, the value of P_1 does affect the spectrum of $P_2(-j \leq m \leq j)$, so that the spectrum of P_2—for P_1 sharp—is both discrete and finite. Accordingly, the proper observables for $\{P_2, Q_2\}$ are the exponential operators $\{U, V\}$ (p. 319ff.), and these are both normal operators of cyclotomic type. Thus, we deal with discrete angles, and the uncertainty relations are precisely those that occurred in our earlier discussion of the finite Weyl systems (see also Carruthers and Nieto [32]).

[1]The discussion of Ref. [32, Section D], which asserts that deviations always occur (even far from the boundary), is incorrect and based on numerical error.

The final case to consider is that in which neither P_1 nor P_2 is sharp and some sort of "minimum uncertainty state" in the complete set of angular momentum variables $\{P_1, P_2, Q_1, Q_2\}$ is to be obtained. Physically, one expects such states to exist, but, because the angles Q_1 and Q_2 do not commute, it is not clear that such states can really exist, short of the classical limit.

One reasonable approach is to exploit the fact that the minimum uncertainty states $\{|\psi_{m\gamma}\rangle\}$ were determinable uniformly with sharp, but arbitrary, values of m for P_2; accordingly, we may form wave packets over the eigenstates of $P_2 = J_3$ of the form

$$|\phi\rangle \equiv \sum_m \alpha_m |\psi_{m\gamma}\rangle, \tag{5.7.126}$$

and seek to minimize the joint dispersion in $\{P_2, Q_2\}$. In this result the states $|\psi_{m\gamma}\rangle$ are the minimum uncertainty states for P_1 and $\cos Q_1$ given by Eq. (5.7.119).

This approach, although reasonable, cannot be successful in simultaneously defining minimum uncertainty states for $\{P_2, \cos Q_2\}$ and $\{P_1, \cos Q_1\}$, since $\cos Q_2$ and $\cos Q_1$ do not commute. Before considering this aspect of the problem, we need to determine the minimum uncertainty states for the observables $\{P_2, \cos Q_2\}$, taking P_1 sharp.

We seek minimum uncertainty states of the form

$$|\psi_j\rangle = \sum_{m=-j}^{j} a_{jm} |jm\rangle, \tag{5.7.127}$$

which satisfy

$$(J_3 + i\gamma \cos Q_2)|\psi_j\rangle = 0, \tag{5.7.128}$$

where γ is a real positive number, $0 < \gamma < \infty$. (For convenience we choose $\langle J_3\rangle = \langle \cos Q_2\rangle = 0$.) If $|\psi_j\rangle$ exists, it will be a minimum uncertainty state; that is, it will yield dispersions satisfying

$$\begin{aligned} \Delta J_3 \Delta(\cos Q_2) &= \tfrac{1}{2}|\langle \sin Q_2\rangle|, \\ \Delta J_3 / \Delta(\cos Q_2) &= \gamma. \end{aligned} \tag{5.7.129}$$

Using $J_3|jm\rangle = m|jm\rangle$ and $\cos Q_2|jm\rangle = (|jm+1\rangle + |jm-1\rangle)/2$, we find that the coefficients in Eq. (5.7.127) must satisfy the recursion relation

$$\frac{i\gamma}{2} a_{j,m-1} + m a_{jm} + \frac{i\gamma}{2} a_{j,m+1} = 0, \tag{5.7.130}$$

with $a_{j,j+1}=a_{j,-j-1}=0$. Starting with $a_{j,j+1}=0$ and iterating Eq. (5.7.130) downward in m leads directly to the result

$$a_{j,j-k}=\sum_{s}(-1)^{s}\frac{(k-s)!}{s!}\binom{j-s}{k-2s}\left(\frac{\gamma}{2i}\right)^{2j-k+2s}, \qquad (5.7.131)$$

where $k=0,1,\ldots,2j$, and the summation is over $s=0,1,\ldots,k/2$ (k even) or $(k-1)/2$ (k odd). [We have arbitrarily chosen $a_{jj}=(\gamma/2i)^{2j}$.]

We must ascertain that a_{jm} given by Eq. (5.7.131) satisfies $a_{j,-j-1}=0$ before concluding that the desired minimum uncertainty states have been found. Setting $k=2j+1$, we find

$$a_{j,-j-1}=\sum_{s}(-1)^{s}\frac{(2j+1-s)!}{s!}\binom{j-s}{2j-2s+1}\left(\frac{\gamma}{2i}\right)^{2s-1}. \qquad (5.7.132)$$

For integral j, one has

$$a_{j,-j-1}=0, \qquad \text{all } \gamma, \qquad (5.7.133)$$

since the binomial coefficient for each term $s=0,1,\ldots,j$ in the sum vanishes. Moreover, in this case, one may show directly from Eq. (5.7.131) that

$$a_{j,-m}=(-1)^{m}a_{j,m}, \qquad j \text{ integral}. \qquad (5.7.134)$$

For half-integral j, however, no term in the right-hand side of Eq. (5.7.132) vanishes ($s=0,1,\ldots,j+\frac{1}{2}$), and, indeed, $a_{j,-j-1}=0$ is just the condition that the set of $2j+1$ equations (5.7.130) has a nontrivial solution. (This last result is also correct for integral j, but is satisfied for all γ.) Thus, in order that minimum uncertainty states of the type (5.7.127) exist for j half-integral, we find [multiplying Eq. (5.7.132) by $\gamma/2i$] that γ must satisfy ~~the *characteristic equation* of degree $2j+1$ given by~~

$$\sum_{s}\frac{(2j+1-s)!}{s!}\binom{j-s}{2j-2s+1}\left(\frac{\gamma}{2}\right)^{2s}=0. \qquad (5.7.135)$$

For $j=\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$, the characteristic equations for γ are, respectively,

$$\begin{gathered}\gamma^{2}-1=0,\\ \gamma^{4}-3\gamma^{2}+9=0,\\ \gamma^{6}-6\gamma^{4}+45\gamma^{2}-225=0.\end{gathered} \qquad (5.7.136)$$

The first equation has the positive root $\gamma=1$, the second equation has no real roots, and the third equation has a single positive root in the interval

$\sqrt{5}<\gamma<\sqrt{6}$. These results show that, for half-integral j, minimum uncertainty states of the type (5.7.127) exist only for isolated values of j and γ.

For example, the state

$$|\psi\rangle=\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle+i\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)$$

is a minimum uncertainty state for $\{J_3,\cos Q_2\}$ having explicitly

$$\langle J_3\rangle=\langle\cos Q_2\rangle=0$$

and

$$\Delta J_3=\Delta(\cos Q_2)=\langle\sin Q_2\rangle=\tfrac{1}{2}.$$

There are no other minimum uncertainty states (5.7.127) *for a spin-$\frac{1}{2}$ system.*

The results obtained above for half-integral j are surprising and, to a certain extent, disheartening, since they imply that in a minimum uncertainty state (when it exists) a simultaneous measurement of ΔJ_3 and $\Delta(\cos Q_2)$ will yield numbers that are independent of the measuring process! Moreover, since γ assumes only isolated or nonphysical values (that is, γ is not positive and real) for half-integral j, whereas it may assume arbitrary real positive values for integral j, one cannot mix states of integral and half-integral angular momentum in forming the states (5.7.126).

We conclude from the preceding results that only for integral j is it meaningful to formulate the problem of finding minimum uncertainty states simultaneously for $\{P_1,\cos Q_1\}$ and $\{P_2,\cos Q_2\}$. That there are no such solutions may be seen directly from the explicit solutions (5.7.119) and (5.7.127) for the separate problems: Any simultaneous solution $|\phi\rangle$ must have the form

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$$|\phi\rangle=\sum_m \alpha_m|\psi_{m\gamma_1}\rangle=\sum_j c_j|\psi_j\rangle. \tag{5.7.137}$$

This requires a relationship of the following type to hold for all (j,m) with j integral:

$$\beta_m(-i)^{j-|m|}I_{j-j_0}(\gamma_1)=c_j a_{jm}(\gamma_2). \tag{5.7.138}$$

In this result β_m is to be independent of j, and c_j is to be independent of m. Since the coefficients $a_{jm}(\gamma_2)$ do not factor into a product of j-independent and m-independent terms, it is impossible to satisfy Eq. (5.7.138).

We conclude: *There exist no simultaneous minimum uncertainty states for the angular momentum observables $\{P_1,\cos Q_1\}$ and $\{P_2,\cos Q_2\}$.*

The basic reason for the nonexistence of simultaneous minimum uncertainty states for $\{P_1, \cos Q_1\}$ and $\{P_2, \cos Q_2\}$ is the failure of $\cos Q_1$ and $\cos Q_2$ to commute, and indeed one sees that the state $|\phi\rangle$ must also satisfy the condition $[\cos Q_1, \cos Q_2]|\phi\rangle = 0$. This condition may be used to give an alternative proof that the only simultaneous minimum uncertainty state $|\phi\rangle$ is $|\phi\rangle = 0$.

Since our attempt to find states with minimum dispersions in all four operators $\{P_i, Q_i\}$ has failed—the ultimate reason being the lack of truly canonical commutation relations in the $\{P_i, Q_i\}$—it would seem appropriate to return to the variables $\{N_i, \phi_i\}$, which do indeed have canonical commutation rules. The point of view being taken is pragmatic, that we wish to define minimum uncertainty relations for states in a (two-dimensional) lattice space (j, m), and the use of a particular coordinate basis; that is, (N_1, N_2) versus $\left(\frac{N_1+N_2}{2}, \frac{N_1-N_2}{2}\right)$, if canonical, should be exploited.

The desired minimum uncertainty states are now simply the direct product of minimum uncertainty states in each of the pairs $\{N_i, \phi_i\}$. These states should then provide the "best" states for the desired joint dispersions in the physical variables, since in the *classical* limit the transformation *is* canonical.[1] Such states do indeed exist.

Such minimum uncertainty states in the two sets of variables $\{N_i, \phi_i\}$ have not so far been systematically examined, and it would be of considerable interest to carry out this construction, to determine if such states are acceptable approximations to minimum uncertainty states in $\{P_i, Q_i\}$.

5. Notes

1. *Essays on the uncertainty principle.* There has recently appeared a collection of essays (Ref. [33]) devoted to commemorating the fiftieth anniversary of Heisenberg's discovery of the uncertainty principle. In these essays the significance of the uncertainty principle for the foundations of quantum physics is examined in great detail, from many viewpoints.

An unusual application of the uncertainty principle (not discussed in Ref. [33]) has been developed by E. C. G. Stueckelberg [34] and discussed in a series of papers with his collaborators. These authors formulate quantum mechanics in a Hilbert space over the real numbers where all symmetry operators (see Topic 1) are necessarily linear. In order for an uncertainty principle to exist, it is necessary that there exist an operator $\mathcal{J}$, acting in the Hilbert space, that is linear, antisymmetric, and commutes with *all* observa-

[1] Technically there is still a difficulty in this approach, owing to the fact that the states of integer and half-integer angular momentum are inherently mixed, in these product states. Presumably this difficulty can be overcome by doubling the lattice spacings, and considering states on the two sublattices.

bles. In effect, the operator $\mathcal{J}$ corresponds to the imaginary unit i of complex Hilbert space; Stueckelberg's analysis thus shows the basic reason why the imaginary unit enters quantum theory. [A similar analysis as to why Hilbert space (for quantum mechanics) is complex has been given by Mackey [24, pp. 107–109]. Although apparently unaware of Stueckelberg's analysis, the conclusions reached by Mackey are much the same.]

There is a kind of "uncertainty principle" operative in the measurement theory for angular momentum. For a *precise* measurement of the z-component of the angular momentum, the measuring system must itself possess unlimitedly large values of angular momentum. An *approximate* measurement can be accomplished with the measuring system possessing bounded values (the bound increases as the precision increases). This result is a special case of the general result (for additively conserved operators with a discrete spectrum) obtained by Wigner [36] and proved in more detail by Araki and Yanase [37]. (This general result has been referred to in the literature as the "Wigner–Araki–Yanase theorem.")

2. *Unbounded operators*. The analysis of operators that are unbounded, and hence not defined on all of Hilbert space, is a problem requiring considerable mathematical finesse (von Neumann [9], Riesz and Sz-Nagy [35]), which cannot be avoided. The requisite techniques are developed very clearly in the Reed–Simon textbooks [7], in a style accessible to physicists.

For a given unbounded operator T, let us denote its domain by $\mathfrak{D}(T)$ (the set of vectors on which the action of T is defined) and its range by $\mathfrak{R}(T)$ [the set of vectors Tg, $g\in\mathfrak{D}(T)$]. One has then the following distinction between Hermitian (equivalently, symmetric) and self-adjoint operators:

Let T be a (densely) defined linear operator on a Hilbert space $\mathcal{H}$ with domain $\mathfrak{D}(T)$. Let $\mathfrak{D}(T^*)$ be the set of $f\in\mathcal{H}$ for which there exists an $f^*\in\mathcal{H}$ with the property $(f,Tg)=(f^*,g)$ for all $g\in\mathfrak{D}(T)$. For each such $f\in\mathfrak{D}(T^*)$, we define the adjoint operator, T^*, by $T^*f=f^*$. The operator T is called *Hermitian* (or *symmetric*) if $T\subset T^*$—that is, if $\mathfrak{D}(T)\subset\mathfrak{D}(T^*)$ and $Tf=T^*f$ for all $f\in\mathfrak{D}(T)$. Equivalently, T is Hermitian if and only if $(f,Tg)=(Tf,g)$, all $f,g\in\mathfrak{D}(T)$. T is called *self-adjoint* if $T=T^*$—that is, if and only if T is Hermitian and $\mathfrak{D}(T)=\mathfrak{D}(T^*)$. In general, one has for symmetric operators the relation

$$T\subset T^{**}\subset T^*.$$

The significance of self-adjoint operators lies in the fact that the spectral theorem (which is essential to quantum theoretic applications) applies only to self-adjoint operators. Further discussion, accessible to physicists, is to be found in Reed and Simon [7] as well as in Jauch [38].

3. *Hardy's theorem and the uncertainty relation*. The mathematical structure underlying the Heisenberg uncertainty relation is that a function f and its Fourier transform $\hat{f}$ cannot both "decrease too fast." A precise version of

this assertion is Hardy's theorem: *If* α *and* β *are positive numbers, if* $f(x) \leqslant (\textit{constant})\exp(-\alpha x^2)$ *on the line, and if* $\hat{f}(\gamma) \leqslant (\textit{constant})\exp(-\beta\gamma^2)$ *on the line, then either* $f=0$, *or* f *is a constant multiple of* $\exp(-\alpha x^2)$, *or else there are infinitely many such functions* f *according as* $\alpha\beta > \pi^2$, $\alpha\beta = \pi^2$, *or* $\alpha\beta < \pi^2$.

From a physical point of view, this theorem is rather obvious (using Heisenberg's result), since the minimum uncertainty states for $\langle x \rangle = \langle p \rangle = 0$ are an explicit construction of the $\alpha\beta = \pi^2$ case, while the excited states of the harmonic oscillator yield the infinity of states for $\alpha\beta < \pi^2$.

Let us mention one further point. Dym and McKean [39] in their fine textbook call Hardy's theorem "a sharpening of Heisenberg's inequality." This is clearly true in a purely mathematical sense, but from another point of view it can be misleading. Heisenberg's result is an assertion about the *physical attributes* of momentum and position involving Planck's constant h. Without this *quantal* relationship the inequality becomes merely an interesting mathematical assertion about an abstract property of functional analysis.

4. *Absolutely continuous functions.* A function f on the interval $[a, b]$ is said to be *absolutely continuous* if, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$$

for every finite collection of disjoint intervals $[x_i', x_i]$ satisfying

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

(See Reed and Simon [7, p. 305].)

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TOPIC 8. SOME INTERRELATIONS BETWEEN ANGULAR MOMENTUM THEORY AND PROJECTIVE GEOMETRY

1. Introductory Survey

~~In physical applications of the theory of angular momentum, the com-~~ position law for two kinematically independent angular momenta, $\mathbf{J}_1+\mathbf{J}_2=\mathbf{J}$, plays a central role; this composition leads to the mathematically equivalent structure, the reduction of the direct product (Kronecker product) of irreps into irreducible components— that is,

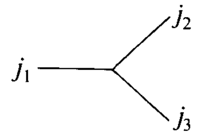
$$[j_i]\otimes[j_k]=\sum_l c_{ikl}[j_l], \tag{5.8.1}$$

where $[j]$ denotes the irrep D^j, and c_{ikl} denotes the intertwining number $c_{ikl}=\mathrm{INT}(j_i\otimes j_k;\ j_l)$ [see Eqs. (2.38) and (2.39)] defined by

$$c_{ikl}=\begin{cases}1 & \text{if } [j_l]\in[j_i]\otimes[j_k],\\ 0 & \text{if } [j_l]\notin[j_i]\otimes[j_k].\end{cases} \tag{5.8.2}$$

It has been an intriguing problem to find significant interpretations of this combination law as a geometry. The desire for such an interpretation is the hope that one may obtain a deeper view of the essential structure. The problem is not well-posed, however, and although much effort has been expended, genuine insights have been meager. In this section we shall review this work briefly, in the hope that it may provide a stimulus for further efforts.

The great success of Feynman's innovation—associating to every perturbation theoretic matrix element in quantum electrodynamics a corresponding diagram—has led to corresponding attempts (by many, mostly unpublished) to associate *diagrams* to angular momentum matrix elements. In the language of Feynman diagrams, the elementary objects are (a) the line (denoting free propagation), and (b) the vertex (representing interaction). There is an immediate transcription into angular momentum theory where (a) the line becomes an angular momentum $\mathbf{J}$, and (b) the vertex becomes the *coupling* of three angular momenta to an invariant:



where $\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_3=\mathbf{0}$.

Clearly, a vertex can be associated with a Wigner coefficient, but there is a problem in the diagrammatic language as to what to do with the magnetic quantum numbers (m_i). If one considers invariant structures, such as the Racah coefficient, this problem does not occur.

One finds in the literature two significant diagrammatic interpretations of the Racah coefficient.[1] The first interpretation associates a Racah coeffi-

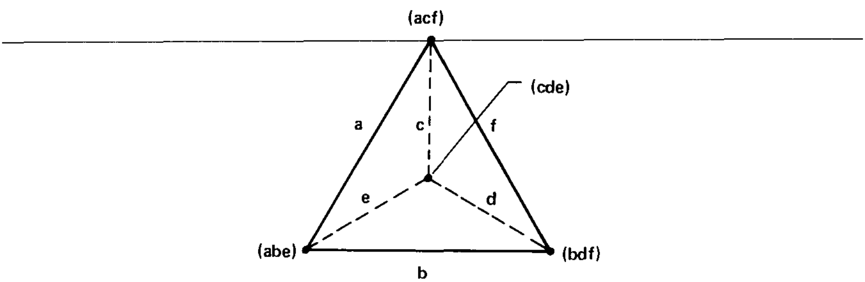


Figure 5.5. The association of a Racah coefficient $W(abcd;\, ef)$ with a tetrahedron in which the "triangles" of the coefficient are represented by vertices.

[1]A third (new) interpretation is developed in Topic 9, p. 394.

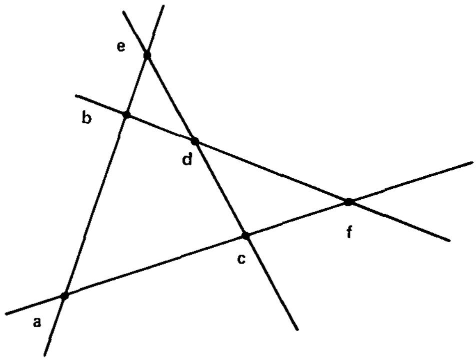


Figure 5.6. The association of a Racah coefficient $W(abcd; ef)$ with a complete quadrilateral.

cient with a tetrahedron—see Fig. 5.5 (Wigner [1, Chapter 27]). The tetrahedron is a direct transcription of the definition of the Racah coefficient as a sum over four Wigner coefficients, each of the four vertices of the tetrahedron corresponding to one of the four Wigner coefficients.

The second interpretation associates a Racah coefficient with a complete quadrilateral (Fano and Racah [2, see Appendices]). This interpretation dualizes the vertex in the plane: three lines meeting in a point (vertex) becomes the dual configuration, three points lying along one line. This leads to the diagram in the plane shown in Fig. 5.6.

It is clear that one can extend such considerations to invariants obtained by coupling more angular momenta. The most extensive developments in this vein have been given by Jucys *et al.* [3]. (This subject is discussed in detail in Topic 12 of the present monograph.) One might take a different view, though, and argue that the Racah invariant (the Racah function) is ~~already the elementary invariant, and one should consider this structure~~ itself as defining a geometry.

We shall consider this latter viewpoint in Section 2 and then develop in Section 3 the projective geometric interpretation found by Robinson [4].

2. The Racah Coefficient as a Complete Quadrilateral

There are three basic multiplication properties of the Racah coefficients: the orthonormality relations, the Racah sum rule, and the B–E identity (see Chapter 3, Section 18, AMQP, and Chapter 4 of the present monograph). In a notation suitable for the present discussion, these properties are the following:[1]

P1: The orthonormality relation:

$$\sum_f (2e+1)(2f+1)W(abcd;ef)W(abcd;e'f)=\delta_{ee'}. \qquad (5.8.3)$$

P2: The Racah sum rule:

$$\sum_q (-1)^{c+e-q}(2q+1)W(adec;qp)W(acde;gq)=(-1)^{g+p-a-d}W(aced;gp) \qquad (5.8.4)$$

P3: The B–E identity:

$$\sum_d (-1)^{c+c'-d}(2d+1)W(bb'cc';de)W(aa'cc';df)W(aa'bb';dg)$$
$$=(-1)^{e+f-g}W(abfe;gc)W(a'b'fe;gc'). \qquad (5.8.5)$$

In view of the fact that Racah coefficients may be considered as discretized Jacobi functions, Racah's relation P2 is the analog to the group property for the multiplication of rotation matrices.[2] (From this point of view P1 is then the group axiom of the existence of an inverse.) A significant interpretation of the third identity, P3, has also been given in Chapter 3: The B–E relation is the necessary and sufficient condition that RW-algebra be associative.

Let us develop now a geometric interpretation of P2.

We take a (real) projective geometry over the plane with the following axioms:

A1: Two points determine a unique line.
A1′: Two lines determine a unique point.
A2: There exist three noncollinear points.

To associate this structure with angular momentum concepts, we make the following interpretation: A nonzero angular momentum is a "point."

Axiom A1 then means that two distinct angular momenta, j_1 and j_2, determine a "line," which is defined as the set of "points" (angular momenta)

[1]We have discussed in Chapter 3, Section 18 and Appendix B, AMQP, how these properties determine the Racah coefficients up to phase conventions.

[2]We show in Topic 12 that Racah's relation $P2$ is also an expression of the fact that the mappings between different coupling schemes for three angular momenta constitute a commutative mapping diagram.

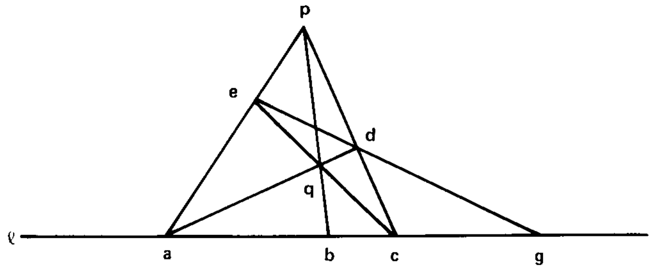


Figure 5.7. Geometric construction of the fourth harmonic point g corresponding to the three given collinear points a, b, and c.

belonging to the Kronecker product $[j_1]\otimes[j_2]$. (This interpretation is imprecise to the extent that the interpretation for zero angular momentum is unclear.)

Consider now the geometric construction known as the determination of the fourth harmonic point g, corresponding to the three given collinear points a, b, and c (see Fig. 5.7).

We choose any point in the plane not on the line l—call it p—and draw the lines (pa), (pb), and (pc). Pick any point on (pb)—call it q—and draw the lines (aq) and (cq). The intersections $(aq)\cap(pc)\equiv d$ and $(cq)\cap(pa)\equiv e$ determine the line (ed), which intersects l in the point g.

The assertion that the point g is independent of the choice of p and q is a theorem in Desarguesian plane projective geometry, and is the lock-incidence axiom of Moufang (non-Desarguesian) projective geometry (Artin [5]).

The feature of interest for angular momentum theory is, however, far less deep, and consists only in recognizing that the configuration just drawn is precisely the configuration that enters in Racah's relation P2. To see this, note that there are *three* Racah coefficients (that is, three complete quadrilaterals) associated with the configuration shown in Fig. 5.7. These are the three complete quadrilaterals associated in Fig. 5.6 with the Racah coefficients:

(*i*) $W(aced;gp)$,
(ii) $W(acde;gq)$,
(iii) $W(adec;qp)$.

These are precisely the three Racah coefficients entering the Racah relation $P2$. This fact suggests that there may be some relationship between the two structures (Racah coefficients and projective geometry). This was remarked on quite early by W. T. Sharp, but the situation is unsatisfactory in that there is no geometric interpretation (proposed) for the summation over q that enters P2.

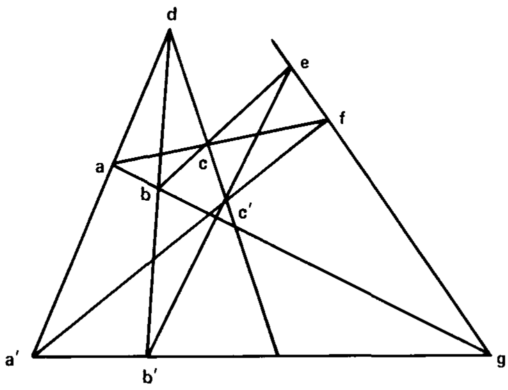


Figure 5.8. Geometric construction of Desargues' theorem.

A similarly suggestive, but incompletely interpreted, fact concerns P3, noted first by Fano and Racah [2]. Let us consider the geometric configuration appropriate to Desargues' theorem (see Fig. 5.8). The Desargues theorem asserts that, if two triangles (abc) and $(a'b'c')$ are perspective from a point d, then the two triangles are perspective from a line; that is, the lines (bc) and $(b'c')$ meet in e, $(ac)\cap(a'c')=f$, $(ab)\cap(a'b')=g$, with e, f, and g, collinear.

Observe now that if we omit the point d in Fig. 5.8, we have two Racah functions defined in the resulting configuration:

$$W(abfe;gc) \qquad \text{and} \qquad W(a'b'fe;gc').$$

Next omit the four points a, a', f, and g. Then we have the Racah coefficient:

$$W(bb'cc';de).$$

Omitting b, b', e, and g yields

$$W(aa'cc';df),$$

and, finally, omitting c, c', e, and f yields

$$W(aa'bb';dg).$$

The fact of interest is that these five Racah coefficients are precisely the five coefficients that enter the relation P3. Once again this result is merely suggestive of a relationship between Racah coefficients and projective geometry, since there is no interpretation of the summation over d that enters in P3.

3. Robinson's Interpretation

In the preceding discussion we have found several suggestive facts relating angular momentum and projective geometry, but we have also found several unsolved difficulties:

(*a*) No interpretation of zero angular momentum (which can occur in the Kronecker product and hence cannot be ignored).

(*b*) No geometric interpretation of the summation that occurs in the laws P2 and P3.

Robinson's starting point is rather different from that given above. He noted that, if one considers the dual structure in the diagrammatic approaches based on combining vertices, then one finds that there exist *three and only three points on a line*. (This results simply from the dual to the vertex.) If one wishes to enforce this fact as an intrinsic element of the structure, then one necessarily deals with the projective geometry PG$(n,2)$. [Thus, one considers that every line has precisely three points (hence, one deals with the finite field F_2) and that the angular momentum diagrams are configurations in an n-dimensional projective space over F_2, denoted by PG$(n,2)$.]

This is a *severe* truncation of the initial angular momentum structure, but it is, as we shall show, a *consistent* interpretation.

First, let us observe that this interpretation removes the difficulties noted above.[1] Since the Kronecker product is now unique, we see that zero angular momentum will occur only when the *same* angular momentum is coupled to itself. We can exclude this self-coupling and accordingly exclude 0. As for the summation difficulty, this too disappears, since there is no longer any sum in the Kronecker product reduction. There is one further advantage of the structure: Since we deal with the number base 2, the sign ambiguity in the Racah coefficients also disappears.

We can proceed now in either of two ways: We can postulate the axioms of PG$(n,2)$ and verify that a consistent interpretation of the three laws P1, P2, and P3 results, or we can use these three laws, use Robinson's interpretation, and verify that we get PG$(n,2)$.

Let us proceed in the first way. We interpret a *point* as a nonzero angular momentum and assume that (*i*) there are at least two distinct points j_i and j_k; (*ii*) two points j_i and j_k determine one and only one line $(j_i j_k)=(j_k j_i)$; and (*iii*) if j_i and j_k are distinct, there exists at least one point $j_l \neq 0$ such that

[1] It is interesting to note that the asymptotic limit of the Racah coefficient (to be discussed in Topic 9) yields, in effect, an alternative elimination (via turning point evaluation) of the summations in P2 and P3.

$c_{ikl} \neq 0$ in Eq. (5.8.1). It follows from the symmetry of the c_{ikl} that the same line is determined by any two of its points. If we assume that (*iv*) there is at least one point not on the line $(j_i j_k)$, then, in order to define a plane and prove the intersection of any two coplanar lines, it is sufficient to assume that (*v*) (Pasch) if l_1, l_2, l_3 are three noncollinear points, and j_1 is a point on the line $(l_2 l_3)$, and j_2 is a point on the line $(l_1 l_3)$, then there is a point j_3 on the line $(l_1 l_2)$ such that j_1, j_2, and j_3 are collinear (see Fig. 5.9).

If we examine Fig. 5.9, we see that the Pasch axiom is exactly the assertion that the Racah coefficient $W(j_1 j_2 l_2 l_1; j_3 l_3)$ *does not vanish.*

Moreover, since we are working with integers in characteristic 2, *we have mapped all nonvanishing Racah coefficients onto the integer* 1.

If we further take as an axiom that (*vi*) there is at least one point *not* in the plane $(l_1 l_2 l_3)$, then this implies, as is well known, Desargues' theorem in space. Had we proceeded from the three laws to the geometry, then the third law (Desargues' theorem) would have implied the extension from the plane $(n=2)$ to space $(n \geqslant 2)$.

Since we have a finite field, there is necessarily (from the Wedderburn theorem) commutativity, and hence Pascal's theorem follows from axiom (*vi*).

Let us consider the configuration in the plane. We get the Fano diagram given by Fig. 5.10:

Every collineation maps points to points and lines to lines. Those collineations that leave the point (111) invariant are symmetries of the Racah coefficient. We can enumerate explicitly all such collineations.

Using the cycle notation for a permutation, we see that six of the transformations on the Fano diagram, leaving the point (111) invariant, are

$$
1 \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (j_1 j_2)(l_1 l_2) \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

$$
(j_2 j_3)(l_2 l_3) \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (j_1 j_3)(l_1 l_3) \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
$$

$$
(j_1 j_2 j_3)(l_1 l_2 l_3) \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (j_1 j_3 j_2)(l_1 l_3 l_2) \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.8.6}
$$

These six operations yield a permutation representation of the group S_3 (permutation group on three objects) as shown by the matrices; they correspond to permuting the columns of the 6-j symbol, or, equivalently, the three opposite pairs of labels in the tetrahedron associated with the Racah coefficient.

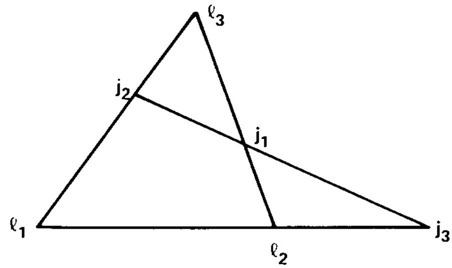


Figure 5.9. Geometric construction of the Pasch axiom.

Further symmetries interchange the $j_i l_i$ by pairs:

$$(j_1l_1)(j_2l_2)\sim\begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix},\qquad (j_2l_2)(j_3l_3)\sim\begin{pmatrix}1&1&1\\0&0&1\\0&1&0\end{pmatrix},$$

$$(j_1l_1)(j_3l_3)\sim\begin{pmatrix}0&0&1\\1&1&1\\1&0&0\end{pmatrix}. \tag{5.8.7}$$

Adjoining these three transformations to the permutations of S_3 above, one obtains 24 of the symmetries of the Racah coefficient.

The explicit matrices that generate these 24 symmetries form a modular representation of the group S_4 (permutation group on four objects), which is reducible, but not completely reducible.

The Regge symmetries of the Racah coefficients are generated from the transformation

$$(-1)^{a+b+c+d}W(abcd;ef)=\begin{Bmatrix}a&b&e\\d&c&f\end{Bmatrix}=\begin{Bmatrix}a&b+x&e-x\\d&c-x&f+x\end{Bmatrix}, \tag{5.8.8}$$

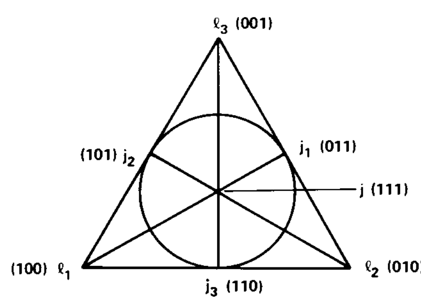


Figure 5.10. The Fano diagram

where $x \equiv (c+e-b-f)/2$. This additional transformation together with the 24 symmetries found by Racah generate the 144 symmetries of the group $S_3 \times S_4$.

What is the analog for the Regge symmetry in the present truncated structure? This is not immediately clear, since, for the field of characteristic two, the variable x, in Eq. (5.8.8) is not well-defined. Since for characteristic two, $+x=-x$, we can simply take the *definition* of the Regge transformation to be

$$\begin{vmatrix} a & b & e \\ d & c & f \end{vmatrix} = \begin{vmatrix} a & b+x & e+x \\ d & c+x & f+x \end{vmatrix}, \tag{5.8.9}$$

where the notation $\begin{vmatrix} a & b & e \\ d & c & f \end{vmatrix}$ means the modulus 2 map of the Racah coefficient $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$, and x is one of the points in PG$(n,2)$.

It can be demonstrated, by direct evaluation, that this is valid for the structure as given by Robinson. Using the symmetries already obtained, one finds the more general result

$$\begin{vmatrix} a & b & e \\ d & c & f \end{vmatrix} = \begin{vmatrix} a+y+z & b+z+x & e+x+y \\ d+y+z & c+z+x & f+x+y \end{vmatrix}, \tag{5.8.10}$$

which is valid for all x, y, $z \in$ PG$(n,2)$, including the zero vector. *The Fano diagram is used to interpret the addition of two points.*

This result contains as special cases the 24 symmetries already obtained earlier. For example, $(j_1 j_2)(l_1 l_2)$ is generated by $x=y=0$, $z=j_3$.

In addition, one obtains degenerate Racah coefficients. These are

$$x=y=0,\ z=j_1\colon \begin{vmatrix} j_1\, j_2\, j_3 \\ l_1\, l_2\, l_3 \end{vmatrix} \to \begin{vmatrix} 0\, j_3\, j_3 \\ j\, l_3\, l_3 \end{vmatrix}, \tag{5.8.11}$$

$$x=j_1,\ y=j_2,\ z=j_3\colon \begin{vmatrix} j_1\, j_2\, j_3 \\ l_1\, l_2\, l_3 \end{vmatrix} \to \begin{vmatrix} 0 & 0 & 0 \\ j & j & j \end{vmatrix}. \tag{5.8.12}$$

Robinson interprets the first relation as a projection onto the line $(l_3 j_3 j)$, and the second as a projection onto the point j.

It follows from this result that the Racah coefficients equivalent to $\begin{vmatrix} j_1\, j_2\, j_3 \\ l_1\, l_2\, l_3 \end{vmatrix}$, as generated by all transformations in Eq. (5.8.10), can all be mapped to the form $\begin{vmatrix} 0 & 0 & 0 \\ j & j & j \end{vmatrix}$.

Thus, it follows that every Racah configuration in the Fano plane—that is, a six-point configuration such that

$$\begin{Bmatrix} j_1\, j_2\, j_3 \\ l_1 l_2 l_3 \end{Bmatrix} \to \begin{vmatrix} j_1\, j_2\, j_3 \\ l_1 l_2 l_3 \end{vmatrix} \neq 0, \tag{5.8.13}$$

can by the symmetry (5.8.10) be mapped uniquely to the seventh point in the Fano plane. This representation by the seventh point permits composition, the resultant being the unique third point defined by the two points composed. We categorize this composition in Section 4.

4. Further Results

In order to apply this geometric model for the Racah coefficient, we must introduce a formal multiplication for the symbols $\begin{vmatrix} j_1\, j_2\, j_3 \\ l_1 l_2 l_3 \end{vmatrix}$.

Robinson shows that the multiplication of two such symbols satisfies the rule

$$\begin{vmatrix} j_1\, j_2\, j_3 \\ l_1 l_2 l_3 \end{vmatrix} \cdot \begin{vmatrix} j_1'\, j_2'\, j_3' \\ l_1' l_2' l_3' \end{vmatrix} = \begin{vmatrix} j_1+j_1' & j_2+j_2' & j_3+j_3' \\ l_1+l_1' & l_2+l_2' & l_3+l_3' \end{vmatrix} (\text{mod } 2). \tag{5.8.14}$$

The proof of this result is given by making the map onto the seventh point followed by composition of points as given at the end of Section 3.

With this result we can now verify that *in the* PG$(n,2)$ *model the three laws* P1, P2, *and* P3 *are valid.*

Proof. Consider first the product of two coefficients:

$$\begin{vmatrix} j_1\, j_2\, j_3 \\ l_1 l_2 l_3' \end{vmatrix} \cdot \begin{vmatrix} j_1\, j_2\, j_3 \\ l_1 l_2 l_3 \end{vmatrix} = \begin{cases} 1 & \text{if } l_3' = l_3, \\ 0 & \text{if not.} \end{cases} \tag{5.8.15}$$

This result follows, since, if $l_3' = l_3$, then we get 1 on the right-hand side, and $\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$ is a degenerate Racah coefficient with the value 1. If $l_3' \neq l_3$, then one of the two coefficients on the left vanishes, since in a plane of PG$(n,2)$ there are only seven points, and if six lie in a Racah configuration, say $\begin{vmatrix} j_1\, j_2\, j_3 \\ l_1 l_2 l_3 \end{vmatrix} = 1$, then necessarily $\begin{vmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3' \end{vmatrix}$, $l_3' \neq l_3$, does *not* lie in such a configuration and hence vanishes. This validates P1.

Consider now the product of two (nonvanishing) Racah coefficients in a plane of PG$(n,2)$ that belong to distinct symmetry classes. (We have seen that each such symmetry class could be defined by the point in the plane that is left invariant under all 24 Racah symmetries.) Since all seven points are equivalent in PG(n,2), we choose any three—say,

$$
\begin{aligned}
j=(1\,1\,1)&\leftrightarrow\begin{vmatrix} j_3 & j_2 & j_1 \\ l_3 & l_2 & l_1 \end{vmatrix},\\
j_1=(0\,1\,1)&\leftrightarrow\begin{vmatrix} j_3 & l_2 & l_1 \\ j_2 & l_3 & j \end{vmatrix},\\
l_1=(1\,0\,0)&\leftrightarrow\begin{vmatrix} j_3 & j_2 & j_1 \\ l_3 & l_2 & j \end{vmatrix}.
\end{aligned}
\tag{5.8.16}
$$

The product of any two of these [using the multiplication law (5.8.14)] is the third one; that is,

$$
\begin{vmatrix} j_3 & l_2 & l_1 \\ j_2 & l_3 & j \end{vmatrix}\cdot\begin{vmatrix} j_3 & j_2 & j_1 \\ l_3 & l_2 & j \end{vmatrix}=\begin{vmatrix} j_3 & j_2 & j_1 \\ l_3 & l_2 & l_1 \end{vmatrix}.
\tag{5.8.17}
$$

This result can be recognized as the corresponding form of Racah's sum rule. Thus, P2 is verified.

In PG(n,2) Desargues' theorem is automatically valid for $n\geqslant 3$, and degenerate for $n=2$. Thus, we can find a configuration in PG(3,2) that corresponds to Desargues' theorem.

We have given in Fig. 5.8 the 3-space form of this theorem; the corresponding diagram for PG(3,2) is given in Fig. 5.11.

Corresponding to every plane of this figure there are seven different symmetry classes of Racah coefficients. By the Regge symmetry, each symmetry class may be uniquely coordinated with one of the seven points in each plane.

Consider the Racah coefficient associated with the central point in each of the four faces of the tetrahedron. We have the following associations:

$$
\begin{aligned}
(1110)&\leftrightarrow\begin{vmatrix} J_1 & J_2 & J_3 \\ J_{23} & J_{13} & J_{12} \end{vmatrix}, &\qquad (1101)&\leftrightarrow\begin{vmatrix} J_{23} & J_{13} & J_3 \\ J'_{13} & J'_{23} & J \end{vmatrix},\\
(1011)&\leftrightarrow\begin{vmatrix} J_{23} & J_2 & J_{12} \\ J'_{12} & J & J'_{23} \end{vmatrix}, &\qquad (0111)&\leftrightarrow\begin{vmatrix} J_1 & J_{13} & J_{12} \\ J & J'_{12} & J'_{13} \end{vmatrix}.
\end{aligned}
\tag{5.8.18}
$$

Finally, there is a "plane" whose midpoint is given by (1111). To this point one associates the Racah coefficient

$$
(1111)\leftrightarrow\begin{vmatrix} J_1 & J_2 & J_3 \\ J'_{23} & J'_{13} & J'_{12} \end{vmatrix}.
\tag{5.8.19}
$$

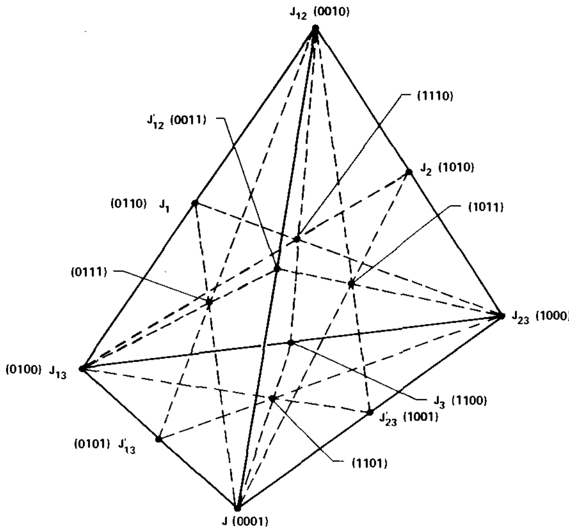


Figure 5.11. Geometric construction corresponding to the projective geometry PG(3,2).

Using the vectorial equality,

$$(1111)+(1110)=(1101)+(1011)+(0111),$$

one now deduces the product relation for the associated Racah functions:

$$\left|\begin{matrix} J_1 & J_2 & J_3 \\ J'_{23} & J'_{13} & J'_{12} \end{matrix}\right| \cdot \left|\begin{matrix} J_1 & J_2 & J_3 \\ J_{23} & J_{13} & J_{12} \end{matrix}\right| = \left|\begin{matrix} J_{23} & J_{13} & J_3 \\ J'_{13} & J'_{23} & J \end{matrix}\right| \cdot \left|\begin{matrix} J_{23} & J_2 & J_{12} \\ J'_{12} & J & J'_{23} \end{matrix}\right| \cdot \left|\begin{matrix} J_1 & J_{13} & J_{12} \\ J & J'_{12} & J'_{13} \end{matrix}\right|. \tag{5.8.20}$$

This relation can be recognized as the relation P3, modified in PG$(n,2)$ by a lack of summation.

Thus, we have verified that each of the defining relations for the Racah functions—that is, P1, P2, and P3—has a valid analog in PG$(n,2)$. [One cannot proceed in the other direction, since, for example, it is equally valid in PG$(n,2)$ that the product of all five Racah functions in Eq. (5.8.20) is unity, a relation that is clearly wrong for general Racah functions.] ■

It is valid to conclude that *any relation for the general* $3n$-j *symbol* $(n\geq 2)$ *is provable in* PG$(n,2)$. This symbolic analysis, even though it greatly truncates the content of the invariant angular functions, can provide useful structural properties of the higher $3n$-j symbols. Some indication of this has been discussed briefly by Robinson [4] for 9-j and 12-j symbols.

5. Wigner and Racah Coefficients as Magic Squares

We have seen that there exist two types of interrelations between angular momentum theory and projective geometry. The first is merely suggestive: The planar representations of P2 and P3 are precisely the geometric *configurations* of the theorem of the complete quadrilateral and Desargues' theorem, respectively. The second relationship is precise and is a well-defined map of the Racah coefficients, and higher $3n$-j invariants, onto the field F_2 and the projective geometry PG(n,2).

There is a third, and very different, type of relationship between the 3-j and 6-j symbols and projective geometry. This is a mapping that associates the 3-j and 6-j symbols with *magic squares*, and then embeds these latter canonically in a projective geometry. In the case of the 3-j symbol, Regge [6] has shown that each symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ may be associated with a square array of the form

$$\begin{array}{|ccc|}\hline -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \\ \hline \end{array}\,. \tag{5.8.21}$$

Each entry in this array is then a nonnegative integer, and the row sums and column sums are all equal to $J \equiv j_1+j_2+j_3$. This form displays conveniently the 72 symmetries of the 3-j symbols as combinations of permutations of rows and columns and transposition, as discussed in Section 12 of Chapter 3 and Appendix D of Chapter 5, AMQP.

Two such arrays that differ only by a symmetry correspond to the same numerical magnitude for the associated 3-j symbol.

Extending Regge's work, Shelepin [7] and Smorodinskii and Shelepin [8] associate each 6-j symbol with a 4×4 array:

$$\begin{array}{|cccc|}\hline j+j_3-j_{12} & j_1+j_{23}-j & j+j_{12}-j_3 & j_2+j_3+j_{12}+\tfrac{1}{2}J \\ j_1+j_{12}-j_2 & j_2+j_3-j_{23} & j_3+j+j_{23}+\tfrac{1}{2}J & j_2+j_{23}-j_3 \\ j_2+j_{12}-j_1 & j_1+j_{12}+j+\tfrac{1}{2}J & j_1+j_2-j_{12} & j_3+j_{23}-j_2 \\ j_1+j_2+j_{23}+\tfrac{1}{2}J & j+j_{23}-j_1 & j_3+j_{12}-j & j_1+j-j_{23} \\ \hline \end{array}\,, \tag{5.8.22}$$

where $J=2(j_1+j_2+j_3+j+j_{12}+j_{23})$. The entries in this array are nonnegative integers, and the row sums and column sums are all equal to J.

The arrays (5.8.21) and (5.8.22) possess a natural, ordered algebraic structure. A square array of nonnegative integers, all of whose row sums and column sums are equal, is called a *magic square*. Let M_n denote the set of $n\times n$ magic squares. If $R=(r_{ij})$ belongs to M_n, then the sum

$$S_R=\sum_{i=1}^{n} r_{ij}=\sum_{j=1}^{n} r_{ij} \tag{5.8.23}$$

will be called the *rank* of R. The set M_n is closed under the operations of matrix addition and multiplication. If A and B are magic squares of ranks S_A and S_B, then $A+B$ and AB are magic squares of ranks $S_{A+B}=S_A+S_B$ and $S_{AB}=S_A S_B$.

The algebraic structure possessed by the 3-j and 6-j symbols by virtue of the embedding in magic squares has been studied by Giovannini and Smith [9]. They show that the set M_n with the two compositions, matrix addition and matrix product, forms a locally finite, partially ordered semiring. There is a canonical embedding of this semiring into a locally finite, partially ordered ring, R_n. (This embedding associates the elements of R_n with the differences $A-B$ of elements of M_n with the proviso that $A-B=C-D\in R_n$ if $A+D=C+B\in M_n$.) If we extend the coefficients of the ring R_n to the rational numbers, the ring may now be embedded in a semisimple algebra Q_n, which is five-dimensional for the 3-j symbols and ten-dimensional for the 6-j symbols. Finally, by considering vector subspaces of the respective algebras, Giovannini and Smith associate a projective geometry with the 3-j and 6-j symbols. A point of the projective geometry is the equivalence class of all magic squares (over the rationals) that are proportional to a given magic square (identified with a given 3-j or 6-j symbol).

Remarks. (a) By means of this construction, an algebraic interpretation of the limit relationship between the 3-j symbols and 6-j symbols given by

$$\lim_{p\to\infty}(-1)^{2p}[2j_{23}+2p+1]^{\frac{1}{2}}\begin{Bmatrix} j_1 & j_2 & j_{12}\\ j_3+p & j+p & j_{23}+p\end{Bmatrix}$$

$$=(-1)^{2(j_1+j_3)}\begin{pmatrix} j_1 & j_2 & j_{12}\\ j_{23}-j & j_3-j_{23} & j-j_3\end{pmatrix} \tag{5.8.24}$$

has been achieved in Ref. [9] (see also Ref. [10] and Chapter 3, Section 18, AMQP). If we add p to the appropriate elements in (5.8.22) and replace by 0 those elements that become infinite in the limit, then the array (5.8.22) is transformed into

$$\boxed{\begin{matrix} 0 & j_1+j_{23}-j & j+j_{12}-j_3 & 0 \\ j_1+j_{12}-j_2 & j_2+j_3-j_{23} & 0 & j_2+j_{23}-j_3 \\ j_2+j_{12}-j_1 & 0 & j_1+j_2-j_{12} & 0 \\ 0 & 0 & j_3+j_{12}-j & j_1+j-j_{23} \end{matrix}}\ . \qquad (5.8.25)$$

Algebraically, the transformation of (5.8.22) into (5.8.25) is a projection of the vector space Q_4 onto a subspace. The remaining nine entries are precisely the entries of the magic square (5.8.21) corresponding to the 3-j symbol $\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_{23}-j & j_3-j_{23} & j-j_3 \end{pmatrix}$. Thus, the image of the projection in Q_4 is a subspace isomorphic to Q_3 as a vector space.

(b) The Regge mapping (5.8.21) and the Shelepin mapping (5.8.22) differ in one important respect. The Regge mapping is one-to-one onto, since every 3×3 integral magic square defines a set of (j_i, m_i), $i=1$, 2, 3, and hence an associated 3-j symbol. By contrast, the Shelepin mapping is *not* an onto mapping. (Giovannini and Smith [9] note that for ranks 1, 2, and 5 there is no associated 6-j symbol that satisfies the four triangle relations with admissible values of the angular momenta.)

(c) The projective geometry associated with the 3-j and 6-j symbols via the magic square mappings is of a quite different nature from the interrelations discussed earlier. In particular, the structural relations P1–P3 for the 6-j coefficients are not directly involved in the Shelepin mapping into the magic squares.

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TOPIC 9. PHYSICAL INTERPRETATION AND ASYMPTOTIC (CLASSICAL) LIMITS OF THE ANGULAR MOMENTUM FUNCTIONS

1. Introductory Remarks

The rotation matrices (representation coefficients), the Wigner and Racah coefficients, and the more general $3n$-j symbols are all quantum mechanical constructs; accordingly, the physical interpretation of these objects—just as for any quantum mechanical quantity—must be found in terms of probability amplitudes whose absolute squares have the meaning of probabilities for definite physical measurements. These physical measurements are, in turn, restricted by the uncertainty relations, which, as we have seen in Topic 7, constitute a fundamental element in the physical interpretation.

In the region of large quantum numbers,[1] classical concepts become increasingly valid; this is the content of the Bohr correspondence principle [1], which is valid in quantum mechanics. Accordingly, in approaching the classical limit, it will be possible to define states for which all three components of the angular momentum vector are confined to narrow ranges around specified values; the range becomes narrower, the closer one approaches the classical limit. This implies not only that the angular momentum constructs will approach an interpretation in terms of ordinary geometric concepts, but that this interpretation is *inherently an average*, performed over a range of the parameters involved.[2]

The fact that the interpretation required by quantum mechanics involves an absolute square (that is, a probability) leads to a distinction between the *asymptotic limit* of a mathematically well-defined function (the probability amplitude), and the *classical limit* of the related physical quantity. The mathematical quantity may, and generally does, have a rapidly oscillating complex phase that drops out of the averaged absolute square of the physical classical limit.

[1]Recall that we measure angular momentum **J** in units of $\hbar$, so that $\hbar \to 0$ (the classical limit) implies that, for finite angular momentum, the quantum numbers (jm) must become large.

[2]The necessity for interpreting the classical limit of the angular momentum functions as an average was discussed in the work of Brussaard and Tolhoek [2] and particularly emphasized by Wigner [3] and by Ponzano and Regge [4].

2. Physical Interpretation of the Rotation Matrix Elements (Representation Coefficients)

The physical interpretation of the matrix elements, $D^j_{m'm}(U)$, of the rotation operator $\mathcal{U}\equiv\exp(-i\theta\hat{n}\cdot\mathbf{J})$ has been discussed in Chapter 3, AMQP. It was shown there that a state having total angular momentum j and projection m along the $\hat{e}_3$-axis is characterized by the ket vector $|jm\rangle$, and that under a rotation of frames given by $\hat{e}_i\to\hat{e}'_i=\sum_j R_{ij}\hat{e}_j$, with $R=(R_{ij})\in SO(3)$, this state vector undergoes the transformation

$$|jm\rangle\to|jm\rangle'=\sum_{m'} D^j_{m'm}(U)|jm'\rangle. \tag{5.9.1}$$

In this transformation $\pm U$ denotes the 2×2 unitary unimodular matrix that maps to R in the homomorphism of $SU(2)$ onto $SO(3)$ (see Chapter 2, AMQP).

This relation supplies the desired physical interpretation. Using the Euler angles $(\alpha\beta\gamma)$ (see Chapter 2, Section 6, AMQP) to specify the relationship between the two frames $(\hat{e}_1,\hat{e}_2,\hat{e}_3)$ and $(\hat{e}'_1,\hat{e}'_2,\hat{e}'_3)$, we find that

$$|D^j_{m'm}(\alpha\beta\gamma)|^2=\left[d^j_{m'm}(\beta)\right]^2 \tag{5.9.2}$$

is the probability that a system in the angular momentum state $|jm\rangle$ in the original frame will be found to be in the angular momentum state $|jm'\rangle$ in the rotated frame.[1] Note that this probability depends only on the angle β between the e_3- and e'_3-axes, and not on the Euler angles α,γ. (These angles describe rotations around the e_3- and e'_3-axes, respectively, and drop out, physically speaking, because of the uncertainty relations.)

Using the relation

$$d_{mm'}(\beta)=(-)^{m-m'}d_{m'm}(\beta), \tag{5.9.3}$$

one verifies that the probability in Eq. (5.9.2) is symmetric in m and m', as, of course, it must be on physical grounds.[2]

Let us consider now the classical limit, and take our system to be characterized in the original frame by a state of angular momentum j maximally oriented along the e_3-axis—that is, $m=j$. The probability that

[1] This physical interpretation was especially emphasized by Güttinger [5] and Wigner [3]. In the discussion of this section, we follow Wigner [3, Chapter 27].

[2] The relations $D^{j*}_{m'm}(U)=(-1)^{m'-m}D^j_{-m',-m}(U)$ and $d^j_{m'm}(\pi-\beta)=(-1)^{j-m'}d_{m',-m}(\beta)$ also lead to physical assertions.

the angular momentum projection along the e_3'-axis has the value $m'=m$ is given by (see Chapter 3, Sections 5 and 6, AMQP)

$$P(m)\equiv\left[d^j_{jm}(\beta)\right]^2=\binom{2j}{j-m}\left(\cos\frac{\beta}{2}\right)^{2(j+m)}\left(\sin\frac{\beta}{2}\right)^{2(j-m)}. \qquad (5.9.4)$$

In the classical limit, one expects the most probable values of m to be distributed around the classical value $m_0=j\cos\beta$. Introducing the probability $P(m_0)$ for this classical value, one finds the relative probability to be

$$\begin{aligned}\frac{P(m)}{P(m_0)}&=\frac{(j-m_0)!(j+m_0)!}{(j-m)!(j+m)!}\left(\tan\frac{\beta}{2}\right)^{2(m_0-m)}\\&=\frac{(j-m_0)!(j+m_0)!}{(j-m)!(j+m)!}\left(\frac{j-m_0}{j+m_0}\right)^{m_0-m},\end{aligned} \qquad (5.9.5)$$

where we have used

$$\left(\tan\frac{\beta}{2}\right)^2=\frac{1-\cos\beta}{1+\cos\beta}=\frac{j-m_0}{j+m_0}.$$

Taking now $j\pm m_0$ to be large compared to $(m-m_0)$—that is, $j\pm m_0\gg(m-m_0)$—we obtain the classical limit (Wigner [3, p. 351])

$$P(m)\sim P(m_0)\exp\left[\frac{-j(m-m_0)^2}{j^2-m_0^2}\right]=P(m_0)\exp\left[-\frac{1}{j}\left(\frac{m-m_0}{\sin\beta}\right)^2\right]. \qquad (5.9.6)$$

Thus, we see that the probability for m is a Gaussian distribution around the classical value $m_0=j\cos\beta$. [The dispersion in the variable $\delta=m-m_0$ is $(j)^{\frac{1}{2}}\sin\beta$, which is approximately $(j)^{\frac{1}{2}}$ by assumption in the derivation.]

The simplicity of this result stems from specializing such that the state in the original frame, $|jj\rangle$, was maximally oriented. A more general classical limit can be derived (Brussaard and Tolhoek [2]); we shall discuss this more general result in Section 10 below.

3. Physical Interpretation of the Wigner Coefficients

The most direct way to obtain a physical interpretation of the Wigner coefficients is to utilize the definition of these coefficients as coupling coefficients (Chapter 3, Section 12, AMQP). Thus, one has the relationship

$$|(j_1j_2)jm\rangle=\sum_{m_1m_2}C^{j_1j_2j}_{m_1m_2m}|j_1m_1\rangle\otimes|j_2m_2\rangle, \qquad (5.9.7)$$

which expresses the coupling of two kinematically independent systems in the angular momentum state $|j_1m_1;\, j_2m_2\rangle \equiv |j_1m_1\rangle \otimes |j_2m_2\rangle$ to the composite state having sharp (total) angular momentum (jm). The physical interpretation of the coefficient $C^{j_1j_2j}_{m_1m_2m}$ implied by this equation is that the absolute square of the matrix element

$$|\langle j_1m_1;\, j_2m_2|(j_1j_2)jm\rangle|^2 = |C^{j_1j_2j}_{m_1m_2m}|^2 \tag{5.9.8}$$

is the probability that a measurement of the 3-components of the angular momenta of the individual parts of the system, originally in the total angular momentum state $|(j_1j_2)jm\rangle$, will find the system in the state $|j_1m_1\rangle \otimes |j_2m_2\rangle$. It is essential to note that this interpretation shows that the *phase* of the coefficient $C^{j_1j_2j}_{m_1m_2m}$ is *not* observable. Consequently, any mathematical definition of this coefficient as a definite function can assign the phase only by a convention.

The classical arrangement that expresses this result is illustrated in Fig. 5.12. For the sake of symmetry, we replace $\mathbf{J}$ by $-\mathbf{J}$, so that the triangle relation becomes $\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J} = \mathbf{0}$.

Classically the triangle $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J})$ is determined by the three lengths of the vectors, but the orientation is determined by the projections m_1 and m_2 (note that $m_1 + m_2 + m = 0$) only to within a rotation of the figure around the 3-axis.

