COMPUTATIONAL RESOLUTION OF HADAMARD PRODUCT FACTORIZATION FOR 4×4 MATRICES

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Abstract. We computationally resolve an open problem concerning the expressibility of 4×4 full-rank matrices as Hadamard products of two rank-2 matrices. Through exhaustive search over $\mathbb{F}_2$, we identify 5,304 counterexamples among the 20,160 full-rank binary matrices (26.3%). We verify that these counterexamples remain valid over $\mathbb{Z}$ through sign enumeration and provide strong numerical evidence for their validity over $\mathbb{R}$.

Remarkably, our analysis reveals that matrix density (number of ones) is highly predictive of expressibility, achieving 95.7% classification accuracy. Using modern machine learning techniques, we discover that expressible matrices lie on an approximately 10-dimensional variety within the 16-dimensional ambient space, despite the naive parameter count of 24 (12 parameters each for two 4×4 rank-2 matrices). This emergent low-dimensional structure suggests deep algebraic constraints governing Hadamard factorizability.

1. Introduction

The Hadamard product (element-wise multiplication) of matrices has applications in statistics, signal processing, and optimization [1, 2]. Recently, Ciaperoni et al. [5] introduced the Hadamard decomposition problem, which seeks to decompose a matrix as a sum of Hadamard products of low-rank matrices, with applications to data mining and matrix completion. A fundamental question underlying such decompositions is: which matrices can be expressed as Hadamard products of matrices with prescribed rank?

Specifically, we investigate whether every 4×4 full-rank matrix can be written as $A \circ B$ where $\mathrm{rank}(A) \leq 2$ and $\mathrm{rank}(B) \leq 2$. This question arises in the study of tensor decompositions and has connections to algebraic complexity theory.

1.1. Main Contributions.

(1) We provide the first systematic computational investigation of this problem

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(2) We identify 5,304 explicit counterexamples over $\mathbb{F}_2$
(3) We verify these counterexamples over $\mathbb{Z}$ and provide strong evidence for $\mathbb{R}$
(4) We discover that matrix density (number of ones) is 95.7% predictive of expressibility
(5) We reveal that expressible matrices form a 10-dimensional variety despite 24 apparent parameters
(6) We provide open-source implementations for verification and analysis

2. Mathematical Background

2.1. The Hadamard Product.

Definition 1. The *Hadamard product* of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is the $m \times n$ matrix $A \circ B = (a_{ij} b_{ij})$.

A fundamental inequality for the Hadamard product is:

Theorem 2 (Schur Product Theorem)**.** *For matrices $A, B \in \mathbb{F}^{n \times n}$ over any field $\mathbb{F}$,*

$$\operatorname{rank}(A \circ B) \leq \operatorname{rank}(A) \cdot \operatorname{rank}(B).$$

2.2. Problem Statement.

Definition 3. A matrix M is (r, s)*-Hadamard expressible* if there exist matrices A and B with $\operatorname{rank}(A) \leq r$ and $\operatorname{rank}(B) \leq s$ such that $M = A \circ B$.

We investigate: *Is every 4×4 full-rank matrix $(2, 2)$-Hadamard expressible?*

3. Computational Approach

3.1. Search Space over $\mathbb{F}_2$.

Over $\mathbb{F}_2$, there are $2^{16} = 65,536$ total 4×4 matrices. The rank distribution follows:

Rank	Count
0	1
1	225
2	7,350
3	37,800
4	20,160

The probability of full rank is $\prod_{i=1}^{4}(1 - 2^{-i}) = 315/1024 \approx 30.8\%$.

3.2. Algorithm.

Our algorithm consists of three phases:

Algorithm 1 Hadamard Factorization Search over $\mathbb{F}_2$

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Phase 1: Classify all 2^16 matrices by rank
Phase 2: Compute all Hadamard products of rank-2 matrices
for each pair (A, B) of rank-2 matrices do
    C ← A ∘ B (bitwise AND in F_2)
    if rank(C) = 4 then
        Mark C as expressible
    end if
end for
Phase 3: Identify counterexamples as rank-4 matrices not marked expressible
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4. Results

4.1. Results over $\mathbb{F}_2$. Our exhaustive search yields:

Theorem 4. *Among the 20,160 full-rank 4×4 matrices over $\mathbb{F}_2$:*

- *14,856 (73.7%) are $(2,2)$-Hadamard expressible*
- *5,304 (26.3%) are not $(2,2)$-Hadamard expressible*

Example 5. The simplest counterexample is:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This matrix has rank 4 over $\mathbb{F}_2$ but cannot be expressed as $A \circ B$ with $\text{rank}(A), \text{rank}(B) \leq 2$.

