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Some Matroids from Discrete Applied Geometry

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Dedicated to Janos Baracs and Henry Crapo on the occasions of their 64th birthdays

Abstract. We present an array of matroids drawn from three sources in discrete applied geometry: (i) static (or first-order) rigidity of frameworks and higher skeletal rigidity; (ii) parallel drawings (or equivalently polyhedral pictures); and (iii) C_r^{r-1}-cofactors abstracted from multivariate splines in all dimensions. The strong analogies (sometimes isomorphisms) between generic rigidity matroids and generic cofactor matroids is one central theme of the chapter. We emphasize matroidal results for the combinatorial 'generic' situations, with geometric techniques used when they contribute combinatorial insights. A second basic theme is the analysis of represented matroids using the duality of row and column dependencies of the representing matrix (generalizing statics and kinematics in rigidity).

Parts I and II concentrate on matroids for geometric graphs (or frameworks) and the role of submodular counts on the edges and vertices. The specific structure of such counts, which generalize the graphic matroid (rigidity on the line), is explored in an Appendix. For matroids based on geometric realizations of higher simplices, Part III emphasizes the intimate role of the homology (statics) of resulting geometric chain complexes and cohomology (kinematics) of the associated cochain complexes. We include a re-presentation of the simplicial homology matroid as the starting point for the geometric complexes and new results on orthogonal homology matroids. A number of unsolved problems and conjectures, both old and new, are presented.

Contents

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1. Introduction

Within the spirit of Gian-Carlo Rota's 'geometric approach to matroid theory' [**Ku**], I offer several related arrays of matroids: matroids for geometric graphs which generalize the graphic matroid (Figure 1.1); and matroids from geometric chain complexes which generalize the simplicial matroids of ordinary homology (Figure 13.1), themselves a generalization of graphic matroids. I am a geometer and the underlying work on these objects has been driven by geometric questions. The work draws heavily on problems and examples from true applications of discrete geometry and matroidal thinking has contributed significant insights to this work. We present these patterns in the hope that they will also contribute to the geometric side of 'combinatorial geometries'.

1.1. The broad themes. A number of problems in discrete applied geometry lead to matrices with rows indexed by the edges of a graph and columns indexed by the vertices. With these matrices, we naturally have matroids on the index sets (edges) given by the independence of the rows. Some of these matroids, such as that for first-order rigidity of frameworks, have been around for at least 150 years. Other matroids, such as parallel drawing, scene analysis and cofactor matroids for splines, have arisen in the last few decades, within research on processing geometry on computers (Computer Aided Geometric Design, computer graphics, robotic vision, . . .).

These matroids from applied geometry have an inherent 'sensual' content. For rigidity, the dependence and rank of a set of edges may be modeled with bars and joints and sensed through the hands and through the eyes. Explicit visual and geometric constructions can verify the rank or dependence of sets (constructions such as dynamic parallel drawings on The Geometer's Sketchpad or Maxwell's reciprocal diagrams). Unfortunately, this sensual character will be absent in our text. We encourage the readers to turn to other writing [**CoW**,**RW**,**CrW5**] and other media, such as physical models and computer programs (Geometer's Sketchpad, Cabri, Structures) for that more complete experience.

Not all matrices produce interesting matroids, nor are all aspects of a large field like first-order rigidity of frameworks matroidal. We focus on those aspects which have basic matroidal content and which generalize to wider families of matroids for which these core examples serve as an introduction. Over two decades, these core matroids have been studied by a number of matroid theorists including: Henry Crapo, Jack Graver, Andras Recski, Brigitte Servatius and Neil White; and specific matroidal features have been identified and developed [**GSS**,**Re4**]. Within the larger theories, we select topics to illustrate some central themes:

1. Matroid theory offers important insights into these structures.

   Techniques such as matroid union, Dilworth truncation, matroid partition, submodular functions, etc. have contributed to the theory and the algorithms (Part I and Appendix §A).
2. Many of the techniques developed for the core examples generalize to other matroids arising from other settings.

   This body of matrix / matroid techniques on graphs has wider applications and has a more unified whole (Parts I, II, III). Already, inductive techniques have been moved from rigidity of frameworks to multivariate splines and related techniques have been moved back, based on the common structure of the matrices (see §10, §11, and §16.5).
3. These matroids raises interesting, unsolved problems and conjectures within matroid theory.

   We will describe several of these conjectures in Part II.
4. These matroids on geometric graphs are a first layer in a rich array of matroids derived from geometric chain complexes (Part III) [**Wh15**].

   We want to promote an investigation this broad family of matroids. Some of these are intimately related to the core areas of matroid theory, such as orthogonality and homology of surfaces (§14), others are related to fields such as multivariate splines (§15) and still others drawn from core combinatorial and geometric properties of polytopes (§16).
5. For graphs and for homologies we emphasize a 'two sided' analysis of the matrix of a represented matroid – the row rank and row dependencies as the circuits of the matroid (statics of frameworks and homology), and the column rank and kernel (kinematics of frameworks and cohomology).

   The interplay of these two complementary or dual approaches to a single matroid is a central theme of this chapter. This duality is implicit in the homology and cohomology of the underlying chain complexes.

1.2. A pattern of matroids on graphs. Part I concentrates on alternate forms of a single matroid: the core example in the white box of Figure 1.1, referred to as plane rigidity, plane parallel drawing and C_1^0-cofactors. On complete graphs (for points affinely spanning the plane), these have bases E with $|E| = 2|V| - 3$. We give such space to this well-studied matroid (and the even simpler matroid on the line – the graphic matroid of the lowest white box) because the three versions have distinct, independently interesting, extensions to higher geometric dimensions (or, for cofactors, higher algebraic powers) in Part II. The exploration of this matroid also establishes the pattern of results and techniques which we strive to approximate in these extensions.

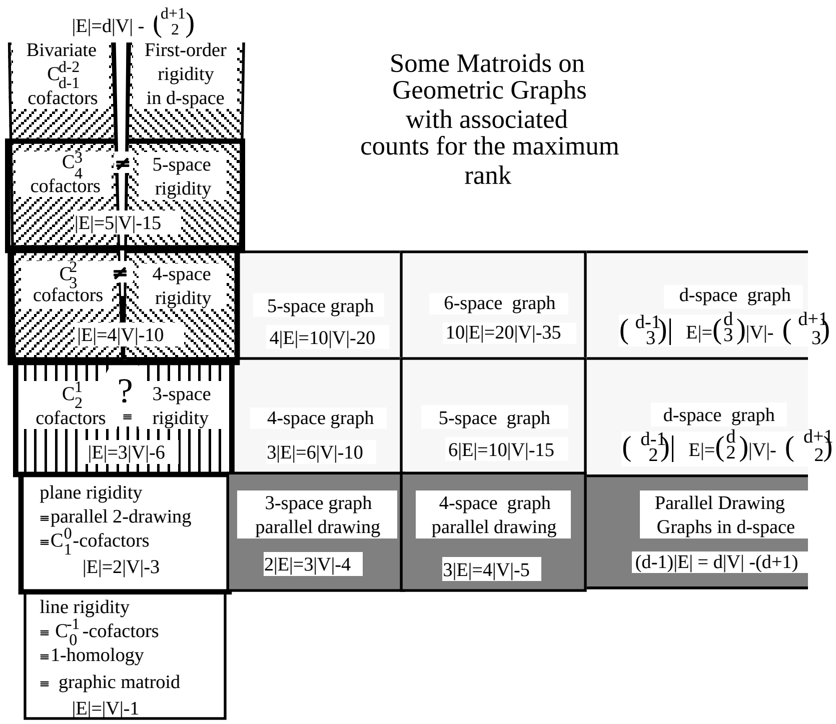

Fig. 1.1. *Some connected families of matroids on graphs realized in d-space.*

Part II describes three strips of Figure 1.1 which extend out for this core example: the parallel drawing hypermatroids (or polymatroids) for geometric graphs in dimensions $d \geq 3$ (and more general parallel drawing and polyhedral picture matroids) (§8), the first-order rigidity matroids for dimensions $d \geq 3$ (§9,§11), and the bivariate C_s^{s-1}-cofactor matroids for graphs in the plane (§10,§11).

Part III describes two related families of matroids on higher skeleta of simplicial complexes in d-space – families spreading in layers 'above' the initial layer of Figure 1.1 (see Figure 13.1). These families generalize the usual matroids for simplicial homology of the underlying complex (§14) in precisely the sense that the rigidity and cofactor matroids generalize the usual graphic matroid.

Parts I and II record a number of examples of matroids for which the rank is calculated by explicit 'counts' on the edges and vertices of a graph, multigraph or hypergraph. In the Appendix §A, we extract a particular pattern of submodular functions and induced matroids on hypergraphs which corresponds to these 'counts' of the edges and vertices.

It is our goal to paint a picture of larger patterns within this range of examples, connections which are easily lost in the detailed presentation of a single part. We hope that this broader picture will contribute to the evolution of these areas and the resolution of the large collection of conjectures which are scattered through the chapter.

It has been difficult to resist many aesthetically pleasing digressions: examples or twists of the theory. I trust the reader will forgive some wandering from the central path to give a richer flavour to the results, the connections, and the problems. In spite of appearances, I have left out more of these 'added remarks' than I have retained!

1.3. Acknowledgments. I owe a large debt to Henry Crapo and Neil White for over two decades of joint projects and informal conversations on areas implicitly and explicitly related to this chapter. In particular, the Appendix records previously unpublished joint work with Neil White. I owe a less obvious, but very deep debt to Janos Baracs for sharing his rich geometric insights, his provocative conjectures and his stimulating examples. Within the Structural Topology Research Group, I shared with Janos and Henry a program to understand the geometry and combinatorics of frameworks in 3-space.

15 years of exchanges, conjectures and conversations with Bob Connelly have also contributed to Parts I and II. Several projects with Tiong-Seng Tay contributed to Parts II and III (particularly §16 which includes results of a joint project with Tiong-Seng and Neil White on 'skeletal rigidity'). Conversations with Luis Billera and joint work with Peter Alfeld on multivariate splines also contributed to §10 and §15 in essential ways.

Finally, I owe a longstanding debt to Gian-Carlo Rota who introduced me to matroid theory and to the attractions of a good analogy. Since my days as a graduate student I have continued to learn both from his particular insights and from his broader vision: his search for patterns and analogies which connect diverse areas of mathematics – and his love of a digression!

All of this material has evolved through many hands and many minds. The community of workers in this field has been an important source of support over the last 25 years. In the end, however, the larger patterns and connections claimed here are my responsibility.

Part I: The Core Plane Matroid

We present a single matroid on the edges of a geometric graph in the Euclidean plane from four points of view. Each of the points of view has its own historical roots and its own applications. These approaches also lead to three distinct 'higher' families of matroids.

1. Statics and stresses of bar frameworks (§2). This is a well-developed theory, with roots in structural engineering of pin-jointed iron trusses and more general frameworks. The key structure is the rigidity matrix, for a graph G and a configuration $\mathbf{p}$ of the points, and the corresponding matroid of the rows. This theory has a combinatorial aspect (what happens for almost all configurations for the vertices) and a geometric aspect: the projective geometry of the algebraic variety of special configurations for the vertices which reduce the rank of the matroid. In keeping with the combinatorial tone of this chapter, we will concentrate on the generic aspects. However, in an interesting twist of proving the general through single special examples, certain generic results are best derived from very special configurations.

2. First-order rigidity of bar frameworks (§3). We will follow the lead of the classical engineers and also analyze the rank of the rigidity matrix in terms of its column rank (instantaneous kinematics). While this two sided 'dual' approach (in the sense of linear algebra) is not common in matroid theory, it is an essential feature of this field. In Part III, we will see that this two sided approach reflects the more general duality of homology and cohomology for chain complexes over fields [**Wh15**].
3. Parallel drawings (and scene analysis) (§4). For graphical analysis of instantaneous kinematics in the last century, engineers developed a geometrically equivalent representation called parallel drawing (or parallel redrawing). For modern studies, this theory has three roles:
   a. It remains easier to handle graphically (with programs such as Sketchpad) because the failures in rank then occur on a large scale as well as an 'infinitesimal scale'.
   b. Parallel drawings are natural geometric objects for certain modern geometric theories, such as Minkowski decomposition of polytopes, reciprocal diagrams, etc. and they have the simplest generalizations to all higher dimensions.
   c. Parallel drawings are the projective polar of the 'projection and lifting' problems for scene analysis of polyhedral objects (see §4, §8).
4. C_1^0-cofactors from bivariate splines (§5). This is a matroid extracted from approximation theory and CAGD. However, it records a far older connection which is implicit in Maxwell's Theorem of 1864 [**Max**,**CrW2**,**3**,**4**]:
   A realization of a planar graph in the plane is dependent in the static rigidity matroid if, and only if, it is the projection of the edges of a spatial polyhedron (on the faces identified by some planar drawing) with at least two distinct planes for faces.

We will close Part I with some other 'plane' matroids from CAD (§6) which highlight further extensions and related unsolved problems and wrap up with a summary of plane results to prepare for the step up to Part II (§7).

The broad theory of rigidity of frameworks includes many other mathematical components, some of them clearly not matroidal (see Remark 3.2.6 and Figure 3.4). We encourage the interested reader to look further into the wider literature in the references [**CoW**,**RW**,**Wh14**].

2. The Plane Rigidity Matroid: Statics

Unlike many presentations of rigidity [**GSS**,**Wh11**], we will begin with the statics (the basic dependencies of the matroid) then follow with the more traditional first-order kinematics in §3. While this approach has less immediate physical motivation, it leads us from the more traditional approach in matroid theory (dependencies) to a dual approach typical of rigidity.

2.1. The matroid from self-stresses. A *plane bar-and-joint framework*, or *plane framework* for short, is a standard graph $G = (V, E)$ (no loops or multiple edges) and a *plane configuration* $\mathbf{p} : V \to \mathbb{R}^2$, with $\mathbf{p}(i) = \mathbf{p}_i$. The framework is also written $G(\mathbf{p})$ and the configuration $\mathbf{p}$ can be treated as a point in $\mathbb{R}^{2|V|}$.

A *dependence* on the plane framework $G(\mathbf{p})$ is an assignment $\omega : E \to \mathbb{R}$, with $\omega\{i, j\} = \omega_{i,j} = \omega_{j,i}$, such that, for each vertex i:

$$\sum_{j|\{i,j\}\in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}.$$

A dependence is also called a *self-stress*, and a *non-trivial self-stress* is a dependence with $\omega_{i,j} \neq 0$ for some $\{i,j\} \in E$. These self-stresses are the row dependencies of the *rigidity matrix* of the framework, $R_G(\mathbf{p})$:

$$[\ldots \quad \omega_{i,j} \quad \ldots] \begin{bmatrix} \vdots & \ddots & \vdots & \ldots & \vdots & \ddots & \vdots \\ \mathbf{0} & \ldots & (\mathbf{p}_i - \mathbf{p}_j) & \ldots & (\mathbf{p}_j - \mathbf{p}_i) & \ldots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ldots & \vdots & \ddots & \vdots \end{bmatrix} = [\mathbf{0} \quad \ldots \quad \mathbf{0}].$$

The corresponding *plane rigidity matroid* on the edges, $\mathcal{R}_2(G;\mathbf{p})$, defines independence of sets by independence of rows of the rigidity matrix. A framework $G(\mathbf{p})$ is *independent* if its edge set is independent in $\mathcal{R}_2(G;\mathbf{p})$, and the *rank of* $G(\mathbf{p})$ is the rank of $\mathcal{R}_2(G;\mathbf{p})$. For the complete graph on n vertices, K_n, we write $\mathcal{R}_2(n;\mathbf{p})$.

Example 2.1.1. Consider the framework $G(\mathbf{p})$ of Figure 2.1A. This has a rigidity matrix as follows, with a column for a dependence ω (which is guaranteed to exist, as we will see below).

$R_G(\mathbf{p})$	ω	v_1	v_2	v_3	v_4
$\{1,2\}$	ω_{12}	$\mathbf{p}_1 - \mathbf{p}_2$	$\mathbf{p}_2 - \mathbf{p}_1$	$\mathbf{0}$	$\mathbf{0}$
$\{1,3\}$	ω_{13}	$\mathbf{p}_1 - \mathbf{p}_3$	$\mathbf{0}$	$\mathbf{p}_3 - \mathbf{p}_1$	$\mathbf{0}$
$\{1,4\}$	ω_{14}	$\mathbf{p}_1 - \mathbf{p}_4$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{p}_4 - \mathbf{p}_1$
$\{2,3\}$	ω_{23}	$\mathbf{0}$	$\mathbf{p}_2 - \mathbf{p}_3$	$\mathbf{p}_3 - \mathbf{p}_2$	$\mathbf{0}$
$\{2,4\}$	ω_{24}	$\mathbf{0}$	$\mathbf{p}_2 - \mathbf{p}_4$	$\mathbf{0}$	$\mathbf{p}_2 - \mathbf{p}_4$
$\{3,4\}$	ω_{34}	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{p}_3 - \mathbf{p}_4$	$\mathbf{p}_3 - \mathbf{p}_4$

The dependence, or self-stress, can be visualized as a set of forces, equal in magnitude and opposite in direction, in the bars (Figure 2.1B,D) which represent tension ($\omega_{i,j} < 0$) or compression ($\omega_{i,j} > 0$) pushing or pulling on the vertex. The equations $\sum_{j|\{i,j\}\in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}$ says these forces are in *equilibrium* at vertex i – an equilibrium verified graphically for the four vertices through the four small vector polygons in Figure 2.1C.

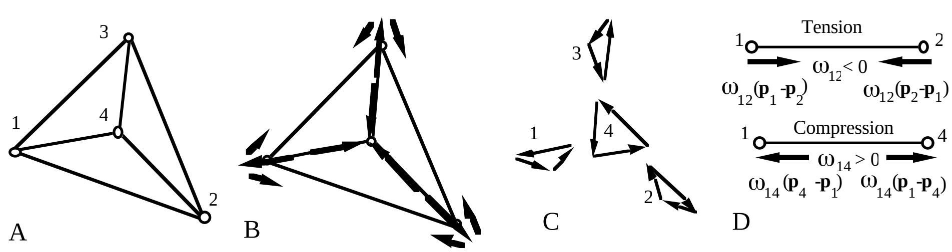

Fig. 2.1. *A dependent framework (A), with forces of a self-stress (B) and their vector equilibria at each vertex (C).*

If we consider a particular configuration $\mathbf{p}_1 = (0,0)$, $\mathbf{p}_2 = (3,0)$, $\mathbf{p}_3 = (0,3)$, and $\mathbf{p}_4 = (1,1)$, then we can verify that the assignment ω given in the second column is a row dependence.

$R_G(\mathbf{p})$	ω	v_1		v_2		v_3		v_4	
$\{1,2\}$	1	−3	0	3	0	0	0	0	0
$\{1,3\}$	1	0	−3	0	0	0	3	0	0
$\{1,4\}$	−3	−1	−1	0	0	0	0	1	1
$\{2,3\}$	1	0	0	3	−3	−3	3	0	0
$\{2,4\}$	−3	0	0	2	−1	0	0	−2	1
$\{3,4\}$	−3	0	0	0	0	−1	2	1	−2

T_x	1	0	1	0	1	0	1	0
T_y	0	1	0	1	0	1	0	1
T_r	0	0	0	3	−3	0	−1	1

Up to a single scalar multiplier, this self-stress is unique. In fact, looking at the equation for any vertex, there is a unique linear combination of the three distinct vectors (up to a single scalar). This matrix has rank 5 and the matroid has a single circuit.

The lower box shows three independent solutions to the equations $R_G(\mathbf{p})\mathbf{x} = \mathbf{0}$ (that is, three rows orthogonal to the rows of $R_G(\mathbf{p})$). These 'equilibrium coefficients' (see Remark 2.1.5) also indicate that the matrix has column rank $\leq 8-3 = 5$. Since the row rank is 5, these added vectors must span the solution space.

What is the rank of the matrix $R_G(\mathbf{p})$, and the matroid $\mathcal{R}(G;\mathbf{p})$, for the complete graph on n vertices, K_n, realized with at least two distinct points? In Example 2.1 and in subsets of these edges, we see that the ranks follow a pattern:

$$n = 4, \text{rank } = 5, \quad n = 3, \text{rank } = 3, \quad n = 2, \text{rank } = 1, \quad n = 1, \text{rank } = 0.$$

In general, for $n > 1$, the rank of $R_{K_n}(\mathbf{p})$ is $2n-3$. Notice that removing vertices and restricting the configuration does not change the rank of a set of edges on the remaining vertices, nor does changing the labels of the vertices and the corresponding points in the configuration: the matroid is 'symmetric' [**Ka2**]. We can speak of the independence and dependence of a set E' of edges at any configuration $\mathbf{p}$ for at least the vertices $V(E')$ of these edges.

We can prove that the rank of a framework $G(\mathbf{p})$, such that $\mathbf{p}$ contains at least two distinct points, is at most $2|V|-3$, by considering the equilibrium coefficients illustrated in Example 2.1.

$$\begin{array}{lcl} T_x &=& [\ \ 1 \quad 0 \quad \ldots \quad 1 \quad 0 \quad \ldots \quad 1 \quad 0\ ] \\ T_y &=& [\ \ 0 \quad 1 \quad \ldots \quad 0 \quad 1 \quad \ldots \quad 0 \quad 1\ ] \\ T_r &=& [\ -x_1 \quad y_1 \quad \ldots \quad -x_i \quad y_i \quad \ldots \quad -x_{|V|} \quad y_{|V|}\ ] \end{array}$$

It is easy to check that these are solutions to the linear equations $R_G(\mathbf{p})\mathbf{x} = \mathbf{0}$, for any graph on these vertices. For T_x and T_y, this is the observation that $(\mathbf{p}_i - \mathbf{p}_j) + (\mathbf{p}_j - \mathbf{p}_i) = \mathbf{0}$. For T_r, we have:

$$(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}_i)^\perp + (\mathbf{p}_j - \mathbf{p}_i)\cdot(\mathbf{p}_j)^\perp = (\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}_i^\perp - \mathbf{p}_j^\perp) = (\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}_i - \mathbf{p}_j)^\perp = 0.$$

where $\mathbf{p}_i^{\perp}$ represents a counterclockwise rotation of $\mathbf{p}_i$ by 90°. (See Remark 2.1.5 for a static interpretation of these solutions and §3 for a kinematic interpretation.)

T_x, T_y, T_r are three independent vectors, for at least two distinct points (check the first four columns in Example 2.1). If we do not have two distinct points, then the rank of $G(\mathbf{p})$ will be 0 for all sets of edges. We have shown the following simple sufficient condition for dependence.

COUNTING LEMMA 2.1.2. *Any non-empty set of edges E with $|E| > 2|V(E)| - 3$ is dependent for every plane configuration $\mathbf{p}$. Equivalently, a set E is independent only if, for all non-empty subsets E'', $|E''| \leq 2|V(E'')| - 3$.*

By induction, we can also demonstrate that some independent graphs have this maximal rank $2|V| - 3$, at general position configurations $\mathbf{p}$ – i.e. they are bases of the matroid $\mathcal{R}_2(|V|; \mathbf{p})$. Given a graph $G = (V, E)$, a *vertex 2-addition of* 0 is the addition of one new vertex, 0, and two new edges $(0, i), (0, j)$ creating the graph $G' = (V', E')$.

A graph $G = (V, E)$, with at least two vertices, is 2-*simple* if there is an ordering of the vertices $\sigma(1), \sigma(2), \ldots, \sigma(|V|)$ such that:

(i) G_2 is the single edge $\{\sigma(1), \sigma(2)\}$;
(ii) for $2 \leq i < |V|$, G_{i+1} is a vertex 2-addition of $\sigma(i+1)$ to G_i;
(iii) $G_{|V|}$ is G.

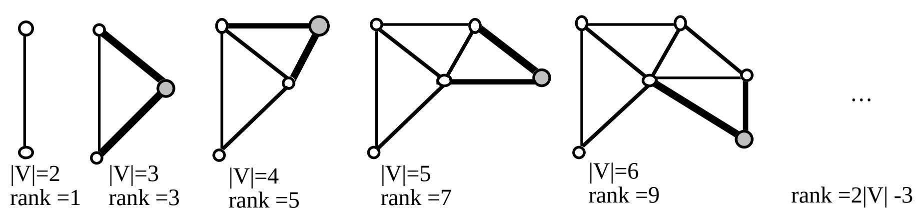

Fig. 2.2. *Building a 2-simple graph as a basis for the plane rigidity matroid $\mathcal{R}_2(n; \mathbf{p})$.*

VERTEX 2-ADDITION LEMMA 2.1.3. *Given a framework $G(\mathbf{p})$ and a vertex 2-addition of 0 creating the framework $G'(\mathbf{p}_0, \mathbf{p})$, with $\mathbf{p}_0, \mathbf{p}_i, \mathbf{p}_j$ not collinear, then*

1. $G'(\mathbf{p}_0, \mathbf{p})$ *is independent if and only if* $G(\mathbf{p})$ *is independent;*
2. rank $R_{G'}(\mathbf{p}_0, \mathbf{p}) =$ rank $R_G(\mathbf{p}) + 2$.

PROOF. Consider the rigidity matrix for $R_{G'}(\mathbf{p}_0, \mathbf{p})$:

$R_{G'}(\mathbf{p}_0, \mathbf{p})$	0	1	$\ldots$	$\lvert V\rvert$
e_1	$\mathbf{0}$			
$\vdots$	$\vdots$		$R_G(\mathbf{p})$	
$e_{\lvert E\rvert}$	$\mathbf{0}$			
$\{0, i\}$	$\mathbf{p}_0 - \mathbf{p}_i$	$\ldots$	$\ldots$	$\ldots$
$\{0, j\}$	$\mathbf{p}_0 - \mathbf{p}_j$	$\ldots$	$\ldots$	$\ldots$

Part 1. The vertex 2-addition requires that G has at least two vertices. Assume there is a dependence in $R_{G'}(\mathbf{p}_0, \mathbf{p})$. For the first two columns, the equilibrium equations are:

$$\omega_{0,i}(\mathbf{p}_0 - \mathbf{p}_i) + \omega_{0,j}(\mathbf{p}_0 - \mathbf{p}_j) = \mathbf{0}.$$

Now $(\mathbf{p}_0 - \mathbf{p}_i)$ and $(\mathbf{p}_0 - \mathbf{p}_j)$ are linearly independent if and only if $\mathbf{p}_0, \mathbf{p}_i, \mathbf{p}_j$ are not collinear, in which case $\omega_{0,i} = \omega_{0,j} = 0$. ω is a non-trivial dependence if and only if this is a non-trivial dependence on $R_G(\mathbf{p})$.
Part 2. As noted above the two added rows are independent of any basis for $R_G(\mathbf{p})$. Thus the rank has increased by 2.

A *general position* plane configuration $\mathbf{p}$ has any set of at most 3 points affinely independent (i.e., no three points are collinear and any two points are distinct).

Static 2-Rigidity Theorem 2.1.4. *For any $n \geq 2$ and any general position configuration $\mathbf{p}$ on n vertices, the edges E of any 2-simple graph G on n vertices are a basis of $\mathcal{R}_2(n;\mathbf{p})$ of rank $2n-3$.*

Proof. If $n = 2$, K_n is a single edge which has rank $1 = 2 \times 2 - 3$. This graph is both 2-simple and a basis of the matroid.

If $n > 2$, we prove by induction that there is a 2-simple graph G whose edges are a basis for K_n of size $2n-3$. Assume G_k is a 2-simple graph for $n = k$, which is a basis for $\mathcal{R}_2(k;\mathbf{p}|_k)$ of rank $2k-3$. Let G_{k+1} be a vertex 2-addition of $k+1$ with edges $(1, k+1), (2, k+1)$. Since $\mathbf{p}$ is in general position, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_{k+1}$ are not collinear, and E_{k+1} is independent of rank $2k - 3 + 2 = 2(k+1) - 3$ by Lemma 2.1.3. By Lemma 2.1.2, this is a maximal independent set in K_{k+1}, so E_{k+1} is a basis. This completes the induction step.

Any framework $G(\mathbf{p})$ for which $R_G(\mathbf{p})$ has rank $2|V| - 3$ (or for which $|V| \leq 1$) is *statically 2-rigid*. We also say that the edge set E is *statically 2-rigid* on $V(E)$ at $\mathbf{p}$. There are other edge sets which are not 2-simple but which satisfy the condition of Lemma 2.1.2 to be independent for some configurations (Figure 2.3A,B). We will see below that these are also statically 2-rigid for some choices of $\mathbf{p}$, including the illustrated configurations.

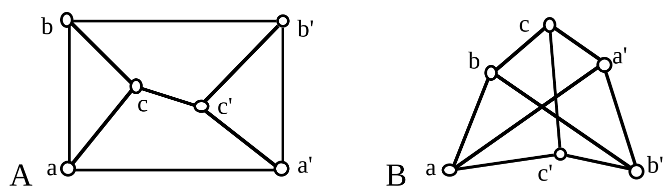

Fig. 2.3. *Graphs which are not 2-simple, but are bases for the 2-rigidity matroid.*

Remark 2.1.5. For a civil engineer, static rigidity of a framework means that the framework will 'resolve all the permitted external loads'. We can interpret our definition in precisely these terms.

The vocabulary describes the the row space of the rigidity matrix. An *equilibrium load* is an assignment $\mathbf{L} : V \to \mathbb{R}^2$ of vectors $\mathbf{L}_i$ to the vertices satisfying the equilibrium equations:

$$(i) \quad \sum_{i \in V} \mathbf{L}_i = \mathbf{0} \quad \text{and} \quad (ii) \quad \sum_{i \in V} \mathbf{L}_i \cdot (\mathbf{p}_i)^{\perp} = 0.$$

These equations say that the forces of the load have: (i) no net translational component; and (ii) no net rotational component. Equivalently, an equilibrium load is any vector orthogonal to the three vectors T_x, T_y, T_r defined above. In particular, the entries of any row of the rigidity matrix for $K_{|V|}(\mathbf{p})$ form an equilibrium load on these vertices. The row space of $R_G(\mathbf{p})$ is a subset of the space of equilibrium loads.

We have already seen that the equilibrium equations have rank 3, for frameworks with at least two distinct vertices. Therefore the space of equilibrium loads has dimension $2|V| - 3$. The only equilibrium load with $|V| = 1$ is the zero load. The only equilibrium loads on two distinct vertices have the form $\lambda(\mathbf{p}_1 - \mathbf{p}_2)$ at 1 and $\lambda(\mathbf{p}_2 - \mathbf{p}_1)$ at 2. Notice that the definition of an equilibrium load depends on the vertices and the configuration, not on the edges of the framework.

A *resolution* of the equilibrium load $\mathbf{L}$ on the framework $G(\mathbf{p})$ is a *stress*: an assignment of scalars $\omega : E \to \mathbb{R}$, such that, for each vertex i:

$$\mathbf{L}_i + \sum_{j|\{i,j\}\in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}.$$

This resolution achieves an equilibrium of the internal forces (tensions and compressions) and the external load at each vertex (Figure 2.4). A self-stress is a resolution of the zero equilibrium load.

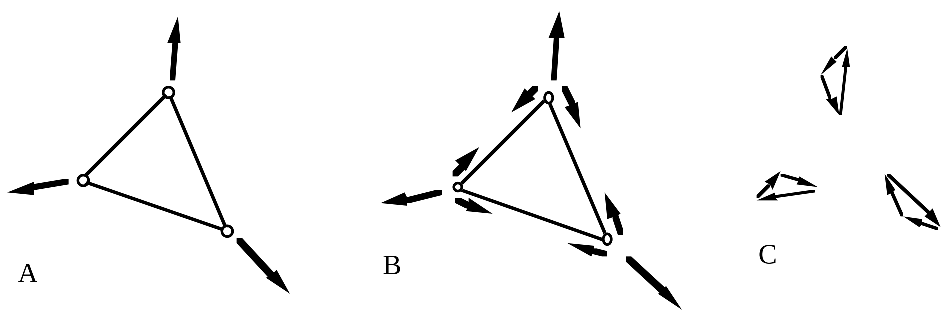

Fig. 2.4. *An equilibrium load on a statically* 2*-rigid framework (A), with its resolution (B) and the visual check of equilibria at the vertices (C).*

For each edge $\{i, , j\} \in E$, we write the corresponding row of the rigidity matrix as $R_{i,j}(\mathbf{p})$. The resolution of an equilibrium load $\mathbf{L}$ is a linear combination of these rows:

$$\mathbf{L} + \sum_{\{i,j\}\in E} \omega_{i,j} R_{i,j}(\mathbf{p}) = \mathbf{0}.$$

By definition, a framework $G(\mathbf{p})$ is statically 2-rigid if the dimension of the row space equals the dimension of the equilibrium loads. Since the row space is contained in the space of equilibrium loads, the two spaces are the same.

COROLLARY 2.1.6. *A framework $G(\mathbf{p})$ is statically 2-rigid if and only if each equilibrium load has a resolution by a stress in the edges of the framework.*

This does correspond to the civil engineer's concept of static rigidity. If the framework is independent then the resolution of an external equilibrium load is unique. However, if the framework is dependent then we can add any multiple of a self-stress to a given resolution to get another resolution. Which resolution actually appears in a physical loaded framework will depend on the elasticity of the materials and any prestress (self-stress) built into the structure [**CoW**].

REMARK 2.1.7. If we choose a configuration $\mathbf{p}$ consisting of distinct collinear points then the matrix and the matroid take on a familiar form.

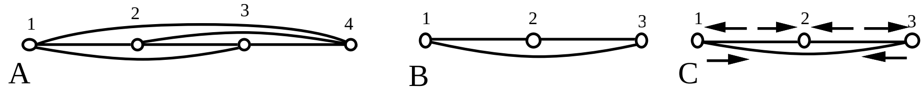

Fig. 2.5. *In a collinear framework (A), every polygon (B) has a dependence (C).*

Consider the framework of Figure 2.5A with all vertices collinear on a line with unit direction $\mathbf{u}$. All entries have the form $\mathbf{p}_i - \mathbf{p}_j = l_{i,j}\mathbf{u}$, where $l_{i,j}$ is a signed length for $i < j$. The rigidity matrix now looks like:

$R_G(\mathbf{p})$	ω	1	2	3	4
$\{1,2\}$	$\frac{1}{l_{1,2}}$	$l_{1,2}\mathbf{u}$	$-l_{1,2}\mathbf{u}$	$\mathbf{0}$	$\mathbf{0}$
$\{1,3\}$	$-\frac{1}{l_{1,3}}$	$l_{1,3}\mathbf{u}$	$\mathbf{0}$	$-l_{1,3}\mathbf{u}$	$\mathbf{0}$
$\{1,4\}$	0	$l_{1,4}\mathbf{u}$	$\mathbf{0}$	$\mathbf{0}$	$-l_{1,4}\mathbf{u}$
$\{2,3\}$	$\frac{1}{l_{2,3}}$	$\mathbf{0}$	$l_{2,3}\mathbf{u}$	$-l_{2,3}\mathbf{u}$	$\mathbf{0}$
$\{2,4\}$	0	$\mathbf{0}$	$l_{2,4}\mathbf{u}$	$\mathbf{0}$	$-l_{2,4}\mathbf{u}$
$\{3,4\}$	0	$\mathbf{0}$	$\mathbf{0}$	$l_{3,4}\mathbf{u}$	$-l_{3,4}\mathbf{u}$

After dividing each row $R_{i,j}$ by the non-zero scalar $l_{i,j}$, all entries are $\mathbf{0}$ or $\pm\mathbf{u}$, and the pattern of the $\mathbf{u}$ entries is the pattern of the usual matrix representation of the cycle matroid of the graph G, as the rows of a matrix over $\mathbb{R}$ [**Wh11**]. In particular, every polygon has a dependence (self-stress), as illustrated by the coefficients for the polygon $(1,2),(2,3),(3,1)$ (Figure 2.5B,C) above.

Proposition 2.1.8. *For a configuration $\mathbf{p}$ of distinct collinear points on the vertices of a graph G, the matroid $\mathcal{R}_2(G;\mathbf{p})$ is the cycle matroid of the graph G. In particular:*

1. *A set E' of edges is independent if and only if it is a forest;*
2. *A set E' of edges is independent if and only if, for all nonempty subsets E'', $|E''| \leq |V(E'')| - 1$;*
3. *A set E' has rank $|V(E')| - 1$ if and only if $G' = (V(E'), E')$ is a connected graph;*
4. *A set E' is a basis for the 2-rigidity matroid on $K_n(\mathbf{p})$ if and only if it is a spanning tree on the n vertices;*
5. *A set E' is a basis for the 2-rigidity matroid on $K_n(\mathbf{p})$ if and only if $|E'| = n - 1$ and for all subsets E'', $|E''| \leq |V(E'')| - 1$.*

Our goal in the next section is a similar characterization of the bases of $\mathcal{R}_2(n;\mathbf{p})$ for the most general or 'generic' plane configurations $\mathbf{p}$. From Remark 2.1.7, we already see that the rank of the matroid depends on the geometric placement of the configuration $\mathbf{p}$. However, for any given graph there is a maximal rank over all plane configurations. Consider the rigidity matrix $R_{K_n}(\mathbf{x})$, where the positions $\mathbf{p}_i$ are replaced by indeterminates (x_i, y_i). The rank of any subset of edges can be determined by maximal non-zero minors on these rows – which are polynomials in these indeterminates. Such polynomials are either zero for all reals or they define an algebraic variety of *singular positions* for which they are zero and are non-zero on the complement – an open dense subset of $\mathbb{R}^{2|V|}$. The union of these singular varieties over the finite number of such non-zero minors in $R_{K_n}(\mathbf{x})$ defines the *singular configurations* on n-vertices. The open dense complement of the singular configurations for n-vertices is the set of *generic configurations for n-vertices*.

If the coordinates of a configuration are algebraically independent reals then the configuration will be generic. Any generic configuration on n-vertices has an extension to a generic configuration on $n+k$ vertices – adding points with coordinates algebraically independent of the existing coordinates – and any restriction of a generic configuration on $n+k$ vertices is a generic configuration on n vertices. We will suppress discussion of these extensions and restrictions of generic configurations.

Each of these generic configurations gives a set of edges on at most n vertices its maximal possible rank and defines the same *generic* 2*-rigidity matroid* on the complete graph on these vertices, $\mathcal{R}_2(n)$. A set E of edges is *generically* 2*-independent* or *independent in* $\mathcal{R}_2(n)$ if E is independent in the framework $(V(E), E)(\mathbf{p})$ for some generic configuration $\mathbf{p}$ (therefore all generic configurations). A set of edges E is *generically* 2*-rigid* if E if the framework $(V(E), E)(\mathbf{p})$ is statically 2-rigid for some generic plane configuration $\mathbf{p}$. A graph G is generically 2-independent (generically 2-rigid) if its edge set E is generically 2-independent (generically 2-rigid).

Notice that an edge set E is generically 2-rigid if and only if the $\mathcal{R}_2(V(E))$-matroid closure of E, written $\langle E\rangle$, is the complete graph on its vertices. Similarly, at any non-collinear configuration $\mathbf{p}$, an edge set is statically 2-rigid if and only if closure of E in $\mathcal{R}_2(V(E); \mathbf{p})$ is the complete graph. However, at a collinear configuration $\mathbf{p}$ on at least 3-vertices, a graph $G = (V, E)$ is *never statically* 2*-rigid*, though a spanning tree E will have $K_{V(E)}$ as its $\mathcal{R}_2(V; \mathbf{p})$ matroid closure.

2.2. Bases of the generic 2-rigidity matroid. In the Counting Lemma 2.1.2, we gave a necessary condition for independent sets E' in the generic 2-rigidity matroid:

$$\textit{for all non-empty subsets } E'',\ |E''| \leq 2|V(E'')| - 3.$$

This is also a sufficient condition for independence.

We begin with an inductive construction for bases of the generic 2-rigidity matroid on K_n. One basic technical 'trick' for establishing independence or rank results for generic 2-rigidity is the following.

Special Position Lemma 2.2.1. *For any set of edges E, the following are equivalent:*

1. *E is independent in the generic 2-rigidity matroid;*
2. *E is independent in $\mathcal{R}_2(V(E); \mathbf{p})$ for one configuration $\mathbf{p}$.*

Proof. This follows from the fact that if a polynomial in $2n$ variables is non-zero at one point (or configuration), then it is non-zero at almost all configurations.

A graph G' is an *edge* 2*-split* of the graph G on $a, b; c$, if $\{a, b\}$ is an edge of G and G' is formed from G by adding a new vertex 0, removing the edge $\{a, b\}$, and adding three new edges $\{0, a\}, \{0, b\}, \{0, c\}$.

Edge 2-Split Theorem 2.2.2. *Assume G' is an edge 2-split of $G = (V, E)$ on $a, b; c$, and $\mathbf{p}$ is a plane configuration with $\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c$ not collinear.*

If $G(\mathbf{p})$ is 2-independent (statically 2-rigid), then $G'(\mathbf{p}_0, \mathbf{p})$ is 2-independent (statically 2-rigid) for almost all choices of $\mathbf{p}_0$, including $\mathbf{p}_0$ a distinct point on $\mathbf{p}_a, \mathbf{p}_b$ (Figure 2.6B).

Conversely, if $G'(\mathbf{p}_0, \mathbf{p})$ is 2-independent (statically 2-rigid) for some choice of $\mathbf{p}_0$, with vertex 0 connected to exactly vertices a, b, c at three non-collinear points then, for some edge e of with endpoints in a, b, c, $E' = E \cup \{e\} - \{(0, a), (0, b), (0, c)\}$ is independent (statically 2-rigid) at $\mathbf{p}$ and G' is an edge 2-split of $G = (V' - 0, E)$.

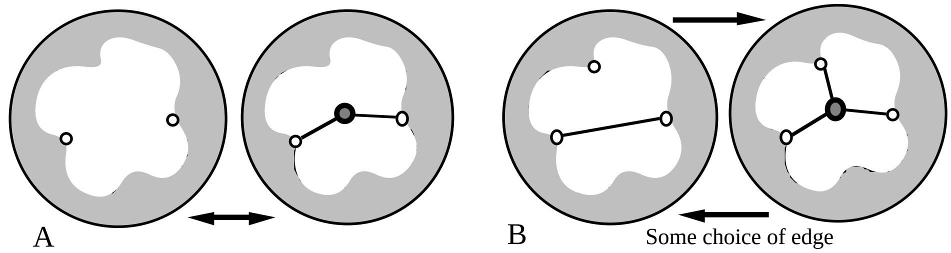

Fig. 2.6. *Two inductive steps which preserve static 2-rigidity and give Henneberg 2-constructions for bases: (A) vertex 2-addition; (B) edge 2-split.*

Proof. We first verify the implications for independence. We choose to place $\mathbf{p}_0$ at a distinct point on the line $\mathbf{p}_a, \mathbf{p}_b$. Up to reordering the vertices and edges, the rigidity matrix for G' has the form:

$R_{G'}(\mathbf{p}_0, \mathbf{p})$	0	a	b	$\dots$
e_1	$\mathbf{0}$	$\dots$	$\dots$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$e_{\lvert V \rvert}$	$\mathbf{0}$	$\dots$	$\dots$	$\dots$
$\{0, a\}$	$\mathbf{p}_0 - \mathbf{p}_a$	$\mathbf{p}_a - \mathbf{p}_0$	$\mathbf{0}$	$\dots$
$\{0, b\}$	$\mathbf{p}_0 - \mathbf{p}_b$	$\mathbf{0}$	$\mathbf{p}_b - \mathbf{p}_0$	$\dots$
$\{0, c\}$	$\mathbf{p}_0 - \mathbf{p}_c$	$\mathbf{0}$	$\mathbf{0}$	$\dots$

Assume there is a non-trivial dependence in $G(\mathbf{p})$. The equations for vertex 0 read:

$$\omega_{0,a}(\mathbf{p}_0 - \mathbf{p}_a) + \omega_{0,b}(\mathbf{p}_0 - \mathbf{p}_b) + \omega_{0,c}(\mathbf{p}_0 - \mathbf{p}_c) = \mathbf{0}.$$

Since $\omega_{0,a}(\mathbf{p}_0 - \mathbf{p}_a)$ and $\omega_{0,b}(\mathbf{p}_0 - \mathbf{p}_b)$ are parallel, and $\omega_{0,c}(\mathbf{p}_0 - \mathbf{p}_c)$ is in a distinct direction, we conclude that $\omega_{0,c} = 0$ and $\omega_{0,a}(\mathbf{p}_0 - \mathbf{p}_a) = -\omega_{0,b}(\mathbf{p}_0 - \mathbf{p}_b) = \omega_{a,b}(\mathbf{p}_a - \mathbf{p}_b)$ for some scalar $\omega_{a,b}$. ($\omega_{a,b} \neq 0$ if $\omega_{0,b} \neq 0$.) If we transfer $\omega_{a,b}$ to $\{a, b\}$, and transfer the scalars on all edges $E' \cap E$ from G' to G, we must have a non-trivial dependence on E. We conclude that if $G(\mathbf{p})$ is independent, then $G'(\mathbf{p}_0, \mathbf{p})$ is independent. Since this works for the special positions of $\mathbf{p}_0$, the same independence will hold for almost all positions $\mathbf{p}_0$ by the Special Position Lemma.

Conversely, assume that $G'(\mathbf{p}_0, \mathbf{p})$ is independent for some choice of $\mathbf{p}_0$, with vertex 0 connected to exactly vertices a, b, c at three non-collinear points. Consider the graphs $G_{a,b}$, $G_{b,c}$ and $G_{a,c}$, formed by deleting vertex 0 and its edges, to create E^* and adding the subscripted edge. (If this edge is already present, we 'double' the edge to get the dependence needed below.) If any one of these is independent at $\mathbf{p}$, we are finished.

Otherwise, assume that these have dependencies, α, β, γ. We have:

$$\alpha_{a,b} R_{a,b} = \sum_{e \in E^*} -\alpha_e R_e \quad \text{with} \quad \alpha_{a,b} \neq 0$$

$$\beta_{b,c} R_{b,c} = \sum_{e \in E^*} -\beta_e R_e \quad \text{with} \quad \beta_{b,c} \neq 0$$

$$\gamma_{a,c} R_{a,c} = \sum_{e \in E^*} -\gamma_e R_e \quad \text{with} \quad \gamma_{b,c} \neq 0$$

In the graph K_4 on $\{0, a, b, c\}$ at $\mathbf{p}'$ which has $E'' = 6 > 2|V| - 3$, we have a dependence ω, which must be non-zero on at least one of the edges at 0, since the noncollinear triangle a, b, c is independent. This gives

$$\omega_{0,a}R_{0,a} + \omega_{0,b}R_{0,b} + \omega_{0,c}R_{0,c} + \omega_{a,b}R_{a,b} + \omega_{b,c}R_{b,c} + \omega_{a,c}R_{a,c} = \mathbf{0}.$$

Substituting from above, we have:

$$\omega_{0,a}R_{0,a} + \omega_{0,b}R_{0,b} + \omega_{0,c}R_{0,c} + \sum_{e \in E^*} -(\alpha_e + \beta_e + \gamma_e)R_e = \mathbf{0}.$$

This is a non-trivial dependence on $R_{G'}(\mathbf{p})$, contradicting our assumption. We conclude that one of $G_{a,b}$, $G_{b,c}$ and $G_{a,c}$ is independent at $\mathbf{p}$, as required.

We now check the static 2-rigidity. An edge 2-split adds one vertex and adds a net of two edges. This exchanges the count $|E| = 2n - 3$ on G with the count $|E'| = 2(n + 1) - 3$ on G'. Since the edge 2-split (or the converse) preserves independence, we conclude that this takes a basis for K_n to a basis for K_{n+1}.

For a graph $G = (V, E)$ with at least two vertices, a *Henneberg* 2*-construction* is an ordering of the vertices $\sigma(1), \sigma(2), \ldots, \sigma(|V|)$ and a sequence of graphs $G_2, \ldots, G_{|V|}$ such that:

(i) G_2 is the single edge $\{\sigma(1), \sigma(2)\}$;

(ii) for $2 \le i < |V|$, G_{i+1} is a 2-addition of vertex $\sigma(i+1)$ to G_i or G_{i+1} is an edge 2-split on G_i which adds a vertex $\sigma(i+1)$;

(iii) $G_{|V|}$ is G.

Henneberg's Theorem 2.2.3 [**He**,**TW2**]. *A graph G with at least two vertices is a basis for $\mathcal{R}_2(n)$ if and only if G has a Henneberg* 2*-construction.*

Proof. Assume that G has a Henneberg 2-construction:

$$G_2, \ldots, G_k, G_{k+1}, \ldots, G_{|V|} = G.$$

G_2 is a a single edge on two vertices – a basis for K_2 at any generic configuration. Assume that G_k is a basis for $K_k(\mathbf{p})$ for some generic $\mathbf{p}$. G_{k+1} is constructed from G_k by either (i) vertex 2-addition or (ii) an edge 2-split. By the Vertex 2-Addition Theorem or the Edge 2-Split Theorem, these make $G_{k+1}(\mathbf{p}')$ generically 2-independent and generically 2-rigid.

Conversely, assume G is a basis for $\mathcal{R}_2(|V|)$. We prove there is a Henneberg 2-construction by induction on the number of vertices. If $|V| = 2$, we are finished with $G_2 = G$.

Otherwise, assume that all basic graphs on $|V| - 1$ vertices have a Henneberg 2-construction and G has $|V|$ vertices with $|V| > 2$. Since $|E| = 2|V| - 3$, we have $2|E| = 4|V| - 6$. Therefore there is a vertex of valence ≤ 3. If any vertex i has valence < 2 then deleting i leaves G' with $|E'| \ge |E| - 1 = 2|V| - 4 = 2|V'| - 2$. We conclude that G' is dependent, which contradicts the independence of G. Therefore, there is a vertex $\sigma(|V|) = 0$ of valence 2 or 3.

Assume 0 has valence 2. G is now a vertex 2-addition of the graph G' obtained by deleting 0. G is basic on $|V| - 1$ vertices. This gives the last step of a Henneberg 2-construction, which turns the 2-construction of G' (guaranteed by induction) into a Henneberg 2-construction of G.

Assume 0 has valence 3. Then, by the Edge 2-Split Theorem, G is an edge 2-split of some basic G', on $|V| - 1$ vertices. Again, this gives the last step of a

Henneberg 2-construction which turns the guaranteed 2-construction of G' into a Henneberg 2-construction of G.

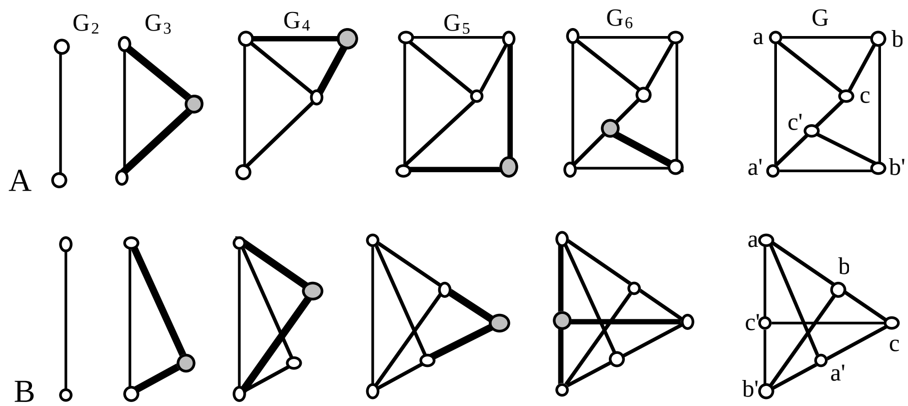

Fig. 2.7. *Henneberg 2-construction for two bases of the generic 2-rigidity matroid on K_6.*

Remark 2.2.4. While we will concentrate on the generic configurations, we note that the geometry of special configurations of these graphs belongs to projective geometry [**CrW1**,**RW**,**WW1**]. For example:

A. For the graph of Figure 2.7A (and Figure 2.3A), the framework $G(\mathbf{p})$ is dependent if and only if, either one of the triangles a, b, c or a', b', c' is collinear or the three lines aa', bb', cc' are concurrent. Equivalently, by Desargues' Theorem of projective geometry, the framework $G(\mathbf{p})$ is dependent if and only if the three points of intersection $ab \wedge a'b'$, $bc \wedge b'c'$, $ac \wedge a'c'$ are collinear [**WW1**].

B. For the bipartite graph $K_{3,3}$ of Figure 2.7B, the framework $G(\mathbf{p})$ is dependent if and only if the six vertices lie on a conic section. Equivalently, by Pascal's Theorem of projective geometry, the framework $G(\mathbf{p})$ is dependent if and only if the three points $ab \wedge a'b'$, $bc \wedge b'c'$, $ac' \wedge a'c$ are collinear [**WW1**].

Laman's Theorem 2.2.5 [**La**]. *A non-empty set of edges E' on $\{1, \dots, n\}$ is a basis for the generic 2-rigidity matroid on K_n if and only if $|E'| = 2n-3$ and for all proper subsets E'', $|E''| \leq 2|V(E'')| - 3$.*

Proof. The Counting Lemma proves the counting condition is necessary for 2-independence.

We prove the sufficiency by induction on n. If $n = 2$ then $|E'| = 2 \times 2 - 3 = 1$ means we have a single edge, which is a generic basis. Assume the sufficiency is true for all $n \leq k$ and that $|E'| = 2(k+1) - 3$. By the same argument used in the proof of Henneberg's Theorem, we have a vertex 0 of valence 2 or 3.

Assume 0 has valence 2. It is simple to check that D, formed by deleting this vertex and its edges, satisfies $|D| = 2k - 3$ and for all nonempty subsets $|D'| \leq 2|V(D')| - 3$, since these are subsets of E'. We conclude that D is a basis for K_k and, by vertex 2-addition, E' is an independent set of rank $2(k+1) - 3$ – a basis for K_{k+1}.

Assume 0 has valence 3, attached to a, b, c. Consider the sets $D_{a,b}, D_{b,c}, D_{a,c}$ formed by deleting 0, and adding the subscripted edge. (If the edge already lies in E', then we work with a double edge.) By a simple count, $|D_e| = 2|V(D_e)| - 3$, for each pair e. If one of these sets satisfies $|D'| \leq 2|V(D')| - 3$, for all non-empty subsets D', then this D_e will satisfy the induction hypothesis. Therefore, this D_e will be a basis for K_k. By the Edge 2-Split Theorem, E' will be a basis for K_{k+1}.

We now seek a contradiction from the assumption that each of $D_{a,b}, D_{b,c}, D_{a,c}$ contains a (minimal) non-empty subset $D'_{a,b}, D'_{b,c}, D'_{a,c}$ with $|D'_e| \geq 2|V(D'_e)| - 2$, and $|D''_e| = 2|V(D''_e)| - 3$ for non-empty subsets. By the induction hypothesis, each of these is a *circuit* in the generic 2-rigidity matroid (removing any one edge leaves an independent set, but the whole set is dependent). (If we have doubled an edge e, this doubled edge will be the circuit – and we have the 'dependence' with opposite scalars on the two copies of the edge, in any configuration with distinct vertices.) Working at a generic configuration $\mathbf{p}$ for k vertices, we have the three row dependencies, with D formed by deleting 0 and its edges:

$$\alpha_{a,b}R_{a,b} = -\sum_{e\in D}\alpha_e R_e \quad \alpha_{a,b} \neq 0;$$
$$\beta_{b,c}R_{b,c} = -\sum_{e\in D}\beta_e R_e \quad \beta_{b,c} \neq 0;$$
$$\gamma_{a,c}R_{a,c} = -\sum_{e\in D}\gamma_e R_e \quad \gamma_{a,c} \neq 0.$$

As in the proof of the Edge 2-Split Theorem, adding in the 3-valent vertex 0 gives a non-trivial dependence on E' – the required contradiction.

A simple observation on these counts shows that a circuit for the generic 2-rigidity matroid has $|E'| = 2|V(E')| - 2$ and $|E''| \leq 2|V(E'')| - 3$ for all proper subsets. This implies the following result.

2-RIGID CIRCUITS COROLLARY 2.2.6. *If E is a circuit in the generic 2-rigidity matroid, then E is generically 2-rigid.*

The Henneberg 2-constructions give inductive constructions for all bases of the generic 2-rigidity matroid. They can be easily adapted to construct either independent sets (add fewer edges in the construction) or generically 2-rigid sets (add extra edges in the construction). There are other inductive techniques which will reappear as results and conjectures for generic 3-rigidity [**TW2**,**Wh9**]. We give two of these results in visual form in Figures 2.8, 2.9.