To proceed with the quantum mechanical interpretation, it is necessary to recall the uncertainty principle. The fact that the measurements associated with Fig. 5.12 are to be carried out on a quantum state of sharp (jm) implies two results: (a) The 3-axis is an axis of symmetry (from m being sharp), and (b) there is symmetry around the classical direction of $\mathbf{J}$ (from j

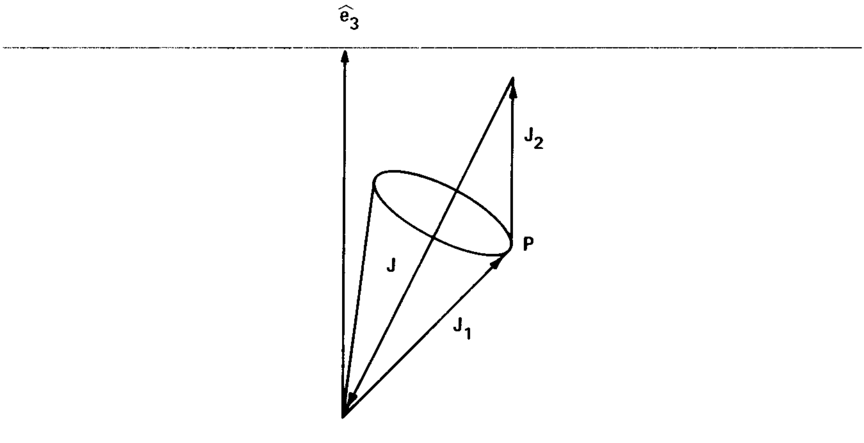

Figure 5.12.

being sharp). *This latter implication shows that the point P must be considered as uniformly probable on the circle shown in Fig.* 5.12. (Note that, although m_1 and m_2 vary as P traverses the circle, the quantum number m remains fixed, as it must.)

These implications from the uncertainty principle (Topic 7, Section 4) provide the key to the physical interpretation.

Since the configuration is invariant to rotations about the 3-axis, it is no restriction to fix a particular orientation. To calculate a probability from the requirement that P can be uniformly distributed on the circle, we first use the geometric relation

$$\cos\theta_{J_1\hat{e}_3} = \cos\theta_{J_1J}\cos\theta_{J\hat{e}_3} + \sin\theta_{J_1J}\sin\theta_{J\hat{e}_3}\cos\phi, \tag{5.9.9}$$

where $\theta_{J_1\hat{e}_3}$ is the angle between the vectors $\mathbf{J}_1$ and $\hat{e}_3$, and similarly for $\theta_{J\hat{e}_3}$; the angle ϕ is defined as the dihedral angle between the $(\mathbf{J}_1, \mathbf{J})$ and $(\mathbf{J}, \hat{e}_3)$ planes.

The requirement that P be uniformly distributed over the circle can be replaced by the more convenient, but equivalent, condition that $d\phi/dt$ be constant, which we then take to be 2π, weighting all points of the circle uniformly. The time dt that a given configuration exists then becomes a measure of the probability of that configuration.

Noting that $j_1\cos\theta_{J_1\hat{e}_3} = m_1$, and differentiating Eq. (5.9.9) with respect to time, we find that

$$dm_1/dt = -j_1\sin\theta_{J_1J}\sin\theta_{J\hat{e}_3}\sin\phi\, d\phi/dt,$$

where we have used the fact that the angles θ_{J_1J} and $\theta_{J\hat{e}_3}$ are fixed.

To interpret this result, we use the geometric fact that the sine of the dihedral angle ϕ can be expressed as the magnitude of the vector product of the normals to the planes. Thus, we have

$$\begin{aligned}\sin\phi &= \|\hat{n}_{J_1J} \times \hat{n}_{J\hat{e}_3}\| \\ &= \left\| \frac{\mathbf{J}_1 \times \mathbf{J}}{\|\mathbf{J}_1 \times \mathbf{J}\|} \times \frac{\mathbf{J} \times \hat{e}_3}{\|\mathbf{J} \times \hat{e}_3\|} \right\|,\end{aligned} \tag{5.9.10}$$

which, in turn, yields the simple result

$$j_1\sin\theta_{J_1J}\sin\theta_{J\hat{e}_3}\sin\phi = \|\mathbf{J}_1 \times \mathbf{J}_2 \cdot \hat{e}_3\|/\|\mathbf{J}\|. \tag{5.9.11}$$

(In these results the notation $\|\mathbf{A}\|$ denotes the length of the vector $\mathbf{A}$.) Note that $\|\mathbf{J}_1 \times \mathbf{J}_2\|$ is just *twice* the area of the triangle $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J})$, so that $\|\mathbf{J}_1 \times \mathbf{J}_2 \cdot \hat{e}_3\|$ is twice the area of the triangle projected onto a plane perpendicular to $\hat{e}_3$.

The probability that a given configuration will have the projection m_1 for j_1 is given by[1]

$$\mathrm{Prob}(m_1)=2\left(\frac{dm_1}{dt}\right)^{-1}=\frac{j}{\pi\|\mathbf{J}_1\times\mathbf{J}_2\cdot\hat{e}_3\|}. \qquad (5.9.12)$$

This result gives the desired probability interpretation of Eq. (5.9.8) in the classical limit. Hence, we have shown: *The classical limit of the Wigner coefficient—averaged over fluctuations—is given by the expression* (Wigner [3, p. 353])

$$(2j+1)^{-1}\left(C^{j_1j_2j}_{m_1,m_2,-m}\right)^2=\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m\end{pmatrix}^2\sim\frac{\delta_{m_1+m_2+m,0}}{2\pi\|\mathbf{J}_1\times\mathbf{J}_2\cdot\hat{e}_3\|}, \qquad (5.9.13)$$

which is the reciprocal of 4π *times the area of the triangle* $(\mathbf{J}_1,\mathbf{J}_2,\mathbf{J})$ *projected onto the* e_2-e_3 *plane.*[2]

The additional symmetry afforded by the 3-j symbol, as opposed to the Wigner coefficient on the left of Eq. (5.9.13), is evident from this result, for the limiting result given by Eq. (5.9.13) shows that the three angular momenta $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}$ all play equivalent roles.

An alternative interpretation The orthonormality of the Wigner coefficients allows one to invert the relation given by Eq. (5.9.7), thus obtaining the result

$$|j_1m_1\rangle\otimes|j_2m_2\rangle=\sum_{jm}C^{j_1j_2j}_{m_1m_2m}|(j_1j_2)jm\rangle. \qquad (5.9.14)$$

It is quite obvious that the use of this expression leads to precisely the same probability interpretation as that expressed by Eq. (5.9.8).

It is, however, *not* obvious that the corresponding *classical arrangement* will, in fact, give the same result, and it is useful to pursue this point further.

The classical arrangement for this situation is, at first glance, much the same as in Fig. 5.12—since the relation $\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}=\mathbf{0}$ still holds—but the uncertainty relations attendant on the present interpretation are quite different, and this implies a different diagram (see Fig. 5.13). From Eq. (5.9.14) one sees that there is now *cylindrical symmetry* for both $\mathbf{J}_1$ and $\mathbf{J}_2$ around their respective 3-axes. These conditions validate the diagram shown in Fig. 5.13. Note that the conditions for this interpretation now keep both

[1] The factor of 2 in Eq. (5.9.12) comes from the fact that a generic value of m_1 occurs twice as the point P traverses the circle.

[2] In going from Eq. (5.9.12) to Eq. (5.9.13), we have divided by the factor $(2j+1)$, which is taken (since j is large) to be $2j$.

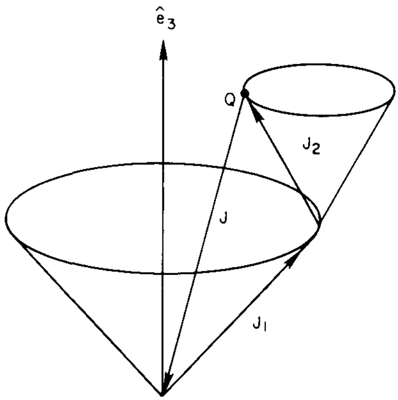

Figure 5.13.

m_1 and m_2 fixed, and it is the magnitude of $\mathbf{J}$ that varies. An equivalent arrangement (since the diagram as a whole may be rotated about the 3-axis) is to fix the orientation of $\mathbf{J}_1$ and allow the point Q to traverse the circle at a uniform rate.

Spherical geometry now yields the relation

$$\mathbf{J}_1\cdot\mathbf{J}_2=j_1 j_2\left(\cos\theta_{J_1\hat{e}_3}\cos\theta_{J_2\hat{e}_3}+\sin\theta_{J_1\hat{e}_3}\sin\theta_{J_2\hat{e}_3}\cos\psi\right), \qquad (5.9.15)$$

where ψ is the dihedral angle between the $(\mathbf{J}_1\hat{e}_3)$ and $(\mathbf{J}_2\hat{e}_3)$ planes; this is the angle that has all its values equiprobable.

Since classically $\mathbf{J}_1\cdot\mathbf{J}_2=\frac{1}{2}(\mathbf{J}^2-\mathbf{J}_1^2-\mathbf{J}_2^2)$, we find by differentiating Eq. (5.9.15) that

$$j\left(\frac{dj}{dt}\right)=-j_1 j_2\sin\theta_{J_1\hat{e}_3}\sin\theta_{J_2\hat{e}_3}\sin\psi\left(\frac{d\psi}{dt}\right).$$

Using the relation

$$\sin\theta_{J_1\hat{e}_3}\sin\theta_{J_2\hat{e}_3}\sin\psi=\frac{\mathbf{J}_1\times\mathbf{J}_2\cdot\hat{e}_3}{j_1 j_2},$$

and normalizing the rate $d\psi/dt=2\pi$, we find that the probability that the configuration has the value j is given by[1]

$$\mathrm{Prob}(j)=2\left(\frac{dj}{dt}\right)^{-1}=\frac{j}{\pi|\mathbf{J}_1\times\mathbf{J}_2\cdot\hat{e}_3|}. \qquad (5.9.16)$$

[1] The factor 2 again occurs because a generic value of j occurs twice as ψ goes from 0 to 2π.

The result expressed by Eq. (5.9.16) is *precisely the same* as that given in Eq. (5.9.12) and verifies that the interpretation of the Wigner coefficient in Eq. (5.9.14) is the same as that given by Eq. (5.9.7). It is rather remarkable that two such different and distinct probability distributions—expressed by Figs. 5.12 and 5.13—should nonetheless imply precisely the same physical result.

4. Physical Interpretation of the Racah Coefficients

The desired interpretation of the Racah coefficient is most directly obtained from the recoupling aspect of the Racah coefficient (discussed in detail in Topic 12, Section 6). The Racah coefficient viewed as a recoupling coefficient relates wave functions corresponding to the different ways of coupling three angular momenta $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}_3$ to a given angular momentum $\mathbf{J}$.

We let the ket symbol[1] $|[(j_1 j_2)_{j'} j_3]_{jm}\rangle$ denote the state vector corresponding to the coupling scheme symbolized by writing

$$\mathbf{J} = (\mathbf{J}_1 + \mathbf{J}_2) + \mathbf{J}_3 = \mathbf{J}' + \mathbf{J}_3. \tag{5.9.17}$$

Similarly, the ket symbol $|[j_1(j_2 j_3)_{j''}]_{jm}\rangle$ denotes the state vector corresponding to the coupling scheme

$$\mathbf{J} = \mathbf{J}_1 + (\mathbf{J}_2 + \mathbf{J}_3) = \mathbf{J}_1 + \mathbf{J}''. \tag{5.9.18}$$

The Racah coefficient $W(j_1 j_2 j j_3; j'j'')$ expresses the transformation between these two schemes by the equation (see Fig. 5.29, Topic 12)

$$|[(j_1 j_2)_{j'} j_3]_{jm}\rangle = \sum_{j''} [(2j'+1)(2j''+1)]^{\frac{1}{2}} \times W(j_1 j_2 j j_3; j'j'')|[j_1(j_2 j_3)_{j''}]_{jm}\rangle. \tag{5.9.19}$$

It follows from this result that the quantum mechanical probability, P, defined by

$$P \equiv (2j'+1)(2j''+1)W^2(j_1 j_2 j j_3; j'j'') = (2j'+1)(2j''+1)\begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j & j'' \end{Bmatrix}^2 \tag{5.9.20}$$

[1] This notation and its relationship to binary trees is developed in Topic 12.

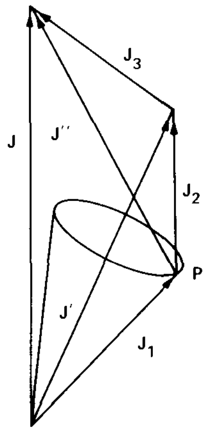

Figure 5.14.

is the probability that a system prepared in a state of the coupling scheme (5.9.17) (which assumes that j_1, j_2, j_3, j', and j have definite magnitudes) will be found (measured) to be in a state of the coupling scheme (5.9.18) (which assumes j_1, j_2, j_3, j'', and j to have definite magnitudes).

A geometric arrangement that expresses these coupling relationships as triangles in space (similar to the construction of Figs. 5.12 and 5.13) is given in Fig. 5.14. This figure expresses the six angular momenta $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$, $\mathbf{J}'$, $\mathbf{J}''$, and $\mathbf{J}$ as the sides of a tetrahedron (or, better, as a 4-simplex in 3-space, since the tetrahedron is generally irregular); the four faces of the simplex are the four triangles of the two coupling schemes.[1]

From Fig. 5.14 one sees that, if five of the six momenta are fixed in magnitude— say, $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$, $\mathbf{J}'$, and $\mathbf{J}$— then the dihedral angle ϕ between the two planes— here $(\mathbf{J},\mathbf{J}',\mathbf{J}_3)$ and $(\mathbf{J}_1,\mathbf{J}_2,\mathbf{J}')$— can vary freely, thus changing $\mathbf{J}''$ in such a way that the point P describes a circle in space.

By analogy to the use of the uncertainty relation in interpreting Fig. 5.12, we see that taking j' to have a sharp magnitude forces the conjugate angle ϕ to have a uniform distribution. In other words, by the uncertainty principle *the point P in Fig.* 5.14 *is to be considered as uniformly probable on the circle.* One recognizes from this result (for the distribution of P) that the present calculation becomes very similar to the previous calculation for the Wigner coefficient.

[1] There is another geometric realization of the 6-j symbol that is important. This construction realizes the 6-j symbol as a complete quadrilateral in the plane. The six angular momenta are represented as *points*, and the triangles (faces) in Fig. 5.14 correspond now to three points lying on a straight line. This realization has been discussed in Topic 8.

The dihedral angle ϕ may be calculated from the normals to the $(\mathbf{J},\mathbf{J}',\mathbf{J}_3)$ and $(\mathbf{J}_1,\mathbf{J}_2,\mathbf{J}')$ planes. Thus, we find

$$\cos\phi=\frac{\mathbf{J}'\times\mathbf{J}}{\|\mathbf{J}'\times\mathbf{J}\|}\cdot\frac{\mathbf{J}'\times\mathbf{J}_1}{\|\mathbf{J}'\times\mathbf{J}_1\|}=\frac{(j')^2(\mathbf{J}_1\cdot\mathbf{J})-(\mathbf{J}'\cdot\mathbf{J}_1)(\mathbf{J}'\cdot\mathbf{J})}{\|\mathbf{J}'\times\mathbf{J}\|\,\|\mathbf{J}'\times\mathbf{J}_1\|}. \tag{5.9.21}$$

Differentiating this result with respect to time, one finds

$$\sin\phi\frac{d\phi}{dt}=\frac{(j')^2 j''(dj''/dt)}{\|\mathbf{J}'\times\mathbf{J}\|\,\|\mathbf{J}'\times\mathbf{J}_1\|}, \tag{5.9.22}$$

where we have used $(d/dt)(\mathbf{J}_1\cdot\mathbf{J})=(d/dt)(\mathbf{J}''\cdot\mathbf{J})=-j''(dj''/dt)$ and noted from Fig. 5.14 that all other terms are constant as P varies.

To determine $\sin\phi$ we use the cross products of the normals to the planes $(\mathbf{J},\mathbf{J}',\mathbf{J}_3)$ and $(\mathbf{J}_1,\mathbf{J}_2,\mathbf{J}')$. We find

$$\sin\phi=\frac{\|(\mathbf{J}'\times\mathbf{J})\times(\mathbf{J}'\times\mathbf{J}_1)\|}{\|\mathbf{J}'\times\mathbf{J}\|\,\|\mathbf{J}'\times\mathbf{J}_1\|}=\frac{j'(\mathbf{J}'\times\mathbf{J})\cdot\mathbf{J}_1}{\|\mathbf{J}'\times\mathbf{J}\|\,\|\mathbf{J}'\times\mathbf{J}_1\|}. \tag{5.9.23}$$

Combining this result with Eq. (5.9.22), we obtain

$$\left[(\mathbf{J}'\times\mathbf{J})\cdot\mathbf{J}_1\right]\frac{d\phi}{dt}=j'j''\frac{dj''}{dt}. \tag{5.9.24}$$

To determine the probability for a given value of j'', we use the relation[1]

$$P(j'')=2\left(\frac{dj''}{dt}\right)^{-1}=\frac{j'j''}{\pi(\mathbf{J}'\times\mathbf{J})\cdot\mathbf{J}_1}, \tag{5.9.25}$$

~~where $d\phi/dt$ has been replaced by 2π.~~

The probability $P(j'')$ has, however, by construction, the value given in Eq. (5.9.20):

$$P(j'')=P\equiv(2j'+1)(2j''+1)\begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j & j'' \end{Bmatrix}^2. \tag{5.9.26}$$

It follows that, in the classical limit where all six angular momenta become large, the 6-j symbol has the limiting value given by

$$\begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j & j'' \end{Bmatrix}^2\sim\frac{1}{4\pi(\mathbf{J}'\times\mathbf{J})\cdot\mathbf{J}_1}. \tag{5.9.27}$$

[1] The factor 2 comes from the fact that a generic value of j'' occurs twice as ϕ varies from 0 to 2π.

Using the fact that the volume of the tetrahedron in Fig. 5.14 is given by $V=\frac{1}{6}(\mathbf{J}'\times\mathbf{J})\cdot\mathbf{J}_1$, one may express this result geometrically:

$$\begin{Bmatrix} j_1\, j_2\, j' \\ j_3 j\, j'' \end{Bmatrix}^2 \sim \frac{1}{24\pi V}. \tag{5.9.28}$$

The classical limit of the square of the 6-j symbol (Racah coefficient) becomes equal to the reciprocal of 24π times the volume of the tetrahedron formed by the six angular momentum vectors whose lengths are the arguments in the coefficient.

Remarks. (*a*) We have emphasized in Chapter 3, AMQP, and in Chapter 4 of the present monograph (and again in Topic 12) that the most important angular momentum function is the Racah coefficient, basing this assertion on the fact that both the Wigner coefficients and the representation functions $d^j_{m'm}$ can be obtained from the Racah coefficient by the appropriate limiting procedure: The Wigner coefficient is the limit of the Racah coefficient, given by Eq. (3.280), Chapter 3, AMQP (see also Eq. (5.8.24), and Refs. [4] and [6]); both the Racah and the Wigner coefficients define in the limit the rotation functions $D^j_{m'm}$ as given explicitly by Eqs. (5.92) and (5.96) in AMQP (see also Refs. [2] and [7]). These limiting results accord with the classical limits found in the present section.

More generally, one sees that there are as many limits for the 6-*j* coefficient as there are partitions of the integer 4 (the number of vertices of the tetrahedron symbolizing the coefficient). Letting the partition [4] correspond to the 6-*j* symbol itself, we see that the partition [3 1] corresponds to the limit where one vertex is removed to large distances, yielding the Wigner coefficient whose classical limit is the area of the triangle (formed by the ~~three remaining vertices) projected onto the direction defined by the "re-~~ moved" vertex. The partition [2 1 1] corresponds to the limit $6\text{-}j\to d^j_{m'm}$, while the partition [1 1 1 1] corresponds to the classical limit itself. This analysis in terms of partitions shows that there is one further limit to consider, corresponding to the partition [2 2]. This limit—two pairs of vertices separated by a large distance—has been discussed in Ref. [4] (Eq. 2.11); it does not appear to be of great interest, other than for systematics.

It would have been possible, therefore, to have restricted the discussion in the sections above to the classical limit of the 6-*j* symbol alone, obtaining the other classical limits by specialization. We have chosen not to do this, since the significance of the uncertainty relations for the physical interpretation are most clearly brought out in the discussion of the Wigner coefficient.

(*b*) The classical limit for the Racah coefficient involved the volume of the tetrahedron formed from the six angular momenta belonging to the given coefficient. We call attention now to a remarkable fact concerning this tetrahedron.

First, let us recall that the general conditions for the 6-j coefficient $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ to be defined (exist) are that the four triads—(abc), (aef), (bdf), (cde)—satisfy the triangle conditions for angular momenta. These conditions imply that the sums defined by

$$\begin{aligned} q_1 &= a+b+c & p_1 &= a+b+d+e \\ q_2 &= a+e+f & p_2 &= a+c+d+f \\ q_3 &= b+d+f & p_3 &= b+c+e+f \\ q_4 &= c+d+e \end{aligned} \tag{5.9.29}$$

are all nonnegative integers and, from the triangle inequalities, satisfy the relations

$$p_i \geqslant q_j, \qquad i=1,2,3;\ j=1,2,3,4. \tag{5.9.30}$$

(From Fig. 5.14 with $j_1=a, j_2=b, j'=c, j_3=d, j=e$, and $j''=f$, one sees that the q_j are the perimeters of the faces of the tetrahedron, and the p_i are the sums of all six lengths, omitting one pair of opposite sides.)

Now let us ascribe physical reality to the tetrahedron whose sides are these six angular momenta (see Note 1). The volume V of this tetrahedron may be given in terms of the six edges by means of Cayley's formula [8]:

$$288V^2 = \det\begin{bmatrix} 0 & d^2 & e^2 & f^2 & 1 \\ d^2 & 0 & c^2 & b^2 & 1 \\ e^2 & c^2 & 0 & a^2 & 1 \\ f^2 & b^2 & a^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}. \tag{5.9.31}$$

The necessary and sufficient condition that the six edges $(a, \ldots, f)$ form a physical solid (tetrahedron) is that $V^2 > 0$.

It is a remarkable fact that *the triangle conditions or, equivalently, Eqs.* (5.9.29) *and* (5.9.30) *for the existence of the 6-j symbol are weaker than the condition that the tetrahedron exist as a realizable solid.*

Proof. To prove this remark, let us consider the sides $(b, \ldots, f)$ to be fixed in length and the side a to be variable; call this side x to emphasize its variability. Assuming the triangle conditions to be satisfied, we see that faces (bdf) and (cde) correspond to realizable triangles. [We assume that the variables are in a "general position" so that the area of each of the triangles, (bdf) and (cde), does not vanish.]

From Eq. (5.9.31), we can determine that $V^2(x^2)$ is a quadratic polynomial in the variable x^2. Hence, there are two roots for which $V^2=0$; these two roots correspond geometrically to the two possible planar positions for the triangles (bdf) and (cde). Since the curvature is negative—that is, $\partial^2 V^2(x^2)/(\partial x^2)^2 = -d^2/72$—the square of the volume is positive ($V^2>0$) for x^2 lying between these two roots; hence, the tetrahedron exists for all x^2 that satisfy the condition $x^2_{\mathrm{min}} \leqslant x^2 \leqslant x^2_{\mathrm{max}}$.

When the tetrahedron is flat ($V^2=0$), the triangles (xef) and (xbc)—which exist from the triangular conditions—will in general have nonvanishing area. It follows that the condition that the tetrahedron be flat ($V^2=0$) is weaker than the triangle conditions, as asserted. ■

These remarks indicate that nonvanishing 6-j symbols can correspond geometrically to *negative* (volume)^2! [In fact, the fundamental Racah coefficient $W(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1)$ corresponds to $V^2=-(\frac{1}{12})^2$.] The meaning of this surprising (and intriguing!) observation will be explored in the next subsection.

5. The Nature of the Classical Limit

The nature of the classical limit has been well investigated in quantum mechanics,[1] and has been particularly clarified in Feynman's path-integral approach (see Feynman and Hibbs [9]), which extended Dirac's discussion of the role of the classical Lagrangian in quantum theory. In the approach to the classical limit, the probability amplitude $\psi(x)$ takes the form

$$\begin{aligned}\psi(x) \sim &(\text{slowly varying magnitude})\\ &\times(\text{rapidly varying phase of modulus } 1),\end{aligned}$$

with the rapidly varying phase corresponding to $\exp[iS(x)]$, where the classical action S is measured in units of $\hbar$ (hence, S is numerically large in the classical limit).

The classical limit considers the absolute square $|\psi(x)|^2$ and hence yields only the slowly varying magnitude. The probability amplitudes considered in the previous sections are real, so that the classical limit is an average of the square, yielding again the slowly varying magnitude.

These considerations are valid in regions where the classical motion is allowed—that is, in regions where the kinetic energy is positive. At the classical turning points for the motion, the kinetic energy vanishes, and the classically defined probability, which is proportional to (velocity)$^{-1}$, becomes (weakly) singular. In classically unallowed regions, the kinetic energy

[1]A mathematically oriented discussion is in Guillemin and Sternberg [8a].

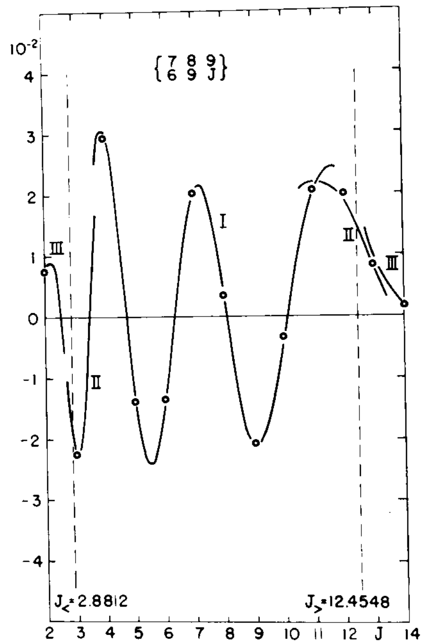

Figure 5.15. Values of the 6-j coefficient (circles) compared with smooth curves corresponding to the asymptotic formulas, Eq. (5.9.33) (classically allowed region I), Eq. (5.9.97) (transition region II), and Eq. (5.9.90) (classically forbidden region III). (From G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients," in *Spectroscopic and Group Theoretical Methods in Physics* (F. Block *et al.*, eds.), pp. 1–58. Wiley (Interscience), New York, 1968. Reprinted by permission.)

becomes negative, and the probability becomes exponentially decreasing ("quantum mechanical tunneling").

The behavior of the 6-j symbol for large values of the six arguments accords exactly with these expected semiclassical characteristics. *The role of the kinetic energy is played for the* 6-j *symbol by the classical* (*volume*)^2. In regions where $V^2 > 0$ ("classically allowed region"), the 6-j coefficient is rapidly oscillating with an envelope for the oscillations being given by Wigner's result, Eq. (5.9.28). In regions where $V^2 < 0$ ("classically forbidden region"), the 6-j coefficient is exponentially decreasing. The transition region is determined by $V^2 = 0$, and, as expected, Eq. (5.9.28) is singular here; the 6-j coefficient is, of course, well-defined in the transition region, and it is only the limitations of the classical limit that lead to this (weak) singularity.

This behavior is illustrated in Figs. 5.15, 5.16, and 5.17, which show at the same time the Ponzano–Regge asymptotic approximations for the 6-j symbol, as we now discuss.

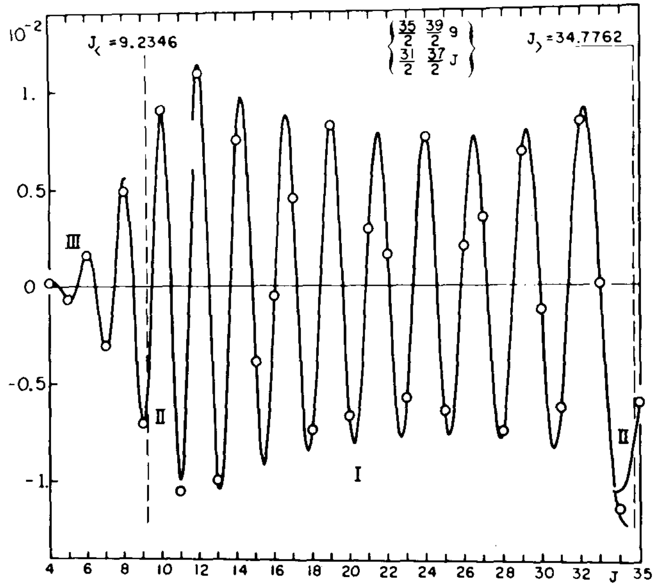

Figure 5.16. For high J the agreement between the asymptotic formulas and the values of the 6-j coefficient improves. (From G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients," in *Spectroscopic and Group Theoretical Methods in Physics* (F. Block *et al.*, eds.), pp. 1–58. Wiley (Interscience), New York, 1968. Reprinted by permission.)

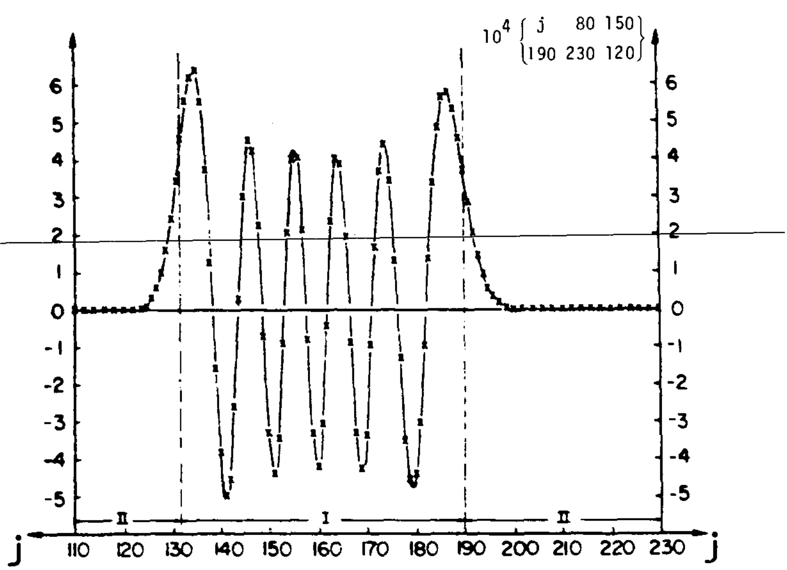

Figure 5.17. The exponential decay of the 6-j coefficient in the classically forbidden region is emphasized here. (From K. Schulten and R. G. Gordon, "Exact recursive evaluation of 3j- and 6j- coefficients for quantum-mechanical coupling of angular momenta," J. Math. Phys. **16** (1975), 1961–1988. Reprinted by permission.)

6. The Ponzano–Regge Asymptotic Relations for the 6-j Symbol

The generalization of the classical results given by Wigner for the 6-j symbol, Eq. (5.9.28), to include the required rapidly oscillating phase was carried out by Ponzano and Regge [4]. These authors also developed asymptotic relations for the 6-j coefficient, suitable for calculating approximate results in all three regions (I, classically allowed; II, transition; III, classically unallowed). These results will be discussed in this section (see Note 2).

The same qualitative physical arguments that are used in discussing the nature of the classical limit suggest at the same time the form of the rapidly oscillating phase factor. In classically allowed regions far from turning points, the phase in the probability amplitude for one-dimensional motion has the form $\exp(ixp/\hbar)$, where x and p are a *canonically conjugate pair*, which are, in this example, position and momentum. On physical grounds, therefore, one expects the phase to involve the conjugate pair {(magnitude of angular momentum), (angle of rotation around direction of angular momentum)}.

It is useful to employ the systematic notation shown in Fig. 5.18. *It is essential to note that the edges of the tetrahedron have been increased by* $\frac{1}{2}$ *over the corresponding value in the* 6-j *symbol* (see Note 1). The angular momentum j_{hk} denotes the side *opposite* to the line joining vertices P_h and P_k; the angle θ_{hk} is the angle between the *outward* normals to the planes h and k (these planes are not shown in Fig. 5.18). (The plane h is the plane opposite vertex h.) Thus, θ_{hk} is indeed an angle of rotation around j_{hk}. For notational purposes, it is also convenient to define $j_{hh}=0$, $j_{kh}=j_{hk}$, $\theta_{hh}=0$ and $\theta_{kh}=\theta_{hk}$.

By symmetry one expects the phase to involve all six angular momenta ~~and angles symmetrically. Hence, one expects the phase factor to be~~ $\exp\left(i\sum_{h<k} j_{hk}\theta_{hk}\right)$, where[1]

$$\sum_{h<k} j_{hk}\theta_{hk}=j_{12}\theta_{12}+j_{13}\theta_{13}+j_{23}\theta_{23}+j_{14}\theta_{14}+j_{24}\theta_{24}+j_{34}\theta_{34}. \quad (5.9.32)$$

In the *classically allowed region* ($V^2>0$) the Ponzano–Regge asymptotic form is

$$\begin{Bmatrix} a\,b\,c \\ d\,e\,f \end{Bmatrix} \sim [12\pi V]^{-\frac{1}{2}}\cos\left[\left(\sum_{h<k} j_{hk}\theta_{hk}\right)+\frac{\pi}{4}\right]. \quad (5.9.33)$$

[1]Ponzano and Regge [4] introduce the summation symbol $\sum_{h,k=1}^{4} j_{hk}\theta_{hk}$ using the symmetrized definitions $j_{hh}=0$, $j_{hk}=j_{kh}$, $\theta_{hk}=\theta_{kh}$. However, on the basis of their actual usage of this symbol in Ref. [4] (pp. 14, 15, 56), we interpret their meaning to be that of Eq. (5.9.32) and not *twice* this result, as implied by their notation.

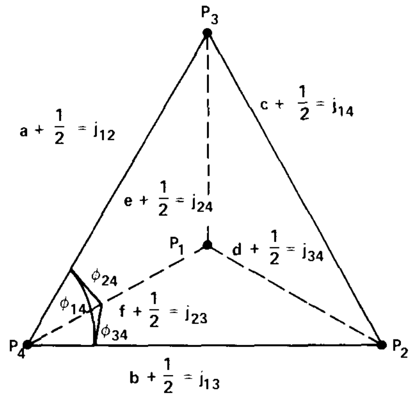

Figure 5.18. Notation used in asymptotic formulas for the 6-j symbol $\left\{\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right\}$.

An explicit definition of the angles θ_{hk} is given by

$$A_h A_k \sin\theta_{hk} = \tfrac{3}{2} V j_{hk}, \qquad h=k=1,2,3,4, \tag{5.9.34}$$

where A_h is the area of the face opposite vertex P_h.

7. Symmetry Properties

It is a remarkable geometric fact that each of the terms in Eq. (5.9.33) (magnitude, phase factor) is separately invariant under the 144 symmetries of the 6-j symbol.

Invariance of the volume. (a) It is geometrically quite obvious that the 24 "tetrahedral" symmetries do leave the volume V invariant. (b) By contrast, under the Regge symmetries generated by the transformation [see Eq. (3.313), AMQP],

$$R: a \to a, \qquad b \to (b+c+e-f)/2, \qquad c \to (b+c+f-e)/2, \tag{5.9.35}$$

$$d \to d, \qquad e \to (b+e+f-c)/2, \qquad f \to (c+e+f-b)/2,$$

it is far from obvious that the volume V is, in fact, invariant. [The proof consists in demonstrating directly that the Cayley determinant (5.9.31) is unchanged. This verification is laborious and uninformative and, hence, omitted.]

Invariance of the phase. The fact that the expression $\sum_{h<k} j_{hk}\theta_{hk}$ is invariant under the 144 symmetries of the 6-j symbol was demonstrated in Appendix D of Ref. [4]; we shall give this proof below. It is once again

obvious that the tetrahedral symmetries leave this form invariant; it is only the Regge symmetries that require proof.

Proof. Let us first recall some geometric relations for the tetrahedron (see Fig. 5.18). We have defined the lengths j_{hk} and the conjugate angles θ_{hk} in terms of the four vertices P_h. It is useful to define the quantities q_h $(h=1,2,3,4)$ and p_{hk} $(h<k)$ by

$$
\begin{aligned}
q_h &\equiv \sum_k j_{hk},\\
p_{12}=p_{34} &\equiv j_{13}+j_{14}+j_{23}+j_{24},\\
p_{13}=p_{24} &\equiv j_{12}+j_{14}+j_{23}+j_{34},\\
p_{14}=p_{23} &\equiv j_{12}+j_{13}+j_{24}+j_{34}.
\end{aligned}
\tag{5.9.36}
$$

We also let ϕ_{hk} denote the angle about the vertex P_k in the plane opposite to vertex P_h. Using the law of sines and defining the quantity Σ_4, we find

$$
\begin{aligned}
2\Sigma_4 \equiv \sin\theta_{12}\sin\theta_{23}\sin\phi_{24} &= \sin\theta_{13}\sin\theta_{23}\sin\phi_{34}\\
&= \sin\theta_{12}\sin\theta_{13}\sin\phi_{14},
\end{aligned}
\tag{5.9.37}
$$

and similarly for Σ_h.

Let us also define, with respect to the vertex P_h, the quantity σ_h given by

$$
\sigma_h \equiv \sum_{k,l\neq h} \theta_{kl}/4, \tag{5.9.38}
$$

so that, for example,

$$
\sigma_1 = \tfrac{1}{2}(\theta_{23}+\theta_{24}+\theta_{34}).
$$

In terms of these variables, Euler (see Hammer [10]) has shown that

$$
\Sigma_h \tan\tfrac{1}{2}\phi_{hk} = \sin\sigma_h \sin(\sigma_h - \theta_{lm}), \tag{5.9.39}
$$

where $(hklm)$ is any permutation of (1234), and, moreover, that

$$
K \equiv \frac{9}{4}\,\frac{V^2}{A_1A_2A_3A_4} = \frac{\Sigma_k}{A_k}, \tag{5.9.40}
$$

so that Σ_k/A_k is independent of k.

A symmetric presentation of the angles ϕ_{hk} in terms of the sides of the tetrahedron is afforded by the form

$$\tan\frac{1}{2}\phi_{hm}=\left[\frac{(p_{hl}-q_l)(p_{hk}-q_k)}{q_h(p_{hm}-q_m)}\right]^{\frac{1}{2}}, \tag{5.9.41}$$

in which $(hklm)$ is a permutation of (1234).

The final auxiliary variables we shall need are the following two arrays (a) and (b):

(a) Augmented Regge array:

$$[r_{st}]\equiv\begin{bmatrix} p_{14}-q_1 & p_{13}-q_1 & p_{12}-q_1 & q_1 \\ p_{14}-q_2 & p_{13}-q_2 & p_{12}-q_2 & q_2 \\ p_{14}-q_3 & p_{13}-q_3 & p_{12}-q_3 & q_3 \\ p_{14}-q_4 & p_{13}-q_4 & p_{12}-q_4 & q_4 \end{bmatrix}. \tag{5.9.42}$$

(This array is the same as the 4×3 array used in discussing the Regge symmetry of the 6-j symbol in Chapter 3, Section 18, AMQP, augmented by the fourth column.)

(b) Angle array:

$$\left[\tfrac{1}{2}\vartheta_{st}\right]=\begin{bmatrix} \sigma_4-\theta_{23} & \sigma_3-\theta_{24} & \sigma_2-\theta_{34} & \sigma_1 \\ \sigma_3-\theta_{14} & \sigma_4-\theta_{13} & \sigma_1-\theta_{34} & \sigma_2 \\ \sigma_2-\theta_{14} & \sigma_1-\theta_{24} & \sigma_4-\theta_{12} & \sigma_3 \\ \sigma_1-\theta_{23} & \sigma_2-\theta_{13} & \sigma_3-\theta_{12} & \sigma_4 \end{bmatrix}. \tag{5.9.43}$$

The essential relation to establish is that

$$\cos\vartheta_{st}=1-288V^2(r_{st})^2\left[\left(\prod_{n=1}^{4}r_{sn}\right)\left(\prod_{n=1}^{4}r_{nt}\right)\right]^{-1} \tag{5.9.44}$$

for $s\neq t$. This result follows directly from Eqs. (5.9.39) and (5.9.41).

To establish the invariance of the phase, we first note that, under the Regge transformation R given by Eq. (5.9.35), the array $[r_{st}]$ is mapped into the new array $[r'_{st}]$ obtained from $[r_{st}]$ by interchanging rows 3 and 4.

$$\begin{aligned} R:[r_{st}]&\to[r'_{st}], \\ r_{st}&\to r_{s't'}. \end{aligned} \tag{5.9.45}$$

Using this transformation in Eq. (5.9.44), one sees that

$$R:\cos\vartheta_{st}\to\cos\vartheta_{s't'}. \tag{5.9.46}$$

To establish that the angles themselves (and not just their cosines) undergo this transformation, that is, to show that

$$R\colon \theta_{st} \to \theta_{s't'}, \tag{5.9.47}$$

Regge and Ponzano argue that this result is true for an almost regular tetrahedron, where the angles θ_{hk} and θ'_{hk} lie in the interval 0 to $\pi/2$. Then by an analytic continuation the result is established generally.

We conclude: *Under the* 144 *symmetries of the* 6-*j symbol, the arrays* $[r_{st}]$ *and* $[\theta_{st}]$ *suffer the same orthogonal transformation, hence, the sum* $\sum_{s<t} j_{st}\theta_{st}$ *is invariant.* ■

The invariance of the volume of the tetrahedron, the product $A_1A_2A_3A_4$, and the phase $\sum_{h<k} j_{hk}\theta_{hk}$ under the symmetry R are surprising, and non-trivial, geometric facts.

8. Proof of the Ponzano–Regge Relation

The strategy to be used in proving the asymptotic formula, Eq. (5.9.33), is to show that this formula satisfies, asymptotically, the three fundamental identities for the Racah coefficient, Eqs. (3.273)–(3.275), Chapter 3, AMQP [Eqs. (5.8.3)–(5.8.5), this volume]. As discussed in Chapter 3, AMQP, these identities (plus the symmetries that have already been shown to be valid) suffice to define the Racah coefficient to within an overall sign (see Note 3).

The three identities all involve a summation over an angular momentum magnitude taken between limits imposed by the triangle conditions. Since the asymptotic limit assumes all angular momenta to be large, we shall consider it valid to replace the sum by an integral. The resulting integrals will then be evaluated asymptotically by the method of stationary phase, a technique that, in physics, is synonymous with the classical approximation itself (Dirac [11], Feynman and Hibbs [9]).

Let us consider first the B–E identity [Eq. (2.69) or (5.8.5)], which may be written in terms of 6-j symbols as

$$\begin{Bmatrix} g & h & j \\ e & a & d \end{Bmatrix}\begin{Bmatrix} g & h & j \\ e' & a' & d' \end{Bmatrix} = \sum_x (-1)^{\Phi_x}(2x+1)\begin{Bmatrix} a & a' & x \\ d' & d & g \end{Bmatrix}\begin{Bmatrix} d & d' & x \\ e' & e & h \end{Bmatrix}\begin{Bmatrix} e & e' & x \\ a' & a & j \end{Bmatrix}, \tag{5.9.48}$$

where $\Phi_x = g+h+j+e+a+d+e'+a'+d'+x$. With the 6-$j$ symbols represented by tetrahedra in 3-space, the geometric configuration corresponding to this identity is then given by Fig. 5.19. This configuration consists of five points, ten lines (angular momenta), ten faces, and five tetrahedra.

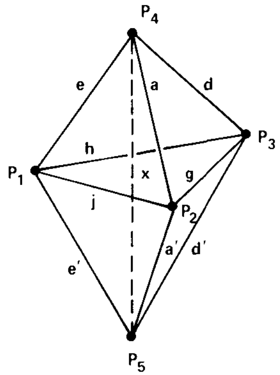

Figure 5.19. The five tetrahedra corresponding to the B-E identity.

Introducing the Ponzano–Regge expression (5.9.33) for the 6-j symbols in the right-hand side of Eq. (5.9.48) and denoting for simplicity the three 6-j symbols by 1, 2, 3, we find

$$\mathcal{G} \equiv \mathrm{RHS} \simeq (12\pi)^{-\frac{3}{2}} \sum_x (2x+1)(-1)^{\Phi_x}\left[V_1(x)V_2(x)V_3(x)\right]^{-\frac{1}{2}}$$
$$\times \cos\Omega_1(x)\cos\Omega_2(x)\cos\Omega_3(x), \tag{5.9.49}$$

where

$$\Omega_i(x)=\left(\sum_{h<k} j_{hk}^{(i)}\theta_{hk}^{(i)}\right)+\frac{\pi}{4}, \tag{5.9.50}$$

in which, for example,

$$\begin{aligned} &j_{12}^{(1)}=a+\tfrac{1}{2}, && j_{13}^{(1)}=a'+\tfrac{1}{2}, && j_{14}^{(1)}=x+\tfrac{1}{2},\\ &j_{34}^{(1)}=d'+\tfrac{1}{2}, && j_{24}^{(1)}=d+\tfrac{1}{2}, && j_{23}^{(1)}=g+\tfrac{1}{2}. \end{aligned} \tag{5.9.51}$$

Thus, the $\{j_{hk}^{(1)}\}$ are those associated in Fig. 5.18 with the 6-j symbol $\begin{Bmatrix} a & a' & x \\ d' & d & g \end{Bmatrix}$. Accordingly, $\theta_{hk}^{(1)}$ is the angle between the outward normals to the two faces that share the edge $j_{hk}^{(1)}$ in the tetrahedron in Fig. 5.19 corresponding to $\begin{Bmatrix} a & a' & x \\ d' & d & g \end{Bmatrix}$ and having volume $V_1(x)=(\mathbf{x}\times\mathbf{d}\cdot\mathbf{a})/6$. The $\{j_{hk}^{(2)},\theta_{hk}^{(2)}\}$, $\{j_{hk}^{(3)},\theta_{hk}^{(3)}\}$, and the volumes $V_2(x)$ and $V_3(x)$ have a similar definition in terms of the tetrahedra corresponding to the 6-j symbols $\begin{Bmatrix} d & d' & x \\ e' & e & h \end{Bmatrix}$ and $\begin{Bmatrix} e & e' & x \\ a' & a & j \end{Bmatrix}$, respectively.

Now let each of the ten angular momenta in Fig. 5.19 be scaled by the (positive) length R— that is,

$$x+\tfrac{1}{2}=R\xi,\, a+\tfrac{1}{2}=R\alpha,\ldots,e+\tfrac{1}{2}=R\varepsilon,\ldots,e'+\tfrac{1}{2}=R\varepsilon'. \quad (5.9.52)$$

We take R to be numerically large and replace the summation over x in Eq. (5.9.49) by an integration over ξ. This step is straightforward and can be justified except for the treatment of the phase factor Φ_x, which must be kept integral-valued. Let us defer this problem for the moment.

Using the scaled volume $V_i'(\xi)$ defined by

$$R^3 V_i'(\xi)=V_i(x), \quad (5.9.53)$$

the summation expression (5.9.49) for $\mathcal{G}$ is now replaced by the integral

$$\mathcal{G}\simeq(12\pi)^{-\frac{3}{2}}\frac{R}{4}^{-\frac{5}{2}}\int\xi\, d\xi\left[V_1'(\xi)V_2'(\xi)V_3'(\xi)\right]^{-\frac{1}{2}}P(\xi). \quad (5.9.54)$$

In this expression, $P(\xi)$ denotes the rapidly varying phase of the integrand, obtained by writing the cosine factors in Eq. (5.9.49) as exponentials and using the scaling transformation (5.9.52):

$$P(\xi)=\sum_{k=1}^{4}\left(e^{iR\Gamma_k(\xi)}+e^{-iR\Gamma_k(\xi)}\right), \quad (5.9.55)$$

---

where $R\Gamma_k(\xi)$ for $k=1,2,3,4$ are defined by

$$R\Gamma_k(\xi)\equiv\Gamma_k(x)\equiv\Omega_1(x)\pm\Omega_2(x)\pm\Omega_3(x)+\pi\Phi_x. \quad (5.9.56)$$

The signs in the second and third terms in this result are $(+,+)$ for $k=1$, $(+,-)$ for $k=2$, $(-,+)$ for $k=3$, and $(-,-)$ for $k=4$.

The crucial remark to make now is that *the stationary phase requirement on the rapidly varying phase P is identical to the requirement that the configuration shown in Fig. 5.19 correspond to five points embedded in 3-space.*

To demonstrate this remarkable result (due to Ponzano and Regge), we first note that the condition that five points lie in 3-space is the requirement

(Cayley [8]) that the 4-space volume V_4 defined by

$$-2^4(4!)^2 V_4^2 = \det\begin{bmatrix} 0 & j^2 & h^2 & e^2 & (e')^2 & 1 \\ j^2 & 0 & g^2 & a^2 & (a')^2 & 1 \\ h^2 & g^2 & 0 & d^2 & (d')^2 & 1 \\ e^2 & a^2 & d^2 & 0 & x^2 & 1 \\ (e')^2 & (a')^2 & (d')^2 & x^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{5.9.57}$$

should *vanish*.

Taking the length x to be variable, we see that Eq. (5.9.57) implies a quadratic equation in x^2 that has two roots, $x_>^2$ and $x_<^2$. These two roots correspond to the two realizable geometric configurations: The first realization is given in Fig. 5.19 and corresponds to the larger root $x_>^2$, where the points P_4 and P_5 lie on opposite sides of the plane containing the triangle (P_1, P_2, P_3); the second realization is given in Fig. 5.20 and corresponds to the smaller root $x_<^2$, where the points P_4 and P_5 lie on the same side of the plane.

Let us consider now the stationary phase condition. The rapidly varying phases are the eight terms $e^{i\Gamma_k(x)}$ and their complex conjugates (c.c.) in Eq. (5.9.55). Under a first-order variation of the lengths of the edges in the expression for $\Omega_i(x)=[\sum_{h<k} j_{hk}^{(i)}\theta_{hk}^{(i)}]+\frac{\pi}{4}$, we may consider all angles $\theta_{hk}^{(i)}$ to be constant.[1] Keeping all lengths except x constant [see Eqs. (5.9.51)] then leads to the variation

$$\frac{\partial \Gamma_k(x)}{\partial x} = \pm\theta_x^1 \pm \theta_x^2 \pm \theta_x^3 + \pi = 0, \tag{5.9.58}$$

where we have defined $\theta_x^i = \theta_{14}^{(i)}$ $(i=1,2,3)$. The angles $\pi-\theta_x^i$ may then be verified to be the internal dihedral angles around the edge x in Fig. 5.19. The signs $(\pm)$ in this result denote the eight possible results coming from Eq. (5.9.56) and the conjugate factor in Eq. (5.9.55). The additional π comes from the phase factor Φ_x.

The stationary phase condition (5.9.58) *is precisely the geometric condition that the sum of the internal dihedral angles* $\pi-\theta_x^i$ *around the edge* x *be a multiple of* 2π. This is simply the condition that the configuration be realizable in 3-space, as asserted.

[1]This important result was shown in an n-dimensional context by Regge [12].

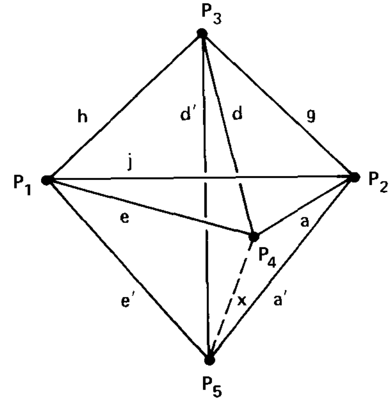

Figure 5.20.

Thus, we see that the stationary phase condition selects out the two realizable configurations (corresponding to the $x^2_>$ and $x^2_<$ roots), and leads to four nonvanishing terms for the integral (each term occurs with its complex conjugate).

It remains only to carry out this evaluation using the method of steepest descents.

Evaluation of the integral. The method of steepest descents [see Refs. [13]–[16]) evaluates asymptotically integrals of the general form

$$J(R) \equiv \int_C g(\xi) e^{iRf(\xi)} d\xi \tag{5.9.59}$$

in the limit where $R \to \infty$. [It is assumed here that $f(\xi)$ and $g(\xi)$ are real functions.] This method yields for the integral the asymptotic result

$$J(R) \sim \sum_j \left(\frac{2\pi}{R|f''(\xi_j)|} \right)^{\frac{1}{2}} g(\xi_j) \exp i\left[Rf(\xi_j) \pm \frac{\pi}{4} \right], \tag{5.9.60}$$

where (a) the ξ_j are the points of stationary phase for which $f'(\xi_j) = 0$, and which lie on the contour C (suitably deformed if necessary), and (b) the phase $\pm\pi/4$ is chosen according to $f''(\xi_j) \gtrless 0$. In the application at hand, the function $g(\xi)$ is given by

$$g(\xi) = \tfrac{1}{4}(12\pi)^{-\frac{3}{2}} R^{-\frac{5}{2}} \xi [V_1'(\xi) V_2'(\xi) V_3'(\xi)]^{-\frac{1}{2}}, \tag{5.9.61}$$

and $Rf(\xi) = \pm\Omega_1(x) \pm \Omega_2(x) \pm \Omega_3(x) + \pi\Phi_x$. There are two roots for the stationary phase condition, $f'(\xi) = 0$, as discussed above.

The real problem in this evaluation concerns the determination of the second derivative, $f''(\xi)$, at the points where $f'(\xi)=0$. We have already determined that $f'(\xi)=0$ is equivalent to the condition[1] $\theta_x^1+\theta_x^2+\theta_x^3+\pi=0(\bmod 2\pi)$.

Thus, we must evaluate the derivative

$$f''(\xi)=\frac{\partial\left(\theta_x^1+\theta_x^2+\theta_x^3\right)}{\partial\xi} \tag{5.9.62}$$

at the points where $\theta_x^1+\theta_x^2+\theta_x^3+\pi=0(\bmod 2\pi)$. This difficult task is carried out in Ref. [4], Appendix E, where it is shown that

$$f''(\xi)=\mp\tfrac{1}{6}(\xi_{\gtrless})^2V_4'V_5'/[V_1'V_2'V_3']_{\xi=\xi_{\gtrless}}, \tag{5.9.63}$$

where V_4' and V_5' denote the volumes of the tetrahedra corresponding to the 6-j symbols in the left-hand side of Eq. (5.9.48).

Using Eqs. (5.9.63) and (5.9.61) in Eq. (5.9.60), we find the integral in Eq. (5.9.54) to have the asymptotic value

$$\begin{aligned}\mathcal{G}&\sim(12\pi)^{-1}(V_4V_5)^{-\frac{1}{2}}\tfrac{1}{4}\left[e^{i(\Omega_4+\Omega_5)}+e^{i(\Omega_4-\Omega_5)}+\text{c.c.}\right]\\&=(12\pi V_4)^{-\frac{1}{2}}\cos\Omega_4(12\pi V_5)^{-\frac{1}{2}}\cos\Omega_5,\end{aligned} \tag{5.9.64}$$

where $\{V_4,\Omega_4\}$ and $\{V_5,\Omega_5\}$ denote, respectively, the volumes and angle expressions of the form (5.9.33) for the tetrahedra corresponding to the 6-j symbols $\begin{Bmatrix} g\,h\,j\\ e\,a\,d\end{Bmatrix}$ and $\begin{Bmatrix} g & h & j\\ e' & a' & d'\end{Bmatrix}$.

Thus, we have verified that the Ponzano–Regge asymptotic result for the 6-j symbol satisfies the B–E identity, Eq. (5.9.48).

~~*Asymptotic evaluation of Racah's identity.* To complete the proof of the~~ Ponzano–Regge result, we next show that the remaining two identities, Racah's sum rule and the orthogonality relation, are also verified. This step was not carried out in Ref. [4], although it is implied to be valid. It is quite instructive to carry out the proof explicitly, for, in the case of Racah's identity, we are led to interesting new viewpoints and results.

The Racah sum rule (identity) may be written in the form

$$\sum_x(-1)^{f+g+x}(2x+1)\begin{Bmatrix} a\,b\,x\\ d\,e\,f\end{Bmatrix}\begin{Bmatrix} d\,e\,x\\ b\,a\,g\end{Bmatrix}=\begin{Bmatrix} a\,e\,f\\ b\,d\,g\end{Bmatrix}. \tag{5.9.65}$$

[1]For each root two terms (a given term and its complex conjugate) contribute to the integral; precisely which term occurs depends on the angular momenta, and this determines the $\pm$ signs in (5.9.58). We choose a particular set of signs arbitrarily.

Just as in the previous case, our procedure in verifying this identity for the Ponzano–Regge asymptotic form will be to introduce the relation (5.9.33) into the left-hand side, convert the sum to an integral, and evaluate by steepest descents. The result will then be shown to be the asymptotic limit of the right-hand side, thus validating the identity.

There is an immediate dilemma that confronts one in attempting to carry out this procedure in explicit detail: *The geometric representation for each of the three* 6-j *symbols that appear in this identity cannot be uniformly realized in* 3-*space*! It is easy to show this: Consider Fig. 5.21, which depicts each of the three 6-j symbols as a tetrahedron. Consider, in particular, sides a and b. In the two tetrahedra involved in the sum (the two on the left in Fig. 5.21) the sides a and b are *joined* at a vertex, but in the tetrahedron on the right, the sides a and b are *opposite*. Clearly, no deformation can make these structures "fit together" in the nice geometric way in which the previous case fitted three tetrahedra together, forming two new tetrahedra.

[There is one possible resolution that can be eliminated. This uses the fact that in a planar representation of the 6-j symbols there *is* a perfectly good representation of these three 6-j symbols simultaneously (as the figure showing the three diagonal points of a quadrangle). But this is no help, since we are forced to use a three-dimensional representation by the Ponzano–Regge asymptotic form itself.]

The resolution of this dilemma lies in the recognition that there exists a *third* representation of the 6-j symbol, as a figure in a real projective plane.[1] Since the sphere in 3-space with opposite points identified is a homeomorph of the real projective plane, it is possible to exhibit this representation of the 6-j symbol as a three-dimensional solid (with a center of symmetry), as shown in Fig. 5.22.

In this figure, each of the six lines (angular momenta) in the 6-j symbol $\begin{Bmatrix} a & e & f \\ b & d & g \end{Bmatrix}$ appears twice; similarly, each of the four triads (aef), (bdf), (adg), and (beg) appears twice (with opposite orientation). As a figure in the projective plane, we have three vertices, six lines, and four faces; it follows that the Euler characteristic is $\chi = 1-3+6-4+1=1$, as required for a one-sided surface.

The way one fits together geometrically this new representation for the 6-j symbol $\begin{Bmatrix} a & e & f \\ b & d & g \end{Bmatrix}$ with the two tetrahedra on the left in Fig. 5.21 is to observe that the line x decomposes the diagram in Fig. 5.22 into precisely four tetrahedra; these are the desired two tetrahedra T_1 and T_2 together with the tetrahedra T_1' and T_2' having orientations opposite to T_1 and T_2.

[1]The other two representations are the Racah–Fano planar realization (as a complete quadrilateral) and the Regge–Wigner representation as a tetrahedron. This third realization does not, to our knowledge, appear in the literature, and seems to be new.

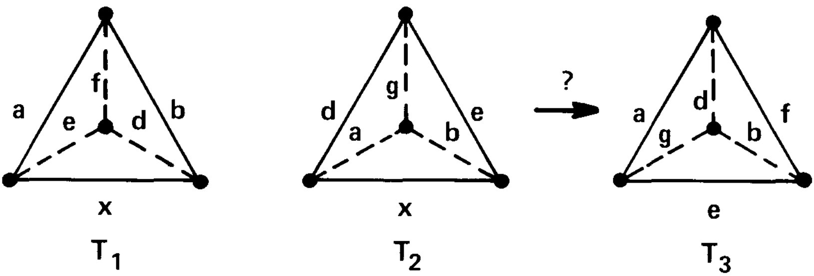

Figure 5.21.

This unusual representation of the Racah coefficient nicely resolves our dilemma, and correctly implies that the stationary phase point is that value of x for which the joining of the two tetrahedra T_1 and T_2 yields a plane (for example, the plane containing the five lines d, e, x, e, and d). This geometric joining constraint thus implies that the angles θ_x^i satisfy the relation

$$\theta_x^1 + \theta_x^2 + \pi \equiv 0 \pmod{2\pi}. \tag{5.9.66}$$

It is quite surprising, in our view, that there does exist a 3-space representation realizing two 6-j symbols in one way, as T_1 and T_2, while the third 6-j symbol is realized in a totally different, yet geometrically compatible way!