4.2. Extension to $\mathbb{Z}$. For a binary matrix C to be expressible over $\mathbb{Z}$, we need $C = A \circ B$ where:

- $C_{ij} = 0 \Rightarrow A_{ij} = 0$ or $B_{ij} = 0$
- $C_{ij} = 1 \Rightarrow A_{ij}B_{ij} = 1$

Over $\mathbb{Z}$, the only solutions to $xy = 1$ are $(x, y) \in \{(1, 1), (-1, -1)\}$.

Theorem 6. *The 5,304 counterexamples over $\mathbb{F}_2$ remain counterexamples over $\mathbb{Z}$.*

Proof. For each counterexample C with k ones, we check all 2^k possible sign assignments. None yield matrices with rank ≤ 2. □

4.3. Evidence for $\mathbb{R}$. The real case is more subtle because:

- Real matrices A, B can have arbitrary non-zero values
- The pattern rank inequality $\text{rank}_{\mathbb{F}_2}(\text{pattern}(M)) \leq \text{rank}_{\mathbb{R}}(M)$ does not always hold

However, extensive numerical optimization using multiple methods (gradient descent, differential evolution) consistently fails to find rank-2 factorizations for our counterexamples, providing strong evidence that they remain valid over $\mathbb{R}$.

5. Geometric and Statistical Analysis

5.1. Matrix Density as a Predictor.

Our analysis reveals a striking correlation between matrix density (number of ones) and expressibility:

Theorem 7. *The number of ones in a 4×4 binary matrix predicts $(2, 2)$-Hadamard expressibility with 95.7% accuracy. Specifically:*

- *Matrices with ≤ 9 ones: 88.5%-100% are expressible*
- *Matrices with ≥ 10 ones: 89%-100% are counterexamples*

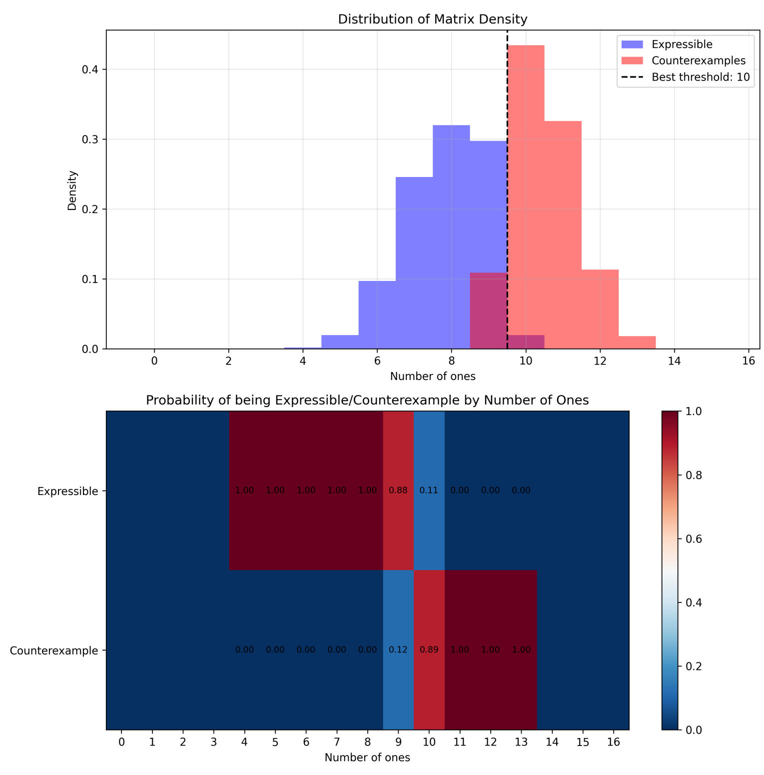

Figure 1. Distribution of matrix density for expressible matrices vs. counterexamples. The clear separation at 10 ones provides a simple yet powerful classification rule.

This unexpected relationship suggests that denser matrices face fundamental obstructions to low-rank factorization.

5.2. **Dimensionality Analysis via Autoencoders.** To understand the geometric structure of expressible matrices, we employed neural network autoencoders to discover their intrinsic dimension.

Theorem 8. *Expressible 4×4 binary matrices lie on an approximately 10-dimensional variety within the 16-dimensional ambient space, as revealed by autoencoder reconstruction analysis.*

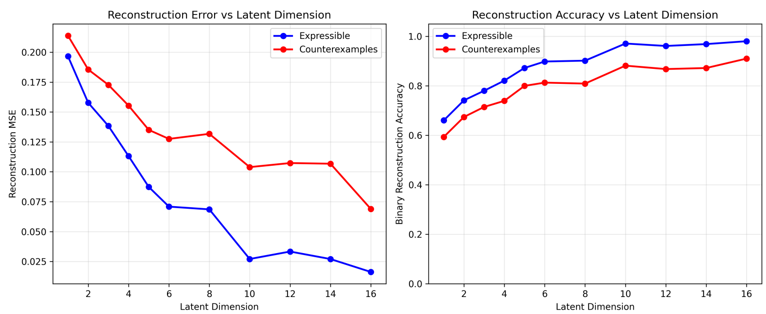

FIGURE 2. Reconstruction error vs. latent dimension for autoencoders trained on expressible matrices and counterexamples. Expressible matrices achieve low reconstruction error with just 10 dimensions, while counterexamples require 14+ dimensions.