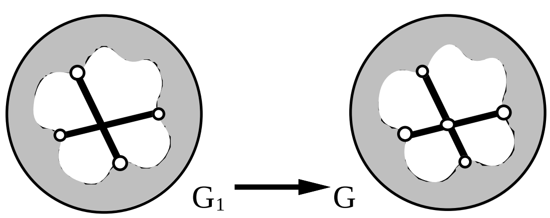

Fig. 2.8. *X-replacement - an inductive technique which replaces two edges by a 4-valent vertex and preserves generic 2-independence and 2-rigidity.*

We close this section with a conjecture about circuits in the generic 2-rigidity matroid [**GSS** Exercise 4.17].

CONNELLY'S CONJECTURE 2.2.7. *For the generic 2-rigidity matroid, every vertex 3-connected circuit has an inductive construction from K_4 using edge 2-splits.*

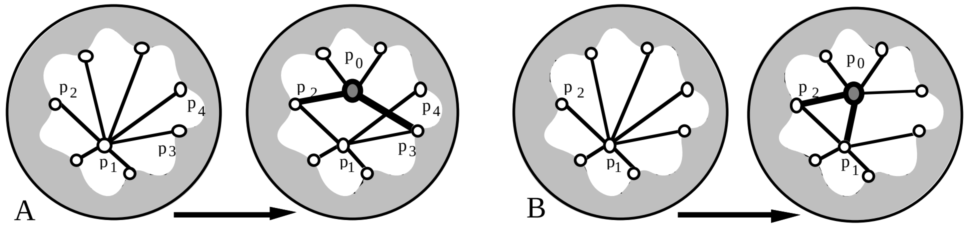

Fig. 2.9. *Two forms of vertex 2-splits – in which a given vertex 1 is split into two copies 0, 1, with the original edges at 1 partitioned between them (plus the heavy added or doubled lines). This inductive step preserves generic 2-independence and 2-rigidity.*

The results of §2 depend on a mixture of arguments based the counts $|E| \leq 2|V(E)| - 3$ and arguments based on the form of the rigidity matrix. In Appendix §A, we will see that these counts alone define a matroid on the edges of a complete graph. (This analysis is based on standard matroid constructions from submodular functions, non-negative on non-empty sets.) A number of the results of this section also depend only on these counts. For example, the existence of a Henneberg 2-construction for each 'count basis' of K_n could be proven directly from the counts. These counting techniques will break down for generic rigidity in 3-space.

The matrix-based techniques, such as vertex 2-addition, edge 2-splits and vertex 2-splits will generalize to higher dimensions, but will prove insufficient to characterize the generic 3-rigidity matroid.

3. The Plane Rigidity Matroid: Kinematics

Since our matroid is defined in terms of the row rank of the rigidity matrix, we can also analyze the matroid through the column rank or kernels of this matrix. In engineering, these solutions are called 'infinitesimal motions' and their study is called infinitesimal or first-order kinematics.

3.1. First-order rigidity in the plane. A *first-order flex* or *infinitesimal motion* of a plane framework $G(\mathbf{p})$ is an assignment of *velocities* to the vertices $\mathbf{u} : V \to \mathbb{R}^2$, such that for each edge $\{i, j\} \in E$ we have $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0$ (Figure 3.1). Equivalently, a first-order motion is a solution to the system of linear equations: $R_G(\mathbf{p})\mathbf{x} = \mathbf{0}$. The equation can also be written as $(\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{u}_i = (\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{u}_j$, and visualized as: the two velocities $\mathbf{u}_i, \mathbf{u}_j$ have the same orthogonal projections onto the edge $\mathbf{p}_i - \mathbf{p}_j$ (Figure 3.1A).

A first-order flex $\mathbf{u}$ is a *trivial first-order flex* if it is a linear combination of the generating first-order flexes:

$$\begin{array}{rcl} T_x & = & [\begin{array}{ccccccccc} 1 & 0 & \dots & 1 & 0 & \dots & 1 & 0 \end{array}] \\ T_y & = & [\begin{array}{ccccccccc} 0 & 1 & \dots & 0 & 1 & \dots & 0 & 1 \end{array}] \\ T_r & = & [\begin{array}{ccccccccc} -x_1 & y_1 & \dots & -x_i & y_i & \dots & -x_{|V|} & y_{|V|} \end{array}] \end{array}$$

In this interpretation, $\alpha T_x + \beta T_y$ is a translation with velocity $\mathbf{u}_i = (\alpha, \beta)$ at each vertex. Similarly, $\mathbf{u}_i = (\mathbf{p}_i)^{\perp}$ is the velocity of a counterclockwise rotation about the origin. The trivial first-order motions are the velocities of a general congruence of the configuration (Figure 3.1A,B).

A plane framework $G(\mathbf{p})$ is *first-order rigid* if every first-order flex is trivial (Figures 2.2, 2.3). Otherwise it is *first-order flexible* (Figure 3.1C,D). In §2.1 we checked that the trivial first-order flexes form a space of dimension 3 on at least two

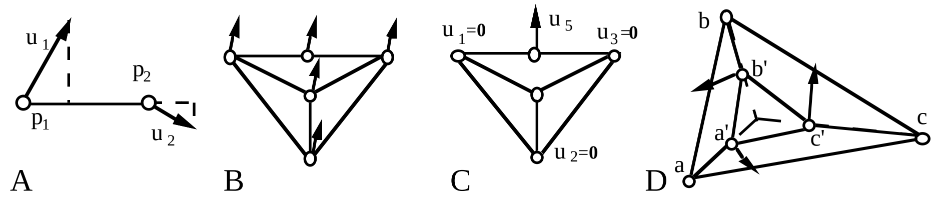

Fig. 3.1. *The arrows indicate the non-zero velocities of trivial first-order motions (A,B) and non-trivial motion (C,D) of frameworks.*

distinct points, and dimension 2 (the translations) on one point. With the Static 2-Rigidity Theorem 2.1.4, this proves the following basic equivalence [**RW**,**Wh4**]:

2-Rigidity Equivalence 3.1.1. *For a plane framework $G(\mathbf{p})$ with at least two vertices, the following are equivalent:*

1. $G(\mathbf{p})$ *is first-order rigid;*
2. $G(\mathbf{p})$ *is statically* 2*-rigid;*
3. *the rigidity matrix* $R_G(\mathbf{p})$ *has rank* $2|V|-3$*.*

Corollary 3.1.2. *For a framework $G(\mathbf{p})$, with the points of $\mathbf{p}$ in general position, a first-order motion is non-trivial if and only if for some pair of vertices h, k (not an edge) $(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) \neq 0$.*

Proof. In §2.1, we checked that the generators for the trivial motions satisfy

$$(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) = (\mathbf{p}_h - \mathbf{p}_k) \cdot \mathbf{u}_h + (\mathbf{p}_k - \mathbf{p}_h) \cdot \mathbf{u}_k = 0$$

for each pair h, k (Figure 3.1 A). This same orthogonality now applies to the entire space of trivial first-order motions. Therefore, any first-order flex with $(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) \neq 0$ is not trivial.

Conversely, if $(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) = 0$ for each pair h, k, then $\mathbf{u}$ is a first-order flex of the complete framework $K_{|V|}(\mathbf{p})$. By the Static 2-Rigidity Theorem 2.1.4, $K_{|V|}(\mathbf{p})$ is statically (therefore first-order) 2-rigid and $\mathbf{u}$ must be trivial.

For any framework $G(\mathbf{p})$, the set of pairs h, k for which $(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) = 0$ for all first-order flexes will be $\langle E \rangle$, the closure of E in the matroid $\mathcal{R}_2(K_V; \mathbf{p})$.

First-Order Flex Test 3.1.3. *For any plane configuration $\mathbf{p}$ for the graph K_n, the following are equivalent:*

1. *the edge $\{h, k\}$ is not in the closure of the set E in $\mathcal{R}_2(n; \mathbf{p})$;*
2. *every self-stress ω on $E \cup \{h, k\}$ is zero on $\{h, k\}$;*
3. *there is a first-order flex $\mathbf{u}$ on $G(\mathbf{p})$, such that $(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) \neq 0$.*

Proof. The equivalence of 1. and 2. is the definition of of the matroid closure. 2. $\Rightarrow$ 3. If $\{h, k\}$ is independent, then adding the row $R_{h,k}$ to the rigidity matrix for E must increase the rank by one or equivalently the nullity is reduced by one. Let $\mathbf{u}$ be one of the first-order flexes removed. Therefore

$$(\mathbf{p}_h - \mathbf{p}_k) \cdot \mathbf{u} = R_{h,k} \cdot \mathbf{u} \neq 0.$$

3. $\Rightarrow$ 2. We prove the contrapositive. If there is a self-stress ω on $E' = E \cup \{h, k\}$, then $\omega_{h,k} R_{h,k} = -\sum_{\{i,j\} \in E} \omega_{i,j} R_{i,j}$. Therefore, for any first-order flex $\mathbf{u}$ of E:

$$\omega_{h,k}(\mathbf{p}_h - \mathbf{p}_k) \cdot \mathbf{u} = \omega_{h,k} R_{h,k} \cdot \mathbf{u} = -\sum_{\{i,j\} \in E} \omega_{i,j} R_{i,j} \cdot \mathbf{u} = -\sum_{\{i,j\} \in E} 0 = 0.$$

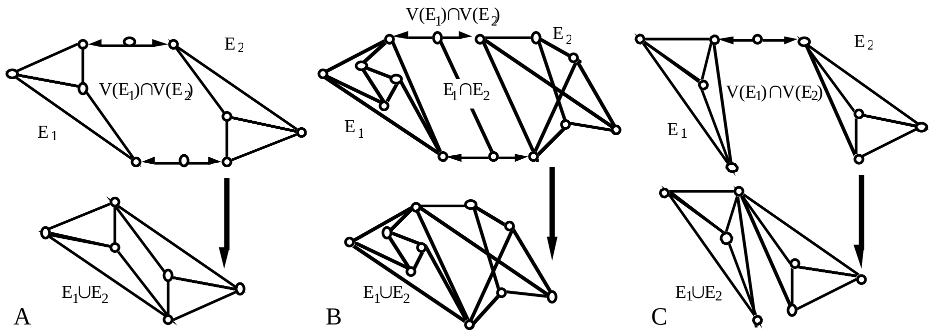

Fig. 3.2. *Gluing 2-rigid sets on two or more points gives a 2-rigid set (A), gluing independent sets on an 2-rigid set gives an independent set (B) and gluing rigid sets on a single point gives a flexible set (C).*

Generic Plane Gluing Lemma 3.1.4. *For two edge sets E_1, E_2,*

1. *if E_1 and E_2 are generically 2-rigid sets and $|V(E_1) \cap V(E_2)| \geq 2$, then the set $E_1 \cup E_2$ is generically 2-rigid (Figure 3.2A);*
2. *if E_1 and E_2 are generically 2-independent sets and the intersection graph $G = (V(E_1) \cap V(E_2), E_1 \cap E_2)$ is generically 2-rigid, then the set $E_1 \cup E_2$ is generically 2-independent (Figure 3.2B);*
3. *if $|V(E_1) \cap V(E_2)| < 2$, then the closure $\langle E_1 \cup E_2 \rangle$ in $\mathcal{R}_2(V(E_1) \cup V(E_2))$ is contained in $K_{V(E_1)} \cup K_{V(E_2)}$ (Figure 3.2C).*

Proof. 1. Assume that E_1 and E_2 are 2-rigid sets at a generic point $\mathbf{p}$ and that $|V(E_1) \cap V(E_2)| \geq 2$. Assume $\mathbf{u}$ is a first-order flex on $G = (V(E_1 \cup E_2); E_1 \cup E_2)$. By assumption,

$$\mathbf{u}|_{V(E_1)} = \alpha_1 T_x + \beta_1 T_y + \gamma_1 T_r \quad \text{and} \quad \mathbf{u}|_{V(E_2)} = \alpha_2 T_x + \beta_2 T_y + \gamma_2 T_r.$$

However, since these agree on at least two shared vertices at distinct points, these coefficients $\alpha_i, \beta_i, \gamma_i$ are equal and $\mathbf{u}$ is this single combination – a trivial first-order flex. We conclude that $E_1 \cup E_2$ is generically 2-rigid.
2. Assume that E_1 and E_2 are generically 2-independent sets and that $G = (V(E_1) \cap V(E_2); E_1 \cap E_2)$ is generically 2-rigid. We prove, by contradiction, that there is no non-empty subset D with $|D| \geq 2|V(D)| - 2$.

Assume there is such a subset D. Consider the sets $D_1 = D \cap E_1$ and $D_2 = D \cap E_2$. If one of these is empty, then D is contained in one side (say E_1) and $|D| \leq 2|V(D)| - 3$ which is a contradiction. If D_1 and D_2 are disjoint then:

$$|D| = |D \cap E_1| + |D \cap E_2| \leq 2|V(D_1)| - 3 + 2|V(D_2)| - 3 \leq 2|V(D)| - 4 < 2|V| - 3.$$

This is also a contradiction.

Therefore $D_{12} = D_1 \cap D_2 \subseteq E_1 \cap E_2$ is nonempty and $E_1 \cap E_2$ is generically 2-independent and 2-rigid on $V(E_1) \cap V(E_2)$, with at least two vertices. We conclude that $|E_1 \cap E_2| = 2|V(E_1) \cap V(E_2)| - 3$. With this property in mind, we extend the nonempty D_{12} to $E_1 \cap E_2$ which may add vertices V' to $V(D)$ and edges E' to D. Since $|D_{12}| \leq 2|V(D_{12})| - 3 \leq 2|V(D) \cap V(E_1 \cap E_2)| - 3$, and

$$|D_{12}| + |E'| = |E_1 \cap E_2| = 2|V(E_1 \cup E_2| - 3 = 2|V(D) \cap V(E_1 \cap E_2)| - 3 + 2|V'|,$$

the net changes satisfy $E' \geq 2|V'|$. This addition to D creates C with

$$|C| = |D| + |E'| \geq 2|V(D)| - 2 + 2|V'| = 2|V(C)| - 2.$$

We find our contradiction in C, using $C_1 = C \cap E_1$ and $C_2 = C \cap E_2$, which are independent subsets of E_1 and E_2 respectively. Note that $C_1 \cap C_2 = E_1 \cap E_2$ so $|C_1 \cap C_2| = 2|V(C_1) \cap V(C_2)| - 3$. We now have

$$\begin{aligned}|C| &= |C_1| + |C_2| - |C_1 \cap C_2| \\ &\leq 2|V(C_1)| - 3 + 2|V(C_2)| - 3 - [2|V(C_1) \cap V(C_2)| - 3] \\ &= 2|V(C)| - 3.\end{aligned}$$

This is the desired contradiction.
3. Assume $|V(E_1) \cap V(E_2)| < 2$ and take a generic plane configuration $\mathbf{p}$. Let $\mathbf{r}$ represent the rotation about the vertex $V(E_1) \cap V(E_2)$ (or a new general position point, if $V(E_1) \cap V(E_2) = \emptyset$). Consider the first-order flex $\mathbf{u}$ which assigns velocity $\mathbf{0}$ to points in $V(E_1)$ and $\mathbf{r}(j)$ to points in $V(E_2)$. For any $\{h, j\} \notin K_{V(E_1)} \cup K_{V(E_2)}$, we have:

$$(\mathbf{p}_h - \mathbf{p}_j) \cdot (\mathbf{u}_h - \mathbf{u}_j) = (\mathbf{p}_h - \mathbf{p}_j) \cdot \mathbf{0} - (\mathbf{p}_h - \mathbf{p}_j) \cdot \mathbf{r}(j) = 0 + (\mathbf{p}_h - \mathbf{p}_j) \cdot \mathbf{r}(j) \neq 0$$

where $(\mathbf{p}_h - \mathbf{p}_j) \cdot \mathbf{r}(j) \neq 0$ because the center of $\mathbf{r}$ is not collinear with any other two points by our assumption of a generic configuration. By the First-Order Flex Test 3.1.3, $\{h, j\}$ is not in $\langle E_1 \cup E_2 \rangle$. We conclude that $\langle E_1 \cup E_2 \rangle \subseteq K_{V(E_1)} \cup K_{V(E_2)}$.

REMARK 3.1.5. Properties 1 and 3, like property 2, can be proven directly from the counts, rather than by the 'matrix related' kinematic arguments used here. They can also be proven at any general position configuration $\mathbf{p}$. For any matroid on the edges of a complete graph, Properties 1 and 3 are the defining properties of an *abstract* 2*-rigidity matroid* [**GSS**] (see also §9.3). A number of properties of rigidity matroids at general position plane configurations, such as vertex 2-addition and the necessity of the counting properties for independent sets, can be proven from these properties and the usual properties of closure, rank etc. in a matroid [**GSS**, pages 86 -ff]. We recommend this reference for an extensive, matroidal analysis of these rigidity-like matroids on graphs. We will give some new related conjectures in Part II.

3.2. Connections to finite rigidity. Alongside static rigidity, with its roots in civil engineering, first-order rigidity has roots in mechanical engineering and in the study of linkages. We briefly describe this connection to indicate another source for this matroid. Published sources for this include [**AR**,**CoW**,**Gl**,**RW**]. This section will *not* give any additional matroidal results.

Given a plane framework $G(\mathbf{p})$, consider other 'equivalent' configurations $\mathbf{q}$ for which the edges of $G(\mathbf{q})$ have the same length as $G(\mathbf{p})$. These are captured by the *rigidity map*: $f_G : \mathbb{R}^{2|V|} \to \mathbb{R}^E$, with:

$$f_G(\mathbf{p}_1, \ldots, \mathbf{p}_{|V|}) = (\ldots, |\mathbf{p}_i - \mathbf{p}_j|, \ldots).$$

The inverse image $B(G, \mathbf{p}) = f_G^{-1}(f_G(\mathbf{p}))$ is the set of all *bar equivalent* frameworks $G(\mathbf{q})$ with $|\mathbf{p}_i - \mathbf{p}_j| = |\mathbf{q}_i - \mathbf{q}_j|$ for all bars $\{i, j\} \in E$. Of course, if the configuration $\mathbf{q}$ is *congruent to* $\mathbf{p}$, that is, there is an isometry T of $\mathbb{R}^2$ with $T(\mathbf{p}_i) = \mathbf{q}_i$ for all $i \in V$, then $G(\mathbf{q})$ is bar equivalent to $G(\mathbf{q})$.

An *analytic flex* of $G(\mathbf{p})$ is an analytic function $\mathbf{p}(t) : [0, 1) \to B(G, \mathbf{p})$ such that $\mathbf{p} = \mathbf{p}(0)$. A plane framework $G(\mathbf{p})$ is *flexible* if there is a *non-trivial* analytic flex $\mathbf{p}(t)$ of $\mathbf{p}$ with $\mathbf{p}(t)$ not congruent to $\mathbf{p}$ for all $t > 0$ (Figure 3.3B,C). Otherwise all flexes are *trivial* and the framework is *rigid*.

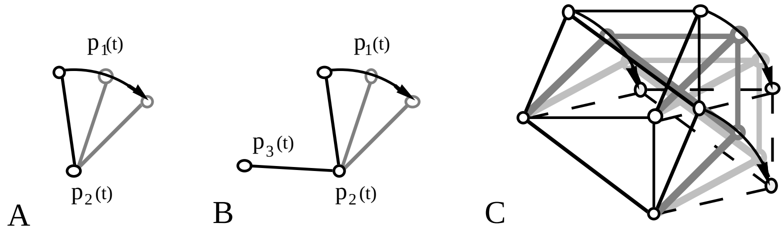

Fig. 3.3. *Shaded versions of a trivial analytic flex (A) and non-trivial flexes (B,C).*

Alternate Rigidity Definitions 3.2.1 [**Gl**]. *For a plane framework $G(\mathbf{p})$ the following conditions are equivalent:*

1. *the framework $G(\mathbf{p})$ is rigid;*
2. *for every continuous path $\mathbf{p}(t) \in \mathbb{R}^{2|V|}$, $0 \leq t < 1$ and $\mathbf{p}(0) = \mathbf{p}$, such that $G(\mathbf{p}(t))$ is bar equivalent to $G(\mathbf{p})$ for all t, $\mathbf{p}(t)$ is congruent to $\mathbf{p}$ for all t;*
3. *there is an $\varepsilon > 0$ such that, if $G(\mathbf{p})$ and $G(\mathbf{q})$ are bar equivalent and $|\mathbf{p}-\mathbf{q}| < \varepsilon$ then $\mathbf{p}$ is congruent to $\mathbf{q}$.*

The rigidity matrix $R_G(\mathbf{p})$ is the Jacobian of the rigidity map f_G, at $\mathbf{p}$, which indicates the connection between rigidity and first-order rigidity. Equivalently, the derivative of an analytic path $\mathbf{p}(t)$ is a first-order flex $\mathbf{p}'$:

$$D_t\big[(\mathbf{p}_i(t) - \mathbf{p}_j(t))^2 = c_{ij}\big]|_{t=0} \quad \Rightarrow \quad 2(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}'_j - \mathbf{p}'_i) = 0.$$

If $\mathbf{p}(t)$ is non-trivial, its derivative is usually a non-trivial first-order flex. (If $\mathbf{p}'$ is trivial, the second derivative will also be a first-order flex. Continue this process until a non-trivial first-order flex is found [**Co2**].) This process gives one proof of:

First-Order Rigid to Rigid Theorem 3.2.2. *If a bar framework $G(\mathbf{p})$ is first-order rigid then $G(\mathbf{p})$ is rigid.*

The converse implication from rigidity to first-order rigidity also holds for generic configurations.

Generic 2-Rigidity Theorem 3.2.3. *For a graph G the following are equivalent:*

1. *G is generically 2-rigid;*
2. *for all configurations $\mathbf{q}$ in some non-empty open set U in $\mathbb{R}^{2|V|}$, the frameworks $G(\mathbf{q})$ are rigid;*
3. *for all $\mathbf{q}$ in an open dense subset of configurations in $\mathbb{R}^{2|V|}$, $G(\mathbf{q})$ is first-order rigid (and rigid);*
4. *$G(\mathbf{p})$ is first-order rigid for some configuration $\mathbf{p}$ in the plane.*
5. *$G(\mathbf{p})$ is rigid for some generic configuration $\mathbf{p}$ in the plane.*

Remark 3.2.4. Generic 2-rigidity, and therefore rigidity at a generic plane configuration $\mathbf{p}$, directly defines a matroid by the condition:

A set of edges E is a basis for $K_{V(E)}$ if, and only if $(V(E), E)(\mathbf{p})$ is a rigid framework, and is minimal among such frameworks.

The comparable definition for a special position $\mathbf{p}$ may not produce a matroid. Consider the configurations in Figure 3.4A,B. Both of these are minimal rigid frameworks, but they have different sizes. They cannot be bases of the same matroid.

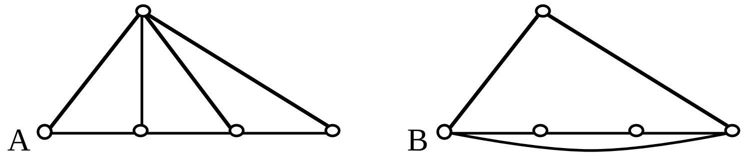

Fig. 3.4. *Two minimal rigid plane frameworks of different sizes on the same non-generic plane configuration* $\mathbf{p}$.

3.3. Counts and trees for generic 2-rigidity. In Remark 2.1.7 we saw that the rigidity matroid on a collinear configuration $\mathbf{p}$, or equivalently, rigidity on the line (1-rigidity), is the cycle matroid of the graph, with independence characterized by

(i) $|E'| \leq \; |V(E')| - 1$ *for all non-empty subsets* E'*;*
(ii) $|E|$ *is a forest.*

There are important traces of this pattern in the 2-rigidity matroid count:

$$|E| = 2|V| - 3 = 2(\;|V| - 1\;)\; - 1.$$

This numerical break-down of the count corresponds to a matroidal construction:

the generic 2*-rigidity matroid is the matroid union of two copies of the graphic matroid on* G*, followed by a Dilworth truncation* (see [**LY**,**Wh7**] and §A).

We will not prove this result here. However, the following example brings out the way in which 'trees and forests' arise in the rigidity matrix of a graph.

Example 3.3.1. Consider the rigidity matrix for a triangle at $\mathbf{p}$:

$$\begin{array}{c} \\ \{1,2\} \\ \{2,3\} \\ \{1,3\} \end{array} \begin{array}{c} \begin{array}{cccccc} 1_x & 1_y & 2_x & 2_y & 3_x & 3_y \end{array} \\ \begin{pmatrix} (x_1 - x_2) & (y_1 - y_2) & (x_2 - x_1) & (y_2 - y_1) & 0 & 0 \\ 0 & 0 & (x_2 - x_3) & (y_2 - y_3) & (x_3 - x_2) & (y_3 - y_2) \\ (x_1 - x_3) & (y_1 - y_3) & 0 & 0 & (x_3 - x_1) & (y_3 - y_1) \end{pmatrix} \end{array}$$

We permute the columns, placing first columns of each vertex together, then second columns together:

$$\begin{array}{c|ccc|ccc} & 1_x & 2_x & 3_x & 1_y & 2_y & 3_y \\ \hline \{1,2\} & (x_1 - x_2) & (x_2 - x_1) & 0 & (y_1 - y_2) & (y_2 - y_1) & 0 \\ \{2,3\} & 0 & (x_2 - x_3) & (x_3 - x_2) & 0 & (y_2 - y_3) & (y_3 - y_2) \\ \{1,3\} & (x_1 - x_3) & 0 & (x_3 - x_1) & (y_1 - y_3) & 0 & (y_3 - y_1) \end{array}$$

Dividing each non-zero row of a block by a scalar, we have two copies of the matrix for the cycle matroid of the graph:

$$\begin{array}{ccc|ccc|c|ccc} & & 1_x & 2_x & 3_x & & 1_y & 2_y & 3_y \\ \{1,2\} & (x_1 - x_2) & 1 & -1 & 0 & (y_1 - y_2) & 1 & -1 & 0 \\ \{2,3\} & (x_2 - x_3) & 0 & 1 & -1 & (y_2 - y_3) & 0 & 1 & -1 \\ \{1,3\} & (x_1 - x_3) & 1 & 0 & -1 & (y_1 - y_3) & 1 & 0 & -1 \end{array}$$

To check whether a general graph with $|E| = 2|V| - 3$ is generically 2-rigid (or a basis of $\mathcal{R}_2(V)$), we can delete three columns from the rigidity matrix (at least one from each of the blocks as above), and take a determinant. For such a determinant to be non-zero, there must be two minors on disjoint edge sets E_x for x and E_y for y, each with non-zero determinant. Since these blocks are copies of the graphic matroid on G (with $|V| - 1$ columns for one side, say x, and $|V| - 2$ columns for the other) these minors will be non-zero only if we have three edge-disjoint trees: T_1 from the x-side (with $|V| - 1$ rows), and T_2, T_3 from the y-side (with $|V| - 2$ rows). The deleted 'columns' will also be connected to edges whose rows have only one non-zero entry left in a block.

For this decomposition, any vertex will touch exactly two of the trees (T_1 and one of T_2 or T_3). In addition, no two non-empty subtrees have the same span, because of our counts: a subset spanned by two trees has $|E'| = |T_1'| + |T_2'| = 2(|V| - 1) > 2|V| - 3$. These three properties define a $3\mathit{Tree}2$ partition of the edges (see Figure 3.5A) [**Cr5**,**Ta5**]: (i) three trees; (ii) exactly two at each vertex; and (iii) no two non-empty subtrees spanning the same set of vertices. The following theorem records (without proof) standard connections between trees and bases for the matroid $\mathcal{R}_2(n)$.

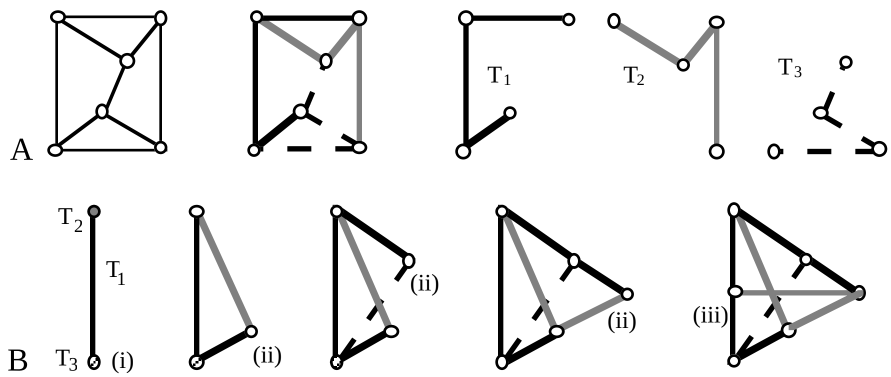

Fig. 3.5. $3\mathit{Tree}2$ *partitions for two bases of the generic* 2*-rigidity matroid on* 6 *vertices.*

Generic 2-Basis Theorem 3.3.2. *An edge set E is a basis of the generic 2-rigidity matroid on $K_{V(E)}$ if and only if:*

1. $|E| = 2|V| - 3$ *and for every nonempty subset E', $|E'| \leq 2|V(E')| - 3$* (Laman's Theorem)*;*
2. $G = (V(E), E)$ *has a Henneberg* 2*-construction* (Henneberg's Theorem)*;*
3. E *has a proper* $3\mathit{Tree}2$ *partition* (Crapo's Theorem [**Cr5**])*;*
4. *for each $\{i, j\} \in E$, the multigraph obtained by doubling the edge $\{i, j\}$ is the union of two spanning trees* (Recski's Theorem [**Re2**]).

Remark 3.3.3. Each of these combinatorial characterizations for bases has an associated algorithm for verifying whether a graph is a basis for the generic 2-rigidity matroid on K_V:

1. Counts: this can be checked by an $O(|V|^2)$ algorithm based on bipartite matchings on an associated graph [**Im**,**Su2**,**3**].
2. 2-construction: existence of a 2-construction can be checked by an $O(2^{|V|})$ algorithm, but a proposed 2-construction can be verified in $O(|V|)$ time.
3. 3Tree2 covering: existence is checked by an $O(|V|^2)$ matroid algorithm [**Cr5**].

4. Double tree partition: all required double-tree partitions can be found by a matroidal algorithm of order $O(|V|^3)$ [**Re2**].

What is the complexity of moving between tree coverings and Henneberg 2-constructions? It is a simple (linear time) process to create a 3Tree2 partition of the edges of a basis directly from any Henneberg 2-construction (Figure 3.5B):

(i) for a single edge, the edge is one tree and the two vertices are the second and third trees;

(ii) for a 2-valent vertex, choose one tree covering each of the attachments (there must be two distinct choices) and place the new edge in the chosen tree;

(iii) for a 3-valent vertex replacing an edge, we add two edges in the same tree as the removed edge (preserving the counts at those ends) and add the third edge, in a different tree but extending some tree at the third vertex.

Notice that this construction allows us to choose one tree as a spanning tree. Unfortunately,we believe there is no direct (polynomial time) extraction of a Henneberg 2-construction from a 3Tree2 decomposition. (There is the exponential process, through the Generic 2-Basis Theorem 3.3.2: for each step removing a 3-valent vertex, we test all three possible insertions of an edge.) Henneberg 2-constructions are harder to find, easier to verify and easier to use than 3Tree2 decompositions.

A final thread for the generic 2-rigidity matroid is connectivity. It is clear that any graph which is 2-rigid is at least vertex 2-connected. If we remove one vertex and the attached edges, the remaining graph is connected (recall Figure 3.2C). On the other hand, the graph in Figure 3.6 is 5-connected but not generically 2-rigid. This failure is verified by a formula for the rank of the matroid adapted from submodular functions.

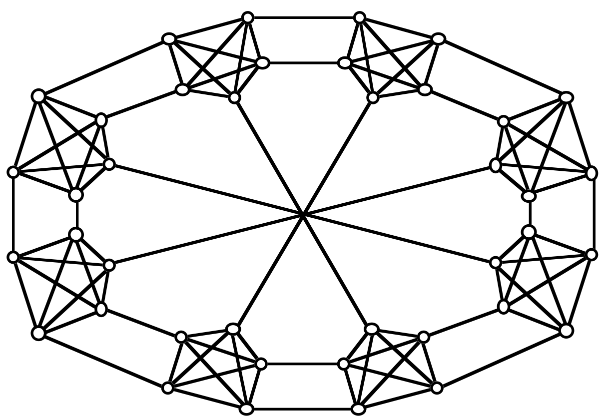

Fig. 3.6. *A* 5*-connected graph which is not generically* 2*-rigid.*

Theorem 3.3.4 [**GSS**, Theorem 4.4.3]. *The rank of an edge set E in the generic* 2*-rigidity matroid is:*

$$r(E) = \min\left\{\sum_{i=1}^{k}\big(2|V(E_i)| - 3\big)\right\}$$

where the minimum is taken over all partitions $\{E_i\}_{i=1}^k$ of E.

The minimum occurs for blocks which are maximal generically 2-rigid components of the graph. In particular, if we partition the graph of Figure 3.6 into 8 blocks for the K_5 subgraphs and leave all other edges in separate blocks, we have

$$r(E) = 8 \times 7 + 20 = 76 < 2 \times 40 - 3 = 2|V| - 3,$$

which verifies that this is not 2-rigid. The following positive result is best possible.

Sufficient Connectivity 3.3.5 [**LY**]. *If a graph G is vertex 6-connected then G is generically 2-rigid.*

4. Parallel Drawings in the Plane

We now introduce a second geometric form for the matroid of §2 and §3. For a graph G at a plane configuration $\mathbf{p}$ we consider an alternate constraint matrix based on maintaining the directions of the edges (§4.1), then show this is isomorphic to the plane rigidity matroid.

This matroid for a 'direction graph' has an immediate generalization to a matroid on the vertex-face incidences on hypergraphs realized as point and line configurations in the plane (§4.2). This form, in turn, has a simple generalization to parallel scenes of point and hyperplane configurations in arbitrary dimensions (§8). Of all the matroids we will describe in higher dimensions and on larger hypergraphs, these parallel drawing matroids (and their projective duals, the polyhedral picture matroids) have the cleanest theory, with analogs of all the geometric and combinatorial results for 2-rigidity.

4.1. Basic concepts of parallel designs. We begin again with a graph G and a plane configuration $\mathbf{p}$ for the vertices. For each edge, the new constraint on configurations $\mathbf{q}$ is 'this edge must retain its direction' rather than 'this edge must retain its length'. For a new configuration $\mathbf{q}$, the constraint for an edge $\{i, j\}$ is :

$$(\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot (\mathbf{q}_i - \mathbf{q}_j) = 0.$$

The solutions of this homogeneous linear system are the *parallel designs* $G(\mathbf{q})$ of the original *plane design* $G(\mathbf{p})$.

A parallel design $G(\mathbf{q})$ for $G(\mathbf{p})$ is *trivial* if $\mathbf{q}$ is a translation or dilation of $\mathbf{p}$ (Figure 4.1 A). Otherwise, the parallel design is non-trivial (Figure 4.1B).

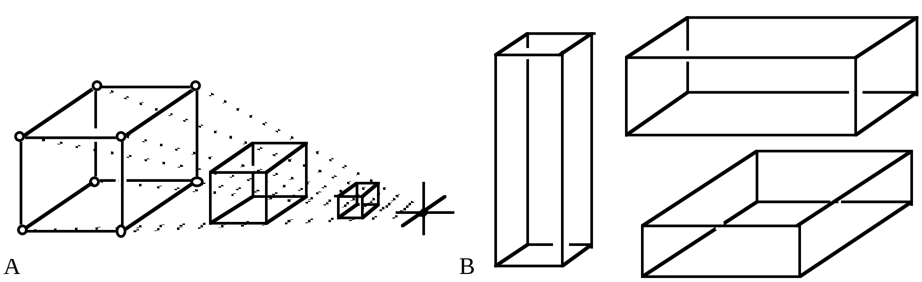

Fig. 4.1. *Some trivial (A) and non-trivial (B) parallel designs for a plane design.*

Example 4.1.1. The constraints for parallel designs create an analog to the rigidity matrix. For the design in Figure 4.2A, the *parallel design matrix* is $P(G, \mathbf{p})$:

	a	b	c	d	e	f
$\{a,b\}$	$(\mathbf{a}-\mathbf{b})^{\perp}$	$(\mathbf{b}-\mathbf{a})^{\perp}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\{a,c\}$	$(\mathbf{a}-\mathbf{c})^{\perp}$	$\mathbf{0}$	$(\mathbf{c}-\mathbf{a})^{\perp}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\{a,d\}$	$(\mathbf{a}-\mathbf{d})^{\perp}$	$\mathbf{0}$	$\mathbf{0}$	$(\mathbf{d}-\mathbf{a})^{\perp}$	$\mathbf{0}$	$\mathbf{0}$
$\{b,c\}$	$\mathbf{0}$	$(\mathbf{b}-\mathbf{c})^{\perp}$	$(\mathbf{c}-\mathbf{b})^{\perp}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\{c,f\}$	$\mathbf{0}$	$\mathbf{0}$	$(\mathbf{c}-\mathbf{f})^{\perp}$	$\mathbf{0}$	$\mathbf{0}$	$(\mathbf{f}-\mathbf{c})^{\perp}$
$\{d,e\}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$(\mathbf{d}-\mathbf{e})^{\perp}$	$(\mathbf{e}-\mathbf{d})^{\perp}$	$\mathbf{0}$
$\{d,f\}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$(\mathbf{d}-\mathbf{f})^{\perp}$	$\mathbf{0}$	$(\mathbf{f}-\mathbf{d})^{\perp}$
$\{e,f\}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$(\mathbf{e}-\mathbf{f})^{\perp}$	$(\mathbf{f}-\mathbf{e})^{\perp}$

$T_{\mathbf{t}}$	$\mathbf{a}+\mathbf{t}$	$\mathbf{b}+\mathbf{t}$	$\mathbf{c}+\mathbf{t}$	$\mathbf{d}+\mathbf{t}$	$\mathbf{e}+\mathbf{t}$	$\mathbf{f}+\mathbf{t}$
T_D	$\frac{\mathbf{a}}{2}$	$\frac{\mathbf{b}}{2}$	$\frac{\mathbf{c}}{2}$	$\frac{\mathbf{d}}{2}$	$\frac{\mathbf{e}}{2}$	$\frac{\mathbf{f}}{2}$
$\mathbf{q}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\frac{\mathbf{x}+\mathbf{d}}{2}$	$\frac{\mathbf{x}+\mathbf{e}}{2}$	$\frac{\mathbf{x}+\mathbf{f}}{2}$

The lower box gives a translation $T_{\mathbf{t}}$, a dilation T_D towards the origin (Figure 4.2A), and a non-trivial parallel design $\mathbf{q}$, where $\mathbf{x}$ is the point of intersection of ad, cf (Figure 4.2B). For most configurations (Figure 4.2B), this parallel drawing would be blocked by the addition of an edge $\{b, e\}$. However for the special position of Figure 4.2C, this parallel design will work on the extended design.

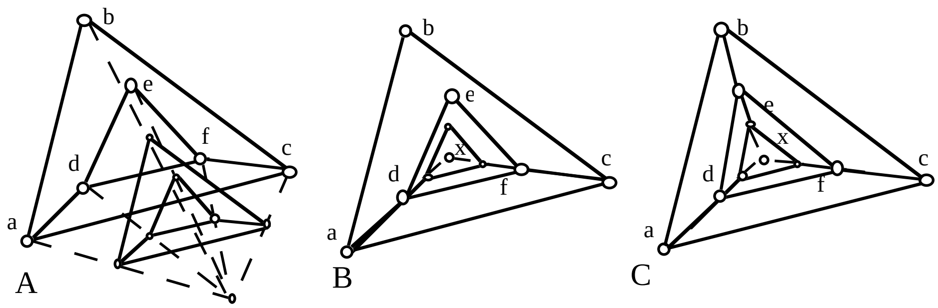

Fig. 4.2. *Trivial (A) and non-trivial (B) parallel designs for a plane design and a special position in which the non-trivial parallel design also exists for the extended object (C).*

We have defined a parallel design matrix $P(G, \mathbf{p})$ and a corresponding matroid. The resemblance to the rigidity matroid is striking – just a small twist in the entries, from $\mathbf{p}_i - \mathbf{p}_j$ to $(\mathbf{p}_i - \mathbf{p}_j)^{\perp}$.

LEMMA 4.1.2. *For a graph G and a plane configuration $\mathbf{p}$, an assignment ω of scalars to the edges is a row dependence of the rigidity matrix if and only if ω is a row dependence of the parallel design matrix.*

PROOF. Compare the conditions for a row dependence of the rigidity matrix and the parallel design matrix for $G(\mathbf{p})$. For each vertex i:

$$\sum_{j|\{i,j\}\in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0} \qquad \Leftrightarrow \qquad \sum_{j|\{i,j\}\in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j)^{\perp} = \mathbf{0}^{\perp} = \mathbf{0}$$

This equivalence of conditions gives the desired equivalence of dependencies.

Corollary 4.1.3. *For any graph G and plane configuration $\mathbf{p}$, the 2-rigidity matroid and the parallel 2-design matroid are the same.*

For kinematics, the connection is almost as easy:

$$\begin{aligned} & (\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{u}_i + (\mathbf{p}_j - \mathbf{p}_i) \cdot \mathbf{u}_j = 0 \\ \Leftrightarrow \quad & (\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot \mathbf{u}_i{}^{\perp} + (\mathbf{p}_j - \mathbf{p}_i)^{\perp} \cdot \mathbf{u}_j{}^{\perp} = 0 \\ \Leftrightarrow \quad & (\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot \mathbf{q}_i + (\mathbf{p}_j - \mathbf{p}_i)^{\perp} \cdot \mathbf{q}_j = 0 \end{aligned}$$

Since $\mathbf{p}$ is itself a (trivial) solution to the system, we can replace $(\mathbf{u}_i)^{\perp}$ by $\mathbf{q} = (\mathbf{u}_i)^{\perp} + \mathbf{p}_i$ as the new parallel design. Visually in Figure 4.3, this means $(\mathbf{u}_i)^{\perp}$ is the change in the configurations: $\mathbf{q}_i - \mathbf{p}_i$. In this translation,

1. the trivial first-order motions correspond to trivial parallel designs;
2. the zero first-order motion corresponds to the parallel design $\mathbf{p}$;
3. a translation by $\mathbf{t}$ corresponds to a translation by $\mathbf{t}^{\perp}$;
4. a rotation about the origin corresponds to a dilation towards the origin;
5. a non-trivial first-order motion corresponds to a non-trivial parallel design.

This correspondence between first-order flexes and parallel designs has its roots in drafting techniques during the last century. Today, we find these parallel drawings are also simpler to construct and detect with dynamic geometry programs, such as The Geometer's Sketchpad or Cabri, than first-order motions.

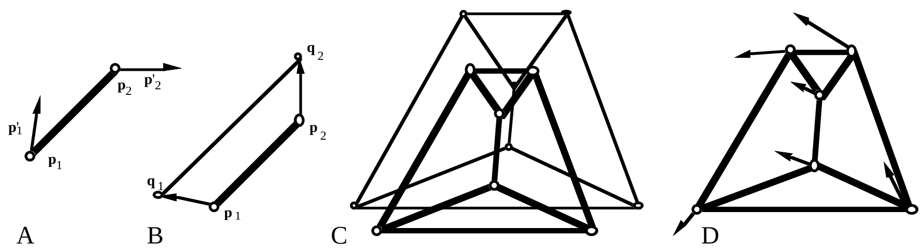

Fig. 4.3. *Turning the velocities of a first-order motion of a framework by $90°$ gives a parallel drawing (A,B) (D,C).*

Theorem 4.1.4. *A plane framework $G(\mathbf{p})$ has a non-trivial first-order flex if and only if the configuration $G(\mathbf{p})$ has a non-trivial parallel design $G(\mathbf{q})$.*

Because of these correspondences, all the combinatorial and geometric results for plane first-order rigidity translate to plane parallel designs.

Remark 4.1.5. If we consider an analytic path $\mathbf{p}(t)$ of parallel designs, and take derivatives at time $t = 0$, we have $(\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot (\mathbf{p}'_i(0) - \mathbf{p}'_j(0)) = 0$. These are the equations we have given. Conversely, given any parallel design $\mathbf{q}$, we can take the affine combinations $\mathbf{p}(t) = (1-t)\mathbf{p} + t\mathbf{q}$ as an analytic path of parallel designs. If $\mathbf{q}$ is non-trivial, then $\mathbf{p}(t)$ is a non-trivial drawing of $\mathbf{p}$ for all $t > 0$. Here there is no distinction between first-order and full 'motions'.

4.2. Parallel 2-scenes

It is a simple matter to generalize the process of parallel drawings from the plane designs in which each edge has only two vertices, to structures of points and longer lines with fixed directions. We introduce a general vocabulary which will extend easily to all dimensions. There are certain arbitrary conventions for writing the equations of lines (and planes in higher dimensions), because we are not using the underlying affine or projective coordinates.

We replace the abstract structure of a graph with a *polyhedral incidence structure* $S = (V, F; I)$: an abstract set of *vertices* V, an abstract set of *faces* F and a set of *incidences* $I \subset V \times F$. The incidence structure can also be viewed as a bipartite *incidence graph* with two set of vertices V, F and the incidences I as edges.

We replace the configuration of points for the vertices of a graph with a 2*-scene* for an incidence structure $S = (V, F; I)$: a pair of location maps, $\mathbf{p} : V \to \mathbb{R}^2$, $\mathbf{p}_i = (x_i, y_i)$ and $\mathbf{P} : F \to \mathbb{R}^3$, $\mathbf{P}^j = (A^j, 1, D^j)$, such that, for each incidence $(i, j) \in I$: $A^j x_i + y_i + D^j = 0$. Notice that the normal to the face j is now $\mathbf{n}^j = (A^j, 1)$. As a convention, we assume that no face is 'vertical' – parallel to the vector $(0, 1)$. Up to a rotation of the original drawing, this will be valid.

A *parallel* 2*-scene* to $S(\mathbf{p}, \mathbf{P})$ is a 2-scene $S(\mathbf{q}, \mathbf{Q})$ such that for each face j, the normals are equal: that is, the $(P^j_1, 1) = (Q^j_1, 1)$ (Figure 4.4B). A *non-trivial parallel* 2*-scene* for $S(\mathbf{p}, \mathbf{P})$ is a parallel 2-scene $S(\mathbf{q}, \mathbf{Q})$, such that the point configuration $\mathbf{q}$ is not a translation or dilation of the configuration $\mathbf{p}$ (Figure 4.4B).

Instead of giving a 2-scene and asking about parallel scenes with the same normals, we can begin directly with an assignment $\mathbf{n}$ of *normals* for the faces: a 2-vector $\mathbf{n}^j = (A^j, 1)$ to each face $j \in F$. A 2-scene $S(\mathbf{p}, \mathbf{P})$ *realizes the normals* $\mathbf{n}$ if for each face $j \in F$ and vertex i, with $\mathbf{P}^j = (A^j, 1, D^j) = (\mathbf{n}^j, D^j)$

$$\mathbf{n}^j \cdot (x_i, y_i) + D^j = A^j x_i + y_i + D^j = 0.$$

Given the normal $\mathbf{n}^j$, this is a homogeneous linear equation in the unknowns x_i, y_i, D^j. The polyhedral incidence structure and the given directions create a linear system of $|I|$ equations in $|F| + 2|V|$ variables. This system is recorded in the $|I|$-by-$(|F| + 2|V|)$ *parallel* 2*-scene matrix* for plane normals $\mathbf{n}$, where the variables are ordered: $[\ldots, D^j, \ldots\ ;\ \ldots, x_i, y_i, \ldots]$.

Example 4.2.1. For the configuration in Figure 4.4A, the parallel 2-scene matrix $M(S, \mathbf{n})$is:

$$
\begin{array}{rc}
 & \begin{array}{cccc|cccccccccccc} a & b & c & d & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & \end{array} \\
\begin{array}{r} (a,2) \\ (a,3) \\ (a,4) \\ (b,1) \\ (b,3) \\ (b,5) \\ (c,4) \\ (c,5) \\ (c,6) \\ (d,1) \\ (d,2) \\ (d,6) \end{array} &
\left(\begin{array}{cccc|cccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & A^a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^a & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & A^b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & A^b & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^b & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^c & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^c & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^c & 1 \\
0 & 0 & 0 & 1 & A^d & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & A^d & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^d & 1
\end{array}\right)
\end{array}
$$

It is clear there is a 2-space of trivial solutions to $M(S, \mathbf{n})\mathbf{x} = \mathbf{0}$:

$$[-\mathbf{p}_0 \cdot \mathbf{n}^a, -\mathbf{p}_0 \cdot \mathbf{n}^b, -\mathbf{p}_0 \cdot \mathbf{n}^c, -\mathbf{p}_0 \cdot \mathbf{n}^d;\ \mathbf{p}_0,\ \mathbf{p}_0,\ \mathbf{p}_0,\ \mathbf{p}_0,\ \mathbf{p}_0,\ \mathbf{p}_0]$$

in which all points lie at the same spot $\mathbf{p}_0$. However a count of columns minus rows (variables minus maximum possible rank) indicates we must have a solution space of dimension at least $|F| + 2|V| - |I| = 4 + 2 \times 6 - 12 = 4$. The configurations for Figure 4.4A,B indicate that, for general directions there is indeed a four space of solutions: (i) pick $\mathbf{p}_1$ (2 choices); (ii) pick $\mathbf{p}_2$ arbitrarily on the line $\mathbf{P}^d$ which is now fixed (1 choice); (iii) pick $\mathbf{p}_6$ arbitrarily on the same line (1 choice). The remaining points and lines can be constructed uniquely from the preassigned directions and these points.

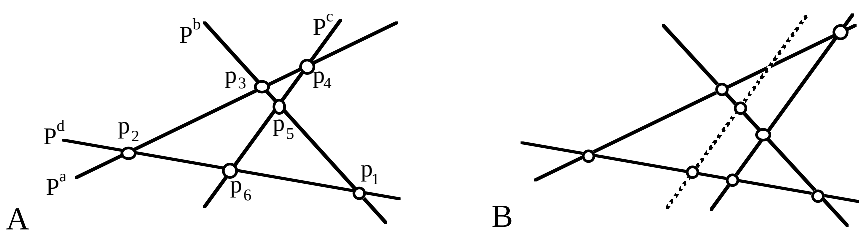

Fig. 4.4. *A 2-scene of incidences of points and lines (A) with a non-trivial parallel 2-scene (B).*

For a generic choice of directions on a general incidence structure, there is a 2-space of trivial parallel 2-scenes with all vertices at the same spot. If the directions were taken from an existing 2-scene $(\mathbf{p}, \mathbf{P})$ with at least two distinct points, then this space will have dimension at least 3: generated by these trivial 2-scenes and dilations of the given configuration, or equivalently, generated by translations and dilations of the the given configuration. These counts give some necessary conditions for independent sets of incidences on (1) a generic set of directions; (2) a generic set of directions with a non-trivial 2-scene. These conditions turn out to be necessary and sufficient [**Wh8**].

A set I' of incidences is *independent* at set of normals $\mathbf{n}$ if the corresponding rows of the parallel 2-scene matrix $M(S, \mathbf{n})$ are linearly independent.

Parallel 2-Scene Theorem 4.2.2. *For a generic set of plane normals* $\mathbf{n}$ *and an incidence structure* $S = (V, F; I)$*, the following are equivalent:*

1. *the set* I *of incidences is independent at* $\mathbf{n}$*;*
2. *for all non-empty subsets* I' *of incidences,* $|I'| \leq 2|V(I')| + |F(I')| - 2$.

The following are also equivalent:

3. *the set of incidences is a circuit at* $\mathbf{n}$*;*
4. $|I| = 2|V| + |F| - 1$ *and for all non-empty subsets* I' *of incidences* $|I'| \leq 2|V(I')| + |F(I')| - 2$.

This characterization of independence in terms of counts has strong similarities to the characterization of the generic 2-rigidity matroid. The *generic* 2*-parallel matroid* on the complete incidence structure $I = V \times F$ is defined by the independence of rows of the parallel 2-scene matrix at a generic set of directions.

A set of incidences is 2*-tight* if its closure in the 2-parallel matroid is the complete incidence matroid on its vertices and faces. Many results from generic 2-rigidity extend directly:

1. all circuits in the generic 2-parallel matroid are 2-tight (analogous to 2-rigidity of circuits);

2. the Gluing Lemma will have an analog for combining 2-tight or independent sets of the 2-parallel matroid to create new 2-tight or independent sets on the combined vertices and faces;
3. there are inductive constructions for bases on a given set of vertices and faces to add an additional face (attached to one existing vertex), or an additional vertex (attached to two existing faces). [A counting argument confirms that each basis must have either a face of valence 1 or a vertex of valence 2 or valence 3. The replacement principles leading to vertices of valence 3 have not been investigated.]

We *conjecture* that a set of incidences will be 2-tight if the bipartite incidence graph is 4-connected (the analog of the Sufficient Connectivity Theorem 3.3.5). One aspect of this analogy to plane rigidity is explored further in appendix §A.

These results primarily apply to configurations with all vertices coincident. For a direct correspondence with the results on parallel drawings of plane graphs in the next section, we state a corollary for configurations with distinct vertices.

Corollary 4.2.3. *For a generic set of plane normals* $\mathbf{n}$ *and an incidence structure* $S = (V, F; I)$*, the following are equivalent:*
1. *the structure has an independent realization with all vertices distinct, unique up to translation and dilation;*
2. $|I| = 2|V| + |F| - 3$ *and for all subsets with at least two incidences* $|I'| \leq 2|V(I')| + |F(I')| - 3$.

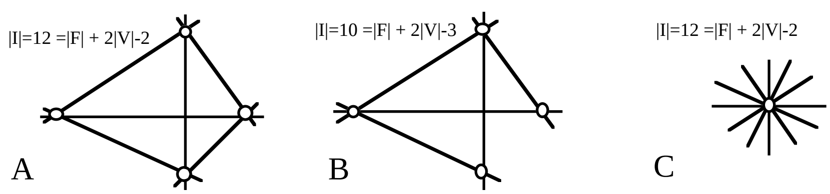

Fig. 4.5. *A 2-scene on a graph which a circuit at a generic configuration (A); independent with one edge removed (B), and also independent with all lines concurrent, for generic directions (C).*

Example 4.2.4. Consider the incidence structure of a complete quadrangle (Figure 4.5A). [As a visual convention, we extend the lines of 2-scene to indicate their normals and distinguish it from a parallel design on a graph.] If the normals for all but one of the edges are assigned, the reader can check that the entire configuration can be constructed from two distinct vertices (Figure 4.5B). The last normal is not generic – it is determined by the previous normals in this position. Equivalently, the last incidence is dependent on the previous incidences for the given normals.

For generic normals, the realizations have all vertices coincident (Figure 4.5C).

4.3. Reduction to plane graphs. Graphs are a very special type of incidence structure, in which each face is incident with exactly two vertices. In this special case, the Parallel 2-Scene Theorem 4.2.2 reduces to a familiar result: Laman's Theorem 2.2.5.

Consider the counts. With $F' = E'$ and $|I'| = 2|E'|$ for all nonempty subsets E', we have:

$$|I'| \leq 2|V(I')| + |F(I')| - 3 \Leftrightarrow 2|E'| \leq 2|V(I')| + |E'| - 3 \Leftrightarrow |E'| \leq 2|V(I')| - 3.$$

If we have generic directions D, the counts in the Parallel 2-scene Theorem guarantee a plane graph $G(\mathbf{p})$ with distinct vertices which is unique up to trivial parallel drawings and independent. This, in turn, guarantees that generic configurations $\mathbf{p}$ give independence and uniqueness, up to trivial parallel 2-scenes. In fact, generic directions and generic configurations are equivalent for these counts.

This equivalence fails for a generic circuit of the generic 2-rigidity matroid, with $|E| = 2|V| - 2$ and $|E| \leq 2|V| - 3$ on proper subsets. Realized at a generic configuration $\mathbf{p}$, the induced directions will *not* be generic directions (see Figure 4.5A)! The Parallel 2-Scene Theorem guarantees that generic directions will have only trivial 2-scenes for such a count.

Example 4.3.1. We illustrate this conversion from the parallel 2-scene matrix for the 2-scene of Figure 4.6A to the standard rigidity matrix for the framework of Figure 4.6B.