Let us now verify these assertions. We introduce the Ponzano–Regge form (5.9.33) for the 6-j coefficients into the left-hand side of Eq. (5.9.65), make the scale transformation (5.9.52), and replace the summation over x by an integration over ξ. The result of these steps is the following approxi-

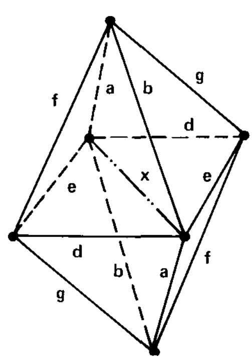

Figure 5.22. The geometric realization of the Racah sum formula.

mation (for large R) to the left-hand side of the Racah identity (5.9.65):

$$\mathrm{LHS} \equiv \mathcal{I} \simeq \tfrac{1}{2}(12\pi R)^{-1} \int \xi \, d\xi \left[V_1'(\xi) V_2'(\xi)\right]^{-\frac{1}{2}} P(\xi), \tag{5.9.67}$$

where

$$P(\xi) = \sum_{k=1}^{2} \left(e^{iR\Gamma_k(\xi)} + \text{c.c.}\right); \tag{5.9.68}$$

$$\begin{aligned} R\Gamma_1(\xi) &\equiv \Gamma_1(x) \equiv \Omega_1(x) + \Omega_2(x) + \pi\Phi_x, \\ R\Gamma_2(\xi) &\equiv \Gamma_2(x) \equiv \Omega_1(x) - \Omega_2(x) + \pi\Phi_x; \end{aligned} \tag{5.9.69}$$

$$\begin{aligned} \Omega_i(x) &= \left(\sum_{h<k} j_{hk}^{(i)} \theta_{hk}^{(i)} \right) + \frac{\pi}{4}, \\ \Phi_x &= f + g + x. \end{aligned} \tag{5.9.70}$$

In particular, we have that $x + \frac{1}{2} = j_{14}^{(1)} = j_{14}^{(2)}$, while the angles $\theta_x^1 \equiv \theta_{14}^{(1)}$ and $\theta_x^2 \equiv \theta_{14}^{(2)}$ are defined, respectively, to be the angles between the outward normals to the faces of the tetrahedra T_1 and T_2 that share the side x. (In general, the $\{j_{hk}^{(1)}, \theta_{hk}^{(1)}\}$ and $\{j_{hk}^{(2)}, \theta_{hk}^{(2)}\}$ in Eq. (5.9.70) are defined in terms of the tetrahedra $T_1 \leftrightarrow \begin{Bmatrix} a & b & x \\ d & e & f \end{Bmatrix}$ and $T_2 \leftrightarrow \begin{Bmatrix} d & e & x \\ b & a & g \end{Bmatrix}$ according to the generic rules given in Fig. 5.18.)

Just as before, the stationary phase point(s) are obtained from the condition

$$\frac{\partial \Gamma_k(x)}{\partial x} = \pm\theta_x^1 \pm \theta_x^2 + \pi = 0. \tag{5.9.71}$$

This is just the geometric joining condition (5.9.66).

It is important to recognize that there is now only *one* stationary phase point, since, when the tetrahedra T_1 and T_2 join, the geometry of the configuration implies that

$$x^2 = a^2 + b^2 + d^2 + e^2 - f^2 - g^2. \tag{5.9.72}$$

[When the phase condition is met, the three planes contain (*abab*), (*fgfg*), and (*dede*) as parallelograms (see Fig. 5.22), and the above result for x^2 follows. Alternatively, we may impose the "joining condition" in a different form by requiring that the four vertices (of each "parallelogram") lie in a plane.] It follows that at the stationary phase point only two of the four exponentials survive—a term and its complex conjugate.

Using the method of steepest descents, we find the asymptotic value of the integral to be

$$\mathcal{I} \simeq (12\pi R)^{-1}\xi_0[V_1'(\xi_0)V_2'(\xi_0)]^{-\frac{1}{2}}$$
$$\times\left(\frac{2\pi}{R|f''(\xi_0)|}\right)^{\frac{1}{2}}\frac{1}{2}\left(e^{iR[f(\xi_0)\pm\frac{\pi}{4}]}+\text{c.c.}\right). \tag{5.9.73}$$

The critical value of x, denoted by $x_0=R\xi_0$ above, is determined by the stationary phase condition $f'(\xi)=\theta_x^1+\theta_x^2+\pi=0$, and results geometrically in the value $x_0=[a^2+b^2+d^2+e^2-f^2-g^2]^{\frac{1}{2}}$.

Just as in the verification of the B–E identity, the difficult part of the steepest descents method is the evaluation of the second derivative $f''(\xi_0)$. This evaluation is carried out below, where it is shown that

$$f''(\xi_0)=-\xi_0^2/6V'(\xi_0). \tag{5.9.74}$$

In obtaining this result, we make use of the fact that *at the stationary point* $x_0^2=a^2+b^2+d^2+e^2-f^2-g^2$ *the volumes of the three tetrahedra* T_1, T_2, *and* T_3 *are all equal.* This property, which may be verified directly from Cayley's formula (with some labor), is a quite unexpected and remarkable geometric property of the joining condition.

Substituting $f''(\xi_0)$ into Eq. (5.9.73) and using the equality of all the volumes, we obtain the desired result:

$$\mathcal{I} \simeq (12\pi V_3)^{-\frac{1}{2}}\cos\Omega_3 \simeq \begin{Bmatrix} a & e & f \\ b & d & g \end{Bmatrix}. \tag{5.9.75}$$

One verifies that the exponentials in Eq. (5.9.73) properly give $2\cos\Omega_3$ by examining the angles in the joined tetrahedra in Fig. 5.22.

Thus, *the Ponzano–Regge form satisfies Racah's identity asymptotically.*

Proof of the form of $f''(\xi_0)$. Let us now give the evaluation of the second derivative of the function $f(\xi)$ at the stationary phase point ξ_0. Since $f'(\xi)=\theta_x^1+\theta_x^2+\pi$, we must evaluate

$$f''(\xi_0)=R\left[\frac{\partial}{\partial x}\left(\theta_x^1+\theta_x^2\right)\right]_{x=x_0}. \tag{5.9.76}$$

(The partial derivative denotes that x is varied while holding $a, b, d, e, f,$ and g fixed.)

An explicit form of the angles θ_x^1 and θ_x^2 can be obtained from the geometry of the tetrahedra T_1 and T_2. The desired relation is

$$\sin\theta_x^i = \frac{3xV_i(x)}{2A_{dex}A_{bax}}, \qquad i=1,2, \tag{5.9.77}$$

where A_{dex} and A_{bax} denote the areas of the triangles (dex) and (bax) (see Fig. 5.22). Differentiating this relation with respect to x yields

$$\cos\theta_x^i\left(\frac{\partial\theta_x^i}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{3xV_i(x)}{2A_{dex}A_{bax}}\right]. \tag{5.9.78}$$

At the stationary phase point x_0, the angles θ_x^i $(i=1,2)$ are related by $\theta_{x_0}^1 = \pi - \theta_{x_0}^2$. Thus, we have the relations

$$\sin\theta_{x_0}^1 = \sin\theta_{x_0}^2, \qquad \cos\theta_{x_0}^1 = -\cos\theta_{x_0}^2.$$

Using this relation in Eq. (5.9.78), we now obtain

$$\left[\cos\theta_x^1\frac{\partial}{\partial x}(\theta_x^1+\theta_x^2)\right]_{x=x_0} = \frac{3x_0}{2A_{dex_0}A_{bax_0}}\left\{\frac{\partial}{\partial x}[V_1(x)-V_2(x)]\right\}_{x=x_0}. \tag{5.9.79}$$

The right-hand side of this result may be simplified further by noting that $V_1^2 - V_2^2 = (V_1+V_2)(V_1-V_2)$ and using the fact that $V_1(x_0)=V_2(x_0)$. Thus, we find

$$\left\{\frac{\partial}{\partial x}[V_1(x)-V_2(x)]\right\}_{x=x_0} = \frac{1}{2V_1(x_0)}\left\{\frac{\partial}{\partial x}[V_1^2(x)-V_2^2(x)]\right\}_{x=x_0}.$$

Since the volumes $V_i^2(x)$ are polynomials in x^2, it is also convenient to change the derivative with respect to x to one with respect to x^2. Carrying out these steps, we find from Eqs. (5.9.76) and (5.9.79) that

$$f''(\xi_0) = \frac{3\xi_0^2}{2V_1'(\xi_0)\cos\theta_{x_0}^1 A_{dex_0}A_{bax_0}}\left\{\frac{\partial}{\partial(x^2)}\left[V_1^2(x)-V_2^2(x)\right]\right\}_{x=x_0}. \tag{5.9.80}$$

The last step in the proof is to show that

$$\left\{\frac{\partial}{\partial(x^2)}\left[V_1^2(x)-V_2^2(x)\right]\right\}_{x=x_0} = -\tfrac{1}{9}A_{dex_0}A_{bax_0}\cos\theta_{x_0}^1, \tag{5.9.81}$$

thus establishing the desired result:[1]

$$f''(\xi_0)=-\xi_0^2/6V_1'(\xi_0). \tag{5.9.82}$$

The proof of Eq. (5.9.81) itself may be given using the geometric fact that

$$-A_hA_k\cos\theta_{hk}=9\frac{\partial V^2}{\partial(j_{rs}^2)}, \tag{5.9.83}$$

where $(hkrs)$ is a permutation of (1234) (see Fig. 5.18 for notation). [Equation (5.9.83) can be demonstrated from Cayley's formula, Eq. (5.9.31).] In the application of Eq. (5.9.83) made here to the tetrahedron T_1, we have

$$-A_{dex_0}A_{bax_0}\cos\theta_{x_0}^1=9\frac{\partial V_1^2(x_0)}{\partial(f^2)}.$$

Thus, one must verify that

$$\frac{\partial V_1^2(x_0)}{\partial(f^2)}=\left\{\frac{\partial}{\partial(x^2)}\left[V_1^2(x)-V_2^2(x)\right]\right\}_{x=x_0}.$$

The proof of this relationship for the joined tetrahedra (see Fig. 5.22) T_1 and T_2 is again a (laborious) application of Cayley's formula for the volumes V_1^2 and V_2^2. ■

Orthonormality relation. The final relation to be verified is the orthonormality condition

$$\sum_x(2x+1)\begin{Bmatrix}a&b&x\\d&e&f\end{Bmatrix}\begin{Bmatrix}a&b&x\\d&e&f'\end{Bmatrix}=(2f+1)^{-1}\delta_{ff'}. \tag{5.9.84}$$

This is the easiest of the three identities to be verified.

Introducing the Ponzano–Regge asymptotic form, the scaling transformation (5.9.52), and replacing the sum by an integral, we find

$$\text{LHS}\simeq(24\pi R)^{-1}\int\xi\,d\xi\left[V_1'(\xi)V_2'(\xi)\right]^{-\frac{1}{2}}P(\xi), \tag{5.9.85}$$

where $P(\xi)$ has the form given by Eqs. (5.9.68)–(5.9.70). There is no stationary phase point now, and the only possible result for the integral, except zero, is to have $f=f'$ so that the rapidly varying phases in the terms

[1]The primes in the notation $V_i'(\xi)$ designate the scaled volumes (5.9.53) and *not* derivatives.

cancel. Accordingly, we find

$$\text{LHS} \simeq (12\pi R)^{-1}\delta_{ff'}\int_{\xi_<}^{\xi_>} \xi\, d\xi [V_1'(\xi)]^{-1}. \tag{5.9.86}$$

To evaluate this integral, we consider the tetrahedron associated with $\begin{Bmatrix} a\,b\,x \\ d\,e\,f \end{Bmatrix}$ and introduce the angle θ_f, the outer dihedral angle around the side f opposite to x. As x varies from $x_<$ to $x_>$ (hence, ξ from $\xi_<$ to $\xi_>$), the angle θ_f varies from π to 0, corresponding to flat tetrahedra at the limits.

Using Eqs. (5.9.77) and (5.9.83), one can show that $-\partial\theta_f/\partial x = fx/6V(x)$, or, equivalently, that

$$-\frac{\partial\theta_f}{\partial\xi} = \frac{f\xi}{6RV_1'(\xi)}. \tag{5.9.87}$$

Thus, the integral becomes

$$\int_{\xi_<}^{\xi_>} \xi\, d\xi [V_1'(\xi)]^{-1} = \frac{6R}{f}\int_0^{\pi} d\theta_f = \frac{6R\pi}{f} \tag{5.9.88}$$

yielding the following value for the integral (5.9.86):

$$\text{LHS} \sim (2f)^{-1}\delta_{ff'} \sim \text{RHS}. \tag{5.9.89}$$

This verifies the orthonormality relation asymptotically and completes the proof that the Ponzano–Regge asymptotic form, Eq. (5.9.33), is valid in the classically allowed region. ■

9. Asymptotic Forms Valid in the Transition and Classically Unallowed Regions

The problem of finding one asymptotic form that smoothly carries the Ponzano–Regge result through the transition region ($V^2 \simeq 0$) into the classically forbidden region ($V^2 < 0$) is one that has been solved only heuristically, by finding approximate forms that fit the known numerical results accurately. We shall only summarize here the results of Ref. [4] with some attempt at motivation, but we hope it is clear, despite this cursory survey, that the problem is an important one and merits further study.

It is clear from the relation [Eq. (5.9.34)]

$$A_h A_k \sin\theta_{hk} = \tfrac{3}{2} V j_{hk}$$

that in the classically forbidden region the angle θ_{hk} becomes complex. [The triangle conditions guarantee that the areas A_h are real and nonnegative, so that, if $V^2<0$, the angle becomes $\theta_{hk}=n\pi+i\,\mathrm{Im}\,\theta_{hk}$.]

Experience with the JWKB approximation[1] in quantum mechanics indicates that the desired form may be found from the connection formulas. This suggests the result

$$\begin{Bmatrix} a\,b\,c \\ d\,e\,f \end{Bmatrix} \sim (48\pi|V|)^{-\frac{1}{2}} \cos\Phi \exp\left[-\Big|\sum_{h<k} j_{hk}\,\mathrm{Im}\,\theta_{hk}\Big|\right], \tag{5.9.90}$$

where

$$\Phi \equiv \sum_{h<k} \left(j_{hk}-\tfrac{1}{2}\right)\mathrm{Re}\,\theta_{hk}, \tag{5.9.91}$$

and the sign of $\mathrm{Im}\,\theta_{hk}$ is determined by the expressions

$$\begin{aligned} A_h A_k \cosh(\mathrm{Im}\,\theta_{hk}) &= -9\cos(\mathrm{Re}\,\theta_{hk})\frac{\partial V^2}{\partial\left(j_{rs}^2\right)}, \\ 2A_h A_k \sinh(\mathrm{Im}\,\theta_{hk}) &= 2\,j_{hk}\cos(\mathrm{Re}\,\theta_{hk})|V|, \end{aligned} \tag{5.9.92}$$

in which $(hkrs)$ is a permutation of (1234).

A qualitative understanding of this result, Eq. (5.9.90)—and especially the determination of the angles θ_{hk}—can be obtained by a study of the tetrahedron in the neighborhood of the transition region, $V^2\simeq 0$.

Using Eq. (5.9.34) we see that, if $A_1A_2A_3A_4\neq 0$, then for $V=0$ all the angles θ_{hk} must be multiples of π. If $V^2=0$, then the four vertices of the tetrahedron lie in a plane; the plane figure formed by the six sides will have either three or four external edges. The result is that $\theta_{hk}=\pi$ if the side j_{hk} is ~~external, and $\theta_{hk}=0$ if the side j_{hk} is internal. [Since we use j_{hk}=(nonzero~~ angular momentum)$+\frac{1}{2}$, the exceptional case where three vertices are collinear (and hence some $A_h=0$) is unphysical.]

As V^2 goes through zero and becomes negative, the angles θ_{hk} take the form $n\pi+i\,\mathrm{Im}\,\theta_{hk}$. These considerations show that the phase function Φ defined by Eq. (5.9.91) is well-defined for $V^2\leqslant 0$. The values for $\mathrm{Re}\,\theta_{hk}$ reached for $V^2=0$ then become constant in the region $V^2<0$. It is interesting to note that not only is Φ a multiple of π, but also that $\cos\Phi$ correctly determines the sign of the 6-j symbol in the classically unallowed region. One such realization (see Figure 5.19) has P_4 and P_5 on opposite sides of the plane through the triangle $(P_1P_2P_3)$; the second realization

[1]This is the semiclassical approximation for quantum mechanics obtained independently by Jeffreys [17], Wentzel [18], Kramers [19] and Brillouin [20] in the late 1920s. See also Jeffreys [21] and Fröman and Fröman [22].

corresponding to the lesser root (see Fig. 5.20) has P_4 and P_5 on the same side of the plane.

Consider now the function $\Omega \equiv \left(\sum_{h<k} j_{hk}\theta_{hk} \right) + \frac{\pi}{4}$. Let us keep all but one of the six angular momenta fixed, calling the variable angular momentum x. As x (in the allowed region) approaches $x_{\gtrless}$, where $V^2(x_{\gtrless})=0$, one finds that

$$\Omega(x_{\gtrless}) = \begin{cases} \Phi - \frac{\pi}{4}, & \text{if the flat tetrahedron has three external edges,} \\ \Phi + \frac{\pi}{4}, & \text{if there are four external edges.} \end{cases} \tag{5.9.93}$$

Ponzano and Regge show (Ref. [4], Appendix F) that for $x \to x_{\lessgtr}$ the function $\Omega(x)$ has the behavior

$$\Omega(x) - \Omega(x_{\lessgtr}) \to \pm \frac{9}{2} \frac{V^3}{A_1 A_2 A_3 A_4}, \tag{5.9.94}$$

where the sign is plus if there are three external edges and minus if there are four external edges. It follows that, as $x \to x_{\lessgtr}$, we have

$$\cos \Omega(x) \to \cos\left[\frac{9V^3}{2A_1A_2A_3A_4} + \Phi - \frac{\pi}{4} \right],$$

and, hence, the asymptotic form given in Eq. (5.9.90) takes the approximate value given by

$$\begin{Bmatrix} x & b & c \\ d & e & f \end{Bmatrix} \simeq (12\pi V)^{-\frac{1}{2}} \cos\left[\frac{9V^3}{2A_1A_2A_3A_4} + \Phi - \frac{\pi}{4} \right] \tag{5.9.95}$$

for $x \to x_{\lessgtr}$.

This result is very suggestive, since the right-hand side has the form of the JWKB approximation to the differential equation

$$\frac{d}{d(V^2)} \frac{d}{d(V^2)} \psi = \left(\frac{27}{4A_1A_2A_3A_4} \right)^2 V^2 \psi, \tag{5.9.96}$$

with V^2 playing the role of the coordinate variable, and the Racah function playing the role of the wave function ψ. (The product $A_1A_2A_3A_4$ is taken to be effectively constant for $x \to x_{\lessgtr}$.)

The advantage of this heuristic approach is that the differential equation allows one to continue the solution, Eq. (5.9.90), valid near $x \to x_{\lessgtr}$, through the turning point into the region $V^2<0$ by using the Airy function[1] solutions to Eq. (5.9.96). By this procedure, one obtains the result

$$\begin{Bmatrix} a\,b\,c \\ d\,e\,f \end{Bmatrix} \simeq \frac{1}{2}(4A_1A_2A_3A_4)^{-\frac{1}{6}} \times\left\{\cos\Phi\, Ai\left[\frac{(3|V|)^2}{(4A_1A_2A_3A_4)^{\frac{2}{3}}}\right]+\sin\Phi\, Bi\left[\frac{(3|V|)^2}{(4A_1A_2A_3A_4)^{\frac{2}{3}}}\right]\right\}, \tag{5.9.97}$$

which is valid near the turning point ($V^2\simeq 0$).

For large values of $|V^2|$, $V^2<0$, this result, Eq. (5.9.97), smoothly goes into the asymptotic form

$$\begin{Bmatrix} a\,b\,c \\ d\,e\,f \end{Bmatrix} \sim (48\pi|V|)^{-\frac{1}{2}}\cos\Phi\exp(-|\mathrm{Im}\,\Omega|)$$

already given above [Eq. (5.9.90)]. [In obtaining this result from Eq. (5.9.97), it is necessary to note that Φ is a multiple of π so that $\sin\Phi=0$, thus eliminating the increasing exponential.]

Equation (5.9.97) when combined with Eq. (5.9.33) suffices to determine by numerically accurate formulas the asymptotic behavior of the Racah function for every range of large physical angular momenta.

10. Asymptotic Forms for the Representation Coefficients (Rotation Matrices)

The representation coefficients—that is to say, the matrix elements of the rotation operator $\mathfrak{U}(\alpha\beta\gamma)$—have been discussed in detail in Chapter 3, Sections 5–9, AMQP. There it was shown that, for the matrix element

$$\langle jm'|\mathfrak{U}(\alpha\beta\gamma)|jm\rangle = e^{-im'\alpha}d^j_{m'm}(\beta)e^{-im\gamma}, \tag{5.9.98}$$

the representation coefficient $d^j_{m'm}(\beta)$ could be expressed in terms of the Jacobi polynomials:

$$d^j_{m'm}(\beta)=\left[\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}\right]^{\frac{1}{2}}\left(\sin\frac{\beta}{2}\right)^{m-m'}\left(\cos\frac{\beta}{2}\right)^{m'+m} \times P_{j-m}^{(m-m',\,m+m')}(\cos\beta). \tag{5.9.99}$$

[1] The two independent Airy functions Ai and Bi are discussed and their values tabulated in Ref. [22a].

The great advantage of this expression is that the existence of the differential equation for the Jacobi polynomials allows an asymptotic form for the representation coefficients to be obtained using the JWKB method. This procedure was discussed by Brussaard and Tolhoek [2]; we shall summarize their results in this section.[1]

We shall carry out this discussion, as in earlier sections, at the physicist's level of rigor, although by using known mathematical results on the Jacobi polynomials (Szegö [23], Erdélyi *et al.* [24]) and on the JWKB approximation (Langer [25]) a higher level of rigor would be possible.

The desired differential equation for the representation coefficient $d^j_{m'm}(\theta)$ is[2]

$$\left[\frac{d^2}{d\theta^2}+\cot\theta\frac{d}{d\theta}+j(j+1)-\frac{m'^2+m^2-2mm'\cos\theta}{\sin^2\theta}\right]d^j_{m'm}(\theta)=0. \tag{5.9.100}$$

Let us now introduce the parameters μ and ν defined by

$$\mu\equiv m'\left(j+\tfrac{1}{2}\right)^{-1},\qquad \nu=m\left(j+\tfrac{1}{2}\right)^{-1}, \tag{5.9.101}$$

and take $j\gg 1$ so that $[j(j+1)]^{\frac{1}{2}}\simeq j+\frac{1}{2}$. In order to put the differential equation (5.9.100) into the standard form for the JWKB method, we change the variable θ to z by making the transformation

$$\cos\theta\equiv\tanh z, \tag{5.9.102}$$

and, for brevity, denote $d^j_{m'm}(\theta)$ by $g(z)$. We then find (taking $j\gg 1$) that $g(z)$ satisfies the differential equation

$$\frac{d^2g(z)}{dz^2}+K^2(z)g(z)=0, \tag{5.9.103}$$

where

$$K^2(z)=\left(j+\frac{1}{2}\right)^2\left[(1-\mu^2)(1-\nu^2)-(\tanh z-\mu\nu)^2\right]. \tag{5.9.104}$$

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[1] Brussaard and Tolhoek [2] also discussed the classical limits for the Wigner and Racah coefficients and were among the first to emphasize the necessity for averaging.

[2] This is the eigenvalue equation for $\mathfrak{J}^2$ [see Eq. (3.119), Chapter 3, AMQP] after the exponential factors in the form (5.9.98) have been removed.

The expression for the effective kinetic energy, $K^2(z)$, shows that the motion has two transition points ("turning points") corresponding to the values z_1 and z_2 given by

$$\cos\theta_1 = \tanh z_1 = \mu\nu - \left[(1-\mu^2)(1-\nu^2)\right]^{\frac{1}{2}},$$
$$\cos\theta_2 = \tanh z_2 = \mu\nu + \left[(1-\mu^2)(1-\nu^2)\right]^{\frac{1}{2}}. \qquad (5.9.105)$$

The physical significance of these two transition points will be discussed further in connection with Fig. 5.23 below. We shall consider only $|\cos\theta|<1$ in the following discussion; hence, we must take $\mu \neq \nu$, since $\mu=\nu$ corresponds to $\cos\theta_2 = 1$.

The essential point to recognize is that Eq. (5.9.103) is in the standard form for applying the JWKB method to a motion with two turning points, since $K^2(z)$ is a quadratic form in z. Expressing $K^2(z)$ as the quadratic form

$$K^2(z) = K_0^2(z-z_1)(z_2-z) \qquad (5.9.106)$$

with $z_1 < z_2$ and $K_0^2 > 0$, we may apply the standard JWKB method, which uses one-third order Bessel functions to approximate the solutions near the turning points [linearly varying $K^2(z)$].

This method shows that the function $g_1(z)$, which is valid for $z \leqslant z_1$, has the form

$$g_1(z) = \frac{2}{\pi}\left[\frac{|\mathrm{Im}\, t_1|}{|K(z)|}\right]^{\frac{1}{2}}$$
$$\times\left[\pi \sin\eta_1 I_{\frac{1}{3}}(|\mathrm{Im}\, t_1|) + \cos\left(\frac{1}{3}\pi - \eta_1\right) K_{\frac{1}{3}}(|\mathrm{Im}\, t_1|)\right], \qquad (5.9.107)$$

where the argument $\mathrm{Im}\, t_1 \equiv \mathrm{Im}\, t_1(z)$ of the Bessel functions is (for $z \leqslant z_1$) given by

$$|\mathrm{Im}\, t_1| = \int_z^{z_1} |K(\xi)|\, d\xi.$$

The function $g_1(z)$ joins smoothly with the function $g_2(z)$, which is defined for $z_1 \leqslant z \leqslant z_2$, and given by

$$g_2(z) = \left[4t_1/3K(z)\right]^{\frac{1}{2}}\left[\cos(\tfrac{1}{3}\pi + \eta_1) J_{\frac{1}{3}}(t_1) + \cos(\tfrac{1}{3}\pi - \eta_1) J_{-\frac{1}{3}}(t_1)\right], \qquad (5.9.108)$$

where the argument t_1 is given by

$$t_1(z)=\int_{z_1}^{z} K(\xi)\, d\xi \qquad \text{with } K(\xi)=+\left[K^2(\xi)\right]^{\frac{1}{2}}.$$

For values of z that are far from the turning point z_1, the functions g_1 and g_2 take on the approximate forms

$$g_1(z)\simeq\left[2\pi|K(z)|\right]^{-\frac{1}{2}}\left(2\sin\eta_1 e^{|\operatorname{Im} t_1|}+\cos\eta_1 e^{-|\operatorname{Im} t_1|}\right), \qquad z\ll z_1, \tag{5.9.109}$$

$$g_2(z)\simeq\left[\pi K(z)/2\right]^{-\frac{1}{2}}\cos\left(t_1+\eta_1-\tfrac{1}{4}\pi\right), \qquad z_1\ll z\leqslant z_2. \tag{5.9.110}$$

Similarly, continuity through the turning point z_2 can be achieved by means of the function $g_2'(z)$, which is defined for $z_1\leqslant z\leqslant z_2$, and given by

$$g_2'(z)=\left[4t_2/3K(z)\right]^{\frac{1}{2}}\left[\cos\left(\tfrac{1}{3}\pi+\eta_2\right)J_{\frac{1}{3}}(t_2)+\cos\left(\tfrac{1}{3}\pi-\eta_2\right)J_{-\frac{1}{3}}(t_2)\right], \tag{5.9.111}$$

with the argument t_2 given by

$$t_2(z)=\int_{z}^{z_2} K(\xi)\, d\xi, \qquad K(\xi)=+\left[K^2(\xi)\right]^{\frac{1}{2}}.$$

The function $g_2'(z)$ then joins smoothly with the function $g_3(z)$, which is defined for $z_2<z$ by

$$g_3(z)=\frac{2}{\pi}\left[\frac{|\operatorname{Im} t_2|}{|K(z)|}\right]^{\frac{1}{2}}\left[\pi\sin\eta_2 I_{\frac{1}{3}}(|\operatorname{Im} t_2|)+\cos\left(\tfrac{1}{3}\pi-\eta_2\right)K_{\frac{1}{3}}(|\operatorname{Im} t_2|)\right], \tag{5.9.112}$$

where the argument of the Bessel functions is given by

$$|\operatorname{Im} t_2|=\int_{z_2}^{z} |K(\xi)|\, d\xi.$$

For large $|z-z_2|$, the solutions in Eqs. (5.9.112) and (5.9.111) behave according to Eqs. (5.9.109) and (5.9.110), respectively, with the replacements $t_1\to t_2$, $\eta_1\to\eta_2$.

There is a consistency condition implied by the two forms $g_2(z)$ and $g_2'(z)$ of the function $g(z)$. For the case where the turning points are well separated, we see that the consistency condition $g_2(z)=g_2'(z)$ can be fulfilled (for $z_1 \leqslant z \leqslant z_2$) if

$$\int_{z_1}^{z_2} K(\xi)\,d\xi = -\eta_1 - \eta_2 + (2N+\tfrac{1}{2})\pi, \qquad N \gg 1, \qquad (5.9.113)$$

which provides a relation between the two parameters η_1 and η_2.

To apply these generic results to the case at hand, it is necessary now to evaluate explicitly the functions t_1 and t_2 and the constants η_1 and η_2. The results are rather complicated in appearance (see Brussaard and Tolhoek [2] and Ponzano and Regge [4], Appendix 6). To gain some appreciation of the physical significance of the results, it is useful to recall that the function $d^j_{m'm}(\theta)$ is, in fact, the limit of the Racah function when all but one of the angular momenta are large. Accordingly, we represent the function $d^j_{m'm}(\theta)$ in terms of the limiting configuration shown in Fig. 5.23.

The functions t_1 and t_2 both involve the indefinite integral

$$\int^z (j+\tfrac{1}{2})\left[1-\mu^2-\nu^2+2\mu\nu\tanh z-(\tanh z)^2\right]^{\frac{1}{2}} dz$$

$$= (j+\tfrac{1}{2})\left[J(z)-\mu A(z)-\nu B(z)\right]. \qquad (5.9.114)$$

To interpret this result we note, from Fig. 5.23, that $\pi - J$, A, and B are the internal dihedral angles, respectively, between the planes with common lines $j+\frac{1}{2}$, $a+m+\frac{1}{2}$, and $b+m'+\frac{1}{2}$. Also, we have the relations $\mu=\cos\alpha$ and $\nu=\cos\beta$.

From Fig. 5.23 we can read off the significance of the turning points z_1 and z_2. The turning point $\theta=\theta_1$ corresponds to the flat tetrahedron with the line $j+\frac{1}{2}$ lying inside the triangle (abc). Hence, $\theta_1=\alpha+\beta$, $A=B=J=0$. The turning point $\theta=\theta_2$ corresponds to the flat tetrahedron with the line $j+\frac{1}{2}$ lying outside the triangle (abc). There are two possibilities (recall that $\mu\neq\nu$):

$$\begin{aligned} &\alpha<\beta:\ \theta_2=\beta-\alpha, && A=J=\pi, && B=0;\\ &\alpha>\beta:\ \theta_2=\alpha-\beta, && B=J=\pi, && A=0. \end{aligned} \qquad (5.9.115)$$

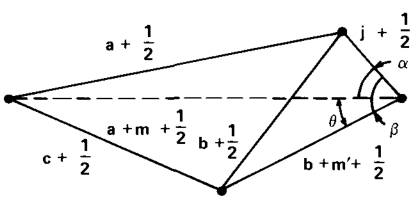

Figure 5.23. Particular case in which j is small with respect to the other edges.

Accordingly, we may now evaluate the functions t_1 and t_2:

$$t_1(\theta)\equiv t_1=(j+\tfrac{1}{2})(J-A\mu-B\nu), \tag{5.9.116}$$

where J, A, and B are given by

$$J=\arccos\left(\frac{\mu\nu-\cos\theta}{|\sin\alpha\sin\beta|}\right), \qquad A=\arccos\left(\frac{\nu-\mu\cos\theta}{|\sin\alpha\sin\theta|}\right),$$

$$B=\arccos\left(\frac{\mu-\nu\cos\theta}{|\sin\beta\sin\theta|}\right), \tag{5.9.117}$$

and similarly for t_2. Furthermore, Eq. (5.9.113) yields now

$$\begin{aligned}-\eta_1-\eta_2+(2N+\tfrac{1}{2})\pi&=\int_{z_1}^{z_2}K(\xi)\,d\xi\\ &=\begin{cases}(j+\frac{1}{2})(\pi-\pi\nu), & \alpha>\beta, \quad (m'>m),\\ (j+\frac{1}{2})(\pi-\pi\mu), & \alpha<\beta, \quad (m'<m).\end{cases}\end{aligned} \tag{5.9.118}$$

To complete the determination of the asymptotic forms, we need to give the values of parameters η_1 and η_2. [We have determined already the sum $\eta_1+\eta_2$ in Eq. (5.9.118).] The desired result is

$$\begin{aligned}\eta_1&=\pi(m-j),\\ \eta_2&=\begin{cases}\pi(m'-m), & \text{if } \alpha>\beta,\\ 0, & \text{if } \alpha<\beta.\end{cases}\end{aligned} \tag{5.9.119}$$

(Note that η_1 and η_2 are determined only mod $2N\pi$.)

These values for η_1 and η_2 rule out the exponentially increasing term in the classically forbidden region, since for physical values of the variables (m, m', j), the values of η_1 and η_2 are integer multiples of π.

These results completely determine the asymptotic form of the $d^j_{m'm}(\theta)$ function, in JWKB approximation.

It is useful, in order to relate these results to the previous sections, to further specialize to the case where $m\ll j$, $m'\ll j$. For this case $K(z)\simeq(j+\frac{1}{2})\sin\theta$, and from Fig. 5.23 the dihedral angles are $A\simeq B\simeq\pi/2$, and $J\simeq\pi-\theta$. It follows that in this case we obtain the asymptotic form:

$$d^j_{m'm}(\theta)\sim\left[\frac{2}{\pi(j+\frac{1}{2})\sin\theta}\right]^{\frac{1}{2}}\cos\left[(j+\tfrac{1}{2})\theta+\frac{\pi}{2}(m'-m)-\frac{\pi}{4}\right], \tag{5.9.120}$$

which is valid for large j and m', $m\ll j$.

Discussion. The limit of the Racah coefficient for five of the six angular momenta large is known to be a rotation matrix element (Racah [7], Edmonds [26]). In particular, for the 6-j coefficient, we have the relation

$$\begin{Bmatrix} c & a & b \\ j & b+m' & a+m \end{Bmatrix} \sim \left[(2a+1)(2b+1)\right]^{-\frac{1}{2}}(-1)^{a+b+c+j+m'} d^{j}_{m'm}(\theta), \tag{5.9.121}$$

where

$$\cos\theta = \frac{a(a+1)+b(b+1)-c(c+1)}{2\left[a(a+1)b(b+1)\right]^{\frac{1}{2}}}, \qquad 0\leqslant\theta\leqslant\pi, \tag{5.9.122}$$

and $a, b, c \gg j, m, m'$ [see Eq. (5.97), AMQP]. This configuration is illustrated in Fig. 5.23.

Since the asymptotic limit of the Jacobi polynomials is known (Szegö [23]), we can determine independently[1] that

$$d^{j}_{m'm}(\theta) \sim \left[\frac{2}{\pi\left(j+\frac{1}{2}\right)\sin\theta}\right]^{\frac{1}{2}} \cos\left[\left(j+\tfrac{1}{2}\right)\theta + \frac{\pi}{2}(m'-m) - \frac{\pi}{4}\right] \tag{5.9.123}$$

for large j and $m', m \ll j$.

It follows that we can determine from Eqs. (5.9.121) and (5.9.123) that the Racah function has the asymptotic form

$$\begin{Bmatrix} c & a & b \\ j & b+m' & a+m \end{Bmatrix} \sim (12\pi V)^{-\frac{1}{2}} \times \cos\left[\left(j+\tfrac{1}{2}\right)\theta + \pi\left(\frac{m'-m}{2} - \frac{1}{4} + a+b+c+j+m'\right)\right] \tag{5.9.124}$$

for $a, b, c, j \gg 1$.

In Fig. 5.23 we now make the identifications $j_{12}=a+\frac{1}{2}$, $j_{13}=c+\frac{1}{2}$, $j_{14}=b+\frac{1}{2}$ $j_{23}=a+m+\frac{1}{2}$, $j_{34}=b+m'+\frac{1}{2}$, and $j_{24}=j+\frac{1}{2}$, so that in this limiting configuration we have

$$\begin{aligned} &\theta_{12} \simeq \theta_{14} \simeq \theta_{23} \simeq \theta_{34} \simeq \pi/2, \\ &\theta_{13} \simeq \pi, \qquad \theta_{24} = \pi - \theta. \end{aligned} \tag{5.9.125}$$

[1]Although η_1 and η_2 can be determined directly in the WKBJ method, Ponzano and Regge actually used this result to determine the η's.

Thus, we find in this limiting configuration that

$$\Omega = \left(\sum_{h<k} j_{hk}\theta_{hk} \right) + \frac{\pi}{4}$$
$$= -\left[\left(j+\frac{1}{2} \right)\theta + \pi\left(\frac{m'-m}{2} - \frac{1}{4} + a+b+c+j+m' \right) \right]$$
$$+2\pi(a+b+c+m'+1). \tag{5.9.126}$$

Substituting this result in Eq. (5.9.124), we obtain precisely the Ponzano–Regge asymptotic form (5.9.33)

This is an important independent verification of the asymptotic form, Eq. (5.9.33), *for the Racah function.*

It is also of interest to note that the approximation

$$j(j+1) \simeq (j+\tfrac{1}{2})^2, \tag{5.9.127}$$

that is, the substitution $j \to j+\frac{1}{2}$, is *essential* in determining the phase given in (5.9.126). [Elsewhere in the asymptotic form the use of $j+\frac{1}{2}$ for j (for example, in determining the volume) is an inessential—but useful!—improvement in the approximation.]

11. Concluding Remarks

The asymptotic properties of the angular momentum functions is clearly of interest in any general discussion of angular momentum theory, yet, granting this, our discussion in the preceding sections has probably been more detailed than might have been expected. There is an underlying motivation for this, which we should like to indicate in these concluding ~~remarks, even though this motivation is frankly speculative.~~

There has long been a feeling among physicists that the quantum mechanics must be further extended by some form of quantization of length. [The physicist Pascual Jordan, who contributed so much to the development of quantum mechanics, admitted (see Refs. [27], [28]) that he has believed this to be an essential future development since 1932.]

Models for the quantization of length in one dimension and two dimensions are not too difficult to obtain, based, for example, on the limit of the cyclic group $Z_n \to R_2$. (Such models have been exploited by Schwinger [29].) In a sense, these structures are too simple; certainly they furnish no possible route for an analogous treatment of $SO(3)$ or $SU(2)$.

It is from this speculative point of view especially interesting that the Racah functions may be viewed as *discretized* angle functions (see Refs. [30], [31]). For example, the Racah function $W(a\nu Jb;\, ab)$ can be considered as a *quantized Legendre polynomial* (Ref. [32]) (see also Chapter 5, Section 8, AMQP).

Let us now reconsider the results obtained in this section from this viewpoint. We have noted already that in the geometric approach to mechanics (Maclane [33]) it is the kinetic energy that plays the role of supplying the Riemannian metric. Thus, a most important suggestion follows from the qualitative JWKB result: *The classical* (*volume*)^2 *plays the role of the kinetic energy in the asymptotic Racah function regime.* This suggests that a possible model for a coordinate free approach to space would regard the 6-j symbol as the elementary 3-simplex. Regge [12] has already sketched the outlines of a simplicial approach to general relativity, thereby avoiding the introduction of coordinates (Misner *et al.* [34][1]). *The use of the Racah function as the basic* 3-*simplex provides a possible model for quantized space.*

Let us note several features that indicate the consistency of such a model. First, the classical limit of the Racah function may be viewed in an interesting alternative way. Let us suppose that three of the vertices of the tetrahedron are held fixed, and that the three (lengths)^2—call them J^2, b^2, c^2—are to be considered as coordinates of the fourth vertex, called P. If we assume that the *a priori* probability for the vertex P to lie in a small volume $dJ_1\,dJ_2\,dJ_3$ [with $\mathbf{J}=(J_1, J_2, J_3)$ being the usual Euclidean coordinates designating P] does not depend on P, then it follows that in terms of the coordinates J^2, b^2, c^2 this same probability is

$$2\mathcal{J}\,dJ^2db^2dc^2, \tag{5.9.128}$$

where $\mathcal{J}$ is the Jacobian

$$\mathcal{J}=\det\left[\frac{\partial(J_1, J_2, J_3)}{\partial(J^2, b^2, c^2)}\right]. \tag{5.9.129}$$

(The factor 2 is a result of the fact that the coordinates J^2, b^2, c^2 do not distinguish orientation, so that two distinct points P correspond to the same set J^2, b^2, c^2.)

The Jacobian may be evaluated and shown to be

$$\mathcal{J}=8|\mathbf{J}\cdot\mathbf{b}\times\mathbf{c}|=48V, \tag{5.9.130}$$

where V is the volume of the tetrahedron. Using the classical limit for the

[1]This comprehensive treatise devotes a chapter (Chapter 42) to the "Regge calculus" for general relativity without coordinates based on Ref. [12], but nowhere mentions the use of the Racah function as the elementary 3-simplex developed in Ref. [4].

Racah function, one sees that the Jacobian thus corresponds to

$$\mathcal{J} \sim \frac{\pi}{2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ b & J & c \end{Bmatrix}^2 . \tag{5.9.131}$$

This observation by Ponzano and Regge supplies a new way to view the classical limit.

Turning the argument around, we can now remark that this result implies the consistency of the model "Racah function as simplex" in the classical limit (since in this limit we recover the uniform a priori probability in space).

This argument also supplies a hint as to why six angles and six conjugates enter into the asymptotic Racah functions. In a coordinate- and metric-free approach, the distinction between *alias* and *alibi* transformations (or equivalently between coordinate frame and point of observation) is lost. In the classical limit, we indeed recover the *point* (vertex P in the above discussion) measured with respect to the *frame* (determined by the fixed triangle). Thus, in this limit, the six pairs of conjugates in the asymptotic Racah function divide up into three conjugate pairs each for the "frame" and the "point."

It is worth while to mention two further implications of such a possible model of "quantized space." The first implication results from reconsidering the proof of the B–E identity. From the present (speculative!) viewpoint, this result may be interpreted as implying that composite blocks of space are three-dimensional only in the large-scale limit.

A second implication comes from the fact that [realizing the 6-j symbols as complete quadrilaterals (see Topic 8) in the plane] the B–E identity has the form of Desargues' configuration. One sees that Desargues' configuration thus becomes a valid Racah relation (without summation, see Topic 8, Section 3) for this model of space only in the large-scale limit. Expressed differently (using a famous result of Hilbert), this shows that the underlying field in this (planar) geometry becomes associative only in the large-scale limit.

It was because of these provocative speculations, using the 6-j symbol as a model for a simplex in 3-space, that we—following Ponzano and Regge—have indulged in such a detailed discussion of the asymptotics of the angular momentum functions, and in particular the Racah coefficient.

Let us note, however, that the asymptotics of the angular momentum functions are of physical importance, per se. For example, in coulomb excitation calculations large numbers of angular momenta enter significantly and the asymptotic angular momentum functions are directly applicable (Alder *et al.* [35], Biedenharn and Brussaard [36]). A quite different example occurs in constructing a soluble ϕ^3 field theory where the asymptotic angular momentum functions enter in an essential way (Amit and Roginsky [37]).

12. Notes

1. *Further comments on the association of a 6-j symbol and a tetrahedron.* The six sides of the tetrahedron are each associated with the magnitude of an angular momentum vector. Since the magnitude j is related to the values of the observable $\mathbf{J}^2$ by $\mathbf{J}^2 \to j(j+1)$, it is not clear, a priori, whether to use j^2 or $j(j+1)$ for the square of an edge in applying Cayley's formula for the volume. Since for j large $[j(j+1)]^{\frac{1}{2}} \simeq j+\frac{1}{2}$ (with an error of order $\hbar^2$), a related question is whether or not to use j or $j+\frac{1}{2}$ in asymptotic formulas. As a general rule, the use of $j+\frac{1}{2}$ is more accurate, and where j appears in an exponent the use of $j+\frac{1}{2}$ is essential.

Ponzano and Regge [4] have chosen for all angular momenta $\{a, b, \ldots\}$ to use the values $\{a+\frac{1}{2}, b+\frac{1}{2}, \ldots\}$. Using this convention, they prove (the proof stems from S. Adler; see footnote 12 in Ref. [4]) that *the volume V is never zero* for angular momenta that satisfy the triangle relations. It must be noted, however, that this result is an artifact of the shift $j \to j+\frac{1}{2}$. The volume V vanishes, for example, if one edge becomes zero; in such a case the shift $j \to j+\frac{1}{2}$ is clearly invalid. Note also that the turning points $(x_>, x_<)$ are affected by this shift, and can be shifted outside the region allowed by the triangular conditions (an example is in Fig. 6, Ref. [4]).

If one applies the Cayley formula to the triangle, the (area)^2 factorizes into Archimede's formula ("Heron's formula"), which exhibits the three triangle conditions as factors. Because of this, and the reasons noted above, we have chosen not necessarily to follow Ponzano and Regge in an a priori shift of the variables, preferring to derive any necessary shift from the asymptotic results themselves. Such a shift is essential in the phase of the ~~cosine, Eq. (5.9.33), and the variables j_{hk} are shifted as shown in Fig. 5.18.~~

2. *Comment on Ref.* [4]. The paper by Ponzano and Regge is a tour de force with an astonishing display of virtuosity in applying geometric arguments. It is a difficult paper to read, but it is strongly recommended for its deep ideas and concepts. Much of our discussion is based directly on this paper.

3. *Rigor of the asymptotic results.* The authors of Ref. [4] were careful not to claim a rigorous proof for their asymptotic results. In our opinion, their work constitutes the essential elements of a valid proof, and only details (such as the proof of Racah's identity given in Section 8) needed explication.

Schulten and Gordon [38] have claimed to give a rigorous proof of the Ponzano–Regge asymptotic results. In point of fact, what is accomplished in Ref. [38] is simply a more careful treatment (by standard methods) of the recursion relation stated (and discussed briefly) in the Ponzano–Regge paper [their Eq. (4.3a)]. In doing so, the insights available in the proofs of Ref. [4] are completely lost.

The work of Ref. [38] is quite valuable, however, from a practical (computationally oriented) viewpoint, for Schulten and Gordon develop in these two papers the best currently available algorithms for computing efficiently and accurately both 3-j and 6-j coefficients.

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TOPIC 10. NONTRIVIAL ZEROS OF THE 3-j AND 6-j SYMBOLS

Survey. Various aspects of the Wigner (3-j) coefficients and Racah (6-j) coefficients have been discussed and developed in AMQP and in the previous sections of this monograph. We have mentioned (see Chapter 3, Appendix D, AMQP, and Chapters 3 and 4 of this monograph) the existence of a class of zeros of these coefficients that have been called "nontrivial" zeros as opposed to the "trivial" zeros resulting from a symmetry (3-j symbol) or a violation of one or more triangle conditions (3-j or 6-j

symbol). These latter zeros are more appropriately called "structural" zeros, and we have discussed at great length in Chapters 3 and 4 their role in determining the structure of the Wigner and Racah coefficients themselves. We have also identified the nontrivial zeros as zeros of the "polynomial part" of a Wigner or Racah coefficient, thus accounting for the *possibility* of such zeros.

The fact that these nontrivial zeros should occur for allowed values of the angular momenta seems rather surprising.

The only systematic studies of these nontrivial zeros have been numerical in approach[1] (Ref. [1], Varshalovich *et al.* [2, p. 226], Bowick [3]), and no general reason for their occurrence is yet known (note, however, the exceptions discussed below). These zeros seem to have an "accidental" aspect, since all generic conditions for the existence of the Wigner or Racah coefficient are satisfied, and there is no apparent reason for the coefficient to be zero. In our view, a systematic study of these "accidental" zeros, their distribution, and related questions might be a rewarding endeavor. We present below a few motivating examples to indicate the basis for our view. The main content of the present section is, however, quite limited in scope. We present a table of the presently known nontrivial zeros, in the hope that this "raw material" may be of value in facilitating further study.

Some motivating examples. An intuitive understanding of any quantum mechanical structure, such as the Wigner and Racah coefficients, almost always involves examining the semiclassical limit (terms up to order $\hbar$, but neglecting $\hbar^2$). This is the regime of large quantum numbers—that is, large angular momenta—and hence of asymptotic forms for these functions. (This has been discussed in Chapter 5, Section 8, AMQP, and in considerable detail for the Racah function in Topic 9 of the present chapter.) Using a heuristic view, one may say that the Wigner and Racah coefficients are a generalization of Jacobi polynomials to arguments that are a form of discretized angle space.

Let us illustrate this view.[2] Consider the Racah function

$$[(2a+1)(2d+1)]^{\frac{1}{2}}W(a2cd;ad)$$

$$=\frac{3x(x+1)-4a(a+1)d(d+1)}{2[a(a+1)(2a-1)(2a+3)d(d+1)(2d-1)(2d+1)]^{\frac{1}{2}}},$$

[1]The present section is based on a paper, Ref. [1], written in collaboration with Professor S. H. Koozekanani. We thank Professor Koozekanani for the privilege of reproducing this work.

[2]See Eqs. (5.95)–(5.98) in AMQP. See also Tables A2 and A3 in the Appendix of Tables, as well as Eqs. (A.17)–(A.20) in Appendix A, Chapter 4.

where

$$x \equiv c(c+1) - a(a+1) - d(d+1). \tag{5.10.2}$$

For large values of a, c, and d (which form a triangle), we have the asymptotic relation

$$(x/2ad) \sim \left[(c^2 - a^2 - d^2)/2ad\right] \equiv \cos\theta. \tag{5.10.3}$$

One recognizes now that the term on the right-hand side in Eq. (5.10.1) corresponds asymptotically to the Legendre polynomial $P_2(\cos\theta)$.

It is clear, therefore, that the Racah function is (in the classical region) an oscillating function of its arguments and necessarily approximates all the zeros of the Legendre (Jacobi) function. (Note, however, that the reflection symmetry of the Legendre function is violated in the Racah function.) Just which zeros[1] are achieved (and how "often"—that is, the distribution) is basically a number-theoretic question, which to our knowledge has never been considered.[2]

First example. From the tables for Racah coefficients in AMQP, one sees that the coefficient $W(\frac{3}{2},2,\frac{3}{2},2;\frac{3}{2},2)$ is (nontrivially) zero. This has the physical implication (see Chapter 7, Section 8, AMQP) that *quadrupole radiation from an aligned state having* $j=\frac{3}{2}$ *to a ground state* $j=\frac{3}{2}$ *is isotropic*, despite the fact that triangle rules would permit a nonisotropic angular distribution. For this particular case the numerator in Eq. (5.10.1) is easily seen to be exactly zero.

One can give an "explanation" of this zero from the quasi-spin model (see Chapter 7, Section 9f, AMQP). Consider two fermions in a $j=\frac{3}{2}$ shell. The interaction energy of a quadrupolar one-body potential in the two-particle state having $J=2$ then has the form

$$\begin{aligned}\langle E\rangle &= \langle(\tfrac{3}{2})^2, v=2, J=2, M|U_0^2|(\tfrac{3}{2})^2, v=2, J=2, M\rangle\\ &\propto W(\tfrac{3}{2},2,\tfrac{3}{2},2;\tfrac{3}{2},2).\end{aligned} \tag{5.10.4}$$

(Here the one-body potential is a quadrupole interaction denoted by U_μ^2; $v=2$ designates the state as having seniority 2.)

Using the concept of quasi-spin (see Section 9h, Chapter 7, AMQP), we may view the same matrix element in a totally different way. The even multipolar interactions transform as vectorial quasi-spin operators. For the particular case given in Eq. (5.10.4) the initial and final states belong to quasi-spin values $S=[(2j+1)/4]-(v/2)=0$, $S_3=[(2j+1)/4]-(n/2)=0$.

[1]Recall that these zeros occur as roots of a well-defined polynomial (see p. 122).

[2]A similar number-theoretic question for the Wigner coefficients, for a physically motivated problem, was considered in Ref. [4].

Thus, the matrix element vanishes because the "triangle rule" is violated ($\mathbf{0} \notin \mathbf{0}+\mathbf{1}$). Note that it is the triangle rule for *quasi-spin* that is violated; for the Racah coefficient itself, all the triangle rules are indeed satisfied so that the zero is nontrivial.

This explanation for the vanishing of $W(\frac{3}{2},2,\frac{3}{2},2;\frac{3}{2},2)$ stems from de-Shalit and Talmi [5, p. 315], who based their result on explicit seniority calculations. The quasi-spin interpretation greatly simplifies the argument.

Second example. An example of quite a different sort—and considerably deeper—is provided by the zero $W(3,5,3,5;3,3)$. This zero is related to the embedding in the orthogonal group $SO(7)$ (in Cartan's notation, B_3) of the exceptional group G_2.

This comes about in the following way. The generators of $SO(7)$ may be realized as tensor operators $\mathbf{T}^1$, $\mathbf{T}^3$, and $\mathbf{T}^5$ with respect to the generators (orbital angular momentum) $L_\mu = T^1_\mu$ of the $SO(3)$ subgroup, where these operators act in a seven-dimensional space specified by $l=3$. Thus, we may take the generators of the group $SO(7)$ to be the Wigner operators given by

$$X^k_\mu = \left\langle \begin{matrix} & k & \\ 2k & & 0 \\ & k+\mu & \end{matrix} \right\rangle \tag{5.10.5}$$

for $k=1,3,5$, and $\mu=k,k-1,\ldots,-k$. Each of these operators then maps the seven-dimensional space with angular momentum basis $\{|3,m\rangle: m=3,2,\ldots,-3\}$ into itself:

$$X^k_\mu|3,m\rangle = C^{3\,k\,3}_{m,\mu,m+\mu}|3,m+\mu\rangle. \tag{5.10.6}$$

Moreover, the commutator of two generators is given by [see Eq. (2.61)]

$$\left[X^k_\mu, X^{k'}_{\mu'}\right] = -2\sum_{k''\mu''} C^{kk'k''}_{\mu\mu'\mu''}[7(2k''+1)]^{\frac{1}{2}} W(3,k,3,k';3,k'')X^{k''}_{\mu''}, \tag{5.10.7}$$

where the summation is over all odd values of k''. This realization of the Lie algebra of $SO(7)$ has the advantage of introducing explicitly Wigner and Racah coefficients into the structure constants.

Consider now the commutation relations for the operators $[X^5_\mu, X^5_\nu]$. *Because the Racah coefficient* $W(3,5,3,5;3,3)$ *is zero*, we see from Eq. (5.10.7) that the operators $X^3_{\mu''}$ do not enter in the right-hand side of Eq. (5.10.7). Since also the commutator $[X^1_\mu, X^k_\nu]$ is a numerical multiple of $X^k_{\mu+\nu}$, it follows that the subset of fourteen generators given by $\{X^1_\mu, X^5_\nu: \mu=1,0,-1;\ \nu=5,4,\ldots,-5\}$ closes under commutation. This is the well-known example (due to Racah [6]) that elucidates the embedding $SO(7)\supset G_2$.

It would be of considerable interest to discuss the remaining exceptional groups by similar explicit results. This problem seems to be surprisingly difficult, however.[1]

Our examples have considered only two of the roughly 1400 zeros of the Racah function given in Table 5.1. Judd [9], in his lectures on atomic theory at Canterbury University, has discussed two additional examples[2] of nontrivial zeros of the Racah coefficient, both vanishings being directly related to vanishings of fractional parentage coefficients in the atomic g-shell. (No such systematic relationship exists in general, however, since Judd points out that a third fractional parentage coefficient vanishes in the same shell, but is not directly connected with any nontrivial 6-j coefficient.)

It is hoped that Tables 5.1 and 5.2 listing nontrivial zeros of the 6-j and 3-j coefficients will stimulate further explanations and/or implications, which are all too few at the present.

Explanation of tables. The nontrivial zeros of the 6-j symbol $\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$ have been calculated for all arguments j_i, $l_i \leqslant 18.5$, $i=1,2,3$. Using the permutational symmetries of the 6-j symbol, we have ordered the arguments j_1, j_2, j_3, l_1, l_2, and l_3 in a "speedometric fashion," with j_1 the slowest changing, and l_3 the most rapidly changing, variables. The results are listed in Table 5.1 with $j_1 \geqslant j_2 \geqslant j_3$ and $j_1 \geqslant \max(l_1, l_2, l_3)$.

An entry in a given row signifies that the 6-j symbol $\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$, where the arguments are the numerical entries in Table 5.1, vanishes. As an example, the first entry in the table is the nontrivial zero $\begin{Bmatrix} 2 & 2 & 2 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{Bmatrix}$ discussed above.

The values of the 6-j symbols were calculated by a computer program, which did its calculations with an arithmetic working only with powers of primes; that is, each number was decomposed into products of prime factors before being manipulated by the computer. Hence, the vanishings found are exact.

The number of entries in Table 5.1 may be reduced considerably by using the Regge symmetries of the 6-j symbol (see Chapter 3, Section 18, and Chapter 5, Appendix D, AMQP). Since it is quite complicated to identify the set of 6-j symbols that are equivalent under the Regge symmetries (in contrast to the permutational symmetries), we have, for convenience, kept all entries in the originally published table (see Bowick [3] for the reduced table).

[1]For F_4 there are some partial results (H. Freudenthal, private communication). Wadzinski [7] has discussed the embedding $U_{26} \supset F_4$. The embedding $SO(26) \supset F_4$ has been given by Dynkin [8] (We are indebted to Prof. B. G. Wybourne for this reference).

[2]These examples are $\begin{Bmatrix} 6 & 4 & 9 \\ 5 & 5 & 2 \end{Bmatrix}$ and $\begin{Bmatrix} 2 & 4 & 5 \\ 5 & 2 & 2 \end{Bmatrix}$.