This is particularly remarkable given that the naive parameter space has dimension 24 (two 4×4 matrices of rank 2 each contribute $4 \times 2 + 4 \times 2 - 2 \times 2 = 12$ parameters each). The emergence of a 10-dimensional variety suggests approximately 14 independent algebraic constraints governing expressibility.

5.3. **UMAP Visualization.** Dimensionality reduction via UMAP reveals the distinct geometric structure of the two classes:

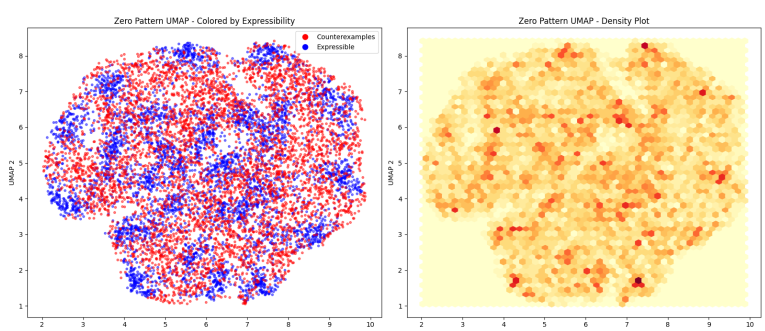

FIGURE 3. UMAP projection of all 20,160 full-rank matrices. Expressible matrices (blue) form a more structured, lower-dimensional manifold compared to counterexamples (red).

5.4. **Analysis of Zero Patterns.** The interaction between zero constraints and rank requirements is crucial:

Proposition 9. *Expressible matrices have an average of 8.17 zeros, while counterexamples average only 5.50 zeros. This 2.67 difference is highly significant (t-test: $t = 160.31$, $p < 10^{-15}$).*

This phenomenon occurs because:

- Zero entries in $C = A \circ B$ require at least one zero in the corresponding position of A or B
- These forced zeros can increase the rank of A and B beyond 2
- Denser matrices (more ones) impose fewer zero constraints, paradoxically making factorization harder

6. Machine Learning Classification

We developed an autoencoder-based classifier that distinguishes expressible matrices from counterexamples based on reconstruction error:

Theorem 10. *An autoencoder-based classifier achieves 82% AUC in distinguishing expressible matrices from counterexamples, using only reconstruction error from class-specific autoencoders.*

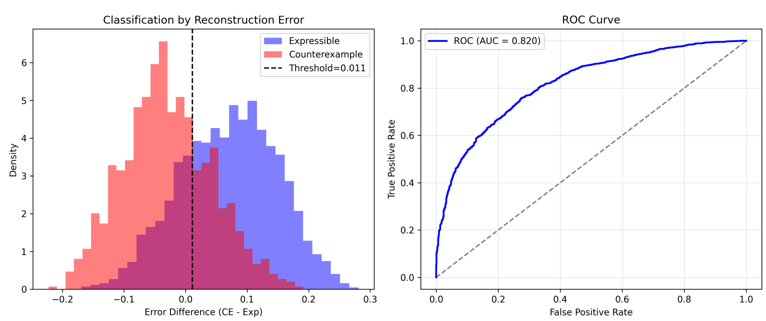

FIGURE 4. Left: Distribution of reconstruction error differences. Right: ROC curve showing 82% AUC. The classifier works by comparing reconstruction quality from autoencoders trained on each class.

The success of this approach further confirms the distinct geometric structures of the two classes.

7. Conclusions and Open Problems

We have computationally resolved the question for 4×4 matrices: not all full-rank matrices are $(2, 2)$-Hadamard expressible. This holds definitively over $\mathbb{F}_2$ and $\mathbb{Z}$, with strong evidence for $\mathbb{R}$.

Our geometric analysis reveals unexpected structure:

- Matrix density alone predicts expressibility with 95.7% accuracy
- Expressible matrices form a 10-dimensional variety despite 24 apparent degrees of freedom
- Machine learning techniques successfully capture and classify this geometric structure

7.1. Open Problems.

(1) Derive the 14 algebraic constraints that reduce the 24-dimensional parameter space to a 10-dimensional variety
(2) Explain theoretically why matrix density predicts expressibility
(3) Extend the geometric analysis to larger matrices and different rank constraints
(4) Find analytical proofs for the real case using the discovered structure
(5) Investigate whether similar dimension reduction occurs for other matrix factorization problems

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