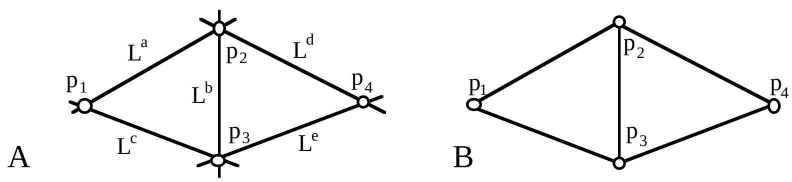

Fig. 4.6. *A plane incidence structure and 2-scene (A) based on a plane graph $G(\mathbf{p})$ (B).*

We begin with the parallel 2-scene matrix:

$$
\begin{array}{c}
\\
(a,1)\\ (a,2)\\ (b,2)\\ (b,3)\\ (c,1)\\ (c,3)\\ (d,2)\\ (d,4)\\ (e,3)\\ (e,4)
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc}
a & b & c & d & e & | & 1 & & 2 & & 3 & & 4 &
\end{array}\\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & | & A^a & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & | & 0 & 0 & A^a & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & | & 0 & 0 & A^b & B^b & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & A^b & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & | & A^c & B^c & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 & A^c & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & | & 0 & 0 & A^d & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 & 0 & 0 & A^d & 1\\
0 & 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 & A^e & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 & 0 & 0 & A^e & 1
\end{pmatrix}
\end{array}
$$

If we row reduce on the columns for the faces, we have:

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
a & b & c & d & e & | & 1 & & 2 & & 3 & & 4 &
\end{array}\\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & | & 0 & 0 & A^a & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & A^b & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 & A^c & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & | & 0 & 0 & 0 & 0 & 0 & 0 & A^d & 1\\
0 & 0 & 0 & 0 & 1 & | & 0 & 0 & 0 & 0 & 0 & 0 & A^e & 1\\
- & - & - & - & - & | & - & - & - & - & - & - & - & -\\
0 & 0 & 0 & 0 & 0 & | & A^a & 1 & -A^a & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & A^b & 1 & -A^b & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & | & A^c & 1^c & 0 & 0 & -A^c & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & A^d & 1 & 0 & 0 & -A^d & -1\\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & A^e & 1 & -A^e & -1
\end{pmatrix}
\end{array}.
$$

Up to scalar multiplication, a row such as

$$[\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad | \quad 0 \quad 0 \quad 0 \quad 0 \quad A^e \quad 1 \quad -A^e \quad -1\,]$$

is the row

$$[\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad | \qquad \mathbf{0} \qquad \mathbf{0} \qquad (\mathbf{p}_3 - \mathbf{p}_4)^{\perp} \quad (\mathbf{p}_4 - \mathbf{p}_3)^{\perp}\,]$$

– the row for the edge $\{3,4\}$ in the parallel drawing matrix for the the graph, with additional zeros for the faces. More generally the bottom right box of this reduced matrix is equivalent to the parallel design matrix of the graph G (or, by taking $\perp$, equivalent to the rigidity matrix of G).

5. The C^0_1-Cofactor Matroid

We present one more version of this matroid for plane graphs. Given a plane graph $G(\mathbf{p})$ with distinct vertices for each edge, an edge $\{i,j\}$ has an associated line $A^{i,j}x+B^{i,j}y+C^{i,j}=0$ and the basic linear form $L^{i,j}(\mathbf{p}) = A^{i,j}x+B^{i,j}y+C^{i,j}$. As a simple convention, we can select the equation:

$$L^{i,j}(\mathbf{p}) = \det \begin{bmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x & y & 1 \end{bmatrix} = -(\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot (x,y) + (\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot \mathbf{p}_j = 0.$$

Notice the antisymmetry: $L^{i,j}(\mathbf{p}) = -L^{j,i}(\mathbf{p})$.

A C^0_1-*cofactor* on a plane graph $G(\mathbf{p})$ is an assignment λ to the edges such that, for each vertex i:

$$\sum_{j|\{i,j\}\in E} \lambda_{i,j} A^{i,j}x + B^{i,j}y + C^{i,j} \equiv 0.$$

These linear equations define the dependencies of a matroid. In fact, we show it is the same matroid we have studied under parallel designs.

PROPOSITION 5.1.1. *Given an plane graph $G(\mathbf{p})$, an assignment λ is a C^0_1-cofactor if and only if λ is a self-stress.*

PROOF. Assume that λ is a C^0_1-cofactor on $G(\mathbf{p})$. For each vertex i, isolating then dropping the constant term gives:

$$\begin{aligned} & \sum_{j|\{i,j\}\in E} \lambda_{i,j} A^{i,j}x + B^{i,j}y + C^{i,j} \equiv 0 \\ \Rightarrow \quad & - \sum_{j|\{i,j\}\in E} \lambda_{i,j} (\mathbf{p}_i - \mathbf{p}_j)^{\perp} \cdot (x,y) = \mathbf{0} \\ \Leftrightarrow \quad & \sum_{j|\{i,j\}\in E} \lambda_{i,j} (\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}. \end{aligned}$$

A C^0_1-cofactor is a self-stress on $G(\mathbf{p})$

For the converse, we need to confirm that:

$$\begin{aligned} & \sum_{j|\{i,j\}\in E} \lambda_{i,j} (\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0} \quad \Leftrightarrow \quad \sum_{j|\{i,j\}\in E} \lambda_{i,j} (A^{i,j}x + B^{i,j}y) = \mathbf{0} \\ ?\Rightarrow \quad & \sum_{j|\{i,j\}\in E} \lambda_{i,j} C^{i,j} = 0. \end{aligned}$$

The key point is that, since the point $\mathbf{p}_i$ is on each of these lines, each $C^{i,j} = -(A^{i,j}x_i + B^{i,j}y_i) = -(A^{i,j}, B^{i,j}) \cdot (x_i, y_i)$. (Of course we get the same value of $C^{i,j}$ at the other vertex (x_j, y_j).) Therefore,

$$\sum_{j|\{i,j\}\in E} \lambda_{i,j} C^{i,j} = \sum_{j|\{i,j\}\in E} \lambda_{i,j}(A^{i,j}, B^{i,j}) \cdot \mathbf{p}_i = \Bigg[\sum_{j|\{i,j\}\in E} \lambda_{i,j}(A^{i,j}, B^{i,j}) \Bigg] \cdot \mathbf{p}_i = 0.$$

We have defined the original matroid of §2, §3 and §4 in a different guise. As mentioned in the introduction, there are two reasons for doing so:

1. this form leads to distinct generalizations in §10, §11 and §15;
2. this form has arisen naturally in applications to polyhedral pictures (see the next remark), and the connections between plane self-stresses and polyhedral pictures are an important part of the theory and history of this field.

Remark 5.1.2. Consider a planar drawing of a planar graph $G(\mathbf{p})$. The 'regions' of the plane defined by the drawing, including the infinite exterior region, form the faces of a spherical polyhedron together with the edges and vertices of the graph. When does this polyhedron lift vertically to a spatial polyhedron, with plane faces meeting in pairs, continuously, over the edges of the plane graph $G(\mathbf{p})$?

Maxwell's Theorem 5.1.3 [**Max**,**CrW2**,**3**]. *A plane framework $G(\mathbf{p})$ on a planar graph is the vertical projection of the edges and vertices of a plane faced spherical polyhedron with distinct planes for the faces at edge $\{i, j\}$ if and only if there is a self-stress on $G(\mathbf{p})$ with a non-zero coefficient $\omega_{i,j}$.*

If the configuration makes a planar drawing of the framework (no crossings of edges), this lifting can be viewed as a piecewise linear, globally continuous (C^0) function with the exterior region considered as an infinite face around the cap, not the 'underneath' of the finite cap. Such functions are called C^0_1 *bivariate splines* and the scalars we have defined are called 'smoothing cofactors' [**Bi**,**ChW**,**Wh10**].

We will not prove this correspondence here. However, we can briefly indicate the correspondence between these cofactors and the underlying polyhedron. Assume we have a polyhedron with planes: $\mathbf{P}^j = A^j x + B^j y + z + D^j$ and $\mathbf{P}^k = A^k x + B^k y + z + D^k$ for faces j, k. If these are to meet over the edge with vertices $\mathbf{p}_h, \mathbf{p}_i$, then for those points we have the same z values:

$$\begin{aligned} z &= -[A^j x_h + B^j y_h + D^j] = -[A^k x_h + B^k y_h + D^k] \\ &= -[A^j x_i + B^j y_i + D^j] = -[A^k x_i + B^k y_i + D^k]. \end{aligned}$$

Equivalently

$$\begin{aligned} (A^j - A^k)x_h + (B^j - B^k)y_h + (D^j - D^k) = 0 \\ (A^j - A^k)x_i + (B^j - B^k)y_i + (D^j - D^k) = 0. \end{aligned}$$

We conclude that

$$(A^j - A^k)x + (B^j - B^k)y + (D^j - D^k) = \lambda_{h,i}(A^{h,i}x + B^{h,i}y + C^{h,i})$$

for some scalar $\lambda_{h,i}$. These scalars will be the C^0_1-cofactors (or self-stress) on the plane graph $G(\mathbf{p})$. The fact that equilibria are achieved around each vertex follows from the telescoping equation for the cycle of faces k incident with i:

$$\sum_k (A^k - A^{k+1})x + (B^k - B^{k+1})y + (D^k - D^{k+1}) = \mathbf{0}.$$

6. Other 'Plane' Matroids

There are matroids related to plane rigidity and parallel drawing which arise out of other plane geometry problems – in particular problems of constraints in plane CAD [**Ow**,**SW**,**Wh12**,**13**]. We point out some similarities and some differences which occur for the underlying 'generic matroids' for these geometric problems. The first and third examples present important unsolved problems.

6.1. Plane incidences. Given a general incidence structure of points and lines (no direction constraints, length constraints etc.), what sets of incidences are independent for some (almost all) realizations as points and lines in the plane (not all lines collinear or all lines concurrent)?

Given an abstract structure $S = (V, F; I)$, the underlying algebraic constraints for 2-scenes are, for each $(i, j) \in I$:

$$A^j x_i + y_i + C^j = 0.$$

Working with these entries as variables, the Jacobian of this system corresponds to the equations of 'infinitesimal' deformations of a given 2-scene $S(\mathbf{p}, \mathbf{P})$:

$$A^j u_i + U^j x_i + w_i + W^j = 0.$$

This will be a system of I equations in $2|V| + 2|F|$ variables – u_i, w_i, U^j, W^j. There are trivial first-order deformations, corresponding to small projective transformations of the configuration $\mathbf{p}, \mathbf{P}$ (which do not take any vertices to infinity or turn any lines vertical). For at least four points affinely spanning the plane, the space of trivial transformations will have dimension 8 [**Wh13**]. Therefore, a necessary condition for independence of sets of at least 4 points spanning the plane, will be:

$$|I'| \leq 2|F| \ + \ 2|V| \ - 8$$

Of course, this 'count' is *not* satisfied for small sets of incidences. For a single incidence we have: $|I| = 1 > 2 + 2 - 8 = 2|F| + 2|V| - 8$.

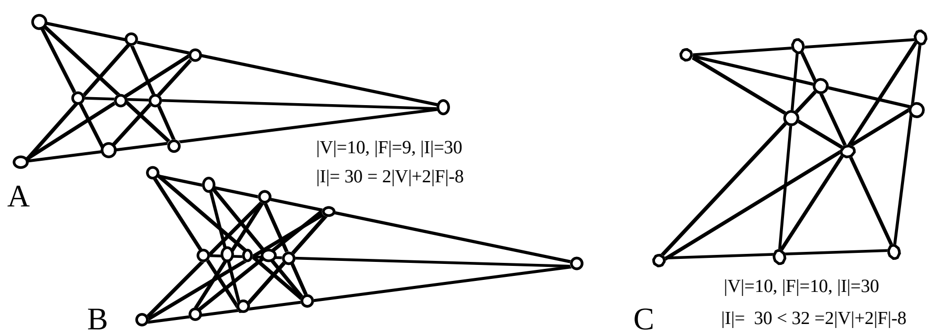

Fig. 6.1. *Projective Theorems such as Pappus' Theorem (A) and Desargue's Theorem (C) state that the final incidence is dependent on the previous incidences, leaving extra incidences preserving deformations (B).*

This counting condition is not enough to define a matroid, or to characterize our independence. All the theorems of projective geometry (Figure 6.1) are extra dependencies of the incidences, with the pattern:

If ... then the last three points will lie on a line; or

If ... then the last three lines will be concurrent.

The incidences in standard 'projective constructions' are independent, but often they are too low a rank to be bases [**Wh13**]. We believe the corresponding 'generic' matroid has no efficient characterization.

CONJECTURE 6.1.1. *There is no polynomial time algorithm to determine the independence of incidence structures in the plane.*

Even without a characterization of independence of incidences, we can conjecture what happens when distance constraints are added to incidence constraints.

CONJECTURE 6.1.2. *Given an incidence structure $S = (V, F; I)$ which is independent for plane incidence, the set of distances E and the incidences I on the vertices are independent at some 2-scene $(\mathbf{p}, \mathbf{P})$ if and only if, for all non-empty sets E' of distances and I' of incidences on the vertices V' and faces F':*

$$|E'| + |I'| \leq 2|V'| + 2|F'| - 3.$$

6.2. Plane directions and lengths. In the spirit of constraints in plane CAD, there is another matroid which extends the plane first-order rigidity matroid. We combine the direction constraints of §4 with the length constraints of §3 in a *double graph* $FG = (V; D, E)$, where the same edge may lie in the direction set D and the length set E (Figure 6.2). At a plane configuration $\mathbf{p}$, these create a constraint matrix, $R_{FG}(\mathbf{p})$, with the two types of rows from the rigidity matrix for lengths and from the parallel design matrix for directions [**SW**,**Wh12**]. With the *generic configurations* for these matrices we define the *generic direction-length matroid* on the double graph.

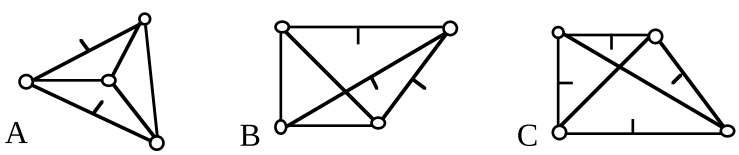

Fig. 6.2. *Examples of double graphs which are generic bases at plane configurations – heavy lines for distance constraints and light lines with spikes for direction constraints.*

THEOREM 6.2.1 [**SW**]. *For a generic plane configuration $\mathbf{p}$ on the vertices V, and a double graph $FG = (V; D, E)$ of length constraints D and direction constraints E, the constraints are a basis for the if and only if:*
1. $|D| + |E| = 2|V(D \cup E)| - 2$;
2. $|D'| + |E'| \leq 2|V(D' \cup E')| - 2$ *for all subsets nonempty subsets* D', E';
3. $|D'| \leq 2|V(D')| - 3$ *for all nonempty* $D' \subseteq D$;
4. $|E'| \leq 2|V(E')| - 3$ *for all nonempty* $E' \subseteq E$.

Within this matroid, we have analogs of all plane rigidity results: both combinatorics [**SW**] and geometry [**Wh12**]. We also have a 'swapping duality' – any result for a design $FG(\mathbf{p})$ also holds for the swapped design $GF(\mathbf{p})$ in which we swap direction constraints D and length constraints E.

While these constraints, as written, are not important in plane CAD, they do imply partial results for angle and length constraints. On the basis of other preliminary results, we *conjecture* that the matroid for angle constraints in the plane is impossible to characterize with a polynomial time algorithm.

6.3. Spherical angles and distances. Closely related to these matroids on plane configurations are matroids for constraints and configurations on the sphere. In particular, the matroids for frameworks of points and bars (on great circles) are isomorphic to plane rigidity matroids under central projection.

THEOREM 6.3.1 [**Wh2**]. *Given a spherical framework* $G(\mathbf{q})$ *and a projection* $\Pi(\mathbf{q}) = \mathbf{p}$ *from the center of the sphere to a plane framework* $G(\mathbf{p})$*, then*

1. $G(\mathbf{q})$ *is first-order rigid on the sphere if and only if* $G(\mathbf{p})$ *is first-order rigid in the plane;*
2. $G(\mathbf{q})$ *is independent on the sphere if and only if* $G(\mathbf{p})$ *is independent in the plane.*

On the sphere we have an additional tool: polarity. Each point $\mathbf{q}_i$ is the normal to a plane $\mathbf{Q}^i$ through the origin and a great circle on the sphere. Such a polarity turns a distance constraint between spherical points $\mathbf{q}_i, \mathbf{q}_j$ into an angle constraint between spherical lines $\mathbf{Q}^i, \mathbf{Q}^j$. Such a combinatorial duality of the abstract structure also preserves the independence, rank etc. of the generic constraint matroid.

It would be desirable to work on the sphere with an incidence structure of great circles and points with added length and angle constraints. However, the incidences alone recreate the problems we described above in §6.1. We have a conjecture for structures with independent incidence constraints.

CONJECTURE 6.3.2. *Given independent incidences* I*, a set of distances* D *and angles* A *is a basis for some (almost all) realizations on the sphere if and only if:*

1. $|D| + |A| + |I| = 2|V| + 2|L| - 3$*;*
2. $|D'| + |A'| + |I'| \leq 2|V'| + 2|L'| - 3$ *for each non-empty subset.*

By appendix §A, the counts of Conjecture 6.3.2 define a matroid on the incidence structure. Our conjecture states that this count-matroid is the same as the generic constraint matroid for the incidence structure on the sphere. Unfortunately, the underlying difficulties of incidences and their possible dependencies block the inductive approaches which have been used for many of the analogous plane results. It is possible that some direct construction, such as Tay's new proof of Laman's Theorem [**Ta5**], can be extended, at least for interesting special cases.

7. Summary of Plane Results

To orient our move up to related matroids in 3-space and higher dimensions, we summarize some critical features of generic 2-rigidity for a graph $G = (V, E)$:

1. The matroid is 'controlled' by the count $2|V| - 3$ in the sense that:
   (a) $2|V| - 3$ is the rank of the complete graph K_n $(n \geq 2)$;
   (b) any set E is independent if and only if $|E'| \leq 2|V'| - 3$ for every non-empty subset E' on vertices V'.
2. The matroid is the matroid union of two copies of the graphic matroid on G, followed by a Dilworth truncation;
3. There is an inductive construction for all bases of the matroid. (It can be easily modified to construct all independent sets.)
4. All circuits of the generic matroid are spanning on their vertices (i.e. their closure is the complete graph on the vertices of the circuit).

5. Some of the combinatorial characterizations have associated $O(|V|^2)$ algorithms for verifying whether a graph is a basis of the generic 2-rigidity matroid.
6. The parallel design matroid for generic plane configurations of a graph is the same as the generic 2-rigidity matroid.
7. C_1^0-cofactors, defined by the algebraic criteria on the lines:
   for each i, $\sum_{\{j|(i,j)\in E\}} \omega_{i,j}(A^{i,j}x + B^{i,j}y + C^{i,j}) = 0$;
   generate the same matroid.

Generalized parallel 2-scenes on $S = (F, V, I)$ give a broader matroid with analogs of results for plane rigidity:

1. The matroid is 'controlled' by the count $|F| + 2|V| - 2$.
2. All circuits of the generic matroid are spanning on their vertices and faces.
3. There are polynomial $O(|I|^2)$ algorithms for detecting bases etc.
4. This general parallel drawing matroid contains all of the others as special cases: Laman's Theorem, Henneberg 2-constructions, etc. are all special cases of results for general plane parallel 2-scenes.

REMARK 7.1.1. There is an extension of plane bar frameworks which generates oriented matroids. In these *tensegrity frameworks*, the edges are partitioned into E_+, E_0, E_- and the dependencies are restricted to self-stresses with the appropriate sign on each edge (with no restriction on E_0) [**CoW**,**RW**]. In structural engineering terms, the edges in E_+ are struts (taking only compression) and in E_- are 'cables' (taking only tension). While there is an extensive theory for such frameworks, the signs of the dependencies at a generic configuration vary between regions of the plane [**WW1**] and there is no single 'generic' behaviour for the signed graph [**RW**]. We will not pursue this extension here.

Finally, we recall that all of the matroids were extracted from applications and the results have been used in these applications. These applications remain a continuing source for problems and a standard to evaluate our results against.

Part II: Higher Dimensions.

We are ready for the other named strips of matroids in Figure 1.1. We begin with the parallel drawing matroids (§8), whose theory is a direct generalization of the results of Part I. However the core of Part II will be the rigidity matroid for 3-space (§9) and the analogous C_2^1-cofactor matroid for plane configurations (§10). As Figure 1.1 indicates, the generic 3-rigidity matroid and the bivariate C_2^1-cofactor matroids are conjectured to be isomorphic (? in Figure 1.1 – see §10.3)

The 'higher' cases: generic rigidity for $d \geq 4$ and the bivariate C_s^{s-1}-cofactor matroids provide important insights back toward these core matroids, through both counterexamples and examples for the analogs of our conjectures for rigidity in 3-space and C_2^1-cofactors. For higher levels, these families of matroids of graphs are known to be distinct (the black gap with $\neq$ in Figure 1.1 – see §11.5), though they are members of the more general class of abstract d-rigidity matroids [**GSS**].

For contrast with these difficulties, we will close Part II with a rigidity related matroid in d-space which has a complete theory as well as potential applications.

8. Parallel Scenes in Higher Dimensions

The definitions and results of §4 generalize directly to 3-space and higher dimensions. §8.1 presents the general definitions for scenes with preassigned normals, then §8.2 specializes to parallel drawings of graphs with preassigned directions for the edges. These results originated in the dual form: scene analysis of polyhedral pictures [**Su3**,**Wh8**] in which assigned normals for faces are dualized to assigned projections for vertices, i.e. the polyhedral pictures of §8.3.

8.1. Parallel d-scenes. For an abstract polyhedral incidence structure $S = (V, F; I)$, a *d-scene* is a pair of location maps: *points* for the vertices $\mathbf{p} : V \to \mathbb{R}^d$, $\mathbf{p}_i = (x_i, \dots, z_i, w_i)$ and *hyperplanes* for the faces $\mathbf{P} : F \to \mathbb{R}^{d+1}$, $\mathbf{P}^j = (A^j, \dots, C^j, 1, D^j)$; such that, for each incidence $(i, j) \in I$:

$$A^j x_i + \ldots + C^j z_i + w_i + D^j = 0.$$

The *normal* to the hyperplane $\mathbf{P}^j$ is $\mathbf{n}^j = (A^j, \dots, C^j, 1)$. (We continue to assume that no hyperplane is 'vertical'.)

A d-scene $S(\mathbf{q}, \mathbf{Q})$ is *parallel* to $S(\mathbf{p}, \mathbf{P})$ if, for each face j, the normals are equal: that is, $(P^j_1, \dots, P^j_{d-1}, 1) = (Q^j_1, \dots, Q^j_{d-1}, 1)$. A *non-trivially parallel d-scene* for $S(\mathbf{p}, \mathbf{P})$ is a parallel 2-scene $S(\mathbf{q}, \mathbf{Q})$, such that the point configuration $\mathbf{q}$ is not a translation or dilation of the configuration $\mathbf{p}$ (Figure 8.1B). Otherwise it is *trivially parallel* (Figure 8.1A).

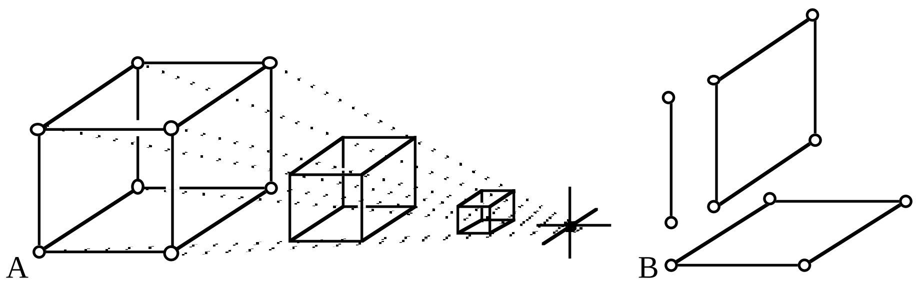

Fig. 8.1. *A 3-scene with trivially parallel scenes (A) and non-trivially parallel 3-scenes (B).*

We can also begin with an assignment $\mathbf{n}$ of *normals* for the faces: a d-vector $\mathbf{n}^j = (A^j, \dots C^j, 1)$ to each face $j \in F$. A d-scene $S(\mathbf{p}, \mathbf{P})$ *realizes the normals* $\mathbf{n}$, if for each face $j \in F$ with $\mathbf{P}^j = (A^j, \dots C^j, 1, D^j) = (\mathbf{n}^j, D^j)$ and vertex i:

$$\mathbf{n}^j \cdot (x_i, \dots, w_i) + D^j = A^j x_i + \ldots + C^j z_i + w_i + D^j = 0.$$

Given the normals $\mathbf{n}^j$, we have a linear system of $|I|$ equations in $|F| + d|V|$ variables recorded by the *parallel d-scene matrix* for d-space normals $\mathbf{n}$, where the variables are ordered: $[\dots, D^j, \dots ; \dots, x_i, \dots, w_i, \dots]$:

$$\begin{array}{c} \\ (1,a) \\ \vdots \\ (i,j) \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccccccccc} a & \dots & j & \dots & m & | & v_1 & \dots & v_i & \dots \end{array} \\
\left(\begin{array}{ccccccccccc}
1 & \dots & 0 & \dots & 0 & | & \mathbf{n}^a & \dots & \mathbf{0} & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots & \ddots \\
0 & \dots & 1 & \dots & 0 & | & \mathbf{0} & \dots & \mathbf{n}^j & \dots \\
\vdots & \ddots & \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots & \ddots
\end{array}\right)\end{array}
\begin{bmatrix} D^a \\ \vdots \\ D^m \\ - \\ \mathbf{p}_1{}^t \\ \vdots \\ \mathbf{p}_i{}^t \\ \vdots \end{bmatrix}
= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}$$

For matroid theory, our primary interest is the *generic d-parallel matroid* on the complete incidence structure $V \times F$, generated by independent rows of the parallel

d-scene matrix at generic normals for the faces. For bases of this matroid, the only realizations at generic normals will be *trivial* – having all points coincident and the parallel d-scenes will be the d-space of translations.

PARALLEL d-SCENES THEOREM 8.1.1 [**Wh5**,**8**]. *For an incidence structure $S = (V, F; I)$, the following are equivalent:*

1. *the incidences I are independent in the generic d-parallel matroid on $V \times F$;*
2. *for all non-empty subsets $I' \subseteq I$, $|I'| \leq d|V(I')| + |F(I')| - d$*

The following are also equivalent:

3. *I is a circuit of the generic d-parallel matroid on $V \times F$:*
4. *$|I| = d|V(I)| + |F(I)| - d + 1$, and for all proper subsets I':* $|I'| \leq d|V(I')| + |F(I')| - d$.

For the translation to parallel drawings of graphs (§8.2), we need a corollary.

COROLLARY 8.1.2 [**Wh5**,**8**]. *For generic normals* $\mathbf{n}$ *in d-space for the faces of an incidence structure $S = (V, F; I)$, the following are equivalent:*

1. *the incidences I are independent in a d-scene with distinct vertices;*
2. *for all subsets I' with $V(I') \geq 2$: $|I'| \leq d|V(I')| + |F(I')| - (d + 1)$.*

EXAMPLE 8.1.3. Consider the twenty four incidences of the cubical structure of Figure 8.1(A). These incidences will be independent for generic normals, leaving a space of parallel 3-scenes of dimension $2|V| + |F| - |I| = 30 - 24 = 6$ (Figure 8.1B). Even the special normals of a symmetric cube will leave the same space of parallel 3-scenes: the incidences are independent for these non-generic normals.

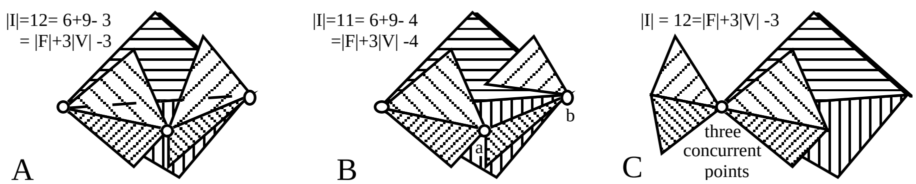

Fig. 8.2. *A dependent incidence structure when realized with distinct vertices but dependent normals (A), an independent subset with generic normals (B), and independent incidences with generic normals but with all vertices coincident (C).*

Consider the twelve incidences of Figure 8.2A. By the Parallel d-Scenes Theorem, this set is a basis of the generic d-parallel matroid and the only realizations for generic directions will have all vertices coincident (Figure 8.2C).

If we remove one incidence, there will be realizations of generic directions with all vertices distinct (Figure 8.2B), by Corollary 8.1.2. However, the additional incidence with distinct vertices restricts the normal of this last face to be orthogonal to the line ab, whose direction is already fixed.

Because this generic d-parallel matroid is combinatorially characterized by these 'submodular counts' (see the §A.1, §A.4), the basic plane results generalize:

1. there are efficient algorithms for independence, dependence, rank etc.;
2. the circuits contain bases for the complete incidence structure on their vertices and faces [**Im**,**Su3**];
3. inductive constructions for the bases of the generic matroid can be given;
4. we *conjecture* that $2d$-vertex connectivity of the incidence graph is sufficient for an incidence structure to be a basis for the complete incidence graph on its vertices and faces.

Remark 8.1.4. For the incidence structure of the faces and vertices of a convex polyhedron or convex d-polytope, the parallel d-scenes are intimately connected to Minkowski sum and Minkowski decomposition of the polytope. A geometric theorem of Shephard [**Sh**] says:

A convex polytope is Minkowski decomposible (is the Minkowski sum of two polytopes with simpler incidence structures) if and only if there are non-trivially parallel d-scenes.

For example, the cube of Figure 8.1A is the Minkowski sum of the bottom square and the line segment of Figure 8.1B.

8.2. Parallel graphs in d-space. Consider a geometric graph $G(\mathbf{p})$ in 3-space. The analogs of the parallel designs of §4.1 are:

configurations $\mathbf{q}$ such that, for each edge $\{h, i\} \in E$, $(\mathbf{p}_h - \mathbf{p}_i)||(\mathbf{q}_h - \mathbf{q}_i)$.

For each edge, this condition gives two constraints on the configuration $\mathbf{q}$ (three equations with one added parameter) corresponding to selecting two planes intersecting in the line $\mathbf{p}_h\mathbf{p}_i$ and using those planes in the spirit of §8.1. For planes with normals $\mathbf{m}_e, \mathbf{n}_e$ through the edge $e = \{h, i\}$, we have the linear constraints:

$$\mathbf{m}_e \cdot \mathbf{q}_h - \mathbf{m}_e \cdot \mathbf{q}_i = 0 \quad \text{and} \quad \mathbf{n}_e \cdot \mathbf{q}_h - \mathbf{n}_e \cdot \mathbf{q}_i = 0$$

on the configurations $\mathbf{q}$. The graph G and these normals $\mathbf{m}, \mathbf{n}$ from the configuration $\mathbf{p}$ yield a $2|E|$-by-$3|V|$ 3-*parallel matrix* for $G(\mathbf{p})$ and define a matroid on the double copies $2E$ of the edges of the graph (or a hypermatroid on the edges E of the graph §A.3). If there are two distinct points, the trivial null space has dimension 4 and the maximum possible rank will be $3|V| - 4$. An essential condition for independent rows is $2|D'| \leq 3|V(D')| - 4$, for all non-empty subsets $D' \subseteq 2E$.

As an convention, we take two copies of the edges of the graph and define a matroid on this set $2E$, based on a generic 3-configuration $\mathbf{p}$ and a generic selection of the two planes through each edge. The independence of the rows of this *generic* 3-*parallel matrix* defines the *generic* 3-*parallel matroid* on the set $2E$.

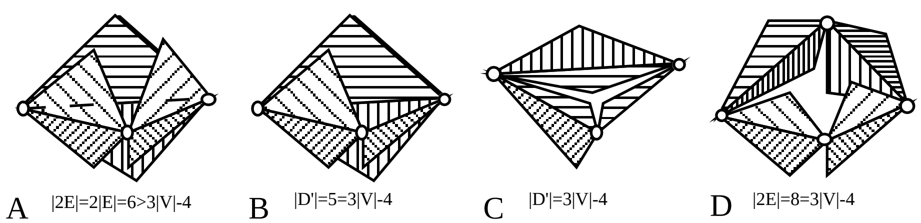

Fig. 8.3. *Graphs in 3-space with up to two planes through each edge which are: (A) always dependent, (B) a generically independent subset of (A), and (C) a dependent choice of planes for (B) in special position, (D) generically independent.*

Example 8.2.1. Consider the structure of Figure 8.3A. This is a graph shown with two planes for each edge. (All non-collinear triangles are generic here.) By the simple count $|2E| > 3|V| - 4$, $2E$ is always dependent, unless we have all points coincident and have generic directions for the normals. By the previous section, the subset D' of Figure B is independent for generic normals, with $|D'| = 3|V| - 4$. However, if we select three planes with the same (vertical) normal for the three edges (Figure 8.3C), these three rows alone will be dependent: if the first two edges $\mathbf{q}_1 - \mathbf{q}_2$, $\mathbf{q}_2 - \mathbf{q}_3$ are horizontal, the third edge $\mathbf{q}_1 - \mathbf{q}_3$ is also horizontal. This indicates that dependence may come from the configuration of the points or from the selected planes for a given configuration.

The skew quadrilateral in Figure 8.3D gives an independent set $2E$ for any choice of pairs of distinct normals for the skew lines. With $|2E| = 3|V| - 4$, the only parallel graphs are dilations and translations of this quadrilateral. On the other hand, a coplanar quadrilateral will be dependent for any allowed choice of the normals, since there are non-trivially parallel graphs within this plane.

We could directly mimic much of §2–§4 to find combinatorial characterizations of the independent sets of edge copies (see §A.4 and [**Wh5**] for portions of this direct development). Instead, we follow the approach of §4.3 and extract the basic results from §8.1. For completeness, we give these results for all dimensions $d \geq 2$.

Given a graph G and a generic configuration $\mathbf{p}$ in d-space, we select normals $\mathbf{n}^r_{i,j}$, $1 \leq r \leq d-1$ for $d-1$ independent, generically selected hyperplanes through each edge $\{i,j\}$. The system of equations: $\mathbf{n}^r_{i,j} \cdot \mathbf{q}_i - \mathbf{n}^r_{i,j} \cdot \mathbf{q}_j = 0$ creates the $(d-1)|E|$-by-$d|V|$ *d-parallel matrix for the graph G* and defines the *generic d-parallel matroid* on the set $(d-1)E$ consisting of $(d-1)$ copies of each edge in E. (The dependence or independence of a subset D of $(d-1)E$ will *not* be changed by permuting the labels of the copies selected for each edge, because we have chosen generic vertices and generic hyperplanes for the $(d-1)$ copies of each edge. Only the number of copies selected for each edge will matter.)

d-Parallel Graphs Theorem 8.2.2 [**Wh5**]. *For the complete graph K_V and a non-empty subset $D \subseteq (d-1)E$, the following are equivalent:*
1. *D is a basis of the generic d-parallel matroid on K_V;*
2. *$|D| = d|V| - (d+1)$ and for all nonempty subsets D',* *$|D'| \leq d|V(D')| - (d+1)$;*
3. *D can be partitioned into $d+1$ edge-disjoint trees, exactly d incident with each vertex, but no d non-empty subtrees span the same subset of vertices.*

Proof. 1. ⇒ 2. The necessity of these counts for a basis follows directly from the observation that, for any set with at least two distinct vertices, there is a 4-space of trivially parallel configurations obtained by translation and dilation.
2. ⇒ 1. Interpret the set D as the faces of an incidence structure $S_D(V, F; I)$. Since each edge copy $e \in D$ converts to a face e with two incident vertices, the count $|D'| \leq d|V(D')| - (d+1)$ is equivalent to the count

$$|I'| = 2|D'| \leq |D'| + d|V(I')| - (d+1) = |F(I')| + d|V(I')| - (d+1),$$

To work with arbitrary subsets I', any faces in $F(I')$ incident with only one vertex in $V(I')$ can be dropped or added without changing the inequality.

By Corollary 8.1.2, for generic normals $\mathbf{n}$ the incidences I are independent in a d-scene $S(\mathbf{p}, \mathbf{P})$ with all vertices distinct. The row reduction of Example 4.3.2, generalized to this setting, directly converts the parallel d-scene matrix into a matrix with independent rows formed by a diagonal block $|F|$-by-$|F|$ identity with zeros below and a lower right $|D|$-by-$d|V|$ block for the rows of the d-parallel matrix for D. The rows of the d-parallel matrix for D are therefore independent. Since the configuration $\mathbf{p}$ defines the lines $\mathbf{P}$ and the normals $\mathbf{n}$, we have one configuration $\mathbf{p}$ for which D gives independent rows. By the usual algebraic arguments for 'generic configurations' (configurations not satisfying the finite set of relevant algebraic equations) this independence holds for all generic configurations.

Since $|D| = d|V| - (d+1)$ is the maximal possible rank with at least two vertices, D is a basis for the generic d-parallel matroid.
2. $\Leftrightarrow$ 3. [**Ha**] shows the equivalence of the counts and the $(d+1)\mathrm{Tree}(d)$-coverings.

The other results for generic 2-rigidity, such as gluing, efficient algorithms for independence, rank, etc., have appropriate generalizations to the d-parallel matroid on a graph G [**Wh5**]. We note that, for all dimensions, gluing across a pair of vertices takes two 'tight' pieces into a 'tight' graph on the union of the vertices.

REMARK 8.2.3. The generic d-parallel matroid on G is very different from the generic d-rigidity matroid on the same graph we will define in §9 and §11. A framework such as the skew quadrilateral of Figure 8.3D has only trivially parallel configurations but have non-trivial first-order motions. We do have a one-way connection, which we note now (also see §16.4).

THEOREM 8.2.4 [**Wh5**]. *If a geometric graph $G(\mathbf{p})$ in d-space, $d \geq 2$ has a non-trivially parallel drawing then the bar framework $G(\mathbf{p})$ has a non-trivial first-order motion in d-space.*

8.3. Polarity to scene analysis. The theorems on parallel d-scenes originated in a projectively polar form: for $d = 3$, our Parallel d-Scene Theorem was Sugihara's Conjecture for polyhedral pictures [**Su3**,**Wh8**]. An assigned normal to each face is replaced by an assigned projection into $(d-1)$-space for each vertex (again $d-1$ choices). These vertices are 'lifted' by the one additional value recording the height ($|V|$ variables), and the $d|F|$ coordinates for non-vertical faces are constrained by the I incidence equations (Figure 8.4). (Also see [**Cr3**,**4**,**CRy**,**CR**] for a related analysis of liftings.)

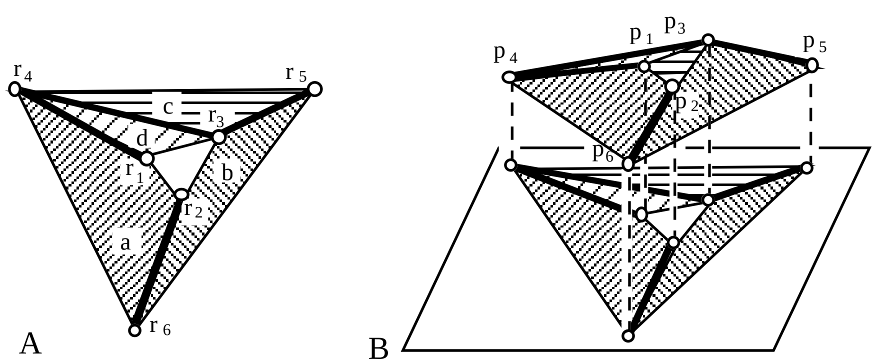

Fig. 8.4. *A 2-picture (A) on an incidence structure, with a lifting to a 3-scene (B).*

In computer science, the study of these 3-scenes lifting a fixed plane projection is called *scene analysis of the polyhedral picture* [**Su3**]. We briefly outline this dual vocabulary and the polarized theorems.

A *$(d-1)$-picture* of an incidence structure S is a location map $\mathbf{r} : V \to \mathbb{R}^{d-1}$, $\mathbf{r}_i = (x_i, y_i, \dots, z_i)$. A *lifting* of a $(d-1)$-picture $S(\mathbf{r})$ is a d-scene $S(\mathbf{p}, \mathbf{P})$, with the vertical projection $\Pi(\mathbf{p}) = \mathbf{r}$. (I.e., if $\mathbf{p}_i = (x_i, \dots, z_i, w_i)$ then $\mathbf{r}_i = (x_i, \dots, z_i) = \Pi(\mathbf{p}_i)$.) A lifting $S(\mathbf{p}, \mathbf{P})$ is *trivial* if all the faces lie in the same plane, and is *sharp* if each pair of faces sharing a vertex have distinct planes. The *lifting matrix* for a picture $S(\mathbf{r})$ is the $|I|$-by-$(|V| + d|F|)$ coefficient matrix $M(S, \mathbf{r})$ of the system of

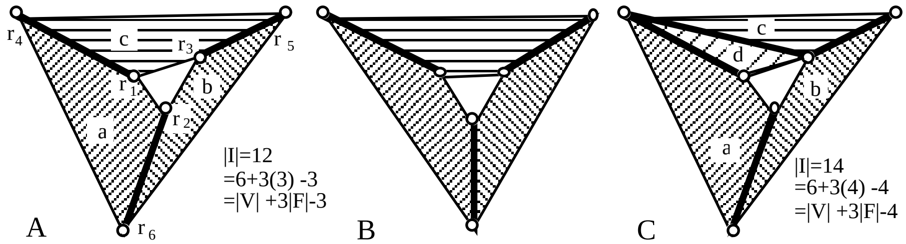

Fig. 8.5. *A generic 2-picture with only trivial liftings (A) but a sharp lifting for the special position (B). With the faces modified (C) the new structure has a sharp lifting for generic pictures.*

equations for liftings of a picture $S(\mathbf{r})$: for each $(i,j) \in I$: $A^j x_i + B^j y_i + \ldots + C^j z_i + w_i + D^j = 0$ where the variables are ordered: $[\ldots, w_i, \ldots ; \ldots, A^j, B^j, \ldots, D^j, \ldots]$.

Independence of the rows of the lifting matrix defines the *lifting matroid* on $S(\mathbf{r})$. Generic points for the picture define the *generic d-lifting matroid*.

Picture Theorem 8.3.1 [**Wh8**]. *The incidences I are independent in the generic d-lifting matroid if and only if subsets $I' \neq \emptyset$, $|I'| \leq |V'| + d|F'| - d$.*

A generic picture of an incidence structure $S = (V, F; I)$ has a sharp lifting if and only if, for all subsets I' with $F(I') \geq 2$, $|I'| \leq |V'| + d|F'| - (d+1)$.

Remark 8.3.2. The translation between liftings of a plane picture and parallel d-scenes for a set of normals is a true geometric polarity in d-space. First we transform the incidence structure $S = (V, F; I)$ to the dual structure $S^* = (V^*, F^*; I^*)$, by setting $V^* = F$, $F^* = V$ and reversing the ordered pairs in I. Given a d-scene $S(\mathbf{p}, \mathbf{P})$ the *polar scene* is $S^*(\mathbf{p}^*, \mathbf{P}_*)$ where the point $\mathbf{p}_i = (x_i, \ldots, z_i, w_i)$ becomes the hyperplane $P^i_* = (x_i, \ldots, z_i, 1, w_i)$ and the hyperplane $P^j = (A^j, \ldots C^j, 1, D^j)$ becomes the point $p^*_j = (A^j, \ldots C^j, D^j)$. This transformation preserves incidences of points and hyperplanes, taking a d-scene on S to a d-scene on S^*, the projections of points to normals of hyperplanes. Therefore it induces an isomorphism of the d-parallel matroid on $S(\mathbf{p}, \mathbf{P})$ and the d-lifting matroid on $S^*(\mathbf{p}^*, \mathbf{P}_*)$.

This transformation is the restriction to our 'affine coordinates' of the projective polarity about the paraboloid with Euclidean coordinates $x^2 + \ldots z^2 + 2w = 0$. This projective polarity also interchanges vertical planes with points at infinity – two sets which we excluded from our d-scenes.

Remark 8.3.3. These generic properties of a structure are computationally robust, in the sense that all small changes in a generic picture with sharp liftings also have sharp liftings (Figure 8.5), and in the sense that small changes in the points of the picture require only small changes in the corresponding lifting. Even special positions of such structures will always have non-trivial liftings, although these may not be sharp. However, up to numerical round off, all pictures 'are generic'. Other structures, which do not have sharp liftings for generic pictures (Figure 8.5A), may have sharp liftings in special positions (Figure 8.5B), but a small change in the position of even one point can destroy this sharpness. (Numerical round off will typically produce such a small change in position.) In practice, the generic results are the essential ingredients for computer analysis of polyhedral pictures [**Su3**].

9. Rigidity of Frameworks in 3-space

We are ready for the generic 3-rigidity matroid for graphs. While the initial form in §9.1 is directly analogous to the generic 2-rigidity matroid in Part I, the problem of a combinatorial characterization of bases in this matroid is substantially harder and is still unsolved (§9.2). We will illustrate the difficulties with some critical examples, some partial results and some conjectures with a matroidal flavour in §9.3, §9.4.

9.1. Statics and first-order kinematics. A 3-*space framework* is a standard graph $G = (V, E)$ and a 3-*configuration* $\mathbf{p} : V \to \mathbb{R}^3$. (The configuration $\mathbf{p}$ is a point in $\mathbb{R}^{3|V|}$.) A *dependence* or *self-stress* on the framework $G(\mathbf{p})$ is an assignment $\omega : E \to \mathbb{R}$, with $\omega\{i,j\} = \omega_{i,j} = \omega_{j,i}$, such that, for each vertex i:

$$\sum_{j|\{i,j\}\in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}.$$

These self-stresses are the row dependencies of the *rigidity matrix* $R_G(\mathbf{p})$:

$$[\dots \quad \omega_{i,j} \quad \dots] \begin{bmatrix} \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & (\mathbf{p}_i - \mathbf{p}_j) & \dots & (\mathbf{p}_j - \mathbf{p}_i) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \end{bmatrix} = [\,\mathbf{0} \quad \dots \quad \mathbf{0}\,].$$

The independence of rows in the rigidity matrix $R_G(\mathbf{p})$ defines the 3-*rigidity matroid* $\mathcal{R}_3(G;\mathbf{p})$ on the edges of the graph with a 3-configuration $\mathbf{p}$. The framework $G(\mathbf{p})$ is 3-*independent* if its edge set is independent in $\mathcal{R}_3(G;\mathbf{p})$, and the *rank* of $G(\mathbf{p})$ is the rank of $\mathcal{R}_3(G;\mathbf{p})$. For the complete graph K_n, we write $\mathcal{R}_3(n;\mathbf{p})$.

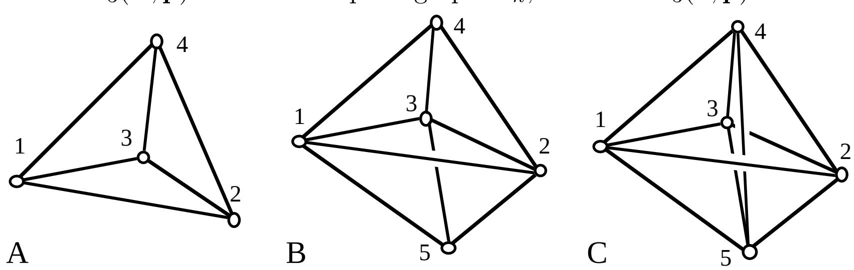

Fig. 9.1. *Independent frameworks in 3-space (A,B) and a circuit in the 3-rigidity matroid (C).*

Example 9.1.1. Consider the 3-space framework $K_4(\mathbf{p})$ of Figure 9.1A.

$R_G(\mathbf{p})$	ω	v_1	v_2	v_3	v_4
$\{1,2\}$	0	$\mathbf{p}_1 - \mathbf{p}_2$	$\mathbf{p}_2 - \mathbf{p}_1$	$\mathbf{0}$	$\mathbf{0}$
$\{1,3\}$	0	$\mathbf{p}_1 - \mathbf{p}_3$	$\mathbf{0}$	$\mathbf{p}_3 - \mathbf{p}_1$	$\mathbf{0}$
$\{1,4\}$	0	$\mathbf{p}_1 - \mathbf{p}_4$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{p}_4 - \mathbf{p}_1$
$\{2,3\}$	0	$\mathbf{0}$	$\mathbf{p}_2 - \mathbf{p}_3$	$\mathbf{p}_3 - \mathbf{p}_2$	$\mathbf{0}$
$\{2,4\}$	0	$\mathbf{0}$	$\mathbf{p}_2 - \mathbf{p}_4$	$\mathbf{0}$	$\mathbf{p}_4 - \mathbf{p}_2$
$\{3,4\}$	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{p}_3 - \mathbf{p}_4$	$\mathbf{p}_4 - \mathbf{p}_3$

Provided the four points are not coplanar, the vectors such as $\mathbf{p}_1 - \mathbf{p}_2$, $\mathbf{p}_1 - \mathbf{p}_3$ and $\mathbf{p}_1 - \mathbf{p}_4$ at vertex 1 are independent, and the only dependence is the zero self-stress. The edges are independent in the 3-rigidity matroid.

If we add an additional 3-valent vertex (Figure 9.1B), the same argument applied at vertex 5 shows the rows are independent, if the points are in general position (no four coplanar). However, if we add one more edge to get K_5, this argument no longer applies (Figure 9.1C). We will see that this is dependent for all 3-configurations and is a circuit for general position 3-configurations. (A configuration is in *general position in d-space* if any set of $\leq d+1$ points is affinely independent.)

What is the rank of the matrix $R_{K_n}(\mathbf{p})$ and of the matroid for a general position 3-configuration? From the previous example and its subsets, we have:
rank $\mathcal{R}_3(5;\mathbf{p}) = 9$, rank $\mathcal{R}_3(4;\mathbf{p}) = 6$, rank $\mathcal{R}_3(3;\mathbf{p}) = 3$, rank $\mathcal{R}_3(2;\mathbf{p}) = 1$.
In general, for $n > 2$ we observe that rank $\mathcal{R}_3(n;\mathbf{p}) = 3v - 6$.

We confirm this rank in two steps. We show this is an upper bound by giving a 6-dimensional solution space of the equations $R_G(\mathbf{p})\mathbf{u} = \mathbf{0}$. In general, the solutions $\mathbf{u} = (\dots, \mathbf{u}_i, \dots)$ are the *first-order flexes*, satisfying the equation for each edge $\{i,j\}$: $(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{u}_i - \mathbf{u}_j) = 0$. The component $\mathbf{u}_i$ is the *velocity* of $\mathbf{p}_i$ in the first-order flex. The guaranteed *trivial first-order flexes* for any graph G on a 3-configuration $\mathbf{p}$ are generated by the six vectors:

$$\begin{bmatrix}\mathbf{i}^t\\ \vdots\\ \mathbf{i}^t\end{bmatrix},\quad \begin{bmatrix}\mathbf{j}^t\\ \vdots\\ \mathbf{j}^t\end{bmatrix},\quad \begin{bmatrix}\mathbf{k}^t\\ \vdots\\ \mathbf{k}^t\end{bmatrix},\quad \begin{bmatrix}(\mathbf{p}_1\times\mathbf{i})^t\\ \vdots\\ (\mathbf{p}_v\times\mathbf{i})^t\end{bmatrix},\quad \begin{bmatrix}(\mathbf{p}_1\times\mathbf{j})^t\\ \vdots\\ (\mathbf{p}_v\times\mathbf{j})^t\end{bmatrix},\quad \begin{bmatrix}(\mathbf{p}_1\times\mathbf{k})^t\\ \vdots\\ (\mathbf{p}_v\times\mathbf{k})^t\end{bmatrix}.$$

For the standard basis $\mathbf{i},\mathbf{j},\mathbf{k}$ of 3-space, the first three solutions generate the *translations* and the final three generate the *rotations* about the origin. Checking a non-collinear triangle (say the first three rows and blocks for the first three vertices in Example 9.1), it is easy to verify that these six solutions are linearly independent. (Note that for two vertices, or a collinear configuration, the rotation about the line through these vertices will give $\mathbf{u}_i = \mathbf{0}$ for all vertices. This will appear as a dependence among the six generating first-order flexes and the space of trivial first-order flexes will have dimension 5.)

Counting Lemma 9.1.2. *A set of at least two edges E with $|E| > 3|V(E)| - 6$ is 3-dependent for every 3-space configuration* $\mathbf{p}$.

Equivalently, a set E is 3-independent at some 3-configuration only if, for all subsets E' with at least two edges, $|E'| \leq 3|V(E')| - 6$.

We demonstrate that some 3-independent graphs have this maximal rank $3|V| - 6$, at general position configurations $\mathbf{p}$ – i.e. they are bases of the matroid $\mathcal{R}_3(|V|;\mathbf{p})$. Given a graph $G = (V,E)$, a *vertex* 3*-addition of* 0 is the addition of one new vertex, 0, and three new edges $(0,i),(0,j),(0,k)$, creating the graph $G' = (V',E')$.

A graph $G = (V,E)$, with at least three vertices, is 3*-simple* if there is an ordering of the vertices $\sigma(1),\sigma(2),\dots,\sigma(|V|)$ such that:
(i) G_3 is the triangle $K_{\{\sigma(1),\sigma(2),\sigma(3)\}}$;
(ii) for $i \geq 3$, G_{i+1} is a vertex 3-addition of $\sigma(i+1)$ to G_i;
(iii) $G_{|V|}$ is G.

Vertex 3-Addition Lemma 9.1.3. *Given a framework $G(\mathbf{p})$ and a vertex* 3*-addition of* 0 *creating the framework $G'(\mathbf{p}_0,\mathbf{p})$, with $\mathbf{p}_0,\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k$ not coplanar:*
1. *$G'(\mathbf{p}_0,\mathbf{p})$ is* 3*-independent if and only if $G(\mathbf{p})$ is* 3*-independent;*
2. rank $\mathcal{R}_3(G';(\mathbf{p}_0,\mathbf{p})) =$ rank $\mathcal{R}_3(G;\mathbf{p}) + 3$.

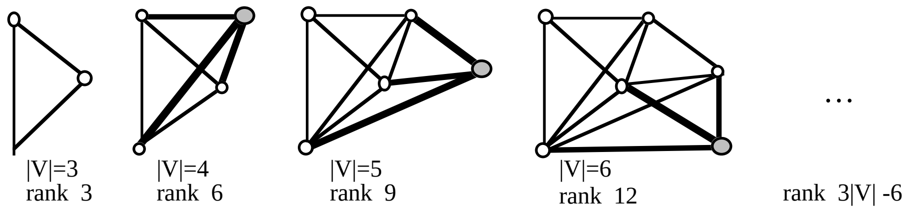

Fig. 9.2. *Building a* 3*-simple graph as a basis for the complete* 3*-rigidity matroid* $\mathcal{R}_3(n; \mathbf{p})$.

PROOF. The proof for the Vertex 2-Addition Lemma 2.1.3 extends directly.

STATIC 3-RIGIDITY THEOREM 9.1.4. *For any* $n \geq 3$ *and any general position* 3*-configuration* $\mathbf{p}$ *on* n *vertices, the edges* E *of any* 3*-simple graph* G *on* n *vertices are a basis of* $\mathcal{R}_3(n; \mathbf{p})$ *of rank* $3n - 6$.

PROOF. If $n = 3$, K_3 is a triangle, which has rank $3 = 3 \times 3 - 6$. This graph is both 3-simple and a basis of the matroid.

If n is greater than 3, we prove, by induction that there is a 3-simple graph G of size $3n - 6$ and its edges are a basis for K_n. Assume G_k is a 3-simple graph for $n = k$, which is a basis for $\mathcal{R}_3(k; \mathbf{p}|_k)$ of rank $3k-6$. Let G_{k+1} be a vertex 3-addition of $k + 1$ with edges $(1, k + 1), (2, k + 1), (3, k + 1)$. Since $\mathbf{p}$ is in general position, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_{k+1}$ are not coplanar and E_{k+1} is 3-independent of rank $(3k - 6) + 3 = 3(k + 1) - 6$. By Lemma 9.1.2, this is a maximal 3-independent set in K_{k+1}, so E_{k+1} is a basis.