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
2.0	2.0	2.0	1.5	1.5	1.5
3.0	2.0	2.0	1.0	2.0	2.0
3.5	3.0	1.5	1.0	1.5	3.0
3.5	3.5	3.0	2.5	1.5	3.0
4.0	3.0	2.0	2.0	3.0	3.0
4.0	3.0	3.0	2.0	3.0	2.0
4.0	3.5	2.5	2.0	2.5	2.5
5.0	4.0	2.0	3.0	4.0	4.0
5.0	4.0	4.0	3.0	4.0	2.0
5.0	4.5	1.5	2.0	2.5	4.5
5.0	4.5	2.5	2.0	1.5	4.5
5.0	4.5	4.5	3.5	3.0	3.0
5.0	5.0	2.0	2.0	2.0	4.0
5.0	5.0	3.0	3.0	3.0	3.0
5.0	5.0	4.0	3.5	1.5	4.5
5.5	4.0	3.5	1.0	3.5	4.0
6.0	4.0	3.0	1.5	3.5	3.5
6.0	4.5	2.5	2.5	4.0	4.0
6.0	5.0	2.0	4.0	5.0	5.0
6.0	5.0	3.0	1.0	3.0	5.0
6.0	5.0	3.0	2.5	3.5	3.5
6.0	5.0	5.0	4.0	5.0	2.0
6.0	6.0	3.0	6.0	5.0	6.0
6.0	6.0	4.0	2.5	2.5	5.5
6.0	6.0	5.0	6.0	3.0	6.0
6.0	6.0	6.0	5.0	4.0	3.0
6.0	6.0	6.0	6.0	5.0	3.0
6.5	5.0	2.5	1.5	3.0	4.5
6.5	5.0	2.5	3.5	5.0	4.5
6.5	5.0	4.5	2.0	4.5	4.0
6.5	5.0	4.5	3.5	5.0	2.5
6.5	6.0	1.5	3.0	3.5	6.0
6.5	6.0	3.5	3.0	1.5	6.0
6.5	6.0	3.5	3.0	2.5	5.0
6.5	6.0	3.5	3.5	4.0	3.5
6.5	6.0	5.5	3.5	5.0	3.5
6.5	6.0	5.5	4.5	3.0	4.5
6.5	6.0	5.5	5.5	3.0	5.5
6.5	6.5	5.0	4.5	1.5	6.0
6.5	6.5	5.0	4.5	3.5	4.0
6.5	6.5	5.0	5.0	5.0	2.5
7.0	4.5	4.5	2.5	4.0	4.0
7.0	5.5	5.5	4.0	4.5	3.5
7.0	6.0	2.0	2.5	3.5	5.5
7.0	6.0	2.0	5.0	6.0	6.0
7.0	6.0	4.0	4.0	6.0	5.0
7.0	6.0	5.0	4.0	4.0	4.0
7.0	6.0	5.0	4.0	6.0	4.0
7.0	6.0	6.0	2.5	5.5	4.5
7.0	6.0	6.0	5.0	6.0	2.0
7.0	6.5	2.5	2.5	3.0	5.0
7.0	6.5	4.5	4.0	5.5	4.5
7.0	6.5	4.5	4.5	5.0	3.0
7.0	6.5	5.5	2.5	5.0	5.0
7.5	5.5	4.0	4.5	5.5	5.0
7.5	5.5	5.0	4.5	5.5	4.0
7.5	6.0	4.5	4.5	5.0	4.5
7.5	6.0	5.5	3.0	5.5	4.0
~~7.5~~	~~6.5~~	~~3.0~~	~~2.0~~	~~4.0~~	~~6.5~~
7.5	6.5	4.0	2.0	3.0	6.5
7.5	6.5	5.0	3.0	5.0	4.5
7.5	7.0	3.5	2.0	3.5	6.0
7.5	7.5	7.0	5.0	3.0	6.5
8.0	5.5	5.5	3.5	5.0	4.0
8.0	6.0	3.0	3.0	5.0	5.0
8.0	6.0	3.0	4.0	6.0	5.0
8.0	6.0	5.0	1.0	5.0	6.0
8.0	6.0	5.0	3.5	4.5	4.5
8.0	6.0	5.0	4.0	6.0	3.0
8.0	6.5	3.5	2.5	3.0	6.0
8.0	6.5	3.5	3.0	4.5	4.5
8.0	7.0	2.0	6.0	7.0	7.0
8.0	7.0	4.0	4.0	5.0	4.0
8.0	7.0	7.0	5.5	4.5	4.5
8.0	7.0	7.0	6.0	7.0	2.0
8.0	7.5	1.5	4.0	4.5	7.5
8.0	7.5	4.5	4.0	1.5	7.5
8.0	7.5	6.5	5.5	3.0	6.0
8.0	8.0	3.0	5.5	5.5	7.5
8.0	8.0	6.0	5.5	1.5	7.5
8.0	8.0	7.0	4.0	4.0	7.0
8.5	7.0	4.5	1.0	4.5	7.0
8.5	7.0	6.5	4.0	6.5	4.0
8.5	7.5	3.0	5.0	6.0	7.5
8.5	7.5	6.0	6.0	3.0	7.5
8.5	8.0	2.5	1.0	2.5	8.0
8.5	8.0	2.5	4.0	3.5	8.0
8.5	8.0	3.5	4.0	2.5	8.0

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
8.5	8.0	5.5	4.0	5.5	5.0
8.5	8.0	6.5	4.5	3.0	7.5
8.5	8.5	3.0	4.0	3.0	7.5
8.5	8.5	5.0	3.5	2.5	8.0
9.0	6.0	4.0	2.0	5.0	5.0
9.0	6.5	5.5	2.0	6.5	6.5
9.0	6.5	6.5	2.0	6.5	5.5
9.0	6.5	6.5	4.5	6.0	4.0
9.0	7.0	6.0	2.0	6.0	6.0
9.0	7.5	2.5	3.0	4.5	6.5
9.0	7.5	2.5	5.0	6.5	6.5
9.0	7.5	3.5	4.0	6.5	7.5
9.0	7.5	5.5	4.5	5.0	5.0
9.0	7.5	6.5	4.0	3.5	7.5
9.0	8.0	2.0	1.5	2.5	7.5
9.0	8.0	2.0	5.5	6.5	7.5
9.0	8.0	2.0	7.0	8.0	8.0
9.0	8.0	3.0	3.0	4.0	6.0
9.0	8.0	3.0	3.0	5.0	8.0
9.0	8.0	3.0	3.5	3.5	7.5
9.0	8.0	4.0	4.5	6.5	6.5
9.0	8.0	5.0	3.0	3.0	8.0
9.0	8.0	6.0	5.0	3.0	7.0
9.0	8.0	6.0	5.0	4.0	6.0
9.0	8.0	6.0	6.0	7.0	3.0
9.0	8.0	8.0	1.5	7.5	7.5
9.0	8.0	8.0	5.5	7.5	3.5
9.0	8.0	8.0	7.0	8.0	2.0
9.0	8.0	8.0	7.5	5.5	5.5
9.0	8.5	4.5	4.0	2.5	7.5
9.0	8.5	4.5	4.5	6.0	6.0
9.0	8.5	8.5	7.0	5.5	4.5
9.0	9.0	4.0	4.0	3.0	7.0
9.0	9.0	4.0	5.0	5.0	5.0
9.0	9.0	5.0	3.5	3.5	7.5
9.0	9.0	5.0	5.0	4.0	6.0
9.0	9.0	5.0	6.0	6.0	4.0
9.0	9.0	5.0	7.5	7.5	3.5
9.0	9.0	8.0	6.0	3.0	8.0
9.0	9.0	8.0	7.0	5.0	5.0
9.0	9.0	8.0	8.5	7.5	3.5
9.5	6.5	5.0	2.5	6.5	6.0
9.5	6.5	6.0	2.5	6.5	5.0
9.5	7.0	3.5	2.0	4.5	6.0
9.5	7.0	3.5	4.5	7.0	5.5
9.5	7.0	5.5	2.5	6.0	5.5
9.5	7.0	5.5	4.5	7.0	3.5
9.5	7.5	4.0	5.0	6.0	6.5
9.5	7.5	6.0	4.5	3.5	7.0
9.5	8.0	4.5	3.5	3.0	7.5
9.5	8.0	4.5	4.5	6.0	4.5
9.5	8.0	4.5	5.0	5.5	6.0
9.5	8.0	7.5	5.0	7.5	4.0
9.5	8.0	7.5	6.0	3.5	7.0
9.5	8.0	7.5	7.0	5.5	6.0
9.5	8.5	4.0	3.5	3.5	7.0
9.5	8.5	5.0	4.5	4.5	6.0
9.5	8.5	8.0	5.5	4.5	6.0
~~9.5~~	~~8.5~~	~~8.0~~	~~8.0~~	~~8.0~~	~~3.5~~
9.5	9.0	1.5	5.0	5.5	9.0
9.5	9.0	5.5	5.0	1.5	9.0
9.5	9.0	6.5	2.5	5.0	8.5
9.5	9.0	7.5	5.5	3.0	7.5
9.5	9.0	7.5	6.5	5.0	5.5
9.5	9.0	7.5	7.5	8.0	2.5
9.5	9.5	6.0	5.5	2.5	8.0
9.5	9.5	6.0	6.0	5.0	5.5
9.5	9.5	6.0	7.0	7.0	3.5
9.5	9.5	7.0	6.5	1.5	9.0
9.5	9.5	9.0	8.0	7.0	3.5
9.5	9.5	9.0	9.0	7.0	4.5
10.0	6.0	5.0	3.0	6.0	6.0
10.0	6.0	6.0	3.0	6.0	5.0
10.0	6.5	5.5	3.0	5.5	5.5
10.0	7.5	3.5	3.5	6.0	6.0
10.0	7.5	7.5	5.5	7.0	4.0
10.0	8.0	3.0	4.5	6.5	6.5
10.0	8.0	4.0	3.5	5.5	5.5
10.0	8.0	7.0	3.0	7.0	6.0
10.0	8.0	8.0	6.0	6.0	5.0
10.0	8.5	4.5	2.0	5.5	8.5
10.0	8.5	5.5	2.0	4.5	8.5
10.0	8.5	7.5	6.0	5.5	5.5
10.0	9.0	2.0	8.0	9.0	9.0
10.0	9.0	4.0	1.0	4.0	9.0
10.0	9.0	4.0	4.5	5.5	5.5
10.0	9.0	5.0	2.0	5.0	8.0
10.0	9.0	6.0	5.5	5.5	5.5

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
10.0	9.0	9.0	7.5	7.5	3.5
10.0	9.0	9.0	8.0	9.0	2.0
10.0	10.0	8.0	5.0	4.0	9.0
10.0	10.0	10.0	8.5	6.5	4.5
10.5	6.5	6.0	1.5	6.5	6.0
10.5	8.0	4.5	4.5	8.0	7.5
10.5	8.0	6.5	1.0	6.5	8.0
10.5	8.0	7.5	4.5	8.0	4.5
10.5	8.5	4.0	2.5	5.5	8.0
10.5	8.5	4.0	3.5	6.5	8.0
10.5	8.5	5.0	3.0	7.0	8.5
10.5	8.5	7.0	3.0	5.0	8.5
10.5	9.0	4.5	2.5	5.0	7.5
10.5	9.0	6.5	6.0	8.5	5.0
10.5	9.0	8.5	2.5	8.0	7.5
10.5	9.0	8.5	3.5	8.0	6.5
10.5	9.0	8.5	6.0	8.5	4.0
10.5	9.0	8.5	8.0	4.5	8.0
10.5	9.5	3.0	4.0	6.0	9.5
10.5	9.5	5.0	3.5	5.5	7.0
10.5	9.5	6.0	3.0	6.0	7.5
10.5	9.5	6.0	4.0	3.0	9.5
10.5	9.5	6.0	4.5	6.5	6.0
10.5	9.5	6.0	6.0	8.0	5.5
10.5	9.5	8.0	3.5	7.5	7.0
10.5	9.5	8.0	4.5	4.5	9.0
10.5	9.5	8.0	5.0	9.0	5.5
10.5	10.5	2.0	5.0	5.0	8.5
10.5	10.5	7.0	8.0	6.0	6.5
10.5	10.5	9.0	7.0	3.0	9.5
11.0	6.0	6.0	2.0	6.0	6.0
11.0	7.0	7.0	4.0	6.0	6.0
11.0	7.5	4.5	5.0	7.5	7.5
11.0	7.5	7.5	5.0	7.5	4.5
11.0	8.0	4.0	3.0	5.0	8.0
11.0	8.0	4.0	4.0	6.0	8.0
11.0	8.0	4.0	5.0	8.0	6.0
11.0	8.0	5.0	1.5	5.5	7.5
11.0	8.0	5.0	3.0	4.0	8.0
11.0	8.0	6.0	4.0	4.0	8.0
11.0	8.0	6.0	5.0	8.0	4.0
11.0	8.5	4.5	3.0	4.5	7.5
11.0	8.5	6.5	5.5	8.0	5.0
11.0	8.5	8.5	6.5	8.0	4.0
11.0	9.0	3.0	2.0	4.0	8.0
11.0	9.0	3.0	3.5	5.5	7.5
11.0	9.0	5.0	4.0	5.0	7.0
11.0	9.0	5.0	5.0	7.0	5.0
11.0	9.0	6.0	1.0	6.0	9.0
11.0	9.0	6.0	5.0	6.0	6.0
11.0	9.0	6.0	6.0	8.0	4.0
11.0	9.0	6.0	5.5	7.5	5.5
11.0	9.0	8.0	4.0	8.0	6.0
11.0	9.5	3.5	3.5	5.0	7.0
11.0	9.5	5.5	4.5	3.0	9.0
11.0	9.5	5.5	6.0	7.5	4.5
11.0	9.5	7.5	4.0	7.5	6.5
11.0	9.5	7.5	5.5	8.0	5.0
11.0	9.5	7.5	7.5	9.0	3.0
11.0	9.5	9.5	7.5	6.0	6.0
11.0	10.0	2.0	4.0	5.0	9.0
11.0	10.0	2.0	5.0	6.0	9.0
11.0	10.0	2.0	9.0	10.0	10.0
11.0	10.0	7.0	5.5	7.5	5.5
11.0	10.0	7.0	6.0	4.0	8.0
11.0	10.0	9.0	5.5	8.5	5.5
11.0	10.0	9.0	7.5	4.5	7.5
11.0	10.0	10.0	4.0	9.0	7.0
11.0	10.0	10.0	5.0	9.0	6.0
11.0	10.0	10.0	9.0	10.0	2.0
11.0	10.5	1.5	6.0	6.5	10.5
11.0	10.5	2.5	2.0	3.5	10.5
11.0	10.5	2.5	4.5	5.0	8.0
11.0	10.5	3.5	2.0	2.5	10.5
11.0	10.5	4.5	4.5	4.0	8.0
11.0	10.5	6.5	6.0	1.5	10.5
11.0	10.5	6.5	6.0	4.5	7.5
11.0	10.5	6.5	6.5	6.0	6.0
11.0	10.5	8.5	7.5	3.0	9.0
11.0	10.5	8.5	8.0	7.5	4.5
11.0	11.0	3.0	3.0	2.0	10.0
11.0	11.0	3.0	4.0	4.0	8.0
11.0	11.0	6.0	4.5	2.5	10.5
11.0	11.0	8.0	7.5	1.5	10.5
11.0	11.0	8.0	7.5	5.5	6.5
11.0	11.0	8.0	8.0	7.0	5.0
11.0	11.0	8.0	9.0	9.0	3.0
11.0	11.0	9.0	8.0	4.0	8.0

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
11.5	7.5	5.0	2.0	5.0	7.5
11.5	8.0	5.5	4.0	7.5	5.0
11.5	8.5	6.0	4.0	5.0	7.5
11.5	8.5	7.0	2.0	8.0	8.5
11.5	8.5	8.0	2.0	7.0	8.5
11.5	8.5	8.0	7.0	8.0	5.5
11.5	9.0	7.5	2.0	7.5	8.0
11.5	9.0	7.5	5.0	7.5	5.0
11.5	9.5	7.0	5.5	4.5	8.0
11.5	9.5	7.0	6.0	7.0	5.5
11.5	9.5	7.0	7.0	7.0	6.5
11.5	9.5	7.0	7.0	9.0	3.5
11.5	10.0	2.5	4.5	6.0	8.5
11.5	10.0	2.5	6.5	8.0	8.5
11.5	10.0	4.5	3.0	6.5	10.0
11.5	10.0	6.5	3.0	4.5	10.0
11.5	10.0	6.5	5.5	5.0	7.5
11.5	10.0	9.5	7.0	9.5	4.0
11.5	10.5	2.0	6.0	7.0	9.5
11.5	10.5	3.0	2.5	2.5	10.0
11.5	10.5	8.0	7.5	7.5	5.0
11.5	11.0	2.5	8.0	8.5	8.0
11.5	11.0	3.5	4.5	5.0	7.5
11.5	11.0	5.5	3.0	5.5	9.0
11.5	11.0	5.5	5.0	2.5	10.0
11.5	11.0	5.5	7.0	7.5	5.0
11.5	11.0	8.5	7.5	5.0	7.5
11.5	11.0	10.5	9.0	7.5	5.0
11.5	11.5	11.0	7.5	4.5	10.0
12.0	8.0	5.0	2.5	6.5	6.5
12.0	8.0	8.0	5.0	7.0	6.0
12.0	8.5	7.5	5.0	6.5	6.5
12.0	9.0	4.0	4.0	7.0	7.0
12.0	9.0	7.0	4.0	8.0	6.0
12.0	9.0	9.0	6.5	7.5	5.5
12.0	9.5	4.5	1.5	5.0	9.0
12.0	9.5	4.5	4.0	6.5	6.5
12.0	9.5	9.5	7.5	9.0	4.0
12.0	10.0	3.0	6.0	8.0	8.0
12.0	10.0	8.0	6.5	6.5	6.5
12.0	10.0	9.0	5.0	9.0	6.0
12.0	10.0	9.0	6.5	9.5	4.5
12.0	10.5	2.5	7.5	9.0	9.0
12.0	10.5	3.5	3.5	6.0	10.0
12.0	10.5	10.5	8.5	8.0	5.0
12.0	10.5	10.5	8.5	10.0	5.0
12.0	11.0	2.0	10.0	11.0	11.0
12.0	11.0	3.0	5.0	7.0	11.0
12.0	11.0	4.0	2.0	5.0	11.0
12.0	11.0	5.0	2.0	4.0	11.0
12.0	11.0	7.0	5.0	3.0	11.0
12.0	11.0	8.0	5.0	8.0	7.0
12.0	11.0	11.0	9.0	7.0	6.0
12.0	11.0	11.0	10.0	11.0	2.0
12.0	11.5	4.5	2.0	4.5	10.5
12.0	11.5	4.5	3.5	5.0	9.0
12.0	11.5	4.5	6.0	6.5	6.5
12.0	11.5	9.5	1.5	9.0	11.0
12.0	11.5	9.5	6.5	9.0	6.0
12.0	11.5	9.5	8.0	9.5	4.5
12.0	11.5	10.5	9.0	6.5	6.5
12.0	12.0	6.0	3.5	3.5	11.5
12.0	12.0	7.0	7.5	6.5	6.5
12.0	12.0	9.0	5.0	4.0	11.0
12.0	12.0	9.0	8.0	4.0	11.0
12.0	12.0	9.0	8.5	6.5	6.5
12.0	12.0	10.0	8.0	3.0	11.0
12.5	8.5	6.0	3.0	8.0	7.5
12.5	8.5	7.0	4.5	7.5	6.0
12.5	9.0	4.5	2.0	4.5	9.0
12.5	9.0	4.5	2.5	6.0	7.5
12.5	9.0	4.5	5.5	9.0	6.5
12.5	9.0	6.5	3.0	7.5	7.0
12.5	9.0	6.5	5.5	9.0	4.5
12.5	9.5	5.0	5.0	9.0	8.5
12.5	9.5	9.0	3.0	9.0	7.5
12.5	9.5	9.0	7.0	9.0	4.5
12.5	10.0	3.5	4.0	5.5	10.0
12.5	10.0	5.5	4.0	3.5	10.0
12.5	10.0	5.5	4.0	4.5	9.0
12.5	10.0	5.5	5.5	8.0	5.5
12.5	10.0	8.5	3.0	8.5	8.0
12.5	10.0	8.5	6.5	6.0	7.5
12.5	10.5	3.0	5.0	7.0	8.5
12.5	10.5	3.0	8.0	10.0	8.5
12.5	10.5	6.0	2.0	7.0	10.5
12.5	10.5	7.0	2.0	6.0	10.5
12.5	11.0	3.5	2.5	5.0	10.5

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
12.5	11.0	3.5	4.5	7.0	10.5
12.5	11.0	3.5	7.0	8.5	7.0
12.5	11.0	3.5	7.0	9.5	11.0
12.5	11.0	4.5	4.0	4.5	9.0
12.5	11.0	6.5	2.0	6.5	10.0
12.5	11.0	6.5	5.0	7.5	7.0
12.5	11.0	6.5	5.5	3.0	10.5
12.5	11.0	6.5	7.5	9.0	4.5
12.5	11.0	9.5	2.5	8.0	10.5
12.5	11.0	9.5	4.5	6.0	10.5
12.5	11.0	9.5	7.0	3.5	11.0
12.5	11.0	9.5	8.5	9.0	4.5
12.5	11.0	10.5	8.0	10.5	4.0
12.5	11.0	10.5	8.5	6.0	7.5
12.5	11.5	4.0	2.5	4.5	10.0
12.5	11.5	4.0	5.0	6.0	7.5
12.5	11.5	4.0	7.0	8.0	6.5
12.5	11.5	7.0	7.0	7.0	6.5
12.5	11.5	9.0	2.5	8.5	10.0
12.5	11.5	9.0	7.5	9.5	5.0
12.5	11.5	10.0	8.5	4.5	9.0
12.5	11.5	10.0	8.5	6.5	7.0
12.5	11.5	10.0	10.0	11.0	2.5
12.5	12.0	1.5	3.5	4.0	11.5
12.5	12.0	1.5	7.0	7.5	12.0
12.5	12.0	7.5	3.5	5.0	11.5
12.5	12.0	7.5	3.5	9.0	11.5
12.5	12.0	7.5	7.0	1.5	12.0
12.5	12.0	8.5	2.5	9.0	11.5
12.5	12.0	9.5	8.5	3.0	10.5
12.5	12.5	5.0	5.5	4.5	9.0
12.5	12.5	8.0	7.5	4.5	9.0
12.5	12.5	8.0	8.5	3.5	11.0
12.5	12.5	8.0	8.5	7.5	6.0
12.5	12.5	9.0	5.0	5.0	11.5
12.5	12.5	9.0	8.5	1.5	12.0
13.0	8.0	6.0	3.5	7.5	7.5
13.0	8.5	6.5	3.5	7.0	7.0
13.0	9.0	5.0	5.5	8.5	8.5
13.0	9.0	8.0	2.5	8.5	7.5
13.0	9.0	9.0	6.0	8.0	6.0
13.0	9.5	8.5	3.5	9.0	7.0
13.0	10.0	8.0	1.0	8.0	10.0
13.0	10.0	8.0	3.5	8.5	7.5
13.0	10.0	8.0	6.0	7.0	7.0
13.0	10.5	3.5	3.0	4.5	10.5
13.0	10.5	3.5	4.0	6.5	8.5
13.0	10.5	3.5	5.0	6.5	10.5
13.0	10.5	3.5	6.5	9.0	8.0
13.0	10.5	3.5	7.5	10.0	8.0
13.0	10.5	4.5	3.0	3.5	10.5
13.0	10.5	6.5	5.0	3.5	10.5
13.0	10.5	6.5	5.5	7.0	7.0
13.0	10.5	9.5	7.5	8.0	6.0
13.0	10.5	9.5	7.5	10.0	6.0
13.0	10.5	10.5	8.0	7.5	6.5
13.0	10.5	10.5	8.5	10.0	4.0
13.0	11.0	3.0	9.0	11.0	9.0
13.0	11.0	4.0	3.0	4.0	10.0
13.0	11.0	4.0	4.0	6.0	8.0
13.0	11.0	5.0	5.0	9.0	11.0
13.0	11.0	9.0	5.0	5.0	11.0
13.0	11.0	9.0	7.5	3.5	10.5
13.0	11.0	9.0	7.5	7.5	6.5
13.0	11.0	9.0	8.0	9.0	5.0
13.0	11.0	9.0	9.0	11.0	3.0
13.0	11.0	10.0	6.0	10.0	6.0
13.0	11.0	10.0	8.0	7.0	7.0
13.0	11.5	4.5	4.0	7.5	11.5
13.0	11.5	7.5	4.0	4.5	11.5
13.0	11.5	9.5	3.5	7.0	11.0
13.0	12.0	2.0	3.0	4.0	11.0
13.0	12.0	2.0	8.0	9.0	11.0
13.0	12.0	2.0	11.0	12.0	12.0
13.0	12.0	5.0	4.5	3.5	10.5
13.0	12.0	5.0	5.0	5.0	9.0
13.0	12.0	5.0	6.5	7.5	6.5
13.0	12.0	8.0	7.0	4.0	10.0
13.0	12.0	8.0	7.5	8.5	7.5
13.0	12.0	8.0	8.0	8.0	6.0
13.0	12.0	9.0	2.5	7.5	11.5
13.0	12.0	9.0	5.0	12.0	12.0
13.0	12.0	12.0	3.0	11.0	10.0
13.0	12.0	12.0	5.0	12.0	9.0
13.0	12.0	12.0	8.0	11.0	5.0
13.0	12.0	12.0	11.0	12.0	2.0
13.0	12.5	1.5	4.5	5.0	12.0
13.0	12.5	2.5	3.0	3.5	10.5
13.0	12.5	5.5	7.5	8.0	6.0
13.0	12.5	6.5	6.0	3.5	10.5
13.0	12.5	8.5	2.5	8.0	11.0
13.0	12.5	8.5	3.5	8.0	10.0
13.0	12.5	8.5	6.0	8.5	7.5
13.0	12.5	8.5	10.0	10.5	3.5
13.0	12.5	11.5	3.0	10.5	10.5
13.0	12.5	12.5	10.5	8.0	6.0
13.0	12.5	12.5	11.0	10.5	3.5
13.0	13.0	6.0	5.0	4.0	10.0
13.0	13.0	7.0	7.0	5.0	9.0
13.0	13.0	12.0	8.5	4.5	11.5
13.5	10.5	5.0	3.0	7.0	9.5
13.5	11.0	4.5	6.0	9.5	10.0
13.5	11.0	5.5	3.0	6.5	9.0
13.5	11.0	7.5	1.0	7.5	11.0
13.5	11.0	9.5	4.0	9.5	8.0
13.5	11.0	9.5	7.0	10.5	5.0
13.5	12.0	11.5	9.0	11.5	4.0
13.5	12.5	3.0	6.0	8.0	12.5
13.5	12.5	3.0	11.0	12.0	8.5
13.5	12.5	5.0	9.0	12.0	11.5
13.5	12.5	8.0	6.0	3.0	12.5
13.5	13.0	2.5	3.0	4.5	13.0
13.5	13.0	4.5	3.0	2.5	13.0
13.5	13.0	6.5	6.0	7.5	8.0
13.5	13.0	7.5	6.0	11.5	13.0
13.5	13.0	7.5	7.0	8.5	7.0
13.5	13.0	10.5	10.5	5.0	10.5
13.5	13.0	11.5	5.0	7.5	13.0
13.5	13.0	11.5	7.0	10.5	7.0
13.5	13.5	7.0	5.5	2.5	13.0
13.5	13.5	11.0	9.0	3.0	12.5
14.0	8.0	8.0	3.5	7.5	7.5
14.0	8.5	8.5	4.5	8.0	7.0
14.0	9.0	8.0	4.5	7.5	7.5
14.0	10.0	5.0	3.5	6.5	9.5
14.0	10.0	5.0	5.0	10.0	7.0
14.0	10.0	7.0	6.0	10.0	5.0
14.0	10.0	10.0	7.0	9.0	6.0
14.0	10.5	4.5	4.5	8.0	8.0
14.0	10.5	4.5	6.5	9.0	10.0
14.0	10.5	5.5	3.5	6.0	9.0
14.0	10.5	8.5	2.0	9.5	10.5
14.0	10.5	9.5	2.0	8.5	10.5
14.0	10.5	9.5	7.5	10.0	5.0
14.0	11.0	4.0	8.0	11.0	8.0
14.0	11.0	5.0	4.5	7.5	7.5
14.0	11.0	6.0	2.5	7.5	10.5
14.0	11.0	6.0	4.5	9.5	10.5
14.0	11.0	6.0	6.0	9.0	6.0
14.0	11.0	6.0	6.0	9.0	10.0
14.0	11.0	8.0	6.0	5.0	10.0
14.0	11.0	8.0	5.0	6.0	9.0
14.0	11.0	8.0	8.0	11.0	4.0
14.0	11.0	9.0	2.0	9.0	10.0
14.0	11.5	6.5	2.5	7.0	10.0
14.0	11.5	6.5	5.0	4.5	10.5
14.0	11.5	8.5	7.0	7.5	7.5
14.0	11.5	11.5	2.5	11.0	10.0
14.0	11.5	11.5	9.5	11.0	4.0
14.0	12.0	3.0	7.5	9.5	9.5
14.0	12.0	5.0	9.0	11.0	12.0
14.0	12.0	6.0	3.0	8.0	12.0
14.0	12.0	6.0	5.0	5.0	10.0
14.0	12.0	6.0	5.0	9.0	10.0
14.0	12.0	7.0	6.0	6.0	9.0
14.0	12.0	7.0	6.0	8.0	9.0
14.0	12.0	8.0	3.0	6.0	12.0
14.0	12.0	11.0	2.5	10.5	10.5
14.0	12.0	11.0	4.5	10.5	8.5
14.0	12.0	11.0	7.0	11.0	6.0
14.0	12.0	11.0	9.0	5.0	10.0
14.0	12.0	11.0	9.0	5.0	12.0
14.0	12.0	12.0	9.5	7.5	7.5
14.0	12.5	2.5	3.5	5.0	11.0
14.0	12.5	2.5	5.0	7.5	10.5
14.0	12.5	2.5	8.0	9.5	10.5
14.0	12.5	2.5	10.5	12.0	11.0
14.0	12.5	5.5	2.0	6.5	12.5
14.0	12.5	6.5	2.0	5.5	12.5
14.0	12.5	6.5	5.0	8.5	9.5
14.0	12.5	6.5	6.5	7.0	8.0
14.0	12.5	7.5	4.5	8.0	9.0
14.0	12.5	7.5	6.5	3.0	12.0
14.0	12.5	7.5	5.5	5.0	10.0
14.0	12.5	7.5	7.5	8.0	7.0
14.0	12.5	7.5	9.0	10.5	4.5

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
14.0	12.5	10.5	4.5	10.0	9.0
14.0	12.5	11.5	7.5	5.0	12.0
14.0	12.5	11.5	9.5	6.0	9.0
14.0	13.0	2.0	12.0	13.0	13.0
14.0	13.0	3.0	3.5	4.5	10.5
14.0	13.0	4.0	3.0	6.0	13.0
14.0	13.0	4.0	3.5	2.5	12.5
14.0	13.0	6.0	2.0	6.0	12.0
14.0	13.0	6.0	3.0	4.0	13.0
14.0	13.0	6.0	8.0	9.0	6.0
14.0	13.0	7.0	3.0	7.0	11.0
14.0	13.0	7.0	6.5	11.5	12.5
14.0	13.0	7.0	9.0	10.0	5.0
14.0	13.0	11.0	9.5	4.5	10.5
14.0	13.0	13.0	11.0	10.0	5.0
14.0	13.0	13.0	12.0	13.0	2.0
14.0	13.5	1.5	2.5	3.0	13.0
14.0	13.5	1.5	5.5	6.0	13.0
14.0	13.5	1.5	8.0	8.5	13.5
14.0	13.5	3.5	3.5	3.0	12.0
14.0	13.5	6.5	6.0	2.5	12.5
14.0	13.5	8.5	2.5	7.0	13.0
14.0	13.5	8.5	5.5	4.0	13.0
14.0	13.5	8.5	8.0	1.5	13.5
14.0	13.5	10.5	9.5	3.0	12.0
14.0	13.5	10.5	9.5	7.0	8.0
14.0	13.5	10.5	10.5	10.0	5.0
14.0	13.5	12.5	12.5	10.0	9.0
14.0	14.0	2.0	2.5	2.5	12.5
14.0	14.0	4.0	6.0	6.0	9.0
14.0	14.0	5.0	5.0	3.0	12.0
14.0	14.0	5.0	7.5	7.5	7.5
14.0	14.0	8.0	7.5	2.5	12.5
14.0	14.0	9.0	9.0	7.0	8.0
14.0	14.0	10.0	7.0	4.0	13.0
14.0	14.0	10.0	9.5	1.5	13.5
14.0	14.0	12.0	7.5	5.5	12.5
14.0	14.0	14.0	12.0	9.0	6.0
14.5	10.5	6.0	4.5	9.5	9.0
14.5	11.0	5.5	5.5	10.0	9.5
14.5	11.5	5.0	10.0	11.0	11.5
14.5	11.5	7.0	4.5	8.5	8.0
14.5	11.5	11.0	4.5	8.5	10.0
14.5	11.5	11.0	10.0	5.0	11.5
14.5	12.0	5.5	3.5	8.0	11.5
14.5	12.0	5.5	7.0	11.5	12.0
14.5	12.0	7.5	3.5	9.0	10.5
14.5	12.0	7.5	6.5	12.0	10.5
14.5	12.0	10.5	3.5	9.0	10.5
14.5	12.0	10.5	4.5	9.0	9.5
14.5	12.0	10.5	5.0	10.5	8.0
14.5	12.0	10.5	6.5	12.0	7.5
14.5	12.0	11.5	3.5	9.0	11.5
14.5	12.0	11.5	7.0	5.5	12.0
14.5	12.5	4.0	1.5	4.5	12.0
14.5	12.5	4.0	5.0	8.0	11.5
14.5	12.5	4.0	8.5	11.5	12.0
14.5	12.5	6.0	8.0	12.0	10.5
14.5	12.5	7.0	1.0	7.0	12.5
14.5	12.5	7.0	5.5	8.5	8.0
14.5	12.5	10.0	5.0	10.0	8.5
14.5	12.5	11.0	4.5	7.5	12.0
14.5	13.0	4.5	5.0	8.5	13.0
14.5	13.0	6.5	3.5	7.0	10.5
14.5	13.0	8.5	5.0	4.5	13.0
14.5	13.0	10.5	3.5	10.0	10.5
14.5	13.0	12.5	1.5	12.0	12.5
14.5	13.0	12.5	5.0	11.5	9.0
14.5	13.0	12.5	6.0	7.5	12.0
14.5	13.0	12.5	8.5	12.0	5.5
14.5	13.0	12.5	10.0	12.5	4.0
14.5	13.5	9.0	6.5	10.5	9.0
14.5	14.0	5.5	5.0	6.5	10.0
14.5	14.0	9.5	4.5	9.0	10.5
14.5	14.0	9.5	9.0	10.5	6.0
14.5	14.0	11.5	5.0	10.5	10.0
14.5	14.0	13.5	9.0	6.5	11.0
14.5	14.5	3.0	8.5	7.5	13.0
14.5	14.5	8.0	10.0	8.0	8.5
14.5	14.5	13.0	9.5	4.5	13.0
14.5	14.5	14.0	8.0	7.0	12.5
15.0	9.5	8.5	1.5	9.0	9.0
15.0	9.5	9.5	5.5	8.0	8.0
15.0	10.0	6.0	3.0	8.0	8.0
15.0	10.0	6.0	5.0	9.0	9.0
15.0	10.5	5.5	6.0	9.5	9.5
15.0	10.5	6.5	3.5	6.0	10.0
15.0	10.5	8.5	5.5	6.0	10.0

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
15.0	11.0	6.0	6.0	8.0	10.0
15.0	11.0	7.0	5.0	8.0	8.0
15.0	11.0	9.0	3.0	11.0	11.0
15.0	11.0	11.0	3.0	11.0	9.0
15.0	11.0	11.0	5.0	10.0	8.0
15.0	11.0	11.0	8.0	10.0	6.0
15.0	11.5	5.5	4.5	9.0	10.0
15.0	11.5	6.5	6.0	7.5	9.5
15.0	11.5	7.5	5.5	7.0	9.0
15.0	11.5	7.5	7.5	11.0	5.0
15.0	11.5	11.5	8.5	9.0	7.0
15.0	12.0	4.0	2.0	4.0	12.0
15.0	12.0	4.0	4.5	7.5	9.5
15.0	12.0	4.0	5.5	7.5	11.5
15.0	12.0	4.0	6.0	9.0	9.0
15.0	12.0	4.0	9.0	11.0	12.0
15.0	12.0	6.0	5.0	9.0	9.0
15.0	12.0	6.0	8.0	11.0	11.0
15.0	12.0	7.0	5.0	11.0	12.0
15.0	12.0	7.0	6.0	8.0	8.0
15.0	12.0	10.0	3.0	10.0	10.0
15.0	12.0	10.0	3.0	11.0	11.0
15.0	12.0	10.0	5.0	9.0	9.0
15.0	12.0	10.0	7.5	11.5	5.5
15.0	12.0	11.0	5.0	7.0	12.0
15.0	12.0	11.0	6.5	11.5	6.5
15.0	12.0	11.0	9.0	4.0	12.0
15.0	12.5	3.5	7.0	9.5	9.5
15.0	12.5	4.5	3.0	6.5	11.5
15.0	12.5	4.5	4.5	7.0	9.0
15.0	12.5	4.5	9.5	12.0	7.0
15.0	12.5	6.5	4.5	8.0	9.0
15.0	12.5	6.5	7.5	10.0	6.0
15.0	12.5	7.5	2.0	8.5	12.5
15.0	12.5	8.5	2.0	7.5	12.5
15.0	12.5	10.5	8.5	5.0	11.0
15.0	12.5	10.5	8.5	8.0	8.0
15.0	12.5	12.5	10.5	12.0	4.0
15.0	12.5	12.5	11.5	11.0	9.0
15.0	13.0	3.0	10.5	12.5	10.5
15.0	13.0	4.0	4.0	7.0	12.0
15.0	13.0	5.0	3.0	6.0	11.0
15.0	13.0	5.0	4.0	4.0	12.0
15.0	13.0	5.0	6.0	8.0	8.0
15.0	13.0	8.0	2.0	8.0	12.0
15.0	13.0	9.0	8.0	8.0	8.0
15.0	13.0	12.0	6.5	6.5	12.5
15.0	13.0	12.0	8.0	12.0	6.0
15.0	13.5	2.5	10.0	11.5	11.5
15.0	13.5	5.5	5.5	6.0	10.0
15.0	13.5	5.5	9.5	11.0	6.0
15.0	13.5	7.5	6.5	4.0	12.0
15.0	13.5	9.5	9.5	6.0	11.0
15.0	13.5	9.5	9.5	10.0	6.0
15.0	13.5	11.5	11.5	10.0	10.0
15.0	13.5	13.5	11.0	8.5	7.5
15.0	13.5	13.5	13.5	10.0	8.0
15.0	14.0	2.0	5.5	6.5	12.5
15.0	14.0	2.0	7.5	8.5	12.5
15.0	14.0	2.0	13.0	14.0	14.0
15.0	14.0	3.0	7.0	9.0	14.0
15.0	14.0	3.0	8.0	8.0	13.0
15.0	14.0	5.0	1.0	5.0	14.0
15.0	14.0	5.0	4.0	6.0	11.0
15.0	14.0	5.0	7.0	8.0	8.0
15.0	14.0	8.0	7.5	9.5	7.5
15.0	14.0	8.0	8.0	9.0	9.0
15.0	14.0	9.0	5.0	9.0	10.0
15.0	14.0	9.0	6.5	9.5	8.5
15.0	14.0	9.0	7.0	3.0	14.0
15.0	14.0	9.0	8.0	4.0	12.0
15.0	14.0	9.0	9.5	6.5	10.5
15.0	14.0	9.0	11.0	12.0	4.0
15.0	14.0	13.0	9.5	6.5	10.5
15.0	14.0	13.0	10.5	3.5	13.5
15.0	14.0	13.0	11.0	8.0	8.0
15.0	14.0	13.0	12.5	9.5	7.5
15.0	14.0	14.0	5.5	12.5	9.5
15.0	14.0	14.0	7.5	12.5	7.5
15.0	14.0	14.0	13.0	14.0	2.0
15.0	14.5	6.5	6.5	5.0	11.0
15.0	14.5	8.5	8.5	7.0	9.0
15.0	14.5	10.5	9.5	6.0	10.0
15.0	14.5	12.5	11.5	10.0	6.0
15.0	14.5	12.5	12.5	9.0	8.0
15.0	14.5	12.5	13.5	9.0	9.0
15.0	15.0	10.0	5.0	5.0	14.0
15.0	15.0	12.0	9.5	4.5	12.5

Table 5.1. Nontrivial Zeros of the 6-j Symbol.

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
15.0	15.0	12.0	10.0	3.0	14.0
15.5	9.5	8.0	2.0	9.0	8.5
15.5	10.5	7.0	3.5	9.5	9.0
15.5	11.0	5.5	3.0	7.5	9.0
15.5	11.0	5.5	5.0	8.5	10.0
15.5	11.0	5.5	5.0	9.5	8.0
15.5	11.0	5.5	6.5	11.0	7.5
15.5	11.0	7.5	1.5	8.0	10.5
15.5	11.0	7.5	3.5	9.0	8.5
15.5	11.0	7.5	6.5	11.0	5.5
15.5	11.0	10.5	5.5	8.0	9.5
15.5	11.5	6.0	5.0	9.0	7.5
15.5	11.5	6.0	5.5	8.5	9.0
15.5	11.5	6.0	9.0	11.0	10.5
15.5	11.5	7.0	4.5	5.5	11.0
15.5	11.5	10.0	5.5	8.5	9.0
15.5	11.5	10.0	8.0	11.0	5.5
15.5	12.0	4.5	3.5	6.0	11.5
15.5	12.0	6.5	4.5	6.0	10.5
15.5	12.0	6.5	5.0	7.5	9.0
15.5	12.0	6.5	6.5	10.0	6.5
15.5	12.0	9.5	1.0	9.5	12.0
15.5	12.0	10.5	8.0	5.5	11.0
15.5	12.5	4.0	4.5	6.5	12.0
15.5	12.5	5.0	3.5	5.5	11.0
15.5	12.5	5.0	7.5	11.5	11.0
15.5	12.5	12.0	9.5	5.5	11.0
15.5	13.0	4.5	9.0	12.5	12.0
15.5	13.0	5.5	5.5	10.0	12.5
15.5	13.0	7.5	6.0	4.5	12.0
15.5	13.0	11.5	5.0	11.5	8.0
15.5	13.5	3.0	4.0	6.0	11.5
15.5	13.5	5.0	4.5	5.5	11.0
15.5	13.5	5.0	7.0	11.0	13.5
15.5	13.5	6.0	4.0	9.0	13.5
15.5	13.5	8.0	8.0	9.0	7.5
15.5	13.5	8.0	9.0	9.0	8.5
15.5	13.5	9.0	4.0	6.0	13.5
15.5	13.5	11.0	7.0	5.0	13.5
15.5	13.5	13.0	8.0	11.0	7.5
15.5	13.5	13.0	10.5	7.5	9.0
15.5	14.0	2.5	5.0	6.5	12.0
15.5	14.0	2.5	12.5	14.0	12.5
15.5	14.0	3.5	4.0	5.5	11.0
15.5	14.0	6.5	6.0	5.5	11.0
15.5	14.0	8.5	7.5	3.0	13.5
15.5	14.0	8.5	8.0	7.5	9.0
15.5	14.0	8.5	10.5	12.0	4.5
15.5	14.0	10.5	5.0	10.5	9.0
15.5	14.0	10.5	7.5	11.0	7.5
15.5	14.0	10.5	9.0	5.5	11.0
15.5	14.0	12.5	7.5	12.0	7.5
15.5	14.0	12.5	10.5	6.0	10.5
15.5	14.0	12.5	10.5	8.0	8.5
15.5	14.0	12.5	11.0	3.5	13.0
15.5	14.0	12.5	11.0	10.5	6.0
15.5	14.0	12.5	12.5	14.0	2.5
15.5	14.0	13.5	11.0	13.5	4.0
~~15.5~~	~~14.5~~	~~10.0~~	~~5.5~~	~~5.5~~	~~14.0~~
15.5	14.5	12.0	10.5	4.5	12.0
15.5	14.5	12.0	10.5	4.5	14.0
15.5	15.0	1.5	9.0	9.5	15.0
15.5	15.0	3.5	1.0	3.5	15.0
15.5	15.0	3.5	5.0	5.5	11.0
15.5	15.0	7.5	4.0	7.5	12.0
15.5	15.0	7.5	5.5	8.0	10.5
15.5	15.0	7.5	7.5	9.0	8.5
15.5	15.0	7.5	10.5	11.0	5.5
15.5	15.0	9.5	9.0	1.5	15.0
15.5	15.0	9.5	9.0	6.5	10.0
15.5	15.0	9.5	9.5	8.0	8.5
15.5	15.0	9.5	12.0	12.5	4.0
15.5	15.0	11.5	8.0	9.5	9.0
15.5	15.0	11.5	9.0	11.5	7.0
15.5	15.0	11.5	10.5	3.0	13.5
15.5	15.0	11.5	12.5	14.0	4.5
15.5	15.0	11.5	13.0	13.5	3.0
15.5	15.0	12.5	9.0	4.5	14.0
15.5	15.0	12.5	11.0	5.5	11.0
15.5	15.5	7.0	4.5	3.5	15.0
15.5	15.5	7.0	9.5	4.5	15.0
15.5	15.5	8.0	9.5	3.5	15.0
15.5	15.5	11.0	10.5	1.5	15.0
15.5	15.5	11.0	10.5	7.5	9.0
15.5	15.5	11.0	11.0	9.0	7.5
15.5	15.5	15.0	10.0	6.0	13.5
15.5	15.5	15.0	12.0	6.0	13.5
16.0	9.0	8.0	2.5	8.5	8.5
16.0	10.0	7.0	4.0	9.0	9.0
16.0	10.5	7.5	4.0	8.5	8.5
16.0	10.5	7.5	5.0	7.5	10.5
16.0	10.5	7.5	5.0	10.5	10.5
16.0	10.5	7.5	8.0	10.5	10.5
16.0	10.5	10.5	5.0	10.5	7.5
16.0	10.5	10.5	6.5	9.0	8.0
16.0	10.5	10.5	8.0	10.5	7.5
16.0	11.0	7.0	2.0	8.0	10.0
16.0	11.0	8.0	4.0	11.0	10.0
16.0	11.0	10.0	4.0	11.0	8.0
16.0	11.0	10.0	6.5	8.5	8.5
16.0	11.0	11.0	7.5	10.5	6.5
16.0	11.5	7.5	5.0	11.5	10.5
16.0	11.5	8.5	2.5	8.0	11.0
16.0	11.5	10.5	5.0	11.5	7.5
16.0	12.0	5.0	5.0	9.0	9.0
16.0	12.0	5.0	8.0	11.0	11.0
16.0	12.0	9.0	4.0	10.0	9.0
16.0	12.0	9.0	5.0	9.0	9.0
16.0	12.0	9.0	6.0	11.0	7.0
16.0	12.0	9.0	8.0	9.0	9.0
16.0	12.0	12.0	9.0	11.0	6.0
16.0	12.5	4.5	9.5	12.0	12.0
16.0	12.5	5.5	5.0	8.5	8.5
16.0	12.5	8.5	6.0	10.5	7.5
16.0	12.5	9.5	8.5	7.0	11.0
16.0	12.5	11.5	4.0	11.5	9.5
16.0	13.0	5.0	4.5	8.5	11.5
16.0	13.0	5.0	8.5	12.5	11.5
16.0	13.0	9.0	1.0	9.0	13.0
16.0	13.0	9.0	5.0	10.0	9.0
16.0	13.0	9.0	7.0	11.0	7.0
16.0	13.0	9.0	7.5	8.5	8.5
16.0	13.0	9.0	8.5	7.5	10.5
16.0	13.0	11.0	4.0	11.0	10.0
16.0	13.0	13.0	8.5	10.5	7.5
16.0	13.0	13.0	10.0	9.0	8.0
16.0	13.0	13.0	11.5	10.5	7.5
16.0	13.5	12.5	10.0	8.5	8.5
16.0	13.5	12.5	12.5	8.0	11.0
16.0	13.5	13.5	11.5	13.0	4.0
16.0	14.0	3.0	9.0	11.0	11.0
16.0	14.0	4.0	8.0	11.0	13.0
16.0	14.0	5.0	10.5	13.5	10.5
16.0	14.0	6.0	4.5	7.5	10.5
16.0	14.0	13.0	9.0	13.0	6.0
16.0	14.5	3.5	10.0	12.5	14.5
16.0	14.5	3.5	10.5	12.0	9.0
16.0	14.5	4.5	6.0	8.5	11.5
16.0	14.5	4.5	5.0	9.5	14.5
16.0	14.5	7.5	8.0	8.5	8.5
16.0	14.5	9.5	6.0	4.5	14.5
16.0	14.5	9.5	9.0	8.5	8.5
16.0	14.5	11.5	8.5	9.0	9.0
16.0	14.5	11.5	11.5	9.0	9.0
16.0	14.5	11.5	12.5	9.0	10.0
16.0	14.5	11.5	13.0	13.5	4.5
~~16.0~~	~~14.5~~	~~12.5~~	~~13.0~~	~~3.5~~	~~14.5~~
16.0	15.0	2.0	14.0	15.0	15.0
16.0	15.0	4.0	4.0	7.0	15.0
16.0	15.0	5.0	5.0	8.0	11.0
16.0	15.0	7.0	4.0	4.0	15.0
16.0	15.0	12.0	12.5	10.5	7.5
16.0	15.0	15.0	11.5	12.5	5.5
16.0	15.0	15.0	11.5	14.5	7.5
16.0	15.0	15.0	12.5	9.5	7.5
16.0	15.0	15.0	14.0	15.0	2.0
16.0	15.5	2.5	4.0	5.5	15.5
16.0	15.5	2.5	8.0	8.5	11.5
16.0	15.5	5.5	4.0	2.5	15.5
16.0	15.5	8.5	9.0	3.5	14.5
16.0	15.5	11.5	5.0	7.5	14.5
16.0	15.5	11.5	11.0	12.5	5.5
16.0	15.5	14.5	11.0	13.5	5.5
16.0	15.5	14.5	12.5	6.0	13.0
16.0	15.5	15.5	11.5	10.0	8.0
16.0	15.5	15.5	13.5	12.0	5.0
16.0	15.5	15.5	15.0	13.5	4.5
16.0	16.0	5.0	10.5	10.5	7.5
16.0	16.0	8.0	6.5	2.5	15.5
16.0	16.0	8.0	9.5	8.5	8.5
16.0	16.0	11.0	8.0	4.0	15.0
16.0	16.0	14.0	10.5	4.5	14.5
16.0	16.0	14.0	12.5	8.5	8.5
16.0	16.0	16.0	14.5	13.5	3.5
16.5	10.5	7.0	2.5	7.5	10.0
16.5	11.0	7.5	3.0	8.5	11.0

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
16.5	11.0	8.5	3.0	7.5	11.0
16.5	11.0	9.5	2.5	11.0	10.5
16.5	11.0	10.5	2.5	11.0	9.5
16.5	11.5	8.0	3.0	8.0	10.5
16.5	11.5	9.0	5.0	11.0	7.5
16.5	11.5	9.0	6.5	10.5	7.0
16.5	11.5	10.0	2.5	10.5	10.0
16.5	12.0	8.5	4.5	11.0	9.5
16.5	12.0	8.5	5.0	10.5	8.0
16.5	12.0	8.5	6.5	10.0	7.5
16.5	12.0	9.5	6.5	7.0	10.5
16.5	12.5	5.0	3.0	7.0	10.5
16.5	12.5	5.0	5.0	8.0	11.5
16.5	12.5	5.0	8.0	12.0	8.5
16.5	12.5	5.0	9.0	12.0	11.5
16.5	12.5	6.0	3.5	8.5	11.0
16.5	12.5	6.0	6.0	11.0	10.5
16.5	12.5	9.0	6.5	5.5	12.0
16.5	12.5	9.0	6.5	7.5	10.0
16.5	12.5	9.0	7.5	10.5	7.0
16.5	12.5	10.0	2.0	11.0	12.5
16.5	12.5	11.0	2.0	10.0	12.5
16.5	12.5	11.0	4.5	11.5	9.0
16.5	13.0	4.5	9.0	12.5	9.0
16.5	13.0	6.5	3.5	8.0	10.5
16.5	13.0	10.5	2.0	10.5	12.0
16.5	13.0	10.5	4.5	11.0	9.5
16.5	13.0	10.5	8.0	6.5	11.0
16.5	13.0	10.5	8.0	7.5	10.0
16.5	13.0	10.5	8.0	12.5	6.0
16.5	13.0	10.5	8.5	10.0	7.5
16.5	13.0	12.5	11.0	10.5	8.0
16.5	13.5	4.0	8.5	10.5	13.0
16.5	13.5	5.0	11.0	13.0	10.5
16.5	13.5	6.0	5.0	7.0	10.5
16.5	13.5	10.0	8.0	5.0	12.5
16.5	14.0	3.5	2.5	5.0	12.5
16.5	14.0	3.5	6.0	8.5	11.0
16.5	14.0	3.5	8.5	11.0	10.5
16.5	14.0	3.5	10.5	13.0	10.5
16.5	14.0	4.5	5.5	9.0	12.5
16.5	14.0	4.5	6.5	8.0	11.5
16.5	14.0	4.5	7.5	11.0	12.5
16.5	14.0	6.5	8.0	10.5	7.0
16.5	14.0	7.5	3.0	9.5	14.0
16.5	14.0	7.5	6.0	9.5	9.0
16.5	14.0	7.5	6.5	7.0	10.5
16.5	14.0	9.5	3.0	7.5	14.0
16.5	14.0	9.5	8.0	7.5	10.0
16.5	14.0	11.5	10.0	10.5	7.0
16.5	14.0	11.5	11.0	9.5	9.0
16.5	14.0	12.5	4.5	9.0	13.5
16.5	14.0	12.5	7.0	12.5	8.0
16.5	14.5	5.0	6.5	7.5	11.0
16.5	14.5	11.0	10.0	10.0	7.5
16.5	14.5	12.0	10.0	7.0	10.5
16.5	14.5	12.0	10.5	3.5	14.0
16.5	14.5	12.0	12.0	11.0	7.5
~~16.5~~	~~15.0~~	~~2.5~~	~~2.0~~	~~3.5~~	~~14.0~~
16.5	15.0	2.5	7.5	9.0	12.5
16.5	15.0	2.5	9.5	11.0	12.5
16.5	15.0	4.5	6.0	7.5	10.0
16.5	15.0	6.5	1.0	6.5	15.0
16.5	15.0	6.5	9.0	10.5	7.0
16.5	15.0	8.5	3.0	8.5	13.0
16.5	15.0	8.5	7.5	5.0	12.5
16.5	15.0	8.5	8.0	10.5	8.0
16.5	15.0	8.5	11.0	12.5	5.0
16.5	15.0	11.5	2.5	10.0	14.5
16.5	15.0	11.5	10.0	4.5	13.0
16.5	15.0	11.5	10.0	7.5	10.0
16.5	15.0	11.5	11.5	12.0	5.5
16.5	15.0	11.5	12.0	10.5	8.0
16.5	15.0	14.5	12.0	14.5	4.0
16.5	15.5	3.0	3.5	5.5	15.0
16.5	15.5	3.0	4.5	6.5	15.0
16.5	15.5	3.0	7.5	8.5	11.0
16.5	15.5	3.0	8.0	10.0	15.5
16.5	15.5	5.0	4.5	2.5	15.0
16.5	15.5	6.0	5.5	3.5	14.0
16.5	15.5	6.0	5.5	7.5	11.0
16.5	15.5	6.0	5.5	9.5	14.0
16.5	15.5	8.0	7.5	5.5	12.0
16.5	15.5	8.0	8.0	7.0	10.5
16.5	15.5	8.0	8.5	4.5	14.0
16.5	15.5	8.0	9.0	9.0	8.5
16.5	15.5	8.0	11.0	12.0	5.5
16.5	15.5	10.0	3.5	7.5	15.0

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
16.5	15.5	10.0	4.5	6.5	15.0
16.5	15.5	10.0	8.0	3.0	15.5
16.5	15.5	11.0	2.5	10.5	14.0
16.5	15.5	11.0	4.5	10.5	12.0
16.5	15.5	11.0	7.0	11.0	9.5
16.5	15.5	11.0	10.5	12.5	6.0
16.5	15.5	11.0	13.0	14.0	3.5
16.5	15.5	15.0	5.5	13.5	11.0
16.5	15.5	15.0	11.0	8.0	10.5
16.5	15.5	15.0	13.0	5.0	12.5
16.5	15.5	15.0	14.5	4.5	14.0
16.5	16.0	1.5	7.5	8.0	15.5
16.5	16.0	3.5	7.5	8.0	10.5
16.5	16.0	5.5	5.5	4.0	13.5
16.5	16.0	5.5	8.5	9.0	8.5
16.5	16.0	7.5	7.0	2.5	15.0
16.5	16.0	7.5	11.0	10.5	8.0
16.5	16.0	14.5	9.0	11.5	9.0
16.5	16.0	14.5	11.5	15.0	6.5
16.5	16.0	15.5	14.0	3.5	14.0
16.5	16.5	4.0	4.5	3.5	14.0
16.5	16.5	7.0	8.5	5.5	13.0
16.5	16.5	7.0	8.5	7.5	10.0
16.5	16.5	9.0	8.5	3.5	14.0
16.5	16.5	10.0	8.0	4.0	14.5
16.5	16.5	10.0	11.5	10.5	7.0
16.5	16.5	13.0	7.5	6.5	15.0
16.5	16.5	13.0	11.0	3.0	15.5
16.5	16.5	13.0	13.5	12.5	5.0
16.5	16.5	14.0	11.0	9.0	9.5
16.5	16.5	14.0	12.5	5.5	12.0
16.5	16.5	14.0	12.5	7.5	10.0
16.5	16.5	14.0	14.0	13.0	4.5
16.5	16.5	14.0	15.0	15.0	2.5
16.5	16.5	15.0	8.0	8.0	14.5
16.5	16.5	15.0	13.5	10.5	7.0
16.5	16.5	15.0	15.5	13.5	5.0
17.0	10.0	8.0	5.0	10.0	10.0
17.0	10.0	10.0	5.0	10.0	8.0
17.0	10.5	7.5	6.0	10.5	10.5
17.0	10.5	10.5	6.0	10.5	7.5
17.0	11.0	9.0	3.0	11.0	10.0
17.0	11.0	9.0	5.0	9.0	9.0
17.0	11.0	9.0	5.5	10.5	7.5
17.0	11.0	10.0	3.0	11.0	9.0
17.0	11.5	8.5	5.5	10.0	8.0
17.0	11.5	9.5	3.0	10.5	9.5
17.0	11.5	11.5	7.5	10.0	8.0
17.0	12.0	6.0	4.0	8.0	11.0
17.0	12.0	6.0	6.5	10.5	10.5
17.0	12.0	6.0	7.0	12.0	8.0
17.0	12.0	8.0	7.0	12.0	6.0
17.0	12.0	9.0	6.0	9.0	9.0
17.0	12.5	6.5	4.0	7.5	10.5
17.0	12.5	10.5	7.5	9.0	9.0
17.0	12.5	10.5	8.5	12.0	6.0
17.0	12.5	12.5	9.0	10.5	7.5
17.0	12.5	12.5	11.5	11.0	9.0
~~17.0~~	~~13.0~~	~~7.0~~	~~7.0~~	~~11.0~~	~~7.0~~
17.0	13.0	7.0	7.0	13.0	12.0
17.0	13.0	12.0	7.0	13.0	7.0
17.0	13.0	13.0	10.0	12.0	6.0
17.0	13.5	4.5	5.0	8.5	10.5
17.0	13.5	4.5	6.0	8.5	12.5
17.0	13.5	4.5	8.0	10.5	12.5
17.0	13.5	4.5	8.0	11.5	9.5
17.0	13.5	7.5	5.5	6.0	12.0
17.0	13.5	7.5	6.5	9.0	9.0
17.0	13.5	11.5	11.5	10.0	10.0
17.0	14.0	4.0	10.0	13.0	10.0
17.0	14.0	5.0	5.0	8.0	10.0
17.0	14.0	8.0	4.0	11.0	14.0
17.0	14.0	11.0	4.0	8.0	14.0
17.0	14.0	11.0	9.0	9.0	9.0
17.0	14.0	12.0	5.0	12.0	10.0
17.0	14.0	12.0	9.5	7.5	10.5
17.0	14.5	6.5	5.0	11.5	14.5
17.0	14.5	8.5	7.0	4.5	13.5
17.0	14.5	8.5	7.0	5.5	12.5
17.0	14.5	8.5	8.5	10.0	8.0
17.0	14.5	11.5	6.0	6.5	14.5
17.0	14.5	11.5	9.5	8.0	10.0
17.0	14.5	14.5	11.5	9.0	9.0
17.0	14.5	14.5	12.0	12.5	5.5
17.0	14.5	14.5	12.5	14.0	4.0
17.0	15.0	3.0	4.0	5.0	15.0
17.0	15.0	3.0	5.0	6.0	15.0
17.0	15.0	3.0	7.0	9.0	12.0

Table 5.1. Nontrivial Zeros of the 6-j Symbol.