Any framework $G(\mathbf{p})$ on $|V| \geq 3$ for which $R_G(\mathbf{p})$ has rank $3|V| - 6$ (or for which $|V| \leq 2$, $G = K_{|V|}$ and $\mathbf{p}$ has distinct points) is *statically* 3*-rigid*. We also say that the edge set E is *statically* 3*-rigid* on $V(E)$ at $\mathbf{p}$. Any framework $G(\mathbf{p})$ for which all first-order motions are trivial motions (are in the space generated by the translations and rotations) is *first-order* 3*-rigid*. Because of the rank of this trivial solution space and the standard result of linear algebra that the dimension of the entire solution space is $3|V| - \text{rank } R_G(\mathbf{p})$, it is easy to check that static 3-rigidity is equivalent to first-order rigidity for a 3-configuration [**Co2**,**RW**,**Wh4**].

There are other edge sets which are not 3-simple but which satisfy the condition of Lemma 9.1.2 to be 3-independent for some configurations (Figure 9.3). These examples are also bases for the 3-rigidity matroid for some choices of $\mathbf{p}$ (see below).

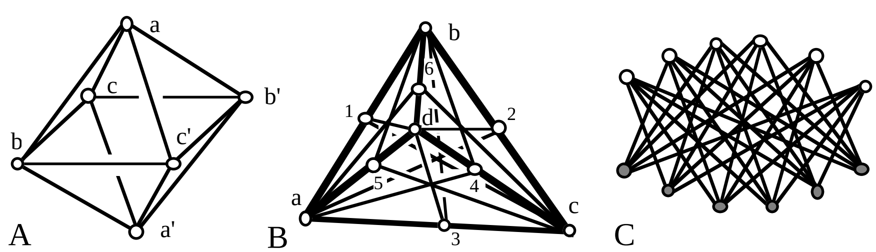

Fig. 9.3. *Graphs which are not* 3*-simple, but which are bases for the* 3*-rigidity matroid.*

At generic 3-configurations (defined either as configurations which achieve the maximum rank of rigidity matrices for graphs on these vertices or as points with algebraically independent coordinates), we create the *generic* 3*-rigidity matroid on the graph*: $\mathcal{R}_3(G)$. For the complete graph on n vertices, this is written $\mathcal{R}_3(n)$.

9.2. A basic problem. We have shown that the generic 3-rigidity matroid on K_V has rank $3|V|-6$, for $|V| \geq 3$, and that any non-empty subset of edges E' with $|E'| > 3|V(E')|-6$ is dependent in this matroid. For plane graphs, these counts characterized the matroid, by characterizing the circuits (see also §A.1). The analogous definition here would be:

a non-empty set E' of edges is a circuit if and only if
(i) $|E'| = 3|V(E')| - 5$ and,
(ii) for all subsets E'' on at least three vertices $V(E'')$:
$|E''| \leq 3|V''| - 6$.

The first clue that this is not sufficient is that this definition does not apply to a single edge (which is 3-independent): $|E| = 1 = 3(2) - 5$. The next example explores this difficulty in more depth.

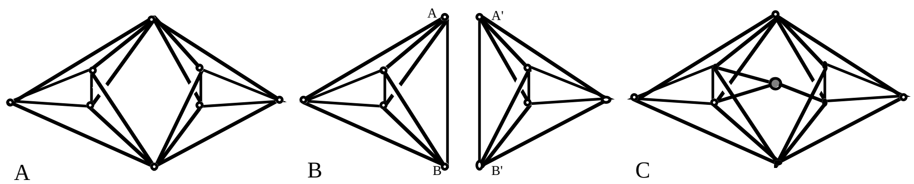

Fig. 9.4. *A non-rigid circuit (A) of the generic 3-rigidity matroid, obtained by circuit exchange of two smaller circuits (B). Insisting on 3-connectivity does not eliminate the dependence (C).*

Example 9.2.1. Figure 9.4A illustrates the 'two bananas' example which has several critical properties:

1. This 3-dependent set is obtained by a 'circuit exchange' on the common edge a, b of two smaller circuits, K_5, which satisfy the criterion: $|E'| = 10 = 3|V(E')| - 5$.
2. This is a 3-circuit (minimal 3-dependent) because deleting any one edge leaves a graph which is a 3-simple graph with one edge deleted.
3. This circuit of the generic 3-rigidity matroid satisfies the Necessary Counts Lemma: all subsets E'' on at least 3-vertices satisfy $|E''| \leq 3|V(E'')| - 6$.
4. This circuit is not 3-rigid: freezing one side permits a rotation about the 'hinge' ab between the bananas.

This example lives in the matroidal properties of the count: $3|V| - 6$. It does not depend on particular properties of our rigidity matrix, but will reappear in any matroid with the basic rank $3n - 6$ on all complete graphs K_n, $n \geq 3$. In §10 we will see a second matroid with the same counts and this same difficulty.

Since the framework of Figure 9.4C is obtained by a vertex 3-addition to the two bananas, it is also 3-dependent. However, this is now 3-connected, confirming that an additional '3-connectivity assumption' will not eliminate our difficulty.

In §2, our second basic tool for characterizing bases in the generic rigidity matroid was the Henneberg 2-constructions. We have a partial result for 3-space. A graph G' is an *edge* 3*-split* of the graph G on $a, b; c, d$, if $\{a, b\}$ is an edge of G and G' is formed from G by adding a new vertex 0, removing the edge $\{a, b\}$, and adding four new edges $(0, a), (0, b), (0, c), (0, d)$ (Figure 9.5B).

Edge 3-Split Theorem 9.2.2 [**TW2**]. *Assume G' is an edge 3-split of $G = (V, E)$ on $a, b; c, d$, and $\mathbf{p}$ is a 3-configuration with $\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \mathbf{p}_d$ not coplanar.*

If $G(\mathbf{p})$ is 3-independent (statically 3-rigid), then $G'(\mathbf{p}_0, \mathbf{p})$ is 3-independent (statically 3-rigid) for almost all $\mathbf{p}_0$, including $\mathbf{p}_0$ a distinct point on $\mathbf{p}_a, \mathbf{p}_b$.

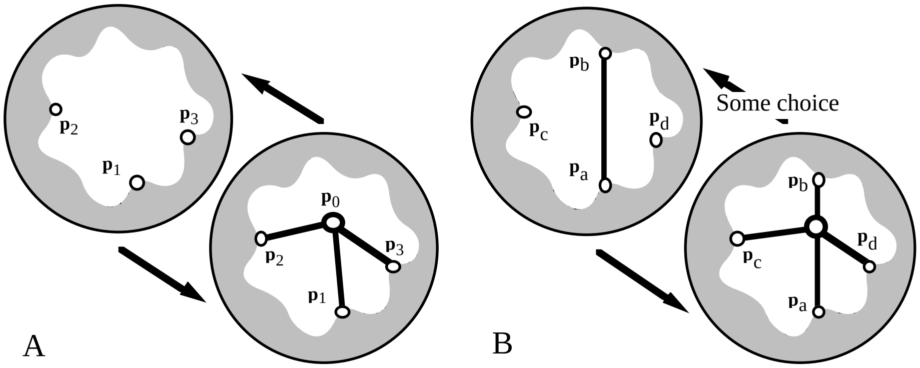

Fig. 9.5. *Two inductive steps which preserve* 3*-rigidity and give Henneberg* 3*-constructions for bases: (A) vertex* 3*-addition; (B) edge* 3*-split.*

Conversely, if $G'(\mathbf{p}_0, \mathbf{p})$ *is* 3*-independent (statically* 3*-rigid) for some choice of* $\mathbf{p}_0$*, with vertex* 0 *connected to exactly vertices* a, b, c, d *at four general position points then, for some edge* e *of vertices in* a, b, c, d*,* $E = E' \cup \{e\} - \{(0,a),(0,b),(0,c),(0,d)\}$ *is independent (statically* 3*-rigid) at* $\mathbf{p}$ *and* G' *is an edge* 3*-split of* $G = (V' - 0, E)$*.*

Proof. One proof follows the same pattern as the proof given for the plane. The only change is that K_4 is now 3-independent and statically 3-rigid.

We outline a second approach in a more matroidal spirit. This is representative of certain 'replacement techniques' or 'small circuits techniques' which are helpful in analyzing the behaviour of particular frameworks [**Wh4**].

We first show that the edge 3-split preserves 3-independence and 3-rigidity. We add the vertex 0 by a three valent addition to vertices a, c, d, choosing 0 to lie on the line a, b, distinct from a and b. This preserves 3-independence, 3-rigidity etc. Now consider the three collinear edges $\{a, b\}, \{a, 0\}, \{b, 0\}$. Realized on a line, this polygon (triangle) forms a circuit. Therefore, replacing any two of these with any other two gives an equivalent rank for the matrix (and matroid). We replace $\{a, b\}, \{a, 0\}$ with $\{a, 0\}, \{b, 0\}$ to complete the edge split.

Assume that $G'(\mathbf{p}_0, \mathbf{p})$ is 3-rigid, but that each of the possible subgraphs G is 3-dependent. Therefore the complete graph on the four vertices a, b, c, d is contained in the closure of $E_1 = E' - \{(0,a),(0,b),(0,c),(0,d)\}$. This complete graph K_4 is 3-independent if the points are not coplanar, so this K_4 extends to a 3-independent set E_2 with the same closure as E_1, $|E_2| = |E_1| = 3(|V|) - 10$. If we now add back the four edges $(0,a),(0,b),(0,c),(0,d)$, we have at least the 3-dependence on the K_5 $0, a, b, c, d$, so the rank of $E_2 \cup \{(0,a),(0,b),(0,c),(0,d)\}$ is now smaller than $|E_2| + 4$. However this has the same rank as $E' = E_1 \cup \{(0,a),(0,b),(0,c),(0,d)\}$ which was assumed to be $|E_1| + 4 = |E_2| + 4$. This contradiction completes the proof.

For a graph $G = (V, E)$ with at least three vertices, a *Henneberg* 3*-construction* is an ordering of the vertices $\sigma(1), \sigma(2), \ldots, \sigma(|V|)$ and a sequence $G_3, \ldots, G_{|V|}$ of graphs such that:

(i) G_3 is the triangle $K_{\{\sigma(1),\sigma(2),\sigma(3)\}}$;
(ii) for $i \geq 3$, G_{i+1} is a 3-addition of vertex $\sigma(i+1)$ to G_i
or G_{i+1} is an edge 3-split on G_i which adds a vertex $\sigma(i+1)$;
(iii) $G_{|V|}$ is G.

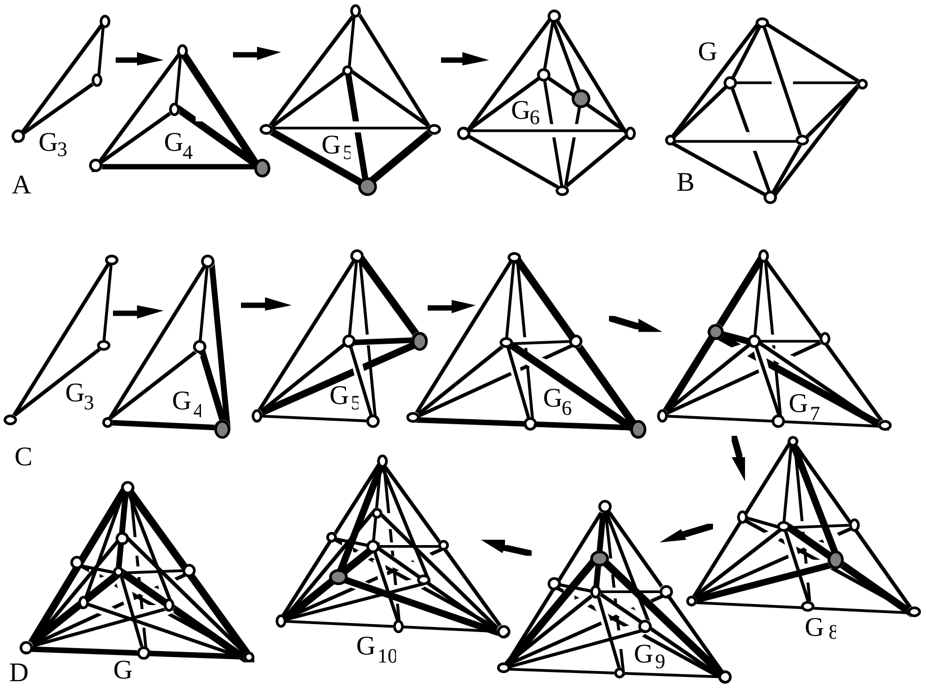

Fig. 9.6. *Henneberg 3-constructions for two bases for generic 3-rigidity matroids.*

Henneberg's Theorem 9.2.3 [**TW2**]. *If G has a Henneberg 3-construction then G is a basis for $\mathcal{R}_3(n)$.*

Proof. The Henneberg 3-construction is a sequence of vertex 3-additions and edge 3-splits. These preserve the generic 3-independence and the generic 3-rigidity.

Remark 9.2.4. Unlike the plane version, this theorem does not run both directions. Some graphs with $|E| = 3|V| - 6$ have all vertices of valence greater than or equal to 5. For example, the graph of an icosahedron or the graph of Figure 9.4C are bases of $\mathcal{R}_3(12)$. These graphs cannot be created by a Henneberg 3-construction. While Henneberg's original work contained a number of important results, it also contained critical errors which misrepresented this case [**He**,**TW2**]. In §9.4 we offer some conjectures for extended 3-constructions to create all bases.

9.3. More partial results. A number of additional plane results do generalize to generic 3-rigidity. Of course, these will remain incomplete as characterizations of the bases and independent sets.

First-Order Flex Test 9.3.1. *For any 3-configuration $\mathbf{p}$ for the graph K_n, the following are equivalent:*

1. *the edge $\{h, k\}$ is not in the closure of the set E, $\langle E\rangle$, in $\mathcal{R}_3(n;\mathbf{p})$;*
2. *every self-stress ω on $E \cup \{h, k\}$ is zero on $\{h, k\}$;*
3. *there is a first-order flex $\mathbf{u}$ on $G = (V, E)$, such that $(\mathbf{p}_h - \mathbf{p}_k)\cdot(\mathbf{u}_h - \mathbf{u}_k) \neq 0$.*

The Plane Generic Gluing Lemma extends to the generic 3-rigidity matroid.

Generic 3-Gluing Lemma 9.3.2. *For two edge sets E_1, E_2,*

1. *if E_1 and E_2 are generically 3-rigid and $|V(E_1) \cap V(E_2)| \geq 3$, the set $E_1 \cup E_2$ is generically 3-rigid (Figure 9.7A);*

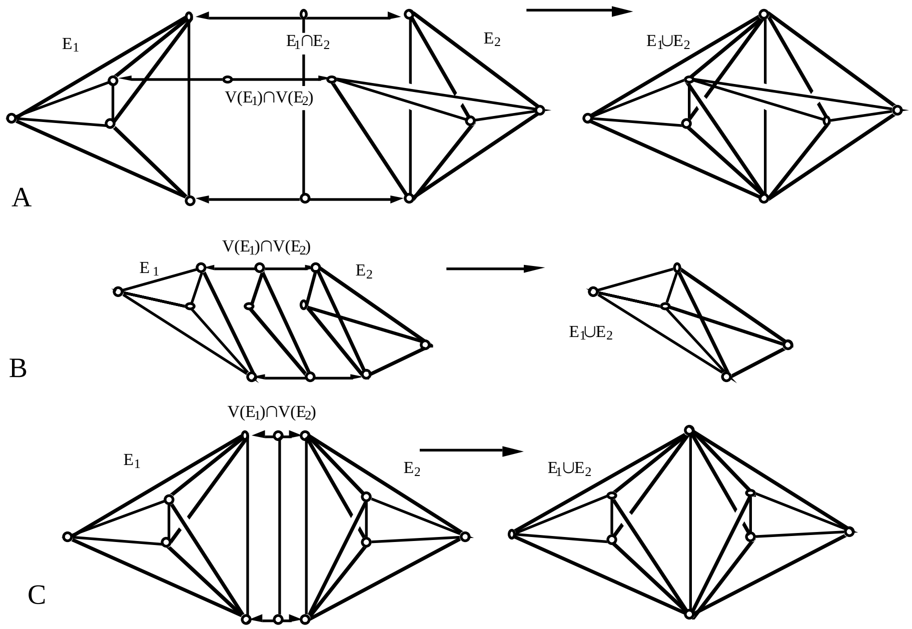

Fig. 9.7. *Gluing spatial frameworks: for* 3*-rigidity (A); for* 3*-independence (B); and for non-rigidity (C).*

2. *if* E_1 *and* E_2 *are generically* 3*-independent and* $E_1 \cap E_2$ *is generically* 3*-rigid on* $V(E_1 \cap E_2)$*, the set* $E_1 \cup E_2$ *is generically* 3*-independent (Figure 9.7B);*
3. *if* $|V(E_1) \cap V(E_2)| < 3$*, then the closure* $\langle E_1 \cup E_2 \rangle$ *in* $\mathcal{R}_3(V(E_1) \cup V(E_2))$ *is contained in* $K_{V(E_1)} \cup K_{V(E_2)}$ *(Figure 9.7C).*

Proof. The basic proofs follow the plane pattern of Lemma 3.1.4. For 3, the desired non-trivial first-order flex is a rotation about the hinge of the two shared vertices (or a general position hinge through the single shared vertex etc.).

Remark 9.3.3. For any matroid on the edges of a complete graph, properties 1 and 3 are equivalent to the defining properties of an *abstract* 3*-rigidity matroid* [**GSS**]. Specifically, an *abstract m-rigidity matroid* is a matroid on the edges of complete graphs with the closure operator $\langle\cdot\rangle$ satisfying two additional properties:

C5. For any sets $E, F \subseteq K_n$, if $|V(E) \cap V(F)| < m$, then $\langle E \cup F\rangle \subset (K_{V(E)} \cup K_{V(F)})$;

C6. If $E, F \subset K_n$ satisfy $\langle E\rangle = K_{V(E)}$ and $\langle F\rangle = K_{V(E)}$ (they are *rigid*) and $|V(E) \cap V(F)| \geq m$, then $\langle E \cup F\rangle = K_{V(E \cup F)}$.

Our Generic Gluing Lemmas 3.1.4 and 9.3.2 show that $\mathcal{R}_2(n)$ is an abstract 2-rigidity matroid and that $\mathcal{R}_3(n)$ is an abstract 3-rigidity matroid.

A number of properties, including the necessary counts $2|V| - 3$ and $3|V| - 6$ follow from these properties [**GSS**]. Consider an abstract 3-rigidity matroid:

1. From C5 with $F = \emptyset$, we see that $\langle E\rangle \subseteq K_{V(E)}$;
2. Analyzing two sides of triangle as E, F in C5, we see that the third edge is independent – so a triangle has rank 3;
3. Analyzing two triangles joined at an edge as E, F in C5, we see that the final edge of a K_4 is independent – so K_4 has rank 6;
4. Using $E = K_N$ and $F = K_4$ (with an overlap of 3 vertices) in property C6, $\langle E \cup F\rangle = K_{V(E\cup F)} = K_{N+1}$ and the rank of $E \cup F$ is $\leq$ rank $E + 3$;

5. By induction from 2 and 4, we see that rank $\mathcal{R}_3(n) = 3n - 6$ for $n \geq 3$. We will return to these abstract 3-rigidity matroids in §9.4 and §10.

There are two special classes of graphs for which we have substantial results: complete bipartite graphs and triangulated surfaces. We state the bipartite result without proof [**BR**,**Wh3**], but follow with a remark which gives the geometric origin of the generic result. Recall that a *complete bipartite graph* is a graph $K_{m,n} = (A \cup B, A \times B)$, where A and B are disjoint sets of cardinality $|A| = m$ and $|B| = n$.

Bipartite Graphs Theorem 9.3.4. *A complete bipartite graph $K_{m,n}$, is generically rigid in 3-space if and only if $m + n \geq 10$ and $m, n > 3$.*

Remark 9.3.5. The geometric basis of this generic result is the following corollary of the static analysis of Bolker and Roth [**BR**].

Quadrics to Bipartite Motions 9.3.6 [**Wh3**]. *For a bipartite graph $K_{m,n}$ and a general position configuration $\mathbf{p}$ in d-space, the framework has only trivial first-order flexes in d-space if and only if the points of m and n each affinely span d-space and the complete set of vertices do not lie on a quadric surface in d-space.*

In 3-space, we need at least 4 vertices to affinely span the space and any 9 points lie on a quadric (satisfy a quadratic equation). The two 'minimal' 3-rigid complete bipartite graphs are $K_{4,6}$, which is also generically 3-independent with $|E| = 24 = 3|V| - 6$, and $K_{5,5}$, which is a generic 3-circuit with $|E| = 25 = 3|V| - 5$.

In the plane, this result gives $K_{3,3}$ as the minimal 2-rigid (and 3-independent) complete bipartite graph. In §11 we will apply this corollary to 4-space.

For graphs of triangulated surfaces (and other graphs), strong results can be proven by induction using a graph construction called a 'vertex split' (Figure 9.8).

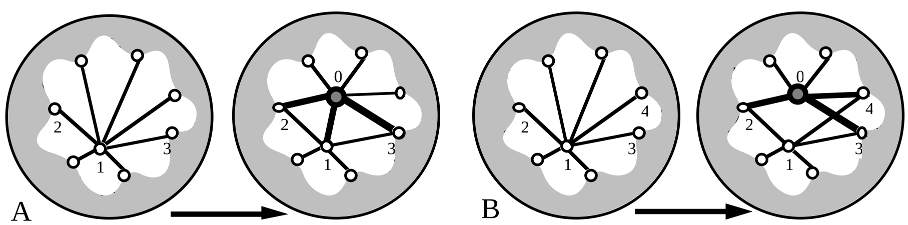

Fig. 9.8. *Vertex 3-splits on two edges (A) and three edges (B), which preserve generic 3-rigidity.*

For a graph G, with a vertex 1 incident to the edges $(1,2), (1,3), \ldots, (1,k)$, $(1,k+1), \ldots, (1,m)$ the *vertex 3-split of 1 on two edges* $(1,2), (1,3)$ is the modified graph G' with edges $(1,4), \ldots, (1,k)$ removed and an added vertex 0 incident with new edges $(0,1), (0,2), (0,3), \ldots, (0,k)$ (Figure 9.8A).

For a graph G, with a vertex 1 incident to the edges $(1,2), (1,3), (1,4), (1,5)$, $\ldots, (1,k), (1,k+1), \ldots, (1,m)$ the *vertex 3-split of 1 on three edges* $(1,2), (1,3)$, $(1,4)$ is the modified graph G' with edges $(1,5), \ldots, (1,k)$ removed and an added vertex 0 incident with new edges $(0,2), (0,3), (0,4), \ldots, (0,k)$ (Figure 9.8B).

For vertex 3-splits on two edges, the following result is proven in [**Wh9**], using a 'limiting special position' argument. For vertex 3-splits on three edges, the result was overlooked, but it has a simpler related argument.

Vertex Splits Theorem 9.3.7. *If the graph G' is a vertex 3-split of a generically 3-rigid graph G on three edges or a vertex 3-split of G on two edges then G' is generically 3-rigid.*

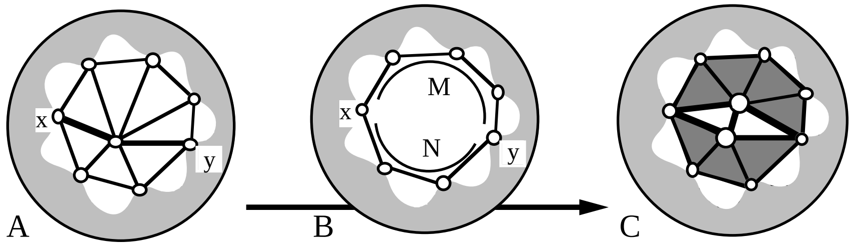

Fig. 9.9. *If a vertex 3-split on two edges of the graph of a triangulated surface (A) follows the cycle of edges at the vertex (B), then the new graph also comes from a triangulated surface (C).*

Applying 3-splits on two edges carefully (Figure 9.9), we take a triangulated surface to a triangulated surface. For example, all triangulated spheres can be created from the tetrahedron by a sequence of vertex 3-splits on 2-edges [**Wh9**].

Triangulated 2-Surfaces Theorem 9.3.8 [**Fo**]. *The graph of any triangulated* 2*-surface is generically* 3*-rigid* (Fogelsanger's Theorem).

In particular, the graph of any triangulated sphere is generically 3*-rigid and* 3*-independent* (Gluck's Theorem [**Gl**]).

Remark 9.3.9. That the graphs of triangulated spheres are also 3-independent follows from Euler's formula. Since $|V| - |E| + |F| = 2$ and $2|E| = 3|F|$ for any triangulated surface:

$$3|E| = 3|V| + 3|F| - 6 \quad \Leftrightarrow \quad |E| = 3|V| - 6.$$

For other closed surfaces, such as a triangulated torus, the counts give $|E| > 3|V| - 6$ and the graphs cannot be independent in the generic 3-rigidity matroid.

While the general result is proven by an induction on the graphs of minimal 2-cycles in homology, the special result for spheres has an older geometric source:

Theorem of Cauchy and Dehn 9.3.10 [**Ca**,**Wh3**]. *The edges and vertices of any strictly convex triangulated sphere form an independent* 3*-rigid framework.*

This result and its relatives are the true mathematical source of rigidity results for the Geodesic Domes promoted by Buckminster Fuller's mystical pronouncements. For non-convex spheres, Connelly has found a special position counterexample which is not only first-order flexible, but actually flexible [**Co1**].

If we add a single additional edge (a 'shaft') to a triangulated sphere whose graph is 4-connected, we create a circuit of the generic 3-rigidity matroid [**Wh6**].

We have a final geometric (and induced combinatorial) result which establishes a critical pattern for 'lifting' first-order results from one dimension to the next.

The *cone graph* $G * u$ is created from $G = (V, E)$ by adding a new vertex u and the $|V|$ edges (u, i) for all vertices $i \in V$. For a 3-configuration $\mathbf{p}$ on V, the *cone projection from* $\mathbf{p}_0$ is a configuration $\mathbf{q} = \Pi_0(\mathbf{p})$ in the plane (placed as a plane P in 3-space) on the vertices $V \backslash 0$, such that $\mathbf{p}_i$ is on the line $\mathbf{q}_i\mathbf{p}_0$ for all $i \neq 0$.

Coning Theorem 9.3.11 [**Wh2**]. *For a* 3*-configuration* $\mathbf{p}$*, with no edge at* $\mathbf{p}_0$ *parallel to the plane* P*, a framework* $G(\Pi_0(\mathbf{p}))$ *is first-order rigid (independent) in the plane if and only if the cone* $(G * 0)(\mathbf{p})$ *is first-order rigid (independent) in* 3*-space.*

This gives an obvious generic corollary, which also extends to 2-circuits by a more subtle geometric argument which we omit.

Generic Coning Corollary 9.3.12. *A graph G is generically 2-rigid (2-independent, a 2-circuit) if and only if the cone graph $G{*}u$ is 3-rigid (3-independent, a 3-circuit, respectively).*

The reader can verify that the same connections exist between bases (trees) and 2-circuits (polygons) and their cones in the plane (2-simple cones and wheels).

9.4. Combinatorial conjectures for 3-space. As we noted in §9.2, a basis for the generic 3-rigidity matroid may have all vertices of valence larger than 4. A quick check of the necessary count: $|E| = 3|V| - 6$ (or equivalently, $2|E| = 6|V| - 12$) guarantees that some vertices have valence 5 or less.

What inductive techniques create 5-valent vertices? An analysis of the rigidity matrix of a 3-rigid, independent graph G at a 5-valent vertex confirms that it can be created from smaller 3-rigid, independent graphs by one of the two constructions: X-replacement; or double V-replacement of Figure 9.10 [**TW2**]. Simple examples confirm that we will require both of these operations, and that a single V-replacement may destroy first-order rigidity (by turning, say, a 3-valent vertex at the point of the V into a 2-valent vertex).

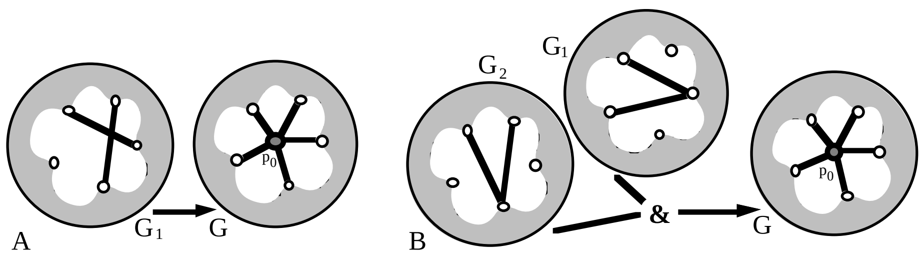

Fig. 9.10. *Two inductive techniques creating 5-valent vertices which are conjectured to preserve generic 3-rigidity: X-replacement (A) and double V-replacement (B).*

Certain special cases of these inductive steps, with additional patterns of edges assumed present, are known to preserve generic 3-rigidity [**GSS**,**TW2**]. For example, from the plane analog of X-replacement (Figure 2.8) and the Generic Coning Corollary, we get X-replacement within the non-cone edges of any cone. Also a form of single V-replacement within triangulated surfaces follows from vertex 3-splits on two edges. However, no general result has been demonstrated.

3-D Replacement Conjecture 9.4.1 [**TW2**]. *The X-replacement in Figure 9.10A takes a graph G_1 which is generically 3-rigid to a graph G which is generically 3-rigid.*

The double V-replacement in Figure 9.10B takes two graphs G_1, G_2 which are generically 3-rigid to a graph G which is generically 3-rigid.

Remark 9.4.2. If these conjectured steps prove correct in 3-space, then we would have inductive techniques to generate the graphs of all generically 3-rigid and all generically 3-independent frameworks in 3-space. This would yield an exact $O(2^{|V|})$ algorithm for bases of the generic 3-rigidity matroid. Of course, a graph can be checked for generic 3-rigidity by a 'brute force' $O(2^{2^{|V|}})$ algorithm:

Assign the points independent variables as coordinates, form the rigidity matrix, then check the rank by symbolic computation.

On the other hand, if numerical coordinates are chosen for the points 'at random' then the rank of this numerical matrix (computed in $O(|E|^3)$ time) will be the generic value, with probability 1.

For the complexity of algorithms on graphs, generic 3-rigidity belongs to an interesting class. The decision problem for generic 3-rigidity has a random polynomial algorithm but no known polynomial time (or even exponential) algorithm. The matroid of §10 will provide a related example.

In a specific sense, it is conjectured that the generic 3-rigidity matroid is the 'freest' matroid of its type. A matroid $\mathcal{M}$ is *maximal in a class of matroids* on the same elements if every set S which is independent in some matroid of this class is independent in $\mathcal{M}$. The whole process of Henneberg 2-constructions and Laman's Theorem implicitly verifies that the generic 2-rigidity matroid is maximal among all abstract 2-rigidity matroids.

THE MAXIMAL CONJECTURE 9.4.3 [**GSS**, Conjecture 2.5.1]. *There is a maximal abstract* 3*-rigidity matroid on the edges of the complete graph* K_n*.*

The generic 3-rigidity matroid $\mathcal{R}_2(n)$ *is this maximal abstract* 3*-rigidity matroid on* K_n*.*

The existence of a maximal matroid illustrates the essential problem of inducing matroids by submodular functions which are negative on singletons (see §A.1).

In the next section, we will see a second candidate for this maximal abstract 3-rigidity matroid. In §11, we will see a pair of counter-examples which demonstrates that generic 4-rigidity is *not* the maximal abstract 4-rigidity matroid.

Related to this Maximal Conjecture is a conjectured formula, due to Andreas Dress, for the rank (or equivalently, the degree of freedom) of a set of edges in the generic 3-rigidity matroid (or at least the maximal abstract 3-rigidity matroid). Effectively, the Dress Conjecture, described in the Unsolved Problems section of this book and [**GSS** §5.7], conjectures the rank function for such a maximal matroid.

Again there are certain easy connectivity results: every generically 3-rigid set on at least 4 vertices is vertex 3-connected as a graph. (This is implicit in the Generic 3-Gluing Theorem.) While a circuit may be only 2-connected in a vertex sense (recall the double bananas), it will be at least 5-connected in an edge sense. Finally, it is easy to adapt the counterexample of Figure 3.6 to give a graph which is 11-connected but not generically 3-rigid. We conjecture that 12-connectivity is sufficient (see §12.1 for some supporting evidence).

SUFFICIENT CONNECTIVITY CONJECTURE 9.4.4. *If a graph G is vertex* 12*-connected then G is generically* 3*-rigid.*

10. The C_2^1-Cofactor Matroid from Bivariate Splines

We will present a second abstract 3-rigidity matroid on graphs which has rank $3n-6$ on complete graphs K_n and shares numerous properties with generic 3-rigidity. While this matroid has its own role in the theory of bivariate splines, we present it here for additional insight into the underlying matroidal structure associated with the generic 3-rigidity matroid. Portions of this theory are explicit and implicit in the papers [**ASW**,**Wh10**,**11**] as well several unpublished manuscripts.

10.1. The definition of the C_2^1-cofactor matroid. Recall the form of a C_1^0-stress from §5. For a graph G and a plane configuration $\mathbf{p}$, this is an assignment of scalars to the edges such that:

$$\sum_{j \mid \{i,j\}\in E} \lambda_{i,j} L^{i,j} = \sum_{j \mid \{i,j\}\in E} \lambda_{i,j}[A^{ij}x + B^{ij}y + C^{ij}] \equiv 0.$$

With a small adjustment in the algebra, the same plane configuration can be used to define the C^1_2-cofactor matroid. We write the quadratic form:

$$\begin{aligned}[L^{i,j}]^2 &= [A^{i,j}x + B^{i,j}y + C^{i,j}]^2 \\ &= (A^{i,j})^2x^2 \ + 2A^{i,j}B^{i,j}xy + (B^{i,j})^2y^2 + 2A^{i,j}C^{i,j}x + 2B^{i,j}C^{i,j}y + (C^{i,j})^2,\end{aligned}$$

for $i < j$ and $[L^{j,i}]^2 = -[L^{i,j}]^2$ for $j > i$, where $L^{i,j} = A^{i,j}x + B^{i,j}y + C^{i,j} = 0$ is the standard equation for the line, as defined in §5. Recall that the definition of $L^{i,j}$ gave $L^{i,j} = -L^{j,i}$. However since $(L^{i,j})^2 = (L^{j,i})^2$, we have to reinsert the antisymmetry into the vectors $[L^{j,i}]^2$.

A *C^1_2-cofactor on the plane graph* $G(\mathbf{p})$ is an assignment of scalars $\lambda_{i,j}$ to the edges $\{i,j\} \in E$ such that for each vertex j:

$$\sum_{j \mid \{i,j\}\in E} \lambda_{ij}[L^{i,j}]^2 \equiv 0.$$

These C^1_2-cofactor equations appear to impose $6|V|$ linear conditions in the $|E|$ variables $\lambda_{i,j}$. However this is misleading. If we focus on the *reduced C^1_2-vector*: $\mathbf{D}^2_{i,j} \ = [\ (A^{ij})^2,\ 2A^{ij}B^{ij},\ (B^{ij})^2]$ for $i < j$ and $\mathbf{D}^2_{j,i} \ = -[\ (A^{ij})^2,\ 2A^{ij}B^{ij},\ (B^{ij})^2]$ for $j > i$, then we can *uniquely* reconstruct $[L^{i,j}]^2$ using the point $\mathbf{p}_i$ (or $\mathbf{p}_j$).

C^1_2-COFACTOR REDUCTION LEMMA 10.1.1. *Given a plane graph $G(\mathbf{p})$, an assignment of scalars $\lambda_{i,j}$ is a C^1_2-cofactor if and only if the scalars satisfy the reduced C^1_2-vector equations:* $\sum_{j \mid \{i,j\}\in E} \lambda_{ij}\mathbf{D}^2_{i,j} = \mathbf{0}$.

PROOF. Since the entries in $\mathbf{D}^2_{i,j}$ are just the first three of the six coefficients in $[L^{i,j}]^2$, it is clear that any C^1_2-cofactor satisfies the reduced C^1_2-vector equations.

Conversely, given $\mathbf{D}^2_{i,j} \ = \ [\ (A^{i,j})^2,\ 2A^{i,j}B^{i,j},\ (B^{i,j})^2]$, $i < j$ and $\mathbf{p}_i = (x_i, y_i)$, we have $A^{i,j}x_i + B^{i,j}y_i + C^{i,j} = 0$ or $C^{i,j} = -(A^{i,j}x_i + B^{i,j}y_i)$, so

$$\begin{aligned}2A^{i,j}C^{i,j} &= 2A^{i,j}(-A^{i,j}x_i - B^{i,j}y_i) = -2(A^{i,j})^2x_i - 2A^{i,j}B^{i,j}y_i;\\ 2B^{i,j}C^{i,j} &= 2B^{i,j}(-A^{i,j}x_i - B^{i,j}y_i) = -2A^{i,j}B^{i,j}x_i - 2(B^{i,j})^2y_i;\\ (C^{i,j})^2 &= (-A^{i,j}x_i - B^{i,j}y_i)^2 = (A^{i,j})^2x_i^2 + 2A^{i,j}B^{i,j}x_iy_i + (B^{i,j})^2y_i^2;\end{aligned}$$

which generate the other three coefficients.

These three 'reconstructions' are linear equations in the coefficients of $\mathbf{D}^2_{i,j}$. Therefore, assuming $\sum_{j \mid \{i,j\}\in E} \lambda_{ij}\mathbf{D}^2_{i,j} = \mathbf{0}$, for the coefficients of x in the extended C^1_2-cofactor equation we have:

$$\begin{aligned}\sum_{j \mid \{i,j\}\in E} \lambda_{ij}2A^{i,j}C^{i,j} &= \sum_{j \mid \{i,j\}\in E} \lambda_{ij}\big[-2(A^{i,j})^2x_i - 2A^{i,j}B^{i,j}y_i\big]\\ &= -2x_1\big[\sum_{j \mid \{i,j\}\in E} \lambda_{ij}(A^{i,j})^2\big] - y_i\big[\sum_{j \mid \{i,j\}\in E} \lambda_{ij}2A^{i,j}B^{i,j}\big] = 0 + 0.\end{aligned}$$

A similar check works for the remaining terms of the cofactor equation. We conclude that the reduced C^1_2-vector equations imply the complete C^1_2-cofactor equations.

With this lemma, we can write the C_2^1-cofactors as row dependencies of an $|E|$-by-$3|V|$ C_2^1*-cofactor matrix*:

$$M_2^1(G;\mathbf{p}) \;=\; \begin{bmatrix} \mathbf{D}_{1,2}^2 & -\mathbf{D}_{1,2}^2 & \dots & 0\ 0\ 0 & \dots & 0\ 0\ 0 & \dots & 0\ 0\ 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0\ 0\ 0 & 0\ 0\ 0 & \dots & \mathbf{D}_{i,j}^2 & \dots & -\mathbf{D}_{i,j}^2 & \dots & 0\ 0\ 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \end{bmatrix}$$

The analogy to the pattern of the 3-rigidity matrix is visible.

The row dependencies of this matrix define the C_2^1*-cofactor matroid* $\mathcal{M}_2^1(G;\mathbf{p})$, with C_2^1*-independence* and C_2^1*-dependence*. In particular, for any generic plane configuration $\mathbf{p}$ we have the *generic* C_2^1*-cofactor matroid* on G, $\mathcal{M}_2^1(G)$ and the generic C_2^1-cofactor matroid $\mathcal{M}_2^1(n)$ on K_n.

Remark 10.1.2. A general sextuple represents a conic section in the plane. Since a scalar multiple does not change the conic represented by an equation, these are viewed as points in projective 5-space. The particular sextuples which represent 'double lines', $[L^{i,j}]^2$ in our cofactor equations, form a special 2-surface in this projective space studied in algebraic geometry – the Veronese surface [**SR** VII.3].

Implicitly, we are working with some basic analogies between this Veronese surface for double lines and the Grassmannian for the lines in projective 3-space which are implicit in the study of 3-frameworks (see §16 and [**Wh15**]). We have not yet explored how much our presentation resembles material buried in the classical studies of these objects. It is likely that useful information for further work will be found in exploring these algebraic connections for all the matroids presented through the rest of the chapter.

Example 10.1.3. Consider the plane graph of Figure 10.1. In this configuration, we have the matrix $M_2^1(G;\mathbf{p})$:

	v_1			v_2			v_3			v_4		
$\{1,2\}$	0	0	1	0	0	-1	0	0	0	0	0	0
$\{1,3\}$	1	0	0	0	0	0	-1	0	0	0	0	0
$\{1,4\}$	1	-2	1	0	0	0	0	0	0	-1	2	-1
$\{2,3\}$	0	0	0	1	2	1	-1	-2	-1	0	0	0
$\{2,4\}$	0	0	0	1	-4	4	0	0	0	-1	4	-4
$\{3,4\}$	0	0	0	0	0	0	4	-4	1	-4	4	-1

These rows are independent and the C_2^1-cofactor matroid on K_4 has rank 6.

In particular, the three rows which attach vertex 1 to the independent triangle $2,3,4$ are clearly independent. The only way these three rows could be dependent is if the three lines used only two distinct slopes. (In approximation theory for C_2^1-splines, vertices with only two slopes are called 'singular' [**ASW**,**Wh10**].)

We will follow the basic steps used in §9.1 to show that the C_2^1-cofactor matroid has rank $3n-6$ on K_n at any general position plane configuration $\mathbf{p}$.

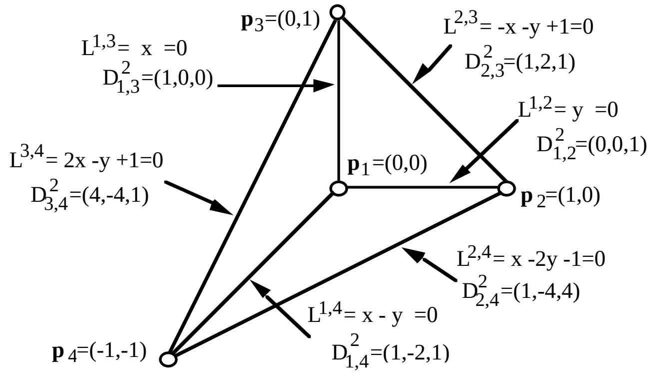

Fig. 10.1. *A sample geometric graph for which the C_2^1-cofactor matrix has independent rows.*

First, consider the C_2^1-*flexes* – solutions to the linear system $M_2^1(G;\mathbf{p})\mathbf{x} = \mathbf{0}$. For configurations on $n \geq 3$ non-collinear points, this contains a subspace of *trivial* C_2^1-*flexes* of dimension 6 generated by the following C_2^1-flexes:

1. the three space of C_2^1-*translations*: $(\mathbf{t}, \dots, \mathbf{t})^t$, for $\mathbf{t} \in R^3$;
2. the flex $(2x_1, y_1, 0, \dots, 2x_i, y_i, 0, \dots, 2x_n, y_n, 0)^t$;
3. the flex $(0, x_1, 2y_1, \dots, 0, x_i, 2y_i, \dots, 0, x_n, 2y_n)^t$;
4. the flex $(x_1^2, 2x_1y_1, y_1^2, \dots, x_i^2, 2x_iy_i, y_i^2, \dots, x_n^2, 2x_ny_n, y_n^2)^t$.

It is easy to see that the C_2^1-translations satisfy all equations of rows:

$$\mathbf{D}_{i,j}^2 \cdot \mathbf{t} + \mathbf{D}_{j,i}^2 \cdot \mathbf{t} = (\mathbf{D}_{i,j}^2 - \mathbf{D}_{i,j}^2) \cdot \mathbf{t} = 0.$$

For the flex in 2:

$$\begin{aligned}&\mathbf{D}_{i,j}^2 \cdot (2x_i, y_i, 0) - \mathbf{D}_{i,j}^2 \cdot (2x_j, y_j, 0)\\ &= 2(A^{i,j})^2 x_i + 2A^{i,j}B^{i,j}y_i - 2(A^{i,j})^2 x_j - 2A^{i,j}B^{i,j}y_j = -2A^{i,j}C^{i,j} + 2A^{i,j}C^{i,j} = 0.\end{aligned}$$

An analogous argument works for 3 and 4. We conclude that these generate a space of C_2^1-flexes which work for the complete graph on the vertices, for all configurations.

What is the dimension of this space? Choosing the three points $(0,0)$, $(1,0)$ and $(0,1)$, we have the C_2^1-flexes:

$$\begin{bmatrix}1\\0\\0\\1\\0\\0\\1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\\0\\1\\0\\0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\\0\\0\\1\\0\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\2\\0\\0\\0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\2\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\\1\end{bmatrix}$$

which are visibly independent. Any other three non-collinear points are affinely equivalent, and the entire algebraic structure is affinely invariant (also projectively invariant – see §11 [**ASW**,**Wh10**]). We conclude that the kernel has dimension at least 6, and the maximum rank for the C_2^1-cofactor matroid on K_n is $3n - 6$.

C_2^1-Cofactor Counting Lemma 10.1.4. *A set E of at least two edges with $|E| > 3|V(E)| - 6$ is C_2^1-dependent for every plane configuration* $\mathbf{p}$.

Equivalently, a set E is C_2^1-independent at some plane configuration only if, for all subsets E'' with at least two edges, $|E''| \leq 3|V(E'')| - 6$.

$\mathcal{M}_2^1$ VERTEX 3-ADDITION LEMMA 10.1.5. *Given a plane graph $G(\mathbf{p})$, and a vertex 3-addition of 0 creating the graph G' and given that the points $\mathbf{p}_0, \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ are in general position in the plane graph $G'(\mathbf{p}_0, \mathbf{p})$, then*

1. *$G'(\mathbf{p}_0, \mathbf{p})$ is C_2^1-independent if and only if $G(\mathbf{p})$ is C_2^1-independent;*
2. *rank $\mathcal{M}_2^1(G'; \mathbf{p}_0, \mathbf{p}) =$ rank $\mathcal{M}_2^1(G; \mathbf{p}) + 3$.*

PROOF. Observe in Example 10.1 that the three rows attaching the new vertex are independent in the three new columns for this vertex.

An induction on 3-simple graphs proves the analog of Theorem 9.1.4.

$\mathcal{M}_2^1$ RIGIDITY THEOREM 10.1.6. *For any $n \geq 3$ and any general position plane configuration $\mathbf{p}$ on n vertices, the edges E of any 3-simple graph G on n vertices are a basis of $\mathcal{M}_2^1(n; \mathbf{p})$ of rank $3n - 6$.*

Any plane graph $G(\mathbf{p})$ for which $M_2^1(G; \mathbf{p})$ has rank $3|V| - 6$ (or with $|V| \leq 2$ distinct points) is *C_2^1-rigid*. We also say that the edge set E is *C_2^1-rigid* on $V(E)$ at $\mathbf{p}$. It is a simple exercise to show that this is equivalent to saying all C_2^1-flexes of $G(\mathbf{p})$ are trivial flexes in the space generated by the list above. An edge set E is *generically C_2^1-rigid* if its closure in the generic C_2^1-cofactor matroid is $K_{V(E)}$. (This is equivalent to being C_2^1-rigid for some plane configuration $\mathbf{p}$.)

COROLLARY 10.1.7. *A plane graph $G(\mathbf{p})$ with vertices in general position is not C_2^1-rigid if and only if there is a C_2^1- flex $\mathbf{u}$ and a pair of vertices h, k (not an edge) such that:*

$$(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) \neq 0.$$

With the same counts and the same induction for 3-simple graphs as for $\mathcal{R}_3(n)$, the double bananas of Figure 9.4 are also a circuit in the C_2^1-cofactor matroid on any general position plane configuration and in the generic C_2^1-cofactor matroid.

REMARK 10.1.8. These C_2^1-cofactors originated in the theory of C_2^1-splines which are functions over decompositions of the plane into polygonal cells, piecewise polynomial of degree at most 2 and globally C^1 [**Bi**,**ChW**,**Wh10**]. For a planar drawing of a connected planar graph, up to addition of a global quadric to the function there is an isomorphism between these C_2^1-splines on the induced decomposition and the C_2^1-cofactors (called 'smoothing cofactors') on the planar graph. To obtain a matroid on all graphs, we have extended the algebra to non-planar graphs.

Recent work of Ripmeester has identified the C_2^1-flexes with 'dual splines' over the vertices and edges, using essentially the same C_2^1-cofactor matrix [**Ri**]. Like the work on smoothing cofactors, this work on dual splines extends to C_s^r-splines for all piecewise polynomials of degree at most s and globally C^r functions over plane polygonal decompositions. Effective use of this duality, from the two sides of the matrix, has already produced some new results for splines [**Ri**]. This duality holds the promise of more progress on the difficult problems of the dimensions of the spaces of multivariate splines for small s and r, or equivalently, the rank of the C_s^r on planar graphs $G(\mathbf{p})$ at special configurations in the plane.

10.2. Partial results for the C_2^1-cofactor matroid. The analogy between the generic C_2^1-cofactor matroid and the generic 3-rigidity matroid is strong. All of the known results for the generic 3-rigidity matroid extend to the generic C_2^1-cofactor matroid, although the underlying geometric results differ in detail and some of the geometric proofs (such as those for bipartite frameworks) must be changed. We conjecture that the two matroids are isomorphic although we have no algebraic or matrix technique to give this isomorphism (see §10.3).

We transfer more results of §9.2 and §9.3 to the generic C_2^1-cofactor matroid.

$\mathcal{M}_2^1$ EDGE 3-SPLIT THEOREM 10.2.1. *Assume G' is an edge 3-split of $G = (V, E)$ on $a, b; c, d$, and $\mathbf{p}$ is a plane configuration with $\mathbf{p}_a, \mathbf{p}_b, \mathbf{p}_c, \mathbf{p}_d$ in general position.*

If $G(\mathbf{p})$ is C_2^1-independent (C_2^1-rigid), then $G'(\mathbf{p}_0, \mathbf{p})$ is C_2^1-independent (C_2^1-rigid) for almost all choices of $\mathbf{p}_0$, including $\mathbf{p}_0$ a distinct point on the line $\mathbf{p}_a\mathbf{p}_b$, not on the line $\mathbf{cd}$.

Conversely, if $G'(\mathbf{p}_0, \mathbf{p})$ is C_2^1-independent (C_2^1-rigid) for some choice of $\mathbf{p}_0$, with vertex 0 connected to exactly vertices a, b, c at three non-collinear points then, for some edge e of vertices in a, b, c, d, $E' = E \cup \{e\} - \{(0, a), (0, b), (0, c), (0, c)\}$ is C_2^1-independent (C_2^1-rigid) at $\mathbf{p}$ and G' is an edge 3-split of $G = (V' - 0, E)$.

PROOF. The proof used for Theorem 9.2.2 extends to C_2^1-independence and C_2^1-rigidity. We add the vertex 0 by a three valent addition to vertices a, c, d, choosing 0 to lie on the line a, b, distinct from a and b. This preserves independence, 3-rigidity etc. Now consider the three collinear edges $\{a, b\}, \{a, 0\}, \{b, 0\}$. Realized on a line, this triangle forms a C_2^1-circuit (all reduced C_2^1-vectors are multiples of the same vector). Therefore, replacing any two of these edges with any other two gives an equivalent rank for the matrix (and matroid). We replace $\{a, b\}, \{a, 0\}$ with $\{a, 0\}, \{b, 0\}$ to complete the edge 3-split.

$\mathcal{M}_2^1$ CONSTRUCTION THEOREM 10.2.2. *If G has a Henneberg 3-construction then the edges of G form a basis for $\mathcal{M}_2^1(|V|)$.*

PROOF. The Henneberg 3-construction is just a sequence of vertex 3-additions and edge 3-splits. These preserve the basic counts and the C_2^1-independence.

The proof of Theorem 3.1.3 extends directly to prove:

C_2^1-FLEX TEST 10.2.3. *For any plane configuration $\mathbf{p}$ for the graph K_n, the following are equivalent:*

1. *the edge $\{h, k\}$ is not in the closure of the set E in $\mathcal{M}_2^1(n; \mathbf{p})$;*
2. *every C_2^1-cofactor λ on $E \cup \{h, k\}$ is zero on $\{h, k\}$;*
3. *there is a C_2^1-flex $\mathbf{u}$ on $G = (V, E)$, such that $(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) \neq 0$.*

GENERIC $\mathcal{M}_2^1$ GLUING LEMMA 10.2.4. *For two edge sets E_1, E_2,*

1. *if E_1 and E_2 are generically C_2^1-rigid and $|V(E_1) \cap V(E_2)| \geq 3$, the set $E_1 \cup E_2$ is generically C_2^1-rigid;*
2. *if $|V(E_1) \cap V(E_2)| < 3$, then the closure in the generic C_2^1-rigidity matroid, $\langle E_1 \cup E_2 \rangle$, is contained in $K_{V(E_1)} \cup K_{V(E_2)}$;*
3. *if E_1 and E_2 are generically 3-independent and $E_1 \cap E_2$ is generically C_2^1-rigid, the set $E_1 \cup E_2$ is generically C_2^1-independent.*

PROOF. 1. Since E_1 is C_2^1-rigid, it has the same closure as a 3-simple graph built onto a triangle in $|V(E_1) \cap V(E_2)|$. Similarly for E_2. Combining these two 3-simple graphs on the common triangle, we have a 3-simple graph which is C_2^1-rigid and is contained in the closure of $E_1 \cup E_2$. We conclude that the closure of $E_1 \cup E_2$ is also $K_{V(E_1) \cup V(E_2)}$, as required.
2. Consider a single edge through two distinct vertices. The C_2^1-matrix has rank 1 and a null space of dimension 5 spanned by the six trivial C_2^1-flexes. Therefore some non-trivial linear combination $F_{\mathbf{p}_1,\mathbf{p}_2}$ of these flexes is $\mathbf{0}$ restricted to these vertices. On the other hand, this set has dimension 6 on every non-collinear triangle, so $F_{\mathbf{p}_i,\mathbf{p}_j}$ gives a non-zero component for all vertices off this line. We have a C_2^1 analog of a 'rotation' about the line through these points, which is zero only for points on this line. Applying the zero C_2^1-flex to the vertices of E_2 and $F_{\mathbf{p}_i,\mathbf{p}_j}$ to the vertices of E_2, where $V(E_1) \cap V(E_2) \subseteq \{i, j\}$, will give the required $\mathbf{u}$ to demonstrate that $\{h, k\}$ is not in the closure for any $h \in V(E_1) - V(E_1) \cap V(E_2)$ and $k \in V(E_2) - V(E_1) \cap V(E_2)$, as required.
3. If we extend each of the independent sets E_1 and E_2 to generically C_2^1-rigid sets on the same vertices, we will add no edges to the C_2^1-rigid set $E_1 \cap E_2$. If $|V(E_1) \cap V(E_2)| \geq 3$, these extended sets E_1' and E_2' will now combine to a rigid set. In this case, a simple counting exercise shows that $E_1' \cup E_2'$ has $|E_1' \cup E_2'| = 3|V(E_1' \cup E_2')| - 6$. Since it is C_2^1-rigid, with $|V| \geq 3$, it must be independent and the original subset is also independent.

If $|V(E_1) \cap V(E_2)| \leq 2$, then we have a modified count $|E_1' \cup E_2'| < 3|V(E_1' \cup E_2')| - 6$, and a corresponding set of non-trivial C_2^1-flexes for the gap. Again this verifies the independence of the enlarged set and the original set.

COROLLARY 10.2.5. *The generic C_2^1-cofactor matroid on K_n is an abstract 3-rigidity matroid.*

For the C_2^1-rigidity of bipartite graphs, we will have a combinatorial proof.

BIPARTITE C_2^1-RIGIDITY THEOREM 10.2.6. *A complete bipartite graph $K_{m,n}$ is generically C_2^1-rigid if and only if $m + n \geq 10$ and $m, n > 3$.*

PROOF. It is a simple task to show that $K_{4,6}$ has a Henneberg 3-construction, using edge 3-splits on all six edges of K_4. Therefore it is C_2^1-rigid. For $K_{5,5}$, there is a modified construction, following a Henneberg 3-construction with the simple addition of the last edge to existing vertices (Figure 10.2), so $K_{5,5}$ is also C_2^1-rigid.

Finally, any larger complete bipartite graph can be created from these two examples by a sequence of vertex 3-additions and adding edges between existing vertices. These will be C_2^1-rigid but not independent.

Vertex 3-splits on two edges were first developed for the C_2^1-cofactor matroid and they preserve C_2^1-rigidity [**ASW**,**Wh10**]. The vertex 3-split on three edges follows by a simpler argument of the same 'row reduction' style as given there.

$\mathcal{M}_2^1$ VERTEX 3-SPLITS THEOREM 10.2.7. *If the graph G' is a vertex 3-split of a generically C_2^1-rigid graph G on 3 edges or a vertex 3-split on 2 edges then G' is generically C_2^1-rigid.*

$\mathcal{M}_2^1$ TRIANGULATED 2-SURFACES THEOREM 10.2.8. *The graph of any triangulated 2-surface is generically C_2^1-rigid.*

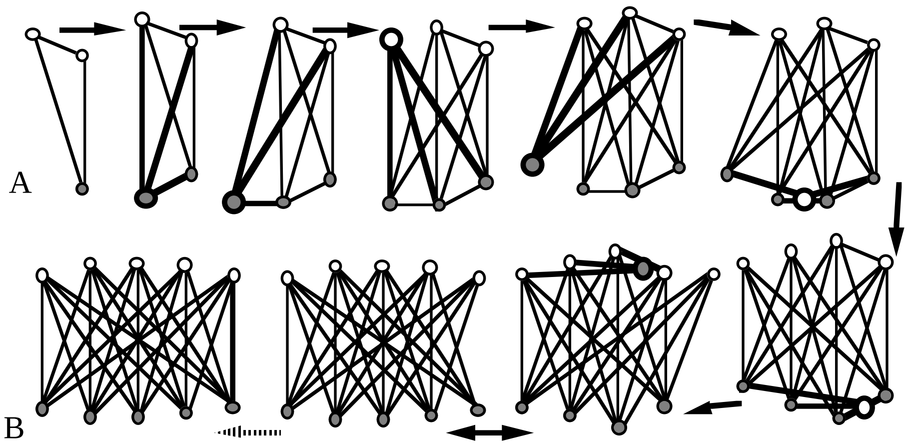

Fig. 10.2. *A Henneberg 3-construction for the bipartite graph $K_{5,5}$ minus one bar (A), and then $K_{5,5}$ by adding one edge (B).*

In particular, the graph of any triangulated sphere is generically C^1_2-rigid and C^1_2-independent (Billera's Theorem [**ASW**,**Bi**,**Wh10**]).

Proof Outline. The following three steps form the basis of Fogelsanger's proof for generic 3-rigidity [**Fo**]:

1. An inductive construction of homology 2-cycles using vertex 3-splits, gluing across triangles and basic tetrahedra;
2. The fact that generic tetrahedra are 3-rigid;
3. The fact that gluing across three vertices and vertex 3-splits preserve generic 3-rigidity.

Using the same base inductive construction of homology 2-cycles, we have all the analogs for generic C^1_2-rigidity to prove that these same objects are C^1_2-rigid.

In §11.4, we will prove a general result, of which the following is a special case.

$\mathcal{M}^1_2$ Coning Corollary 10.2.9. *A graph G is C^0_1-rigid (C^0_1-independent) at the plane configuration $\mathbf{p}$ if and only if the cone graph $(G * u)$ is C^1_2-rigid (C^1_2-independent) at $\mathbf{p}_u, \mathbf{p}$, for all $\mathbf{p}_u$ in general position relative to $\mathbf{p}$.*

10.3. Combinatorial C^1_2-cofactor conjectures. With one exception, we will extend the conjectures of §9.4 to the generic C^1_2-cofactor matroid. The exception becomes a theorem.

$\mathcal{M}^1_2$ X-Replacement Theorem 10.3.1. *The X-replacement in Figure 9.10A takes a generically C^1_2-rigid graph G_1 to a graph G which is generically C^1_2-rigid.*

Proof. Take a generic plane configuration $\mathbf{p}$ for G_1, with the edges $\{1,2\}$, $\{3,4\}$ to be replaced. We do a 3-addition attaching the vertex 0 to vertices 1, 3 and 5 and placing $\mathbf{p}_0$ at the point of intersection of $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$. The resulting graph is C^1_2-rigid.

In this position, we use collinear substitution to replace $\{0,1\}$,$\{1,2\}$ with $\{0,1\}$,$\{0,2\}$ and to replace $\{0,3\}$,$\{3,4\}$ with $\{0,3\}$,$\{0,4\}$. This completes the proof.

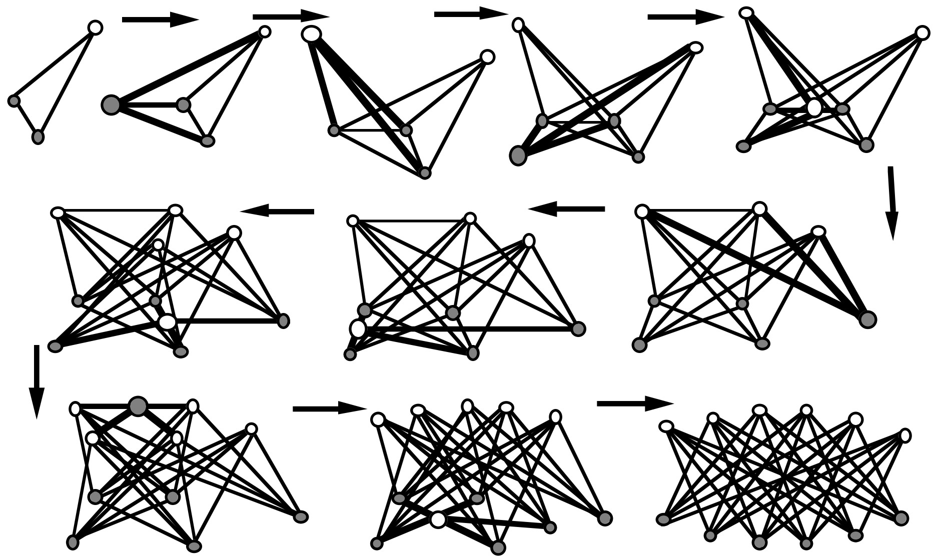

Fig. 10.3. *An extended 3-construction for the graph $K_{6,6}$ minus six edges.*

With this added inductive technique, we can verify other bases of the C_2^1-cofactor matroid. Figure 10.3 gives such an extended 3-construction, in which the last step is an X-replacement. [This extended 3-construction can be geometrically modified to work for generic 3-rigidity, by ensuring that the two 'crossing edges' for the X-replacement are created as intersecting lines, and kept intersecting as the other vertices are moved around. This is the special case of X-replacement in which the proof of Theorem 10.3.1 also adapts to 3-rigidity.]

The following table compares generic 3-rigidity and generic C_2^1-rigidity.

	Generic 3-Rigidity	Generic C_2^1-Rigidity
rank K_n, $n \geq 3$	$3n-6$	$3n-6$
vertex 3-addition	Yes	Yes
edge 3-split	Yes	Yes
3-constructions	Yes	Yes
abstract 3-rigidity	Yes	Yes
vertex 3-split	Yes	Yes
simplicial 2-surfaces	Rigid	Rigid
3-coning	Yes	Yes
3-X-replacement	Conjectured	Yes
double V-replacement	Conjectured	Conjectured
Dress Conjecture	Conjectured	Conjectured
$K_{4,6}$	Basis	Basis
$K_{5,5}$	Circuit	Circuit
maximal abstract 3-rigidity matroid	Conjectured	Conjectured

All of our experience with this 'connection' indicates that the generic C_2^1-cofactor matroid is isomorphic to the generic 3-rigidity matroid.

$\mathcal{M}_2^1$ 3-RIGIDITY ISOMORPHISM CONJECTURE 10.3.2. *For every n, the generic C_2^1-cofactor matroid on K_n is the maximal abstract 3-rigidity matroid. In particular, the generic C_2^1-cofactor matroid on K_n is isomorphic to the generic 3-rigidity matroid on K_n.*

11. Higher Dimensions

We have seen the nearly complete combinatorial theory of the generic 2-rigidity matroid and the isomorphic generic C_1^0-cofactor matroid break down into a mix of partial results and conjectures for the generic 3-rigidity matroid and the (possibly isomorphic) generic C_2^1-cofactor matroid. Up one step, the generic 4-rigidity matroid and the generic C_3^2-cofactor matroid cause even more difficulties. The analogs of most of the conjectures of §9.4 have counterexamples in the generic 4-rigidity matroid. Surprisingly, we have not found counterexamples in the generic C_3^2-cofactor matroid which is provably not isomorphic and remains a candidate for the maximal abstract 4-rigidity matroid.

As the chart of Figure 1.1 suggests, these matroids have analogs for higher dimensions (rigidity) and for higher algebraic powers (cofactors). We present the general pattern, then turn to the critical counterexamples in 4-space.

11.1. Generic rigidity in d-space. A *d-space framework* is a standard graph $G = (V, E)$ and a *d-configuration* $\mathbf{p} : V \to \mathbb{R}^d$. A *dependence* or *self-stress* on the framework $G(\mathbf{p})$ is an assignment $\omega : E \to \mathbb{R}$, with $\omega\{i, j\} = \omega_{i,j} = \omega_{j,i}$, such that, for each vertex i:

$$\sum_{j \mid \{i,j\} \in E} \omega_{i,j}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}.$$

As before, these self-stresses are the row dependencies of the *rigidity matrix* of the framework, $R_G(\mathbf{p})$.

The independence of rows in this matrix defines the *d-rigidity matroid on G* at a d-configuration $\mathbf{p}$, written $\mathcal{R}_d(G; \mathbf{p})$, on the edges of the graph. The framework $G(\mathbf{p})$ is *d-independent* if its edge set is independent in $\mathcal{R}_d(G; \mathbf{p})$, and the *rank* of $G(\mathbf{p})$ is the rank of $\mathcal{R}_d(G; \mathbf{p})$. Working at some (all) generic configurations in d-space (for example configurations with algebraically independent coordinates) we have the *generic d-rigidity matroid* $\mathcal{R}_d(G)$ and $\mathcal{R}_d(n)$ for the complete graph K_n.

We anticipate the rank $d|V| - \binom{d+1}{2}$ for $\mathcal{R}_d(n; \mathbf{p})$ at a general position d-configuration $\mathbf{p}$ on n vertices, where $n \geq d$.

Given a graph $G = (V, E)$, a *vertex d-addition of* 0 is the addition of one new vertex, 0, and d new edges $(0, i_1), \ldots, (0, i_d)$ creating the graph $G' = (V', E')$.

VERTEX d-ADDITION LEMMA 11.1.1. *Given a framework $G(\mathbf{p})$ and a vertex d-addition of* 0 *creating G', the framework $G'(\mathbf{p}_0, \mathbf{p})$, with $\mathbf{p}_0, \mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_d}$ in general position in d-space, then*

1. *$G'(\mathbf{p}_0, \mathbf{p})$ is d-independent if and only if $G(\mathbf{p})$ is d-independent;*
2. rank $\mathcal{R}_d(G'; \mathbf{p}_0, \mathbf{p}) =$ rank $\mathcal{R}_d(G; \mathbf{p}) + d$.

PROOF. The core of the argument is that the vectors $\mathbf{p}_0 - \mathbf{p}_{i_1}, \ldots, \mathbf{p}_0 - \mathbf{p}_{i_d}$ are linearly independent in d-space when the points are in general position in d-space. Moreover, these vectors are the only non-zero entries in the columns under vertex 0, so their independence is sufficient for the independence of their rows.

The *first-order d-flexes* are the solutions $\mathbf{u} = (\ldots, \mathbf{u}_i, \ldots) \in \mathbb{R}^{d|V|}$ to the equations $R_G(\mathbf{p})\mathbf{u} = \mathbf{0}$. That is, for each edge $\{i, j\}$: $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0$. The guaranteed *trivial first-order d-flexes* for any graph G on a d-configuration $\mathbf{p}$ are generated by the d translations and the $\binom{d}{2}$ rotations about 'axes' given by setting 2 of the coordinates zero. (The resulting velocities at a point $\mathbf{p}_i$ are found by the minors of a fixed set of $d-2$ points spanning this axis along with the point $\mathbf{p}_i$ – a d-space generalization of 'cross product'.) It is a direct calculation to show that, for configurations in d-space on at least d points affinely spanning a hyperplane, these generators are independent and the space of trivial first-order d-flexes has dimension $\binom{d+1}{2}$ (see below). Any framework $G(\mathbf{p})$ for which all first-order motions are trivial motions is *first-order d-rigid*.

K_d LEMMA 11.1.2. *For the graph K_d and a general position d-configuration $\mathbf{p}$ on d vertices:*

1. *the framework $K_d(\mathbf{p})$ is d-independent;*
2. *the space of trivial d-flexes has dimension $\binom{d+1}{2}$ on this configuration;*
3. *$K_k(\mathbf{p})$ is d-rigid for $k \leq d$.*

PROOF. These proofs are by induction through K_k, $1 \leq k \leq d$.
1. For K_1, with no edges and one vertex, the matrix is vacuously d-independent. K_{k+1} comes from K_k by a vertex k-addition – essentially a vertex d-addition with some rows removed (clearly preserving d-independence).
2. The d translations $\mathbf{e}_i$, $1 \leq i \leq d$ applied to a vertex are independent first-order d-flexes. Assume $K_k(\mathbf{p})$ has a $\frac{k(2d-k+1)}{2}$ dimensional space of trivial d-flexes. As we add the vertex $k+1$, we will add the $d-k$ trivial rotations about $\{\mathbf{p}_i\}_{i \leq k}$, which were dependent on previous flexes on the k vertices but give independent velocities when extended to $\mathbf{p}_{k+1}$. For K_d, this gives the independence of the full set of $\binom{d+1}{2}$ trivial first-order d-flexes.
3. For general position configurations $\mathbf{p}$, the complete graph K_k has rank $\binom{k}{2}$ from 1, so the space of first-order d-flexes has dimension $dk - \binom{k}{2}$. From 2 the space of trivial first-order d-flexes has dimension $\frac{k}{2}[2d-k+1] = dk - \binom{k}{2}$, so all first-order d-flexes are trivial for $k \leq d$.

d-COUNTING LEMMA 11.1.3. *A set E of edges on at least d vertices with $|E| > d|V(E)| - \binom{d+1}{2}$ is d-dependent for every d-configuration $\mathbf{p}$.*

Equivalently, a set E on $n \geq d$ vertices is d-independent at some d-configuration $\mathbf{p}$ only if, for all subsets E' with at least d vertices, $|E'| \leq d|V(E')| - \binom{d+1}{2}$.

A graph $G = (V, E)$, with at least d vertices, is *d-simple* if there is an ordering of the vertices $\sigma(1), \sigma(2), \ldots, \sigma(|V|)$ such that:

(i) G_d is th complete graph on $\{\sigma(1), \ldots, \sigma(d)\}$;
(ii) for $i \geq d$, G_{i+1} is a vertex d-addition of $\sigma(i+1)$ to G_i;
(iii) $G_{|V|}$ is G.

STATIC d-RIGIDITY THEOREM 11.1.4. *For any $n \geq d$ and any general position d-configuration $\mathbf{p}$ on n vertices, the edges E of any d-simple graph G on n vertices are a basis of $\mathcal{R}_d(n; \mathbf{p})$ of rank $dn - \binom{d+1}{2}$.*

PROOF. For $n = d$ this is the K_d Lemma. If $n > d$, a simple induction on the number of vertices using vertex d-addition completes the proof.

Any d-framework $G(\mathbf{p})$ for which $R_G(\mathbf{p})$ has rank $d|V| - \binom{d+1}{2}$, (or for which $|V| \leq d$, $G = K_V$ and $\mathbf{p}$ is in general position) is *statically d-rigid*. We also say that the edge set E is *statically d-rigid* on $V(E)$ at $\mathbf{p}$. It is easy to check that static d-rigidity is equivalent to first-order d-rigidity.

COROLLARY 11.1.5. *A framework $G(\mathbf{p})$ with vertices affinely spanning d-space is not first-order d-rigid (equivalently statically d-rigid) if and only if there is a first order flex $\mathbf{u}$ and a pair of vertices h, k (not an edge) such that:*

$$(\mathbf{p}_h - \mathbf{p}_k) \cdot (\mathbf{u}_h - \mathbf{u}_k) \neq 0.$$

EXAMPLE 11.1.6. For $d \geq 4$, we have several variants of the 2-bananas example of Figure 9.4. By the count $|E| = \binom{d+2}{2} > d \times (d+2) - \binom{d+1}{2}$, K_{d+2} is dependent at all configurations – and, in fact, is a circuit of the d-rigidity matroid. (Deleting one edge leaves a vertex d-addition of a d-simple K_{d+1}.) As with the 'two bananas' in 3-space, circuit exchange with two copies of K_{d+1} across a single shared edge gives a dependent set (by matroid circuit exchange). Since all subsets with one edge removed are d-simple, this is a circuit.

If we glued up to $d-1$ vertices together, we would have other dependent and flexible frameworks (see the Gluing Lemma below).

For d-constructions we need the analog of edge splits for d-space. A graph G' is an *edge d-split* of the graph G on $c_1, c_2; c_3, \ldots, c_{d+1}$, if $\{c_1, c_2\}$ is an edge of G and G' is formed from G by adding a new vertex 0, removing the edge $\{c_1, c_2\}$, and adding $d+1$ new edges $\{0, c_1\}, \ldots, \{0, c_{d+1}\}$.

EDGE d-SPLIT THEOREM 11.1.7. *Assume G' is an edge d-split of G on c_1, c_2; $c_3, \ldots, c_{d+1}$, and $\mathbf{p}$ is a d-configuration with $\mathbf{p}_{c_1}, \mathbf{p}_{c_2}, \ldots, \mathbf{p}_{c_{d+1}}$ in general position.*

If $G(\mathbf{p})$ is d-independent (statically d-rigid), then $G'(\mathbf{p}_0, \mathbf{p})$ is d-independent (statically d-rigid) for almost all choices of $\mathbf{p}_0$.

Conversely, if $G'(\mathbf{p}_0, \mathbf{p})$ is d-independent (statically d-rigid) for some choice of $\mathbf{p}_0$, with vertex 0 connected to exactly vertices $c_1, c_2, \ldots, c_{d+1}$ at $d+1$ general position points then, for some edge e among the vertices $c_1, \ldots, c_{d+1}$, $E' = E \cup \{e\} - \{(0, c_1), \ldots, (0, c_{d+1})\}$ is d-independent (statically d-rigid) at $\mathbf{p}$ and G' is an edge d-split of $G = (V' - 0, E)$.

PROOF. We add the vertex 0 by a d-valent addition to vertices $c_1, c_3, \ldots, c_{d+1}$, choosing 0 to lie on the line c_1, c_2, distinct from $\mathbf{p}_{c_1}$ and $\mathbf{p}_{c_1}$. This preserves d-independence, d-rigidity etc. Now consider the three collinear edges $\{c_1, c_2\}$, $\{c_1, 0\}$, $\{c_2, 0\}$. Realized on a line, this triangle forms a circuit. Therefore, replacing any two of these with any other two gives an equivalent rank for the matrix (and matroid). We replace $\{c_1, c_2\}, \{c_1, 0\}$ with $\{c_1, 0\}, \{c_2, 0\}$ to complete the edge d-split.

The converse follows by an argument analogous to the Edge 3-Split Theorem, based on the observation that the K_{d+1} among $c_1, \ldots, c_{d+1}$ is statically d-rigid, and the K_{d+2} among $0, \ldots, c_{d+1}$ is a d-circuit in general position.

For a graph $G = (V, E)$ with at least d vertices, a *d-construction* is an ordering of the vertices $\sigma(1), \sigma(2), \ldots, \sigma(|V|)$ and a sequence $G_d, \ldots, G_{|V|}$ of graphs such that:

(i) G_d is the complete graph on on $\{\sigma(1), \ldots, \sigma(d)\}$;
(ii) for $i \geq d$, G_{i+1} is a vertex d-addition of vertex $\sigma(i+1)$ to G_i or G_{i+1} is an edge d-split on G_i which adds a vertex $\sigma(i+1)$;
(iii) $G_{|V|}$ is G.

d-CONSTRUCTION THEOREM 11.1.8 [**TW2**]. *If a graph G on n vertices has a d-construction then G a basis for $\mathcal{R}_d(n)$.*

As expected, the generic d-rigidity is an abstract d-rigidity matroid. We state the appropriate Gluing Lemma.

GENERIC d-GLUING LEMMA 11.1.9. *For two edge sets E_1, E_2,*

1. *if E_1 and E_2 are generically d-rigid and $|V(E_1) \cap V(E_2)| \geq d$, the set $E_1 \cup E_2$ is generically d-rigid;*
2. *if $|V(E_1) \cap V(E_2)| < d$, then the closure $\langle E_1 \cup E_2 \rangle$ in $\mathcal{R}_d(V(E_1) \cup V(E_2))$ is contained in $K_{V(E_1)} \cup K_{V(E_2)}$;*
3. *if E_1 and E_2 are generically d-independent and $E_1 \cap E_2$ is generically d-rigid, the set $E_1 \cup E_2$ is generically d-independent.*

PROOF. The proofs follow the patterns we established for the generic 3-rigidity matroid and for the generic C_2^1-cofactor matroid.

COROLLARY 11.1.10. *Generic d-rigidity is an abstract d-rigidity matroid.*

However, we will see below that generic d-rigidity is not the 'most general' abstract d-rigidity matroid!

In closing this section, we note that vertex d-splits, on $d-1$ or d edges, preserve generic d-rigidity [**Wh9**], that triangulated $(d-1)$-hypersurfaces are generically d-rigid [**Fo**], and that a graph G is generically $(d-1)$-rigid ($(d-1)$-independent) if and only if the cone graph $G * u$ is generically d-rigid [**Wh2**].

REMARK 11.1.11. The generic d-ridigity of triangulated $(d-1)$-surfaces is a direct proof of a combinatorial result: the Lower Bound Theorem for edges of manifolds (without boundary) [**Ka1**]. Specifically, the counts for generic d-rigidity show that a triangulated manifold on n vertices (including any simplicial $(d-1)$-sphere) satisfies s $|E| \geq dn - \binom{d+1}{2}$.

While this bound is exact for 2-spheres in 3-space, it is not, in general, exact for 3-spheres in 4-space or analogs in higher spaces. [**Ka1**] gives details of which spherical polytopes make this bound exact.

11.2. Bipartite graphs and X-replacement for $\mathcal{R}_d(G; \mathbf{p})$. Key to this subsection are some basic bipartite graphs.

d-SPACE BIPARTITE GRAPHS 11.2.1. *A complete bipartite graph $K_{m,n}$, with $m > 1$, is generically d-rigid if and only if $m + n \geq \binom{d+2}{2}$ and $m, n > d$.*

PROOF. It is easy to verify that if $1 < m \leq d$ then the bipartite framework is under counted and therefore not generically d-rigid.

The bound of $m + n \geq \binom{d+2}{2}$ follows from the fact that any $\binom{d+2}{2} - 1$ points lie on a quadric surface in d-space, so the resulting framework cannot be rigid by [**Wh3**]. With $m + n \geq \binom{d+2}{2}$, we can avoid such a quadric and obtain d-rigidity, by the geometric result in Remark 9.3.5.

EXAMPLE 11.2.2. Consider the complete bipartite graph $K_{6,7}$. It has the required number of edges for generic 4-rigidity: $|E| = 42 = 4 \times 13 - 10 = 4|V| - \binom{5}{2}$.

However with the $m + n = 13 < 15$ vertices, there must be a quadric surface through these points. [If we write a second degree polynomial in the four variables there are 15 coefficients which satisfy one homogeneous linear equation for each of the $m+n$ points. The solution space to these equations is a *family of quadric surfaces through the points* of dimension at least $15-(m+n)$.] The 4-frameworks on this graph cannot be statically 4-rigid. We conclude that this graph is 4-dependent at all configurations in 4-space.

In fact, the subgraph $K_{6,6}$ is a generic 4-circuit. With 12 vertices, a generic configuration for the twelve points lies on a family of quadrics of dimension $15-12 = 3$. This means that generic configurations leave a 13-dimensional space of first-order flexes [**Wh3**], and the rank of the ridigity matrix is $4|V|-13 = 35 = |E|-1$. Therefore it is generically 4-dependent. Since any proper subset has a modified 4-construction (a 4-construction with edges omitted) which can be extracted from Figure 11.1, each proper subset is generically 4-independent.

The graph $K_{6,6}$ (or its relatives such as $K_{6,7}$ and $K_{7,7}$) provide a counterexample to the 4-dimensional versions of a number of the conjectures of §9.4.

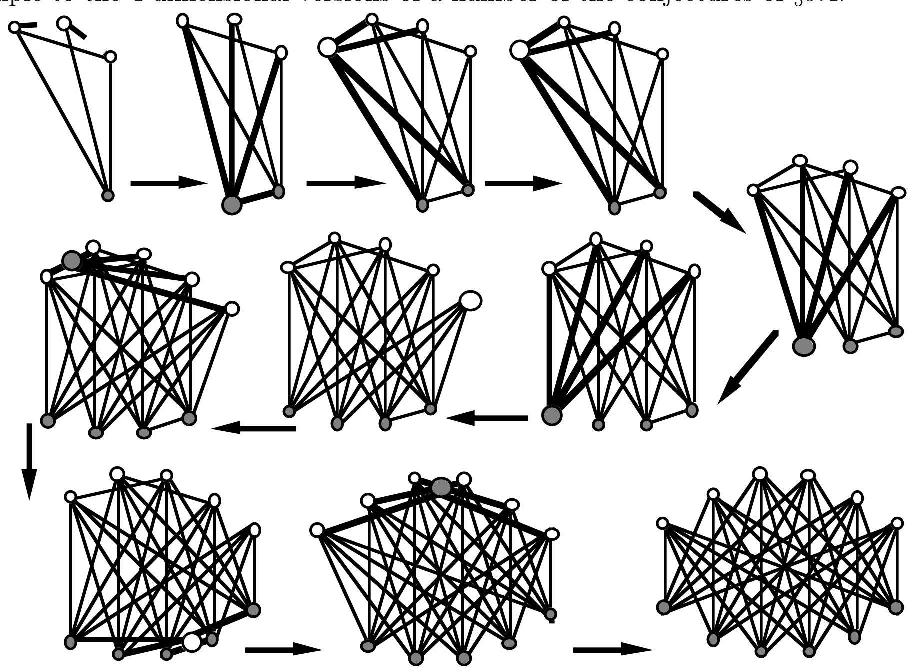

Fig. 11.1. *An extended 4-construction for $K_{6,6}$, in which the last step is an X-replacement.*

Consider the the extended 4-construction of Figure 11.1. We begin with a generically 4-independent subgraph of K_4, namely K_4 with two incident edges omitted. Up to the last step, the construction steps of vertex 4-addition and edge 4-splits will preserve this generic 4-independence. Therefore the second last graph is generically 4-independent and only the last one, $K_{6,6}$, is 4-dependent. *The 4-dimensional version of X-replacement fails!*

The fact that X-replacement works in the generic 2-rigidity matroid and fails for generic 4-rigidity places the conjectured X-replacement for generic 3-rigidity in perspective as *the* borderline case. Recall that X-replacement was proven for the generic C_2^1-cofactor matroid in §10.4. In §11.4, we will prove that a related abstract 4-rigidity matroid does satisfy X-replacement. The problem is *not* a matroidal difficulty for general abstract 4-rigidity matroids, such as that created by the Gluing Lemmas and non-rigid circuits (see §11.5).

The difficulty is ‘geometric’ – embedded in the geometric roots of the generic 4-rigidity matroid. Essentially the problem is that distance, and distance preserving maps, rely on quadric equations. Quadric surfaces are both bad for bipartite frameworks and numerous in 4-space.

These examples and counterexamples generalize to generic d-rigidity for $d \geq 4$.

EXAMPLE 11.2.3. Repeated coning of $K_{6,6}$ generates a generically d-dependent set of edges in each dimension $d \geq 4$. Moreover, the extended 4-construction of $K_{6,6}$ in Figure 11.1 also extends: with K_d replacing K_4; vertex d-addition replacing vertex 4-addition; edge d-splits replacing edge 4-splits; and d-X-replacement in place of 4-X-replacement. Since the end result is a generically d-dependent set, and vertex d-replacement and edge d-splits preserve generic d-independence, we conclude that the d-analog of X-replacement fails to preserve generic d-independence.

EXAMPLE 11.2.4. A second source of higher dimensional counterexamples will be $K_{d+2,d+2}$ in $d \geq 4$. Some simple counts of vertices, edges and quadric surfaces show that $K_{d+2,d+2}$ is a circuit for generic d-rigidity.

Specifically, $|E| = (d{+}2)^2$, $|V| = 2(d{+}2)$, and a quadric is defined by $\binom{d+2}{2}$ independent linear equations from $\binom{d+2}{2}$ points. Therefore for a generic d-configuration, the vertices lie on a space of $\binom{d+2}{2} - 2(d+2)$ quadrics, giving the same space of non-trivial first-order flexes. The maximal set of d-independent edges will have size

$$\begin{aligned} d[2(d+2)] - \binom{d+1}{2} - \left[\binom{d+2}{2} - 2(d+2)\right] &= 2(d+1)(d+2) - (d+1)^2 \\ &= (d+1)^2 + 2(d+1) = (d+2)^2 - 1. \end{aligned}$$

We conclude that, with $|E| = (d+2)^2$, the set contains a single generic d-circuit. That the circuit is the entire set follows from a modified count of a subset.

We anticipate that each of these graphs would have an extended d-construction, using one step of the d-analog of X-replacement. These, and their cones, would provide additional counterexamples to X-replacement in d-space.

11.3. Higher C^{s-1}_s-cofactors. The algebraic construction which defined the C^0_1-cofactor and C^1_2-cofactor matroids can be extended to the family of C^{s-1}_s-cofactor matroids, analogous to the $(s+1)$-rigidity matroids (recall Figure 1.1). For $i < j$, we write the s-form:

$$[L^{i,j}]^s = [A^{i,j}x + B^{i,j}y + C^{i,j}]^s \quad \text{and} \quad [L^{j,i}]^s = -[A^{i,j}x + B^{i,j}y + C^{i,j}]^s = -[L^{i,j}]^s,$$

where $L^{i,j} \equiv A^{i,j}x + B^{i,j}y + C^{i,j} = 0$ is the standard equation for the line, as defined in §5. Notice that, if s is odd, $(-L^{i,j})^s = -[L^{i,j}]^s$ and the antisymmetry is built in from the ground. For convenience in tracking these signs, we define

$$\mathit{Sign}(i,j) = \begin{cases} 1 & \text{if } i < j \\ -1 & \text{if } i > j \\ 0 & \text{if } i = j \end{cases} \quad \text{and} \quad \mathit{Sign}_s(i,j) = \begin{cases} 1 & \text{if } s \text{ is odd and } i \neq j \\ 1 & \text{if } s \text{ is even and } i < j \\ -1 & \text{if } s \text{ is even and } i > j \\ 0 & \text{if } i = j \end{cases}$$

which gives $[L^{i,j}]^s = \mathit{Sign}_s(i,j)(L^{i,j})^s$ for all i, j, s.

A C_s^{s-1}*-cofactor on the plane graph* $G(\mathbf{p})$ is an assignment of scalars $\lambda_{i,j} = \lambda_{j,i}$ to the edges $\{i,j\} \in E$ such that for each vertex j:

$$\sum_{j \ | \ \{i,j\} \in E} \lambda_{ij}[L^{i,j}]^s \equiv 0.$$

These C_s^{s-1}-cofactor equations impose $\binom{s+2}{2}|V|$ linear conditions in the $|E|$ variables $\lambda_{i,j}$. However, we can again focus on the *reduced* C_s^{s-1}*-vector* formed by the $s+1$ terms of degree s in x, y: $\mathbf{D}_{i,j}^s \ = \ \left[\ (A^{ij})^s,\ s(A^{ij})^{s-1}B^{ij}, \dots ,\ (B^{ij})^s\right]$, for $i < j$ and $\mathbf{D}_{j,i}^s \ = -\mathbf{D}_{i,j}^s$ for $i > j$. We can uniquely reconstruct all entries of $[L^{i,j}]^s$ using the point $\mathbf{p}_i$ (or $\mathbf{p}_j$) and the reduced vector $\mathbf{D}_{i,j}^s$.

Reduced C_s^{s-1}-Cofactor Lemma 11.3.1. *Given a plane graph* $G(\mathbf{p})$*, an assignment of scalars* $\lambda_{i,j}$ *is a* C_s^{s-1}*-cofactor if and only if the scalars satisfy the reduced* C_s^{s-1}*-cofactor equations:* $\sum_{j \ | \ \{i,j\} \in E} \lambda_{ij}\mathbf{D}_{i,j}^s = \mathbf{0}$.

Proof. Since the entries in $\mathbf{D}_{i,j}^s$ are the first $s+1$ of the $\binom{s+2}{2}$ entries in $[L^{i,j}]^s$, it is clear that any C_s^{s-1}-cofactor satisfies the reduced cofactor equations.

Conversely, assume we are given the $(s+1)$-vector:

$$\mathbf{D}_{i,j}^s \ = \ \left[\ (A^{ij})^s,\ s(A^{ij})^{s-1}B^{ij}, \dots ,\ (B^{ij})^s\right],$$

$i < j$ and $\mathbf{p}_i = (x_i, y_i)$. With $A^{i,j}x_i + B^{i,j}y_i + C^{i,j} = 0$ and $C^{i,j} = -(A^{i,j}x_i + B^{i,j}y_i)$, for $0 < k$ and $k + l \leq s$ we have

$$\begin{aligned}
\binom{s}{k,l}(C^{i,j})^k(A^{i,j})^{s-k-l}(B^{i,j})^l &= \binom{s}{k,l}(-1)^k\left[A^{i,j}x_i + B^{i,j}y_i\right]^k(A^{i,j})^{s-k-l}(B^{i,j})^l \\
&= (-1)^k\binom{s}{k,l}\sum_{m=0}^{m=k}\binom{k}{m}(A^{i,j})^{s-m-l}(B^{i,j})^{m+l}x_i^{k-m}y_i^m \\
&= \sum_{m=0}^{m=k}\binom{s}{m+l}(A^{i,j})^{s-l-m}(B^{i,j})^{m+l}(-1)^k\left[\frac{\binom{s}{k,l}\binom{k}{m}}{\binom{s}{m+l}}x_i^{k-m}y_i^m\right] \\
&= \sum_{n=l}^{n=k+l}\binom{s}{n}(A^{i,j})^{s-n}(B^{i,j})^n(-1)^k\left[\frac{\binom{s}{k,l}\binom{k}{n-l}}{\binom{s}{n}}x_i^{k+l-n}y_i^{n-l}\right].
\end{aligned}$$

This is clearly a linear expression in the given entries $\mathbf{D}_{i,j}^s(n) = \binom{s}{n}(A^{i,j})^{s-n}(B^{i,j})^n$, $0 \leq n \leq s$ and in the coefficients which are computed from $\mathbf{p}_i$: $\frac{\binom{s}{k,l}\binom{k}{n-l}}{\binom{s}{n}}x_i^{k+l-n}y_i^{n-l}$ $= f_{k,l}(n, \mathbf{p}_i)$, $0 \leq n \leq s$. These equations give recipes for reconstructing all the entries of $[L^{i,j}]^s$ from $\mathbf{D}_{i,j}^s$ and $\mathbf{p}_i$.

Because each reconstruction is linear in the entries of $\mathbf{D}_{i,j}^s$ with fixed coefficients for each i, if $\sum_j \lambda_{i,j}\mathbf{D}_{i,j}^s = \mathbf{0}$, then $\sum_j \lambda_{i,j}[L^{i,j}]^s \equiv 0$ and the reduced cofactor equation is satisfied.

With this lemma, we can write the C_s^{s-1}-cofactors as row dependencies of the C_s^{s-1}*-cofactor matrix* which is $|E|$-by-$(s+1)|V|$:

$$M_s^{s-1}(G;\mathbf{p}) \;=\; \begin{bmatrix} \mathbf{D}^s_{1,2} & -\mathbf{D}^s_{1,2} & \dots & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{D}^s_{i,j} & \dots & -\mathbf{D}^s_{i,j} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \end{bmatrix}$$

The analogy to the $(s+1)$-rigidity matrix is again apparent. The independence of rows in this matrix defines the C_s^{s-1}-*cofactor matroid*, $\mathcal{M}_s^{s-1}(G;\mathbf{p})$, on the plane geometric graph $G(\mathbf{p})$ and the matroid $\mathcal{M}_s^{s-1}(n;\mathbf{p})$ on the complete graph $K_n(\mathbf{p})$. Working at generic plane configurations $\mathbf{p}$, this also defines the *generic* C_s^{s-1}-*cofactor matroid* on the graph G, $\mathcal{M}_s^{s-1}(G)$ and defines $\mathcal{M}_s^{s-1}(n)$ on K_n.

The core theory of this matroid has not been presented elsewhere. To give the central structure of this family of matroids, without a long build-up or hand-waving, we will use a central technique for transferring results up from C_s^{s-1} to C_{s+1}^s: coning. For the rigidity matroids, coning has been used both geometrically and generically for some time [**Wh2**]. While the analogous coning result was conjectured for the generic cofactor matroids several years ago, we offer the first proof.

For the clearest insight into this transfer, we will use a projective version of the cofactor matroid, including the invariance of the matroid under projective transformations on the configuration [**Wh10**]. In particular, we will want to place the vertex of the cone onto the line at infinity in order to 'pull' the matrix rows into a simpler pattern. Any reader willing to accept the Coning Theorem without proof can skip to Lemma 11.3.4.

Recall that for any point $\mathbf{p}_i = (x_i, y_i)$ the affine coordinates are $\overline{\mathbf{p}}_i = (x_i, y_i, 1)$. More generally, *homogeneous projective coordinates* for the point, as a column vector, are

$$\tilde{\mathbf{p}}_i = \lambda_i(x_i, y_i, 1)^t = (\lambda_i x_i, \lambda_i y_i, \lambda_i)^t = (u, v, w)^t,$$

giving a point with *weight* $\lambda_i \neq 0$. Coordinates of the form $(u_i, v_i, 0)^t$ will represent a *point at infinity* where the parallel lines with normal $(-v_i, u_i)$ meet. (The triple $(0,0,0)^t$ does not represent any point in the projective plane.) Working with these homogeneous coordinates, the linear form corresponding to an edge i,j is also homogenized: $\widetilde{L}^{i,j} = A^{i,j}u + B^{i,j}v + C^{i,j}w$. Writing $\mathbf{U} = (u,v,w)^t$ for the column of variables, the form is $\widetilde{L}^{i,j}(\tilde{\mathbf{p}}) = |\tilde{\mathbf{p}}_i\tilde{\mathbf{p}}_j\mathbf{U}|$. With these homogeneous forms, each projective plane configuration $\tilde{\mathbf{p}}$ defines a corresponding space of C_s^{s-1}-cofactors.

In these homogeneous coordinates, a general plane projective transformation T is represented by an invertible 3-by-3 matrix $[T]$: $T(\tilde{\mathbf{p}}) = [T]\tilde{\mathbf{p}}$. The following general result is implicit in [**Wh10**].

$\mathcal{M}_s^{s-1}$ PROJECTIVE INVARIANCE THEOREM 11.3.2. *Given a graph G, a projective plane configuration $\tilde{\mathbf{p}}$, and a projective transformation T, the plane graphs $G(\tilde{\mathbf{p}})$ and $G(T(\tilde{\mathbf{p}}))$ have isomorphic spaces of C_s^{s-1}-cofactors.*

PROOF. Consider any C_s^{s-1}-cofactor λ for $G(\tilde{\mathbf{p}})$. For each i:

$$0 \equiv \sum_{j|\{i,j\}\in E} Sign_s(i,j)\lambda_{i,j}(\widetilde{L}^{i,j})^s = \sum_{j|\{i,j\}\in E} Sign_s(i,j)\lambda_{i,j}|\tilde{\mathbf{p}}_i\tilde{\mathbf{p}}_j\mathbf{U}|^s$$

If we apply the projective transformation T, we have:

$$\begin{aligned} 0 &\equiv \sum_{j|\{i,j\}\in E} Sign_s(i,j)\lambda_{i,j}\big|[T]^{-1}[T][\tilde{\mathbf{p}}_i\tilde{\mathbf{p}}_j\mathbf{U}]\big|^s \\ &= \sum_{j|\{i,j\}\in E} Sign_s(i,j)\lambda_{i,j}|T|^{-s}\big|[T]\tilde{\mathbf{p}}_i[T]\tilde{\mathbf{p}}_j[T]\mathbf{U}\big|^s \end{aligned}$$

Now replacing the variable vector $\mathbf{U}$ by the vector $[T]^{-1}\mathbf{U}$ (a classical 'linear substitution') has the effect of applying a single induced linear transformation $\widetilde{T}$ to the larger set of coefficients for these forms of degree s. This does not effect the equivalence to 0. Therefore, after dividing by the constant $|T|^{-s}$ we have:

$$\begin{aligned} 0 &\equiv \sum_{j|\{i,j\}\in E} Sign_s(i,j)\lambda_{i,j}\big|[T]\tilde{\mathbf{p}}_i[T]\tilde{\mathbf{p}}_j[T][T]^{-1}\mathbf{U}\big|^s \\ &= \sum_{j|\{i,j\}\in E} Sign_s(i,j)\lambda_{i,j}\big|[T]\tilde{\mathbf{p}}_i[T]\tilde{\mathbf{p}}_j\mathbf{U}\big|^s. \end{aligned}$$

This is the equation for a C^{s-1}_s-cofactor on $G(T(\tilde{\mathbf{p}}))$.

We have a linear injection from the space of C^{s-1}_s-cofactors for $G(\tilde{\mathbf{p}})$ to the space of C^{s-1}_s-cofactors for $G(T(\tilde{\mathbf{p}}))$. Since T is invertible, this induced map is also invertible, giving the desired isomorphism of the spaces of cofactors.

If we restrict our attention to finite affine points $\overline{\mathbf{p}}_i$ and $\overline{\mathbf{q}}_i = \frac{1}{\lambda_i}T(\overline{\mathbf{p}}_i)$, then this isomorphism of C^{s-1}_s-cofactors can be scaled to a linear transformation of the non-homogeneous C^{s-1}_s-cofactors, taking $\lambda_{i,j}$ to $\lambda_{i,j}(\lambda_i\lambda_j)^s$.

We have defined the general transformation for the non-reduced equations. The Reduced Cofactor Lemma only works for the equations of finite points, as we implicitly used the points $(x_i, y_i, 1)$ in the computations. Accordingly, in the next proof, we will record the equations for finite points with the reduced vectors, but those for the single infinite point are recorded with the entire vector of coefficients of the s-forms.

$\mathcal{M}^{s-1}_s$ Coning Theorem 11.3.3. *Given a graph G with cone $G*0$ and a plane configuration $\mathbf{p}$ with a point $\mathbf{p}_0$ in general position relative to the points in $\mathbf{p}$, the space of C^{s-1}_s-cofactors for $G(\mathbf{p})$ is isomorphic to the space of C^{s}_{s+1}-cofactors for the cone $G*0(\mathbf{p},\mathbf{p}_0)$.*

Proof. For convenience, we switch to the corresponding projective forms and take a linear transformation placing the cone vertex at $\tilde{\mathbf{p}}_0 = (0,1,0)$ and leaving all other points finite. For all finite pairs of points $i<j$, the line is non-vertical with slope $m_{i,j}$ (by our general position assumption for $\mathbf{p}_0$). Therefore, for $0<i<j$ we can write: $L^{i,j} = (a_{i,j})(m_{i,j}x + y + b_{i,j})$ with $a_{i,j}\neq 0$. The corresponding reduced cofactor is:

$$\mathbf{D}^s_{i,j} = (a_{i,j})^s[(m_{i,j})^s, \dots, \binom{s}{k}(m_{i,j})^k, \dots, 1].$$

For convenience, we absorb the non-zero scalar $(a_{i,j})^s$ into the coefficients of every cofactor, and write the general reduced C^{s-1}_s-vector for $i,j\neq 0$ as:

$$\mathbf{D}^s_{i,j} = Sign(i,j)[(m_{i,j})^s, \dots, \binom{s}{k}(m_{i,j})^k, \dots, 1].$$

The C^s_{s+1}-vector is (up to a non-zero scalar we absorb into the cofactor):

$$\mathbf{D}^{s+1}_{i,j} = \mathit{Sign}(i,j)[(m_{i,j})^{s+1}, (s+1)(m_{i,j})^s, \dots, \binom{s+1}{k}(m_{i,j})^k, \dots, 1].$$

For the edges $i,0$ attached to $(0,1,0)$ the linear form for the line, in affine coordinates, is $L^{0,i} = x + c_{0,i}$ and the corresponding reduced C^s_{s+1}-vector is:

$$\mathbf{D}^s_{i,0} = -[1, 0, \dots, 0, \dots, 0].$$

Finally, for the equations at the single infinite vertex $\tilde{\mathbf{p}}_0$, we have the coefficients of the full form:

$$[L^{0,i}]^{s+1} = [1, 0, \dots, 0, c_{0,i}, 0, \dots, (c_{0,i})^{s+1}]$$

Consider any C^s_{s+1}-cofactor λ^* on $G * 0(\tilde{\mathbf{p}}, \tilde{\mathbf{p}}_0)$. For each finite point $\tilde{\mathbf{p}}_i$:

$$\begin{aligned}
\mathbf{0} &= \sum_{j|\{i,j\}\in E} \lambda^*_{i,j}\mathbf{D}^{s+1}_{i,j} \quad + \ \lambda^*_{i,0}\mathbf{D}^{s+1}_{i,0} \\
&= \sum_{j|\{i,j\}\in E} \mathit{Sign}(i,j)\lambda^*_{i,j}[(m_{i,j})^{s+1}, (s+1)(m_{i,j})^s, \dots, \binom{s+1}{k}(m_{i,j})^k, \dots, 1] \\
&\qquad - \lambda^*_{i,0}[1, 0, \dots, 0, \dots, 0]
\end{aligned}$$

Dropping the first coordinate (and the last vector which is now $\mathbf{0}$) we have:

$$\mathbf{0} = \sum_{j|\{i,j\}\in E} \mathit{Sign}(i,j)\lambda^*_{i,j}[(s+1)(m_{i,j})^s, \dots, \binom{s+1}{k}(m_{i,j})^k, \dots, 1].$$

After multiplying the entry for $(m_{i,j})^k$ by $\frac{\binom{s}{k}}{\binom{s+1}{k}}$, we have

$$\mathbf{0} = \sum_{j|\{i,j\}\in E} \mathit{Sign}(i,j)\lambda^*_{i,j}[(m_{i,j})^s, \dots, \binom{s}{k}(m_{i,j})^k, \dots, 1] = \sum_{j|\{i,j\}} \lambda^*_{i,j}\mathbf{D}^s_{i,j}.$$

These scalars are a C^{s-1}_s-cofactor λ on $G(\tilde{\mathbf{p}})$.

Conversely, assume that λ is a C^{s-1}_s-cofactor on $G(\tilde{\mathbf{p}})$:

$$\mathbf{0} = \sum_{j|\{i,j\}\in E} \mathit{Sign}(i,j)\lambda_{i,j}[(m_{i,j})^s, \dots, \binom{s}{k}(m_{i,j})^k, \dots, 1] = \sum_{j|\{i,j\}\in E} \lambda_{i,j}\mathbf{D}^s_{i,j}.$$

After multiplying the entry for $(m_{i,j})^k$ by $\frac{\binom{s+1}{i}}{\binom{s}{i}}$ we have:

$$\mathbf{0} = \sum_{j|\{i,j\}\in E} \mathit{Sign}(i,j)\lambda_{i,j}[(s+1)(m_{i,j})^s, \dots, \binom{s+1}{k}(m_{i,j})^k, \dots, 1].$$

We now add the initial coordinate for $\mathbf{D}^{s+1}_{i,j}$ to each vector and define

$$\lambda^*_{i,0} = \sum_{j|\{i,j\}\in E} \mathit{Sign}(i,j)\lambda_{i,j}(m_{i,j})^{s+1},$$

giving:

$$\begin{aligned}\mathbf{0} = &\sum_{j|\{i,j\}\in E} Sign(i,j)\lambda_{i,j}[(m_{i,j})^{s+1}, (s+1)(m_{i,j})^{s}, \dots, \binom{s+1}{k}(m_{i,j})^{k}, \dots, 1] \\ &\qquad - \lambda^{*}_{i,0}[1, 0, \dots, 0, \dots, 0] \\ = &\sum_{j|\{i,j\}\in E\cup} \lambda_{i,j}\mathbf{D}^{s+1}_{i,j} \quad + \ \lambda_{i,0}\mathbf{D}^{s+1}_{i,0}.\end{aligned}$$

These scalars satisfy the C^s_{s+1}-cofactor equation at i for all finite points $\tilde{\mathbf{p}}_i$.

What remains to prove is that these induced scalars $\lambda^*_{i,0}$ satisfy the C^s_{s+1}-cofactor equation at $\tilde{\mathbf{p}}_0$. The basic idea is simple and applies to arbitrary plane graphs. Any set of scalars which satisfy the C^s_{s+1}-cofactor equations at all by one vertex, also satisfy them at the last vertex. Consider a set of scalars $\lambda^*_{i,j}$ such that, for all vertices i except the last one (0):

$$0 \equiv \sum_{j|\{i,j\}\in E} \lambda^*_{i,j}[\widetilde{L}^{i,j}]^{s+1}$$

For each edge i, j (including the ones to 0) we have:

$$\lambda^*_{i,j}[\widetilde{L}^{i,j}]^{s+1} + \lambda^*_{i,j}[\widetilde{L}^{j,i}]^{s+1} \equiv 0.$$

Adding over all edges, and regrouping at each vertex i we have:

$$\begin{aligned}0 &\equiv \sum_{i\neq 0}\left(\sum_{k|\{i,k\}\in E} \lambda^*_{i,k}[\widetilde{L}^{i,k}]^{s+1}\right) + \sum_{k|\{0,k\}\in E} \lambda^*_{0,k}[\widetilde{L}^{0,k}]^{s+1} \\ &\equiv \sum_{i\neq 0} 0 + \sum_{k|\{0,k\}\in E} \lambda^*_{0,k}[\widetilde{L}^{0,k}]^{s+1} \quad \equiv \quad \sum_{k|\{0,k\}\in E} \lambda^*_{0,k}[\widetilde{L}^{0,k}]^{s+1}.\end{aligned}$$

We conclude that the C^s_{s+1}-cofactor equation is satisfied at the final vertex, as required, and the extended λ^* is a C^s_{s+1}-cofactor on $G * 0(\tilde{\mathbf{p}}, \tilde{\mathbf{p}}_0)$.

We have shown that each C^s_{s+1}-cofactor λ^* on $G * 0(\tilde{\mathbf{p}}, \tilde{\mathbf{p}}_0)$ restricts to a C^{s-1}_s-cofactor on $G(\tilde{\mathbf{p}})$, and each C^{s-1}_s-cofactor on $G(\tilde{\mathbf{p}})$ extends uniquely to a C^s_{s+1}-cofactor λ^* on $G*0(\tilde{\mathbf{p}}, \tilde{\mathbf{p}}_0)$, as required. By a projective transformation, this applies to any cone $G * 0(\mathbf{p}, \mathbf{p}_0)$, with $\mathbf{p}_0$ distinct from every point in $\mathbf{p}$ and not-collinear with the pair of points in $\mathbf{p}$.

The generic C^{s-1}_s-cofactor matroid has rank $(s+1)n - \binom{s+2}{2}$ on K_n, like the analogous generic $(s+1)$-rigidity matroid. The verification follows from coning. For completeness, we also introduce the companion space of C^{s-1}_s-flexes.

The *C^{s-1}_s-flexes of a plane graph* $G(\mathbf{p})$ are the solutions of the system of equations $M^{s-1}_r(G, \mathbf{p})\mathbf{u} = \mathbf{0}$. The space of *trivial C^{s-1}_s-flexes* is spanned by the $s+1$ 'translations': $\mathbf{F}_{0,l} = (\mathbf{e}_l, \dots, \mathbf{e}_l)$ for the standard basis vectors $\mathbf{e}_l$ for $\mathbb{R}^{s+1}$ and by the $\binom{s+1}{2}$ 'rotations'. This space is generated by the vectors $\mathbf{F}_{k,l}(\mathbf{p})$, $0 \leq l$, $0 < k$, $k + l \leq s - 1$, whose entries under each vertex are the coefficient vectors used in the proof of Lemma 11.3.1: $\mathbf{f}_{k,l}(\mathbf{p}_i) = (\dots, f_{k,l}(j-1, \mathbf{p}_i), \dots)$, $1 \leq j \leq s+1$. Because $\mathbf{D}^s_{i,j} \cdot \mathbf{f}_{k,l}(\mathbf{p}_i) = \mathbf{D}^s_{i,j} \cdot \mathbf{f}_{k,l}(\mathbf{p}_j)$, for each k, l, we have $\mathbf{D}^s_{i,j} \cdot \mathbf{f}_{k,l}(\mathbf{p}_i) - \mathbf{D}^s_{i,j} \cdot \mathbf{f}_{k,l}(\mathbf{p}_j) = 0$. These will be C^{s-1}_s-flexes for any graph. (If we included the case $(C^{i,j})^0$ in the proof of Lemma 11.3.1, we get the entries of the 'translations', so our notation is consistent.) A plane graph $G(\mathbf{p})$, or a set of edges E, is C_s^{s-1}-*rigid* if all C_s^{s-1}-flexes are trivial.

K_n LEMMA 11.3.4. *For a general position plane configuration* $\mathbf{p}$ *on* $n \geq s+1$ *vertices:*

1. *the plane graph* $K_{s+1}(\mathbf{p})$ *is* C_s^{s-1}*-independent;*
2. *the space of trivial* C_s^{s-1}*-flexes has dimension* $\binom{s+2}{2}$ *on this configuration;*
3. *for* $n \geq s$, $M_s^{s-1}(n;\mathbf{p})$ *has rank* $(s+1)n - \binom{s+2}{2}$.

PROOF. These proofs are by induction on s. By the results of §5 and §10.1, all of the above results hold for $s = 1$ and $s = 2$. Assume they hold for $s = r$.
1. Since K_{r+1} is C_r^{r-1}-independent at $\mathbf{p}$, by the Coning Theorem 11.3.3, $(K_{r+1}) * u = K_{r+2}$ is C_{r+1}^r-independent at $(\mathbf{p}, \mathbf{p}_u)$ for all general position plane configurations $(\mathbf{p}, \mathbf{p}_u)$. We have the required induction step.
2. By 1, the rank of $M_{r+1}^r(r+2;\mathbf{p})$ is $\binom{r+2}{2}$. Therefore the space of C_{r+1}^r-flexes on K_{r+2} has dimension $(r+1)(r+2) - \binom{r+2}{2} = \binom{r+2}{2}$. It is an exercise to verify that the $\binom{r+2}{2}$ trivial C_{r+1}^r-flexes are independent and hence generate this space.
3. Assume that, for $m \geq r$, $M_r^{r-1}(m;\mathbf{p})$ has rank $sm - \binom{s+1}{2}$. This means that the space of C_r^{r-1}-cofactors on $K_m(\mathbf{p})$ has dimension $\binom{m}{2} - (sm - \binom{s+1}{2})$. By the Coning Theorem 11.3.3, the space of C_{r+1}^r-cofactors on $(K_m) * u(\mathbf{p}, \mathbf{p}_u) = K_{m+1}(\mathbf{p})$ has the same dimension and the rank of $M_{r+1}^r(m+1;\mathbf{p})$ is:

$$\binom{m+1}{2} - \binom{m}{2} + \left(rm - \binom{r+1}{2}\right) = (r+1)(m+1) - \binom{r+2}{2}.$$

for all $m+1 \geq r+1$. This completes the induction step.

$\mathcal{M}_s^{s-1}$ COUNTING COROLLARY 11.3.5. *A set of at least* $s+1$ *vertices, with* $|E| > (s+1)|V(E)| - \binom{s+2}{2}$ *is* C_s^{s-1}*-dependent for every plane configuration* $\mathbf{p}$.