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
17.0	15.0	3.0	12.0	14.0	12.0
17.0	15.0	5.0	4.0	3.0	15.0
17.0	15.0	6.0	5.0	3.0	15.0
17.0	15.0	6.0	5.0	4.0	14.0
17.0	15.0	6.0	5.0	10.0	15.0
17.0	15.0	6.0	6.0	7.0	11.0
17.0	15.0	6.0	8.0	10.0	8.0
17.0	15.0	8.0	7.0	6.0	12.0
17.0	15.0	10.0	5.0	6.0	15.0
17.0	15.0	11.0	11.0	12.0	6.0
17.0	15.0	14.0	10.0	14.0	6.0
17.0	15.0	14.0	11.5	7.5	10.5
17.0	15.0	14.0	12.5	7.5	13.5
17.0	15.5	5.5	5.0	4.5	13.5
17.0	15.5	9.5	4.0	9.5	12.5
17.0	15.5	9.5	7.0	10.5	9.5
17.0	15.5	9.5	8.5	3.0	15.0
17.0	15.5	9.5	12.0	13.5	4.5
17.0	15.5	13.5	11.5	6.0	12.0
17.0	16.0	2.0	4.5	5.5	14.5
17.0	16.0	2.0	10.5	11.5	14.5
17.0	16.0	2.0	15.0	16.0	16.0
17.0	16.0	4.0	4.0	4.0	14.0
17.0	16.0	7.0	7.0	6.0	12.0
17.0	16.0	7.0	8.0	8.0	10.0
17.0	16.0	10.0	9.0	4.0	14.0
17.0	16.0	10.0	9.0	5.0	13.0
17.0	16.0	10.0	10.0	9.0	9.0
17.0	16.0	10.0	11.0	11.0	7.0
17.0	16.0	13.0	7.0	7.0	15.0
17.0	16.0	13.0	11.5	4.5	13.5
17.0	16.0	13.0	11.5	8.5	9.5
17.0	16.0	13.0	13.0	13.0	5.0
17.0	16.0	14.0	13.5	13.5	4.5
17.0	16.0	16.0	4.5	14.5	12.5
17.0	16.0	16.0	10.5	14.5	6.5
17.0	16.0	16.0	15.0	16.0	2.0
17.0	16.5	1.5	10.0	10.5	16.5
17.0	16.5	4.5	5.0	4.5	13.5
17.0	16.5	4.5	7.0	7.5	10.5
17.0	16.5	6.5	10.0	10.5	7.5
17.0	16.5	8.5	8.0	4.5	13.5
17.0	16.5	10.5	10.0	1.5	16.5
17.0	16.5	11.5	7.0	15.5	15.5
17.0	16.5	12.5	11.5	3.0	15.0
17.0	16.5	16.5	14.0	10.5	7.5
17.0	17.0	3.0	4.5	4.5	13.5
17.0	17.0	8.0	8.0	5.0	13.0
17.0	17.0	8.0	12.0	12.0	6.0
17.0	17.0	9.0	9.0	6.0	12.0
17.0	17.0	12.0	11.5	1.5	16.5
17.0	17.0	15.0	13.5	4.5	13.5
17.0	17.0	16.0	11.0	6.0	15.0
17.5	11.0	8.5	5.5	10.0	9.5
17.5	12.5	7.0	7.0	12.0	12.5
17.5	12.5	12.0	7.0	7.0	12.5
17.5	13.0	10.5	3.0	12.5	13.0
17.5	13.0	12.5	3.0	10.5	13.0
17.5	13.5	8.0	2.5	9.5	13.0
17.5	13.5	8.0	5.5	12.5	13.0
17.5	14.0	6.5	1.5	7.0	13.5
17.5	14.0	6.5	4.0	9.5	13.0
17.5	14.0	6.5	7.0	12.5	13.0
17.5	14.0	7.5	4.5	11.0	13.5
17.5	14.0	8.5	2.5	9.0	12.5
17.5	14.0	11.5	3.0	11.5	12.0
17.5	14.5	9.0	2.0	10.0	14.5
17.5	14.5	10.0	2.0	9.0	14.5
17.5	14.5	14.0	2.5	12.5	14.0
17.5	15.0	7.5	4.0	8.5	12.0
17.5	15.0	9.5	2.0	9.5	14.0
17.5	15.0	9.5	7.0	9.5	10.0
17.5	15.0	9.5	10.5	14.0	6.5
17.5	15.0	13.5	2.5	13.0	13.5
17.5	15.0	13.5	5.5	13.0	10.5
17.5	15.0	13.5	8.0	13.5	8.0
17.5	15.0	13.5	9.5	14.0	6.5
17.5	15.0	13.5	10.0	14.5	9.0
17.5	15.0	14.5	10.0	10.5	9.0
17.5	15.5	6.0	5.5	9.5	13.0
17.5	15.5	9.0	4.5	9.5	12.0
17.5	15.5	10.0	5.5	10.5	11.0
17.5	15.5	13.0	5.5	12.5	11.0
17.5	15.5	14.0	10.0	10.0	9.5
17.5	15.5	15.0	4.0	12.0	14.5
17.5	15.5	15.0	11.5	14.5	7.0
17.5	15.5	15.0	14.5	14.5	5.0
17.5	16.0	4.5	7.0	10.5	16.0
17.5	16.0	6.5	5.5	9.0	12.5
17.5	16.0	8.5	10.5	13.0	7.5
17.5	16.0	10.5	7.0	4.5	16.0
17.5	16.0	10.5	7.0	15.5	16.0
17.5	16.0	11.5	6.0	15.5	16.0
17.5	16.0	12.5	7.5	6.0	15.5
17.5	16.0	15.5	6.0	11.5	16.0
17.5	16.0	15.5	7.0	10.5	16.0
17.5	16.0	15.5	7.5	12.0	15.5
17.5	16.0	15.5	10.5	13.0	7.5
17.5	16.0	15.5	13.0	15.5	4.0
17.5	16.5	5.0	2.0	6.0	16.5
17.5	16.5	6.0	2.0	5.0	16.5
17.5	16.5	9.0	7.0	10.0	10.5
17.5	16.5	13.0	6.0	8.0	15.5
17.5	16.5	14.0	1.5	13.5	16.0
17.5	16.5	14.0	4.0	13.0	13.5
17.5	16.5	14.0	7.0	13.0	10.5
17.5	16.5	14.0	9.0	6.0	15.5
17.5	16.5	14.0	9.5	13.5	8.0
17.5	16.5	14.0	11.0	14.0	6.5
17.5	16.5	16.0	15.0	8.0	11.5
17.5	17.0	3.5	14.5	16.0	15.5
17.5	17.0	5.5	2.0	5.5	16.0
17.5	17.0	11.5	8.0	11.5	10.0
17.5	17.0	11.5	10.0	12.5	11.0
17.5	17.5	11.0	7.0	5.0	16.5
17.5	17.5	14.0	13.5	7.5	14.0
17.5	17.5	15.0	11.5	4.5	16.0
18.0	10.0	9.0	4.0	10.0	10.0
18.0	10.0	10.0	4.0	10.0	9.0
18.0	10.5	9.5	4.0	9.5	9.5
18.0	11.0	11.0	6.0	10.0	9.0
18.0	11.5	10.5	6.0	9.5	9.5
18.0	12.0	7.0	3.5	9.5	9.5
18.0	12.0	7.0	8.0	12.0	12.0
18.0	12.0	12.0	8.0	12.0	7.0
18.0	12.5	12.5	8.5	11.0	8.0
18.0	13.5	5.5	5.5	10.0	10.0
18.0	13.5	6.5	7.5	13.0	11.0
18.0	13.5	7.5	3.0	9.5	12.5
18.0	14.0	6.0	2.0	7.0	13.0
18.0	14.0	6.0	5.5	9.5	9.5
18.0	14.0	6.0	7.0	8.0	14.0
18.0	14.0	7.0	8.0	12.0	12.0
18.0	14.0	8.0	3.0	9.0	12.0
18.0	14.0	8.0	7.0	6.0	14.0
18.0	14.0	11.0	1.0	11.0	14.0
18.0	14.0	11.0	8.5	9.5	9.5
18.0	14.0	11.0	8.5	13.5	6.5
18.0	14.0	14.0	10.5	10.5	8.5
18.0	14.0	14.0	11.0	13.0	6.0
18.0	14.5	9.5	8.0	9.5	9.5
18.0	14.5	9.5	11.0	13.5	6.5
18.0	14.5	14.5	13.0	13.5	6.5
18.0	15.0	4.0	8.0	11.0	11.0
18.0	15.0	6.0	5.0	10.0	14.0
18.0	15.0	6.0	6.0	11.0	14.0
18.0	15.0	7.0	7.0	7.0	13.0
18.0	15.0	13.0	6.0	13.0	10.0
18.0	15.0	13.0	10.5	9.5	9.5
18.0	15.0	15.0	12.0	12.0	7.0
18.0	15.0	15.0	12.0	14.0	7.0
18.0	15.5	4.5	4.5	8.0	14.0
18.0	15.5	7.5	4.0	10.5	15.5
18.0	15.5	8.5	7.5	11.0	9.0
18.0	15.5	8.5	8.0	10.5	10.5
18.0	15.5	8.5	11.0	12.5	7.5
18.0	15.5	10.5	4.0	7.5	15.5
18.0	15.5	12.5	6.0	12.5	10.5
18.0	15.5	12.5	9.0	13.5	7.5
18.0	15.5	15.5	11.0	12.5	7.5
18.0	15.5	15.5	11.5	14.0	6.0
18.0	15.5	15.5	13.5	15.0	4.0
18.0	16.0	3.0	10.5	12.5	12.5
18.0	16.0	5.0	2.5	6.5	15.5
18.0	16.0	5.0	5.5	10.5	15.5
18.0	16.0	5.0	9.0	13.0	16.0
18.0	16.0	7.0	3.0	9.0	16.0
18.0	16.0	9.0	3.0	7.0	16.0
18.0	16.0	9.0	8.5	11.5	8.5
18.0	16.0	13.0	2.5	11.5	15.5
18.0	16.0	13.0	6.5	7.5	15.5
18.0	16.0	13.0	9.0	5.0	16.0
18.0	16.0	13.0	10.0	15.0	8.0
18.0	16.0	15.0	5.0	14.0	12.0
18.0	16.0	15.0	6.0	14.0	11.0
18.0	16.0	15.0	7.5	12.5	10.5

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
18.0	16.0	15.0	7.5	15.5	10.5
18.0	16.0	15.0	10.0	13.0	8.0
18.0	16.0	15.0	11.0	15.0	6.0
18.0	16.0	16.0	13.0	10.0	9.0
18.0	16.5	2.5	12.5	14.0	14.0
18.0	16.5	3.5	7.0	9.5	15.5
18.0	16.5	3.5	15.0	15.5	15.5
18.0	16.5	5.5	2.5	6.0	15.0
18.0	16.5	5.5	4.5	7.0	13.0
18.0	16.5	5.5	8.0	9.5	9.5
18.0	16.5	7.5	5.0	8.5	12.5
18.0	16.5	12.5	2.5	12.0	15.0
18.0	16.5	13.5	10.5	14.0	7.0
18.0	16.5	14.5	5.0	13.5	12.5
18.0	16.5	14.5	7.5	12.0	11.0
18.0	16.5	15.5	13.0	9.5	9.5
18.0	17.0	2.0	16.0	17.0	17.0
18.0	17.0	3.0	9.0	11.0	17.0
18.0	17.0	4.0	5.0	8.0	17.0
18.0	17.0	8.0	3.0	8.0	15.0
18.0	17.0	8.0	5.0	4.0	17.0
18.0	17.0	8.0	6.0	9.0	12.0
18.0	17.0	9.0	4.0	9.0	14.0
18.0	17.0	11.0	9.0	3.0	17.0
18.0	17.0	11.0	9.0	12.0	9.0
18.0	17.0	13.0	14.0	7.0	14.0
18.0	17.0	14.0	6.0	13.0	12.0
18.0	17.0	15.0	6.5	11.5	15.5
18.0	17.0	15.0	9.0	14.0	9.0
18.0	17.0	15.0	13.5	15.5	4.5
18.0	17.0	17.0	14.5	12.5	6.5
18.0	17.0	17.0	16.0	17.0	2.0
18.0	17.5	10.5	11.0	9.5	9.5
18.0	17.5	12.5	12.0	9.5	9.5
18.0	17.5	13.5	13.5	6.0	14.0
18.0	17.5	14.5	9.5	15.0	10.0
18.0	17.5	17.5	12.5	14.0	7.0
18.0	18.0	2.0	15.5	15.5	15.5
18.0	18.0	11.0	6.0	6.0	17.0
18.0	18.0	11.0	13.0	10.0	10.0
18.0	18.0	12.0	9.0	4.0	17.0
18.0	18.0	13.0	11.0	10.0	10.0
18.0	18.0	13.0	13.5	7.5	12.5
18.0	18.0	14.0	12.0	3.0	17.0
18.0	18.0	14.0	12.5	6.5	14.5
18.0	18.0	16.0	10.0	7.0	16.0
18.0	18.0	18.0	15.5	11.5	7.5
18.0	18.0	18.0	16.0	14.0	5.0
18.5	12.5	8.0	4.0	11.0	10.5
18.5	13.0	6.5	3.5	9.0	10.5
18.5	13.0	6.5	5.0	8.5	13.0
18.5	13.0	6.5	7.5	13.0	8.5
18.5	13.0	6.5	8.0	11.5	13.0
18.5	13.0	6.5	8.0	12.5	11.0
18.5	13.0	8.5	4.0	10.5	10.0
18.5	13.0	8.5	5.0	6.5	13.0
18.5	13.0	8.5	7.5	13.0	6.5
18.5	13.0	11.5	8.0	6.5	13.0
18.5	13.5	6.0	2.5	6.5	13.0
18.5	13.5	11.0	9.0	13.0	6.5
18.5	14.0	6.5	6.5	12.0	11.5
18.5	14.0	7.5	5.0	7.5	12.0
18.5	14.0	7.5	7.5	12.0	7.5
18.5	14.0	13.5	4.0	13.5	11.0
18.5	14.0	13.5	10.5	13.0	6.5
18.5	14.0	13.5	11.0	13.5	9.0
18.5	14.5	6.0	6.0	10.0	14.5
18.5	14.5	6.0	6.5	8.5	13.0
18.5	14.5	6.0	7.5	12.5	12.0
18.5	14.5	10.0	6.0	6.0	14.5
18.5	14.5	10.0	7.5	6.5	13.0
18.5	15.0	4.5	3.0	6.5	13.0
18.5	15.0	4.5	5.0	7.5	14.0
18.5	15.0	6.5	6.5	8.0	12.5
18.5	15.0	7.5	2.5	9.0	14.5
18.5	15.0	8.5	6.5	6.0	13.5
18.5	15.0	8.5	8.0	10.5	9.0
18.5	15.0	8.5	10.0	13.5	6.0
18.5	15.0	10.5	1.0	10.5	15.0
18.5	15.0	12.5	4.0	12.5	12.0
18.5	15.0	12.5	9.0	12.5	8.0
18.5	15.0	12.5	9.0	13.5	9.0
18.5	15.5	5.0	6.0	10.0	13.5
18.5	15.5	5.0	10.0	14.0	13.5
18.5	15.5	7.0	3.5	9.5	15.0
18.5	15.5	7.0	4.5	10.5	15.0
18.5	15.5	7.0	8.0	14.0	15.5
18.5	15.5	8.0	2.5	8.5	14.0

j_1	j_2	j_3	ℓ_1	ℓ_2	ℓ_3
18.5	15.5	8.0	6.5	6.5	13.0
18.5	15.5	8.0	6.5	10.5	10.0
18.5	15.5	8.0	10.0	13.0	6.5
18.5	15.5	9.0	7.5	7.5	12.0
18.5	15.5	9.0	8.0	9.0	10.5
18.5	15.5	9.0	9.0	11.0	8.5
18.5	15.5	9.0	11.0	14.0	5.5
18.5	15.5	13.0	7.5	13.5	9.0
18.5	15.5	14.0	3.5	11.5	15.0
18.5	15.5	14.0	4.5	10.5	15.0
18.5	15.5	14.0	6.0	15.0	11.5
18.5	15.5	14.0	8.0	7.0	15.5
18.5	15.5	14.0	10.5	14.5	8.0
18.5	15.5	14.0	12.0	13.0	6.5
18.5	15.5	15.0	10.5	12.5	8.0
18.5	16.0	3.5	7.5	9.0	15.5
18.5	16.0	5.5	5.0	6.5	13.0
18.5	16.0	9.5	8.0	4.5	15.0
18.5	16.0	9.5	8.0	15.5	15.0
18.5	16.0	11.5	9.0	11.5	9.0
18.5	16.0	11.5	9.0	12.5	10.0
18.5	16.0	11.5	11.0	11.5	11.0
18.5	16.0	14.5	9.0	14.5	8.0
18.5	16.0	14.5	12.0	6.5	13.0
18.5	16.0	15.5	12.5	9.0	10.5
18.5	16.5	4.0	3.0	6.0	15.5
18.5	16.5	4.0	6.5	9.5	15.0
18.5	16.5	4.0	7.5	10.5	15.0
18.5	16.5	4.0	11.0	14.0	15.5
18.5	16.5	6.0	6.0	11.0	16.5
18.5	16.5	8.0	1.0	8.0	16.5
18.5	16.5	8.0	3.5	8.5	14.0
18.5	16.5	8.0	6.0	8.0	12.5
18.5	16.5	8.0	7.5	10.5	10.0
18.5	16.5	8.0	11.0	13.0	6.5
18.5	16.5	10.0	8.5	15.5	14.0
18.5	16.5	11.0	6.0	6.0	16.5
18.5	16.5	11.0	9.5	6.5	13.0
18.5	16.5	11.0	10.5	13.5	7.0
18.5	16.5	12.0	7.5	12.5	10.0
18.5	16.5	13.0	3.5	12.5	14.0
18.5	16.5	13.0	6.0	14.0	12.5
18.5	16.5	15.0	6.0	10.0	15.5
18.5	16.5	15.0	12.5	7.5	12.0
18.5	16.5	15.0	12.5	9.5	10.0
18.5	16.5	15.0	14.0	8.0	12.5
18.5	16.5	15.0	14.0	15.0	4.5
18.5	17.0	2.5	15.0	16.5	15.0
18.5	17.0	3.5	10.5	13.0	16.5
18.5	17.0	4.5	3.0	5.5	15.0
18.5	17.0	6.5	5.0	8.5	12.0
18.5	17.0	8.5	4.5	9.0	13.5
18.5	17.0	10.5	9.5	3.0	16.5
18.5	17.0	10.5	9.5	7.0	12.5
18.5	17.0	10.5	10.5	10.0	9.5
18.5	17.0	10.5	10.5	13.0	7.5
18.5	17.0	10.5	13.5	15.0	4.5
18.5	17.0	12.5	4.5	12.0	13.5
18.5	17.0	12.5	12.0	14.5	6.0
18.5	17.0	14.5	12.5	6.0	13.5
18.5	17.0	16.5	3.0	15.5	15.0
18.5	17.0	16.5	6.5	15.0	11.5
18.5	17.0	16.5	7.5	15.0	10.5
18.5	17.0	16.5	11.0	15.5	7.0
18.5	17.0	16.5	14.0	6.5	13.0
18.5	17.0	16.5	14.0	16.5	4.0
18.5	17.5	2.0	8.0	9.0	15.5
18.5	17.5	2.0	14.0	15.0	16.5
18.5	17.5	11.0	10.0	6.0	13.5
18.5	17.5	13.0	10.5	10.5	10.0
18.5	17.5	13.0	13.0	7.0	13.5
18.5	17.5	14.0	12.5	4.5	15.0
18.5	17.5	17.0	13.0	7.0	13.5
18.5	18.0	1.5	11.0	11.5	18.0
18.5	18.0	2.5	5.0	6.5	18.0
18.5	18.0	6.5	5.0	2.5	18.0
18.5	18.0	7.5	8.0	6.5	13.0
18.5	18.0	9.5	3.5	7.0	17.5
18.5	18.0	10.5	10.0	6.5	13.0
18.5	18.0	11.5	8.5	14.0	12.5
18.5	18.0	11.5	10.5	12.0	10.5
18.5	18.0	11.5	11.0	1.5	18.0
18.5	18.0	11.5	12.0	10.5	9.0
18.5	18.0	13.5	12.5	3.0	16.5
18.5	18.0	13.5	12.5	9.0	10.5
18.5	18.0	13.5	13.5	12.0	7.5
18.5	18.0	13.5	14.0	9.5	11.0
18.5	18.0	15.5	6.0	11.5	15.0
18.5	18.0	15.5	8.0	11.5	15.0

Table 5.2. Nontrivial Zeros of the 3-j Symbol.

j_1	j_2	j_3	m_1	m_2	m_3	j_1	j_2	j_3	m_1	m_2	m_3
3	3	2	2	−2	0	10	10	4	9	−4	−5
5	4	2	3	−2	−1	11	8	5	8	−6	−2
6	5	3	5	−4	−1	11	8	5	8	−4	−4
6	4	4	2	−2	0	10	9	5	4	0	−4
5	5	4	4	−3	−1	8	8	8	6	−5	−1
6	6	3	5	−5	0	23/2	7	13/2	15/2	−5	−3/2
8	6	3	6	−4	−2	9	8	8	−5	4	1
9	8	2	5	−4	−1	9	9	7	5	0	−5
8	6	5	6	−5	−1	12	9	5	11	−8	−3
9	8	3	4	−4	0	11	9	6	9	−8	−1
9	7	4	8	−6	−2	11	9	6	0	1	−1
15/2	15/2	5	11/2	−7/2	−2	11	9	6	6	−1	−5
8	6	6	5	−5	0	10	9	7	−7	8	−1
8	7	5	6	−6	0	25/2	25/2	2	15/2	−15/2	0
19/2	15/2	4	1/2	−3/2	1	13	12	2	7	−6	−1
19/2	13/2	5	1/2	−3/2	1	25/2	21/2	4	9/2	−9/2	0
11	10	2	6	−5	−1	11	10	6	9	−9	0
11	11	1	3	−3	0	13	21/2	7/2	1	−1	0
11	9	3	4	−3	−1	25/2	11	7/2	3/2	0	−3/2
21/2	21/2	3	17/2	−17/2	0						

The nontrivial zeros of the 3-j symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are given in Table 5.2 above for all values of (j_1, j_2, j_3) and (m_1, m_2, m_3) such that $j_1 + j_2 + j_3 \leq 27$ and $-j_i \leq m_i \leq j_i$. These results are taken from Varshalovich *et al.* [2] and Bowick [3]. Here we have listed only the reduced results given by Bowick. (Each coefficient that is zero thus has associated with it 72 other zero coefficients obtained by application of all the symmetries of the 3-j symbol.)

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Topic 11. The Relationship between Generalized Hypergeometric Functions and the Racah–Wigner Coefficients

The generalized hypergeometric series, denoted by ${}_pF_q$, is defined on p real or complex numerator parameters $a_1, a_2, \ldots, a_p$, q real or complex denominator parameters $b_1, b_2, \ldots, b_q$, and a single variable z by

$$
{}_pF_q\begin{pmatrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{pmatrix}; z\Big) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{5.11.1}
$$

where

$$
(a)_n = a(a+1)\ldots(a+n-1), \qquad (a)_0 = 1,
$$

denotes a rising factorial. If one of the numerator parameters is a negative integer, the series terminates and the function is a polynomial in z. In the general case, the series is not defined if any of the b parameters is a negative integer. In the applications to be made here, one of the b parameters is always a negative integer. However, *all* the numerator parameters are negative integers, and the series always terminates before the troublesome zero in the denominator occurs. Thus, there is no difficulty with the negative b parameter, and it is customary to retain the same ${}_pF_q$ notation for this negative b parameter case.

The key expression for relating the Racah function to hypergeometric functions is

$$
\frac{(\beta_1+1)!\,{}_4F_3\left(\begin{matrix} \alpha_1-\beta_1, \alpha_2-\beta_1, \alpha_3-\beta_1, \alpha_4-\beta_1 \\ -\beta_1-1, \beta_2-\beta_1+1, \beta_3-\beta_1+1 \end{matrix}; 1\right)}{(\beta_2-\beta_1)!(\beta_3-\beta_1)!(\beta_1-\alpha_1)!(\beta_1-\alpha_2)!(\beta_1-\alpha_3)!(\beta_1-\alpha_4)!}
$$

$$
= \sum_s \frac{(-1)^{\beta_1+s}(s+1)!}{(s-\alpha_1)!(s-\alpha_2)!(s-\alpha_3)!(s-\alpha_4)!(\beta_1-s)!(\beta_2-s)!(\beta_3-s)!}, \tag{5.11.2}
$$

where the β_i and α_i are integers satisfying

$$
\begin{gathered}
\beta_1 \geqslant \max(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \\
\beta_1 \geqslant 0, \qquad \beta_2 \geqslant \beta_1, \qquad \beta_3 \geqslant \beta_1.
\end{gathered} \tag{5.11.3}
$$

The finite summation over s extends over all terms such that the factorial factors are nonnegative, or, using $1/n! = 0$ for $n = -1, -2, \ldots$, we may regard the sum as extending over all integers. Equation (5.11.2) is obtained by a straightforward application of the general definition (5.11.1), subject to the parameter ranges given by Eqs. (5.11.3).

Comparing Eq. (A.2), Chapter 4, for the Racah coefficient with Eq. (5.11.2), we obtain the following expression for the 6-j symbol and Racah function in terms of the ${}_4F_3$ function evaluated at $z=1$:

$$\begin{Bmatrix} a\,b\,e \\ d\,c\,f \end{Bmatrix} = (-1)^{a+b+c+d} W(abcd;ef)$$
$$= \Delta(abe)\Delta(cde)\Delta(acf)\Delta(bdf)$$
$$\times \frac{(-1)^{\beta_1}(\beta_1+1)!\,{}_4F_3\left(\begin{matrix} \alpha_1-\beta_1, \alpha_2-\beta_1, \alpha_3-\beta_1, \alpha_4-\beta_1 \\ -\beta_1-1, \beta_2-\beta_1+1, \beta_3-\beta_1+1 \end{matrix}; 1 \right)}{(\beta_2-\beta_1)!(\beta_3-\beta_1)!(\beta_1-\alpha_1)!(\beta_1-\alpha_2)!(\beta_1-\alpha_3)!(\beta_1-\alpha_4)!}, \tag{5.11.4}$$

where

$$\beta_1 = \min(a+b+c+d, a+d+e+f, b+c+e+f). \tag{5.11.5}$$

The parameters β_2 and β_3 are identified in either way with the pair remaining in the 3-tuple

$$(a+b+c+d, a+d+e+f, b+c+e+f) \tag{5.11.6}$$

after deleting β_1, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ may be identified with any permutation of the 4-tuple

$$(a+b+e, c+d+e, a+c+f, b+d+f). \tag{5.11.7}$$

Noting that

$$\sum_{i=1}^{4} \alpha_i = \sum_{i=1}^{3} \beta_i = 2(a+b+c+d+e+f), \tag{5.11.8}$$

we see that the a and b parameters occurring in ${}_4F_3$, Eq. (5.11.4), satisfy the relation

$$\sum_{i=1}^{3} b_i = 1 + \sum_{i=1}^{4} a_i; \tag{5.11.9}$$

that is, the ${}_4F_3$ series is Saalschützian. [If we adopt the standard definition of the ${}_pF_q$ functions, Eq. (5.11.1), we are forced into the above subtleties of identifying β_1 as in Eq. (5.11.5) to avoid parameter values of b_2 and b_3, which are sometimes positive, sometimes negative.]

An expression for the Wigner coefficients in terms of generalized hypergeometric functions (at $z=1$) may be obtained by using the limit relation [see Eq. (3.280) in AMQP]

$$\begin{aligned} C^{bdf}_{\alpha\beta\gamma} &= \lim_{j\to\infty} W^{bdf}_{\alpha\beta\gamma}(j+\gamma) \\ &= \delta_{\alpha+\beta,\gamma} \lim_{j\to\infty} \left[(2f+1)(2j+2\alpha+1)\right]^{\frac{1}{2}} W(j, b, j+\gamma, d;\, j+\alpha, f). \end{aligned} \tag{5.11.10}$$

We identify $a=j$, $b=b$, $c=j+\gamma$, $d=d$, $e=j+\alpha$, and $f=f$ in Eqs. (5.11.4)–(5.11.7). The labels α_i and β_i appearing in these equations may then be written as $\beta_i=2j+\delta_i$, where

$$\delta_1 = \min(b+\alpha+d+\beta,\, d-\beta+f+\gamma,\, b+\alpha+f+\gamma), \tag{5.11.11}$$

and δ_2 and δ_3 are the pair of integers remaining in the triple

$$(b+\alpha+d+\beta,\, d-\beta+f+\gamma,\, b+\alpha+f+\gamma) \tag{5.11.12}$$

after removing δ_1. We similarly may write $\alpha_i=2j+\varepsilon_i$ $(i=1,2,3)$, where $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is any permutation of

$$(b+\alpha,\, d+\alpha+\gamma,\, f+\gamma), \tag{5.11.13}$$

and $\alpha_4=b+d+f$. Using these results, we obtain

$$\begin{aligned} W^{bdf}_{\alpha\beta\gamma}(j+\gamma) &= \delta_{\alpha+\beta,\gamma}(2f+1)^{\frac{1}{2}}(-1)^{b+d+\gamma+\delta_1}\Delta(bdf) \\ &\quad \times \left[(b+\alpha)!(b-\alpha)!(d+\beta)!(d-\beta)!(f+\gamma)!(f-\gamma)!\right]^{\frac{1}{2}} \\ &\quad \times \left[\frac{(2j+2\alpha+1)^{\frac{1}{2}}(2j+\delta_1-b-d-f+1)_{b+d+f+1}}{\left[(2j+\alpha-b)_{2b+1}(2j+\alpha-\gamma-d)_{2d+1}(2j+\gamma-f)_{2f+1}\right]^{\frac{1}{2}}}\right] \\ &\quad \times \frac{{}_4F_3\left(\begin{matrix}\varepsilon_1-\delta_1, \varepsilon_2-\delta_1, \varepsilon_3-\delta_1, b+d+f-\delta_1-2j \\ -2j-\delta_1-1, \delta_2-\delta_1+1, \delta_3-\delta_1+1\end{matrix}; 1\right)}{(\delta_2-\delta_1)!(\delta_3-\delta_1)!(\delta_1-\varepsilon_1)!(\delta_1-\varepsilon_2)!(\delta_1-\varepsilon_3)!}. \end{aligned} \tag{5.11.14}$$

In obtaining this result, we have rewritten the j-dependent factors in rising factorial notation in anticipation of taking the limit, which we now effect:

The factor in square brackets limits to unity, while the ${}_4F_3$ factor limits to the ${}_3F_2$ factor obtained by deleting the j-dependent parameters. We thus obtain

$$C^{bdf}_{\alpha\beta\gamma}=\delta_{\alpha+\beta,\gamma}\big[(2f+1)(b+\alpha)!(b-\alpha)!(d+\beta)!(d-\beta)!(f+\gamma)!(f-\gamma)!\big]^{\frac{1}{2}}$$
$$\times\frac{(-1)^{b+d+\gamma+\delta_1}\Delta(bdf)\,{}_3F_2\left(\begin{matrix}\varepsilon_1-\delta_1,\varepsilon_2-\delta_1,\varepsilon_3-\delta_1\\ \delta_2-\delta_1+1,\delta_3-\delta_1+1\end{matrix};1\right)}{(\delta_2-\delta_1)!(\delta_3-\delta_1)!(\delta_1-\varepsilon_1)!(\delta_1-\varepsilon_2)!(\delta_1-\varepsilon_3)!},\tag{5.11.15}$$

where we again summarize the significance of the parameters ε_i and δ_i: The parameters $(\delta_1,\delta_2,\delta_3)$ may be any permutation of

$$(b+\alpha+d+\beta,\ d-\beta+f+\gamma,\ b+\alpha+f+\gamma),\tag{5.11.16}$$

except that δ_1 is the smallest integer in this set; the parameters $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ may be any permutation of

$$(b+\alpha,\ d+\alpha+\gamma,\ f+\gamma).\tag{5.11.17}$$

Note that we always have $\delta_1\geqslant\varepsilon_i$.

[The direct expansion of Eq. (5.11.15), using definition (5.11.1), yields the results given by Eq. (3.170), Chapter 3, AMQP, in the form $(-1)^{b+d-f}C^{bdf}_{-\alpha,-\beta,-\gamma}$.]

An interesting question now occurs: Given the relations (5.11.15) and (5.11.4) between Wigner coefficients and the ${}_3F_2$ function and between Racah coefficients and the ${}_4F_3$ function, respectively, which of the symmetry relations are known in consequence of the properties of the hypergeometric functions (which have a very long history of development)?

As an example of these historical considerations, let us examine the transpositional or Regge symmetry of Eq. (3.182) Chapter 3, AMQP, which was not known to physicists until 1958:

$$\begin{aligned}
&b\to(b+\alpha+d+\beta)/2,\\
&d\to(b-\alpha+d-\beta)/2,\\
&f\to f,\\
&\alpha\to(b+\alpha-d-\beta)/2,\\
&\beta\to(b-\alpha-d+\beta)/2,\\
&\gamma\to b-d.
\end{aligned}\tag{5.11.18}$$

If we consider the case $\delta_1=b+\alpha+d+\beta$ in Eq. (5.11.16), then the relevant hypergeometric factor in Eq. (5.11.15) may be chosen as

$$ {}_3F_2\left(\begin{matrix} -d-\beta, -b+\alpha, -b-d+f \\ -b-\beta+f+1, -d+\alpha+f+1 \end{matrix} ; 1 \right), \tag{5.11.19} $$

and the transformation of the a_i and b_i parameters in ${}_3F_2$ corresponding to (5.11.18) is simply the exchange $-d-\beta \leftrightarrow -b+\alpha$, all other parameters being left alone. Thus, from the viewpoint of hypergeometric function theory, the Regge symmetry is "trivial," since the hypergeometric series (5.11.1) is clearly invariant under permutations of its numerator parameters. [The remaining multiplicative factors in Eq. (5.11.15) are invariant under the transformation (5.11.18) for the given δ_1.]

It is easy to verify that the four generator symmetries (hence, all seventy-two symmetries) of the Wigner coefficients given by Eqs. (3.180) of Chapter 3, AMQP, are consequences of the following properties of ${}_3F_2\left(\begin{matrix} -a, -b, -c \\ d+1, e+1, \end{matrix} ; 1 \right)$, where $a, b, c, d, e, a-c$, and $b-c$ are all nonnegative integers:

(i) The ${}_3F_2$ series is invariant under permutations of a, b, and c among themselves and under the exchange of d and e.

(ii) The ${}_3F_2$ series satisfies the transformation law (Thomae [1], Bailey [2], Askey [3]):

$$ {}_3F_2\left(\begin{matrix} -a, -b, -c \\ d+1, e+1, \end{matrix} ; 1 \right) \Big/ a!b!c!d!e! $$
$$ = \frac{(-1)^c\, {}_3F_2\left(\begin{matrix} -c-d, -c-e, -c \\ -c+a+1, -c+b+1, \end{matrix} ; 1 \right)}{(c+d)!(c+e)!c!(a-c)!(b-c)!}. \tag{5.11.20} $$

~~Thus, all seventy-two of the Wigner coefficient symmetries follow from Eq.~~ (5.11.15) and well-known properties of the hypergeometric functions.

Consider now the 6-j symbol [see Eqs. (5.11.4)–(5.11.7)]. One easily sees that the 144 symmetries of the 6-j symbol imply only the trivial invariances of the ${}_4F_3$ series under the permutations of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and under the exchange of β_2 and β_3 [the only subtlety involved here is to recognize that β_1 is the *same* for all six permutations of $(a+d, b+c, e+f)$]. Conversely, if one makes a particular identification of $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ (three cases to be considered in identifying β_1), then the trivial invariances of the ${}_4F_3$ series under permutations of numerator parameters and the b_2, b_3 denominator parameters, together with the determination of the transformations of the triangle factors, imply the full 144 symmetries of the 6-j symbol. Thus, the symmetry properties of the ${}_4F_3$ series involved in the 144 Regge symmetries of the 6-j symbol are all well-known.

Remarks. (*a*) The relationship between Wigner and Racah coefficients and the ${}_3F_2$ and ${}_4F_3$ hypergeometric series, respectively, has been pointed out independently by several authors {Erdélyi [4] (Racah coefficients); Rose [5] (Wigner and Racah coefficients); Jahn and Howell [6] (Racah coefficients)}. The results given by these authors are incomplete in that the formulas are not valid over the full domain of definition of the quantum numbers occurring in a Wigner or Racah coefficient.

(*b*) At about the time of the writing of the present section, discussing the relationship of Wigner and Racah coefficients to hypergeometric series, was completed, we learned of the comprehensive work of Rao and Venkatesh [7–9], which reaches conclusions similar to ours. We wish to acknowledge several informative conversations and correspondences with Drs. Rao and Venkatesh.

(*c*) Professor Askey has kindly communicated to us that he and his students have observed that the identification of Racah coefficients with the ${}_4F_3$ series may be utilized to define *orthogonal polynomials* (in one variable) that obey orthogonality relations (discrete and integral types). He notes that many interesting sets of (known) orthogonal polynomials are contained as limiting cases of these relations (see Wilson [10]). These results are significant to note in relationship to the present volume, since they illustrate nicely our viewpoint that physically motivated concepts often suggest significant developments in mathematics.

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TOPIC 12. COUPLING OF n ANGULAR MOMENTA: RECOUPLING THEORY

1. Introduction

In Chapter 3, Sections 11 and 12, AMQP, the theory of the coupling of the states of two angular momenta to states of sharp total angular momentum [symbolized by $\mathbf{J}=\mathbf{J}(1)+\mathbf{J}(2)$] was initiated. Despite the rich structure and wide applicability of the associated concepts (Wigner coefficients, tensor operators, Racah coefficients, etc.) that arise in this theory, the scope of the theory developed there is still limited to rather simple physical systems or models of physical systems (those systems that may be thought of as being composed of two "parts" that may be interacting).

In this present Topic we indicate the extensions of the theory of coupling of two angular momenta that are required for dealing with composite systems made up of n kinematically independent parts, each part carrying its own angular momentum.

We shall deal principally with what is appropriately called *the theory of binary coupling of angular momenta*, since it is a theory in which one couples angular momenta sequentially, in pairs. Such a theory prejudices the indistinguishability of like particles, since one must select an identified sequence of pairwise couplings. Accordingly, the theory deals with the relationships between different *coupling schemes*—that is, between distinct sequences of pairwise couplings. This is the content of recoupling theory.

[Methods for coupling n angular momenta "democratically" have thus far been developed only for $n=3$ (Chakrabarti [1], Lévy-Leblond and Lévy-Nahas [2]). We discuss alternatives to binary coupling theory in Notes 2 and 3.]

The subject of the coupling of n angular momenta is a difficult one, and the literature is extensive. Several monographs (Jucys *et al.* [3], El Baz and Castel [4], Lehman and O'Connell[1] [6]) deal almost exclusively with techniques for implementing "graphical" methods for carrying out summations over the projection quantum numbers in products of Wigner coefficients. (Further discussions may also be found in Jucys and Bandzaitis [7], Brink and Satchler [8], Sandars [9], and Shelepin [10].)

Our goal here is to relate coupling methods to standard results from *graph theory*. The subject divides rather naturally into three parts: (a) the classification of coupling schemes and its relationship to the theory of *pure binary trees* (Sections 1–5); (b) the elementary operations underlying the structure of transformation coefficients (Sections 6–7); and (c) the classification of transformation coefficients and its relationship to the theory of *trivalent trees* and *cubic graphs* (Sections 8–9).

[1]This short NBS technical report is an elaboration of a technique suggested by Danos [5].

We have already developed in Chapter 3, Note 8, AMQP, the mathematical framework for the present discussion, using the viewpoint of the rotations themselves (kinematic independence, tensor product space, transformations under rotations, etc.). We address here the problem of constructing the basic total angular momentum multiplets[1] $|jm\rangle$ from the basic angular momentum muliplets $|j_i m_i\rangle$ $(i=1,\ldots,n)$ associated with the n kinematically independent parts, and develop the relationships between different coupling schemes.

The problem of constructing the total angular momentum $\mathbf{J}$ of a composite system possessing constituent angular momenta $\mathbf{J}(\alpha)$ $(\alpha=1,2,\ldots,n)$ is symbolized by

$$\mathbf{J}=\mathbf{J}(1)+\mathbf{J}(2)+\cdots+\mathbf{J}(n). \tag{5.12.1}$$

Thus, we assume identical Lie algebras,

$$\left[J_i(\alpha), J_j(\alpha)\right]=ie_{ijk}J_k(\alpha), \qquad \alpha=1,2,\ldots,n, \tag{5.12.2}$$

where each component $J_i(\alpha)$ $(i=1,2,3)$ of $\mathbf{J}(\alpha)$ commutes with each component $J_i(\beta)$ $(i=1,2,3)$ of $\mathbf{J}(\beta)$ $(\alpha\neq\beta)$. [$\mathbf{J}(\alpha)$ is the generator of independent rotations of part α (see Chapter 3, Note 8, AMQP).] The total angular momentum components defined by $J_i=\sum_\alpha J_i(\alpha)$ $(i=1,2,3)$ then also satisfy the $SU(2)$ Lie algebra commutation relations

$$\left[J_i, J_j\right]=ie_{ijk}J_k. \tag{5.12.3}$$

($\mathbf{J}$ is the generator of overall rotations of the composite system.) The finite-dimensional Hilbert space on which the preceding operators act is the set of tensor product vectors

$$|j_1m_1;\, j_2m_2;\ldots;\, j_nm_n\rangle\equiv|j_1m_1\rangle\otimes|j_2m_2\rangle\otimes\cdots\otimes|j_nm_n\rangle. \tag{5.12.4}$$

We denote by $\mathcal{H}(j_1j_2\ldots j_n)$ the space of dimension $\prod_{\alpha=1}^{n}(2j_\alpha+1)$ spanned by the orthonormal vectors (5.12.4) corresponding to $m_\alpha=j_\alpha, j_\alpha-1,\ldots,-j_\alpha$ $(\alpha=1,2,\ldots,n)$.

The action on the states (5.12.4) of an arbitrary linear combination of angular momenta components

$$\mathcal{Q}=\sum_{\alpha i}a_i^\alpha J_i(\alpha)\equiv\sum_{\alpha=1}^{n}\mathbf{1}(1)\otimes\cdots\otimes\underbrace{\left(\sum_i a_i^\alpha J_i(\alpha)\right)}_{\text{position}\,\alpha}\otimes\cdots\otimes\mathbf{1}(n) \tag{5.12.5}$$

[1]We suppress all other physical labels (α) in the notation for state vectors.

is defined by

$$\mathscr{Q}|j_1m_1;\, j_2m_2;\ldots;\, j_nm_n\rangle=\Big(\sum_i a_i^1 J_i(1)|j_1m_1\rangle\Big)\otimes|j_2m_2\rangle\otimes\cdots\otimes|j_nm_n\rangle$$

$$+|j_1m_1\rangle\otimes\Big(\sum_i a_i^2 J_i(2)|j_2m_2\rangle\Big)\otimes\cdots\otimes|j_nm_n\rangle$$

$$+\cdots+|j_1m_1\rangle\otimes\cdots\otimes|j_{n-1}m_{n-1}\rangle\otimes\Big(\sum_i a_i^n J_i(n)|j_nm_n\rangle\Big). \tag{5.12.6}$$

We assume the standard action of each $J_i(\alpha)$ $(i=1,2,3)$ on the corresponding kets $|\, j_\alpha m_\alpha\rangle$ (see Chapter 3, AMQP, particularly Note 8).

The problem is to determine all linear combinations

$$|(j_1j_2\cdots j_n)jm\rangle=\sum_{\substack{\text{all } m_\alpha\\ \text{with } \sum_\alpha m_\alpha=m}} C^{j_1j_2\cdots j_nj}_{m_1m_2\cdots m_nm}|j_1m_1;\, j_2m_2;\ldots;\, j_nm_n\rangle \tag{5.12.7}$$

such that the components of $\mathbf{J}$ have the standard action on these new basis vectors.

All such linear combinations (5.12.7) are already simultaneous eigenvectors of the n commuting (Hermitian) operators (angular momentum of part α) given by

$$\mathbf{J}^2(\alpha)=J_1^2(\alpha)+J_2^2(\alpha)+J_3^2(\alpha),$$
$$\mathbf{J}^2(\alpha)|(j_1j_2\cdots j_n)jm\rangle=j_\alpha(j_\alpha+1)|(j_1j_2\cdots j_n)jm\rangle, \tag{5.12.8}$$

where $\alpha=1,2,\ldots,n$. The requirements that $\mathbf{J}^2$ and J_3 also be diagonal on the states (5.12.7) and, more specifically, that $\mathbf{J}$ have the standard action on these states lead to conditions on the coefficients in the linear combination. Since we have, however, a totality of $2n$ Hermitian operators

$$\{\mathbf{J}^2(\alpha),\, J_3(\alpha);\, \alpha=1,2,\ldots,n\} \tag{5.12.9}$$

that are diagonal on the states (5.12.4), and we have only $n+2$ operators in the set

$$\{\mathbf{J}^2,\, J_3;\, \mathbf{J}^2(\alpha):\alpha=1,2,\ldots,n\}, \tag{5.12.10}$$

it follows that there are many solutions, in general, to our problem $(n>2)$.

2. The Multiplicity Problem and Recoupling Theory

In considering the problem posed above for $n>2$, we immediately encounter a new feature of rotations, which is best illustrated by an example. The representation of $SU(2)$ given by the matrix direct product

$$D^2(U)\otimes\left[D^1(U)\otimes D^2(U)\right] \tag{5.12.11}$$

is reducible. One carries out the reduction in the following way, using the rules for addition of two angular momenta [Clebsch–Gordan series for $SU(2)$]:

$$\begin{aligned}
[2]\otimes([1]\otimes[2]) &= [2]\otimes([1]\oplus[2]\oplus[3])\\
&= [2]\otimes[1]\oplus[2]\otimes[2]\oplus[2]\otimes[3]\\
&= [0]\oplus 3[1]\oplus 3[2]\oplus 3[3]\oplus 2[4]\oplus[5].
\end{aligned}$$

Observe that there is a multiplicity of occurrence of certain irreps of $SU(2)$ contained in the reducible representation (5.12.11).

The multiple occurrence of $D^j(U)$ greatly complicates the problem of reducing the matrix direct product $\otimes^{\alpha} D^{j_\alpha}(U)$. From the point of view of basis vectors of the type (5.12.7), this means that, for each j that is repeated n_j times, there must exist n_j perpendicular spaces, each of dimension $2j+1$, whose basis vectors must be enumerable by supplying the vectors (5.12.7) with additional indices (k):

$$\{|(j_1 j_2 \cdots j_n)(k)jm\rangle : m=j,\, j-1,\ldots,-j\}. \tag{5.12.12}$$

The index or index set (k) may or may not be related to the eigenvalues of $n-2$ additional independent Hermitian operators $\{K_\lambda\colon \lambda=1,2,\ldots,n-2\}$, ~~constructed as polynomial forms[1] in the components $\{J_i(\alpha)\}$, which mutu-~~ ally commute among themselves as well as with the $n+2$ operators in the set (5.12.9), thus giving a totality of $2n$ commuting operators.

The property that any index set (k) must possess is that, for fixed $j_1 j_2 \cdots j_n$, and j, its range of values must enumerate precisely n_j perpendicular vectors of highest weight $(m=j)$:

$$\{|(j_1 j_2 \cdots j_n)(k)jj\rangle\}. \tag{5.12.13}$$

[We may then obtain the full indexing of an orthonormal basis of $\mathcal{H}(j_1 j_2 \cdots j_n)$ by lowering the state vector (5.12.13) to (5.12.12) by the

[1] We shall see, in fact, that this is possible.

standard application of J_{-}^{j-m}.] The indices n_j must satisfy

$$\sum_j (2j+1)n_j = \prod_\alpha (2j_\alpha+1). \tag{5.12.14}$$

Even if one is successful in introducing an orthonormal basis into the space $\mathcal{H}(j_1 j_2 \cdots j_n)$ by use of an index set (k), it will, in general, be possible to introduce still another index set $[l]$, which serves the same purpose as (k), but which has a different significance. For example, $[l]$ might be associated with the eigenvalues of a second set $\{L_\lambda: \lambda = 1,2,\ldots, n-2\}$ of independent commuting Hermitian operators which are not expressible in terms of the K_λ and the basic set (5.12.10). We would thus be led to a basis

$$\{|(j_1 j_2 \cdots j_n)[l]\, jm\rangle\}. \tag{5.12.15}$$

By construction, the basis vectors (5.12.12) and (5.12.15) are all of the form (5.12.7). Hence, they are simultaneous eigenvectors of $\mathbf{J}^2(\alpha)$ with eigenvalues $j_\alpha(j_\alpha+1)$ $(\alpha=1,2,\ldots,n)$; of $\mathbf{J}^2$ with eigenvalue $j(j+1)$; and of J_3 with eigenvalue m. This result implies that there exists a unitary transformation of dimension n_j that connects the bases (5.12.12) and (5.12.15):

$$|(j_1 j_2 \cdots j_n)[l]\, jm\rangle = \sum_{(k)} U_{[l](k)}(j_1 j_2 \cdots j_n; j)|(j_1 j_2 \cdots j_n)(k)\, jm\rangle, \tag{5.12.16}$$

where the transformation coefficients

$$U_{[l](k)}(j_1 j_2 \cdots j_n; j) \tag{5.12.17}$$

connecting the $[l]$-scheme to the (k)-scheme are independent of m.

Much of the theory of the coupling of n angular momenta $(n \geqslant 3)$ is addressed to the problem of introducing additional operator sets so as to obtain a unique determination of the basis vectors of $\mathcal{H}(j_1 j_2 \cdots j_n)$ and to the explicit calculation of the transformation coefficients between such schemes. The development of the properties and interrelations between transformation coefficients constitute the subject of *recoupling theory*. The most extensive treatment of what might be called conventional (binary) schemes is given by Jucys *et al.* [3].

3. The Enumeration of Binary Coupling Schemes[1]

Conventional schemes proceed by *pairwise coupling* of the angular momenta in the set

$$\{\mathbf{J}(1), \mathbf{J}(2), \ldots, \mathbf{J}(n)\},$$

using at each step the $SU(2)$ Wigner coefficients for the coupling of two angular momenta. The enumeration of all pairwise coupling schemes is the well-known *problem of parentheses* (Comtet [11]). The problem is to count all possible ways of introducing parentheses ( ), [ ], { },... into the sum

$$\mathbf{J}_1 + \mathbf{J}_2 + \cdots + \mathbf{J}_n \tag{5.12.18}$$

such that each subsum is binary. [We now write $\mathbf{J}_i = \mathbf{J}(i)$ to avoid ambiguity in the use of ( ).]

For $n = 2, 3$, and 4 the solutions are (the last pair of parentheses is usually omitted)

$$\begin{aligned}
n=2:&\quad \mathbf{J}_1 + \mathbf{J}_2;\\
n=3:&\quad (\mathbf{J}_1 + \mathbf{J}_2) + \mathbf{J}_3\\
&\quad \mathbf{J}_1 + (\mathbf{J}_2 + \mathbf{J}_3);\\
n=4:&\quad (\mathbf{J}_1 + \mathbf{J}_2) + (\mathbf{J}_3 + \mathbf{J}_4)\\
&\quad [(\mathbf{J}_1 + \mathbf{J}_2) + \mathbf{J}_3] + \mathbf{J}_4\\
&\quad [\mathbf{J}_1 + (\mathbf{J}_2 + \mathbf{J}_3)] + \mathbf{J}_4\\
&\quad \mathbf{J}_1 + [(\mathbf{J}_2 + \mathbf{J}_3) + \mathbf{J}_4]\\
&\quad \mathbf{J}_1 + [\mathbf{J}_2 + (\mathbf{J}_3 + \mathbf{J}_4)].
\end{aligned} \tag{5.12.19}$$

(Observe that the order $1, 2, \ldots, n$ has been preserved in these enumerations: We consider below the problem of permuting the $\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_n$.)

In place of the binary bracketing of the sum (5.12.18), it is convenient to consider the bracketing of the *ordered product* (hence, the j_i are considered as noncommuting)

$$j_1 j_2 \cdots j_n \tag{5.12.20}$$

[1]We wish to acknowledge the contributions to Sections 3, 4, 5, 8, and 9 made by Dr. Paul R. Stein of the Los Alamos National Laboratory in his informal lectures on *graph theory*. The information gained from these lectures and from discussions with Dr. Stein allowed us to relate the binary coupling theory of angular momenta to some classical counting problems arising in combinatorics.

of the quantum labels j_i corresponding to the (fixed) eigenvalues $j_i(j_i+1)$ of $\mathbf{J}_i^2$.[1] The two enumeration problems are in one-to-one correspondence as illustrated here for the scheme $n=4$ in Eqs. (5.12.19):

$$(j_1 j_2)(j_3 j_4), [(j_1 j_2) j_3] j_4, [j_1(j_2 j_3)] j_4,$$

$$j_1[(j_2 j_3) j_4], \; j_1[j_2(j_3 j_4)]. \tag{5.12.21}$$

We call an element in the set (5.12.21) a *binary bracketing* of $j_1 j_2 j_3 j_4$.

Let us denote a generic binary bracketing of the product (5.11.20) by

$$(j_1 j_2 \cdots j_n)^B. \tag{5.12.22}$$

An important property of any particular bracketing scheme is the *unique specification* by each symbol of the intermediate angular momenta—it is not necessary to introduce explicit symbols for the intermediate angular momenta. For example, the symbol $[(j_1 j_2) j_3] j_4$ already carries fully the information that the coupling scheme is

$$\mathbf{J}_1 + \mathbf{J}_2 = \mathbf{J}_{12}, \qquad \mathbf{J}_{12} + \mathbf{J}_3 = \mathbf{J}_{123}, \qquad \mathbf{J}_{123} + \mathbf{J}_4 = \mathbf{J}. \tag{5.12.23}$$

The number of ways, a_n, of introducing parentheses into the product (5.12.20) such that each subproduct is binary, that is, the number of elements in the set

$$\{(j_1 j_2 \cdots j_n)^B : B \text{ is a binary bracketing}\}, \tag{5.12.24}$$

is given by the *Catalan* (1838) *numbers*

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}, \qquad n = 2, 3, \ldots. \tag{5.12.25}$$

The Catalan numbers a_n thus give the number of distinct coupling schemes that preserve the order $1, 2, \ldots, n$ of the angular momenta in $\mathbf{J}_1 + \mathbf{J}_2 + \cdots + \mathbf{J}_n$.

[1] It is important to recognize the symbolic significance of these binary bracketings in enumerating coupling schemes for the angular momentum operators $\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_n$ themselves. Thus, the j_i should not be thought of here as assuming specific numerical values, since the one-to-one correspondence $j_i \leftrightarrow \mathbf{J}_i$ is essential. No particular problem arises as long as the order $12 \cdots n$ is maintained as in (5.12.21), but later, when we consider permutations of the j_i, in $j_1 j_2 \cdots j_n$ (see p. 443), the one-to-one correspondence $j_i \leftrightarrow \mathbf{J}_i$ can be lost if some of the j_i are taken to be numerically equal.

Let us next consider permutations of the angular momenta in the sum (5.12.1) or (5.12.18):

$$\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2+\cdots+\mathbf{J}_n=\mathbf{J}_{i_1}+\mathbf{J}_{i_2}+\cdots+\mathbf{J}_{i_n}, \tag{5.12.26}$$

where $(i_1 i_2 \cdots i_n)$ is a permutation of $(1,2,\ldots,n)$. For clarity it is useful to write Eq. (5.12.26) in the tensor product notation (5.12.5):

$$\begin{aligned}\mathbf{J}&=\mathbf{J}_1\otimes\mathbf{1}(2)\otimes\cdots\otimes\mathbf{1}(n)+\cdots+\mathbf{1}(1)\otimes\cdots\otimes\mathbf{1}(n-1)\otimes\mathbf{J}_n\\&=\mathbf{1}(1)\otimes\cdots\otimes\mathbf{J}_{i_1}\otimes\cdots\otimes\mathbf{1}(n)+\cdots+\mathbf{1}(1)\otimes\cdots\otimes\mathbf{J}_{i_n}\otimes\cdots\otimes\mathbf{1}(n).\end{aligned} \tag{5.12.27}$$

This more detailed way of writing Eq. (5.12.26) shows clearly that the reordering of terms does not alter the operator $\mathbf{J}$ and that its action still takes place on the vector space $\mathcal{H}(j_1 j_2\cdots j_n)$ with basis (5.12.4).

What has changed, however, is the *coupling instruction*, which is symbolized by writing, for example,

$$\mathbf{J}_1\otimes\mathbf{1}(2)+\mathbf{1}(1)\otimes\mathbf{J}_2 \text{ versus } \mathbf{1}(1)\otimes\mathbf{J}_2+\mathbf{J}_1\otimes\mathbf{1}(2).$$

The former signifies that one couple according to the rule

$$|(12)(j_1 j_2)jm\rangle\equiv\sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m}|j_1 m_1\rangle\otimes|j_2 m_2\rangle,$$

whereas the latter signifies that one couple according to the rule[1]

$$|(21)(j_2 j_1)jm\rangle\equiv\sum_{m_1 m_2} C^{j_2 j_1 j}_{m_2 m_1 m}|j_1 m_1\rangle\otimes|j_2 m_2\rangle.$$

Using the symmetry relation $C^{j_1 j_2 j}_{m_1 m_2 m}=(-1)^{j_1+j_2-j}C^{j_2 j_1 j}_{m_2 m_1 m}$ of the Wigner coefficient, one finds

$$|(21)(j_2 j_1)jm\rangle=(-1)^{j_1+j_2-j}|(12)(j_1 j_2)jm\rangle. \tag{5.12.28}$$

For two angular momenta, the effect of interchanging the order of the angular momenta is simply a phase change in the coupled state vectors.

[1]The "extra" labels (12) and (21) in these state vectors are introduced in order that the vectors corresponding to all allowed numerical assignments of the quantum numbers $j_1 j_2 jm$ be denoted by distinct symbols. These labels denote that the coupling instructions are $\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2$ and $\mathbf{J}=\mathbf{J}_2+\mathbf{J}_1$ (see footnote, p. 441).

It is convenient, initially, to count all $n!$ permutations of the $\mathbf{J}_i$ in $\mathbf{J}$ as leading to distinct coupled states, accounting later for the simplification implied by Eq. (5.12.28).

Counting in this manner, we find: The total number c_n of distinct coupling schemes of n angular momenta is enumerated by the symbols in the set

$$\left\{ (j_{i_1} j_{i_2} \cdots j_{i_n})^B : \begin{array}{l} B \text{ is a binary bracketing;} \\ i_1 i_2 \cdots i_n \text{ is a permutation of } 1,2,\ldots,n \end{array} \right\}. \tag{5.12.29}$$

This number is

$$c_n = n! a_n = (n)_{n-1}. \tag{5.12.30}$$

All $n!$ permutations are to be counted in enumerating the elements of the set (5.12.29), since it is the *placement* of the indices $(j_1 j_2 \cdots j_n)$ that matters, and not the values of the individual j_k (see footnote, p. 441). In other words, the j_k serve as generic symbols for an enumeration process, and their values are irrelevant.

4. Binary Couplings and Binary Trees

There is a one-to-one correspondence between the elements in the set $\{(j_1 j_2 \cdots j_n)^B$: B is a binary bracketing$\}$ and a certain class of "*binary trees*" (Comtet [11]), sometimes called *pure binary trees*. Since trees carry information about coupling schemes, it is useful to illustrate the concept with several examples. (We discuss in Note 1 the method used by Jucys *et al.* [3] in associating "diagrams" to sets of angular momentum labels.)