Equivalently, a set E *is independent in the generic* C_s^{s-1}*-cofactor matroid only if, for all subsets* E' *on at least* $s+1$ *vertices,* $|E'| \leq (s+1)|V(E')| - \binom{s+2}{2}$.

We transfer some additional results from the d-rigidity matroids.

$\mathcal{M}_s^{s-1}$ VERTEX ADDITION LEMMA 11.3.6. *Given a plane graph* $G(\mathbf{p})$ *and a vertex* $(s+1)$*-addition of* 0 *attached to* $c_1, \dots c_{s+1}$ *creating* G' *and the plane graph* $G'(\mathbf{p}_0, \mathbf{p})$*, with* $\mathbf{p}_0$ *not collinear with any pair of points in* $\mathbf{p}_{c_1}, \dots, \mathbf{p}_{c_{s+1}}$*, then*

1. $G'(\mathbf{p}_0, \mathbf{p})$ *is independent in the* C_s^{s-1}*-cofactor matroid if and only if* $G(\mathbf{p})$ *is independent in the* C_s^{s-1}*-cofactor matroid;*
2. rank $\mathcal{M}_s^{s-1}(G';\mathbf{p}_0,\mathbf{p}) =$ rank $\mathcal{M}_s^{s-1}(G;\mathbf{p}) \ + \ (s+1)$.

PROOF. Without loss of generality, we can assume that none of the lines $\mathbf{p}_0, \mathbf{p}_{c_i}$ is vertical (after a Euclidean transformation). By the non-collinearity assumption, the $s+1$ lines determined by pairs $(\mathbf{p}_0, \mathbf{p}_{c_i})$ have distinct slopes m_i. These lines give $L^{0,c_i} = a_k(m_i x + y + b_i)$ and reduced C_s^{s-1}-cofactors: $\mathbf{D}_{0,k}^s = (a_k)^{s+1}[m_i^s + \dots \binom{s}{k} m_i^k + \dots 1]$. Ignoring the (non-zero) row multipliers $(a_i)^{s+1}$, the added the rows of the matrix for these edges, under the vertex 0 are:

$$\begin{bmatrix} (m_1)^s & \dots & \binom{s}{k}(m_1)^k & \dots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ (m_i)^s & \dots & \binom{s}{k}(m_i)^k & \dots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ (m_{s+1})^s & \dots & \binom{s}{k}(m_{s+1})^k & \dots & 1 \end{bmatrix}$$

Up to column multipliers $\binom{s}{k}$, this is the Vandermonde matrix, which has independent rows if and only if the m_i are all distinct. We conclude that for general position plane configurations, these rows are independent additions to any corresponding matrix $M_s^{s-1}(G;\mathbf{p})$, adding $s+1$ to the rank.

An induction on $(s+1)$-simple graphs gives the following Corollary.

$(s+1)$-Simple Graph Theorem 11.3.7. *For any $n \geq s+1$ and any general position plane-configuration $\mathbf{p}$ on n vertices, the edges E of any $(s+1)$-simple graph G on n vertices are a basis of $\mathcal{M}_s^{s-1}(n;\mathbf{p})$ of rank $(s+1)n - \binom{s+2}{2}$.*

It is now easy to prove that any plane graph $G(\mathbf{p})$ is C_s^{s-1}-rigid if and only if:
1. $|V| \geq s+1$ and $M_s^{s-1}(G;\mathbf{p})$ has rank $(s+1)|V| - \binom{s+2}{2}$; or
2. $|V| \leq s$ and $G = K_V$ with $\mathbf{p}$ in general position.

Corollary 11.3.8. *A plane graph $G(\mathbf{p})$ with vertices in general position is not C_s^{s-1}-rigid if and only if there is a C_s^{s-1}-flex $\mathbf{u}$ and a pair of vertices h,k (not an edge) such that: $\mathbf{D}^s_{h,k} \cdot (\mathbf{u}_h - \mathbf{u}_k) \neq 0$.*

With the same counts and the same induction as for 3-simple graphs, the double bananas of Figure 9.4 also cone to a circuit for any general position plane configuration and a circuit for the generic C_s^{s-1}-cofactor matroid.

Remark 11.3.9. These C_s^{s-1}-cofactors originated in the theory C_s^{s-1}-splines: functions over decompositions of the plane into polygonal cells which are piecewise polynomial of degree at most s, globally C^{s-1} [**Bi**,**CW**,**Wh10**]. For a planar drawing of a connected planar graph, up to addition of a single global quadric to the function, there is an isomorphism between these C_s^{s-1}-splines on the induced decomposition and the C_s^{s-1}-cofactors on the planar graph. For our matroidal purposes, we have extended the algebra to non-planar graphs.

The analogy between generic $(s+1)$-rigidity and generic C_s^{s-1}-rigidity is already established. We will transfer additional results of §11.1 to the generic C_s^{s-1}-cofactor matroid, sometimes using alternate proofs.

$\mathcal{M}_s^{s-1}$ Edge Split Theorem 11.3.10. *Assume G' is an edge $(s+1)$-split of $G=(V,E)$ on $c_1,c_2;c_3,\dots,c_{s+2}$, and $\mathbf{p}$ is a plane configuration with $\mathbf{p}_{c_1},\dots,\mathbf{p}_{c_{s+2}}$ in general position. If $G(\mathbf{p})$ is C_s^{s-1}-independent (C_s^{s-1}-rigid), then $G'(\mathbf{p}_0,\mathbf{p})$ is C_s^{s-1}-independent (C_s^{s-1}-rigid) for almost all choices of $\mathbf{p}_0$, including $\mathbf{p}_0$ a distinct point on $\mathbf{p}_{c_1},\mathbf{p}_{c_2}$, not collinear with any other pair $\mathbf{p}_{c_i},\mathbf{p}_{c_j}$, $3 \leq j \leq s+2$.*

Conversely, if $G'(\mathbf{p}_0,\mathbf{p})$ is C_s^{s-1}-independent (C_s^{s-1}-rigid) for some choice of $\mathbf{p}_0$, with vertex 0 connected to exactly vertices $\sigma(1),\dots,\sigma(s+2)$ at $s+1$ points in general position then, for some edge e with endpoints in $c_1,c_2;c_3,\dots,c_{s+2}$, $E' = E\cup\{e\} - \{(0,c_i\}$ is C_s^{s-1}-independent (C_s^{s-1}-rigid) at $\mathbf{p}$ and G' is an edge $(s+1)$-split of $G=(V'-0,E)$.

Proof. The proof used for Theorem 11.1.9 and Theorem 10.2.1 extends to C_s^{s-1}-independence and C_s^{s-1}-rigidity.

$\mathcal{M}_s^{s-1}$ Construction Theorem 11.3.11. *If G has an $(s+1)$-construction then the edges of G form a basis for $\mathcal{M}_s^{s-1}(|V|)$.*

Example 11.3.12. The complete bipartite graph $K_{d+1,\binom{d+1}{2}}$ has such a d-construction. We form the $(d+1)$-simplex by a single vertex d-addition, then do an edge d-split on each of these $\binom{d+1}{2}$ edges. We conclude that $K_{d+1,\binom{d+1}{2}}$ is both generically d-rigid and generically C_{d-1}^{d-2}-rigid.

C_s^{s-1}-Flex Test 11.3.13. *For any plane configuration $\mathbf{p}$ for the graph K_n, the following are equivalent:*

1. *the edge $\{h,k\}$ is not in the C_s^{s-1}-closure $\langle E\rangle$ in $\mathcal{M}_s^{s-1}(n;\mathbf{p})$ of the set E;*
2. *every C_s^{s-1}-cofactor λ on $E\cup\{h,k\}$ is zero on $\{h,k\}$;*
3. *there is a C_s^{s-1}-flex $\mathbf{u}$ on $G=(V,E)$, such that $\mathbf{D}_{h,k}^s\cdot(\mathbf{u}_h-\mathbf{u}_k)\neq 0$.*

Generic $\mathcal{M}_s^{s-1}$ Gluing Lemma 11.3.14. *For two edge sets E_1, E_2,*

1. *if E_1 and E_2 are generically C_s^{s-1}-rigid and $|V(E_1)\cap V(E_2)|\geq(s+1)$, the set $E_1\cup E_2$ is generically C_s^{s-1}-rigid;*
2. *if $|V(E_1)\cap V(E_2)|\leq s$, then the closure $\langle E_1\cup E_2\rangle$ in $\mathcal{M}_s^{s-1}(G)$ is contained in $K_{V(E_1)}\cup K_{V(E_2)}$;*
3. *if E_1 and E_2 are generically C_s^{s-1}-independent and $E_1\cap E_2$ is generically C_s^{s-1}-rigid, the set $E_1\cup E_2$ is generically C_s^{s-1}-independent.*

Proof. The proof of the Generic C_2^1-Gluing Theorem 10.24 extends directly.

Corollary 11.3.15. *The generic C_s^{s-1}-cofactor matroid is an abstract $(s+1)$-rigidity matroid.*

Remark 11.3.16. Techniques such as vertex splits, and the resulting C_s^{s-1}-rigidity of simplicial s-surfaces (analogs of Fogelsanger's Theorem [**Fo**]), extend to these matroids.

The generic C_s^{s-1}-rigidity of simplicial s-surfaces is an alternate direct proof of the Lower Bound Theorem for edges of manifolds (without boundary) [**Ka1**]. It gives the same information for these specific structures. With this coning result, the extensions of [**Ka1**] to simplicial manifolds with boundary also transfer to these cofactor matroids.

11.4. X-replacement and bipartite graphs in $\mathcal{M}_s^{s-1}(n)$. The appropriate form of X-replacement preserves C_s^{s-1}-rigidity for all $s\geq 1$. Specifically, given a graph G_1 with edges $\{i_1,i_2\},\{i_3,i_4\}$ and vertices $i_5,\dots i_{d+2}$, G is a *d-X-replacement on G_1* if G has one added vertex i_0 and replaces the edges $\{i_1,i_2\},\{i_3,i_4\}$ with the $d+2$ edges $\{i_0,i_j\}$, $1\leq j\leq d+2$.

$\mathcal{M}_s^{s-1}$ X-Replacement Theorem 11.4.1. *An $(s+1)$-X-replacement takes a generically C_s^{s-1}-rigid graph G_1 to a graph G which is generically C_s^{s-1}-rigid.*

Proof. Take a generic plane configuration $\mathbf{p}$ for G_1, with the edges $\{i_1,i_2\}$, $\{i_3,i_4\}$ to be replaced. We do a vertex $(s+1)$-addition for the additional vertex i_0, attached to vertices i_1, i_3 and $i_5,\dots,i_{s+3}$ and placed at the point of intersection of $\mathbf{p}_{i_1}\mathbf{p}_{i_2}$ and $\mathbf{p}_{i_3}\mathbf{p}_{i_4}$. This is C_s^{s-1}-rigid by the $\mathcal{M}_s^{s-1}$ Vertex $(s+1)$-Addition Theorem.

In this position, we use collinear substitution to replace $\{i_0, i_1\}, \{i_1, i_2\}$ with $\{i_0, i_1\}, \{i_0, i_2\}$ and to replace $\{i_0, i_3\}, \{i_3, i_4\}$ with $\{i_0, i_3\}, \{i_0, i_4\}$. This is now the C_s^{s-1}-rigid graph of the $(s+1)$-X-replacement, which completes the proof.

With this added inductive technique, we can verify directly that the bipartite counterexamples for the generic $(s+1)$-rigidity matroid are not a problem for the generic C_s^{s-1}-cofactor matroid. Figure 11.2 gives an extended 4-construction of $K_{6,7}$, using X-replacement for the last two steps. We conclude that $K_{6,7}$ is a basis for the generic C_3^2-rigidity matroid.

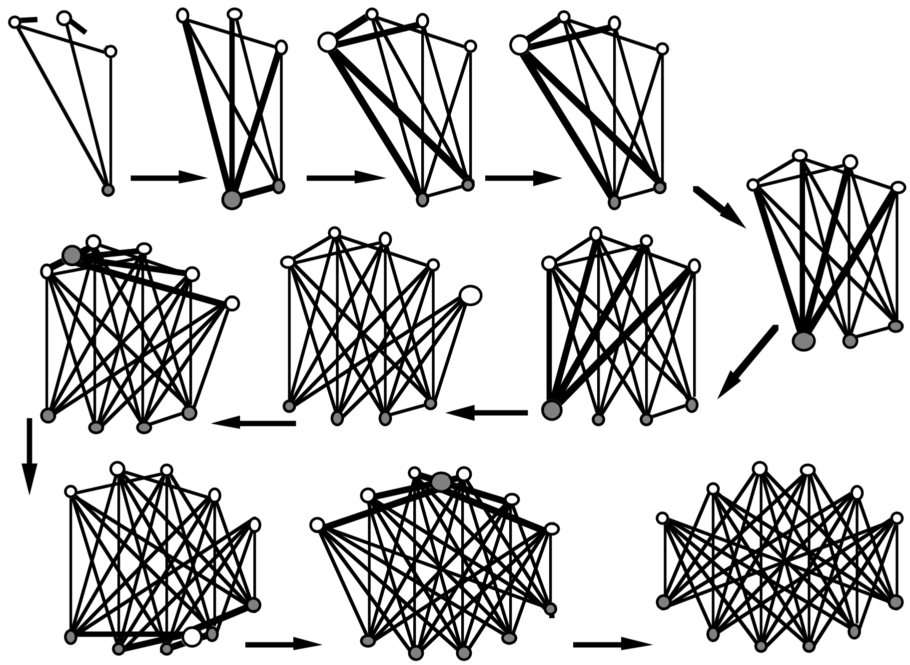

Fig. 11.2. *An extended 4-construction for $K_{6,7}$, in which the last two steps are X-replacement.*

$\mathcal{M}_3^2$ Bipartite Theorem 11.4.2. *The complete bipartite graphs $K_{5,10}$ and $K_{6,7}$ are generically C_3^2-rigid and C_3^2-independent.*

Every complete bipartite graph $K_{m,n}$, $m \le n$ with $m = 5, n \ge 10$ or $m \ge 6, n \ge 7$ is generically C_3^2-rigid.

Proof. Example 11.3.1 gives a 4-construction for $K_{5,10}$ and Figure 11.2 gives an extended 4-construction for $K_{6,7}$. These verify the C_3^2-rigidity of these bipartite graphs. Larger complete bipartite graphs contain these, and can be created by a sequence of vertex 4-additions followed by adding the edges still missing.

Notice that the 'smaller' bipartite graphs, such as $K_{4,n}$, $K_{5,9}$ and $K_{6,6}$ are under counted, with $|E| < 4|V| - 10$. Example 11.3.1 also shows that $K_{s+2,\binom{s+2}{2}}$ is C_s^{s-1}-rigid. The extended 4-construction for $K_{6,7}$ generalizes, through coning, to an extended $(s+1)$-construction for $(s-3)$-fold cones of $K_{6,7}$. Therefore these are generically C_s^{s-1}-rigid and C_s^{s-1}-independent, but not $(s+1)$-independent. We expect that extended constructions (including X-replacement) will prove that $K_{7,10}$ is generically C_4^3-rigid, $K_{8,14}$ is generically C_5^4-rigid, etc. We also expect that extended

$(s+1)$-constructions will verify that $K_{s+3,s+3}$ is generically C_s^{s-1}-independent for all $s \geq 3$, setting aside all the counterexamples of §11.2. This evidence supports the following conjecture for general s.

$\mathcal{M}_s^{s-1}$ BIPARTITE CONJECTURE 11.4.3. *A complete bipartite graph $K_{m,n}$, $m, n \geq 2$, is C_s^{s-1}-rigid if and only if $m \times n \geq (s+1)(m+n) - \binom{s+2}{2}$.*

Because these cofactor matroids, in this generality, do not have even the 'applications' of generic d-rigidity, they have not been studied in detail. It is possible that some surprises lie buried, as they did in the bipartite frameworks of d-rigidity.

11.5. Comparisons of abstract d-rigidity matroids. §11.1 and §11.3 demonstrated that we have two families of abstract d-rigidity matroids which share a number of properties. §11.2 and §11.4 demonstrated that they are not isomorphic for $d \geq 4$. The results of §11.4 confirm that the difficulties for 4-rigidity are not rooted in the structure of abstract d-rigidity matroids. Independently, [**Th**] verified this by constructing a specific abstract 4-rigidity matroid in which $K_{6,6}$ was independent. Therefore, generic d-rigidity is not a maximal abstract d-rigidity matroid for $d \geq 4$. We summarize the current situation for $d = 4$.

	Generic 4-Rigidity	Generic C_3^2-Rigidity
rank K_n, $n \geq 4$	$4n-10$	$4n-10$
vertex 4-addition	Yes	Yes
edge 4-split	Yes	Yes
4-constructions	Yes	Yes
abstract 4-rigidity	Yes	Yes
vertex 4-split	Yes	Yes
simplicial 4-polytopes	Rigid	Rigid
4-coning	Yes	Yes
4-X-replacement	No	Yes
$K_{5,10}$	Basis	Basis
$K_{6,6}$	Circuit	Independent
$K_{6,7}$	Dependent	Basis
maximal abstract 4-rigidity matroid	No	Conjectured

The Dress Conjecture in the chapter on Unsolved Problems has analogs for higher dimensions [**GSS**]. Because of examples such as $K_{6,6}$, the Dress conjecture and its relatives fail for generic d-rigidity for $d \geq 4$. However we *conjecture* that these extensions are correct for generic C_s^{s-1}-rigidity. This suggests there still is a 'maximal' abstract d-rigidity matroid. In the absence of counterexamples, we offer the following conjecture.

THE MAXIMAL MATROID CONJECTURE 11.5.1. *There is a maximal abstract d-rigidity matroid on the complete graph K_n for $d \geq 2$.*

For $d \geq 2$, the generic $C_{(d-1)}^{(d-2)}$-cofactor matroid $\mathcal{K}_{(d-1)}^{(d-2)}(n)$is the maximal abstract d-rigidity matroid on K_n.

It seems ironic that the abstract d-rigidity matroids, in their full generality, may be a better description of the cofactor matroids than the d-rigidity matroids!

We close with a connectivity conjecture for which the results of the next section will give some support.

CONJECTURE 11.5.2. *If a graph G is $2\binom{d+1}{2}$-connected in a vertex sense, then:*
1. *G is generically d-rigid;*
2. *G is generically C_{d-1}^{d-2}-rigid.*

The relationships among the matroids in Parts I and II were displayed in Figure 1.1. Looking back, we see both the close parallels and the subtle divergence between the generic rigidity matroids and generic cofactor matroids on graphs. The parallel drawing matroids are shown as an offshoot from the plane rigidity, in another direction. The additional boxes, with question marks for their 'names', are hypermatroids on graphs (with the indicated ranks on complete graphs of sufficient size) which will appear in Part III as lower homologies of the rigidity matroids on larger facets (see §16.4). Their direct geometric interpretation and specific combinatorial properties have never been investigated, though their place in this schematic diagram hints at the analogies to rigidity in higher dimensions which should be pursued. To condense the diagram, we completely omitted other hypermatroids on graphs which are offshoots of the higher cofactor matroids in several other directions, including the C_d^s-cofactors mentioned earlier and the lower homologies of the multivariate cofactor matroids of §15. While we believe these hypermatroids deserve an initial investigation, their enduring interest will depend on our interest in the matroids in Part III which bring them to light.

12. d-Space Structures Which Work!

We have seen the increasing difficulties of combinatorially characterizing the bases of the generic d-rigidity matroids for $d \geq 3$. In contrast, several related structures have simple, basically complete combinatorial theories and simpler matroids than d-rigidity. Since these give more solid underpinnings for work, even in 3-space, we briefly outline the associated matroids (and hypermatroids).

12.1. Bar-and-body frameworks in d-space. A bar-and-body framework ([**Ta1**,**2**,**WW3**,**Wh7**]) begins with a multigraph $\overline{G} = (V, \overline{E})$. A vertex $i \in V$ of the graph is interpreted as a large rigid *body*. Each edge e has two identified vertices, $v_1(e)$, $v_2(e)$, which identify the two bodies which this 'bar' will join. Geometrically, a bar is described by giving an ordered pair of attaching points (universal joints) in d-space $(\mathbf{p}_e, \mathbf{q}_e)$, with $\mathbf{p}_e$ on $v_1(e)$ and $\mathbf{q}_e$ on $v_2(e)$. Each body is 'expandable' to contact all the points which are attach these bars to the body. Notice that there is no provision for specifying that two edges attached to some body share a common point of attachment.

Under first-order motions in d-space, each body will undergo a trivial first-order motion. This space has dimension $\binom{d+1}{2}$ and can be coordinatized by the *first-order screw centers* S^i - vectors of size $\binom{d+1}{d-1} = \binom{d+1}{2}$. (In projective Grassmann-Cayley algebra, these are $(d-1)$-tensors in projective d-space [**DRS**,**Wht4**,**WW3**].) Each bar $\mathbf{p}_e, \mathbf{q}_e$ can be written as a vector of length $\binom{d+1}{2}$ the *Plücker coordinates* $\mathbf{b}_e$ of the segment $\mathbf{p}_e \wedge \mathbf{q}_e$. These are formed by taking all 2-by-2 minors of the 2-by-$(d+1)$ matrix $\begin{bmatrix} (\mathbf{p}_e)_1 & \dots & (\mathbf{p}_e)_d & 1 \\ (\mathbf{p}_e)_1 & \dots & (\mathbf{p}_e)_d & 1 \end{bmatrix}$. If we use appropriate bases for the space of screws, the constraint equation for a bar $\mathbf{b}_e$ is [**WW3**]:

$$\mathbf{b}_e \cdot S^{v_1(e)} = \mathbf{b}_e \cdot S^{v_2(e)} \quad \Leftrightarrow \quad \mathbf{b}_e \cdot \left(S^{v_1(e)} - S^{v_2(e)}\right) = 0.$$

As this writing indicates, this first-order constraint depends only on the line of the bar, not the specific points. (Changing the points along the line, but keeping them distinct, just multiplies the bar vector by a non-zero scalar reflecting the change in length (size) and direction (sign) from the change of points.) If we looked at long range flexes, as in mechanical engineering rather than first-order flexes of structural engineering, the choice of points along the line would matter a great deal!

In summary, a *bar-and-body framework in d-space* is a multigraph $\overline{G} = (V, \overline{E})$ and an assignment of *bar vectors* $\mathbf{b}_e$ (2-extensors $\mathbf{p}_e \wedge \mathbf{q}_e$ to the edges $e \in \overline{E}$. Together the bar vectors form a *line configuration* $\mathbf{b}$. A *first-order flex of* $\overline{G}(\mathbf{b})$ is an assignment of $\binom{d+1}{2}$-vectors S^i to the vertices such that for each edge e, $\mathbf{b}_e \cdot (S^{v_1(e)} S^{v_2(e)} = 0$. A first-order flex is *trivial* if each vertex has the same screw center: $S^1 = \ldots = S^{|V|}$. This space of trivial first-order flexes has dimension $\binom{d+1}{2}$. A bar-and-body framework $\overline{G}(\mathbf{b})$ is *first-order rigid* if all first-order flexes are trivial.

These linear equations define the *bar-and-body rigidity matrix* $R_{\overline{G}}(\mathbf{b})$:

$$\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & \ldots & v_1(e) & \ldots & v_2(e) & \ldots & |V| \end{array} \\
\begin{array}{c} \vdots \\ e \\ \vdots \end{array} & \begin{pmatrix} \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \ldots & \mathbf{b}_e & \ldots & -\mathbf{b}_e & \ldots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \end{pmatrix}
\end{array}$$

The kernel of $R_{\overline{G}}(\mathbf{b})$ is the space of first-order flexes and the row dependencies of $R_{\overline{G}}(\mathbf{b})$ are the dependencies of the *bar-and-body matroid* on $\overline{G}(\mathbf{b})$.

Example 12.1.1. Consider joining two bodies (say an object to the ground) in 3-space (Figure 12.1A). Since $6|V| - 6 = 6$, we will need 6 bars. For example, the 'top' and 'bottom' triangle of an octahedron (full rigid bodies) are attached by 6 bars (Figure 12.1B). In generic position (or any convex realization) this structure is first-order rigid both as a bar and joint framework and as a bar-and-body framework.

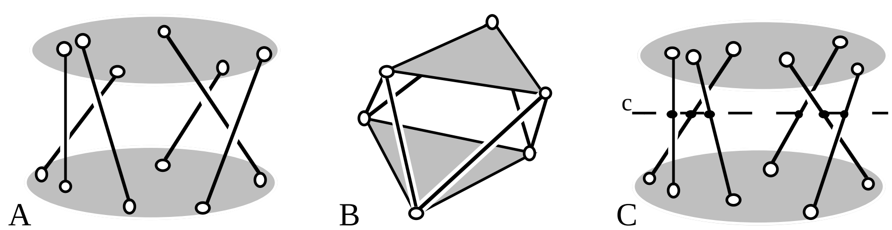

Fig. 12.1. *A generic bar-and-body framework (A), a special position of the same graph (B) which is first-order rigid, and a special position which is not first-order rigid in 3-space (B).*

If these six bars happen to all intersect a single line $\mathbf{c}$ (Figure 12.1C), then this line will be the axis of a non-trivial rotation of one body relative to the second body. The corresponding six rows of the bar-and-body rigidity matrix will be dependent.

In general, these rows will be dependent if and only if some 6-tuple $\mathbf{S}$ satisfies $\mathbf{b}_k \cdot \mathbf{S} = 0$, $1 \leq k \leq 6$. Such a sextuple represents a general screw $\mathbf{S}$ in space, and these lines are in the linear line complex of null lines of the screw [**Kl**,**WW1**].

If we take a generic line assignment $\mathbf{b}$ in d-space (i.e. use generic points for the two ends of each bar) the bar-and-body rigidity matrix $R_{\overline{G}}(\mathbf{b})$ defines the *generic* *body d-rigidity matroid* on $\overline{E}$. For a fixed set of vertices, we can recognize that any set of more than $\binom{d+1}{2}$ edges between two bodies will be dependent in this matroid. For simplicity, we can assume that no graph has more than $\binom{d+1}{2}$ edges between any pair of bodies, so the multigraph $\binom{d+1}{2}K_n$, with $\binom{d+1}{2}$ edges between each pair of vertices, is the context for this matroid. The following theorem characterizes these matroids, in analogy to the characterization of line rigidity in §2.1.

TAY'S THEOREM 12.1.2 [**Ta1**,**2**,**WW3**]. *For a multigraph $\overline{G} = (V\overline{E})$ the following are equivalent:*

1. *for some line assignment* $\mathbf{b}$ *in d-space, the bar-and-body framework* $\overline{G}(\mathbf{n})$ *is first-order rigid, with rank* $\binom{d+1}{2}|V| - \binom{d+1}{2}$*;*
2. *for almost all bar assignments* $\mathbf{b}$ *in d-space, the bar-and-body framework* $\overline{G}(\mathbf{b})$ *is first-order rigid;*
3. *the graph* $\overline{G}$ *is generically body d-rigid;*
4. *the multigraph* $\overline{G}$ *contains* $\binom{d+1}{2}$ *edge-disjoint spanning trees.*

The following are also equivalent:

1. *the set* $E' \subseteq \overline{E}$ *is independent in the generic body d-rigidity matroid;*
2. *for all subsets* $E'' \subseteq E'$, $|E''| \leq \binom{d+1}{2}|V(E''| - \binom{d+1}{2}$*;*
3. *the set* E'' *can be covered by* $\binom{d+1}{2}$ *forests.*

One proof for this theorem connects our rigidity matrix to the matrix for the matroid union of $\binom{d+1}{2}$ copies of the graphic matroid on the normal graph G which underlies the multigraph $\overline{G}$ (see [**Wh7**] and §A).

REMARK 12.1.3. A bar-and-body framework in which the attaching vertices of the bars affinely span d-space can be modeled as a bar-and-joint framework, with each body replaced by a basis for the d-rigidity matroid on these joints (say an appropriate d-simple framework). The first-order d-rigidity, flexibility or d-independence of the two structures will be the same. Thus our theorem does give explicit results for those special graphs which are models of such frameworks.

However, if we take a general graph as a bar framework, this will force certain attachments to share a single vertex on a 'body'. We get no new information for these general bar frameworks.

However, we do have enticing results on connectivity.

COROLLARY 12.1.4 [**Wh7**]. *A multigraph* $\overline{G}$ *is generically body d-rigid if the multigraph is* $2\binom{d+1}{2}$ *connected in an edge sense.*

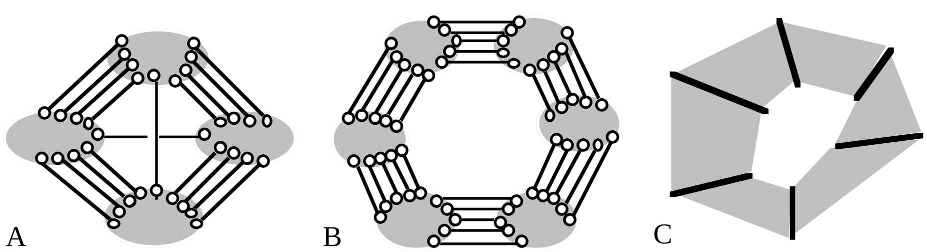

Fig. 12.2. *Two generically body 3-rigid multigraphs (A,B), and a corresponding body-and-hinge framework (C) whose graph is a hexagonal cycle.*

12.2. Body-and-hinge frameworks in 3-space. A body-and-hinge framework can be described as an interesting special case of a bar-and-body framework with important applications [**CrW1**,**TW1**,**Wh7**]. For simplicity, we just outline the process for 3-space. However all these results generalize to arbitrary dimensions.

The basic idea is that a set of rigid bodies is connected, in pairs, along 'hinges' (lines 3-space). The bodies each move, preserving the contacts along the hinges. Central to our description is the observation that the rigid motions in space can be represented by 'screw centers' S^i (weighted line segments in projective geometry – and their sums). If two bodies share two hinges on distinct lines, they will be locked into a single rigid unit, and might as well be called a single body. For this reason, we restrict ourselves to ordinary graphs, not multigraphs.

A *body-and-hinge framework* in 3-space is a graph $G = (V, E)$ and an assignment $\mathbf{h}$ of directed lines $\mathbf{h}_{ij}$ in 3-space, written as 2-extensors (6-vectors) in the Grassmann algebra for 3-space, to the directed edges, with $\mathbf{h}_{i,j} = -\mathbf{h}_{j,i}$.

A *first-order flex* for a body-and-hinge framework $G(\mathbf{h})$ is an assignment of screw centers of motion $\mathbf{S}^i$ (sum of 2-extensors in the Grassmann algebra) to each vertex such that, for every edge $\{i, j\}$: $\mathbf{S}^i - \mathbf{S}^j = \alpha_{i,j}\mathbf{h}_{i,j}$ for some scalar $\alpha_{i,j}$. A body-and-hinge framework $G(\mathbf{h})$ is *first-order rigid* if all first-order flexes are *trivial*, with all screw centers $\mathbf{S}_i = \mathbf{C}$ for a fixed center $\mathbf{C}$.

A *generic hinge assignment* in 3-space is achieved by assigning two generic points in 3-space, $\mathbf{p}_{i,j}, \mathbf{q}_{i,j}$, to each edge and then taking the Plücker coordinates of the line through these points as the hinge $\mathbf{h}_{i,j}$. A graph G is *generically hinge* 3*-rigid* if the body-and-hinge framework $G(\mathbf{h})$ is first-order rigid for some generic hinge assignment.

Implicitly, each hinge $\mathbf{h}_{ij}$ can be replaced by a set of 5 independent bars, each intersecting the hinge line producing an *equivalent* bar-and-body framework on the multigraph $5G$ with 5 copies of each edge of G (Figure 12.2 C,B). We can also see this count $5|E|$ since each hinge equation $\mathbf{S}^i - \mathbf{S}^j = \alpha_{i,j}\mathbf{h}_{i,j}$ is six linear equations in the unknowns $\mathbf{S}^i, \mathbf{S}^j, \alpha_{i,j}$. Row reducing to eliminate the auxiliary unknowns $\alpha_{i,j}$ leaves $5|E|$ equations for the unknown centers $\mathbf{S}^i, \mathbf{S}^j$.

Unfortunately, if we take a generic line assignment $\mathbf{b}$ for the $5|E|$ bars, the five lines for each hinge will not redefine an underling hinge. They will define a general 'screw' $\mathbf{S}_{i,j}$ which is the sum of the 6-tuples for two lines, but not a line. This screw will not satisfy the underlying quadratic Plücker equation [**Kl**,**Wh7**]. Fortunately, this can be worked around with some algebraic geometry [**Wh7**] and the bar-and-body results do translate to hinges.

3-Hinge Theorem 12.2.1 [**TW1**]. *For a graph G the following are equivalent:*
1. *For some hinge assignment* $\mathbf{h}$*, the body and hinge framework* $G(\mathbf{h})$ *is first-order rigid;*
2. *For almost all hinge assignments* $\mathbf{h}$*, the body and hinge framework* $G(\mathbf{h})$ *is first-order rigid;*
3. *The graph* G *is generically hinge* 3*-rigid;*
4. *If each edge of the graph is replaced by five copies, the resulting multigraph* $5G$ *contains six edge-disjoint spanning trees.*

Corollary 12.2.2. 3*-edge-connectivity of* G *is sufficient for generic hinge* 3*-rigidity of a graph* G*.*

Remark 12.2.3. There are two special geometries for hinge structures which occur in practice. A *panel-hinge framework* has all hinges of each body coplanar (as in the plane faces of a polyhedron with hinges for the edges). The geometrically polar structures, with all hinges of each body concurrent in a single point, are the *molecular frameworks*. The model here is a chemical molecule in which each atom and its attached bonds is a body and the bond lines are the hinges. See [**Fr1**,**2**] for a study of such frameworks in connection with models of glass.

We can define *generic molecular framework* by assigning generic points to the atoms, and then assigning the corresponding hinge the line joining the two vertices for the atoms it 'bonds'. These define the *generic molecular* 3*-rigidity matroid on* G. There are numerous examples (related to spherical polyhedra) which support the following conjecture [**TW1**].

Molecular Framework Conjecture 12.2.4. *The generic hinge* 3*-rigidity matroid on* G *is the same as the generic molecular* 3*-rigidity matroid. In particular, a graph* G *is generically hinge* 3*-rigid if and only if* G *is generically molecular* 3*-rigid.*

III. Matroids for Geometric Homologies.

13. Some Background

In the previous sections the matroids were defined for geometric graphs (1-skeleta of simplicial complexes). Many of the combinatorial results were also based on results for graphs (trees, unions of trees, inductions on graphs, etc.). The parallel drawing matroids for hyperplanes and points are an interesting exception. They were defined for points and hyperplanes, but they were presented in terms of the bipartite incidence graph (incidences were 'edges' joining vertices and faces). For over a decade, Henry Crapo has worked on an underlying homology (actually cohomology) for these structures - unraveling the subtle relationships for incidence sets which are not bases of the matroid [**Cr1**,**Cr3**,**4**,**CR2**,**CRy**]. These are 'geometric homologies', homologies based on both the abstract combinatorics of the incidence structure and on the geometry of the configuration of the points (or planes in the case of parallel drawings). While we will not present this developing theory here, this 'geometry of the circuits' of the underlying represented matroid is essential to a complete understanding of the underlying matroid [**Cr1**,**Cr4**,**CR2**].

In the last decade, several additional families of matroids based on geometric homologies have been defined for higher skeleta of simplicial complexes and other cell complexes realized in geometric spaces (real d-space or projective d-space). Figure 13.1 places these matroids into a pattern of 'family relationships', corresponding to the relationships we now see in the original Figure 1.1. Even this diagram includes only a fraction of the rich variety of matroids and homologies which are implicit in fields such as simplicial homology, multivariate splines and skeletal rigidity.

As we move to higher skeleta, the matroids of interest, which will be presented as matrices over the reals, have ranks which implicitly depend on all of the lower skeleta. This connection to the lower skeleta is explicitly unraveled through homology theory: a sequence of matrices (matroids) for each pair of adjacent skeleta. The layers of Figure 13.1 and Figure 1.1 are stacked with 3-homology above 2-homology above 1-homology, etc. The lower homologies of the other structures may also appear in the layers below this structure (perhaps on a diagonal), and other matroids could be described to fill each layer out into several 'quadrants' of geometric matroids (see §16.4).

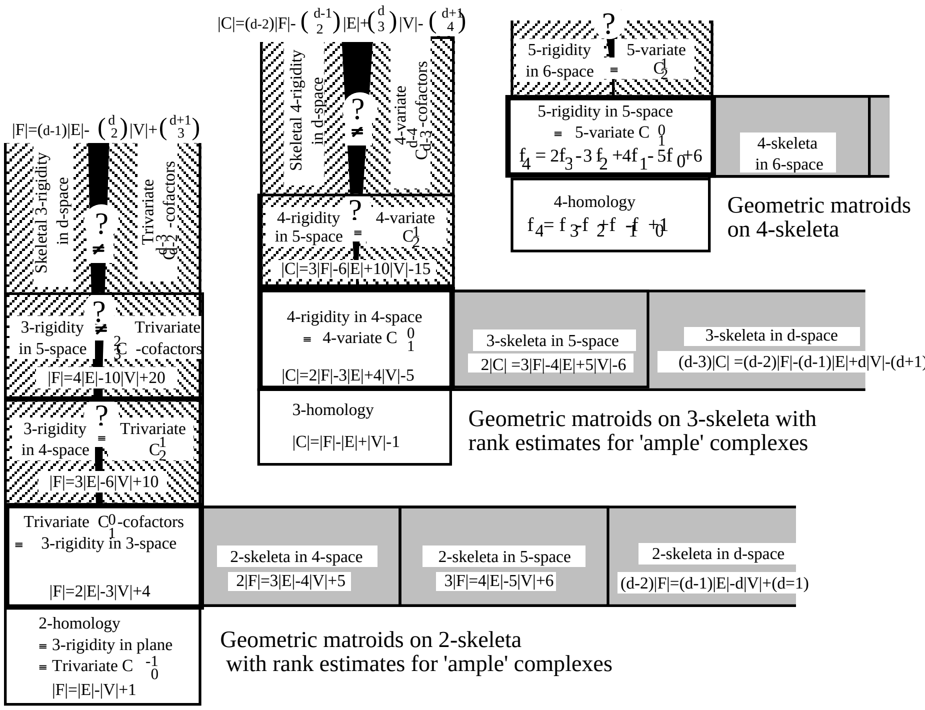

Fig. 13.1. *A schematic diagram for families of matroids associated with geometric homologies.*

The second applied geometric homology to be explicitly described arose from multivariate splines [**Bi**]. In conjectured ranks of the form:

$$\operatorname{rank}(F) \leq 2|E| - 3|V| + 4 \quad \text{or} \quad \operatorname{rank}(C) \leq 3|F| - 6|E| + 10|V| - 15,$$

we see an alternating sum of terms for the lower skeleta (faces, edges, vertices, and a constant term). Billera anticipated that this sum was the Euler characteristic of a chain complex in homology. With this insight, he developed a chain complex and applied the tools of homology to develop some important results (see §15).

Homology was introduced into this applied context (approximation theory and Computer Aided Geometric Design) because homology describes essential relationships. Any other approach is ad hoc, inadequate, or both. Initial work in the plane can overlook this connection, as we did in previous sections. As the dimension of the space increases, the structure given by a chain complex homology becomes essential to keeping the relationships straight, at least implicitly [**ASW**].

Recent work on combinatorial and geometric proofs of the g-theorem has lead to a theory of 'skeletal rigidity' for higher skeleta of simplicial polytopes (and general simplicial complexes) [**Lee1**,**2**,**TWW1**,**2**,**3**]. Implicitly, and sometimes explicitly, this is based on a second geometric homology. Again, an analysis of the lower homologies is central to estimates of the rank of the matroid associated with the highest cycles, which are the objects of primary interest (§16).

We will devote the next three sections to three related examples of homologies and their matroids: ordinary simplicial homology and the k-cycle matroid (§14), multivariate splines and the cofactor matroids (§15), and rigidity of higher skeleta and their rigidity matroids (§16). There are a number of important unsolved problems in these areas and the general theory is undeveloped. The methods and the insights of the previous sections, based on graphs and related matroid theory, give valuable information for these structures - but they are not sufficient. Techniques from homology (such as spectral sequences) have also been used [**TWW3**], but good solutions will require new ideas.

14. Simplicial Homology Matroids

The applied problems of §15 and §16 will emphasize important gaps in our understanding of the matroids for the standard simplicial homology of higher simplicial complexes. Clearly, this theory is sparse in comparison with the extensive results and techniques for graph theory (homology of the 1-skeleton). In addition, this structure is essential to understanding the matroid of the later geometric generalizations.

We begin with the basic simplicial matroid, or k-cycle matroid, based on standard simplicial homology [**CL**,**CR1**]. Previous presentations in matroid theory have focused on the top two homologies (see Remark 14.1.7). Our presentation will emphasize the important role of the lower homologies of the complex, because of my experiences with the extensions and applications of the the following sections. For graphs, with only two possibly non-trivial homologies to consider, these two approaches coincide. For larger simplices, attention to the lower homologies gives critical information.

14.1. The simplicial k-cycle matroids. The basic references for this section are [**CL**,**CR1**] along with a standard book on algebraic topology [**Mu**].

A *simplicial complex* Δ is a family of *simplices*: subsets of a finite set $V = \{v_1, v_2, \dots, v_n\}$ such that every subset of a simplex is also a simplex. For convenience, we use the linear ordering of the vertices and assume that each simplex is written in lexicographic order: $\{v_{i_0}, \dots, v_{i_j}\},\ i_0 < i_1 < \dots < i_j$.

For each $-1 \leq j \leq n$, we have the subcomplex of $\Delta^j = \{\sigma \in \Delta \bigm| |\sigma| \leq j+1\}$ and the j-*simplices* $\Delta^{(j)} = \{\sigma \in \Delta \bigm| |\sigma| = j+1\}$. The cardinality of this set is $f_j = |\Delta^{(j)}|$.

Warning. We index simplices by the conventions of homology theory, not the indexing commonly used in matroid theory. In [**CL**,**CR1**], our j-simplex, a set of $j+1$ elements, is called a $(j+1)$-simplex and our $\Delta^{(j)}$ is becomes $\Delta^{(j+1)}$. For combinatorists, our choice is somewhat awkward, since combinatorists count entries not topological dimension. However, as we transfer results, techniques, and references from algebraic topology, it seems appropriate to work within their long established conventions.

The j-*chains* $C_j(\Delta)$ of the simplicial complex are the formal sums

$$\mathbf{c} = \sum_{\sigma \in \Delta^{(j)}} c_\sigma \cdot \sigma,$$

with coefficients $c_\sigma \in \mathbb{R}$. Effectively, the j-chains are elements of a vector space $\mathbb{R}^{f_j}$. [Everything we do will apply to an arbitrary field F, but we will not record this fact.] The critical maps of homology are the sequence of boundary maps on these chains: $\partial_j : C_j(\Delta) \to C_{j-1}(\Delta)$ defined on simplices by

$$\partial_j\{v_{i_0}, \dots, v_{i_j}\} = \sum_{l=0}^{j} (-1)^l \{v_{i_0}, \dots, \widehat{v_{i_l}}, \dots, v_{i_j}\}$$

where $\widehat{v_l}$ indicates the deletion of v_l. This map on simplices is extended linearly to all j-chains. It is a standard (and critical) result that $\partial_{j-1}\partial_j = 0$ for all $0 \leq j < n$. This whole package of chains and operators is the *chain complex*:

$$\mathcal{C}_k(\Delta):\ \mathbf{0} \rightarrow C_k(\Delta) \xrightarrow{\partial_k} C_{(k-1)}(\Delta) \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \xrightarrow{\partial_{-1}} \mathbf{0}.$$

Since $C_{-1}(\Delta) = \{c_\emptyset \cdot \emptyset\}$, $\mathcal{C}_k$ is an *augmented chain complex*, in the vocabulary of algebraic topology.

Remark 14.1.1. It is often convenient to explicitly use the underlying *oriented simplices*. Taking all orders of the vertices of a simplex, these ordered simplices are placed into two orientation classes depending on whether their ordered sequence of indices represent an even or odd permutation. The equivalence class of an ordered simplex is denoted $[v_{i_0}, \ldots, v_{i_k}]$ or $[\sigma]$ and the other equivalence class, differing by an odd permutation, is denoted $-[\sigma]$.

To simplify writing the boundary operator, we define $Sign(\rho,\sigma)$ for simplices $\rho \in \Delta^{(i-1)}$, $\sigma \in \Delta^{(i)}$, by:

$$Sign(\rho,\sigma) = \begin{cases} +1 & \text{if, for some } x \in V,\ [\rho, x] = [\sigma] \\ -1 & \text{if, for some } x \in V,\ [\rho, x] = -[\sigma] \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, for each $\sigma \in \Delta^{(i)}$, $\partial_i[\sigma] = \sum_{\rho \ \mid \ \rho\in\Delta^{(i-1)}} Sign(\rho,\sigma)\cdot[\rho]$.

An i-chain $\mathbf{c} = \sum_{\sigma\in\Delta^{(i)}} c_\sigma \cdot [\sigma]$ is an *i-cycle* if $\partial\mathbf{c} = \mathbf{0}$. Explicitly, the space $Z_i(\Delta)$ of i-cycles is the solution space for the system of linear equations:

$$\begin{aligned} \partial\Big(\sum_{\sigma\in\Delta^{(i)}} c_\sigma \cdot [\sigma]\Big) &= \sum_{\sigma\in\Delta^{(i)}} \Big(\sum_{\rho\in\Delta^{(i-1)}} Sign(\rho,\sigma) c_\sigma \cdot [\rho]\Big) \\ &= \sum_{\rho\in\Delta^{(i-1)}} \Big(\sum_{\sigma\in\Delta^{(i)}} Sign(\rho,\sigma) c_\sigma\Big)\cdot[\rho] = \mathbf{0}. \end{aligned}$$

In terms analogous to stresses in frameworks, these k-cycles are the row dependencies of the f_k-by-f_{k-1} *k-cycle matrix*: $M_k(\Delta)$, with $M_k(\Delta)[\sigma,\rho] = [Sign(\rho,\sigma)]$ for each $\sigma \in \Delta^{(k)}$, $\rho \in \Delta^{(k-1)}$. These k-cycles are the dependencies of the *simplicial k-cycle matroid* $\mathcal{M}_k(\Delta)$, also called the simplicial matroid of Δ^k in [**CL**].

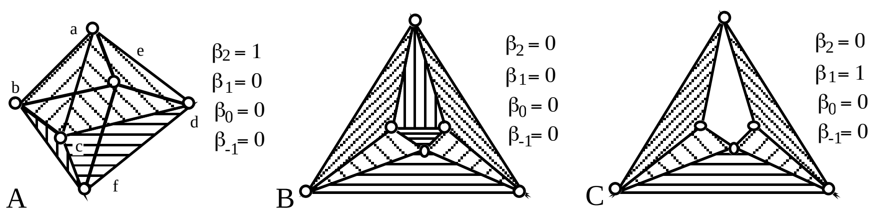

Fig. 14.1. *Three 2-simplicial complexes with their Betti numbers for simplicial homology.*

Example 14.1.2. Consider the simplicial complex Δ^2 of Figure 14.1A: the triangles, edges and vertices of an 'octahedron'. $M_2(\Delta^2)$ is shown below, with blanks for the zero entries:

$M_2(\Delta)$	ω	ab	ac	ad	ae	bc	be	bf	cd	cf	de	df	ef
abc	1	1	−1			1							
abe	−1	1			−1		1						
acd	1		1	−1					1				
ade	1			1	−1						1		
bcf	−1					1		−1		1			
bef	1						1	−1					1
cdf	−1								1	−1		1	
def	−1										1	−1	1

The second column lists the coefficients of a 2-cycle (row dependence) ω, confirming that the rank of this matroid is at most 7. With any one face removed, say $\{d, e, f\}$ in Figure 14.1B, it is easy to check that the remaining rows are independent. The rank of the original matroid is 7, with the disk of Figure 14.1B as a basis. Up to scalar multiplication, this 2-cycle ω is the unique circuit of the 2-cycle matroid.

Finally, the 'ring' of Figure 14.1C is also independent, of rank 5. In all cases we have not required the complete graph of all edges on the six vertices. The presence, or absence, of edges which do not appear in the faces will simply change the number of 'all zero' columns in the submatrix for the faces we are considering but they will not change the k-cycles or the matroid.

The rank of the simplicial k-cycle matroid can be investigated through the row space and the k-cycles, by the analog of the 'statics of rigidity'. To complete the vocabulary for these analyses, we define two more spaces related to the boundary operators. The space of *i-boundaries* $B_i(\Delta)$ is the image of ∂_{i+1}. That is, $\mathbf{c} \in B_i(\Delta)$ if there is some chain $\mathbf{d} \in C_{i+1}(\Delta)$ with $\partial_{i+1}(\mathbf{d}) = \mathbf{c}$. Since $\partial_i\partial_{i+1} = 0$, i-boundaries are i-cycles and $B_i(\Delta) \subseteq Z_i(\Delta)$. The *$i$-homology* of Δ is the space: $\widetilde{H}_i(\Delta) = Z_i(\Delta)/B_i(\Delta)$, and the *reduced Betti number* is $\beta_i(\Delta) = \dim(\widetilde{H}_i(\Delta))$. We note that $B_k(\Delta^k) = \mathbf{0}$, so $\widetilde{H}_k(\Delta^k) = Z_k(\Delta^k)$. (Since we have an augmented complex, this is the *reduced homology* and we write $\widetilde{H}_i(\Delta)$ rather than $H_i(\Delta)$, although they are the same for $i > 0$.)

From our definition of the simplicial k-cycle matroid, we know that

$$\text{rank } \mathcal{M}_k(\Delta) = f_k(\Delta) - \dim(Z_k(\Delta)) = f_k(\Delta) - \beta_k(\Delta^k).$$

The following basic identity for the homology of a general chain complex will lead to other computations for the rank.

THEOREM 14.1.3 [**Mu**]. *For any chain complex $\mathcal{C}$:*

$$\mathcal{C}: \ \mathbf{0} \to C_r \xrightarrow{\partial_r} C_{(r-1)} \xrightarrow{\partial_{r-1}} C_{(r-2)} \xrightarrow{\partial_{r-2}} \cdots \qquad \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow \mathbf{0}.$$

the following identity defines the Euler characteristic of the chain complex:

$$\chi(\mathcal{C}) = \sum_{i=-1}^{r} (-1)^i \dim(C_i) = \sum_{i=-1}^{r} (-1)^i \beta_i(\mathcal{C}).$$

COROLLARY 14.1.4. *The rank of the k-cycle matroid $\mathcal{M}_k(\Delta)$ is:*

$$\text{rank } \mathcal{M}_k(\Delta) = f_k(\Delta) - \beta_k(\Delta^k) = \sum_{i=-1}^{k-1}(-1)^{k-1+i} f_i(\Delta) - \sum_{i=-1}^{k-1}(-1)^{k-1+i}\beta_i(\Delta).$$

PROOF. We observed above that rank $\mathcal{M}_k(\Delta) = f_k(\Delta) - \beta_k(\Delta^k)$. From the Euler characteristic:

$$\sum_{i=-1}^{k-1}(-1)^{k-1+i} f_i(\Delta) - \sum_{i=-1}^{k}(-1)^{k-1+i}\beta_i(\Delta^k) = f_k(\Delta) - \beta_k(\Delta^k) = \text{rank } \mathcal{M}_k(\Delta).$$

EXAMPLE 14.1.5. For any non-empty complex, $\partial_0 v_i = 1 \cdot [\emptyset]$, so $\beta_{-1}(\Delta) = 0$.

For any connected complex ($\Delta^{(1)}$ is a connected graph) $\beta_0(\Delta^k) = 0, k \geq 1$.

More generally, $\beta_0(\Delta^k)$ is the number of components of Δ^1, minus one. Therefore, for any non-empty connected graph $G = (V, E)$ and its simplicial complex,

$$\text{rank } \mathcal{M}_1(G) = |E| - \beta_1(G) = \Big[|V| - \beta_0(G)\Big] - \Big[1 - \beta_{-1}(G)\Big] = |V| - 1.$$

The 1-cycle matroid of $G = \Delta^1$ is the standard cycle matroid of the graph, with polygons as 1-cycles.

Notice that, if $f_{k+1}(\Delta) \neq 0$ then $\emptyset \neq B_k(\Delta) \subseteq Z_k(\Delta^k)$. Therefore $\Delta^{(k)}$ will not be simplicially k-independent.

EXAMPLE 14.1.6. If we return to the complexes of Figure 14.1, we find that the Betti numbers and Euler characteristic are:

A. The characteristic is $\chi = 8 - 12 + 6 - 1 = 1$ and $\beta_2 = 1$, $\beta_0 = 0$, $\beta_{-1} = 0$. We conclude that $\beta_1 = 0$ as well.
B. The characteristic is $\chi = 7 - 12 + 6 - 1 = 0$ and $\beta_2 = 0$, $\beta_0 = 0$, $\beta_{-1} = 0$ so $\beta_1 = 0$ as well. (The 1-cycle around the boundary of the disc is the 1-boundary of the chain sum of all triangles.)
C. The characteristic is $\chi = 5 - 11 + 6 - 1 = -1$ and $\beta_2 = 0$, $\beta_0 = 0$, $\beta_{-1} = 0$. We conclude that $\beta_1 = 1$, representing the 1-cycle around the hole, which is not the boundary of a 2-chain.

REMARK 14.1.7. Assume $\Delta(n)$ is the complete simplicial complex on $n \geq k+1$ vertices (i.e. $\Delta(n) = 2^n$), then $\beta_i((\Delta(n))^k) = 0$ for all $i \leq k-1$ [**Mu**]. This gives

$$\text{rank } \mathcal{M}_k(\Delta(n)) = \sum_{i=-1}^{k-1}(-1)^{k-1+i} f_i(\Delta(n)) = \binom{n-1}{k},$$

since $f_i(\Delta(n)) = \binom{n}{i+1}$ and $\sum_{i=-1}^{k-1}(-1)^{k-1+i}\binom{n}{i+1} = \binom{n-1}{k}$.

A direct analysis of the complete simplicial complex also gives the rank $\binom{n-1}{k}$ [**CL**]. The set of simplices $X_a = \{\sigma \in (\Delta(n))^k \mid a \in \sigma\}$ for a fixed $a \in V$ forms a basis for $\mathcal{M}_k(\Delta(n))$, with cardinality $\binom{n-1}{k}$.

In [**CL**,**CR1**], all calculations for sets X of k-simplices are done within this complete simplicial complex on n points. Specifically, for a subset $X \subseteq \Delta^{(k)}$, all lower chains are indexed by the complete complex $\Delta^{k-1}(n)$, on the n vertices, forming a subcomplex $X(n)$. With this definition, $\beta_i(X(n)) = 0$ for $i < k-1$, since these calculations are unchanged from the complete simplicial complex. Since $\sum_{i=-1}^{k-1}(-1)^{k-1+i}f_i(X(\Delta(n))) = \binom{n-1}{k}$, we have

$$\text{rank } \mathcal{M}_k(X(n)) = |X| - \beta_k(X(n)) = \binom{n-1}{k} - \beta_{k-1}(X(n)).$$

In [**CR1**], these two equivalent calculations are the definition of the rank function of the i-cycle matroid.

Of course, the rank of the matroid $\mathcal{M}_k(\Delta)$ is unchanged by adding additional lower simplices (and their subsets), to Δ^{k-1}. (The row rank of the cycle matrix is unchanged by adding additional zero columns.) This does change both the counts $f_i(\Delta)$ for lower faces and the lower Betti numbers $\beta_i(\Delta)$. Because we will use the details of the lower homologies and the f_i to calculate (or estimate) the rank of $\mathcal{M}_k(\Delta)$, we examine the set $X \subseteq \Delta^{(k)}$ through the the subcomplex $\langle\langle X\rangle\rangle = \{Y|\ Y \subseteq X_i, X_i \in X\}$. For any set $X \subseteq \Delta^{(k)}$ we now have:

$$\text{rank}(X) = |X| - \beta_k\langle\langle X\rangle\rangle = \sum_{i=-1}^{k-1}(-1)^{k-1+i}f_i\langle\langle X\rangle\rangle - \sum_{i=-1}^{k-1+i}(-1)^{k+i}\beta_i\langle\langle X\rangle\rangle.$$

A set $X \subseteq \Delta^{(k)}$ is *k-adequate* if $\beta_i = 0, i < k-1$, in which case:

$$\text{rank}(X) = |X| - \beta_k\langle\langle X\rangle\rangle = \sum_{i=-1}^{k-1}(-1)^{k-1+i}f_i\langle\langle X\rangle\rangle - \ \beta_{k-1}\langle\langle X\rangle\rangle.$$

For these k-adequate sets, the rank calculations and other related constructions will be most direct.