We start with a single *point* called the *root* of the tree. The root then bifurcates to two points (as illustrated in Fig. 5.24) at level 1, joined to the

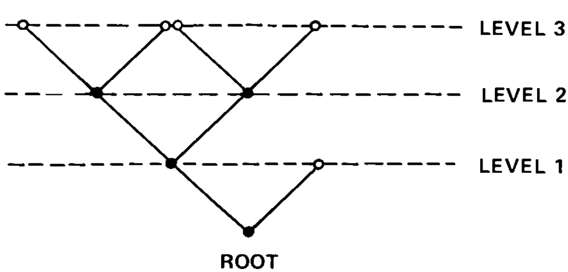


Figure 5.24.

root by lines. Each of the points at level 1 then either bifurcates to level 2 or is a *terminal point.* This process is continued at each level; that is, a point either bifurcates or terminates, until a total of $n-1$ bifurcations (counting the root) have occurred:

The number of *unlabeled* (points not labeled) *binary trees* for the cases $n=3$ and $n=4$ are shown in Figs. 5.25A and B. The points (nodes, vertices) in these binary trees are either *bifurcation points* (solid circles) or *terminal points* (open circles). Because the root is a distinguished point, the bifurcation points are also classified as the *root* and the *internal points* (all bifurcation points except the root). Each of these points has a direct significance for the coupling problem, as we now explain.

The relationship of binary trees to the problem of parentheses is obtained by first labeling the terminal points of a tree and then assigning to each bifurcation point the bracketed label of the points to which it is joined, maintaining the order of the labeling symbols. Thus, if we assign the labels j_1, j_2, j_3, and j_4 to the terminal points of the trees in Fig. 5.25B in a clockwise sense, we obtain all the bracketed expressions (5.12.21). We illustrate the method in detail in Fig. 5.26 and display below each diagram in Fig. 5.25B the bracketing that is obtained by applying the method to the same clockwise labeling of terminal points.

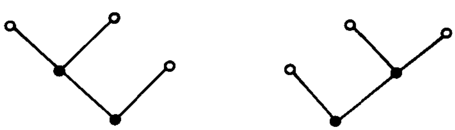


Figure 5.25A. Binary trees with three terminal points ($n=3$).

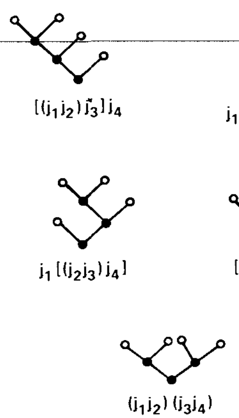


Figure 5.25B. Binary trees with four terminal points ($n=4$).

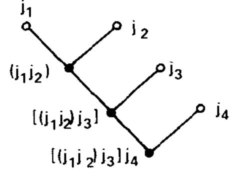


Figure 5.26.

One should observe that the formation of bracketed products is both *noncommutative and nonassociative*—for example, $(j_1j_2)j_3 \neq j_3(j_1j_2)$ and $(j_1j_2)j_3 \neq j_1(j_2j_3)$—reflecting the fact that the state vectors corresponding to the addition schemes $(\mathbf{J}_1+\mathbf{J}_2)+\mathbf{J}_3$ and $\mathbf{J}_3+(\mathbf{J}_1+\mathbf{J}_2)$ need not be equal (noncommutivity), nor need the state vectors corresponding to the addition schemes $(\mathbf{J}_1+\mathbf{J}_2)+\mathbf{J}_3$ and $\mathbf{J}_1+(\mathbf{J}_2+\mathbf{J}_3)$ be equal (nonassociativity).

By making all $n!$ possible assignments of the n symbols $j_1j_2\cdots j_n$ to the terminal points of the set of unlabeled binary trees containing $n-1$ bifurcation points, we obtain a set of labeled trees that is in one-to-one correspondence with the set of all coupling schemes of n angular momenta.

5. Modification of the Enumeration Problem

In quantum physics one cannot distinguish between state vectors that differ only by phase factors. Accordingly, we do not wish to distinguish between coupled bases of the space $\mathcal{H}(j_1j_2\cdots j_n)$ that differ only by phase factors. We now address this problem.

The notation $(j_{i_1}j_{i_2}\cdots j_{i_n})^B$, where $i_1i_2\cdots i_n$ is a permutation of $1,2,\ldots,n$ and B is a binary bracketing of n symbols, fully specifies a coupling scheme for n angular momenta. However, it is useful in the notation for *state vectors* to introduce explicitly the $n-2$ intermediate angular momenta $(k)=(k_1k_2\cdots k_{n-2})$ corresponding to the bracketing B, the final angular momentum j, and its projection m. For the purpose of this discussion, we introduce the following notation for a state vector:[1]

$$|(i_1i_2\cdots i_n)(j_{i_1}j_{i_2}\cdots j_{i_n})^B(k_1k_2\cdots k_{n-2})jm\rangle. \qquad (5.12.31)$$

The connection with binary trees is obvious: The angular momenta $j_{i_1}j_{i_2}\cdots j_{i_n}$ are the labels of the *terminal points*, the labels $k_1k_2\cdots k_{n-2}$ are the labels of

[1]It may appear redundant to include $(i_1i_2\cdots i_n)$ as well as $(j_{i_1}j_{i_2}\cdots j_{i_n})^B$ in the notation for a state vector. To avoid confusion, it is desirable, however, in the notation for a state vector, to label distinct vectors by distinct symbols for all possible assignments of the quantum numbers. It then becomes necessary to specify separately the coupling instruction symbolized by writing $\mathbf{J}=\mathbf{J}_{i_1}+\mathbf{J}_{i_2}+\cdots+\mathbf{J}_{i_n}$ (see footnotes, pp. 441, 442).

the *internal points*, and j is the label of the *root*. For example, in Fig. 5.26 we are now writing $k_1=(j_1 j_2)=j_{12}$, $k_2=[(j_1 j_2)j_3]=j_{123}$, and $j=[(j_1 j_2)j_3]j_4=j_{1234}$, where $k_1(k_1+1)$, $k_2(k_2+1)$, and $j(j+1)$ are, respectively, the eigenvalues of the intermediate and final angular momenta:

$$\mathbf{J}_{12}^2=\left(\sum_{i=1}^{2}\mathbf{J}_i\right)\cdot\left(\sum_{i=1}^{2}\mathbf{J}_i\right),$$
$$\mathbf{J}_{123}^2=\left(\sum_{i=1}^{3}\mathbf{J}_i\right)\cdot\left(\sum_{i=1}^{3}\mathbf{J}_i\right), \tag{5.12.32}$$
$$\mathbf{J}^2=\left(\sum_{i=1}^{4}\mathbf{J}_i\right)\cdot\left(\sum_{i=1}^{4}\mathbf{J}_i\right).$$

Our problem now is to determine which of the state vectors, c_n in number, enumerated by the notation (5.12.31) are trivially related by a phase, or rather to determine those that are not necessarily so related.

The problem posed above may be solved by the following procedure. Consider the set of labeled trees corresponding to the elements in the set (5.12.29). Each tree in the set may be characterized in the following manner: It has terminal points labeled by a permutation of $(j_1 j_2 \cdots j_n)$; internal points labeled by a set of coupled angular momentum quantum numbers $k_1 k_2 \cdots k_{n-2}$; and a root labeled by j. Corresponding to this labeled tree, we have a set of mutually commuting diagonal observables:

$$\{\mathbf{J}_\alpha^2(\alpha=1,2,\ldots,n);\mathbf{K}_\lambda^2(\lambda=1,2,\ldots,n-2),\,\mathbf{J}^2\}, \tag{5.12.33}$$

where $\mathbf{K}_\lambda^2$ denotes the square [see Eq. (5.12.32)] of the coupled angular momentum operator having eigenvalue $k_\lambda(k_\lambda+1)$. When J_3 is adjoined to the set (5.12.33), we have a complete set of commuting Hermitian observables, and, up to phases dependent only on $j_1 j_2 \cdots j_n$, each set of simultaneous eigenvalues of these operators defines a *unique*, normalized, simultaneous eigenvector. We conclude that the labeled binary trees that correspond to the same set of operators (5.12.33) also correspond to state vectors that differ at most by phases.

The essential property needed for identifying the labeled binary trees that correspond to the same set of operators (5.12.33) is the *invariance of the eigenvalue* of $\mathbf{K}_\lambda^2$ under all permutations of the angular momentum labels assigned to the terminal points of the *binary subtree* for which the bifurcation point labeled by k_λ serves as root, this result applying to each $\lambda=1,2,\ldots,n-1$ ($\mathbf{K}_{n-1}^2\equiv\mathbf{J}^2$). Notice now that each bifurcation point corresponds to the formation of a single new *binary subproduct* (for example, see Fig. 5.26), and from the above invariance property, it follows that the eigenvalues of the operator set (5.12.33) are invariant under the exchange of the order of elements in this binary subproduct, this result being true at each of the $n-1$ bifurcation points. Thus, there are 2^{n-1} operations (exchange or do not exchange the two elements in each of the $n-1$ binary subproducts at the $n-1$ bifurcation points) that leave the eigenvalues of the operator set (5.12,33) invariant. But the 2^{n-1} labels thus assigned to the $n-1$ bifurcation points must, in fact, arise exactly once from some permuted assignment of $j_1 j_2 \cdots j_n$ to the terminal points of some binary tree. Thus, we may partition the set (5.12.29) into a number of subsets equal to

$$d_n = c_n / 2^{n-1} = (2n-3)!! \tag{5.12.34}$$

such that each subset contains 2^{n-1} elements yielding state vectors (5.12.31) having the same intermediate angular momenta $k_1 k_2 \cdots k_{n-2}$ (hence, the state vectors are related by phase factors), and such that the different subsets correspond to different intermediate angular momenta $k_1 k_2 \cdots k_{n-2}$.

The counting problem we have solved above may be viewed in still another useful way. First, let us restate our result. We have solved the problem of partitioning the set

$$\left\{ \left(j_{i_1} j_{i_2} \cdots j_{i_n} \right)^B : \begin{array}{l} B \text{ is a binary bracketing;} \\ i_1 i_2 \cdots i_n \text{ is a permutation of } 1,2,\ldots,n \end{array} \right\}$$

into equivalence classes under the equivalence relation

$$\left(j_{i_1} j_{i_2} \cdots j_{i_n} \right)^B \sim \left(j_{l_1} j_{l_2} \cdots j_{l_n} \right)^{B'} \tag{5.12.35}$$

if $(j_{l_1} j_{l_2} \cdots j_{l_n})^{B'}$ can be obtained from $(j_{i_1} j_{i_2} \cdots j_{i_n})^B$ by commuting binary subproducts—that is, by using $(ab)=(ba)$, $c(ab)=(ab)c$, etc. The solution above shows there are $(2n-3)!!$ equivalence classes, each containing 2^{n-1} elements.

A second method of counting leads us to consider first the Wedderburn–Etherington problem of commuting parentheses (Comtet [11], p. 67). This view is useful because it counts the number of different "types" of bracketed symbols. This number, in turn, tells us the number of "types" of coupling schemes.

The *type of a bracketed symbol* $(j_{i_1} j_{i_2} \cdots j_{i_n})^B$ is obtained by setting each symbol j_i equal to a common symbol—say, x. Thus, the types of the symbols in Fig. 5.25B are $[(x^2)x]x$, $x[x(x^2)]$, $x[(x^2)x]$, $[x(x^2)]x$, $(x^2)(x^2)$.

Allowing commutation of binary subproducts, there are only two types, $[(x^2)x]x$ and $(x^2)(x^2)$, of bracketed symbols corresponding to the trees shown in Fig. 5.27.

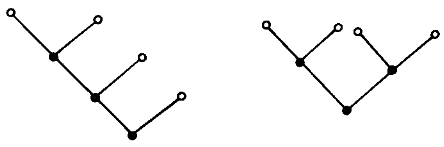


Figure 5.27.

The Wedderburn–Etherington number b_n is the number of binary trees having n terminal points and corresponding to bracketed symbols of distinct types, *where commutation of binary products is allowed.* The number b_n is also the number of unlabeled trees obtained from the set of unlabeled binary trees by defining two trees T' and T to be equivalent whenever T' can be brought into the same "shape" as T by any sequence of operations constructed from the following fundamental operation: Take a subtree having any bifurcation point as its root and reflect the subtree through the vertical line passing through its root, leaving the remaining part of the tree unaltered.

One should check that the three trees of Fig. 5.25B not included in Fig. 5.27 are equivalent to the left-hand figure of Fig. 5.27 under the geometric equivalence relation described above. For $n=5$, we have $b_5=3$, and the three inequivalent trees are as shown in Fig. 5.28. The three types of bracketed expressions are indicated below each diagram.

Binary trees obtained by admitting the commutation operation of binary subproducts (or by admitting the geometric operation described above in which we do not distinguish left from right around each bifurcation point) are called *nonoriented binary trees.*[1] The number of such unlabeled trees with n terminal points is the Wedderburn–Etherington number b_n.

A closed form for the numbers $b_n (n=1,2,\dots)$ is not known, although recursion relations may be given for generating them (Comtet [11]). We list the first few:

n	1	2	3	4	5	6	7	8	9	10
b_n	1	1	1	2	3	6	11	23	46	98

The number d_n [see Eq. (5.12.34)] equals the number of elements in the set

$$\left\{ \left(j_{i_1} j_{i_2} \cdots j_{i_n}\right)^B : \begin{array}{l} B \text{ is a bracketing corresponding to} \\ \text{a nonoriented tree; } i_1 i_2 \cdots i_n \text{ is} \\ \text{a permutation of } 1,2,\dots,n \end{array} \right\}.$$

[1]This terminology is not fully standardized. The term "oriented" trees sometimes refers to trees having arrows going between the points; in this usage a nonoriented tree is a tree without arrows.

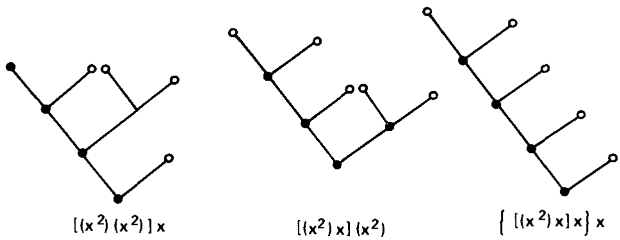


Figure 5.28.

Equivalently, we may obtain d_n by making all $n!$ assignments of $j_1 j_2 \cdots j_n$ to the terminal points of the set of nonoriented trees and counting directly the distinct symbols $(j_{i_1} j_{i_2} \cdots j_{i_n})^B$.

For example, associated with the two trees in Fig. 5.27, we have the two bracketings $[(ab)c]d$ and $(ab)(cd)$, which can be filled in with j_1, j_2, j_3, and j_4 in the following distinct ways (commutation allowed):

Type $[(x^2)x]x$

$$[(j_1 j_2)j_3]j_4,\ [(j_2 j_3)j_1]j_4,\ [(j_3 j_1)j_2]j_4$$

$$[(j_1 j_2)j_4]j_3,\ [(j_2 j_4)j_1]j_3,\ [(j_4 j_1)j_2]j_3$$

$$[(j_1 j_3)j_4]j_2,\ [(j_3 j_4)j_1]j_2,\ [(j_4 j_1)j_3]j_2$$

$$[(j_2 j_3)j_4]j_1,\ [(j_3 j_4)j_2]j_1,\ [(j_4 j_2)j_3]j_1$$

Type $(x^2)(x^2)$

$$(j_1 j_2)(j_3 j_4),\ (j_1 j_3)(j_2 j_4),\ (j_2 j_3)(j_1 j_4)$$

Thus, there are fifteen nontrivial coupling schemes for four angular momenta, and they are classified into two types.

We list also the bracket types for $n=5$ and $n=6$, noting the number of ways of "filling in" each type:

$$n=5:\ \underset{15}{\big[(x^2)(x^2)\big]x},\ \underset{30}{\big[(x^2)x\big](x^2)},\ \underset{60}{\big\{\big[(x^2)x\big]x\big\}x};$$

$$n=6:\ \underset{45}{\big[(x^2)(x^2)\big](x^2)},\ \underset{90}{\big[(x^2)x\big]\big[(x^2)x\big]},\ \underset{90}{\big\{\big[(x^2)(x^2)\big]x\big\}x},$$

$$\underset{180}{\big\{\big[(x^2)x\big](x^2)\big\}x},\ \underset{180}{\big\{\big[(x^2)x\big]x\big\}(x^2)},\ \underset{360}{\big\langle\big\{\big[(x^2)x\big]x\big\}x\big\rangle x}.$$

We have, incidently, proved an interesting relation by viewing the modified counting problem in two different ways: Let $p_n(b)$ denote the number of distinct ways of filling in one of the bracket types B corresponding to an unoriented binary tree having n terminal points (the numbers appearing below the bracketings above). The general formula we have proved is

$$\sum_{b=1}^{b_n} p_n(b)=(2n-3)!!. \tag{5.12.36}$$

6. The Basic Structures Underlying Transformation Coefficients

The results obtained above reveal interesting combinatoric aspects of coupling theory, but the main problem in the coupling of n angular momenta is the classification of all transformation coefficients connecting the $(2n-3)!!$ different binary coupling schemes.

The state vector (5.12.31) can be expressed in the form given by

$$\begin{aligned} |(i)(j_{(i)})^B(k)jm\rangle &= |(i_1i_2\cdots i_n)(j_{i_1}j_{i_2}\cdots j_{i_n})^B(k_1k_2\cdots k_{n-2})jm\rangle \\ &\equiv \sum_{\substack{m_i \\ \text{with } \sum_{i=1}^{n} m_i=m}} C^{(i)}_{(k)}\begin{pmatrix} j_{i_1}j_{i_2}\cdots j_{i_n} \\ m_{i_1}m_{i_2}\cdots m_{i_n}\end{pmatrix}^B |j_1m_1;j_2m_2;\ldots;j_nm_n\rangle, \end{aligned} \tag{5.12.37}$$

where the C-coefficient is a well-defined product of $n-1$ $SU(2)$ Wigner coefficients in which the detailed placement of the quantum numbers depends on the bracketing B and the ordering $(i)=(i_1i_2\cdots i_n)$ of the angular momentum operators in $\mathbf{J}=\mathbf{J}_{i_1}+\mathbf{J}_{i_2}+\cdots+\mathbf{J}_{i_n}$.

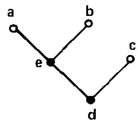
Using the abbreviated notation of Eq. (5.12.37), we find that the transformation coefficient between two such schemes is given by[1]

$$\begin{aligned} &\langle (i')(j_{(i')})^{B'}(k')j|(i)(j_{(i)})^B(k)j\rangle \\ &= \sum_{\substack{m_i \\ \text{with } \sum_i m_i=m}} C^{(i')}_{(k')}\begin{pmatrix} j_{i'_1}\ j_{i'_2}\cdots\ j_{i'_n} \\ m_{i'_1}m_{i'_2}\cdots m_{i'_n}\end{pmatrix}^{B'} C^{(i)}_{(k)}\begin{pmatrix} j_{i_1}\ j_{i_2}\cdots\ j_{i_n} \\ m_{i_1}m_{i_2}\cdots m_{i_n}\end{pmatrix}^{B}. \end{aligned} \tag{5.12.38}$$

Let us digress a moment on notations. Devising a suitable notation for coupled angular momentum states is a vexing problem. One requires a

[1] The book by Jucys *et al.* [3] is the pioneering work initiating graphic methods for reducing general sums over products of Wigner coefficients such as in Eq. (5.12.38). The C-coefficients in this result are called *generalized Clebsch–Gordan coefficients* by these authors.

notation such as given in Eq. (5.12.37) for general discussions, yet a more informative notation is required when specific bracketings (specific coupling schemes) are discussed. *It is the labeled tree itself, however, that provides the most accessible and detailed information.* We have, therefore, devised a notation that reflects most directly the information contained in a labeled tree. An example will make the general case transparent:



$$\leftrightarrow \left[(ab)_e c\right]_d .$$

This notation has the important property of keeping the basic bracketing $[(ab)c]$ prominent by lowering the intermediate coupling label e and the final coupling label d to inferior subscript roles. The symbol $[(ab)_e c]_d$ thus replaces $(j_1 j_2 j_3)^B(k_1)j=(abc)^B(e)d$ in the general notation (5.12.37) for a state vector. The coupling scheme is clearly $\mathbf{A}+\mathbf{B}=\mathbf{E}$, $\mathbf{E}+\mathbf{C}=\mathbf{D}$.

[One should note the formal structural analogy between Rota's double tableau forms (see Chapter 5, Appendix A, AMQP) and the transformation coefficients (5.12.38). *The former are polynomial forms defined on double standard Weyl tableaux; the latter are functions defined on pairs of labeled trees.*]

We shall consider carefully all coupling schemes for $n=3$, for, as we shall see, in a certain sense (to be explained below) the results are *definitive* for all binary couplings. We have already seen that commutation of elements in a binary product induces a phase transformation on a state vector. Hence, it is sufficient to consider transformations between the three coupling schemes $(ab)c$, $(bc)a$, and $(ca)b$ corresponding to the assignment of the even permutations of a, b, and c to the single binary tree of type $(x^2)x$.

Supplying intermediate coupling symbols as described above, we obtain the following state vectors:[1]

$$\begin{aligned}
|(\mathbf{ABC})\left[(ab)_d c\right]_{jm}\rangle &= \sum_{\alpha\beta} C^{a\,b\,d}_{\alpha,\beta,\alpha+\beta} C^{d\,c\,j}_{\alpha+\beta,m-\alpha-\beta,m} \\
&\quad \times |a\alpha;\, b\beta;\, c, m-\alpha-\beta\rangle, \\
|(\mathbf{BCA})\left[(bc)_e a\right]_{jm}\rangle &= \sum_{\beta\gamma} C^{b\,c\,e}_{\beta,\gamma,\beta+\gamma} C^{e\,a\,j}_{\beta+\gamma,m-\beta-\gamma,m} \\
&\quad \times |a, m-\beta-\gamma;\, b\beta;\, c\gamma\rangle, \qquad (5.12.39) \\
|(\mathbf{CAB})\left[(ca)_f b\right]_{jm}\rangle &= \sum_{\gamma\alpha} C^{c\,a\,f}_{\gamma,\alpha,\gamma+\alpha} C^{f\,b\,j}_{\gamma+\alpha,m-\gamma-\alpha,m} \\
&\quad \times |a\alpha;\, b, m-\gamma-\alpha;\, c\gamma\rangle.
\end{aligned}$$

[1]We have used a slight variation of the general notation (5.12.37) in replacing $(i_1 i_2 i_3)=(123)$, (231), (312) ($n=3$ in the present case) by the angular momentum operator symbols themselves—that is, by (**ABC**), (**BCA**), (**CAB**), where $\mathbf{A}\to a$, $\mathbf{B}\to b$, $\mathbf{C}\to c$. Again we remark that extra symbols such as these are required in order to assign distinct symbols to distinct vectors as we let a, b, c run over all possible values $0, \frac{1}{2}, 1, \ldots$.

Using next the relationship between Racah coefficients and Wigner coefficients [see Eq. (3.266), Chapter 3, AMQP], we calculate the following transformation coefficients:

$$\begin{aligned}
&\langle(\mathbf{ABC})[(ab)_d c]_j|(\mathbf{BCA})[(bc)_e a]_j\rangle \\
&\quad=(-1)^{a+e-j}[(2d+1)(2e+1)]^{\frac{1}{2}}W(abjc;de), \\
&\langle(\mathbf{BCA})[(bc)_e a]_j|(\mathbf{CAB})[(ca)_f b]_j\rangle \\
&\quad=(-1)^{b+f-j}[(2e+1)(2f+1)]^{\frac{1}{2}}W(bcja;ef), \qquad (5.12.40)\\
&\langle(\mathbf{CAB})[(ca)_f b]_j|(\mathbf{ABC})[(ab)_d c]_j\rangle \\
&\quad=(-1)^{c+d-j}[(2f+1)(2d+1)]^{\frac{1}{2}}W(cajb;fd),
\end{aligned}$$

where we follow the practice of dropping the projection quantum number m in a transformation coefficient, since the coefficient is independent of this label. Note that the second two results above may be obtained from the first one by making the appropriate permutations of labels.

Let us now interpret the results expressed by Eqs. (5.12.40). To do this we introduce the sets of basis vectors defined in the following manner:

$$\begin{aligned}
(\mathbf{ABC})_{(ab)c}&=\{|(\mathbf{ABC})[(ab)_d c]_{jm}\rangle:[d]\in[a]\otimes[b] \text{ and } [c]\otimes[j]\}, \\
(\mathbf{BCA})_{(bc)a}&=\{|(\mathbf{BCA})[(bc)_e a]_{jm}\rangle;[e]\in[b]\otimes[c] \text{ and } [a]\otimes[j]\}, \\
(\mathbf{CAB})_{(ca)b}&=\{|(\mathbf{CAB})[(ca)_f b]_{jm}\rangle:[f]\in[c]\otimes[a] \text{ and } [b]\otimes[j]\}.
\end{aligned} \qquad (5.12.41)$$

In these sets the quantum numbers (jm) are specified with $[j]\in[a]\otimes[b]\otimes[c]$, and are suppressed in the notation for the sets themselves. Observe that there are the same number of elements in each of these three sets so that the basis vectors may be enumerated by sets of indices

$$d_1, d_2, \ldots, d_q;\ e_1, e_2, \ldots, e_q;\ f_1, f_2, \ldots, f_q; \qquad (5.12.42)$$

respectively.

Using relations (5.12.40), we may now define the map $R\colon(\mathbf{ABC})_{(ab)c}\to(\mathbf{BCA})_{(bc)a}$ by the following rule:

$$|(\mathbf{ABC})\big[(ab)_{d_\lambda}c\big]_{jm}\rangle \xrightarrow{R} |(\mathbf{BCA})\big[(bc)_{e_\lambda}a\big]_{jm}\rangle = R|(\mathbf{ABC})\big[(ab)_{d_\lambda}c\big]_{jm}\rangle$$
$$=\sum_{\mu}(-1)^{a+e_\lambda-j}\big[(2d_\mu+1)(2e_\lambda+1)\big]^{\frac{1}{2}}W(abjc;\,d_\mu e_\lambda)|(\mathbf{ABC})\big[(ab)_{d_\mu}c\big]_{jm}\rangle, \tag{5.12.43}$$

where $\lambda,\mu=1,2,\ldots,q$.

The two additional maps S: $(\mathbf{BCA})_{(bc)a}\to(\mathbf{CAB})_{(ca)b}$ and $T\colon(\mathbf{CAB})_{(ca)b}\to(\mathbf{ABC})_{(ab)c}$ are similarly defined by making the appropriate cyclic permutations of labels in Eq. (5.12.43). The maps R, S, and T are, of course, just alternative ways of expressing the relations (5.12.40). Our purpose in making this reinterpretation is so that we can relate our results to a standard concept in mapping diagrams.

Consider the diagram of maps between sets given by

$$\begin{tikzcd}
& (\mathbf{CAB})_{(ca)b} & \\
(\mathbf{ABC})_{(ab)c} \arrow[ur, "T^{-1}"] \arrow[rr, "R"'] & & (\mathbf{BAC})_{(ba)c} \arrow[ul, "S"']
\end{tikzcd} \tag{5.12.44}$$

Then *the composition map* $S\circ R\colon(\mathbf{ABC})_{(ab)c}\to(\mathbf{CAB})_{(ca)b}$ *is equal to the map* $T^{-1}\colon(\mathbf{ABC})_{(ab)c}\to(\mathbf{CAB})_{(ca)b}$; *that is,*

$$S\circ R=T^{-1}. \tag{5.12.45}$$

(Mapping diagrams that satisfy this condition are called *commutative*.)

Proof. The proof is a straightforward application of the definition of the maps R, S, and T, and the Racah sum rule satisfied by the Racah coefficients [see Eq. (3.274), Chapter 3, AMQP]:

$$\sum_{e}(-1)^{b+c-e}(2e+1)W(acjb;\,fe)W(abjc;\,de)=(-1)^{d+f-a-j}W(cajb;\,fd). \tag{5.12.46}$$

We have also used the symmetry and orthogonality of the Racah coefficients in establishing the desired result. Conversely, *the Racah sum rule is* *implied by the symmetry and orthogonality of the Racah coefficients and the requirement that the mapping diagram* (5.12.44) *be commutative.* ■

We may give a significant generalization of transformations of the type (5.12.43) by recognizing that each of the angular momentum operators **A**, **B**, and **C**, may, in fact, be sums of other angular momentum operators. We next carry out this generalization. Let (x), (y), and (z) denote given binary bracketings of one or more angular momenta [for example, $(x)=(j_1 j_2)$, $(y)=[(j_3 j_4)j_5]$, etc.]. Let a, b, and c denote the respective "total" or "resultant" angular momenta associated with (x), (y), and (z) so that $(x)_a$, $(y)_b$, and $(z)_c$ denote specific coupling schemes.[1] [We define $(x)_a=a$ for $x=a$ so that the special cases represented by Eqs. (5.12.40) are included in the present generalization.]

Consider then a basis vector of the type (5.12.37) that contains in its symbol the subcoupling scheme $\{[(x)_a(y)_b]_d(z)_c\}_k$ corresponding to the coupling of irreps $[a]$ and $[b]$ to irrep $[d]$ followed by that of $[d]$ and $[c]$ to $[k]$. We denote this basis vector by

$$|\cdots\{[(x)_a(y)_b]_d(z)_c\}_k\cdots\rangle. \tag{5.12.47}$$

There are only two basic operations in binary coupling theory, as we shall now describe:

Commutation (transposition) of symbols:

$$(x)_a(y)_b \to (y)_b(x)_a. \tag{5.12.48}$$

Association of symbols:

$$[(x)_a(y)_b](z)_c \to (x)_a[(y)_b(z)_c]. \tag{5.12.49}$$

In one-to-one correspondence with each of these transformations of symbols, we associate the following substitutions of basis vectors in Hilbert space:[2]

$$\begin{aligned}|\cdots[(x)_a(y)_b]_d\cdots\rangle &\to (-1)^{a+b-d}|\cdots[(y)_b(x)_a]_d\cdots\rangle\\ &=|\cdots[(x)_a(y)_b]_d\cdots\rangle.\end{aligned} \tag{5.12.48$'$}$$

[1] This notation for a generic coupling scheme conflicts with that used earlier for a falling factorial; this latter numerical quantity is not used in this Topic, and we hope that no confusion results.

[2] One may also formulate these transformations as set transformations on basis vectors [as in Eq. (5.12.43)], but the notation becomes rather cumbersome. For the actual calculation of transformation coefficients, it is easier to use Eqs. (5.12.48′) and (5.12.49′) directly.

$$
\begin{aligned}
|\cdots\{[(x)_a(y)_b]_d(z)_c\}_k\cdots\rangle \to \sum_e [(2d+1)(2e+1)]^{\frac{1}{2}} W(abkc;de) \\
\times|\cdots\{(x)_a[(y)_b(z)_c]_e\}_k\cdots\rangle \\
=|\cdots\{[(x)_a(y)_b]_d(z)_c\}_k\cdots\rangle.
\end{aligned}
\tag{5.12.49'}
$$

Using the two operations of types (5.12.48) and (5.12.49) (and noting that the symbols in these two transformations are generic—that is, the $(x)_a$ and $(y)_a$ need not coincide), we can generate all twelve ways of permuting and bracketing $(x)_a(y)_b(z)_c$. Correspondingly, using Eqs. (5.12.48′) and (5.12.49′), we generate all possible transformations between the twelve sets of basis functions.

This fact leads us to the *fundamental theorem* in binary coupling of angular momenta: *The transformation coefficient between each pair of binary coupling schemes is expressible in terms of sums over products of Racah coefficients* (6-j coefficients). (In making this statement, we have ignored the presence of dimension and phase factors.)

Proof. All bracketings of n symbols are identical when commutation and association are admitted as equivalence relations into bracketings. This result implies that there always exists a sequence of commutations and associations that carries any bracketing into any other bracketing; hence, there always exists a corresponding sequence of phase and Racah coefficient formations that carries each state vector in the coupled basis corresponding to the first bracketing into a state vector in the coupled basis corresponding to the second bracketing. ■

Let us observe that the rules given above are *constructive* and that one ~~may write out any transformation coefficient directly in terms of Racah~~ coefficients *without any reference to the underlying Wigner coefficients* simply by identifying a sequence of commutations and associations that carries the bracketing of angular momenta corresponding to the initial coupling scheme to the bracketing of angular momenta corresponding to the final coupling scheme.

To assist in carrying out such calculations, we have given a more pedestrian interpretation of a diagram of commutative maps in Fig. 5.29. In this diagram a typical vector in a coupled basis is symbolized by the vertex of the triangle, and the line joining a pair of vertices is associated with the transformation coefficient defined by the inner product of the two vectors. [This diagram is then applicable also, with the same joining coefficient, to the more general case obtained by replacing a, b, c, respectively by $(x)_a$, $(y)_b$, $(z)_c$.]

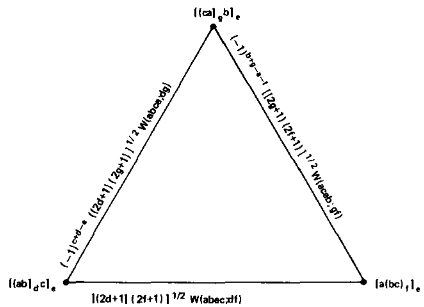


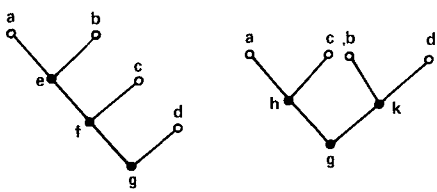
Figure 5.29.

Let us now consider several illustrative examples of using these rules in the evaluation of transformation coefficients [since we shall always consider generic basis vectors, we ignore the symbol $(i_1 i_2 \cdots i_n)$ in the general notation (5.12.37)].

Example 1. Consider the evaluation of the transformation coefficient

$$\langle \{[(ab)_e c]_f d\}_g | [(ac)_h (bd)_k]_g \rangle. \tag{5.12.50}$$

This coefficient corresponds to the pair of labeled trees



where we have labeled explicitly the internal points (coupled angular momenta). One may write out explicitly the coefficient in terms of Wigner coefficients directly from the corresponding pair of trees:

$$\begin{aligned} &\langle \{[(ab)_e c]_f d\}_g | [(ac)_h (bd)_k]_g \rangle \\ &= \sum_{\alpha+\beta+\gamma+\delta=m} C^{a\,b\,e}_{\alpha,\beta,\alpha+\beta} C^{e\,c\,f}_{\alpha+\beta,\gamma,\alpha+\beta+\gamma} C^{f\,d\,g}_{\alpha+\beta+\gamma,\delta,m} \\ &\quad \times C^{a\,c\,h}_{\alpha,\gamma,\alpha+\gamma} C^{b\,d\,k}_{\beta,\delta,\beta+\delta} C^{h\,k\,g}_{\alpha+\gamma,\beta+\delta,m}. \end{aligned} \tag{5.12.51}$$

On the other hand, the following sequence of commutations and associations carries the bracketing $(ac)(bd)$ to $[(ab)c]d$:

$$(ac)(bd) \xrightarrow{R} [(ac)b]d \xrightarrow{\phi} [b(ac)]d \xrightarrow{R} [(ba)c]d \xrightarrow{\phi} [(ab)c]d, \tag{5.12.52}$$

where ϕ and R serve to remind us that a phase or Racah coefficient (with, perhaps, a phase) enters into the corresponding state vector transformation. In detail, the following factors are supplied by the four steps in the sequence (5.12.52) (see Fig. 5.29):
Steps 1 and 2, corresponding to

$$[(ac)_h(bd)_k]_g \to \{[(ac)_h b]_f d\}_g \to \{[b(ac)_h]_f d\}_g,$$

introduce the factors

$$[(2f+1)(2k+1)]^{\frac{1}{2}} W(hbgd; fk) \qquad \text{and} \qquad (-1)^{b+h-f}.$$

Steps 3 and 4, corresponding to

$$\{[b(ac)_h]_f d\}_g \to \{[(ba)_e c]_f d\}_g \to \{[(ab)_e c]_f d\}_g,$$

introduce the factors

$$[(2e+1)(2h+1)]^{\frac{1}{2}} W(bafc; eh) \qquad \text{and} \qquad (-1)^{a+b-e}.$$

Thus, by manipulating directly the symbols in the left-hand side of Eq. (5.12.51) according to the rules given by Eqs. (5.12.48′) and (5.12.49′), we obtain

$$\begin{aligned} &\langle \{[(ab)_e c]_f d\}_g | [(ac)_h (bd)_k]_g \rangle \\ &\quad = (-1)^{e+h-a-f} [(2f+1)(2k+1)]^{\frac{1}{2}} W(hbgd; fk) \\ &\qquad \times [(2e+1)(2h+1)]^{\frac{1}{2}} W(bafc; eh). \end{aligned} \tag{5.12.53}$$

We emphasize again that it is quite remarkable that the elementary Hilbert space transformation rules associated with commutation and association allow one to obtain all transformation coefficients directly in terms of Racah coefficients without reference to the underlying Wigner coefficient structure.

Example 2. We shall now prove directly that the 9-j symbols, which are often defined in terms of Wigner coefficients [see Eqs. (3.250) and (3.318), Chapter 3, AMQP], may also be written as a sum over a product of three 6-j symbols.

We first observe, using the Wigner coefficient definition of the 9-j symbol, that

$$\begin{bmatrix} a\,b\,e \\ c\,d\,f \\ h\,k\,g \end{bmatrix} = \langle [(ab)_e(cd)_f]_g | [(ac)_h(bd)_k]_g \rangle . \qquad (5.12.54)$$

The sequence

$$(ac)(bd) \xrightarrow{R} [(ac)b]d \xrightarrow{\Phi} [b(ac)]d \xrightarrow{R} [(ba)c]d \xrightarrow{\Phi} [(ab)c]d \xrightarrow{R} (ab)(cd) \qquad (5.12.55)$$

carries the final bracketing scheme in Eq. (5.12.54) to the initial scheme. Again, we detail the steps to illustrate how a summation makes its appearance due to the occurrence of an intermediate bracketing type in the sequence (5.12.55) that is distinct from either the initial or final one:
Steps 1 and 2, corresponding to

$$[(ac)_h(bd)_k]_g \rightarrow \{[b(ac)_h]_l d\}_g ,$$

introduce the factor (using Fig. 5.29)

$$(-1)^{h+b-l}[(2l+1)(2k+1)]^{\frac{1}{2}} W(hbgd; lk).$$

Steps 3 and 4, corresponding to

$$[b(ac)_h]_l \rightarrow [(ab)_e c]_l ,$$

---

introduce the factor

$$(-1)^{a+b-e}[(2e+1)(2h+1)]^{\frac{1}{2}} W(balc; eh).$$

Step 5, corresponding to

$$\{[(ab)_e c]_l d\}_g \rightarrow [(ab)_e(cd)_f]_g ,$$

introduces the factor

$$[(2l+1)(2f+1)]^{\frac{1}{2}} W(ecgd; lf).$$

The complete transformation is given by

$$
\begin{aligned}
&\langle [(ab)_e(cd)_f]_g | [(ac)_h(bd)_k]_g \rangle \\
&\quad = [(2e+1)(2f+1)(2h+1)(2k+1)]^{\frac{1}{2}} \\
&\qquad \times \sum_l (-1)^{a+l-e-h}(2l+1)W(hbgd;lk)W(balc;eh)W(ecgd;lf) \\
&\quad = [(2e+1)(2f+1)(2h+1)(2k+1)]^{\frac{1}{2}} \\
&\qquad \times \sum_l (-1)^{2l}(2l+1)\begin{Bmatrix} a\,c\,h \\ l\,b\,e \end{Bmatrix}\begin{Bmatrix} b\,d\,k \\ g\,h\,l \end{Bmatrix}\begin{Bmatrix} e\,f\,g \\ d\,l\,c \end{Bmatrix} \\
&\quad = [(2e+1)(2f+1)(2h+1)(2k+1)]^{\frac{1}{2}}\begin{Bmatrix} a\,b\,e \\ c\,d\,f \\ h\,k\,g \end{Bmatrix}. \qquad (5.12.56)
\end{aligned}
$$

Using the symmetry of the 9-j symbol under permutations of its columns, we obtain the result given by Eq. (3.319), Chapter 3, AMQP.

7. The Classification Problem for Transformation Coefficients

A number of interesting points are brought out by the preceding examples, which may be described in terms of a "path" between two bracketings of the same n symbols. We first define a *step* as a transformation between any two associations of three symbols, allowing commutation. Thus, the transformations $[(x)(y)](z) \to (x)[(y)(z)]$, $[(x)(y)](z) \to [(z)(x)](y)$, and $[(z)(x)](y) \to (x)[(y)(z)]$ are steps, whereas $[(x)(y)](z) \to [(y)(x)](z)$ is not. A *path* is any sequence of steps between two bracketings of the same n symbols. Equation (5.12.52) defines a path containing two steps; Eq. (5.12.55) defines one containing three steps.

The essential characteristic of a step is: Each step in the bracketing symbol in the right-hand side (left-hand side) of a transformation coefficient induces a Racah coefficient transformation in Hilbert space. For example, we have

$$
\begin{aligned}
&\langle (\ldots)^B | (\ldots \{[(x)_a(y)_b]_d(z)_c\}_e \cdots) \rangle \\
&\quad = \sum_f [(2d+1)(2f+1)]^{\frac{1}{2}} W(abec;df) \\
&\qquad \times \langle (\ldots)^B | (\ldots \{(x)_a[(y)_b(z)_c]_f\}_e \cdots) \rangle. \qquad (5.12.57)
\end{aligned}
$$

A path P containing p steps (a path of length p) between bracketing schemes B and B' will thus produce an expression for

$$\langle (\ldots)^{B} | (\ldots)^{B'} \rangle$$

in terms of a multiple summation over a product of p Racah coefficients (assuming we have made no simplifications because of recognized identities among the Racah coefficients). However, in general, there will exist distinct paths $P_1, P_2, \ldots$ of the same or different lengths $p_1, p_2, \ldots$ connecting the same two bracketing schemes. Thus, the expression of the transformation coefficient (5.12.57) in terms of Racah coefficients by using different paths $P_1, P_2, P_3, \ldots$ will produce a number of relations between Racah coefficients:

$$\begin{aligned} \langle (\ldots)^{B} | (\ldots)^{B'} \rangle_{P_1} &= \langle (\ldots)^{B} | (\ldots)^{B'} \rangle_{P_2} \\ &= \langle (\ldots)^{B} | (\ldots)^{B'} \rangle_{P_3} = \ldots . \end{aligned} \tag{5.12.58}$$

As an example of this phenomenon, consider the transformation coefficient in Eq. (5.12.53), now evaluated in terms of Racah coefficients by using the following path of length 3:

$$[(ab)c]d \xrightarrow{R} [(ab)d]c \xrightarrow{\phi} c[(ab)d] \xrightarrow{R} c[a(bd)] \xrightarrow{R} (ca)(bd) \xrightarrow{\phi} (ac)(bd). \tag{5.12.59}$$

Explicit evaluation of the left-hand side of Eq. (5.12.53), using this path, yields the Biedenharn–Elliott (B–E) relation between Racah coefficients [see Eq. (2.69), Chapter 2].

Still another phenomenon is illustrated by the following example, which shows that the 9-j symbol may arise as the transformation coefficient between still other coupling schemes:

$$\begin{aligned} &\langle \{[(ab)_e c]_f d\}_g | \{[(ad)_h c]_k b\}_g \rangle \\ &= (-1)^{b+c+d-a-g}[(2e+1)(2f+1)(2h+1)(2k+1)]^{\frac{1}{2}} \begin{Bmatrix} a & b & e \\ h & k & c \\ d & g & f \end{Bmatrix}. \end{aligned} \tag{5.12.60}$$

This result is obtained from the three-step transformation (we do not note explicitly the phase inducing transformations)

$$[(ab)c]d \xrightarrow{R} [(ab)d]c \xrightarrow{R} [(ad)b]c \xrightarrow{R} [(ad)c]b. \tag{5.12.61}$$

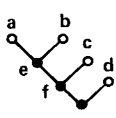
Thus, although one understands the general procedure for reducing any transformation coefficient to a sum over 6-j symbols, there is no *general theory* (in existence) for "classifying" all transformation coefficients.

This problem is more conveniently phrased in the language of the transformations themselves (as opposed to the matrix of a transformation). We have seen that all transformations are generated by just two types given by Eqs. (5.12.48′) and (5.12.49′). Let us call any combination of a transformation of type (5.12.48′) with one of type (4.12.49′) an *elementary transformation*. Then *the transformation between each pair of binary coupling schemes is a product* (*no summations*) *of elementary transformations*. The existence of the Racah sum rule and the B–E identity imply two fundamental relationships of the form $AA'=A''$ and $A_1A_2A_3=A_1'A_2'$, where A, A', A_1, etc., are elementary transformations. Thus, one is dealing with an algebra of compositions of transformations in which certain basic relations are to be enforced. Clearly, such relations may be used to reduce certain products of elementary transformations to products containing fewer terms. Various questions concerning this structure may be formulated: Are all relations between elementary transformations of the form $A_1A_2\ldots=A_1'A_2'\ldots$ consequences of the two basic relations implied by the Racah sum rule and the B–E identity? When is a transformation "irreducible"? That is, when can it not be written as a product of fewer elementary transformations? When are two irreducible transformations equal? How does one give a precise meaning to the term "classification of all transformations between binary coupling schemes"? General answers to questions such as these are not yet known, at least within the framework of the algebraic concepts formulated in this section.

It is apparent that the study of transformation coefficients and the numerous relations between them is a complicated one, requiring systematic development. Jucys *et al.* [3] have made important progress in this direction, using graphical methods (the relationship between these graphical methods and the algebraic method outlined above has not been systematically worked out). We turn next to a discussion of these graphical methods, emphasizing the relationship to graph theory itself (see, for example, the book by Harary [12]).

8. A Mapping of Pairs of Labeled Binary Trees to Cubic Graphs

Each coupling scheme for n angular momenta (each labeled tree) may be mapped to the set of triples that specify the "triangle conditions" corresponding to the $n-1$ binary couplings in the scheme: For example, we have



$$\longrightarrow \{(abe),(ecf),(fdg)\}. \qquad (5.12.62)$$

[The triangle associated with a *labeled fork* 

of the tree is (lmn).]

Let us next interpret the set (5.12.62) in terms of the concept of a *graph* (defined generally below). We first introduce the set of *points* P defined by

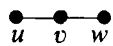
$$P \equiv \{u, v, w\}, \tag{5.12.63}$$

where we have written $u=(abe)$, $v=(ecf)$, and $w=(fdg)$. We next introduce the set of *lines* L defined by a set of (unordered) pairs of points of P:

$$L \equiv \{(u, v), (v, w)\}. \tag{5.12.64}$$

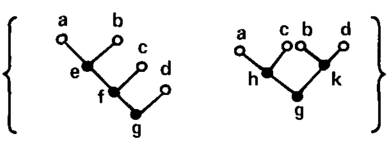
The lines (u, v) and (v, w) are defined in terms of the points of P to be those pairs of points containing a common label in the triples representing the points.

The set of points P together with the lines L constitute a graph G having three points and two lines. The graph G is usually represented by a planar diagram (also called a graph):



(5.12.65)

Let us next apply the same idea to the pair of trees representing the transformation coefficient (5.12.50):

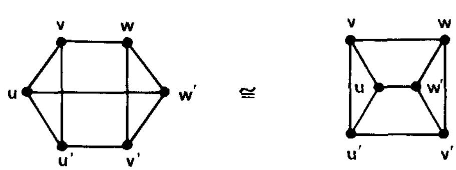


$$\longrightarrow \left\{ \begin{matrix} (abe), (ecf), (fdg) \\ (ach), (hkg), (bdk) \end{matrix} \right\} = P. \tag{5.12.66}$$

~~Writing $u=(abe)$, $v=(ecf)$, $w=(fdg)$, $u'=(ach)$, $v'=(hkg)$, and $w'=(bdk)$,~~ we see that the lines in P are the (unordered) pairs of points given by

$$L = \left\{ \begin{matrix} (u, v), (u, u'), (u, w'), (v, w), (v, u') \\ (w, v'), (w, w'), (u', v'), (v', w') \end{matrix} \right\}. \tag{5.12.67}$$

The graph G defined by the points P and the lines L may be represented by the planar diagrams:



(5.12.68)

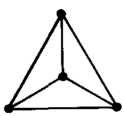
[We have shown two equivalent ways of representing the graph (see the definition of *graph isomorphism* (denoted $\cong$) given below).]

The graph illustrated in (5.12.68) has six points and nine lines, there being three lines *incident* to each point. It is called a regular graph of degree 3.

This illustrative example suggests a relationship between *pairs of labeled binary trees and certain cubic graphs* (see definition below). We develop this relationship, in general, below. The motivation for this line of development may be represented schematically as

$$\text{transformation coefficient} \longrightarrow \text{pair of binary trees} \longrightarrow \text{cubic graph.}$$

The idea is then to utilize the properties of the cubic graphs to classify the set of transformation coefficients themselves, the prototype for this structure being the association of the 6-j symbol with a tetrahedron, which may be represented as the planar diagram (cubic graph).



Before turning to these developments, it is convenient to introduce the definition of a graph together with a limited glossary of graph terminology (Harary [12], Korfhage [13]).

A (finite) graph G is a set P containing p elements together with a prescribed set L of q (unordered) pairs of distinct elements of P. [G is often called a (p,q) graph.]

Point of G: an element of P.

Line of G: an element of L [the line $l=\{p_1, p_2\}$ is said to join points p_1 and p_2 of G].

Adjacent points: any pair of points defining a line of G.

Incident point and line: The point p_1 and the line $l=\{p_1, p_2\}$ are said to be incident with one another, as are p_2 and l.

Adjacent lines: two distinct lines incident with a common point.

Degree of a point: number of lines incident with the point. (Euler's theorem: The sum of the degrees of all the points is twice the number of lines.)

End point: a point of degree 1.

Regular graph: a graph in which all points have the same degree.

Cubic graph: a regular graph in which the points are of degree 3.

Multigraph: a graph in which more than one line can join the same two points.

Labeled graph: a graph [such as (5.12.68)] in which the points and/or lines are assigned distinct labels which usually have some sort of significance.

Unlabeled graph: a graph in which no labels appear (except possibly for identification of lines and/or points).

Walk in G: an alternating sequence of points and lines $p_1 l_1 p_2 l_2 \ldots p_{n-1} l_{n-1} p_n$ such that line l_i is incident with points p_i and p_{i+1}, $i = 1, 2, \ldots, n-1$. The walk may also be represented by the line sequence $l_1 l_2 \ldots l_{n-1}$ or the point sequence $p_1 p_2 \ldots p_n$.

Trail: a walk with no repeated lines.

Cycle: a walk such that $p_1 = p_n$ with no other repeated points.

Path: a walk with no repeated points.

Length of a path: number of lines in the path.

Distance between two points: length of the shortest path joining the points, if any, otherwise the distance is infinite.

Hamiltonian graph: a graph in which the longest cycle contains all the vertices (points).

Isomorphic graphs: two graphs G and H are isomorphic ($G \cong H$) if there exists a one-to-one correspondence between their points which preserves adjacency.

Connected graph: a graph in which every pair of points is joined by a path.

Subgraph of G: a graph having all its points and lines in G.

Acyclic graph: a graph containing no cycles.

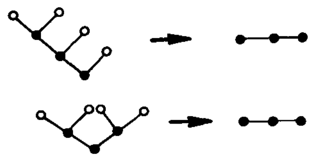
Tree: a connected acyclic graph.

Consider a single labeled binary tree corresponding to the coupling of n angular momenta [n end points labeled by n (distinct) angular momenta, 1 root labeled by the total angular momentum, $n-2$ internal points labeled by (distinct) coupled angular momenta]. We now read off the labeled binary tree the (angular momenta) triangles corresponding to each of the labeled forks ($n-1$ triples). These triples[1] are now taken to be the set of points P of a graph G in which the lines of G are defined to be the set L of pairs of ~~points of P whose triples have a common label. Then: *G is a connected*~~ $(n-1, n-2)$ *graph*; *that is* (Harary [12], p. 33), *G is a tree T. Furthermore, the degree of no point of T exceeds* 3, *so that T is a trivalent tree* (Cayley [14]).

Consider next the set of trivalent trees obtained from the correspondence of labeled binary trees to trees as described above. The geometric operations (see p. 448) used to define equivalence of two (unlabeled) binary trees preserves the triangles as read off the set of forks in the labeled binary tree. Hence, all labeled binary trees related by such geometric operations define the same trivalent tree T. Furthermore, the distinct labelings of a binary tree of the same type (see p. 447) preserves adjacency of points in the corresponding trivalent tree T. Thus, the correspondence given above associates a single trivalent tree with each type (and all its equivalents) of binary tree.

[1]We shall regard a triple (abc) as unordered.

Distinct types of binary trees may, however, be associated with the same trivalent tree. For example, we have



(5.12.69)

(These correspondences are obtained by labeling the points of each of the binary trees with distinct symbols, reading off the triangles, and then drawing the trivalent tree diagram using the point-to-line relation defined by the triangles as described above. Observe that the labeling of a binary tree serves to define the point-to-line relation in the trivalent tree T. Once the diagram of T has been given, the labeling may be discarded. In this way one obtains a correspondence between the set of unlabeled binary trees having *n end points* and the set of trivalent trees having $n-1$ *points*.)

In Fig. 5.30, we list all the trivalent trees containing up to nine points. [We have taken these diagrams from Harary [12].) Cayley [14] gives generating formulas for determining the number t_p of trivalent trees having p points. The first few numbers are (one always has $t_p \leq b_{p+1}$, where b_{p+1} is the Wedderburn–Etherington number):

p	2	3	4	5	6	7	8	9	10	11
t_p	1	1	2	2	4	6	11	18	37	66 .

To prove that every trivalent tree T having p points corresponds to at least one binary tree having $p+1$ end points (terminal points), we choose any end point of T as the root of the binary tree and join a line and an adjacent point to it; to each of the other endpoints of T we join two lines and two corresponding adjacent points; and to the points of T of degree 2 we join a single line and adjacent point. The resulting graph is then a binary tree having $p+1$ end points, and it corresponds to the trivalent tree T having p points.

Let us next turn to a characterization of the graph G obtained from the set of angular momentum triangles by reading off the labeled forks of a *pair* of labeled binary trees corresponding to a transformation coefficient [see the example given by Eq. (5.12.66)]. Thus, we consider a pair $\{B, B'\}$ of labeled binary trees, B and B', such that (a) each binary tree has n end points labeled by a permutation of $j_1 j_2 \cdots j_n$ and a root labeled by j; (b) the $n-2$ internal points of B are labeled by $b_1 b_2 \cdots b_{n-2}$, and those of B' by

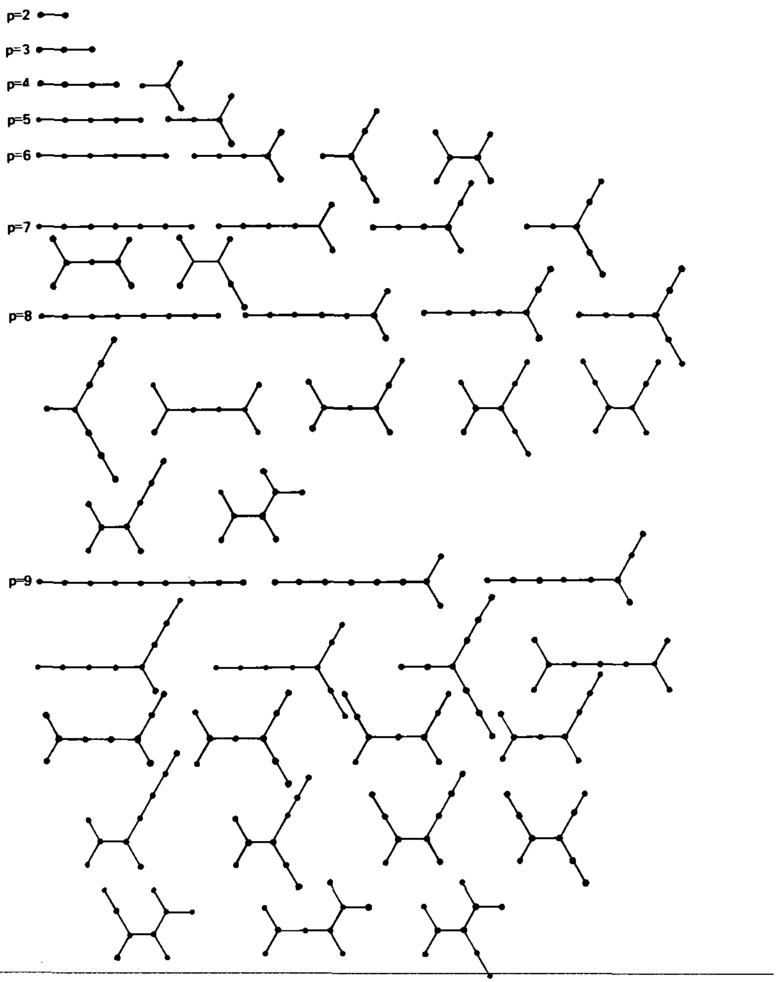


Figure 5.30. Trivalent trees.

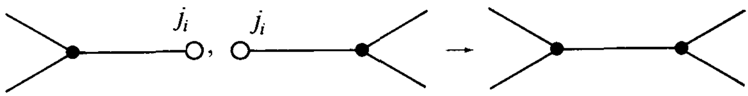
$b_1'b_2' \cdots b_{n-2}'$. Let us introduce the notation:

P = set of triangles of the binary tree B,
= set of points of the corresponding trivalent tree T;
P' = set of triangles of the binary tree B',
= set of points of the corresponding trivalent tree T'.

Then the graph $G = G(T, T')$ is defined to be the graph with points $P \cup P'$ and with lines consisting of (a) the set of lines of T, (b) the set of lines of

T', and (c) *the set of lines joining T and T' defined by the following rules*: (i) a line joins a point $p \in P$ and a point $p' \in P'$ whenever the triples defining p and p' contain a single common label; and (ii) two lines (multigraph) join a point $p \in P$ and a point $p' \in P'$ whenever the triples defining p and p' contain two common labels (the labeling of internal points by distinct symbols implies that, at most, two labels will be the same in any pair of triples $p \in P$, $p' \in P'$).

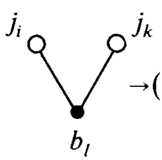
The planar diagram of the graph $G(T, T')$ (possibly a multigraph) described above is obtained by tying together the points of B and B' (end points and roots) that have the same label. This result may be symbolized by



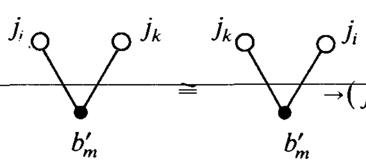
(end points disappear when joined), (5.12.70)

(a line joins the two roots).

A multigraph will occur only if B and B' contain labeled forks of the respective forms:



$\rightarrow (j_i j_k b_l) = p \in P,$ (5.12.71)



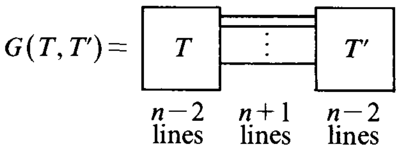
$\rightarrow (j_i j_k b'_m) = p' \in P'.$

The joining procedure described above then results in two lines joining points $p \in P$ and $p' \in P'$:

p  p'. (5.12.72)

In this case it can be shown that the transformation coefficient corresponding to the pair of labeled binary trees $\{B, B'\}$ reduces to a transformation coefficient corresponding to a pair of labeled binary trees containing $n-1$ end points. We may, therefore, without loss of generality, restrict our considerations to pairs of labeled binary trees *without common forks* [labeled forks of the type (5.12.71)].