Example 14.1.8. Given a simplicial complex Δ and a new vertex a, the *complete cone with apex a* is the complex $a * \Delta = \{\psi \bigm| \psi \in \Delta \text{ or } \ \psi = \{a\} \cup \pi, \pi \in \Delta\}$. A standard result of topology gives $\beta_i(a * \Delta) = 0$ for all i [**Mu**]. For $i < k$, $\beta_i((a*\Delta)^{k-1}) = \beta_i(a*\Delta) = 0$, so cones will be k-adequate, for all k. Any complete cone $a * \Delta^{k-1}$ is k-independent in $\mathcal{M}_k$. (Note that the explicit basis given above for $\Delta(n)$ was a cone $a * \Delta^{k-1}$.)

Example 14.1.9. The complexes of Figure 14.1A,B have rank $\mathcal{M}_2(\Delta) = 7 = |E| - |V| + 1$. For any triangulated sphere Δ^2, we also have $\beta_1 = 0$ (this is a consequence of 'simply connected' topology), so rank $\mathcal{M}_2(\Delta^2) = |E| - |V| + 1$. This estimate will apply to all connected simplicial complexes with $\beta_1(\Delta) = 0$.

For other complexes the estimate $|E| - |V| + 1$ for the rank may be either too high or too low.

1. For the complex of Figure 14.1C, we have

$$\text{rank } \mathcal{M}_2(\Delta^2) = 5 = 11 - 6 < |E| - |V| + 1.$$

   This change from $|E| - |V| + 1$ occurs because $\beta_2(\Delta) = 0$, $\beta_0 = 0$, $\beta_{-1} = 0$ but $\beta_1(\Delta) = 1$. A generating 1-cycle which is not a 1-boundary is the polygon around the interior hole.
2. If we take two triangulated spheres sharing no vertices, we have: $\beta_1(\Delta) = 0$, $\beta_0(\Delta) = 1$ (two connected components), and $\beta_{-1}(\Delta) = 0$ so rank $\mathcal{M}_2(\Delta) = |E| - |V| + 1 + 1 > |E| - |V| + 1$.
3. For a triangulated torus Δ, with $\beta_1(\Delta) = 2$, $\beta_0(\Delta) = 0$, we have:
   rank $\mathcal{M}_2(\Delta) = |E| - |V| + 1 - 2 < |E| - |V| + 1$.

As these examples illustrate, it is not possible to test independence in the simplicial 2-cycle matroid by a simple inequality on our 'estimate' $|E|-|V|+1$, as we tested the 1-cycle matroid with the estimate $|E| \leq |V|-1$.

REMARK 14.1.10. We have not found literature on the computational complexity of $\beta_k(\Delta)$ (and therefore the rank of the simplicial k-cycle matroid) of a simplicial complex. The simplicial k-cycle matrix certainly gives a deterministic algorithm, of order $O([f_k(\Delta)]^2)$. However this approach will jump to an exponential algorithm in the more general 'generic geometric' extensions in the next two section. One reason why characterizations in terms of graphs and their related 'counts' were so useful for plane rigidity is that these characterizations yield efficient polynomial time combinatorial algorithms for the geometric extension to plane frameworks.

14.2. Cohomology, kinematics and gluing. There is an alternate analysis of rank $\mathcal{M}_k(\Delta)$, using the column space and the solution space of the simplicial k-cycle matrix. This is the direct analog of the kinematics of rigidity, with their infinitesimal motions and trivial motions. As we shall see, these are the traditional cocycles and coboundaries of the cochain complex on Δ [**Mu**]. We present a simplified form, appropriate to work over a field. We implicitly draw on special cases of the work in [**TWW1**,**2**,**3**] which is described further in §16.

The k-cochains $C^k(\Delta)$ are also formal sums of simplices in $\Delta^{(k)}$. In principle, the k-cochains lie in the dual space $(\mathbb{R}^{f_k})^*$. In practice, these are isomorphic to the k-chains. For each $\rho \in \Delta^{(i)}$, the i^{th} *coboundary operator* δ_i is defined by:

$$\delta_i[\rho] = \sum_{\sigma \in \Delta^{(i+1)}} Sign(\rho, \sigma) \cdot [\sigma].$$

extended linearly to all i-cochains. The *cochain complex* is then

$$\begin{aligned}\mathcal{C}^k(\Delta^k):\ \mathbf{0} \xleftarrow{\delta_k} C^k(\Delta) \xleftarrow{\delta_{k-1}} C^{k-1}(\Delta) \xleftarrow{\delta_{k-2}} \cdots \\ \cdots \xleftarrow{\delta_1} C^1(\Delta) \xleftarrow{\delta_0} C^0(\Delta^) \xleftarrow{\delta_{-1}} C^{-1}(\Delta) \leftarrow \mathbf{0}.\end{aligned}$$

We verify that our explicit definition coincides with the usual implicit definition [**Mu**] and, implicitly, that $\delta_i\delta_{i-1} = 0$. Given an i-chain $\mathbf{c}$ and an i-cochain $\mathbf{d}$, we write the bilinear form as $\langle \mathbf{c}, \mathbf{d}\rangle = \sum_{\sigma\in\Delta^{(i)}} c_\sigma \cdot d_\sigma$.

LEMMA 14.2.1. *For all chains $\mathbf{c} \in C_i$ and cochains $\mathbf{d} \in C^{i-1}$:*

$$\langle\ \partial_i \mathbf{c}, \mathbf{d}\rangle\ =\ \langle \mathbf{c}, \delta_{i-1}\mathbf{d}\rangle.$$

PROOF. By the linearity of ∂_i and δ_{i-1} and the bilinearity of $\langle\ ,\ \rangle$, it suffices to check this for a general $\sigma \in \Delta^{(i)}$ and $\rho \in \Delta^{(i-1)}$

$$\begin{aligned}\langle\ \partial_i[\sigma], [\rho]\ \rangle\ &=\ \Big\langle \sum_{\pi\in\Delta^{(i-1)}} Sign(\pi,\sigma)\cdot[\pi],\ [\rho]\ \Big\rangle\ =\ Sign(\rho,\sigma) \\ &=\ \Big\langle\ [\sigma], \sum_{\psi\in\Delta^{(i)}} Sign(\rho,\psi)\cdot[\psi]\ \Big\rangle =\ \langle\ [\sigma], \delta_{i-1}[\rho]\ \rangle.\end{aligned}$$

For a simplicial complex Δ, the kernel of δ_i is the *i-cocycles*, $Z^i(\Delta)$, and the image of δ_{i-1} is the *i-coboundaries*, $B^i(\Delta)$. Since $\delta_i\delta_{i-1} = 0$, $B^i(\Delta) \subseteq Z^i(\Delta)$

and we have the *cohomology spaces* of the augmented cochain complex, $\widetilde{H}^i(\Delta^k) = Z^i(\Delta^k)/B^i(\Delta^k)$, and the corresponding *Betti numbers* $\beta^i(\Delta^k) = \dim \widetilde{H}^i(\Delta^k)$.

As explicit equations, the conditions for an i-cocycle have the form:

$$\delta \mathbf{d} = \sum_{\rho \in \Delta^{(i-1)}} d_\rho \cdot [\rho] = \sum_{\sigma \in \Delta^{(i)}} \left(\sum_{\rho \in \Delta^{(i-1)}} Sign(\rho, \sigma) d_\rho \right) \cdot [\sigma] = 0.$$

These i-cocycles are the solutions to the system of equations: $M_i(\Delta)\mathbf{x} = \mathbf{0}$, where $M_i(\Delta)$ is the i-cycle matrix of the previous section, with i-boundaries as rows, the $(i+1)$-cycles as row dependencies, the $(i+1)$-coboundaries as columns and the $(i+1)$-cocycles as column dependencies. The *i-cocycle matrix* with the i-cocycles as row dependencies is then $M^i = (M_{i+1})^t$

Example 14.2.2. Consider again the cell complex Δ^1 of (Figure 14.1A,B), the edges and vertices of an 'octahedron' . The matrix with the 1-coboundaries as rows, the 0-boundaries as columns, the 0-cocycles as row dependencies and the 1-cycles as column dependencies is $M_1(\Delta)^t = M^0(\Delta)$:

	λ	ab	ac	ad	ae	bc	be	bf	cd	cf	de	df	ef
a	1	1	1	1	1								
b	1	-1				1	1	1					
c	1		-1			-1			1	1			
d	1			-1					-1		1	1	
e	1				-1		-1				-1		1
f	1							-1		-1		-1	-1

The column λ is a sample 0-cocycle (which is unique up to choice of a scalar). In the language of 'kinematics of graphs on a line', this 0-cocycle is the unique (up to scalar multiplication) trivial motion – a translation. In this example, we can also see that the 1-coboundaries are all orthogonal to the rows of the 2-cycle matrix for Δ of Example 14.1.2 (the 1-boundaries of Δ).

This orthogonality of i-boundaries and i-coboundaries is a special case of the more central orthogonality of the next lemma. This orthogonality gives explicit form to our analogy of cohomology and kinematics and can lead to a general matroidal orthogonality (§14.3).

Lemma 14.2.3. *For any simplicial complex Δ,*

$$Z_i(\Delta) = (B^i(\Delta))^\perp \text{ and } Z^i(\Delta) = (B_i(\Delta))^\perp,$$

where $^\perp$ denotes perpendicular subspace in the sense of $\langle\ ,\ \rangle$.

Proof. For any i-chain $\mathbf{c}$, we have:

$$\begin{aligned} \mathbf{c} \in Z_i(\Delta) &\iff \partial \mathbf{c} = 0 \iff \forall \rho \in \Delta^{i-1} \langle\ \partial_i \mathbf{c}, [\rho]\ \rangle = 0 \\ &\iff \forall \rho \in \Delta^{i-1} \langle\ \mathbf{c}, \delta_{i-1}[\rho]\ \rangle = 0 \iff \mathbf{c} \in (B^i(\Delta))^\perp. \end{aligned}$$

An identical argument works for $Z^i(\Delta) = (B_i(\Delta))^\perp$.

Corollary 14.2.4 [**Mu**]. *For any finite simplicial complex Δ, working over a field F, the homologies and cohomologies are isomorphic and $\beta^i(\Delta^k) = \beta_i(\Delta^k)$.*

We can use this orthogonality to complete the analogy with the kinematics of frameworks. The $(k-1)$-cocycles, $Z^{k-1}(\Delta)$, are the *k-motions* of a complex Δ (solutions to the equations $M_k(\Delta)\mathbf{x} = \mathbf{0}$). In the same spirit, the *trivial k-motions* will be the motions of a complete complex on the vertices, restricted to the $(k-1)$-simplices of Δ – that is the $(k-1)$-coboundaries of the complete complex $\Delta(n)$ on these vertices. For any $(k-2)$-simplex $\pi \in \Delta(n)^{(k-2)}$ and $(k-1)$-simplex $\sigma \in \langle\langle X\rangle\rangle^{(k-1)}$, we have $\langle\ [\rho], \delta_{k-2}[\pi]\ \rangle \neq 0$ if and only if $\pi \subseteq \rho$. Therefore, in $\langle\langle X\rangle\rangle$, $\delta_{k-2}[\rho] \neq \mathbf{0}$ if and only if $\rho \in \langle\langle X\rangle\rangle^{(k-2)}$. The trivial k-motions of $\langle\langle X\rangle\rangle$ are the $(k-1)$-coboundaries $B^{k-1}\langle\langle X\rangle\rangle$.

Completing this analogy to kinematics, a set X of k-simplices will be *simplicially k-rigid* if and only if all k-motions are k-trivial motions. That is, the set X is simplicially k-rigid if $Z^{k-1}\langle\langle X\rangle\rangle = B^{k-1}\langle\langle X\rangle\rangle$, or equivalently, if $\beta^{k-1}\langle\langle X\rangle\rangle = \beta_{k-1}\langle\langle X\rangle\rangle = 0$.

In the earlier sections, a simple but critical task was to determine (count) the exact dimension of the space of trivial motions. The importance of these estimates will be further clarified in the next sections and §A. Our previous calculations give one formula for this dimension.

Proposition 14.2.5. *For any set X of k-simplices, the space of trivial k-motions, $B^{k-1}\langle\langle X\rangle\rangle$, has dimension $T^k(X) = \sum_{i=-1}^{k-2}(-1)^{k-2+i}[f_i\langle\langle X\rangle\rangle - \beta^i\langle\langle X\rangle\rangle]$.*

Proof. In the complex $\langle\langle X\rangle\rangle^{k-1}$, all $k-1$ faces are cocycles and $\beta^{k-1}\langle\langle X\rangle\rangle^{k-1} = f_{k-1}\langle\langle X\rangle\rangle - \dim B^{k-1}\langle\langle X\rangle\rangle$. Also, by the Euler characteristic of this cochain complex:

$$\sum_{i=-1}^{k-1}(-1)^{k-1+i}[f_i\langle\langle X\rangle\rangle^{k-1} - \beta^i\langle\langle X\rangle\rangle^{k-1}] = 0.$$

Therefore,

$$T^k(X) = f_{k-1}\langle\langle X\rangle\rangle - \beta^{k-1}\langle\langle X\rangle\rangle^{k-1} = \sum_{i=-1}^{k-2}(-1)^{k-2+i}[f_i\langle\langle X\rangle\rangle - \beta^i\langle\langle X\rangle\rangle].$$

Remark 14.2.6. Our calculations for rank $\mathcal{M}_k\langle\langle X\rangle\rangle$ also yield a count:

$$|X| - \beta_k\langle\langle X\rangle\rangle = f_{k-1}\langle\langle X\rangle\rangle - \beta_{k-1}\langle\langle X\rangle\rangle - \Big(\sum_{i=-1}^{k-2}(-1)^{k-2+i}[f_i\langle\langle X\rangle\rangle - \beta_i\langle\langle X\rangle\rangle]\Big)$$

Therefore, the space of trivial motions has dimension:

$$\begin{aligned} T^k\langle\langle X\rangle\rangle &= f_{k-1}\langle\langle X\rangle\rangle - \text{rank } \mathcal{M}_k\langle\langle X\rangle\rangle - \beta_{k-1}\langle\langle X\rangle\rangle \\ &= \sum_{i=-1}^{k-2}(-1)^{k-2+i}[f_i\langle\langle X\rangle\rangle - \beta_i\langle\langle X\rangle\rangle]. \end{aligned}$$

Since $\beta_i\langle\langle X\rangle\rangle = \beta^i\langle\langle X\rangle\rangle$, these two calculations agree. Representative 'non-trivial motions' now correspond to $Z^{k-1}(\Delta)/B^{k-1}(\Delta) = \widetilde{H}^{k-1}(\Delta)$.

There is a corresponding 'static interpretation' of $\widetilde{H}_{k-1}\langle\langle X\rangle\rangle$. As for frameworks, the row space of $M_k\langle\langle X\rangle\rangle$ – the space $B_{k-1}\langle\langle X\rangle\rangle$ of $(k-1)$-boundaries – becomes the *resolvable k-loads*. The entire space of *equilibrium k-loads* is the orthogonal space to the trivial k-motions, that is, the space of $(k-1)$-cycles $Z_{k-1}\langle\langle X\rangle\rangle$. The space $\widetilde{H}_{k-1}(\Delta)$ is a space of *representative unresolved equilibrium loads* (the equilibrium loads modulo the resolved loads).

Completing the static analogy, a set X is *statically rigid* if all equilibrium k-loads are resolvable, that is, if $Z_{k-1}\langle\langle X\rangle\rangle = B_{k-1}\langle\langle X\rangle\rangle$ or $\beta_{k-1}\langle\langle X\rangle\rangle = 0$. As happens with such definitions for finite objects, since $\beta_{k-1}\langle\langle X\rangle\rangle = \beta^{k-1}\langle\langle X\rangle\rangle$ we have the immediate corollary [**TWW1**]:

Corollary 14.2.7. *For a set of k-simplices X, the following are equivalent:*
1. *the set X is simplicially k-rigid;*
2. *the set X is statically k-rigid;*
3. $\beta^{k-1}\langle\langle X\rangle\rangle = 0$*;*
4. $\beta_{k-1}\langle\langle X\rangle\rangle = 0$.

What is the possible closure of a set $X \subseteq \Delta^{(k)}$ in $\mathcal{M}_k(\Delta)$? Looking at the matrix, it is clear that the only possible simplices have their boundary in the chains of $\Delta^{(k-1)}$. We denote the set of all such simplices (the analog of $K_{V(E)}$ in rigidity) as $[\Delta(k-1)]^{(k)}$.

Cycle-Cocycle Test 14.2.8. *For any simplicial complex Δ, the following are equivalent:*
1. *the simplex $\rho \in [\Delta(k-1)]^{(k)}$ is not in the closure of X in $\mathcal{M}_k([\Delta(k-1)]^{(k)})$;*
2. *every k-cycle ω on $X \cup \{\rho\}$ is zero on ρ;*
3. *there is a $(k-1)$-cocycle $\mathbf{u}$ on $\langle\langle X\rangle\rangle$, such that $\langle \partial[\rho],\ \mathbf{u}\rangle \neq 0$.*

Proof. Essentially the same proofs used for first-order flexes in rigidity and C_r^{r-1}-flexes applies. The equivalence of 1 and 2 is the definition of the matroid.
(2. $\Rightarrow$ 3.) Assume ρ is not dependent in the matroid. Therefore, adding this row to the k-cycle matrix for X we increase the row dimension and reduce the nullity. If $\mathbf{u}$ is one of the solutions removed, we must have: $\langle \partial[\rho],\ \mathbf{u}\rangle \neq 0$.
(3. $\Rightarrow$ 2.) We use the contrapositive. Noting that the rows of the k-homology matrix are the boundaries, any k-cycle with $\omega_\rho \neq 0$ gives:

$$\omega_\rho[\rho] = \sum_{\sigma\in\Delta^{(k)}} \omega_\sigma[\sigma] \quad \Rightarrow \quad \partial[\rho] = -\frac{1}{\omega_\rho}\sum_{\sigma\in\Delta^{(k)}} \omega_\sigma \partial[\sigma]$$

and therefore, any $(k-1)$-cocycle $\mathbf{u}$ gives

$$\langle \partial[\rho],\ \mathbf{u}\rangle = -\frac{1}{\omega_\rho}\sum_{\sigma\in\Delta^{(k)}} \omega_\sigma \langle \partial[\sigma],\ \mathbf{u}\rangle = -\frac{1}{\omega_\rho}\sum_{\sigma\in\Delta^{(k)}} \omega_\sigma 0 = 0.$$

Example 14.2.9. If a complex has $\beta_i(\Delta) = 0$ for all $-1 \leq i$, it is called *acyclic*. If an acyclic complex Δ has dimension d (that is, the maximal simplices are d-simplices) then Δ^d is simplicially $(d+1)$-independent, simplicially $(d+1)$-rigid and $(d+1)$-adequate. Since the reduced Euler characteristic is $\sum_{i=-1}^{d}(-1)^i f_i(\Delta) = \sum_{i=-1}^{d}(-1)^i\beta_i(\Delta) = 0$, the dimension of the space of trivial simplicial d-motions is

$$T^d(\Delta) = \sum_{i=-1}^{d-2} (-1)^{d+i} f_i(\Delta) = f_{d-1}(\Delta) - f_d(\Delta).$$

For the $(d-1)$-simplices of an acyclic Δ^d, Δ^{d-1} is still d-adequate and d-rigid, but it is not d-independent. The reduced Euler characteristics of Δ^d and Δ^{d-1} give:

$$\beta_{d-1}(\Delta^{d-1}) = \sum_{i=-1}^{d-1} (-1)^{d-1+i} f_i(\Delta) = f_d(\Delta).$$

The $(d-1)$-cycle space is generated by the (independent) $(d-1)$-boundaries of the faces $f_d(\Delta)$.

For the k-simplices $k < d$ of an acyclic d-complex, Δ^k is simplicially $(k+1)$-adequate and $(k+1)$-rigid. The reduced Euler characteristics of Δ^d and Δ^k give

$$\beta_k(\Delta^k) = \sum_{i=-1}^{k} (-1)^{k+i} f_i(\Delta) = \sum_{i=k+1}^{d} (-1)^{k+i} f_i(\Delta).$$

For an acyclic Δ, $\beta_k(\Delta^k)$ must be positive for $k < d$, since there are k-cycles (boundaries of $(k+1)$-faces) but these boundaries are not used to compute the Betti number of Δ^k.

Example 14.2.10. If a set of k-simplices is not k-rigid, we may not find additional k-simplices to make it k-rigid unless we add new $k-1$ simplices (Figure 14.2). For example, consider the annulus of Figure 14.2A, with $\beta_1 = 1$. This annulus is 2-independent ($\beta_2 = 0$) and 2-adequate ($\beta_0 = 0$, $\beta_{-1} = 0$ for any non-empty connected complex) but there is no 'additional simplex' on these edges.

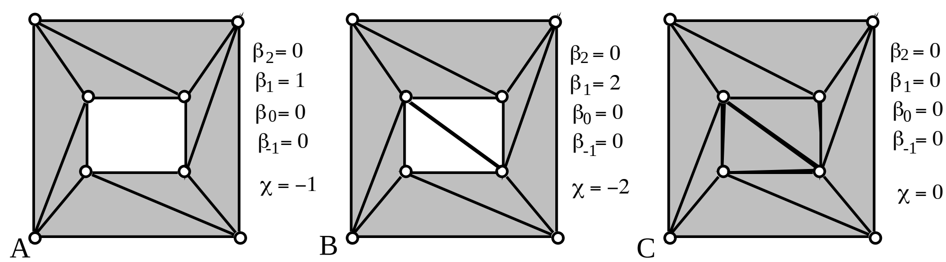

Fig. 14.2. *A non 2-rigid complex (A) with added edge (B) and faces(C) for simplicial 2-rigidity.*

This situation is very different from the rigidity of a plane framework, where we could always find 'missing edges' on the existing vertices which would produce rigidity. Effectively, all sets of vertices which span the plane are 'adequate' in that situation (see §15).

For the rigidity of frameworks, we computed the behaviour of $X \cup Y$ from the behaviour of X, Y and the form of $X \cap Y$, using Gluing Lemmas. In homology, a related set of connections are based on the Mayer-Vietoris sequence of homology theory [**Mu**]. We illustrate with a few simple properties.

k-Homology Gluing Lemma 14.2.11. *Assume X, Y are sets of k-simplices.*

1. *If X and Y are k-independent and $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ is simplicially k-rigid, then $X \cup Y$ is k-independent;*

2. $\text{rank}(X) + \text{rank}(Y) = \text{rank}(X \cup Y) + \text{rank}(X \cap Y)$ *if and only if* $\beta_{k-1}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) = 0$*;*
3. *If* $\langle\langle X\rangle\rangle$*,* $\langle\langle Y\rangle\rangle$ *and* $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ *are* k*-adequate then* $\langle\langle X \cup Y\rangle\rangle$ *is* k*-adequate;*
4. *If* X *and* Y *are simplicially* k*-rigid and* $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ *is* k*-adequate, then* $X \cup Y$ *is simplicially* k*-rigid;*
5. *If* X *and* $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ *are acyclic, then* $X \cup Y$ *and* Y *have equal Betti numbers and share any properties such as simplicial* k*-independence, simplicial* k*-rigidity, simplicial* k*-adequacy etc.*

Proof. For general chain subcomplexes Δ, Ψ, the Mayer-Vietoris sequence is an exact sequence of the form [**Mu**]:

$$\ldots \to \widetilde{H}_i(\Delta \cap \Psi) \to \widetilde{H}_i(\Delta) \oplus \widetilde{H}_i(\Psi) \to \widetilde{H}_i(\Delta \cup \Psi) \\ \to \widetilde{H}_{i-1}(\Delta \cap \Psi) \to \widetilde{H}_{i-1}(\Delta) \oplus \widetilde{H}_{i-1}(\Psi) \to \ldots$$

We set $\Delta = \langle\langle X\rangle\rangle$, $\Psi = \langle\langle Y\rangle\rangle$, and $\Delta \cup \Psi = \langle\langle X\rangle\rangle \cup \langle\langle Y\rangle\rangle = \langle\langle X \cup Y\rangle\rangle$. Although $(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle)^{(k)} = X \cap Y = \langle\langle X \cap Y\rangle\rangle^{(k)}$, for $i < k$ we may have $\langle\langle X \cap Y\rangle\rangle^{(i)} \subsetneq \langle\langle X\rangle\rangle^{(i)} \cap \langle\langle Y\rangle\rangle^{(i)} = \Delta \cap \Psi$. The exact sequence now takes the form:

$$\mathbf{0} \to \widetilde{H}_k(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) \to \widetilde{H}_k\langle\langle X\rangle\rangle \oplus \widetilde{H}_k\langle\langle Y\rangle\rangle \to \widetilde{H}_k\langle\langle X \cup Y\rangle\rangle \\ \to \widetilde{H}_{k-1}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) \to \ldots \\ \ldots \to \widetilde{H}_i(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) \to \widetilde{H}_i\langle\langle X\rangle\rangle \oplus \widetilde{H}_i\langle\langle Y\rangle\rangle \to \widetilde{H}_i\langle\langle X \cup Y\rangle\rangle \\ \to \widetilde{H}_{i-1}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) \to \widetilde{H}_{i-1}\langle\langle X\rangle\rangle \oplus \widetilde{H}_{i-1}\langle\langle Y\rangle\rangle \to \ldots$$

Case 1. Assume X and Y are k-independent. Then the top homologies are zero $\widetilde{H}_k\langle\langle X\rangle\rangle = \widetilde{H}_k\langle\langle Y\rangle\rangle = \widetilde{H}_k(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) = 0$. The top of this sequence includes the section:

$$\ldots \mathbf{0} \oplus \mathbf{0} \to \widetilde{H}_k\langle\langle X \cup Y\rangle\rangle \to \widetilde{H}_{k-1}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) \to \ldots$$

Clearly, if $\beta_{k-1}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) = 0$, then $\beta_k\langle\langle X \cup Y\rangle\rangle = 0$ and the $X \cup Y$ is simplicially k-independent.

Case 2. As above, the Mayer-Vietoris sequence gives:

$$\beta_k(X) + \beta_k(Y) = \beta_k(X \cup Y) + \beta_k(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle)$$

if and only if $\beta_{k-1}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) = 0$. The rest follows from $|X| + |Y| = |X \cap Y| + |X \cup Y|$.

Case 3. Assume $\langle\langle X\rangle\rangle$, $\langle\langle Y\rangle\rangle$ and $\langle\langle X \cap Y\rangle\rangle$ are k-adequate. For each $i < k-1$:

$$\ldots \mathbf{0} \oplus \mathbf{0} \to \widetilde{H}_i\langle\langle X \cup Y\rangle\rangle \to \mathbf{0} \ldots$$

This gives $\beta_i\langle\langle X \cup Y\rangle\rangle = 0, i < k-1$ as required.

Case 4. Assume $\langle\langle X\rangle\rangle$ and $\langle\langle Y\rangle\rangle$ are simplicially k-rigid and $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ is k-adequate. The second band of the Mayer-Vietoris sequence gives:

$$\ldots \mathbf{0} \oplus \mathbf{0} \to \widetilde{H}_{k-1}\langle\langle X \cup Y\rangle\rangle \to \widetilde{H}_{k-2}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) \to \ldots .$$

Since $\beta_{k-2}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle) = 0$ by the k-adequacy, we conclude that $X \cup Y$ is k-rigid.

Case 5. Assume $\langle\langle X\rangle\rangle$ and $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ are acyclic. The typical band of the Mayer-Vietoris sequence gives:

$$\ldots \mathbf{0} \to \mathbf{0} \oplus \widetilde{H}_i\langle\langle Y\rangle\rangle \to \widetilde{H}_i\langle\langle X \cup Y\rangle\rangle \to \mathbf{0} \ldots .$$

This gives the desired isomorphism of homologies. Since all the listed properties are defined from the Betti numbers, they coincide for $\langle\langle Y\rangle\rangle$ and $\langle\langle X\rangle\rangle \cup \langle\langle Y\rangle\rangle$.

Figure 14.3 illustrates that, with acyclic complexes glued across inadequate intersections, we can create and track non-zero lower homologies with the Mayer-Vietoris sequence.

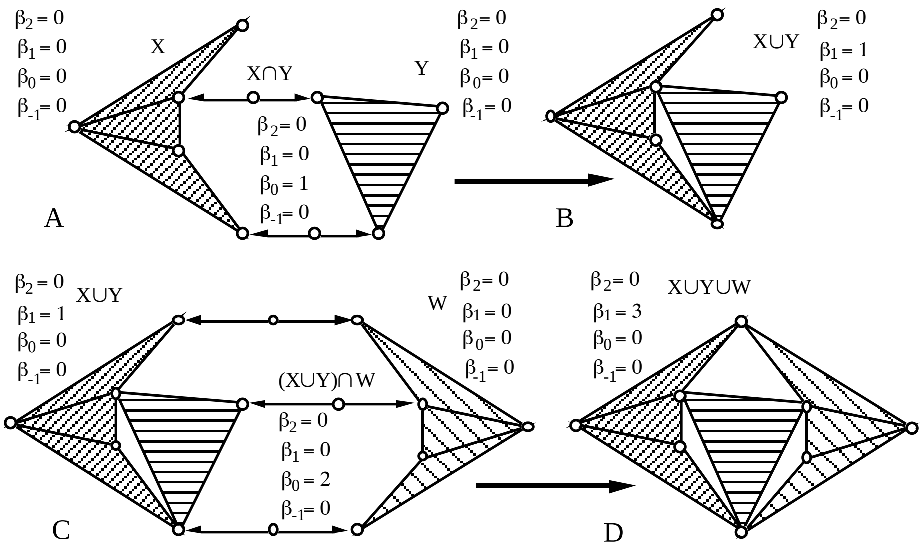

Fig. 14.3. *Gluing together three* 2*-acyclic complexes* X*,* Y *and* W*, creating complexes which are not simplicially* 2*-rigid.*

Example 14.2.12. In topology, suspensions 'transfer' Betti numbers up one dimension. Specifically, given a complex Δ the *suspension* is the union of two cones $(a * \Delta) \cup (b * \Delta)$ with $a \neq b$. Since the intersection is $(a * \Delta) \cap (b * \Delta) = \Delta$ and the two cones have $\beta_i(a*\Delta) = \beta_i(b*\Delta) = 0$ for all i, we find that $\beta_i((a*\Delta)\cap(b*\Delta)) = \beta_{i-1}(\Delta)$ for all i. In particular, the k-cycles of $\big(a * (\Delta^{k-1})\big) \cap \big(b * (\Delta^{k-1})\big)$ are isomorphic to the $(k-1)$-cycles of Δ^{k-1}. The rank of the matroid will jump:

$$\text{rank } \mathcal{M}_k(a * (\Delta^{k-1}) \cap b * (\Delta^{k-1})) = \text{rank } \mathcal{M}_{k-1}(\Delta) + |\Delta^{(k-1)}|.$$

Remark 14.2.13. We see that the preferred complexes for our calculations are the *simplicially k-ample* complexes, with $\beta_i = 0, -1 \leq i \leq k-1$. (These are the complexes whose rank estimates were given in homology boxes of Figure 13.1.) There are standard complexes which are simplicially k-ample:

1. Any simplicial d-ball will have $\beta_i = 0$ for all i, in the reduced homology. This will be k-ample for all k.
2. More generally, any acyclic complex (Example 14.2.8) is k-ample for all k.
3. Any simplicial d-sphere is k-ample for all $k \leq d$. (A d-sphere has $\beta_i = 0$ for all $i < d$, and $\beta_d = 1$.) In the same manner as Example 14.2.8, we can also estimate the dimension of the k-cycle space of a d-sphere, using the fact that the reduced Euler characteristic is $(-1)^d$. This will give a k-cycle space

of dimension:

$$\sum_{i=k+1}^{d} (-1)^{k+1+i} f_i(\Delta) \quad + (-1)^{d-k}.$$

4. a Cohen-Macaulay complex Δ is homeomorphic to a wedge of d-spheres, with $\beta_i(\Delta) = 0, i \leq d-1$. This will also be k-ample for all $k \leq d$.

These Cohen-Macaulay complexes (which includes spheres and balls) have their own important place in the literature of partially ordered sets and algebraic combinatorics, in part because of properties of their face lattice and the algebraic form of related constructions such as their face ring [**St2**]. The combinatorics of these complexes will recur for k-skeletal rigidity (§16.2).

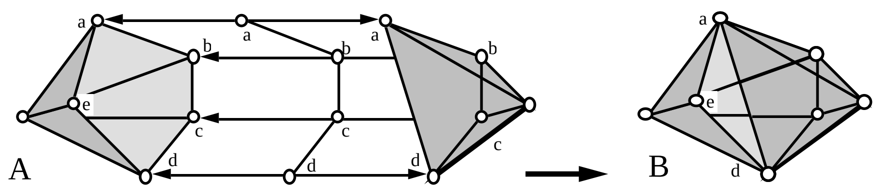

Fig. 14.4. 2-*homology gluing across a* 2-*adequate set.*

Remark 14.2.14. For the rigidity and cofactor matroids of Parts I and II, the Gluing Lemmas were directly connected to two defining properties of the abstract d-rigidity matroids. What would be the analogs here?

Recall that the 1-homology matroid was rigidity on the line. The reader can easily check that this is an abstract 1-rigidity matroid. Any connected graph is 1-rigid and any non-empty set of vertices is 1-adequate. Any non-empty connected graph is 1-ample. C5 asked that gluing two 1-rigid (and 1-ample) sets across any set with at least one vertex (i.e. a 1-adequate set) is 1-rigid (and 1-ample). Case 4 of Lemma 14.2.10 gives the k-rigidity analog of C5 and Cases 3 and 4 combine to give the k-ample analog for C5. (In §15 and §16, we will see that the assumed intersections of C5 really are the 'adequate' sets for these examples and that the rigid sets are also ample.)

For an analog of C6, we face real difficulties. Consider the 2-rigid complexes of Figure 14.4A. These are glued across the 2-ample strip *abcd*, to create a new 2-rigid complex of Figure 14.4B. Therefore the triangle *ade* is in the closure of the union, but not in the closure of either piece. However, if we simply add an new, unattached vertex f to both complexes their union is now 2-inadequate! The triangle *ade* is now a counterexample to the analog of C6 for k-inadequate intersections. Unlike the matroids of Parts I and II, our 'inadequate sets' are not closed under subsets! We are still looking for a good analog of C6 for k-homology matroids or the rigidity and cofactor matroids of §15 and §16.

Example 14.2.15. Figure 14.5 indicates three of the simplest 'Henneberg style extensions' which preserve the Betti numbers of 2-simplicial complexes by the Gluing Lemma. For extensions in homology, we could attach two complexes across any acyclic complex. In polyhedral combinatorics, the extensions of Figure 14.4A,B are two basic 'shelling' operations for 2-complexes (see below) – in which a new

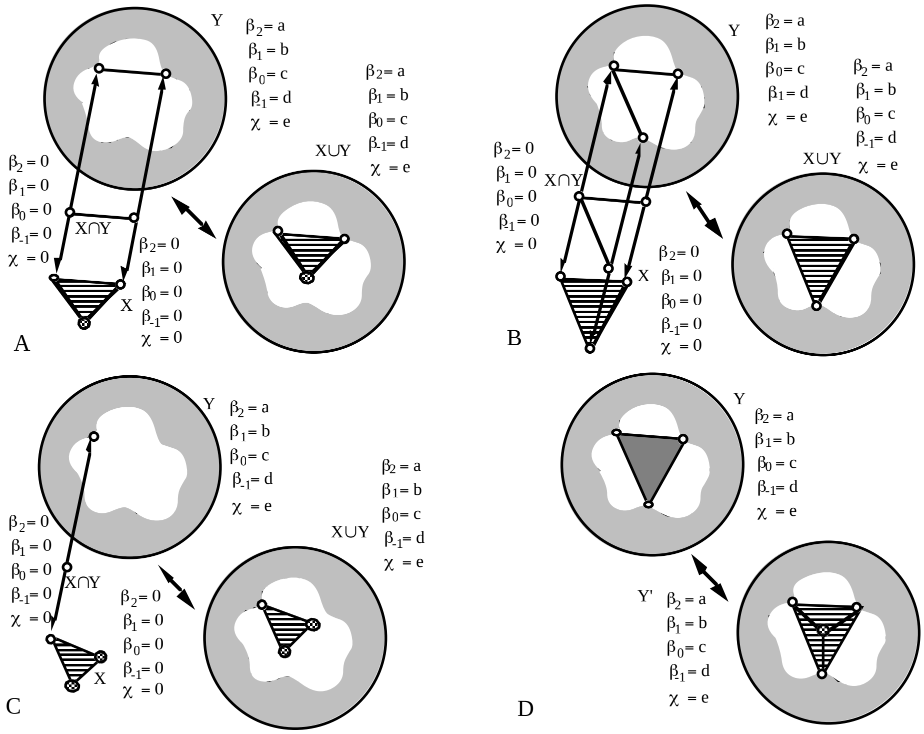

Fig. 14.5. *Standard extensions of a* 2*-complex by (A) vertex* 1*-addition: add one new vertex, two new edges and one triangle; (B) edge* 1*-addition: add a new edge and one new triangle; (C) face* 1*-addition: add two new vertices, three new edges and a new triangle; (D) face splits: replace a triangle by a subdivision into three triangles.*

2-simplex $\langle\langle X\rangle\rangle$ is attached to an existing complex $\langle\langle Y\rangle\rangle$ so that the common intersection $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ is a 1-disc in these examples. Since this makes $\langle\langle X\rangle\rangle$ and $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ acyclic, we conclude that $\beta_i(\langle\langle X\rangle\rangle \cup \langle\langle Y\rangle\rangle) = \beta_i\langle\langle Y\rangle\rangle$ for all i.

We also have a *triangle split* – in which we replace an existing triangle abc by a new vertex 0 and three triangles $0ab$, $0bc$ and $0ca$ (D). This is verified by adding 0 and $0ab$ (step A), then edge $0c$ and triangle $0bc$ (step B). Finally, we appeal to the small 2-cycle of all faces of the tetrahedron $0abc$. From this circuit, we can 'exchange' the triangle abc for the face $0ca$.

Remark 14.2.16. In the literature of combinatorics and homology, an often used technique is *shelling* [**Bj**]: an inductive construction of pure k-complexes by a sequence of 'gluing on' single n-faces. Explicitly, a *shelling order* is an order for the n-simplices of Δ such that, for each j, the intersection of $\langle\langle\sigma\rangle\rangle_{j+1}$ with $\Delta_j = \cup_{i\leq j}\langle\langle\sigma\rangle\rangle_i$ is the union of $(n-1)$-simplices, or equivalently [**Bj**]:

for each pair of facets σ_i, σ_j, $1 \leq i < j$ there is a facet $\sigma_k, k < j$ and a vertex $v \in \sigma_j$ such that $\sigma_i \cap \sigma_j \subset \sigma_k \cap \sigma_j = \sigma_j - v$.

Consider gluing 'up' the shelling order of a shellable n-complex Δ, for the k-cycle matroids. Since the original simplices are k-acyclic and all the intersection complexes $\sigma_{j+1} \cap \Delta_j$ are k-adequate, the shellable complex Δ^k is k-ample – with all Betti numbers zero, except possibly the top $\beta_k(\Delta)$. If $k = n$, the top Betti number will be the number of times $\sigma_{j+1} \cap \Delta_j$ is all $(n-1)$-simplices of σ_{j+1} (i.e. the intersection had $\beta_{n-1}(\sigma_{j+1} \cap \Delta_j) = 1$).

REMARK 14.2.17. In the traditional combinatorial approach to the k-simplicial matroid, in which $X(n)^{(k-1)}$ is the complete $(k-1)$-complex on the vertices, the Mayer-Vietoris sequence is singularly uninformative. While all lower homologies are 0 for $i < k-1$, the second layer of the exact sequence is:

$$\ldots \to \widetilde{H}_{k-1}(X(n) \cap Y(n)) \to \widetilde{H}_{k-1}X(n) \oplus \widetilde{H}_{k-1}Y(n) \to \widetilde{H}_{k-1}(X \cup Y)(n) \to \mathbf{0} \ldots$$

In general, with such a large set of $(k-1)$ faces and a small set of k faces, it is unlikely that any of the β_{k-1} are zero. The critical information contained in features such as $\beta_{k-1}(\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle)$ is lost in this overlay of 'irrelevant' faces.

In closing this subsection, we note several basic steps which recur in §15, §16:

1. a top boundary operator takes k-chains to $(k-1)$-chains, creating a k-cycle matrix and a matroid with these dependencies;
2. this top operation and matrix extend to a full augmented chain complex with the lower homologies and the Euler characteristic as tools for analysis as an analog to the 'statics of frameworks';
3. the dual (orthogonal) cohomology gives an equivalent cochain complex, with equal Betti numbers and Euler characteristic, and a matroid analysis analogous to the 'kinematics of frameworks';
4. the Mayer-Vietoris sequence provides an inductive 'gluing' process, in which ample and rigid complexes play a distinctive role;
5. particular combinatorial (and geometric) constructions (here, coning and suspension) play a special role in connecting the matroids of related complexes;
6. special topological structures (spheres, balls etc.) guarantee nice properties for the matroid and its analysis (ample, adequate etc.).

14.3. Orthogonality and cohomology. Previous work on simplicial matroids has connected the orthogonal matroid of a k-cycle matroid and some cycle matroid of the dual complex [**Cr2**,**CL**]. When and how this connection occurs is clarified through the cohomology.

Given a matroid $\mathcal{M}$ on the finite set A, the *orthogonal matroid* $\mathcal{M}^*$ on A is defined by [**NW**, Proposition 2.1.4]:

B is a basis of $\mathcal{M}^*$ if and only if $A - B$ is a basis of $\mathcal{M}$.

For matroids represented as the rows of a matrix, this orthogonality takes on a more specific form [**Cr**, 5.4.1]

If M and N are both matrices over the field F, with rows indexed by the set A, such that every column of M is orthogonal to every column of N, and such that the columns of M and N, taken together, have rank $|A|$, then the matroid of the rows of M and the rows of N are orthogonal.

Our observations for homology and cohomology contain this orthogonality. We will make this explicit in a more general setting. While other sections concentrate on the field $\mathbb{R}$, for orthogonality there has also been interest in the combinatorics for $F = Z/2$ [**We**]. This section is written for a general field.

We have constructed a matroid from the real (or rational) homology of a simplicial complex. In a number of geometric applications, we are interested in non-simplicial objects, such as the facets of general convex polytopes. These examples are special cases of the more general CW-complexes of algebraic topology [**Mu**]. They generate an homology theory over any field, with i-cycles, i-boundaries and

i-Betti numbers, and induce the *j-cycle matroid* on the j-faces of these C-W complexes (see, for example, [**Cr2**, §5.7]).

For polyhedra, we could induce this same j-cycle matroid by taking a simplicial subdivision of the original (say a barycentric subdivision) and defining a set of cells to be j-independent if and only if this subdivision is j-independent in the simplicial j-cycle matroid. (The Betti numbers of a complex are unchanged by subdivision, as they are topological invariants.) However, for the geometric applications and extensions, this subdivision would change the independence. In those contexts it will be essential to work directly with the non-simplicial faces of the initial complex (see [**CrW4**]). We briefly outline how this works, giving a description for the orthogonal matroids for selected j-cycle matroids.

For any abstract complex Δ, with i-faces $\Delta^{(i)}$ (possibly not simplices) and the largest faces in $\Delta^{(k)}$, we have the chain complex of linear maps over the field F:

$$\begin{aligned}\mathcal{C}(\Delta):\ \mathbf{0} \to \oplus_{\psi\in\Delta^{(k)}} F \xrightarrow{\partial_k} \cdots \\ \xrightarrow{\partial_{i+2}} \oplus_{\rho\in\Delta^{(i+1)}} F \xrightarrow{\partial_{i+1}} \oplus_{\sigma\in\Delta^{(i)}} F \xrightarrow{\partial_i} \oplus_{\pi\in\Delta^{(i-1)}} F \xrightarrow{\partial_{i-1}} \\ \cdots \xrightarrow{\partial_1} \oplus_{v\in\Delta^{(0)}} F \xrightarrow{\partial_0} F \xrightarrow{\partial_{-1}} \mathbf{0}\end{aligned}$$

with boundary operators on $\sigma \in \Delta^{(i)}$, $\partial_i[\sigma] = \sum_{\pi\in\Delta^{(i-1)}} \epsilon_{\pi,\sigma} \cdot [\pi]$, where $\epsilon_{\pi,\sigma} \in \{-1, 0, 1\}$ represents the sign describing the orientation of π as a subface of σ. As always, the boundary operators must satisfy $\partial_i\partial_{i+1} = 0$ for all i. For each j we define the *j-cycle matroid* $\mathcal{M}_j(\Delta)$ on $\Delta^{(j)}$, defined by taking the minimal supports of j-cycles $Z_j(\Delta)$ as the circuits: $\mathbf{c} \in Z_j(\Delta)$ if and only if $\partial_j\mathbf{c} = 0$.

We also have the corresponding cochain complex on Δ:

$$\begin{aligned}\mathcal{C}(\Delta):\ \mathbf{0} \xleftarrow{\delta_k} \oplus_{\psi\in\Delta^{(k)}} F \xleftarrow{\delta_{k-1}} \cdots \\ \xleftarrow{\delta_{i+1}} \oplus_{\rho\in\Delta^{(i+1)}} F \xleftarrow{\delta_i} \oplus_{\sigma\in\Delta^{(i)}} F \xleftarrow{\delta_{i-1}} \oplus_{\pi\in\Delta^{(i-1)}} F \xleftarrow{\delta_{i-2}} \\ \cdots \xleftarrow{\delta_1} \oplus_{v\in\Delta^{(0)}} F \xleftarrow{\delta_0} \oplus_{v\in\Delta^{(0)}} \xleftarrow{\delta_{-1}} F \leftarrow \mathbf{0}.\end{aligned}$$

with $\delta_{i+1}\delta_i = 0$ for all i. This gives the *j-cocycle matroid* $\mathcal{M}^j(\Delta)$ on $\Delta^{(j)}$, taking the minimal supports of j-cocycles $Z^j(\Delta)$ as circuits of the matroid.

With the bilinear form on j-chains $C_j = \oplus_{\rho\in\Delta^{(j)}} F$ and j-cochains C^j:

$$\left\langle \sum_{\rho\in\Delta^{(i)}} c_\rho \cdot [\rho]\ ,\ \sum_{\rho\in\Delta^{(i)}} d_\rho \cdot [\rho] \right\rangle = \sum_{\rho\in\Delta^{(i)}} c_\rho d_\rho,$$

we also have the condition (sometimes the definition of δ_j [**Mu**]) that, for all $\mathbf{c} \in C_{j+1}(\Delta)$ and $\mathbf{d} \in C^j(\Delta)$

$$\langle\ \partial_{j+1}\mathbf{c}\ ,\ \mathbf{d}\ \rangle = \langle\ \mathbf{c}\ ,\ \delta_j\mathbf{d}\ \rangle.$$

By the argument used in §14.2, $Z_j(\Delta) = (B^j(\Delta))^\perp$, where $B^j(\Delta)$ is the space of j-coboundaries, $\delta_{j-1}C^{j-1}$. The j-cycle matroid is represented by the rows of the matrix $M_j(\Delta)$ with the coboundaries $\delta[\pi], \pi \in \Delta^{(j-1)}$ as columns. The j-cocycle matroid can be represented by the rows of the matrix $M^j(\Delta)$, in which the columns are the j-boundaries of simplices in $\rho \in \Delta^{(j+1)}$.

Recall that for $\rho \in \Delta^{(j+1)}$ and $\pi \in \Delta^{(j-1)}$,

$$\langle\ \partial_{j+1}\rho,\ \delta_{j-1}\pi\ \rangle = \langle\ \partial_j\partial_{j+1}\rho,\ \pi\ \rangle = \langle \mathbf{0},\ \pi\ \rangle = 0$$

so the columns of $M_j(\Delta)$ and $M^j(\Delta)$ are mutually orthogonal, for *all* complexes. We have the initial condition for an orthogonality between these matroids.

When is rank $M_j(\Delta) + \text{rank } M^j(\Delta) = |\Delta^{(j)}| = f_j$? Using the observations that $\dim Z_j(\Delta) = \dim B_j(\Delta) + \beta_j(\Delta)$ and that $\dim B_j(\Delta) + \dim Z^j(\Delta) = f_j$, by the orthogonality:

$$\begin{aligned}\text{rank } M_j(\Delta) + \text{rank } M^j(\Delta) &= f_j - \dim Z_j(\Delta) + f_j - \dim Z^j(\Delta)\\ &= 2f_j - [\dim B_j(\Delta) + \beta_j(\Delta) + \dim Z^j(\Delta)] = f_j - \beta_j(\Delta).\end{aligned}$$

We conclude that:

COCYCLE ORTHOGONALITY 14.3.1. *For any simplicial complex Δ and any integer j, the j-cocycle matroid $\mathcal{M}^j(\Delta)$ is the orthogonal matroid of the j-cycle matroid $\mathcal{M}_j(\Delta)$ if and only if $\beta_j(\Delta) = 0$ (or equivalently, $\beta^j(\Delta) = 0$).*

This explains a wide range of orthogonal pairs of matroids coming from homology and cohomology.

EXAMPLE 14.3.2. For any homology k-sphere Δ, $\beta_j(\Delta) = 0$ for all $j < k$. Therefore, for $j < k$ we have an orthogonal pair of matroids $\mathcal{M}_j(\Delta)$ and $\mathcal{M}^j(\Delta)$.

EXAMPLE 14.3.3. For planar graphs G, the 'dual graph' G^* is the graph of the dual complex Δ^*, in which we write the 2-faces as 'vertices' and the edges as 'dual edges'. This turns the 1-cocycle matroid of the spherical complex Δ into the 1-cycle matroid of Δ^* (see below). This orthogonality is also implicit in the Euler characteristic: $|F| - |E| + |V| - 1 = 1 \quad \text{or} \quad |E| = (|V| - 1) + (|F| - 1)$.

REMARK 14.3.4. When $\beta_j(\Delta) = m > 0$, the j-cohomology is still a partial representation of the orthogonal matroid. We need only add m additional columns, the generators for the j-homology, so that the extended column space generates the j-cycle space. The row space of this extended matrix now represents the orthogonal matroid $M_j(\Delta)^*$.

When is the orthogonal matroid to an j-cycle matroid on Δ the j-cycle matroid for a 'dual complex' Δ^* [**Cr2**, pages 91–94]? The basic Poincaré duality of homology theory gives one answer [**Mu**]. The 'dual block decomposition of Δ' uses a barycentric subdivision of the original simplices (or cells), and then identifies 'dual blocks' (unions of these smaller simplices) for each of the original faces. This creates a polyhedral complex, which is probably not a simplicial complex. Without going into details, the proof of the Poincaré duality gives [**Mu**, page 379]:

COROLLARY 14.3.5. *If a cell complex Δ is a compact n-manifold, then the j-cocycle matroid of Δ is the $(n - j)$-cycle matroid of a dual block complex Δ^*.*

The duality construction holds for the broader class of 'homology k-manifolds' – a class which includes all k-manifolds. Putting these two pieces together, we get the general result:

MANIFOLD ORTHOGONALITY 14.3.6. *If Δ is an abstract CW-complex whose space is a compact k-manifold with $\beta_j(\Delta) = 0$, then the j-cycle matroid of Δ, $M_j(\Delta)$, and the $(k - j)$-cycle matroid of a dual block complex Δ^*, $M_{k-j}(\Delta^*)$, are orthogonal matroids.*

EXAMPLE 14.3.7. For the graph of a complex decomposing a 3-sphere Δ, the orthogonal matroid is the 1-cocycle matroid of Δ or, equivalently, the 2-cycle matroid on the dual complex Δ^* for the 3-sphere. In the same spirit, the 2-cycle matroid of the 3-sphere is the 1-cycle or graphic matroid of the dual Δ^*.

For a polyhedral decomposition Δ of a 4-sphere, the orthogonal of the 2-cycle matroid is the 2-cycle matroid on the dual decomposition Δ^*. This orthogonality is also reflected in the Euler characteristic for the 4-sphere:

$$f_4 - f_3 + f_2 - f_1 + f_0 - 1 = 1 \quad \text{or} \quad f_2 = \ (f_1 - f_0 + 1) \ + \ (f_3 - f_4 + 1).$$

This is a true analog for the graph and dual graph on the 2-sphere, an analogy which extends to the j-cycles in a $2j$-sphere.

REMARK 14.3.8. The complete simplicial complex $(K_n)^{(n-1)}$ is an $(n-2)$-sphere, with the dual complex K_n^* isomorphic to K_n. In the duality, $(i-1)$-simplices (i-sets) go to $(n-i-1)$-simplices ($(n-i)$-sets). Therefore, the $(i-1)$-cycle matroid, usually written $S_i^n[F]$, has the $(i-1)$-cocycle matroid as its orthogonal matroid. Since this is the $(n-i-1)$-cycle matroid of the dual, we have the well known result that $S_{n-i}^n[F]$ is the orthogonal matroid to $S_i^n[F]$ [**Cr2**,**CL**].

REMARK 14.3.9. We could choose a subcomplex Δ' of Δ and work with the relative homology $H_i(\Delta, \Delta')$ [**Mu**] and a corresponding *relative k-cycle matroid*. Most of the theory of Section 14 extends immediately, including these orthogonality results, now for 'relative homology n-manifolds' with $\beta_i(\Delta, \Delta') = 0$.

The use of relative homology is central to a careful theory for existing work with multivariate splines [**Bi**,**ASW**]. Relative homology is also basic to the theory of 'pinned frameworks', in which some vertices are fixed (with velocity zero).

15. Multivariate Cofactor Matroids

This section has roots in approximation theory and has applications back to approximation theory (more than one might expect). Nevertheless, the chain complex here is not an object which workers in that field would normally describe. The first change, carried out by Billera [**Bi**], converted the original problem (defining polynomials over full d-cells in d-space, with required continuity across hyperplanes), into the homology of a chain complex on the interior faces of the original complex. This used a short exact sequence of chain complexes and the related long exact sequence of homologies. This change is closely related to the 'smoothing cofactors' in multi-variate splines [**Bi**,**ChW**,**Wh10**]. It has the specific advantage of reducing the computations by one dimension, which is very useful for small dimensions such as $d = 2$, $d = 3$, where most of the work is concentrated.

We make two additional changes, as we did in §10, §11. We work with the entire complex, rather than the 'interior' faces of some higher complex (a relative homology). This is not essential, but will simplify the comparisons with the other chain complexes. Finally, we work with a general $(d-1)$-simplicial complex realized in d-space (line segments in the plane, triangles in 3-space), rather than the interior $(d-1)$-faces of a d-ball, as is usual in approximation theory. Beyond the obvious generality this provides, techniques such as projection [**ASW**] illustrate the use of more general complexes as intermediate stages in the analysis of the splines.

15.1. Trivariate C_1^0-cofactors. Consider an abstract oriented 2-simplicial chain complex Δ: oriented faces $F = \Delta^{(2)}$, oriented edges $E = \Delta^{(1)}$ and vertices

$V = \Delta^{(0)}$, with the usual boundary operators. Add a 3-configuration $\mathbf{p}$ in $\mathbb{R}^3$, which assigns points to the vertices $\mathbf{p}(v_i) = \mathbf{p}_i = (p_{i,1}, p_{i,2}, p_{i,3})$, assigns lines to the edges and assigns planes to the faces, with the condition that the points $\mathbf{p}_i$ for any 2-simplex actually span a plane in $\mathbb{R}^3$. Thus each oriented face $\sigma = (v_1, v_2, v_3)$, induces an equation for its plane:

$$l_\sigma(\mathbf{p}) = A^\sigma x + B^\sigma y + C^\sigma z + D^\sigma = \det \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & 1 \\ p_{2,1} & p_{2,2} & p_{2,3} & 1 \\ p_{3,1} & p_{3,2} & p_{3,3} & 1 \\ x & y & z & 1 \end{bmatrix} = 0.$$

A 3-*stress* on $\Delta(\mathbf{p})$ is an assignment of scalars ω_σ to the faces such that for each edge $\rho \in E$:

$$\sum_{\sigma \in F} Sign(\rho, \sigma)\omega_\sigma l_\sigma(\mathbf{p}) = \sum_{\sigma \in F} Sign(\rho, \sigma)\omega_\sigma (A^\sigma x + B^\sigma y + C^\sigma z + D^\sigma) \equiv 0.$$

This is a row dependence of an $|F|$-by-$4|E|$ matrix (with four columns for the entries $x, y, z, 1$ under each ρ). These 3-stresses define the dependencies of the C^0_1-*cofactor matroid on* $\Delta(\mathbf{p})$, $\mathcal{C}^0_1(\Delta; \mathbf{p})$.