The graph $G(T, T')$ corresponding to labeled binary trees B and B' each having n end points and without common forks is a *cubic graph* having $2(n-1)$ points. The graph $G(T, T')$ may be denoted symbolically by



(5.12.73)

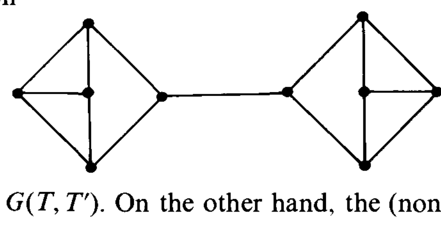
This diagram serves to illustrate that $G(T, T')$ is obtained as the joining of two trivalent trees, T and T' (each having $n-1$ points) by $n+1$ lines of the form $\{p, p'\}$, $p \in T$, $p' \in T'$.

Conversely, let G be a cubic graph whose $2(n-1)$ points can be covered by two disjoint (no common points) trivalent trees T and T' (the points of neither T nor T' define cycles in G), each having $n-1$ points. Then G is the map of a pair of labeled binary trees, $B \to T$ and $B' \to T'$, each having n end points. Hence, G is a cubic graph of the type $G(T, T')$.

Proof. Label the *lines* of T by $b_1 b_2 \cdots b_{n-2}$, the *lines* of T' by $b'_1 b'_2 \cdots b'_{n-2}$, and the *lines* joining T and T' by $j_1 j_2 \cdots j_n j$. Now assign to each of the $2(n-1)$ *points* of G the (unordered) triple of labels corresponding to the labels of the three lines incident to the point. Then $j_1 j_2 \cdots j_n$ are the labels of the end points of binary trees B and B' with root j and internal points $b_1 b_2 \cdots b_{n-2}$ and $b'_1 b'_2 \cdots b'_{n-2}$ such that the set of triangles of B defines T and the set of triangles of B' defines T'. ■

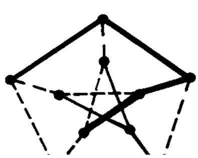
This result, relating a class of cubic graphs and pairs of trivalent trees, characterizes the subset of cubic graphs that is associated with the transformation coefficients arising in the binary coupling theory of n angular momenta.

~~*Not all cubic graphs are of the type* $G(T, T')$. For example, the (nonhamil-~~ tonian) cubic graph



(5.12.74)

is not of the type $G(T, T')$. On the other hand, the (nonhamiltonian) cubic graph



(5.12.75)

is of the type $G(T, T')$ (T is represented by the broad lines and points, T' by the thin lines and points, and the joining lines by dashed lines). The examples given below (Section 9) of cubic graphs of the type $G(T, T')$ are hamiltonian—the second example above shows that not all graphs of the type $G(T, T')$ are hamiltonian.

To our knowledge the set of cubic graphs of the type $G(T, T')$ has not been enumerated for general $2(n-1)$ points, although methods of construction have been discussed in the literature (Gutman and Budrite [15]; Levinson and Chiplis [16]).

A direct method of construction would be to consider all ways of joining pairs of trivalent trees (for given p) from Fig. 5.30 to obtain a cubic graph. The problem of identifying isomorphic cubic graphs[1] would appear to render this direct procedure difficult and impractical. For those cases where the set of all cubic graphs of $2(n-1)$ points have been listed, one can, of course, determine by examination which ones are of the type $G(T, T')$. The listings of cubic graphs are, however, quite limited (see the *cover* of Harary [12] and pp. 43, 44 of Korfhage [13].

We shall not consider the general set $\{G(T, T')\}$ of cubic graphs further, but shall illustrate by specific examples the use of such graphs in the classification of transformation coefficients.

9. Classification of Transformation Coefficients by Use of Cubic Graphs

Let us introduce the notation $\{B|B'\}$ to denote the transformation coefficient corresponding to a pair of binary trees B and B' that are labeled in the following manner: B and B' have end points labeled by (distinct) permutations of $j_1 \cdots j_n$; roots labeled by j; and internal points labeled by $b_1 \cdots b_{n-2}$ and $b'_1 \cdots b'_{n-2}$. Consider the set of transformation coefficients given by

$$\left\{ \{B|B'\} : \begin{array}{l} B \text{ and } B' \text{ are labeled binary trees as described} \\ \text{above and without common forks} \end{array} \right\}. \tag{5.12.76}$$

Define two transformation coefficients in this set to be *equivalent* if they can be expressed in forms (say, in terms of either 3-j coefficients or 6-j coefficients) that differ only by phase factors, dimension factors [multiplicative factors of the type $(2k+1)^{\frac{1}{2}}$], and permutations of their labels.

The remarkable result, which can be proved (by expressing the transformation coefficient explicitly in terms of 3-j coefficients), is: *Two transforma-*

[1] Isomorphic cubic graphs define symmetries of the corresponding transformation coefficient.

tion coefficients are equivalent,

$$\{B_1|B_1'\} \sim \{B_2|B_2'\}, \tag{5.12.77}$$

if and only if the corresponding cubic graphs are isomorphic; that is,

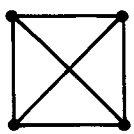
$$G(T_1, T_1') \cong G(T_2, T_2'), \tag{5.12.78}$$

where these graphs are constructed from the pair of binary trees ($B_i \to T_i$, $B_i' \to T_i'$) *by the procedure described in the last section.*

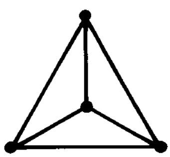
As an example of this lemma, consider all coupling schemes for three angular momenta. One finds that, up to permutation of labels, all transformation coefficients possess the same set of angular momentum triangles (which now define the points and lines of a cubic graph $\cong$ tetrahedron):[1]

$$P = \{(abe), (cde), (acf), (bdf)\},$$

which has the diagram



$\cong$



This result expresses the fact that there exists only one 6-j coefficient.

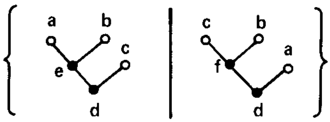
The study of inequivalent transformation coefficients thus becomes synonymous with the study of nonisomorphic cubic graphs of type $G(T, T')$.

Using results from Jucys *et al.* [3], Harary [12], and Korfhage [13], we list below all the cubic graphs on 4, 6, and 8 points [for these cases *all* cubic graphs are of type $G(T, T')$]. The diagram to the left is arranged in the form of a pair of trivalent trees, which cover the cubic graph (T = top row of points and lines, T' = bottom row of points and lines). Below each diagram we have introduced a bracket notation for a *representative transformation coefficient* in terms of a pair of labeled binary trees. [The general notations and definitions for 3n-j coefficients of the first and second kinds (Jucys *et al.* [3]) are given below. Various isomorphic cubic graphs are shown for the purpose of comparing the results given in Jucys *et al.* with those found in books on graph theory. We have also drawn three of the planar diagrams as diagrams in 3-space, since this type of representation of transformation coefficients is also found in the literature (Edmonds [17]; Judd [18]).]

[1]Observe that we now label the vertices of the tetrahedron by the "angular momentum triangles," and the lines by the common labels of the incident points.

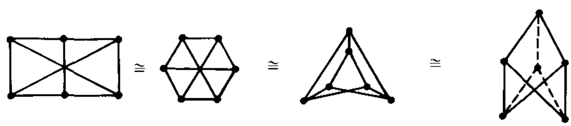
FOUR POINTS

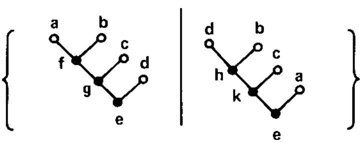
1: 



$$=(-1)^{2c+2e}\left[(2e+1)(2f+1)\right]^{\frac{1}{2}}\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}.$$

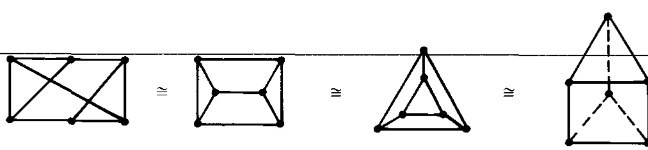
SIX POINTS

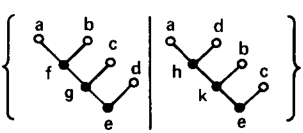
1: 



$$=(-1)^{d+g-a-k}\left[(2f+1)(2g+1)\right.$$

$$\left.\times(2h+1)(2k+1)\right]^{\frac{1}{2}}\begin{Bmatrix} b & a & f \\ d & e & g \\ h & k & c \end{Bmatrix};$$

2: 

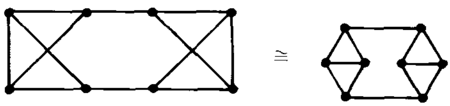


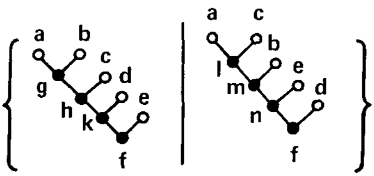
$$=(-1)^{b+c+g+h-f-k}$$

$$\times\left[(2f+1)(2g+1)(2h+1)(2k+1)\right]^{\frac{1}{2}}$$

$$\times\begin{Bmatrix} f & d & k \\ h & b & a \end{Bmatrix}\begin{Bmatrix} f & d & k \\ e & c & g \end{Bmatrix}.$$

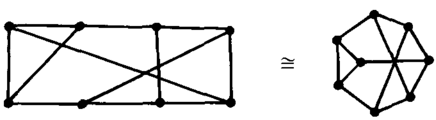
EIGHT POINTS

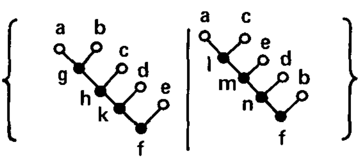




$$=(-1)^{\phi}\left[(2g+1)(2l+1)(2k+1)(2n+1)\right]^{\frac{1}{2}}\delta_{mh}\begin{Bmatrix} a\,b\,g \\ m\,c\,l \end{Bmatrix}\begin{Bmatrix} d\,m\,k \\ e\,f\,n \end{Bmatrix},$$

$\phi=b+e+l+n-c-d-g-k;$



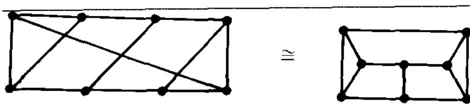


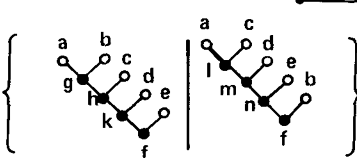
$=$

$$=(-1)^{\phi}\left[(2g+1)(2h+1)(2k+1)(2l+1)(2m+1)(2n+1)\right]^{\frac{1}{2}}$$

$$\times\begin{Bmatrix} a\,b\,g \\ h\,c\,l \end{Bmatrix}\begin{Bmatrix} b\,h\,l \\ f\,k\,e \\ n\,d\,m \end{Bmatrix},$$

$\phi=b+k+l+m-c-g-h-n;$



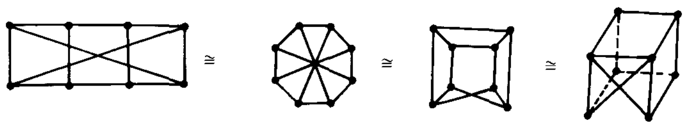


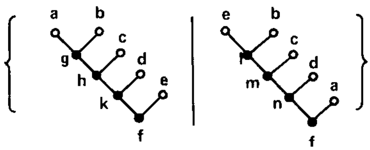
$=$

$$=(-1)^{\phi}\left[(2g+1)(2h+1)(2k+1)(2l+1)(2m+1)(2n+1)\right]^{\frac{1}{2}}$$

$$\times\begin{Bmatrix} a\,b\,g \\ h\,c\,l \end{Bmatrix}\begin{Bmatrix} b\,l\,h \\ d\,k\,m \end{Bmatrix}\begin{Bmatrix} b\,m\,k \\ e\,f\,n \end{Bmatrix},$$

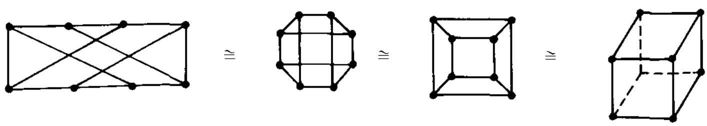
$\phi=b+e+m+n-c-d-h-k-l;$

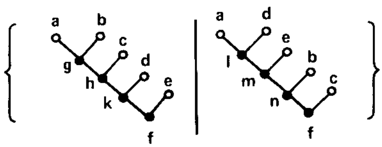
4: 



$$=(-1)^{k+e-n-a}[(2g+1)(2h+1)(2k+1)(2l+1)(2m+1)(2n+1)]^{\frac{1}{2}}$$

$$\times\begin{Bmatrix} a & & g & & h & & k & \\ & b & & c & & d & & f \\ e & & l & & m & & n & \end{Bmatrix};$$

5: 



$$=(-1)^{b+c+g+n-h-m}[(2g+1)(2h+1)(2k+1)(2l+1)(2m+1)(2n+1)]^{\frac{1}{2}}$$

$$\times\begin{bmatrix} c & & h & & k & & f & \\ & g & & d & & e & & n \\ b & & a & & l & & m & \end{bmatrix}.$$

Before turning to the discussion of the results shown above, let us define the new symbols that appear. General expressions for $3n$-j coefficients of the first and second kind have been defined by Jucys *et al.* [3]:

$3n$-j coefficient of the first kind:

$$\begin{Bmatrix} a_1 & & a_2 & & \dots & a_n & \\ & b_1 & & b_2 & \dots & & b_n \\ c_1 & & c_2 & & \dots & c_n & \end{Bmatrix} \equiv \sum_z (2z+1)(-1)^{S+(n-1)z}$$

$$\times\begin{Bmatrix} a_1 & c_1 & z \\ c_2 & a_2 & b_1 \end{Bmatrix}\begin{Bmatrix} a_2 & c_2 & z \\ c_3 & a_3 & b_2 \end{Bmatrix}\cdots\begin{Bmatrix} a_{n-1} & c_{n-1} & z \\ c_n & a_n & b_{n-1} \end{Bmatrix}\begin{Bmatrix} a_n & c_n & z \\ a_1 & c_1 & b_n \end{Bmatrix} \tag{5.12.79}$$

$3n$-j coefficient of the second kind:

$$\begin{bmatrix} a_1 & & a_2 & & \cdots & a_n & \\ & b_1 & & b_2 & \cdots & & b_n \\ c_1 & & c_2 & & \cdots & c_n & \end{bmatrix} \equiv \sum_z (2z+1)(-1)^{S+nz}$$

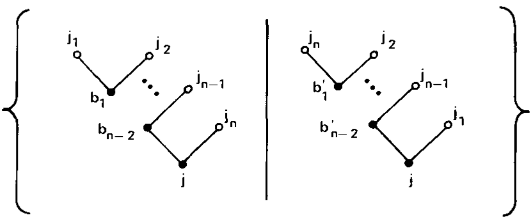
$$\times \begin{Bmatrix} a_1 & c_1 & z \\ c_2 & a_2 & b_1 \end{Bmatrix} \begin{Bmatrix} a_2 & c_2 & z \\ c_3 & a_3 & b_2 \end{Bmatrix} \cdots \begin{Bmatrix} a_{n-1} & c_{n-1} & z \\ c_n & a_n & b_{n-1} \end{Bmatrix} \begin{Bmatrix} a_n & c_n & z \\ c_1 & a_1 & b_n \end{Bmatrix}, \tag{5.12.80}$$

where

$$S = \sum_{i=1}^{n} (a_i + b_i + c_i).$$

Coupling schemes for n angular momenta giving $3(n-1)$-j coefficients are:

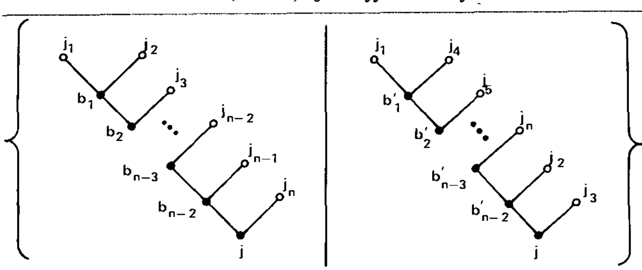
$3(n-1)$-j coefficient of the first kind:



$$= (-1)^{b_{n-2}+j_n-b'_{n-2}-j_1} \left[\prod_{\alpha=1}^{n-2} (2b_\alpha + 1)(2b'_\alpha + 1) \right]^{\frac{1}{2}}$$

$$\times \begin{Bmatrix} j_1 & & b_1 & & \cdots & b_{n-3} & & b_{n-2} & \\ & j_2 & & j_3 & \cdots & & j_{n-1} & & j \\ j_n & & b'_1 & & \cdots & b'_{n-3} & & b'_{n-2} & \end{Bmatrix}; \tag{5.12.81}$$

$3(n-1)$-j coefficient of the second kind:



$$(-1)^{j_2+j_3+b_1-b'_{n-3}+b'_{n-2}-b_2} \left[\prod_{\alpha=1}^{n-2} (2b_\alpha + 1)(2b'_\alpha + 1) \right]^{\frac{1}{2}}$$

$$\times \begin{bmatrix} j_3 & & b_2 & & b_3 & & \cdots & b_{n-2} & & j & \\ & b_1 & & j_4 & & j_5 & \cdots & & j_n & & b'_{n-2} \\ j_2 & & j_1 & & b'_1 & & \cdots & b'_{n-4} & & b'_{n-3} & \end{bmatrix}. \tag{5.12.82}$$

For $n=1$, 2, and 3, these coefficients reduce to standard coefficients according to the following relations:

$$\begin{Bmatrix} a & & \\ & & c \\ b & & \end{Bmatrix} = (-1)^{a+b-c}\varepsilon_{abc},$$

$$\begin{bmatrix} a & & \\ & & c \\ b & & \end{bmatrix} = \delta_{c0}[(2a+1)(2b+1)]^{\frac{1}{2}},$$

$$\begin{Bmatrix} a & & b & \\ & e & & f \\ d & & c & \end{Bmatrix} = (-1)^{a+b+c+d}\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix},$$

$$\begin{bmatrix} a & & b & \\ & e & & f \\ d & & c & \end{bmatrix} = \delta_{ef}\frac{(-1)^{a+c-b-d}}{2e+1}\varepsilon_{abe}\varepsilon_{dce}, \tag{5.12.83}$$

$$\begin{Bmatrix} b & & c & & f & \\ & a & & j & & e \\ d & & h & & i & \end{Bmatrix} = \begin{Bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{Bmatrix},$$

$$\begin{bmatrix} b & & c & & f & \\ & a & & j & & e \\ d & & h & & i & \end{bmatrix} = (-1)^{2b+2d}\begin{Bmatrix} a & j & e \\ i & d & h \end{Bmatrix}\begin{Bmatrix} a & j & e \\ f & b & c \end{Bmatrix}.$$

Let us now consider several features of the examples of cubic graphs and transformation coefficients shown in the figures above.

(a) The transformation coefficients corresponding to the cubic graphs on six or eight points illustrate a general result on the factorizing of transformation coefficients (Jucys *et al.* [3]). *If the cubic graph $G(T, T')$ corresponding to a transformation coefficient $\{B|B'\}$ can be covered by two disjoint subgraphs joined by either two or three lines, then the transformation coefficient factorizes into two transformation coefficients defined on fewer angular momenta.* (This factorization occurs in the second example shown above for cubic graphs having six points; it occurs in the first three examples shown above for cubic graphs having eight points.)

(b) No significance should be attached to the occurrence of only one type of binary tree in these examples—this feature fails for ten points (see below).

(c) Observe that a consequence of the factorization property—stated in (a) above—is that the number of "new coefficients" (coefficients that do not factorize) arising from the nonisomorphic cubic graphs of type $G(T, T')$ with $2(n-1)$ points is less than the number of such cubic graphs.

It is instructive to consider the cubic graphs of the type $G(T, T')$ for the case of ten points (15-j coefficients), since a number of new features arise. There are nineteen nonisomorphic cubic graphs with ten points (Korfhage [13]), but not all these cubic graphs are of type $G(T, T')$, as the example

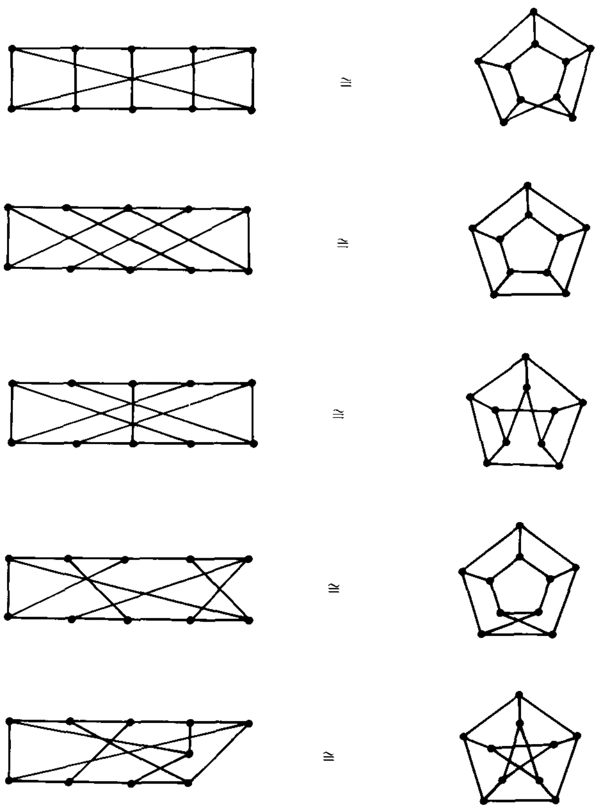
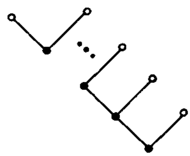


Figure 5.31.

(5.12.74) shows. The number of cubic graphs on ten points that are also of ~~the type $G(T, T')$ does not appear (to our knowledge) in the literature,~~ although the number of such cubic graphs that cannot be covered by two disjoint subgraphs joined by two or three lines is given by Jucys *et al.* as five. Thus, *there are five new transformation coefficients that appear in the coupling of six angular momenta.* We give in Fig. 5.31 the corresponding five nonisomorphic cubic graphs, first as the joining of two disjoint trivalent trees, and then in isomorphic forms as found in Harary [12] and Korfhage [13]. (The first and second graphs correspond, respectively, to 15-j coefficients of the first and second kinds; symbols for the remaining three new 15-j coefficients may be found in Jucys *et al.* [3], together with their definitions in terms of 6-j coefficients.)

The new features brought out by the cubic graphs on ten points include: (a) not all cubic graphs are of type $G(T, T')$; (b) not all cubic graphs of type $G(T, T')$ are hamiltonian; and (c) not all cubic graphs of type $G(T, T')$

can be covered by a pair of binary trees of type



Remarks. (*a*) It is quite remarkable that the structure of a transformation coefficient is determined by the abstract cubic graph associated with the "set of angular momentum triangles." The general theory is incomplete, however, since there are no known (general) formulas giving the number of cubic graphs of type $G(T, T')$, and advances thus far have been by direct enumeration, this technique having been carried about as far as practical by Jucys *et al.* [3, 7]. [The total number of nonisomorphic cubic graphs[1] on $2p$ points for $p=2$, 3, 4, 5, and 6 is 1, 2, 5, 19, and 87; the corresponding number of cubic graphs of type $G(T, T')$ that cannot be covered by two disjoint subgraphs joined by two or three lines is 1, 1, 2, 5, and 18; that is, the number of 6-j, 9-j, 12-j, 15-j, and 18-j coefficients is 1, 1, 2, 5, and 18. It is also known (Jucys *et al.* [3, 7]) that the number of 21-j and 24-j coefficients is 84 and 576.]

(*b*) Numerous relationships between $3n$-j coefficients have been tabulated in Jucys *et al.* [3, 7]; see also Judd [18].

(*c*) The 9-j coefficients were first introduced by Wigner [19]; additional properties were developed by Jahn and Hope [20] and Arima *et al.* [21]. The 12-j coefficients of the first kind were introduced by Jahn (see Ref. [20]) and studied further by Ord-Smith [22]; The 12-j coefficients of the second kind were introduced by Elliott and Flowers [23] and studied by Sharp [24]. Generating formulas for the 9-j, 12-j, and 15-j coefficients have been given by Wu [25] and Huang and Wu [26].

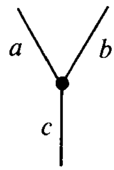
(*d*) We wish to acknowledge several useful discussions with Dr. C. ~~Patterson in interpreting the relationship between cubic graphs and the~~ graphical methods of Jucys. We are also indebted to Prof. J. Paldus for a critical reading of this Topic.

10. Notes

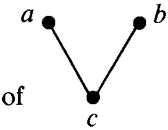
1. *The graphical methods of the Jucys school.* The problem of parentheses suggests in a natural way the relationship between angular momentum coupling schemes and binary trees given in Sections 4 and 5. This then leads to the map between pairs of binary trees and cubic graphs given in Section 8. Notice, however, that one may label uniquely the corresponding cubic

[1] These numbers appear in Korfhage [13] and are attributed to R. W. Robinson in Ref. [12, p. 195].

graph either by its *points* (triples of angular momenta) *or by its lines* (the common labels in pairs of triples). When we use "line labeling," the "structural element" in diagrams representing recoupling coefficients becomes



in place of



,

where (abc) is an "angular momentum triangle." The graphical methods of Jucys use the first of these diagrams and give rules, including procedures for handling phases and dimension factors, for joining such elements into the cubic graphs representing transformation coefficients. These methods have been extended and applied to various problems by Vilenkin, Kuznetsov, and Smorodinskii, among others (see Ref. [27]–[30]).

2. *Democratic coupling of three angular momenta.* Chakrabarti [1] and Lévy-Leblond and Lévy-Nahas [2] have considered a coupling scheme for the addition of three angular momenta, $\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_3$, which is characterized by the simultaneous diagonalization of the six Hermitian operators

$$\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}_3^2,\ K, \mathbf{J}^2,\ J_3=\mathbf{J}\cdot\hat{e}_3, \tag{5.12.84}$$

where the operator K is defined by

$$K \equiv \mathbf{J}_1 \times \mathbf{J}_2 \cdot \mathbf{J}_3. \tag{5.12.85}$$

The commutator of K with the $\mathbf{J}_i$ is given by

$$[K, \mathbf{J}_1] = i[\mathbf{J}_2(\mathbf{J}_3\cdot\mathbf{J}_1) - \mathbf{J}_3(\mathbf{J}_1\cdot\mathbf{J}_2)] \tag{5.12.86}$$

(and cyclically). Using this result, one finds that K is a rotational invariant:

$$[K, \mathbf{J}] = \mathbf{0}. \tag{5.12.87}$$

The tensor product space $\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}=\mathcal{H}(j_1 j_2 j_3)$ is mapped into itself by the action of K and $\mathbf{J}$, and it must be possible to split $\mathcal{H}(j_1 j_2 j_3)$ into $(2j+1)$-dimensional irreducible subspaces with respect to $\mathbf{J}$, where these subspaces are spanned by basis vectors of the form

$$|k(j_1 j_2 j_3)_{jm}\rangle \equiv \sum_{\sum_i m_i = m} C^{j_1 j_2 j_3 j}_{m_1 m_2 m_3 m}(k)|\, j_1 m_1\rangle\otimes|\, j_2 m_2\rangle\otimes|\, j_3 m_3\rangle. \tag{5.12.88}$$

The angular momentum operator $\mathbf{J}$ is to have the standard action on this basis, and k denotes an (real) eigenvalue of K:

$$K|k(j_1j_2j_3)_{jm}\rangle = k|k(j_1j_2j_3)_{jm}\rangle. \tag{5.12.89}$$

A proof is given in Ref. [2] that the set of operators (5.12.84) is complete; that is, the operator K serves to distinguish fully between those subspaces of $\mathcal{H}(j_1j_2j_3)$ with basis[1]

$$\{|k(j_1j_2j_3)_{jm}\rangle : m=j, j-1, \ldots, -j\}, \tag{5.12.90}$$

which are carrier spaces of the same representation $D^j(U)$ of the $SU(2)$ diagonal subgroup of $SU(2)\times SU(2)\times SU(2)$.

The coefficients occurring in Eq. (5.12.88) are Wigner coefficients for the complete reduction of an irrep of the direct product group $SU(2)\times SU(2)\times SU(2)$ into irreps of the diagonal $SU(2)$ subgroup.

The individual angular momenta are treated alike in the coupling scheme (5.12.88). In particular, when the three angular momenta are equal, $j_1=j_2=j_3=n$, the eigenvectors $|k(nnn)_{jm}\rangle$ belong to subspaces that are irreducible with respect to the action of the symmetric group.

The coupling scheme (5.12.88) has many nice properties with respect to the symmetric group S_3 (see Refs. [1] and [2]). The difficulty with the method is that the eigenvalues of K are generally not rational functions of j_1, j_2, and j_3 and have only been calculated numerically (special cases excepted). This suggests that it will not be possible to obtain closed-form general expressions for the Wigner coefficients. We refer to Refs. [1] and [2] for further properties of the eigenvalues $\{k\}$ and the corresponding eigenvectors.

3. *Other methods of spanning the space* $\mathcal{H}(j_1j_2\cdots j_n)$. In spanning the space $\mathcal{H}(j_1j_2\cdots j_n)$, we have considered in this Topic only pairwise couplings of the angular momenta. Aside from three angular momenta (Note 2), there has been little work relating to nonbinary coupling schemes. Alternative techniques for constructing bases (of sharp j) of the space $\mathcal{H}(j_1j_2\cdots j_n)$ usually use the properties of the symmetric group S_n at the outset, following the classic work of Weyl [31]. It is beyond the scope of the present Topic to discuss these general techniques (see, however, Sections 5 and 9 of Chapter 7, AMQP), and we refer to the literature (Refs. [32]–[41]) for such discussions and further references on the subject.

[1] For specified $(j_1j_2j_3)$ the set of eigenvalues $\{k\}$ must then consist of $n(j_1j_2j_3j)$ distinct values, where $n(j_1j_2j_3j)$ denotes the number of occurrences of irrep $[j]$ in the direct product $[j_1]\otimes[j_2]\otimes[j_3]$.

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Appendix of Tables

The tables in this Appendix summarize the properties of Wigner and Racah operators that occur frequently in applications. They are repeated from Volume 8, AMQP. The use of the tables is self-explanatory.

Table A1. The Tensor Harmonics

DEFINITION AND GENERAL PROPERTIES

$$T^k_\mu \equiv \left\langle \begin{matrix} & 0 & \\ k & & -k \\ & \mu & \end{matrix} \right\rangle \left[\prod_{s=1}^{k} (4J^2 + 1 - s^2) \right]^{1/2}$$

$$T^k_k = (-1)^k[(2k)!/k!k!]^{1/2} J^k_+$$

$$T^{k\dagger}_\mu = (-1)^\mu T^k_{-\mu}$$

$$\sum_\mu (-1)^\mu T^k_\mu T^k_{-\mu} = \prod_{s=1}^{k} (4J^2 + 1 - s^2)$$

SPECIAL CASES

k	μ	T^k_μ
0	0	1
1	1	$-\sqrt{2}J_+$
	0	$2J_3$
	-1	$\sqrt{2}J_-$
2	2	$\sqrt{6}J^2_+$
	1	$-\sqrt{6}J_+(2J_3 + 1)$
	0	$2(3J^2_3 - J^2)$
	-1	$\sqrt{6}J_-(2J_3 - 1)$
	-2	$\sqrt{6}J^2_-$

Table A1. *(continued)*

k	μ	T^k_μ
3	3	$-2\sqrt{5}\,J_+^3$
	2	$2\sqrt{30}\,J_+^2\,(J_3 + 1)$
	1	$-2\sqrt{3}\,J_+(5J_3^2 - J^2 + 5J_3 + 2)$
	0	$4(5J_3^2 - 3J^2 + 1)J_3$
	-1	$2\sqrt{3}\,J_-\,(5J_3^2 - J^2 - 5J_3 + 2)$
	-2	$2\sqrt{30}\,J_-^2\,(J_3 - 1)$
	-3	$2\sqrt{5}\,J_-^3$
4	4	$\sqrt{70}\,J_+^4$
	3	$-2\sqrt{35}\,J_+^3\,(2J_3 + 3)$
	2	$2\sqrt{10}\,J_+^2\,(7J_3^2 - J^2 + 14J_3 + 9)$
	1	$-\sqrt{5}\,J_+(28J_3^3 - 12J^2J_3 + 42J_3^2 - 6J^2 + 38J_3 + 12)$
	0	$70J_3^4 - 60J^2J_3^2 + 6(J^2)^2 + 50J_3^2 - 12J^2$
	-1	$\sqrt{5}\,J_-(28J_3^3 - 12J^2J_3 - 42J_3^2 + 6J^2 + 38J_3 - 12)$
	-2	$2\sqrt{10}\,J_-^2(7J_3^2 - J^2 - 14J_3 + 9)$
	-3	$2\sqrt{35}\,J_-^3(2J_3 - 3)$
	-4	$\sqrt{70}\,J_-^4$

Table A2. The $\Delta = [00]$ Wigner Operators

DEFINITION AND GENERAL PROPERTIES

$$\left\langle \begin{matrix} 2j & & 0 \\ & j+m+\mu & \end{matrix} \right| \left\langle \begin{matrix} & 0 & \\ k & & -k \\ & \mu & \end{matrix} \right\rangle \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = C^{j\ \ k\ \ \ j}_{m\ \mu\ m+\mu} = (-1)^k\, C^{\ \ j\ \ \ k\ \ \ \ j}_{-m,-\mu,-m-\mu}$$

$$= (-1)^k \left[\frac{(k+\mu)!}{k!} \frac{(k-\mu)!}{k!} \right]^{1/2} \left[\frac{[j-m]_\mu}{[2j]_k} \frac{(j+m+1)_\mu}{(2j+2)_k} \right]^{1/2}$$

$$\times \sum_{s=0}^{k-\mu} (-1)^s \binom{k}{s} \binom{k}{\mu+s} [j+m]_s [j-m-\mu]_{k-\mu-s}$$

$$\left\langle \begin{matrix} & 0 & \\ k & & -k \\ & \mu & \end{matrix} \right\rangle = (-1)^k \left[\frac{(k+\mu)!}{k!} \frac{(k-\mu)!}{k!} \right]^{1/2} J_+^\mu\, P_\mu^k(J^2, J_3) \prod_{s=1}^{k} \left[4J^2 + 1 - s^2 \right]^{-1/2}$$

where $\mu = 0, 1, \ldots k; \quad k = 0, 1, \ldots, 2j$

$$\left(jm \middle| P_\mu^k(J^2, J_3) \middle| jm \right) \equiv \sum_{s=0}^{k-\mu} (-1)^s \binom{k}{s} \binom{k}{\mu+s} [j+m]_s [j-m-\mu]_{k-\mu-s}$$

SPECIAL CASES OF THE POLYNOMIALS

$$P_k^k(J^2, J_3) = 1$$

$$P_{k-1}^k(J^2, J_3) = k(1 - k - 2J_3), \quad k \geq 1$$

$$P_{k-2}^k(J^2, J_3) = 3(k-1)\binom{k}{3} - kJ^2 + \binom{2k}{2} [(k-2)J_3 + J_3^2] \quad , \quad k \geq 2$$

$$P_{k-3}^k(J^2, J_3) = -8\binom{k-1}{2}\binom{k}{4} + 2\binom{k}{2}(k-3)J^2$$

$$+ \binom{k}{2}\left[-\frac{2}{3}(3k^3 - 18k^2 + 32k - 13) + 4J^2\right] J_3$$

$$- \binom{2k}{3}\left[\frac{3}{2}(k-3)J_3^2 + J_3^3\right] \quad , \quad k \geq 3$$

$$P_{k-4}^k(J^2, J_3) = 30\binom{k-1}{3}\binom{k}{5} - \binom{k}{2}(k^3 - 9k^2 + 26k - 22)J^2 + \binom{k}{2}(J^2)^2$$

$$+ \frac{1}{3}\binom{k}{2}(k-4)[2k^4 - 17k^3 + 51k^2 - 59k + 21) - 6(2k-3)J^2]J_3$$

$$+ \frac{1}{3}\binom{k}{2}[(6k^4 - 57k^3 + 182k^2 - 211k + 69) - 6(2k-3)J^2]J_3^2$$

$$+ \binom{2k}{4}[2(k-4)J_3^3 + J_3^4] \quad , \quad k \geq 4$$

Table A3. The Racah Operators

DEFINITION AND GENERAL PROPERTIES

Matrix element form:

$$\langle j_1+\Delta, j_2+\Delta', j| \left\{\begin{matrix} & \Delta & \\ k & & -k \\ & \Delta' & \end{matrix}\right\} |j_1 j_2 j\rangle$$

$$= \langle j, m, \mu+\Delta-\Delta', \kappa+\Delta-\Delta' | \left\{\begin{matrix} & \Delta & \\ k & & -k \\ & \Delta' & \end{matrix}\right\} |jm\rangle$$

$$= [(2j_1+2\Delta+1)(2j_2+1)]^{1/2} W(j, j_1, j_2+\Delta', k; j_2, j_1+\Delta)$$

$$= (-1)^{j+j_1+j_2+\Delta'+k}[(2j_1+2\Delta+1)(2j_2+1]^{1/2} \left\{\begin{matrix} j & j_1 & j_2 \\ j_2+\Delta' & k & j_1+\Delta \end{matrix}\right\}$$

where $\mu=j_1-j_2$ and $\kappa=j_1+j_2+1$ with ranges $\mu=j,j-1,\ldots,-j;\kappa=j+1,j$

Operator form:

$$\left\{\begin{matrix} & \Delta' & \\ k & & -k \\ & \Delta' & \end{matrix}\right\} = N^k_{\Delta'\Delta}(K,H)P^k_{\Delta'\Delta}(\vec{J}^2, K_3, H_3)\ [\bar{D}^k_\Delta(H_3+K_3)\underline{D}^k_{-\Delta'}(H_3-K_3)]^-$$

where

$$N^k_{\Delta'\Delta}(K,H) = (K_\delta)^{|\Delta-\Delta'|}(H_{\delta'})^{|\Delta+\Delta'|}$$

$$\delta = \text{sign}(\Delta-\Delta'), \delta' = \text{sign}(\Delta+\Delta')$$

$$\bar{D}^k_\Delta(H_3+K_3) = [(H_3+K_3-k+\Delta)_{k+\Delta}(H_3+K_3+2\Delta+1)_{k-\Delta}]^{1/2}$$

$$\underline{D}^k_{-\Delta'}(H_3-K_3) = [(H_3-K_3+1)_{k+\Delta'}(H_3-K_3-k+\Delta')_{k-\Delta'}]^{1/2}$$

Special cases of the polynomials $P^k_{\Delta'\Delta}$ are given below.

Operator actions:

$J_+|jm\mu\kappa\rangle=[(j-m)(j+m+1)]^{1/2}|j,m+1,\mu,\kappa\rangle$

$J_-|jm\mu\kappa\rangle=[(j+m)(j-m+1)]^{1/2}|j,m-1,\mu,\kappa\rangle$

$J_3|jm\mu\kappa\rangle=m|jm\mu\kappa\rangle$

$K_+|jm\mu\kappa\rangle=[(j-\mu)(j+\mu+1)]^{1/2}|j,m,\mu+1,\kappa\rangle$

$K_-|jm\mu\kappa\rangle=[(j+\mu)(j-\mu+1)]^{1/2}|j,m,\mu-1,\kappa\rangle$

$K_3|jm\mu\kappa\rangle=\mu|jm\mu\kappa\rangle$

$H_+|jm\mu\kappa\rangle=[(\kappa-j)(\kappa+j+1)]^{1/2}|j,m,\mu,\kappa+1\rangle$

$H_-|jm\mu\kappa\rangle=[(\kappa+j)(\kappa-j-1)]^{1/2}|j,m,\mu,\kappa-1\rangle$

$H_3|jm\mu\kappa\rangle=\kappa|jm\mu\kappa\rangle$

operators in the sets $\{J_+,J_3,J_-\}$ and $\{K_+,K_3,K_-\}$ satisfy standar ular momentum commutation relations; those in the set $\{H_+,H_3,H_-\}$ isfy hyperbolic commutation rules. Operators from different sets mutually commuting.

Relations between operators:

$\vec{J}^2=J_3(J_3+1)+J_-J_+=K_3(K_3+1)+K_-K_+$

$= H_3(H_3-1)-H_+H_-$

$H_3+K_3=(4\vec{J}_1^2+1)^{1/2}\rightarrow 2j_1+1$

$H_3-K_3=(4\vec{J}^2+1)^{1/2}\rightarrow 2j_2+1$

$2\vec{J}_1\cdot\vec{J}_2=\vec{J}^2-(H_3^2+K_3^2-1)/2$

Table A3. (*continued*)

SPECIAL CASES OF THE POLYNOMIALS $(\Delta_1=k+\Delta,\ \Delta_2=k-\Delta)$

$$P^k_{k\Delta}=1,\ \Delta=k,k-1,\ldots,-k$$

$$P^k_{k-1,\Delta}=-2k\vec{J}^2+\Delta_1K_3(K_3-\Delta_2+1)+\Delta_2(H_3-1)(H_3+\Delta_1)$$

$$-(k-1)\leqslant\Delta\leqslant k-1$$

$$\begin{aligned}P^k_{k-2,\Delta}&=k(2k-1)\vec{J}^4-(2k-1)\vec{J}^2[\Delta_1K_3(K_3-\Delta_2+2)\\&+\Delta_2H_3(H_3+\Delta_1-2)-2\Delta_1\Delta_2+2k]+\frac{1}{2}\Delta_1(\Delta_1-1)K_3^4\\&-\Delta_1(\Delta_1-1)(\Delta_2-1)K_3^3+\frac{1}{2}\Delta_1(\Delta_1-1)(\Delta_2^2-7\Delta_2+5)K_3^2\\&+\frac{1}{2}\Delta_1(\Delta_1-1)(3\Delta_2^2-7\Delta_2+2)K_3+\frac{1}{2}\Delta_2(\Delta_2-1)H_3^4\\&+(\Delta_1-2)\Delta_2(\Delta_2-1)H_3^3+\frac{1}{2}\Delta_2(\Delta_2-1)(\Delta_1^2-7\Delta_1+5)H_3^2\\&-\frac{1}{2}\Delta_2(\Delta_2-1)(3\Delta_1^2-7\Delta_1+2)H_3+\Delta_1\Delta_2K_3^2H_3^2\\&-\Delta_1\Delta_2(\Delta_2-2)K_3H_3^2+\Delta_1\Delta_2(\Delta_1-2)K_3^2H_3\\&-\Delta_1(\Delta_1-2)\Delta_2(\Delta_2-2)K_3H_3+\Delta_1(\Delta_1-1)\Delta_2(\Delta_2-1)\end{aligned}$$

$$-(k-2)\leqslant\Delta\leqslant k-2$$

Table A3. (*continued*)

The polynomials $P_0^k \equiv P_{0,0}^k$:

Invariant construction:

$$P_0^k = \sum_{\mu} (-1)^{\mu} \, T_{\mu}^{k}(\vec{J}_1) T_{-\mu}^{k}(\vec{J}_2)$$

Recursion relation:

$$P_0^{k+1} = \frac{2k+1}{k+1} \, [4\,\vec{J}_1 \cdot \vec{J}_2 + k(k+1)] P_0^k$$

$$- \frac{k}{k+1} \, [4\,\vec{J}_1^2 - (k^2-1)] \, [4\,\vec{J}_2^2 - (k^2-1)] P_0^{k-1}, \quad P_0^0 = 1$$

$$P_0^1 = 4\vec{J}_1 \cdot \vec{J}_2$$

$$P_0^2 = 4[6(\vec{J}_1 \cdot \vec{J}_2)^2 + 3\vec{J}_1 \cdot \vec{J}_2 - 2\vec{J}_1^2 \vec{J}_2^2]$$

$$P_0^3 = 4^2 [10(\vec{J}_1 \cdot \vec{J}_2)^3 + 20(\vec{J}_1 \cdot \vec{J}_2)^2$$

$$-2(\vec{J}_1 \cdot \vec{J}_2)(3\vec{J}_1^2 \vec{J}_2^2 - \vec{J}_1^2 - \vec{J}_2^2 - 3) - 5\vec{J}_1^2 \vec{J}_2^2]$$

$$P_0^4 = 4^2 [70(\vec{J}_1 \cdot \vec{J}_2)^4 + 350(\vec{J}_1 \cdot \vec{J}_2)^3 + 2(\vec{J}_1 \cdot \vec{J}_2)^2 (-30\vec{J}_1^2 \vec{J}_2^2 +$$

$$+ 25\vec{J}_2^2 + 195) + 2(\vec{J}_1 \cdot \vec{J}_2)(-85\vec{J}_1^2 \vec{J}_2^2 + 30\vec{J}_1^2 + 30\vec{J}_2^2 + 45$$

$$+ 3\vec{J}_1^2 \vec{J}_2^2 (2\vec{J}_1^2 \vec{J}_2^2 - 4\vec{J}_1^2 - 4\vec{J}_2^2 - 27)$$

List of Symbols

Many of the symbols used in this volume are given below together with their meanings. Specialized symbols occurring in some Notes and Appendices are left out. Considerable overlap with symbols listed in AMQP is given for continuity.

Signs of Relation

$\equiv$	definition; identically equal to; congruent (in modulo relations)
$\simeq$	approximately equal to
$\approx$	order of magnitude
$\rightarrow$	tends to; approaches; yields; is replaced by
$\sim$	asymptotically equal to, approximately equal to for a large value of a variable; equivalent to (in equivalence relations)
$\cong$	isomorphic to
$\propto$	proportional to
$<$	less than
$>$	greater than
$\leqslant$	less than or equal to
$\geqslant$	greater than or equal to
$\subset$	a subset of, contained in
$\supset$	contains as a subset
$\in$	an element of
$\notin$	not an element of
$\leftrightarrow$	exchange; one-to-one correspondence
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
$\perp$	perpendicular

Operations

$\times$	times; direct product of groups
(σ)	semidirect product of groups, automorphism σ

$\times$	cross (vector) product of vectors in $\mathbb{R}^3$; coupling of irreducible tensor operators
$\cdot$	dot or scalar product of vectors
$\pm$	plus or minus
$\mp$	minus or plus
$\lvert z \rvert$	absolute value
$\oplus$	direct sum of vector spaces
$\otimes$	tensor product of vector spaces or operators; direct or Kronecker product of matrices
$\circ$	composition of functions
$\cup$	union of sets
$\cap$	intersection of sets
$\blacksquare$	conclusion of proof
A^*	complex conjugation of a matrix A
$\tilde{A}$	transposition of a matrix A
$A^\dagger$	Hermitian conjugation of a matrix A
A^{-1}	inverse of a matrix A
$\operatorname{tr} A$	trace of a matrix A
$\det A$	determinant of a matrix A
∂	partial derivative
∇	gradient operator in $\mathbb{R}^3$
∇^2	Laplacian in $\mathbb{R}^3$
∇_4	gradient operator in $\mathbb{R}^4$
∇_4^2	Laplacian in $\mathbb{R}^4$

Sets and Groups

$\mathbb{Z}, \mathbb{Z}^+$	set of all integers, set of nonnegative integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	set of real n-tuples; n-dimensional real vector space
$\mathbb{C}^n$	set of complex n-tuples; n-dimensional complex vector space
$\hat{\mathbb{C}}^{2j+1}$	projective space corresponding to $\mathbb{C}^{2j+1}$
$E(3)$	Euclidean 3-space
Q	four-dimensional vector space of quaternions
$\varnothing$	empty set
S^2	unit sphere in $\mathbb{R}^3$
S^3	unit sphere in $\mathbb{R}^4$

$SO(n)$	group of $n\times n$ real, orthogonal matrices of determinant $+1$, $n=2,3,\ldots$
$O(n)$	group of $n\times n$ real, orthogonal matrices, $n=2,3,\ldots$
$SU(n)$	group of $n\times n$ unitary matrices of determinant $+1$, $n=2, 3,\ldots$
$U(n)$	group of $n\times n$ unitary matrices, $n=1,2,\ldots$
$SU(2)\times SU(2)$	direct product of $SU(2)$ with $SU(2)$
$SU(2)*SU(2)$	realization of the group $SU(2)\times SU(2)$ in which the Casimir operators of the two groups coincide
$SL(n,\mathbb{C})$, $SL(n,\mathbb{R})$	group of complex (real) $n\times n$ matrices of determinant 1
$SU(m,n)$	subgroup of $SL(m+n,\mathbb{C})$ that leaves invariant the Hermitian form $\|z_1\|^2+\cdots+\|z_m\|^2-\|z_{m+1}\|^2-\cdots-\|z_{m+n}\|^2$
$SO(m,n)$	subgroup of $SL(m+n,\mathbb{R})$ that leaves invariant the (real) quadratic form $x_1^2+\cdots+x_m^2-x_{m+1}^2-\cdots-x_{m+n}^2$
G_2, F_4	exceptional Lie group of rank 2 (rank 4)
$PG(n,2)$	n-dimensional projective space over the field $F_2=\{0,1\}$
S_n	group of permutations of n objects, symmetric group
G	generic group; graph
G/K	right cosets of G with respect to a subgroup K
Z_2	two element group containing the inversion and the identity
T	group of rotations mapping a regular tetrahedron into itself
T_h	direct product group $T\times Z_2$
T_d	group of rotation-inversions mapping a regular tetrahedron into itself
O	group of rotations mapping a regular octahedron into itself
O_h	group of rotation-inversions mapping a regular octahedron into itself
C_n	cyclic group of rotations about an n-fold symmetry axis
$\mathbf{T}^J$	irreducible tensor operator of rank J with elements indexed by $M=J,\ldots,-J$
$\{x\in X: P(x)\}$	set of x in X with property $P(x)$

$f: X \to Y$	function (or mapping) f with domain X and range in Y
$f: x \to y = f(x)$	function f mapping $x \in X$ to $y \in Y$

Vectors and Operations in Euclidean Three-Space

$\vec{a}, \vec{b}, \ldots$	generic vectors with common origin in Euclidean 3-space, $E(3)$
$\vec{r}, \vec{s}, \vec{x}, \vec{y}, \ldots$	position vectors in $E(3)$ ($\vec{r}$ is used synonymously with $\vec{x}$, z-axis with x_3-axis)
$\vec{p}$	linear momentum vector in $E(3)$
$\hat{n}, \hat{e}_i, \ldots$	unit vectors in $E(3)$
$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$	right-handed triad of orthogonal unit vectors in $E(3)$, inertial frame
$\vec{x} = \Sigma_i x_i \hat{e}_i$	representation of a vector $\vec{x}$ in terms of the components (x_1, x_2, x_3) and the frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$
$\mathbf{x} = (x_1, x_2, x_3)$	representation of a vector $\vec{x}$ as a 3-tuple in $\mathbb{R}^3$ ($\mathbf{x}$ is also called a vector)
$\mathbf{p} = (p_1, p_2, p_3)$	representation of linear momentum vector $\vec{p}$ as a 3-tuple in $\mathbb{R}^3$
$\vec{a} \cdot \vec{b} = \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$	dot or scalar product of two vectors
$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\hat{e}_1 + (a_3 b_1 - a_1 b_3)\hat{e}_2 + (a_1 b_2 - a_2 b_1)\hat{e}_3$	cross or vector product of vector $\vec{a}$ with vector $\vec{b}$
$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$	cross or vector product of vector $\mathbf{a}$ with vector $\mathbf{b}$
$a = \lVert \mathbf{a} \rVert = (\mathbf{a} \cdot \mathbf{a})^{1/2}$	length of vector $\mathbf{a}$
$\vec{L} = \vec{x} \times \vec{p} = \vec{r} \times \vec{p}$, $\mathbf{L} = \mathbf{x} \times \mathbf{p} = \mathbf{r} \times \mathbf{p}$	angular momentum about point O of a (point) particle located at position $\vec{x} = \vec{r}$ and having linear momentum $\vec{p}$ (equivalently at position $\mathbf{x}$ with linear momentum $\mathbf{p}$)
$\mathcal{R}(\phi, \hat{n}): \vec{r} \to \vec{r}\,' = \mathcal{R}(\phi, \hat{n})\vec{r}$	action of a rotation by angle ϕ about direction $\hat{n}$ on a vector $\vec{r}$
$R: \mathbf{x} \to \mathbf{x}' = R\mathbf{x}$	action of a rotation $R \in SO(3)$ on a vector $\mathbf{x} \in \mathbb{R}^3$
$\mathcal{I}: \mathbf{x} \to \mathbf{x}' = -\mathbf{x}$	action of spatial inversion $\mathcal{I}$ on a vector $\mathbf{x} \in \mathbb{R}^3$
$\boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}$	Cartan spinor associated with an isotropic vector $\mathbf{x} \in \mathbb{C}^3$, $\mathbf{x} \cdot \mathbf{x} = 0$
$R: \boldsymbol{\xi} \to \boldsymbol{\xi}' = \pm U\boldsymbol{\xi}$	rotation of a Cartan spinor corresponding to the rotation $\mathbf{x}' = R\mathbf{x}$ of $\mathbb{R}^3$
$(\hat{\xi}_{+1}, \hat{\xi}_0, \hat{\xi}_{-1})$, $(\hat{\eta}_{+1}, \hat{\eta}_0, \hat{\eta}_{-1})$	spherical basis vectors in $\mathbb{R}^3$

Rotations and Rotation Operators

$\mathcal{R}$	generic rotation about a common origin in $E(3)$
$\mathcal{R}^{-1}$	inverse rotation to $\mathcal{R}$
$\mathcal{R}(\phi, \hat{n})$	rotation by angle ϕ about direction $\hat{n}$
E	identity rotation
$\mathcal{R}_{2\pi}, \mathcal{R}_{4\pi}$	rotations by 2π and 4π about any axis
$\mathrm{R}=(R_{ij})$	generic 3×3 real, proper orthogonal matrix corresponding to rotation $\mathcal{R}$
$R(\phi, \hat{n})$	parametrization of R in terms of angle of rotation ϕ about direction $\hat{n}$
$R(\alpha\beta\gamma)$	parametrization of R in terms of Euler angles
$R(\alpha_0, \boldsymbol{\alpha})$	parametrization of R in terms of Euler–Rodrigues parameters $(\alpha_0, \boldsymbol{\alpha})\in S^3$
$U=(u_{ij})$	generic 2×2 unitary or unitary unimodular matrix; element of quantal rotation group $SU(2)$ corresponding to rotation $R\in SO(3)$
$U(\psi, \hat{n})$	parametrization of U (unimodular) in terms of angle of rotation ψ about direction $\hat{n}$
$U(\alpha\beta\gamma)$	parametrization of U (unimodular) in terms of Euler angles
$U(\alpha_0, \boldsymbol{\alpha})$	parametrization of U (unimodular) in terms of Euler–Rodrigues parameters $(\alpha_0, \boldsymbol{\alpha})\in S^3$
$\mathcal{U}$	rotation operator; unitary operator realization in Hilbert space of rotation $U\in SU(2)$
$\mathcal{U}^{-1}$	inverse rotation operator
$\mathcal{U}(\psi, \hat{n})=e^{-i\psi\hat{n}\cdot\mathbf{J}}$	parametrization of $\mathcal{U}$ in terms of angle of rotation ψ about direction $\hat{n}$
$\mathcal{U}(\alpha\beta\gamma)$	parametrization of $\mathcal{U}$ in terms of Euler angles
$\mathcal{U}(\alpha_0, \boldsymbol{\alpha})$	parametrization of $\mathcal{U}$ in terms of Euler–Rodrigues parameters $(\alpha_0, \boldsymbol{\alpha})\in S^3$

Domains of Parameters of Rotations

$\{(\phi, \hat{n}): 0\leqslant\phi\leqslant\pi, \hat{n}\cdot\hat{n}=1, (\pi, \hat{n})\equiv(\pi, -\hat{n})\}$	parameters of a real, proper orthogonal rotation of vectors in $\mathbb{R}^3$ in terms of the angle of rotation ϕ about direction $\hat{n}$

$\{(\psi, \hat{n}): 0 \leqslant \psi \leqslant 2\pi, \hat{n} \cdot \hat{n} = 1\}$ parameters of a unitary unimodular rotation of spinors in $\mathbb{C}^2$ in terms of the angle of rotation ψ about direction $\hat{n} \in \mathbb{R}^3$

$\{(\alpha\beta\gamma): 0 \leqslant \alpha < 2\pi, 0 \leqslant \beta \leqslant \pi, 0 \leqslant \gamma < 2\pi\}$ Euler angle parameters of a rotation of vectors in $\mathbb{R}^3$

$\{(\alpha\beta\gamma): 0 \leqslant \alpha < 2\pi, 0 \leqslant \beta \leqslant \pi$ or $2\pi \leqslant \beta \leqslant 3\pi, 0 \leqslant \gamma < 2\pi\}$ Euler angle parameters of a unitary unimodular rotation of spinors in $\mathbb{C}^2$

$\{(\alpha_0, \boldsymbol{\alpha}): \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, (-\alpha_0, -\boldsymbol{\alpha}) \equiv (\alpha_0, \boldsymbol{\alpha})\}$ Euler–Rodrigues parameters of a real, proper orthogonal rotation of vectors in $\mathbb{R}^3$

$\{(\alpha_0, \boldsymbol{\alpha}): \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\}$ Euler–Rodrigues parameters of a unitary unimodular rotation of spinors in $\mathbb{C}^2$

Hilbert Space

$\mathcal{H}$	generic Hilbert space
$\mathcal{H}_j$	$(2j+1)$-dimensional Hilbert space, which is a carrier space for an irreducible representation of $SU(2)$
$\|(\alpha)jm\rangle, \|jm\rangle$, $m = j, \ldots, -j$	orthonormal basis of $\mathcal{H}_j$ on which the angular momentum $\mathbf{J}$ has the standard action
$\|\psi\rangle, \|\Psi\rangle, \|\phi\rangle, \|\Phi\rangle, \ldots$	ket vector notation for vectors in Hilbert space
$\langle\psi\|, \langle\Psi\|, \langle\phi\|, \langle\Phi\|, \ldots$	bra vector notation for vectors in the dual space
$\psi, \Psi, \phi, \Phi, f, g, \ldots$	function notation for vectors in Hilbert space
$(\psi, \phi), \langle\psi, \phi\rangle, \langle\psi\|\phi\rangle, \ldots$	inner or scalar product of vectors in Hilbert space
$e_1, e_2, \ldots$	unit vectors in $\mathcal{H}$

$\mathbf{f} = \{f\alpha: f$ is a fixed element of $\mathcal{H}$; $\alpha \in \mathbb{C}$ with $|\alpha| = 1\}$ definition of ray $\mathbf{f}$ with representative $f \in \mathcal{H}$

$\mathbf{f}, \mathbf{g}, \ldots$	rays in $\mathcal{H}$
$\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots$	unit rays in $\mathcal{H}$
$(\mathbf{f}, \mathbf{g}) \equiv \|(f, g)\|$	inner product of rays in $\mathcal{H}$
$P(f, g) \equiv \|\langle f\|g\rangle\|^2 = P(g, f)$	transition probability between states $\|f\rangle$ and $\|g\rangle$
$\\|\psi\\| = (\psi, \psi)^{\frac{1}{2}}$	norm of a vector (function) ψ
$\langle\mathbf{x}\|\psi\rangle = \psi(\mathbf{x})$	bra-ket vector notation for the value in $\mathbb{C}$ of a ket vector $\|\psi\rangle$ with domain $\mathbb{R}^3$
$\mathbf{1}, 1$	unit operator
$\mathcal{O}$	generic operator in Hilbert space

$T: \mathcal{H} \to \mathcal{H}'$	symmetry map (one-to-one onto map of unit rays)
$T(\mathbf{f})$	extension of T to an arbitrary ray $\mathbf{f}$
$\mathcal{U}: \mathcal{H} \to \mathcal{H}'$	unitary or anti–unitary extension of T to vectors in Hilbert space
$f \xrightarrow{\mathcal{U}} f' = \mathcal{U}(f) \in \mathcal{H}'$	image in $\mathcal{H}'$ of $f \in \mathcal{H}$ under $\mathcal{U}$
$\langle \mathcal{U}(f), \mathcal{U}(g) \rangle$	inner product of vectors in $\mathcal{H}'$
$\mathcal{O}_R$	symmetry map in $\mathcal{H}$ corresponding to a rotation $R \in SO(3)$ in $\mathbb{R}^3$
$\mathcal{O}_R \mathcal{O}_S = \omega_{R,S} \mathcal{O}_{RS}$	multiplication rule for symmetry maps $\{\mathcal{O}_R\}$
$\mathcal{O}_U$	symmetry map in $\mathcal{H}$ corresponding to a unitary rotation $U \in SU(2)$ (quantal rotation group)
$\mathcal{O}_U \mathcal{O}_V = \mathcal{O}_{UV}$	group multiplication law for symmetry maps $\{\mathcal{O}_U\}$
$(\mathcal{O}_U f)(R\mathbf{x}) = f(\mathbf{x})$	action of symmetry map $\mathcal{O}_U$ on functions defined on $\mathbb{R}^3$, where $\pm U \to R$ in the homomorphism of $SU(2)$ onto $SO(3)$
$\mathcal{U}^{(j)} \xi_m = \sum_{m'} D^j_{m'm}(U) \xi_{m'}$	action of unitary operator $\mathcal{U}^{(j)}$ on spin-j basis vectors
$\Psi(\mathbf{x}) = \sum_m \psi_m(\mathbf{x}) \xi_m$	generic state vector for spin-j particle
$(\mathcal{O}_U \Psi)(\mathbf{x}) = \sum_m \psi_m(R^{-1}\mathbf{x})(\mathcal{U}^{(j)} \xi_m)$	action of symmetry map $\mathcal{O}_U$ on state vector Ψ
$(\Psi, \Phi) = \sum_m \int \psi_m^*(\mathbf{x}) \phi_m(\mathbf{x})\, d\mathbf{x}$	inner product of state vectors Ψ and Φ of a spin-j particle
$\mathcal{T}$	time reversal operator
$\mathcal{T}\mathbf{x} = \mathbf{x}\mathcal{T}$, $\mathcal{T}\mathbf{p} = -\mathbf{p}\mathcal{T}$, $\cancel{\mathcal{T}\mathbf{J} = -\mathbf{J}\mathcal{T}}$	action of time reversal operator on ~~physical observables **x**, **p**, and **J**~~
$K_0 \Psi = \Psi^*$	action of antilinear, anti-unitary complex conjugation operator K_0 on state vectors
$\mathcal{T} = \mathcal{U} K_0$	decomposition of time reversal operator into a unitary operator $\mathcal{U}$ and the conjugation operator K_0
$U^* = U_0^\dagger U U_0$, $U_0 = -i\sigma_2$	inner automorphism of $SU(2)$ corresponding to complex conjugation
$D^{j*}(U) = A^\dagger(\mathcal{U}_0) D^j(U) A(\mathcal{U}_0)$	unitary equivalence of complex conjugate irrep $[j]^*$ and irrep $[j]$ of $SU(2)$
$A(\mathcal{U}_0) = cD^j(U_0)$, $\lvert c \rvert = 1$	general form of the unitary matrix $A(\mathcal{U}_0)$
$\mathcal{T}\lvert(\alpha)jm\rangle = (-1)^{j-m}\lvert(\alpha)j, -m\rangle$	action of time reversal operator on an angular momentum eigenket

$\mathcal{O}_U \mathcal{T} = \mathcal{T} \mathcal{O}_U$	commutation property of time reversal operator and symmetry operators $\{\mathcal{O}_U\}$ of the quantal rotation group
$\overline{\mathcal{H}}$	projective space corresponding to $\mathcal{H}$
$V^{\perp}$	orthogonal complement in $\mathcal{H}$ of a vector space $V \subset \mathcal{H}$
$\lambda: V \to V'$	semilinear map of a vector space $V \subset \mathcal{H}$ into a vector space $V' \subset \mathcal{H}$
$D(u), D(a)$	representations of unitary (u) and anti-unitary (a) elements of a group G in an n-dimensional (complex) Hilbert space
$M(u), M(a)$	real representation of G realized over $2n$-dimensional real Hilbert space
$I_{\mathrm{FS}}(\lambda) \equiv \left[\int_G dg \,\mathrm{tr}\left(D^{\lambda}(g^2) \right) \right] \times (\text{group volume})^{-1}$	Frobenius–Schur invariant
$\mathcal{O}_g: \psi(\mathbf{x}, t) \to \psi'(\mathbf{x}', t') = e^{if(x_1, t)}\psi(\mathbf{x}, t)$, $f(x_1, t) = \frac{m}{\hbar}\left(vx_1 + \frac{v^2 t}{2} \right)$	transformation of a wave function under the operator $\mathcal{O}_g$ corresponding to a Galilean boost g
$\langle(\alpha')j'm'\|(\alpha)jm\rangle$	bra-ket notation for inner product of angular momentum basis vectors
$\langle(\alpha')j'm'\|\mathcal{O}\|(\alpha)jm\rangle$	matrix elements of an operator, $\mathcal{O}$, in the angular momentum basis
$\sum_j \oplus \mathcal{H}_j$	direct sum of Hilbert spaces $\mathcal{H}_{\frac{1}{2}}, \mathcal{H}_1, \ldots$
$(j, m), (J, M), (l, m),$ $(L, M), (S, M_S), (k, \mu), \ldots$	angular momentum labels of vectors or operators in Hilbert space
$\mathcal{H} \otimes \mathcal{H}$	tensor product of generic Hilbert spaces
$\phi \otimes \psi$	tensor product of vectors $\phi \in \mathcal{H}$ and $\psi \in \mathcal{H}$
$S \otimes T$	tensor product of operator S acting in $\mathcal{H}$ with operator T acting in $\mathcal{H}$
$\mathcal{H}_{j_1}(1) \otimes \mathcal{H}_{j_2}(2)$	tensor product of angular momentum carrier spaces $\mathcal{H}_{j_1}(1)$ and $\mathcal{H}_{j_2}(2)$
$\mathbf{1}(i)$	unit operator in $\mathcal{H}_{j_i}(i)$
$U: T^J_M \to \mathcal{O}_U T^J_M \mathcal{O}_U^{-1} = \sum_{M'} D^J_{M'M}(U) T^J_{M'}$	transformation of an irreducible tensor operator $\mathbf{T}^J$ under the action of $U \in SU(2)$
$\|j_1 m_1;\ j_2 m_2\rangle = \|j_1 m_1\rangle \|j_2 m_2\rangle = \|j_1 m_1\rangle \otimes \|j_2 m_2\rangle$	uncoupled angular momentum basis of $\mathcal{H}_{j_1}(1) \otimes \mathcal{H}_{j_2}(2)$
$\|(j_1 j_2) jm\rangle$	coupled angular momentum basis of $\mathcal{H}_{j_1}(1) \otimes \mathcal{H}_{j_2}(2)$

$\langle \Psi \| \mathcal{O} \| \Phi \rangle$	generic matrix element of an operator $\mathcal{O}$ between states $\|\Psi\rangle$ and $\|\Phi\rangle$
$e^{-i\phi\hat{n}\cdot\mathbf{L}}\Psi(\mathbf{x}) = \Psi(R^{-1}(\phi, \hat{n})\mathbf{x})$	action of orbital rotation operator on analytic functions defined on $\mathbb{R}^3$
$(U, V) \to \mathcal{O}_{(U,V)}$	representation of $SU(2)\times SU(2)$ by unitary operators acting in $\mathcal{H}\otimes\mathcal{H}$
$\mathcal{O}_{(U,V)}\|(j_1 j_2)jm\rangle = \sum_{j'm'} D^{(j_1 j_2)}_{j'm';jm}(U,V)\|(j_1 j_2)j'm'\rangle$	action of $\mathcal{O}_{(U,V)}$ on coupled angular momentum basis of $\mathcal{H}\otimes\mathcal{H}$
$(U,V) \to D^{(j_1 j_2)}(U,V) = D^{j_1}(U)\otimes D^{j_2}(V)$	direct product irrep of $SU(2)\times SU(2)$
$D^{(j_1 j_2)}_{j'm';jm}(U,U) = \delta_{j'j}\varepsilon_{j_1 j_2 j} D^{j}_{m'm}(U)$	reduction property of irrep $D^{(j_1 j_2)}$ under the restriction $(U,V)\to(U,U)$ to the diagonal subgroup
$[j] = D^j(U)$	irrep of $SU(2)$
$[j_1]\otimes[j_2] = \Sigma_j \oplus \varepsilon_{j_1 j_2 j}[j]$	Clebsch–Gordan series for $SU(2)$
$\mathfrak{D}(T)$	domain of a densely defined operator on a Hilbert space $\mathcal{H}$
$\mathfrak{D}(T^*) = \{f\in\mathcal{H}$: there exists an	$h\in\mathcal{H}$ such that $(f,Tg)=(h,g)$ for all $g\in\mathfrak{D}(T)\}$
T^*	adjoint of T; operator defined on $\mathcal{H}$ by $T^*f = h$, each $f\in\mathfrak{D}(T^*)$
T Hermitian (symmetric)	$\mathfrak{D}(T)\subset\mathfrak{D}(T^*)$ and $Tf = T^*f$ for all $f\in\mathcal{H}$
$T = T^*$	self-adjoint operator; T is symmetric and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$

Angular Momentum and Tensor Operators

$\mathbf{L} = (L_1, L_2, L_3)$	orbital angular momentum operator with action defined on vectors in Hilbert space; generators of the group $SO(3)$; basis of the Lie algebra of $SO(3)$; total orbital angular momentum operator
$\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2 = \mathbf{L}\cdot\mathbf{L}$	square of the orbital angular momentum; Casimir operator for the group $SO(3)$
$\mathbf{J} = (J_1, J_2, J_3)$	generic angular momentum operator with action defined on vectors in Hilbert space; generators of the group $SU(2)$; basis of the Lie algebra of $SU(2)$; total angular momentum
$\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 = \mathbf{J}\cdot\mathbf{J}$	square of the angular momentum $\mathbf{J}$; Casimir operator for the group $SU(2)$

$\mathbf{L}\times\mathbf{L}=i\hbar\mathbf{L}$	vector form of the commutation relations for orbital angular momentum $\mathbf{L}$
$\mathbf{J}\times\mathbf{J}=i\mathbf{J}$	vector form of the commutation relations for the dimensionless generators of the quantum rotation group, $SU(2)$
$\mathfrak{L}=(\mathfrak{L}_1,\mathfrak{L}_2,\mathfrak{L}_3),\mathfrak{L}^2$	differential operator realization of orbital angular momentum
$\mathcal{J}=(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3)$ $\mathcal{K}=(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3)$ $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$	differential operator realizations of angular momentum
$J_\pm=J_1\pm iJ_2,\ L_\pm=L_1\pm iL_2$, etc.	raising and lowering operators; complex extension of the Lie algebra of $SO(3)$ or $SU(2)$
$J_{+1}\equiv-\frac{1}{\sqrt{2}}J_+,\ J_0\equiv J_3,\ J_{-1}\equiv\frac{1}{\sqrt{2}}J_-$	spherical components of angular momentum
$\mathbf{1}^{(j)}$	$(2j+1)\times(2j+1)$ unit matrix
$\sigma_0=\mathbf{1}^{(\frac{1}{2})}$	2×2 unit matrix
$\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$	Pauli matrices
$\mathbf{x}\cdot\boldsymbol{\sigma}=x_1\sigma_1+x_2\sigma_2+x_3\sigma_3$,	Cartan map $\mathbf{x}\to\mathbf{x}\cdot\boldsymbol{\sigma}$ of points $\mathbf{x}\in\mathbb{R}^3$ onto 2×2 traceless Hermitian matrices
$\hat{n}\cdot\boldsymbol{\sigma}/2$	component of spin $\boldsymbol{\sigma}/2$ in direction $\hat{n}$
$\mathbf{a}\cdot\mathbf{J}=a_1J_1+a_2J_2+a_3J_3$	general element in the Lie algebra of $SU(2)$; dot product of a vector $\mathbf{a}\in\mathbb{R}^3$ with the angular momentum operator $\mathbf{J}$
$\mathbf{a}\times\mathbf{J}=(a_2J_3-a_3J_2,a_3J_1-a_1J_3,a_1J_2-a_2J_1)$	vector product of a vector $\mathbf{a}\in\mathbb{R}^3$ and the angular momentum operator $\mathbf{J}$
$\mathbf{J}^2\lvert jm\rangle=j(j+1)\lvert jm\rangle$ $J_3\lvert jm\rangle=m\lvert jm\rangle$ $J_\pm\lvert jm\rangle=[(j\mp m)(j\pm m+1)]^{\frac{1}{2}}\lvert jm\pm1\rangle$	standard action of angular momentum operators in the space $\mathcal{H}_j$
$\mathbf{J}(i),\mathbf{J}_i$	angular momentum of "part" i of a composite system, $i=1,2,\ldots$
$T^{k_1}_{\mu_1}(1)\otimes\mathbf{1}(2),\mathbf{1}(1)\otimes T^{k_2}_{\mu_2}(2)$	tensor product of operators with action in the tensor product space $\mathcal{H}_{j_1}(1)\otimes\mathcal{H}_{j_2}(2)$
$\mathcal{T}^k_\mu(\mathbf{J})$	tensor harmonic; μth component of an irreducible tensor operator of rank k belonging to the enveloping algebra of the Lie algebra of $SU(2)$
C_i,Ω	operations of forming the commutator of J_i, respectively, $\mathbf{J}^2$ with a tensor operator

$J_i^{(j_1)}\otimes\mathbf{1}^{(j_2)}, \mathbf{1}^{(j_1)}\otimes J_i^{(j_2)}$	matrix representations of the operators $J_i\otimes\mathbf{1}(2)$ and $\mathbf{1}(1)\otimes J_i$ in the space $\mathcal{H}_{j_1}(1)\otimes\mathcal{H}_{j_2}(2)$
(V_{+1}, V_0, V_{-1})	spherical components of a vector operator $\mathbf{V}$
$\mathbf{T}^J, \mathbf{S}^J$	generic symbol for irreducible tensor operator of rank J
T_M^J: $M=J, J-1,\ldots,-J$	components of $\mathbf{T}^J$ (also called irreducible tensor operators)
$\overline{\mathbf{T}}^J, \overline{T}_M^J$	generic symbol for a conjugate irreducible tensor operator of rank J
$[K_1, K_2]=-iK_3$, $[K_3, K_1]=iK_2$, $[K_2, K_3]=iK_1$	commutation rules for hyperbolic angular momentum $\mathbf{K}=(K_1, K_2, K_3)$; generators of the noncompact group $SU(1,1)$
$\mathbf{K}^2=-K_1^2-K_2^2+K_3^2$	Casimir operator of $SU(1,1)$
$[K_3, K_\pm]=\pm K_\pm$, $[K_+, K_-]=-2K_3$	commutation relations for hyperbolic angular momentum in raising-lowering operator form ($K_\pm=K_1\pm iK_2$)
$\mathbf{K}^2=K_3(K_3+1)-K_-K_+=K_3(K_3-1)-K_+K_-$	Casimir operator of $SU(1,1)$ in terms of raising-lowering operators
$K_\pm=e^{\mp i\theta}\left(i\frac{\partial}{\partial\theta}\pm x\frac{\partial}{\partial x}-\frac{x}{2}\pm\frac{1}{2}\right)$, $K_3=i\frac{\partial}{\partial\theta}$	realization of hyperbolic angular momentum in radial integral problem
$[H_3, H_\pm]=\pm H_\pm$, $[H_+, H_-]=-2H_3$	notation for hyperbolic angular momentum occurring in Chapter 4 and Topic 6

$H_+\Phi_{jm}=[(m-j)(m+j+1)]^{\frac{1}{2}}\Phi_{j,m+1}$, $H_-\Phi_{jm}=[(m-j-1)(m+j)]^{\frac{1}{2}}\Phi_{j,m-1}$, $H_3\Phi_{jm}=m\Phi_{jm}$;	action of hyperbolic angular momentum operators on associated Laguerre functions (see definition of Φ_{jm} under *Functions*)

$$\mathbf{H}^2\Phi_{jm}=[H_3(H_3-1)-H_+H_-]\Phi_{jm}=j(j+1)\Phi_{jm},$$

$$H_-\Phi_{j,j+1}=0$$

$H_1=(\mathbf{p}^2-\mathbf{x}^2)/4$, $H_2=(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x})/4$, $H_3=(\mathbf{p}^2+\mathbf{x}^2)/4$	N-dimensional harmonic oscillator realization of hyperbolic angular momentum; generators of $SU(1,1)$ noninvariance group of oscillator

$\mathbf{T}^{(J_1J_2)}$	$SU(2)\times SU(2)$ irreducible tensor operator
$\{T^{J_1J_2}_{M_1M_2}: M_i = J_i, J_i-1, \ldots, -J_i\}$	components of $\mathbf{T}^{(J_1J_2)}$ classified by transformation properties under $SU(2)$ subgroups $\{(U,\mathbf{1})\}$ and $\{(\mathbf{1},U)\}$
$\left\{ {}^{(J_1J_2)}T^J_M: \begin{matrix} J=\lvert j_1-J_2\rvert,\ldots,J_1+J_2 \\ M=J, J-1,\ldots,-J \end{matrix} \right\}$	components of $\mathbf{T}^{(J_1J_2)}$ classified by transformation properties under the diagonal subgroup (U,U)
${}^{(J_1J_2)}T^J_M = \sum\limits_{M_1M_2} C^{J_1J_2J}_{M_1M_2M} T^{J_1J_2}_{M_1M_2}$	relationship between the two sets of components of $\mathbf{T}^{(J_1J_2)}$
$[\mathbf{S}^{k_1}\times\mathbf{T}^{k_2}]^k$	irreducible tensor operator of rank k obtained by Wigner coupling of two irreducible tensor operators having a common Hilbert space as domain
$[\mathbf{S}^{k_1}\times\mathbf{T}^{k_2}]^k_\mu$	μth component of $[\mathbf{S}^{k_1}\times\mathbf{T}^{k_2}]^k$
$\mathbf{T}^{k_1}(1)\otimes\mathbf{T}^{k_2}(2)$	tensor product of two irreducible tensor operators with action in $\mathcal{H}_{j_1}(1)\otimes\mathcal{H}_{j_2}(2)$
$\langle(\alpha')j'm'\lvert T^J_M\rvert(\alpha)jm\rangle$	matrix element in an angular momentum basis of an irreducible tensor operator
$\langle(\alpha')j'\lVert\mathbf{T}^J\rVert(\alpha)j\rangle$	Condon–Shortley notation for a reduced matrix element; matrix element of the $SU(2)$ invariant operator corresponding to $\mathbf{T}^J$
$J_+\to-\frac{1}{2}a^2, J_-\to\frac{1}{2}\bar{a}^2,$ $J_3\to\frac{1}{4}(a\bar{a}+\bar{a}a)$	symplecton realization of the generators of $SU(2)$
$\mathcal{P}^m_j(a,\bar{a}) = \dfrac{\alpha_j}{2^{j-m}}\binom{2j}{j+m}^{\frac{1}{2}}\sum\limits_s \dfrac{\bar{a}^{j-m-s}a^{j+m}\bar{a}^s}{(j-m-s)!s!}$ $(j=0,\frac{1}{2},1,\ldots;m=j,j-1,\ldots,-j)$	characteristic eigenpolynomials for symplecton realization of $SU(2)$
$\mathrm{adj}:(a,\bar{a})\to(\bar{a},-a)$	adjoint operation for symplecton
$(\mathcal{P}^m_j)^{\mathrm{adj}}=(-1)^{j-m}\mathcal{P}^{-m}_j$	transformation of characteristic eigenpolynomials under the adjoint operation
$\mathbf{S}^{(\delta)}\Omega=\mathbf{S}^{(\delta)}\omega_\delta$	characteristic equation for the commutation operation $\Omega\mathbf{T}=[\mathbf{T},\mathbf{J}^2]$
$\mathbf{S}^{(\delta)}\equiv\mathbf{T}P_\delta,\ \delta=k,k-1,\ldots,-k$	characteristic tensors defined by projection operator method
$P_\lambda = \prod\limits_{\substack{\delta=-k \\ \delta\neq\lambda}}^{k} \dfrac{\Omega-\omega_\delta}{\omega_\lambda-\omega_\delta}$	projection operator associated with Ω
$\omega_\delta=-\delta(\delta+dim)$	definition of invariant operator eigenvalues of Ω
$dim\lvert\, jm\rangle=(2j+1)\lvert\, jm\rangle$	definition of dimension operator

Angular Momentum Coefficients

ε_{abc} characteristic function defined by the Clebsch–Gordan series; 1 if irrep $[c]$ is contained in the direct product $[a]\otimes[b]$, and otherwise 0

$C^{j_1 j_2 j}_{m_1 m_2 m}$ Wigner coefficient; Clebsch–Gordan coefficient; vector addition coefficient (see Section 12, Chapter 3, AMQP)

$INT(J\otimes j;\ j+\Delta)=\varepsilon_{j,J,j+\Delta}\equiv\begin{cases}1 & \text{for } 2j\geqslant J-\Delta,\\ 0 & \text{for } 2j<J-\Delta\end{cases}$ intertwining number; number of times irrep $[j+\Delta]$ is contained in $[J]\otimes[j]$

$\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & -m\end{pmatrix}\equiv(-1)^{j_1-j_2+m}(2j+1)^{-\frac{1}{2}}C^{j_1 j_2 j}_{m_1 m_2 m}$ 3-j coefficient [see Eq. (3.182), AMQP]

$\begin{bmatrix} j_1+m_1 & j_2+m_2 & j-m\\ j_1-m_1 & j_2-m_2 & j+m\\ j-j_1+j_2 & j+j_1-j_2 & -j+j_1+j_2\end{bmatrix}$ Regge array for a Wigner coefficient [see Eq. (3.181), AMQP]

$\Delta(abc)\equiv\left[\dfrac{(a+b+c+1)!}{(a+b-c)!(a-b+c)!(-a+b+c)!}\right]^{\frac{1}{2}}$ triangle coefficient corresponding to triangle (abc)

$\nabla(abc)=1/\Delta(abc)$ reciprocal of triangle coefficient

$W(abcd;ef)$ Racah coefficient [see Eq. (3.290), AMQP]

$\begin{Bmatrix} a & b & e\\ d & c & f\end{Bmatrix}=(-1)^{a+b+c+d}W(abcd;\,ef)$ 6-j coefficient

~~$W^{abc}_{\rho\sigma\tau}(j)=[(2c+1)(2j-2\sigma+1)]^{\frac{1}{2}}W(j-\tau,a,j,b;\,j-\sigma,c)$ Racah coeffi-~~cient with dimension factors

$\begin{Bmatrix} j_1 & j_2 & j\\ k_1 & k_2 & k\\ j_1' & j_2' & j'\end{Bmatrix}$ 9-j coefficient [see Eq. (3.251), AMQP]

$\begin{bmatrix} j_1 & j_2 & j\\ k_1 & k_2 & k\\ j_1' & j_2' & j'\end{bmatrix}$ 9-j coefficient with dimension factors [see Eq. (3.250), AMQP]

$\langle j_1+\Delta_1, j_2+\Delta_2, j+\Delta|\begin{bmatrix} J_1 J_2 J\\ \Delta_1\Delta_2\Delta\end{bmatrix}|j_1 j_2 j\rangle$ matrix element of the 9-j invariant operator

$=\Bigg[\begin{matrix} j_1 & j_2 & j\\ J_1 & J_2 & J\end{matrix}$

$C^{jJj+\Delta}_{m,M,m+M}$	generalized $SU(2)$ Wigner coefficient (see p. 70)
$\bar{C}^{jJj+\Delta}_{m,M,m+M} \equiv (-1)^{u_1} \mathcal{C}^{jJj+\Delta}_{m,M,m+M}$	generalized $SU(2)$ Wigner coefficient (see Topic 6)
$\mathcal{W}^{j_1 J j_1+\Delta}_{j_2-j,\Delta',j_2-j+\Delta'}(j_2+\Delta')$	generalized $SU(2)$ Racah coefficient (see p. 123)
$C\left[\begin{pmatrix} j_1 j_2 \\ j\ m \end{pmatrix} \begin{pmatrix} k_1 k_2 \\ k\ \mu \end{pmatrix} \begin{pmatrix} l_1 l_2 \\ l\ \nu \end{pmatrix}\right]$	$SU(2) \times SU(2)$ Wigner coefficient (see p. 91 ff.)
$[j] = D^j(U)$	irrep of $SU(2)$
$[j_1] \otimes [j_2] = \Sigma_j \oplus \varepsilon_{j_1 j_2 j}[j]$	Clebsch–Gordan series for $SU(2)$
$\begin{Bmatrix} abc \\ def \end{Bmatrix} \sim [12\pi V]^{-\frac{1}{2}} \cos\Omega$ $\Omega = \left[\left(\sum_{h<k} j_{hk}\theta_{hk}\right) + \frac{\pi}{4}\right]$	Ponzano–Regge asymptotic form of 6-j coefficient in terms of volume and angles of the (irregular) tetrahedron (see Topic 9)
$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} \to \begin{vmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{vmatrix} \neq 0$	Robinson's association of a 6-j coefficient with an element of the projective space $PG(n, 2)$ (see Topic 8)

Functions

$D^j_{m'm}(\psi, \hat{n})$	matrix elements of rotation matrix D^j of the quantal rotation group $SU(2)$ expressed in terms of the rotation angle ψ and the direction $\hat{n}$
$\cancel{D^j_{m'm}(U)}$	~~functions defined by irreps ("irrep func-~~ tions") of $SU(2)$ realized as homogeneous polynomials in the elements u_{ij} of $U \in SU(2)$
$D^j_{m'm}(\alpha\beta\gamma)$	irrep functions of $SU(2)$ expressed in terms of Euler angles
$D^j_{m'm}(\alpha_0, \boldsymbol{\alpha})$	irrep functions of $SU(2)$ expressed in terms of Euler–Rodrigues parameters
$D^j_{m'm}(x_0, \mathbf{x})$	extension of irrep functions of $SU(2)$ to arbitrary points $(x_0, \mathbf{x}) \in \mathbb{R}^4$ (quaternionic variables)
$\mathfrak{D}^l_{m'm}(R)$	irrep functions of $O(3)$ expressed as homogeneous polynomials in the elements R_{ij} of $R \in O(3)$

$d^j_{m'm}(\beta)$	irrep functions of $SU(2)$ depending only on the Euler angle parameter β
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomials in x
$Y_{lm}(\beta\alpha)$	spherical harmonics on S^2
$\mathcal{Y}_{lm}(\mathbf{x})$	solid harmonics in $\mathbf{x}\in\mathbb{R}^3$
$\mathcal{Y}_l^m\equiv(i)^l\mathcal{Y}_{lm}$	solid harmonics phased according to the time reversal operator convention for angular momentum eigenkets
$P_l^m(\cos\beta)$	associated Legendre polynomials in $\cos\beta$
$P_l(\cos\beta)$	Legendre polynomials in $\cos\beta$
$D^{j*}_{m'm}(U), D^{j*}_{m'm}(\alpha\beta\gamma), D^{j*}_{m'm}(x_0,\mathbf{x})$	complex conjugated set of values of irrep functions of $SU(2)$
$D^j(U), D^j(\psi,\hat{n}), D^j(\alpha\beta\gamma)$	$(2j+1)\times(2j+1)$ unitary matrix irrep of $SU(2)$
$D^{j_1}(U)\otimes D^{j_2}(U)$ $D^{j_1}(\psi,\hat{n})\otimes D^{j_2}(\psi,\hat{n})$	matrix direct product of irreps of $SU(2)$
$\mathcal{Y}^{(ls)jm}(\mathbf{x})$	tensor solid harmonics in $\mathbf{x}\in\mathbb{R}^3$
$\chi^j(U)=\operatorname{tr}D^j(U)$	trace (character) of irrep j of $SU(2)$
$L_k^\alpha(x)=\sum_{s=0}^{k}\frac{[\alpha+k]_{k-s}(-x)^s}{(k-s)!s!}$	associated Laguerre polynomials (see pp. 304–306)
$\mathfrak{L}_k^\alpha(x)=\left[\frac{k!}{\Gamma(a+k+1)}\right]^{\frac{1}{2}}x^{\alpha/2}e^{-x/2}L_k^\alpha(x)$	associated Laguerre functions
$\Phi_{jm}(x)\equiv\mathfrak{L}^{2j-1}_{m-j-1}(x)$	associated Laguerre functions labeled in terms of hyperbolic angular momentum labels
$R_{nl}(r)=(-1)^{n-l-1}\frac{2}{n^2}[x^{-\frac{1}{2}}\Phi_{ln}(x)]_{x=2r/n}$	radial hydrogen atom functions
$\mathfrak{R}_{nl}(r)=\sqrt{2}\left[x^{-(N-2)/4}\Phi_{jm}(x)\right]_{x=r^2}$, $m=(2n+N)/4,\ m-j-1=(n-l)/2$	N-dimensional harmonic oscillator radial functions
$j_l(kr), h_l^{\text{out}}(kr), J_\nu(kr)$ $I_\nu(x), K_\nu(x)$	Bessel functions [regular spherical, outgoing, spherical (irregular), of the first kind, hyperbolic (finite at origin), hyperbolic (zero at infinity)]
$\Gamma(z)$	gamma function
${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}$,	generalized hypergeometric function with p numerator and q denominator parameters

Notations Associated with Wigner and Racah Operators

$\begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix}$, $m_{12} \geqslant m_{11} \geqslant m_{22}$ — $U(2)$ Gel'fand pattern

$j=(m_{12}-m_{22})/2$, $m=m_{11}-(m_{12}+m_{22})/2$ — $SU(2)$ labels (jm) associated with a $U(2)$ Gel'fand pattern

$\left\{ \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle : m_{11}=m_{22}, m_{22}+1, \ldots, m_{12} \right\}$ — basis of $U(2)$ irrep space

$|jm\rangle \equiv \left| \begin{pmatrix} 2j & & 0 \\ & j+m & \end{pmatrix} \right\rangle$ — relation between standard ket notation and Gel'fand pattern notation

$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle$ — $SU(2)$ unit tensor operator, Wigner operator; action defined in angular momentum representation space $\mathcal{H}=\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_1 \oplus \cdots$

$[\Delta] \equiv \begin{bmatrix} \Delta_1 & & \Delta_2 \\ & W_1 & \end{bmatrix}$, — shift pattern of an $SU(2)$ Wigner operator

$[\Delta_1 \Delta_2]=[J+\Delta\ J-\Delta], [W_1 W_2]=[J+M\ J-M]$

$\left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = [(j+m)!(j-m)!]^{-\frac{1}{2}} a_1^{j+m} a_2^{j-m} \left| \begin{matrix} 0 & & 0 \\ & 0 & \end{matrix} \right\rangle$ — boson realization of angular momentum ket vectors

$\left\langle \begin{matrix} & \tau & \\ 1 & & 0 \\ & i & \end{matrix} \right\rangle$, $i, \tau=0,1$ — fundamental Wigner operators

$\mathbf{W}^{abc}_{\rho\sigma\tau}$ — Racah invariant operator in $\mathcal{H}$ corresponding to coupling of unit tensor operators

$W^{abc}_{\rho\sigma\tau}(j)=\delta_{\rho+\sigma,\tau}[(2c+1)(2j-2\sigma+1)]^{\frac{1}{2}} W(j-\tau, a, j, b; j-\sigma, c)$ — eigenvalue of a Racah invariant on the state $|jm\rangle$

$\left\langle \begin{matrix} m_{12}+\Delta_1 & & m_{22}+\Delta_2 \\ & m_{11}+W_1 & \end{matrix} \middle| \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \middle| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle$

$= \# \times (\text{NPCF}) \times D^{-1} \times (\text{polynomial})$ — canonical form of a Wigner coefficient

PCF, NPCF, DPCF — pattern calculus factor, numerator pattern calculus factor, denominator pattern calculus factor

$p_{ij}=m_{ij}+j-i$ — partial hooks

$P_k(\Delta_1, \Delta_2, W_1, W_2; z_1, z_2)$, $z_1=p_{11}-p_{12}=-(j-m+1)$, $z_2=p_{11}-p_{22}=j+m$ — polynomial part of a Wigner coefficient

$D_{[\Delta_1\Delta_2]}(p_{12}, p_{22})$ — denominator of a Wigner coefficient

$\mathcal{N}_{(J-\Delta-1)/2}$ — characteristic null space of a Wigner operator

$\mathbf{I}^J_\Delta, \mathbf{I}^c_\tau, \mathbf{I}^{ab}_{\rho\sigma}$ — characteristic operators associated with normalization of Wigner and Racah operators

$I^c_\tau(j)=\varepsilon_{j,c,j+\tau}$, $I^{ab}_{\rho\sigma}(j)=\varepsilon_{j+\sigma,a,j+\rho+\sigma}\varepsilon_{j,b,j+\sigma}$ — characteristic values of operators $\mathbf{I}^J_\Delta$, etc. on state $|jm\rangle$

$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^\dagger$ — conjugate $SU(2)$ Wigner operator

$\left\langle \begin{matrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \\ & M_{11} & \end{matrix} \right\rangle$ — $U(2)$ Wigner operator

$\begin{pmatrix} M_{12} & & M_{22} \\ & M_{11} & \end{pmatrix}$ — Gel'fand pattern label of a $U(2)$ Wigner operator

$W_1=M_{11},\ W_2=M_{12}+M_{22}-M_{11}$ — weight $[W_1W_2]$ of a $U(2)$ Gel'fand pattern

$\begin{pmatrix} & \Gamma_{11} & \\ M_{12} & & M_{22} \end{pmatrix}$ — operator pattern label of a $U(2)$ Wigner operator

$\Delta_1=\Gamma_{11},\ \Delta_2=M_{12}+M_{22}-\Gamma_{11}$ — shift pattern $[\Delta_1\Delta_2]$ of an operator pattern

$$T^{J_1J_2J}_{\Delta_1\Delta_2M} = \sum_{M_1M_2} C^{J_1J_2J}_{M_1M_2M} \left\langle \begin{matrix} & J_1+\Delta_1 & \\ 2J_1 & & 0 \\ & J_1+M_1 & \end{matrix} \right\rangle \otimes \left\langle \begin{matrix} & J_2+\Delta_2 & \\ 2J_2 & & 0 \\ & J_2+M_2 & \end{matrix} \right\rangle$$
$SU(2)\times SU(2)\times SU(2)$ tensor operator with shift properties: $j_1\to j_1+\Delta_1,\ j_2\to j_2+\Delta_2$

$$\left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\} = (-1)^{J+\Delta'} \sum_M \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \otimes \left\langle \begin{matrix} & J-\Delta' & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle^\dagger$$
definition of an $SU(2)$ (unit) Racah operator

$$[\Delta] \equiv \begin{bmatrix} \Delta_1 & & \Delta_2 & \\ & \Delta'_1 & & \Delta'_2 \end{bmatrix} = \begin{bmatrix} J+\Delta & & J-\Delta & \\ & J+\Delta' & & J-\Delta' \end{bmatrix}$$
shift pattern of an $SU(2)$ Racah operator

$$\left\{ \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix} \right\} |(j_1j_2)jm\rangle = [(2j_1+2\Delta+1)(2j_2+1)]^{\frac{1}{2}}$$
$$\times W(j, j_1, j_2+\Delta', J;\ j_2, j_1+\Delta)|(j_1+\Delta, j_2+\Delta')jm\rangle$$
action of a Racah operator on the coupled angular momentum basis of $\mathcal{H}\otimes\mathcal{H}$

$\left\{\begin{matrix} & \rho & \\ 1 & & 0 \\ & \sigma & \end{matrix}\right\}$, $\rho, \sigma=0, 1$	fundamental Racah operators
$\begin{pmatrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{pmatrix}$, $m_{12} \geqslant m_{11} \geqslant m_{22} \geqslant m_{21}$	extended Gel'fand pattern
$j_1=(m_{12}-m_{22})/2$, $j_2=(m_{11}-m_{21})/2$, $j=(m_{12}+m_{22}-m_{11}-m_{21})/2$	angular momentum labels in terms of entries m_{ij} in an extended Gel'fand pattern
$\left\lvert\begin{matrix} 2j_1 & & 0 & \\ & j_1+j_2-j & & j_1-j_2-j \end{matrix}\right\rangle = \lvert j_1 j_2 j\rangle$	notation for the equivalence class of basis vectors $\{\lvert(j_1 j_2)jm\rangle : m=j, \cdots, -j\}$
$\left\{\begin{matrix} & a+\rho & \\ 2a & & 0 \\ & a+\sigma & \end{matrix}\right\}$	Racah operator with action defined in the tensor product space $\mathcal{H}\otimes\mathcal{H}$

$$\left\langle \begin{matrix} m_{12}+\Delta_1 & m_{22}+\Delta_2 \\ m_{11}+\Delta_1' & m_{21}+\Delta_2' \end{matrix} \middle| \left\{\begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+\Delta' & \end{matrix}\right\} \middle| \begin{matrix} m_{12} & & m_{22} & \\ & m_{11} & & m_{21} \end{matrix} \right\rangle$$

$$=[(2j_1+2\Delta+1)(2j_2+1)]^{\frac{1}{2}} W(j, j_1, j_2+\Delta', J; j_2, j_1+\Delta)$$

$= \#(\mathrm{NPCF}\times D^{-1}\times D'^{-1}\times P_k(\Delta; p)$ general matrix element of a Racah operator in standard and canonical form

$P_k(\Delta; p)$	polynomial part of a canonical Racah coefficient
$D'_{[\Delta_1'\Delta_2']}(p_{11}, p_{21})$	second denominator function in a Racah coefficient
$N^J_{\Delta'\Delta}(\mathbf{K},\mathbf{H})$, $P^J_{\Delta'\Delta}(\mathbf{J}^2, K_3, H_3)$, $\overline{D}^J_{\Delta}(H_3+K_3)$, $\underline{D}^J_{\Delta}(H_3-K_3)$	operator realization of the "parts" of a canonical Racah operator in terms of the generators of $SU(2)*SU(2)*SU(1,1)$ (numerator pattern calculus factor, polynomial, first denominator, second denominator)
$\lvert jm\mu\kappa\rangle = \left[\dfrac{2j+1}{(\kappa-j-1)!(\kappa+j)!}\right]^{\frac{1}{2}}(\det A)^{\kappa-j-1} D^j_{m\mu}(A)\lvert 0\rangle$	boson operator realization of the basis of $\mathcal{H}\otimes\mathcal{H}$ on which $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{H}$ have the standard action
$e_{\text{tail}} = \begin{cases} 0 & \text{if tail is in top row,} \\ 1 & \text{if tail is in bottom row} \end{cases}$	rule of the pattern calculus
$\begin{array}{cc} \boxed{\begin{matrix} \alpha_1^1 & \alpha_1^2 \\ \alpha_2^1 & \alpha_2^2 \end{matrix}} & \begin{matrix} j+m \\ j-m \end{matrix} \\ j+m' \; j-m' & \end{array}$	$\boxed{\alpha}$-array notation occurring in the irrep functions (elements of the rotation matrix) of $SU(2)$

$\begin{array}{c} \boxed{\begin{array}{cc} \alpha_1^1 & \alpha_1^2 \\ \alpha_2^1 & \alpha_2^2 \end{array}} \begin{array}{c} W_1 \\ W_2 \end{array} \\ \Delta_1 \quad \Delta_2 \end{array}$	$\boxed{\alpha}$-array notation occurring in the expression for a general Wigner operator as a sum of monomials in the fundamental operators
$\begin{bmatrix} abc \\ \rho\sigma\tau \end{bmatrix}$	9-j invariant operator with action defined in tensor product space $\mathcal{H}\otimes\mathcal{H}$
$\mathrm{W}\left[\begin{pmatrix} a & b \\ \rho & \sigma \end{pmatrix}\begin{pmatrix} a' & b' \\ \rho' & \sigma' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ \rho'' & \sigma'' \end{pmatrix}\right]$	$SU(2)\times SU(2)$ Racah invariant operator

Boson Calculus

H	harmonic oscillator Hamiltonian; generic Hamiltonian
p, q	conjugate linear momentum and position operators satisfying $[p, q] = -i\hbar$
$a, \bar{a}$	creation and destruction operators, respectively, for harmonic oscillator satisfying $[\bar{a}, a] = 1$; boson and conjugate boson operators
m, ω	mass, angular frequency parameters of a harmonic oscillator
$N = a\bar{a}$	number operator
$\Delta p, \Delta q$	dispersion in momentum, position
$\mathcal{H}$	Hilbert space of states of the harmonic oscillator
$a = (a_1, a_2, \ldots, a_n)$	n-component boson operator a
$\mathcal{H}_n = \mathcal{H}\otimes\cdots\otimes\mathcal{H}$	Hilbert space of state vectors in n bosons $(a_1, \ldots, a_n)$
$X \to \mathcal{L}_X = \sum_{i,j=1}^{n} X_{ij} a_i \bar{a}_j$	Jordan-Schwinger map of an $n\times n$ matrix X into a boson operator $\mathcal{L}_X$ in $\mathcal{H}_n$
$\mathcal{E} = \mathcal{L}_1 = \sum_{i=1}^{n} a_i \bar{a}_i$	Euler operator in $\mathcal{H}_n$; number operator
$\|0\rangle$	vector (vacuum ket) in $\mathcal{H}$ such that $\bar{a}_i\|0\rangle = 0$, $i = 1, 2, \ldots, n$
$\langle 0\| F^*(\bar{a})G(a)\|0\rangle = \langle F\|G\rangle = [F^*(\partial/\partial\zeta)G(\zeta)]_{\zeta=0}$	inner product in $\mathcal{H}$ of state vectors $\|G\rangle = G(a)\|0\rangle$ and $\|F\rangle = F(a)\|0\rangle$
$J_i = \mathcal{L}_{\frac{1}{2}\sigma_i} = \frac{1}{2}(\tilde{a}\sigma_i\bar{a})$	Jordan-Schwinger boson operator realization of generator $\mathbf{J} = (J_1, J_2, J_3)$ of $SU(2)$

$\mathcal{H}^{(2j)}$	subspace of vectors in $\mathcal{H}_2$ of the form $P(a_1, a_2)\|0\rangle$, where $P(a_1, a_2)$ is a homogeneous polynomial of degree $2j$
$\|jm\rangle = P_{jm}(a_1, a_2)\|0\rangle$ $(m = j, \ldots, -j)$	standard orthonormal angular momentum ket vectors spanning $\mathcal{H}^{(2j)}$
$P_{jm}(a_1, a_2) = [(j+m)!(j-m)!]^{-\frac{1}{2}} a_1^{j+m} a_2^{j-m}$	boson polynomials defining the basis vectors $\|jm\rangle$
$J_+ = a_1\bar{a}_2, J_- = a_2\bar{a}_1$, $J_3 = (a_1\bar{a}_1 - a_2\bar{a}_2)/2$	realization of the $SU(2)$ generators in terms of the boson $a = (a_1, a_2)$
$\mathcal{T}_U, T_U$	unitary operator in $\mathcal{H}_2$ giving the standard unitary irrep $\mathcal{T}_U \to D^j(U)$ of $SU(2)$ when acting in the invariant subspace $\mathcal{H}^{(2j)}$
$\mathbf{a}^i = (a_1^i, a_2^i, \ldots, a_n^i)$ $(i = 1, 2, \ldots, n)$	n-component boson operators
$A = (a_i^j)$ $(i, j = 1, \ldots, n)$	$n \times n$ matrix with n^2 bosons as elements; matrix boson
$X \to \mathfrak{L}_X = \text{trace}(\tilde{A}X\bar{A})$	generalized Jordan map of an $n \times n$ matrix X into boson operator acting in $\mathcal{H}_{n^2}$; generator of left translations
E_{ij} $(i, j = 1, \ldots, n)$	basis of the set of boson operator maps $\{\mathfrak{L}_X \colon X = (x_{ij}), x_{ij} \in \mathbb{C}\}$; left boson operator realization of the Weyl basis of the Lie algebra of $U(n)$
$Y \to \mathfrak{R}_Y = \text{trace}(\tilde{Y}\tilde{A}\bar{A})$	generalized Jordan map of an $n \times n$ matrix Y into boson operator acting in $\mathcal{H}_{n^2}$; generator of right translations
$E^{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, n)$	basis of the set of boson operator maps $\{\mathfrak{R}_Y \colon Y = (y_{\alpha\beta}), y_{\alpha\beta} \in \mathbb{C}\}$; right boson operator realization of the Weyl basis of the Lie algebra of $U(n)$
$\det A = a_{12\cdots n}^{12\cdots n}$	determinant of the $n \times n$ matrix boson A
$L_U \colon A \to UA = L_U(A)$, $R_V \colon A \to A\tilde{V} = R_V(A)$	left and right translations of the $n \times n$ matrix boson A by unitary matrices U and V
$\mathcal{S}_U, \mathcal{T}_V$	unitary operator realizations in $\mathcal{H}_{n^2}$ of left and right translations L_U and R_V
$J_+ = E_{12}$, $J_- = E_{21}$, $J_3 = \frac{1}{2}(E_{11} - E_{22})$	generators of unimodular left translations $\mathcal{S}_U$ in $\mathcal{H}_4$, $U \in SU(2)$
$K_+ = E^{12}$, $K_- = E^{21}$, $K_3 = \frac{1}{2}(E^{11} - E^{22})$	generators of unimodular right translations $\mathcal{T}_V$ in $\mathcal{H}_4$, $V \in SU(2)$

$\mathcal{E}=\Sigma_k E^{kk}=\Sigma_k E_{kk}$ — generator of phase transformations in $\mathcal{H}_4$; number operator, Euler operator

$H_+=a_1^1a_2^2-a_2^1a_1^2$,
$H_-=\bar{a}_1^1\bar{a}_2^2-\bar{a}_2^1\bar{a}_1^2$,
$H_3=\frac{1}{2}(a_1^1\bar{a}_1^1+a_2^1\bar{a}_2^1+a_1^2\bar{a}_1^2+a_2^2\bar{a}_2^2+2)$ — boson realization of hyperbolic angular momentum; realization of the generators of $SU(1,1)$ in the Hilbert space $\mathcal{H}_4$

$D^j_{m'm}(A)$ — boson polynomial obtained from the representation functions $D^j_{m'm}(U)$ by the replacement $u_{ij}\to a_i^j$

$$|jm\mu\kappa\rangle\equiv\left|\left(\frac{\kappa+\mu-1}{2},\frac{\kappa-\mu-1}{2}\right)jm\right\rangle=|(j_1j_2)jm\rangle$$
$$=\left[\frac{2j+1}{(\kappa-j-1)!(\kappa+j)!}\right]^{\frac{1}{2}}(\det A)^{\kappa-j-1}D^j_{m\mu}(A)|0\rangle$$
— explicit basis of $\mathcal{H}_4$ on which the boson operators **J**, **K**, and **H** have the standard action

Notations Associated with Special Topics

$(\hat{e}_r,\hat{e}_\theta,\hat{e}_\phi)$ — triad of unit orthogonal spherical basis vectors

$\mathbf{B}=g(\hat{e}_r/r^2)$ — magnetic field of monopole g located at the origin

$R_a, R_b, R_{ab}=R_a\cap R_b$ — hemispherical regions covering the sphere of fixed radius r with overlap region R_{ab}

$$\mathbf{A}=\begin{cases}\mathbf{A}_a=\dfrac{g(1-\cos\theta)}{r\sin\theta}\hat{e}_\phi, \text{ each } P\in R_a\\[2ex] \mathbf{A}_b=\dfrac{-g(1+\cos\theta)}{r\sin\theta}\hat{e}_\phi, \text{ each } P\in R_b\end{cases}$$
— definition of vector potential yielding $\mathbf{B}=\nabla\times\mathbf{A}$

$\mathbf{A}_a-\mathbf{A}_b=2g\nabla\phi$, each $P\in R_{ab}$ — difference of vector potentials is a gauge transformation in the overlap region

$eg/\hbar c\equiv\mu=\frac{1}{2}$(integer) — Dirac quantization condition

$\boldsymbol{\pi}\equiv\mathbf{p}-(e/c)\mathbf{A}$ — Hermitian sectional operator

$\mathbf{J}=\mathbf{J}_a=\mathbf{x}\times(\mathbf{p}-\frac{e}{c}\mathbf{A}_a)-\mu\hat{e}_r=\mathbf{L}-\mu\mathbf{K}$, in R_a
$\mathbf{J}'=\mathbf{J}_b=\mathbf{x}\times(\mathbf{p}-\frac{e}{c}\mathbf{A}_b)-\mu\hat{e}_r=\mathbf{L}-\mu\mathbf{K}'$, in R_b — definition of total angular momentum of electron in monopole field

$\mathbf{L}=\mathbf{x}\times\mathbf{p}$
$\mathbf{K}=\frac{1}{g}(\mathbf{x}\times\mathbf{A}_a)+\hat{e}_r$
$\mathbf{K}'=\frac{1}{g}(\mathbf{x}\times\mathbf{A}_b)+\hat{e}_r$ — operators used in defining the total angular momentum

$\mathbf{J}^2=\left[\mathbf{x}\times\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\right]^2+\mu^2=J_1^2+J_2^2+J_3^2$ total angular momentum operator squared

$\psi_b(\mathbf{x})=S_{ba}(\mathbf{x})\psi_a(\mathbf{x})$, each $\mathbf{x}\in R_{ab}$ relation between wave functions in the overlap region

$\mathbf{J}_b S_{ba}=S_{ba}\mathbf{J}_a$, in R_{ab} relation between angular momentum operators in the overlap region

$S_{ba}=e^{-2i\mu\phi}$ transition function

$\langle\theta\phi|jm\mu\rangle=\begin{cases} D^{j\,*}_{m,-\mu}(\phi,\theta,-\phi), \text{ in } R_a \\ D^{j\,*}_{m,-\mu}(\phi\theta\phi), \text{ in } R_b \end{cases}$ sectional eigenkets of the angular momentum $\mathbf{J}$

$\frac{d\sigma}{d\Omega}=|f(k,\theta)|^2$ differential scattering cross section

$f(k,\theta)=k^{-1}\sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta)$ scattering amplitude in partial wave expansion

$-\sin\delta_l=\int_0^\infty j_l(kr')\phi_l(kr')V(r')r'^2\,dr'$ phase shift

$\alpha_l(k)$ poles of the scattering matrix in the complex l-plane

$(n'l'|r^k|nl)\equiv\int_0^\infty r^{N-1}dr\,\mathfrak{R}_{n'l'}(r)r^k\mathfrak{R}_{nl}(r)$ radial integral for harmonic oscillator

$U=\exp(i\lambda p/\hbar)$, $V=\exp(i\mu x/\hbar)$ Weyl's unitary operators corresponding to canonical operators p and x

$UV=e^{(i\lambda\mu/\hbar)}VU$ Heisenberg commutation relation in Weyl form

$\langle A\rangle=\langle\psi|A|\psi\rangle$ expectation value of an observable A in the state $|\psi\rangle$

$(\Delta A)^2=\langle\psi|(A-\langle A\rangle)^2|\psi\rangle$ dispersion (squared) of an observable A in the state $|\psi\rangle$

$(\Delta p_i)^2\equiv\langle\psi|(p_i-\langle p_i\rangle)^2|\psi\rangle$, $(\Delta x_i)^2\equiv\langle\psi|(x_i-\langle x_i\rangle)^2|\psi\rangle$ dispersion in momentum p_i and conjugate position x_i

$\Delta p_i\Delta x_i\geqslant\frac{1}{2}\hbar$, Heisenberg uncertainty relation

$\psi(a,b,\mu;x)=(\pi\mu)^{-\frac{1}{4}}\exp-\left[\frac{(x-a)^2}{2\mu}+ibx\right]$ family of minimum uncertainty states of p, x

$AC[0,2\pi)$ space of absolutely continuous, square-integrable functions on the interval $[0,2\pi)$

$\Delta L_i, \langle L_i\rangle$ dispersion in the angular momentum operator L_i; expectation value of L_i

$[L_3, \sin\phi] = -i\cos\phi$, $[L_3, \cos\phi] = i\sin\phi$	commutation relations for $L_3 = -i\partial/\partial\phi$ and the cosine and sine functions of the canonical angle variable ϕ
$\psi_{m\mu}(\phi) = N^{-\frac{1}{2}} e^{im\phi + \mu\cos\phi}$	family of minimum uncertainty states for $L_3, \sin\phi$
$UV = \varepsilon VU$	commutation rule for "conjugate" unitary transformations (U, V) in a finite-dimensional Hilbert space ($\varepsilon = N$th root of unity): discrete Weyl system
$\{\langle a^k \vert : k = 1, 2, \ldots, N\}$	Schwinger's bra-vector basis
$\langle a^k \vert V = \langle a^{k+1} \vert$, $k = 1, 2, \ldots, N$,	canonical action of V
$\langle v^k \vert = N^{-\frac{1}{2}} \sum_{l=0}^{N-1} \langle a^l \vert e^{-2\pi ikl/N}$	relation of eigenbras of V to basis $\{\langle a^k \vert\}$
$\langle v^k \vert U = \langle v^{k-1} \vert$	canonical action of U
$(\mathbf{L}^2)^{\frac{1}{2}}, \psi = -\sin^{-1}[\cos\theta/(1 - \mathrm{L}_3^2/\mathbf{L}^2)^{\frac{1}{2}}]$, $\mathrm{L}_3, \chi = -\sin^{-1}[\mathrm{L}_1/(\mathbf{L}^2 - \mathrm{L}_3^2)^{\frac{1}{2}}]$	classical canonically conjugate angular momentum variables
$2P_1 + 1 = (4\mathbf{L}^2 + 1)^{\frac{1}{2}}$, $P_2 = L_3$ Q_1, Q_2	quantum mechanical canonically conjugate angular momentum operators P_1, Q_1 and P_2, Q_2 for Q_1, Q_2 "angle" variables
$\left\langle \begin{matrix} & 1+\Delta & \\ 2 & & 0 \\ & 1+\mu & \end{matrix} \right\rangle = e^{i(\Delta Q_1 + \mu Q_2)} H_\mu^{(\Delta)}(P_1, P_2)$	polar decomposition of vector Wigner operators
$N_1 = a_1\bar{a}_1$, $N_2 = a_2\bar{a}_2$	number operators
$P_1 = (N_1 + N_2)/2$, $P_2 = (N_1 - N_2)/2$	boson realization of angular momentum operators P_1, P_2
$e^{iQ_2} = a_1\bar{a}_2[(N_1 + 1)N_2]^{-\frac{1}{2}}$, $e^{iQ_1} = a_1 a_2[(N_1 + 1)(N_2 + 1)]^{-\frac{1}{2}}$	boson realization of angle operators Q_1, Q_2
$S_i = a_i(N_i + 1)^{-\frac{1}{2}} = (N_i)^{-\frac{1}{2}} a_i$, $\bar{S}_i = \bar{a}_i(N_i)^{-\frac{1}{2}} = (N_i + 1)^{-\frac{1}{2}} \bar{a}_i$	boson shift operators
$e^{iQ_1} = S_1 S_2$, $e^{-iQ_1} = \bar{S}_1 \bar{S}_2$, $e^{iQ_2} = S_1 \bar{S}_2$, $e^{-iQ_2} = \bar{S}_1 S_2$	realization of angle operator in terms of boson shift operators
$\vert m, \cos q_1\rangle$ $\vert m, \sin q_1\rangle$	eigenstates of the angle operators $\cos Q_1$ and $\sin Q_1$
$\mathcal{H}(j_1 j_2 \cdots j_n)$	tensor product space $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_n}$
$\{(j_1 j_2 \cdots j_n)^B$: B is a binary bracketing$\}$	generic notation for a coupling scheme
$\vert (i_1 i_2 \cdots i_n)(j_{i_1} j_{i_2} \cdots j_{i_n})^B (k_1 k_2 \cdots k_{n-2}) jm\rangle$	generic notation for a state vector in $\mathcal{H}(j_1 \cdots j_n)$ in a specific coupling scheme

$\langle (i')(j_{(i')})^{B'}(k')j \vert (i)(j_{(i)})^{B}(k)j\rangle$	generic notation for a transformation coefficient between coupling schemes
$\vert(\mathbf{ABC})[(ab)_d c]_{jm}\rangle$	state vector in the coupling scheme $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{J}$
$\langle \{[(ab)_e c]_f d\}_g \vert [(ac)_h (bd)_k]_g\rangle$ $\langle [(ab)_e (cd)_f]_g \vert [(ac)_h (bd)_k]_g\rangle$	typical notations for transformation coefficients
$\begin{Bmatrix} a_1 & & a_2 & & \cdots & a_n & \\ & b_1 & & b_2 & \cdots & & b_n \\ c_1 & & c_2 & & \cdots & c_n & \end{Bmatrix}$	$3n$-j coefficient of the first kind
$\begin{bmatrix} a_1 & & a_2 & & \cdots & a_n & \\ & b_1 & & b_2 & \cdots & & b_n \\ c_1 & & c_2 & & \cdots & c_n & \end{bmatrix}$	$3n$-j coefficient of the second kind

Miscellaneous Mathematical and Physical Symbols

$\hbar$	Dirac's notation for Planck's constant h divided by 2π; basic unit of angular momentum
$e=-1.60\times 10^{-19}$ coulomb	charge of the electron in rationalized MKS units
$\delta_{ij}, \delta_{i'j}, \delta_{m'm}$	Kronecker delta
e_{ijk}	$+1$ (-1) for ijk an even (odd) permutation of 123; otherwise 0
$\mathrm{sign}(\tau-i)$, $\tau, i=0,1$	$+1$ for $\tau\geqslant i$; -1 for $\tau<i$
$(a_1, a_2, \ldots, a_n)$	ordered n-tuple
$\mathbf{1}, 1$	unit operators
$\mathbf{1}_n$	$n\times n$ unit matrix
$\mathbf{1}^{(j)}$	$(2j+1)\times(2j+1)$ unit matrix
$[A,B]=AB-BA$	commutator of operators or matrices A and B
$[A,B]_{(k)}\equiv[A,[A,B]_{(k-1)}]$ $[A,B]_{(0)}\equiv B$	multiple commutator of operator A with operator B
$[A,B]_{PB}=\sum_i\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)$	Poisson bracket of observables A and B
$\binom{z}{a}=z(z-1)\cdots(z-a+1)/a!$	binomial function
$(z)_a=z(z+1)\cdots(z+a-1)$	Pochhammer's symbol for a rising factorial
$[z]_a=z(z-1)\cdots(z-a+1)$	falling factorial
$1, i, j, k$	quaternionic basis elements
$q=q_0 1+q_1 i+q_2 j+q_3 k$	general quaternion

$\bar{q}=q_0 1-q_1 i-q_2 j-q_3 k$	conjugate quaternion to q
$N(q)=\bar{q}q=q_0^2+q_1^2+q_2^2+q_3^2$	norm of a quaternion q
$N(q)=1$	unimodular quaternion
$dS(\hat{n})$	differential surface area of S^2 at the point $\hat{n}\in S^2$
$d\omega=d\phi\sin\theta\, d\theta$	differential surface area of S^2 at the point $\hat{n}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$
$d\Omega, d\Omega_U$	differential surface area of S^3 at the point $(x_0,\mathbf{x})\leftrightarrow U(x_0,\mathbf{x})\in SU(2)$
$A^\dagger, A^*, \bar{A}$	conjugate of an operator
$z^*, \bar{z}$	complex conjugate of $z\in\mathbb{C}$
$\mathcal{Q}=$ RW-algebra	algebra of $SU(2)$ Wigner operators
$su(1,1)$	Lie algebra of the Lie group $SU(1,1)$
$\delta(\mathbf{x}-\mathbf{x}')$	Dirac delta function
$U(\mathbf{a})f(\mathbf{x})=f(\mathbf{x}-\mathbf{a})$	action of the unitary displacement operator $U(\mathbf{a})=e^{-i\mathbf{a}\cdot\mathbf{p}/\hbar}$ on a function $f:\mathbf{x}\to f(\mathbf{x})$

Author Index

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