The way we have written the 3-stress conditions is misleading. It suggests there are four equations (for $x, y, z, 1$), but these actually reduce to two equations for each edge. All of the planes share the common line of this edge and two of the four conditions are redundant. For example, if the line is $x = 0, y = 0$, then all planes through the line have a linear form:

$$l_\sigma(\mathbf{p}) = (A^\sigma x + B^\sigma y + C^\sigma z + D^\sigma) \equiv (A^\sigma x + B^\sigma y)$$

and we just need the first two entries under this edge. A similar 'reduction' occurs for each edge. Any plane through this line is a linear combination of two distinct planes through the line. If all edges are in general position relative to the coordinate axes, the coefficients of this linear combination can be (uniquely and linearly) computed using the first two coordinates of the new plane and the coordinates of the two points on the line. Any linear combination of the rows which is verified for these first two columns will also work for the remaining (last two) columns.

There is a more abstract, less arbitrary way to describe this. We treat every entry in the columns of ρ as an element of the 2-dimensional subspace of planes through the line : the linear part $I^1_\rho(\mathbf{p})$ of the ideal $I_\rho(\mathbf{p}) = \{f \mid f(\mathbf{p}_i) = 0, i \in \rho\}$ [**Bi**]. This is a vector space of dimension 2 over $\mathbb{R}$. In this spirit (either pattern) we have the matrix equation for the 3-stresses:

$$\omega M^0_1(\Delta^2; \mathbf{p}) = [\ldots \quad \omega_\sigma \quad \ldots] \begin{bmatrix} & \vdots & \\ \ldots & Sign(\phi, \sigma)\sigma(\mathbf{p})|_{I(\rho)} & \ldots \\ & \vdots & \end{bmatrix} = \mathbf{0}$$

with the $|F|$-by-$2|E|$ matrix $M^0_1(\Delta^2; \mathbf{p})$.

This matrix defines a clear matroid $\mathcal{C}^0_1(\Delta^2; \mathbf{p})$ on the rows, with its rank, independence etc. From the work on rigidity, we have learned the value of working on the 'other side of such a matrix' and of exploring the lower homologies and cohomologies of the new chain complex. The complete chain complex is:

$$\mathcal{K}^0_1(\Delta^2; \mathbf{p}) : \mathbf{0} \to \oplus_{\sigma \in F} I^1_\sigma(\mathbf{p}) \xrightarrow{\partial_2} \oplus_{\rho \in E} I^1_\rho(\mathbf{p}) \xrightarrow{\partial_1} \oplus_{v \in V} I^1_v(\mathbf{p}) \xrightarrow{\partial_0} I^1(\emptyset) \xrightarrow{\partial_{-1}} \mathbf{0},$$

where for any $\psi \in \Delta^2$, $I^1_\psi(\mathbf{p}) = \{f \mid f(\mathbf{p}_i) = 0, i \in \rho,\ f \text{ of degree } \le 1\}$ and for each $\psi \in \Delta^{(i)}$:

$$\partial_i(c_\psi \cdot [\psi]) = c_\psi \partial_i \cdot [\psi] = \sum_{\theta \in \Delta^{(i-1)}} Sign(\theta, \psi) c_\psi \cdot [\theta].$$

As usual, the boundary operator is extended linearly to all i-chains. This is well-defined since, if $Sign(\tau, \psi) \neq 0$, then $I^1_\psi(\mathbf{p}) \subset I^1_\tau(\mathbf{p})$ and c_ψ is well defined in I^1_τ.

What is the Euler characteristic of this chain complex? Since $I^1_\sigma(\mathbf{p}) = \{cl_\sigma(\mathbf{p}) \mid c \in \mathbb{R}\}$, $\dim C_2(\mathcal{K}^0_1(\Delta; \mathbf{p})) = |F|$. The dimension of $C_1(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 2|E|$, the dimension of $C_0(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 3|V|$, since the space of polynomials zero at a point (say the origin) has dimension 3, and the dimension of $C_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 4$, since all polynomials are zero on the empty set. The Euler characteristic is:

$$\chi(\mathcal{K}^0_1(\Delta^2)) = |F| - 2|E| + 3|V| - 4 = \beta_2(\mathcal{K}) - \beta_1(\mathcal{K}^0_1) + \beta_0(\mathcal{K}^0_1) - \beta_{-1}(\mathcal{K}^0_1).$$

Using this Euler characteristic, the rank of the matroid is

$$\begin{aligned}\text{rank } \mathcal{C}^0_1(\Delta^2;\mathbf{p}) &= |F| - \beta_2(\mathcal{K}^0_1(\Delta^2; \mathbf{p})) \\ &= 2|E| - 3|V| + 4 - \beta_1(\mathcal{K}^0_1(\Delta; \mathbf{p})) + \beta_0(\mathcal{K}^0_1(\Delta; \mathbf{p})) - \beta_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})).\end{aligned}$$

There are simple geometric conditions for $\beta_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 0$ and $\beta_0(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 0$.

PROPOSITION 15.1.1. *For a 2-complex Δ, $\beta_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 0$ if and only if there are two distinct points $\mathbf{p}_i \neq \mathbf{p}_j$, $i, j \in V$.*

PROOF. For all Δ we have a fixed set of (-1)-cycles $C_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = P_1$, the polynomials of degree at most 1. When is $B_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = P_1$? Given one point $\mathbf{p}_i$, the boundaries of chains in $I^1_i(\mathbf{p})$ are the space of all linear forms, zero on $\mathbf{p}_i$. This generates 3 of the 4 dimensions needed. If we have a second distinct point $\mathbf{p}_j$, then the boundary of chains $I^1_j(\mathbf{p})$ will include a linear form zero on $\mathbf{p}_j$ but non-zero on $\mathbf{p}_i$, showing that $B_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = P_1$, and $\beta_{-1}(\mathcal{K}^0_1(\Delta)) = 0$.

Conversely, if $\Delta(\mathbf{p})$ includes only one point, then $B_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = I^1_i(\mathbf{p}) \neq P_1$, and $\beta_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 1$. (If $\Delta = \emptyset$, then $\beta_{-1}(\mathcal{K}^0_1(\Delta; \mathbf{p})) = 4$.)

PROPOSITION 15.1.2. *For the geometric complex $\Delta^2(\mathbf{p})$, $\beta_0(\mathcal{K}^0_1(\Delta^2; \mathbf{p})) = 0$ if and only if the geometric graph $\Delta^{(1)}(\mathbf{p})$ has only trivial parallel drawings in 3-space.*

PROOF. This is easiest to analyze using a matrix whose rows are the image of ∂_1 on the chains $c_{(i,j)} \cdot [i, j]$, with $c_{(i,j)} \in I^1_{[i,j]}(\mathbf{p})$. This space is spanned by two specific linear forms, say $l_{(i,j),1}(\mathbf{p}), l_{(i,j),2}(\mathbf{p})$, and the matrix has the form:

$$\begin{array}{c} \\ \vdots \\ (i,j)_1 \\ (i,j)_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} \dots & . & v_i & \dots & v_j & . & \dots \end{array} \\ \begin{pmatrix} \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots \\ \cdots & \mathbf{0} & l_{(i,j),1}(\mathbf{p}) & \cdots & -l_{(i,j),1}(\mathbf{p}) & \mathbf{0} & \cdots \\ \dots & \mathbf{0} & l_{(i,j),2}(\mathbf{p}) & \cdots & -l_{(i,j),2}(\mathbf{p}) & \mathbf{0} & \cdots \\ \ddots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

These entries, now in $I^1_i(\mathbf{p})$ and $I^1_j(\mathbf{p})$, can be written with the first three coefficients of the linear forms (the normals of two planes through $\mathbf{p}_i, \mathbf{p}_j$). We now recognize this as the matrix for parallel drawings of the geometric graph $\Delta^{(1)}(\mathbf{p})$.

This matrix has only the trivial parallel drawings if and only if the row space is all ‘equilibrium loads’ (all 0-cycles). We conclude that $\beta_0(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 0$ if and only if the geometric graph $\Delta^{(1)}(\mathbf{p})$ has only trivial parallel drawings in 3-space.

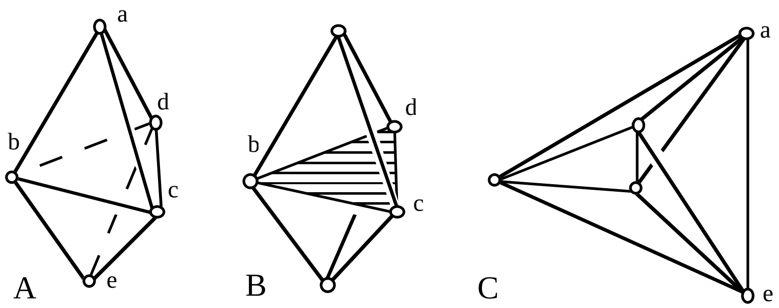

Fig. 15.1. *Some small 2-complexes for the matroid $\mathcal{C}_1^0$ in 3-space.*

Example 15.1.3. Consider the complexes of Figure 15.1. In Figure 15.1A, $|F| = 6, |E| = 9, |V| = 5$, we have $\chi(\mathcal{K}_1^0(\Delta^2;\mathbf{p})) = -1$. We also have $\beta_{-1} = 0$ and $\beta_0 = 0$, provided $\mathbf{p}$ spans at least a plane. If $\mathbf{p}$ affinely spans 3-space, then one of the edges, such as ab, has two distinct planes. A simple inspection shows that, in the 2-cycle equation, the two coefficients for these faces must be zero on both simplices. This zero coefficient will pass across all edges, giving all coefficients 0 and we conclude $\beta_2(\mathcal{K}_1^0(\Delta^2;\mathbf{p})) = 0$. Since $\beta_0(\mathcal{K}_1^0(\Delta^2;\mathbf{p})) = 0$ and $\beta_{-1}(\mathcal{K}_1^0(\Delta^2;\mathbf{p})) = 0$, the Euler characteristic gives $\beta_1(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 1$. (If all points in $\mathbf{p}$ are coplanar, but not collinear, then $\beta_2(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 1$, and $\beta_1(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 2$.)

In Figure 15.1B, we add the additional face bcd. This gives the characteristic $\chi(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 0$. If $\mathbf{p}$ is in general position (no four points coplanar) then $\beta_2(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 0$ (the coefficients of a C_1^0 dependence at an edge with exactly two non-coplanar faces must be zero) so $\beta_1(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 0$ as well. The geometric complex is C_1^0-*acyclic*.

Finally, in Figure 15.1C, we add the edge ab and three planes abc, abd, abe, giving the complete 2-complex on 5 points. The Euler characteristic is now $\chi(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 1$. We assume that the vertices are in general position, satisfying the unique affine dependence $\lambda_a\mathbf{a} + \lambda_b\mathbf{b} + \lambda_c\mathbf{c} + \lambda_d\mathbf{d} + \lambda_e\mathbf{e} = 0$. The 2-cycles are row dependencies of the matrix. The matrix below, with the $(\mathbf{p})$'s omitted for space, gives such an ω :

	ω	ab	ac	ad	ae	bc	bd	be	cd	ce	de
abc	$\lambda_a\lambda_b\lambda_c$	l_{abc}	$-l_{abc}$			l_{abc}					
abd	$\lambda_a\lambda_b\lambda_d$	l_{abd}		$-l_{abd}$			l_{abd}				
acd	$\lambda_a\lambda_c\lambda_d$		l_{acd}	$-l_{acd}$					l_{acd}		
bce	$\lambda_b\lambda_c\lambda_e$					l_{bce}		$-l_{bce}$		l_{bce}	
bde	$\lambda_b\lambda_d\lambda_e$						l_{bde}	$-l_{bde}$			l_{bde}
cde	$\lambda_c\lambda_d\lambda_e$								l_{cde}	$-l_{cde}$	l_{cde}
bcd	$\lambda_b\lambda_c\lambda_d$					l_{bcd}	$-l_{bcd}$		l_{bcd}		
abe	$\lambda_a\lambda_b\lambda_e$	l_{abe}			$-l_{abe}$			l_{abe}			
ace	$\lambda_a\lambda_c\lambda_e$		l_{ace}		$-l_{ace}$					l_{ace}	
ade	$\lambda_a\lambda_d\lambda_e$			l_{ade}	$-l_{ade}$						l_{ade}

At each edge we have three distinct planes - and at most one set of scalars for a 2-cycle. Therefore $\beta_2(\mathcal{K}_1^0(\Delta;\mathbf{p})) \leq 1$. Since $\beta_1(\mathcal{K}_1^0(\Delta;\mathbf{p})) \geq 0$, and $\beta_2(\mathcal{K}_1^0(\Delta;\mathbf{p})) - \beta_1(\mathcal{K}_1^0(\Delta,\mathbf{p})) = 1$, we conclude that $\beta_2(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 1$ and $\beta_1(\mathcal{K}_1^0(\Delta;\mathbf{p})) = 0$. The column under ω gives the actual coefficients of the 2-cycle, unique up to a single scalar multiple.

This 2-cycle in the matroid $\mathcal{C}_1^0(\Delta;\mathbf{p})$ is a typical projection of a 4-polytope. A general theorem of [**CW4**], which examines one source of this matroid in scene analysis, says that the projection of any 4-polytope (in fact of any oriented 3-manifold) which spans 4-space has $\beta_2(\mathcal{K}_1^0(\Delta^2;\mathbf{p})) \geq 1$.

In keeping with our analogies with rigidity and homology, we have the following terminology:

1. a set X of 2-simplices is $\mathcal{K}_1^0$-*independent* if $\beta_2(\mathcal{K}_1^0\langle\langle X\rangle\rangle) = 0$;
2. a set X of 2-simplices is $\mathcal{K}_1^0$-*rigid* if $\beta_1(\mathcal{K}_1^0\langle\langle X\rangle\rangle) = 0$;
3. a set X of 2-simplices is $\mathcal{K}_1^0$-*adequate* if $\beta_0(\mathcal{K}_1^0\langle\langle X\rangle\rangle) = \beta_{-1}(\mathcal{K}_1^0\langle\langle X\rangle\rangle) = 0$;
4. a set X of 2-simplices is $\mathcal{K}_1^0$-*acyclic* if $\beta_i(\mathcal{K}_1^0\langle\langle X\rangle\rangle) = 0$ for all i.

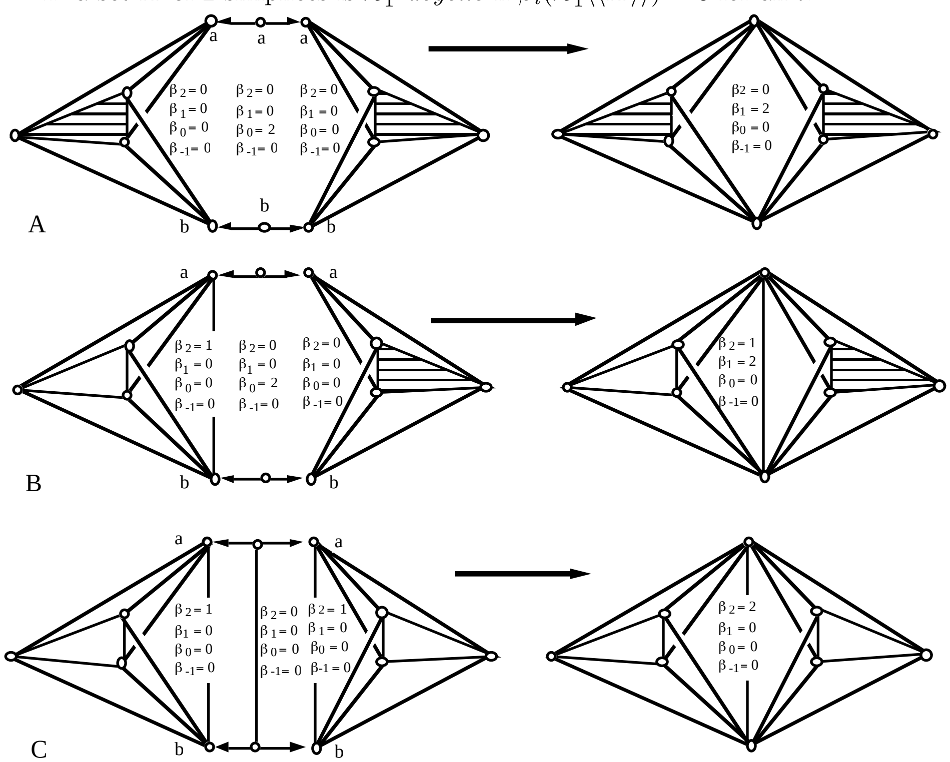

Fig. 15.2. *Gluing 2-complexes in the complex $\mathcal{K}_1^0$ in 3-space, with associated Betti numbers.*

Example 15.1.4. We can use the Mayer-Vietoris sequence for this homology to analyze the three sums of Figure 15.2 A,B,C. The pieces 'glued' are those of Figure 15.1B,C. The basic sequence in homology is:

$$\mathbf{0} \to H_2(\Delta\cap\Psi) \to H_2(\Delta)\oplus H_2(\Psi) \to H_2(\Delta\cup\Psi) \to H_1(\Delta\cap\Psi) \to H_1(\Delta)\oplus H_1(\Psi)$$
$$\to H_1(\Delta\cup\Psi) \to H_0(\Delta\cap\Psi) \to H_0(\Delta)\oplus H_0(\Psi) \to H_0(\Delta)\cup H_0(\Psi)\dots \quad .$$

The resulting β_i are indicated in the figures.

Example 15.1.5. The standard results of homology on cones do *not* extend directly. Observe that the complex X of Figure 15.1C is a cone $(w * K_4)^2$, but has $\beta_2 = 1 \neq 0$ and $\beta_1 = \beta_0 = \beta_{-1} = 0$. What we can observe in this example is that this cone $(w * K_4)^2$ has the same Betti numbers for $\mathcal{K}_1^0$ as the original tetrahedron $(K_4)^2$ has for simplicial homology (see Conjecture 15.3.2).

We offer one further application of the gluing principle to this complex.

Acyclic Gluing Lemma 15.1.6. *If $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ is $\mathcal{K}_1^0$-acyclic, then $\beta_i\langle\langle X \cup Y\rangle\rangle = \beta_i\langle\langle X\rangle\rangle + \beta_i\langle\langle Y\rangle\rangle$.*

If $\langle\langle X\rangle\rangle$ and $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ are both $\mathcal{K}_1^0$-acyclic, then $\beta_i\langle\langle X \cup Y\rangle\rangle = \beta_i\langle\langle Y\rangle\rangle$.

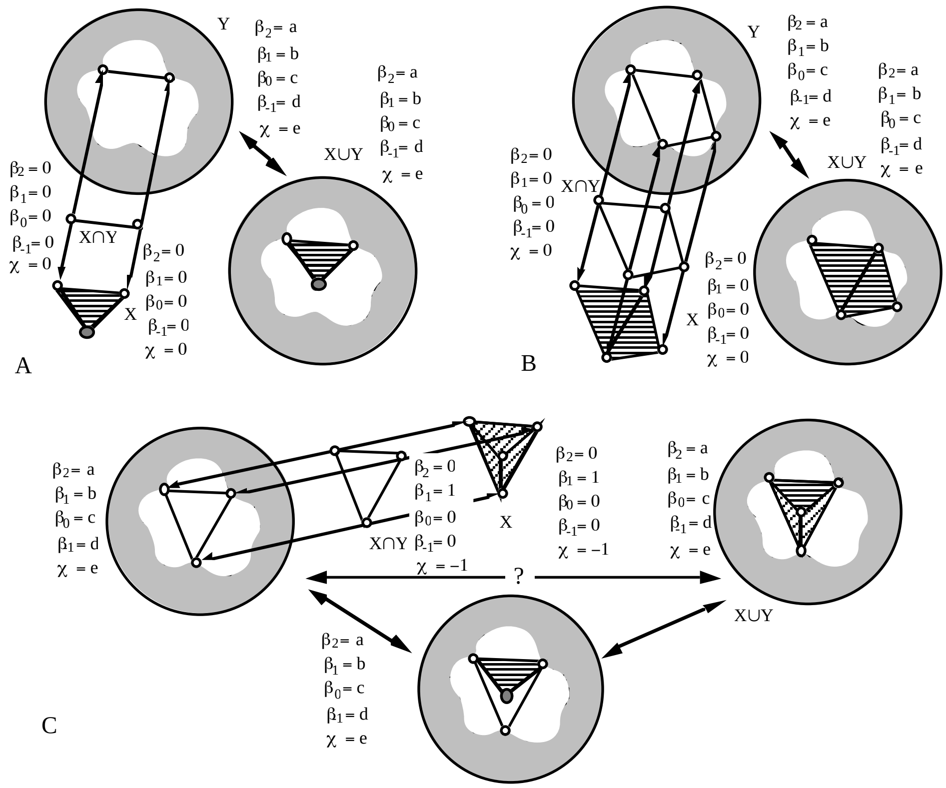

Fig. 15.3. *Standard extensions of a 2-complex in $\mathcal{K}_1^0$ by (A) vertex 1-addition: add one new vertex, two new edges and one triangle; (B) edge 2-addition: add a new edge with two new (non-coplanar) triangles; (C) add a new pyramid of one vertex, three edges and three triangles.*

Example 15.1.7. Figure 15.3 illustrates three standard 'Henneberg-style extensions' for a 2-complex which preserve all Betti numbers in $\mathcal{K}_1^0$:

A. *vertex* 1-*addition*: adding a new triangle, sharing a single existing edge (adding a vertex and two new edges);
B. *edge* 2-*addition*: adding a new edge with two attached triangles. (Notice that the stated $\beta_0 = 0$ holds only if the four points (or two triangles) are not coplanar: a coplanar quadrilateral has non-trivial parallel drawings!)
C. The addition of a triangular pyramid can be seen either as step (A) followed by step (B) *or* as a gluing in which the added $\langle\langle X\rangle\rangle$ and shared $\langle\langle X\rangle\rangle\cap\langle\langle Y\rangle\rangle$ have the matching Betti numbers, non-zero in only one location.

Some of these extension principles must be proven by direct matrix arguments, not by general 'gluing'. In particular, if C is analyzed in one step, the correspondence of Betti numbers for $\langle\langle X\rangle\rangle$ and $\langle\langle X\rangle\rangle \cap \langle\langle Y\rangle\rangle$ in item (C) is *not* sufficient to guarantee then the original and the final complexes have matching Betti numbers. Consider the analog for plane rigidity: attaching a 2-valent vertex $X = \{ab, ac\}$, with one degree of freedom ($\beta_0\langle\langle X\rangle\rangle = 1$ as we will see below), across two vertices b, c to Y, will preserve the Betti numbers (independence, rigidity, etc.) of Y if and only if the three points are not collinear – a condition which is not detected in the Betti numbers (independence or non-rigidity) of X!

15.2. Cohomology for trivariate C^0_1-cofactors. For this homology ('statics and stresses') there is a corresponding cohomology ('kinematics'). With the given coefficients for chains, we need 'dual' coefficients for the cochains in order to define the central bilinear form $\langle \mathbf{c}, \mathbf{d}\rangle \in \mathbb{R}$ on i-chains. Equivalently, we want the $(i-1)$-cocycles to be 'kinematic solutions' to the matrix equation: $M^0_1(\Delta; \mathbf{p})\mathbf{x} = \mathbf{0}$.

Treating the polynomial entries (linear forms) $Ax + By + Cz + D$ as vectors (A, B, C, D), the coefficients of our chains must also be 4-vectors (with an appropriate equivalence relation). For this we need the extended affine coordinates (or homogeneous projective coordinates) for the points $\overline{\mathbf{p}}_i = (p_{i,1}, p_{i,2}, p_{i,3}, 1)$. We write $V_4/\overline{\sigma}$ for the space of 4-vectors with the equivalence relation $\overline{\mathbf{s}} \equiv \overline{\mathbf{t}} \bmod \overline{\sigma}$ if $\overline{\mathbf{s}} - \overline{\mathbf{t}} = \sum_{i\in\sigma} e_i\overline{\mathbf{p}}_i$ for some scalars e_i.

With this notation, the C^0_1-*cochain complex* on $\Delta(\mathbf{p})$ is:

$$\mathcal{K}^0_1(\Delta; \mathbf{p}):\ \mathbf{0} \leftarrow \oplus_{\rho\in F} V_4/\overline{\rho} \xleftarrow{\delta_1} \oplus_{\pi\in E} V_4/\overline{\pi} \xleftarrow{\delta_0} \oplus_{a_i\in V} V_4/\overline{\mathbf{a}}_i \xleftarrow{\delta_{-1}} V_4 \leftarrow \mathbf{0}.$$

For a general i-simplex ρ, with coefficient c_ρ,

$$\delta_i(c_\rho \cdot [\rho]) = \sum_{\sigma=\rho x} Sign(\rho, \sigma) c_\rho \cdot [\sigma].$$

This is extended linearly to i-cochains. The maps are well-defined on the equivalence classes, since, if $\pi \subset \sigma$, then $\overline{\mathbf{s}} \equiv \overline{\mathbf{t}} \bmod \overline{\pi}$ implies $\mathbf{s} \equiv \overline{\mathbf{t}} \bmod \overline{\sigma}$ (the space is larger). In an obvious fashion, the signs guarantee that $\delta_i\delta_{i-1} = 0$.

The elements of the kernel of δ_i are the *i-cocycles* $Z^i(\mathcal{K}^0_1(\Delta; \mathbf{p}))$, and the elements of the image of δ_{i-1} are the *i-coboundaries*, $B^i(\mathcal{K}^0_1(\Delta; \mathbf{p}))$. We have the usual (reduced) cohomology spaces of the cochain complex:

$$\widetilde{H}^i(\mathcal{K}^0_1(\Delta; \mathbf{p})) = Z^i(\mathcal{K}^0_1(\Delta; \mathbf{p}))/B^i(\mathcal{K}^0_1(\Delta; \mathbf{p})).$$

and the corresponding *Betti numbers* $\beta^i(\mathcal{K}^0_1(\Delta; \mathbf{p})) = \dim \widetilde{H}^i(\mathcal{K}^0_1(\Delta; \mathbf{p}))$.

Given an i-chain $\mathbf{c}$, with $c_\pi \in I^1_\pi(\mathbf{p})$ and an i-cochain $\mathbf{d}$, with $d_\pi \in V_4/\overline{\pi}$, we have a natural bilinear form $\langle\, \mathbf{c}, \mathbf{d}\,\rangle = \sum_{\pi\in\Delta^{(i)}} c_\pi \cdot d_\pi$. (The process of 'substituting' d_π into the linear form c_π is written $\cdot d_\pi$.) This is well-defined since, if $\overline{\mathbf{s}} = \overline{\mathbf{t}} \bmod \overline{\pi}$ and $c_\pi \in I^1_\pi(\mathbf{p})$, then

$$c_\pi \cdot \overline{\mathbf{s}} = c_\pi \cdot \overline{\mathbf{t}} + c_\pi \cdot [\sum_{i\in\pi} e_i\overline{\mathbf{p}}_i] = c_\pi \cdot \overline{\mathbf{t}} + \sum_{i\in\pi} e_i[c_\pi \cdot \overline{\mathbf{p}}_i] = c_\pi \cdot \overline{\mathbf{t}} \bmod \overline{\pi}$$

since all forms in I^1_π are zero on points $\mathbf{p}_i$, $i \in \pi$, by definition.

With this bilinear form, the usual duality for an $(i+1)$-chain $\mathbf{c}$ and an i-cochain $\mathbf{d}$ holds: $\langle\ \partial_{i-1}\mathbf{c},\ \mathbf{d}\ \rangle = \langle\ \mathbf{c},\ \delta^i\mathbf{d}\ \rangle$ and the resulting isomorphisms of homology and cohomology spaces and equality of i-Betti numbers, Euler characteristic etc. follow. We also have the orthogonality of i-boundaries and i-cocycles: $(B_i(\Delta, \mathbf{p}))^\perp =$

$Z^i(\Delta, \mathbf{p})$ and the orthogonality of i-coboundaries and i-cycles: $(B^i(\Delta, \mathbf{p}))^\perp = Z_i(\Delta, \mathbf{p})$.

In the pattern established in Section 14.2, we define:

1. the *trivial* C_1^0*-flexes on* $\Delta(\mathbf{p})$ – the 1-coboundaries $B^1(\Delta, \mathbf{p})$;
2. the C_1^0*-flexes on* $\Delta(\mathbf{p})$ – the 1-cocycles $Z^1(\Delta, \mathbf{p})$; and
3. the *non-trivial* C_1^0*-flexes on* $\Delta(\mathbf{p})$ (the C_1^0-flexes modulo the trivial C_1^0-flexes) – the 1-cohomology $H^1(\Delta, \mathbf{p})$.

Notice that the space of 1-cocycles *is* the solution space of the matrix equation $M_1^0(\Delta; \mathbf{p})\mathbf{x} = \mathbf{0}$. This follows from the orthogonality of the rows of $M_1^0(\Delta; \mathbf{p})$ (the 1-boundaries) and the 1-cocycles for the chain complex.

The estimate $3|V| - 4$ is the dimension of the 1-coboundaries (trivial flexes), provided that $\beta^0 = \beta_0 = 0$ and $\beta^{-1} = \beta_{-1} = 0$, that is, provided we have at least two distinct points and a geometric graph which has only trivial parallel drawings.

It is also possible to 'kinematically interpret' the estimate $3|V| - 4$. The coboundaries of the space $V_{d+1}/i(\mathbf{p}) \cdot [i]$ of cochains on a single vertex form a space of dimension 3. However, these $3|V|$ distinct coboundaries have 4 dependencies, corresponding to the four dimensions of 0-coboundaries of the (-1)-cochains, provided that $\beta^0 = 0, \beta^{-1} = 0$.

C_1^0-Flex Test 15.2.1. *For any simplicial* 2*-complex* Δ *with an allowed* 3*-configuration* $\mathbf{p}$ *for its vertices, and a* 2*-simplex* $\psi \in [\Delta(1)]^{(2)}$ *the following are equivalent:*

1. ψ *is in the* C_1^0*-cofactor matroid closure of* $\Delta^{(2)}$ *in* $[\Delta(1)]^{(2)}$*;*
2. *there is a* C_1^0*-dependence* ω *on* $[\Delta(1)]^{(2)}$ *with* $\omega_\psi \neq 0$*;*
3. *all* C_1^0*-flexes* $\mathbf{u}$ *on* $\Delta(\mathbf{p})$ *satisfy* $\partial_2[\psi] \cdot \mathbf{u} = 0$.

Proof. The equivalence of 1 and 2 the definition of the C_1^0-matroid and the closure operator.

(2. $\Rightarrow$ 3.) If ψ is dependent on $\Delta^{(2)}(\mathbf{p})$, then the standard argument shows that any C_1^0-flex $\mathbf{u}$ which is orthogonal to $B_1(\Delta, \mathbf{p})$ is orthogonal to $\partial\psi$.

(2. $\Rightarrow$ 3.) If $\psi \in [\Delta(1)]^{(2)}$ is not dependent on $\Delta(\mathbf{p})$, then adding this row to the matrix $M_1^0(\Delta; \mathbf{p})$ will increase the rank by one and remove some C_1^0-flex $\mathbf{u}$, since they have the same set of 1-simplices. This contradiction completes the proof.

Remark 15.2.2. From this corollary, we realize that a C_1^0-rigid complex $\Delta(\mathbf{p})$ has $[\Delta(1)]^{(2)}$ as its closure. However, from §14.2 we anticipate that the fact that the closure of $\Delta(\mathbf{p})$ is $[\Delta(1)]^{(2)}$ is *not* sufficient for C_1^0-rigidity. This will hold only if $[\Delta^{(1)}]^{(2)}$ is C_1^0-rigid.

Remark 15.2.3. All of the definitions were given for a specific abstract complex Δ^k and a specific configuration $\mathbf{p}$. As usual, the ranks of the matrices can be checked by determinants in the entries – algebraic functions of the coordinates of the configuration. Therefore we have *generic* values for the rank, generic Betti numbers, and an open dense subset of generic configurations. These define the *generic* C_1^0*-cofactor matroid* $\mathcal{K}_1^0(\Delta^2)$.

There are geometric 'special configurations' for a complex Δ which also reduce the rank from this generic value. We have only seen very degenerate forms (for example, all points coplanar) because we have only examined very small complexes. Example 16.1.1 will illustrate this occurrence. (Recall that for plane rigidity, the first interesting special positions occurred for six vertices.)

As the corresponding box in the chart of Figure 13.1 indicates, we will return to this matroid in the §16.1, as skeletal 3-rigidity .

15.3. Trivariate C^{s-1}_s-cofactor matroids. As we did for plane graphs, we can replace the linear form $A^\sigma x + B^\sigma y + C^\sigma z + D^\sigma$ as coefficients on $[\sigma]$ by the s-form $[l_\sigma(\mathbf{p})]^s = [A^\sigma x + B^\sigma y + C^\sigma z + D^\sigma]^s$ to create a new matroid. We define a C^{s-1}_s-*dependence* on $\Delta^2(\mathbf{p})$ as scalars ω_ρ satisfying, for each $\rho \in \Delta^{(1)}$,

$$\sum_{\sigma\in\Delta^{(2)}} Sign(\rho,\sigma)\omega_\sigma[A^\sigma x + B^\sigma y + C^\sigma z + D^\sigma]^s = 0.$$

The independent sets of the C^{s-1}_s-*cofactor matroid* on $\Delta^2(\mathbf{p})$, $\mathcal{C}^{s-1}_s(\Delta;\mathbf{p})$ are those with only the trivial C^{s-1}_s-dependence. These are the row dependencies of a matrix:

$$\begin{array}{c} \\ \vdots \\ \sigma \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & \rho & \cdots \end{array} \\ \left(\begin{array}{ccc} & \vdots & \\ \cdots & Sign(\rho,\sigma)[l_\sigma(\mathbf{p})]^s & \cdots \\ & \vdots & \end{array} \right) \end{array}.$$

Of course, these equations define the 2-cycles of a larger chain complex. For each $\rho \in \Delta^2$, we define the ideal generated by all s'th powers of elements of $I^1_\rho(\mathbf{p})$, $I^s_\rho(\mathbf{p}) = \langle f^s(\mathbf{p}) \mid f(\mathbf{p}) \in I^1_\rho(\mathbf{p})\rangle$. The corresponding chain complex is [**Bi**]:

$$\mathcal{K}^{s-1}_s(\Delta;\mathbf{p}) : \mathbf{0} \to \oplus_{\sigma\in F} I^s_\sigma(\mathbf{p}) \xrightarrow{\partial_2} \oplus_{\rho\in E} I^s_\rho(\mathbf{p}) \xrightarrow{\partial_1} \oplus_{v\in V} I^s_v(\mathbf{p}) \xrightarrow{\partial_0} I^s_\emptyset \xrightarrow{\partial_{-1}} \mathbf{0},$$

where, for each $\psi \in \Delta^{(i)}$:

$$\partial_i(c_\psi\cdot[\psi]) = c_\psi\cdot\partial_i[\psi] = \sum_{\theta\in\Delta^{(i-1)}} Sign(\theta,\psi)c_\psi\cdot[\theta].$$

and the boundary operator is extended linearly to all i-chains. This is well-defined since, if $Sign(\theta,\psi) \neq 0$ then $I^s_\psi \subset I^s_\theta(\mathbf{p})$. ($I^s_\emptyset$ is the set of all polynomials of degree at most s, also written P_s.)

If ψ is an i-simplex, then $I^s_\psi(\mathbf{p})$ has dimension $\binom{s+2-i}{2-i}$ [**Bi**]. This gives the Euler characteristic for $\mathcal{K}^{s-1}_s(\Delta;\mathbf{p})$:

$$\chi(\mathcal{K}^{s-1}_s(\Delta;\mathbf{p})) = |F| - (s+1)|E| + \binom{s+2}{2}|V| - \binom{s+3}{3}.$$

If the complex $\Delta^2(\mathbf{p})$ is C^{s-1}_s-ample – i.e. $\beta_i(\mathcal{K}^{s-1}_s(\Delta;\mathbf{p})) = 0$ for $i \leq 1$ – then the rank for $\mathcal{C}^{s-1}_s(\Delta,\mathbf{p})$, recorded in Figure 13.1, is:

$$\text{rank } \mathcal{C}^{s-1}_s(\Delta,\mathbf{p}) = |F| - \beta_2(\mathcal{K}^{s-1}_s(\Delta;\mathbf{p})) = (s+1)|E| - \binom{s+2}{2}|V| + \binom{s+3}{3}.$$

In this generality, there has been very little work on the homologies of this chain complex. The definitions of cohomology and 'kinematics' do extend with C^s_{s-1}-flexes, C^s_{s-1}-rigidity and a C^s_{s-1} Flex Test, in the spirit of §15.2 (using 'powers of affine points in the space of the simplex' for an equivalence relation on the appropriate vectors). The results for gluing across C^{s-1}_s-acyclic complexes also extend.

We offer a simple result and a conjecture to connect $\mathcal{K}_s^{s-1}(\Delta^2)$ and $\mathcal{K}_{s+1}^s(w*\Delta^2)$. This is directly analogous to the 'coning' of §11.3 and it is likely that the proof given there extends, but this has not yet been verified.

LEMMA 15.3.1. *Given any simplicial* 2*-complex* Δ^2*, the truncated cone* $(w*\Delta)^2$ *satisfies* $\chi(\mathcal{K}_s^{s-1}(\Delta^2;\mathbf{p})) = \chi(\mathcal{K}_{s+1}^s((w*\Delta)^2;\mathbf{p}_w,\mathbf{p}))$

PROOF. This is a pure counting problem. If Δ has the number of faces $|F|$, $|E|$, $|V|$ and 1, then $(w*\Delta)^2$ has the numbers $|F|+|E|$, $|E|+|V|$, $|V|+1$, and 1. In the Euler characteristic calculation we have:

$$\begin{aligned}
&\chi(\mathcal{K}_{s+1}^s((w*\Delta)^2))\\
&\qquad = (|F|+|E|) - (s+2)(|E|+|V|) + \binom{s+3}{2}(|V|+1) - \binom{s+4}{3}\\
&\qquad = |F| - (s+1)|E| + \left[\binom{s+3}{2} - (s+2)\right]|V| - \binom{s+4}{3} + \binom{s+3}{2}\\
&\qquad = |F| - (s+1)|E| + \binom{s+2}{2}|V| - \binom{s+3}{3} \quad = \chi(\mathcal{K}_s^{s-1}(\Delta)).
\end{aligned}$$

CONJECTURE 15.3.2. *Given any simplicial* 2*-complex* Δ *with at least one* 2*-simplex, and a* 3*-configuration* $\mathbf{p}$*, the truncated cone* $(w*\Delta)^2$ *satisfies*

$$\beta_i(\mathcal{K}_s^{s-1}(\Delta;\mathbf{p})) = \beta_i(\mathcal{K}_{s+1}^s((w*\Delta)^2;\mathbf{p}_w,\mathbf{p}))$$

for all $-1 \le i \le 2$ *and all choices of* $\mathbf{p}_w \in R^3$ *in general position relative to* $\mathbf{p}$*.*

REMARK 15.3.3. The case $s=0$ actually yields $I_\psi^0(\mathbf{p}) = \mathbb{R}$ for each simplex. We have returned to the ordinary simplicial 2-cycle matroid of §14.

There is a second geometric situation in which we encounter the simplicial matroid. If all the points of $\mathbf{p}$ lie in a plane, then for all $\psi \in \Delta^{(k)}$, $I_\psi^s(\mathbf{p})$ is a multiple of a fixed single linear form l^s. With $Sign(\rho,\sigma)l^s$ placed in all non-zero entries in the matrix, we see that the 2-cycles are the same as the 2-cycles of regular homology. Thus these 'special positions' reduce all of these cofactor matroids to the simplicial 2-cycle matroid. While this isomorphism of 2-cycles is clear, notice that all lower counts (and lower homologies) will change. An analogous geometric reduction will also apply to §15.4.

15.4. Multivariate C_s^{s-1}-cofactor matroids. A related matroid exists for any d-simplicial complex Δ^d realized by a configuration $\mathbf{p}$ in $(d+1)$-space, for all $d \ge 1$. (As usual, we assume that the points $\mathbf{p}_i$ for any d-simplex span a hyperplane of $\mathbb{R}^{d+1}$.) We have the spaces $I_\psi^s(\mathbf{p}) = \langle f^s(\mathbf{p}) \mid f(\mathbf{p}_j)=0,\ \forall j \in \psi, f \text{ linear}\rangle$ and the corresponding chain complex is:

$$\begin{aligned}
\mathcal{K}_s^{s-1}(\Delta;\mathbf{p})\colon \mathbf{0} \to \oplus_{\sigma\in\Delta^{(d)}} I_\sigma^s(\mathbf{p}) \xrightarrow{\partial_d} \oplus_{\rho\in\Delta^{(d-1)}} I_\rho^s(\mathbf{p}) \xrightarrow{\partial_{d-1}} \oplus_{\psi\in\Delta^{(d-2)}} I_\psi^s(\mathbf{p}) \xrightarrow{\partial_{d-2}}\\
\dots \xrightarrow{\partial_1} \oplus_{v\in\Delta^{(0)}} I^s v(\mathbf{p}) \xrightarrow{\partial_0} I_\emptyset^s \xrightarrow{\partial_{-1}} \mathbf{0},
\end{aligned}$$

where, for each $\psi \in \Delta^{(i)}$:

$$\partial_i(c_\psi \cdot [\psi]) = c_\psi \partial_i \psi = \sum_{\theta\in\Delta^{(i-1)}} Sign(\theta,\psi) c_\psi \cdot [\theta].$$

and the boundary operator is extended linearly to all i-chains.

The d-cycles define the C_s^{s-1}-*cofactor matroid* $\mathcal{C}_s^{s-1}(\Delta^d, \mathbf{p})$, with

$$\text{rank } \mathcal{C}_s^{s-1}(\Delta^{(d)}(\mathbf{p})) = |\Delta^{(d)}| - \beta_d(K_s^{s-1}(\Delta^d, \mathbf{p})).$$

If ψ is an i-simplex, then $I_\psi^s(\mathbf{p})$ has dimension $\binom{s+d-i}{s} = \binom{s+d-i}{d-i}$. (This is computed by considering the special case where the i-simplex is defined by setting $x_j = 0$, $j \leq d-i$, and looking at the homogeneous polynomials of degree s in these $d-i$ variables [**Bi**].) This gives the Euler characteristic for $\mathcal{K}_s^{s-1}(\Delta^d; \mathbf{p})$

$$\chi(\mathcal{K}_s^{s-1}(\Delta^d; \mathbf{p})) = \sum_{i \leq d} (-1)^i \binom{s+d-i}{d-i} f_i(\Delta^d).$$

In particular, if $\Delta^d(\mathbf{p})$ is C_s^{s-1}-*ample* – i.e. $\beta_i(\mathcal{K}_s^{s-1}(\Delta; \mathbf{p})) = 0$ for $i \leq d-1$ – then the rank for $C_s^{s-1}(\Delta, \mathbf{p})$, recorded in Figure 13.1, is:

$$\text{rank } C_s^{s-1}(\Delta, \mathbf{p}) = |F| - \beta_d(\mathcal{K}_s^{s-1}(\Delta; \mathbf{p})) = \sum_{i \leq d-1} (-1)^{i+d-1} \binom{s+d-1-i}{d-1-i} f_i(\Delta^d).$$

The few general results and conjectures we gave for trivariate cofactor matroids extend to these multivariate cofactor matroids. Based on a strong analogy with the chain complexes in the next section, we will make additional conjectures in §16.5.

Remark 15.4.1. For bivariate C_1^0-cofactors, Propositions 5.1.1 and 5.1.2 gave connections between parallel drawings in 3-space and the lower Betti numbers of the cofactor chain complex. In precisely the same way, we have:

Proposition 15.4.2. *For a k-complex Δ and a $(k+1)$-configuration $\mathbf{p}$:*
1. $\beta_{-1}(\mathcal{K}_1^0(\Delta; \mathbf{p})) = 0$ *if and only if there are two distinct points $\mathbf{p}_i \neq \mathbf{p}_j$, $i, j \in V$.*
2. $\beta_0(\mathcal{K}_1^0(\Delta; \mathbf{p})) = 0$ *if and only if the geometric graph $\Delta^{(1)}(\mathbf{p})$ has only trivial parallel drawings in $(k+1)$-space.*

Remark 15.4.3. All these cofactor matroids can also be extended to general polyhedral complexes Φ, provided the configuration $\mathbf{p}$ generates a unique flat of dimension i for each i-face (that is, we have a piecewise linear realization). It is clear that the rest of the chain complex $\mathcal{K}_s^{s-1}(\Phi^d; \mathbf{p})$, which is based on the well-defined vector spaces $I_\psi^s(\mathbf{p})$, generalizes immediately. While the existence of these objects is understood in approximation theory, the only cases we have seen investigated correspond to the matroids $\mathcal{K}_1^0(\Phi^d; \mathbf{p})$ which are directly related to scene analysis and parallel drawings as in §8.3.

Finally, from multivariate splines (including bivariate splines) there are even more challenging chain complexes (and associated polymatroids) $\mathcal{K}_s^r(\Phi^d; \mathbf{p})$, for all integers $r < s$. The reader is referred to [**Bi**] for the definitions.

16. Skeletal Rigidity

As Figure 13.1 anticipates, there is a second family of matroids which share the rank estimates (and many other properties) of the multivariate C_s^{s-1}-cofactor matroids. These are direct generalizations of the first-order rigidity matroids for graphs. All of the results (and some of the conjectures) in this section come from a continuing joint project with Tiong-Seng Tay and Neil White [**TWW1**,**2**,**3**]. The origins of these matroids are the search for geometric and combinatorial proofs of fundamental results in polyhedral combinatorics, such as the h-theorem and the g-theorem for simplicial polytopes (see §16.4 and [**Ka1**,**Lee1**,**2**]). These matroids generalize the geometry of rigid frameworks and retain the potential for both nice geometric reproofs of known combinatorial results about the faces of simplicial convex polytopes and generalizations of these results to homology spheres and other combinatorial complexes.

The analogy with the matroids $\mathcal{K}_s^{s-1}(\Delta^r; \mathbf{p})$ is my own responsibility. Generalizing the conjectures of §10.3, §11.5, we conjecture a strong relationship between the generic behaviour of the cofactor matroids and the matroids of this section. At a minimum, this analogy is a fruitful source of conjectures and insights for both families of matroids (see §16.5).

There are at least five different matrices which capture the initial matroid we want to describe [**TWW1**,**2**]. However, efficient presentation of these matroids in the context of homology works best with an exterior algebra on the underlying affine (or projective) coordinates. We will use a modern form of this algebra: the Grassmann-Cayley algebra [**DRS**,**Wht4**,**TWW1**].

We offer a brief description of the notation for §16. This is essentially a projective idea [**TWW1**,**3**] but we simplify to an affine presentation. As we use the algebra here, it is notation for the lexicographically ordered minors of a non-square matrix. We use affine coordinates for the points in d-space $\overline{\mathbf{p}}_i = (p_{i,1}, p_{i,2}, \dots, p_{i,d}, 1)$. Given two affine points $\overline{\mathbf{x}}, \overline{\mathbf{y}}$, the 2-*extensor* $\overline{\mathbf{x}} \vee \overline{\mathbf{y}}$ is the $\binom{d+1}{2}$-tuple of 2-by-2 minors of the matrix $[\overline{\mathbf{x}}\ \overline{\mathbf{y}}]$ (recall §12). Similarly, the 3-*extensor* $\overline{\mathbf{x}} \vee \overline{\mathbf{y}} \vee \overline{\mathbf{w}}$ is the $\binom{d+1}{3}$-tuple of 3-by-3 minors of $[\overline{\mathbf{x}}\ \overline{\mathbf{y}}\ \overline{\mathbf{w}}]$.

The field (the reals) or 0-*extensors* can be written $V_{d+1}^{(0)}$; the points or *1-extensors* as the vector space $V_{d+1}^{(1)}$, and the space of j-*tensors* – sums of j-extensors – as $V_{d+1}^{(j)}$, where $\dim V_{d+1}^{(j)} = \binom{d+1}{j}$.

For a given oriented simplex σ of dimension i, $\overline{\sigma}$ is the exterior product of its vertices, an $(i+1)$-extensor. This is well defined on the orientation class $[\sigma]$, since $\vee$ is alternating: $\mathbf{x} \vee \mathbf{y} = -\mathbf{y} \vee \mathbf{x}$. We will work with an equivalence relation *modulo the kernel of* $\overline{\sigma}$ on the j-tensors defined by:

$$P \overset{\overline{\sigma}}{=} S \quad \text{if and only if} \quad P \vee \overline{\sigma} = S \vee \overline{\sigma}$$

where $\vee$ is extended linearly to tensors as sums of extensors. In particular, we denote this space of all j-tensors in a projective space of dimension d, mod a simplex σ of dimension $\leq d-j$ as $V_{d+1}^{(j)}/\ker \overline{\sigma}$. We note that, if $P \overset{\overline{\sigma}}{=} P'$ and $Q \overset{\overline{\sigma}}{=} Q'$, then $P + Q \overset{\overline{\sigma}}{=} P' + Q'$ since

$$(P+Q) \vee \overline{\sigma} = P \vee \overline{\sigma} + Q \vee \overline{\sigma} = P' \vee \overline{\sigma} + Q' \vee \overline{\sigma} = (P' + Q') \vee \overline{\sigma}$$

and $P \vee Q \overset{\overline{\sigma}}{=} P' \vee Q'$, since

$$P \vee Q \vee \overline{\sigma} = P \vee Q' \vee \overline{\sigma} = (-1)^{j+k} Q' \vee P \vee \overline{\sigma} = (-1)^{j+k} Q' \vee P' \vee \overline{\sigma} = P' \vee Q' \vee \overline{\sigma}.$$

Therefore the equivalence relation is well-defined within the Cayley algebra. We also note that $V_{d+1}^{(0)}/\ker \overline{\sigma} = \mathbb{R}$, and that, if $\sigma \in \Delta^{(i)}$ and P is a extensor of step $> d+1-i$, then $P \overset{\overline{\sigma}}{=} \mathbf{0}$, automatically. This space, $V_{d+1}^{(j)}/\ker \overline{\sigma}$, has dimension $\binom{d-i}{j}$, as a vector space over the reals.

We begin with an alternate presentation of the C^0_1-cofactor matroid for 2-complexes in 3-space. The principle value of this change is the distinct array of generalizations which flow from this 'rigidity' approach, as we observed in Part II.

16.1. 3-rigidity in 3-space. Consider a simplicial complex Δ^2 with an affine 3-configuration $\overline{\mathbf{p}}$. A 3-*stress* on $\Delta(\overline{\mathbf{p}})$ is an assignment of scalars ω_σ, $\sigma \in F$ such that, for each edge $\rho \in E$:

$$\sum_{\sigma|\rho v=\sigma} \omega_\sigma \overline{\mathbf{v}} \cdot [\rho] \stackrel{\overline{\rho}}{=} 0,$$

where $\overline{\mathbf{u}} \stackrel{\overline{\rho}}{=} \overline{\mathbf{v}}$, $\rho = \{a,b\}$ means that $\overline{\mathbf{u}} \vee \overline{\mathbf{a}} \vee \overline{\mathbf{b}} = \overline{\mathbf{v}} \vee \overline{\mathbf{a}} \vee \overline{\mathbf{b}}$ or, equivalently,

$$[\overline{\mathbf{u}}\,\overline{\mathbf{a}}\,\overline{\mathbf{c}}\,\overline{\mathbf{x}}] = \begin{bmatrix} u_1 & u_2 & u_3 & 1 \\ a_1 & a_2 & a_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ x & y & z & 1 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & 1 \\ a_1 & a_2 & a_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ x & y & z & 1 \end{bmatrix} = [\overline{\mathbf{v}}\,\overline{\mathbf{a}}\,\overline{\mathbf{c}}\,\overline{\mathbf{x}}]\,.$$

(In this form, we already see the underlying connection to the 3-stresses as defined for C^0_1-cofactors.) The geometric meaning of this calculation is:

Does the weighted affine point $\sum_{\sigma|\rho v=\sigma} \omega_\sigma \overline{\mathbf{v}} = \left(\sum_{\sigma|\rho v=\sigma} \omega_\sigma\right)\overline{\mathbf{w}}$ *lie on the affine line* $\overline{\mathbf{a},\mathbf{b}}$ *(the line of* ρ*)?* [**Lee1,TWW1**]

The 3-stresses are the row dependencies of the *rigidity matrix* $R_{3,3}(\Delta;\overline{\mathbf{p}})$:

$$R_{3,3}(\Delta,\overline{\mathbf{p}}) = \begin{array}{c} \\ \vdots \\ \rho v \\ \vdots \end{array} \begin{array}{c} \cdots \quad \rho \quad \cdots \\ \left(\begin{array}{ccc} & \vdots & \\ \cdots & \overline{\mathbf{v}} & \cdots \\ & \vdots & \end{array}\right) \end{array}.$$

where the entries under ρ are computed in the equivalence classes of $\stackrel{\overline{\rho}}{=}$ (vectors of the vector space $V^{(2)}_{d+1}/\ker\overline{\rho}$). These row dependencies define the *skeletal* 3-*rigidity matroid* $\mathcal{R}_{3,3}(\Delta;\mathbf{p})$. For a generic 3-configuration (or using the maximal ranks of the rigidity matrices over all 3-configurations) we have the *generic skeletal* 3-*rigidity matroid* $\mathcal{R}_{3,3}(\Delta)$.

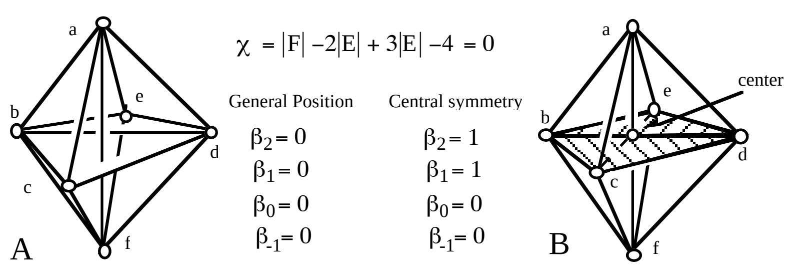

Fig. 16.1. *A* 2*-simplicial complex in* 3*-space which is* 3*-independent in general position (A) but has a* 3*-stress in special centrally symmetric position (B).*

Example 16.1.1. Consider the extended complex of Figure 16.1A, where all four triangles to the central 'shaft' a, f and the triangles bcd, bde are added to the faces of the octahedron. The 3-rigidity matrix is:

	ω	ab	ac	ad	ae	af	bc	bd	be	bf	cd	cf	de	df	ef
abc	1	$\overline{\mathbf{c}}$	$\overline{\mathbf{b}}$				$\overline{\mathbf{a}}$								
abe	1	$\overline{\mathbf{e}}$			$\overline{\mathbf{b}}$				$\overline{\mathbf{a}}$						
acd	1		$\overline{\mathbf{d}}$	$\overline{\mathbf{c}}$							$\overline{\mathbf{a}}$				
ade	1			$\overline{\mathbf{e}}$	$\overline{\mathbf{d}}$								$\overline{\mathbf{a}}$		
bcf	1						$\overline{\mathbf{f}}$			$\overline{\mathbf{c}}$		$\overline{\mathbf{b}}$			
bef	1								$\overline{\mathbf{f}}$	$\overline{\mathbf{e}}$					$\overline{\mathbf{b}}$
cdf	1										$\overline{\mathbf{f}}$	$\overline{\mathbf{d}}$		$\overline{\mathbf{c}}$	
def	1												$\overline{\mathbf{f}}$	$\overline{\mathbf{e}}$	$\overline{\mathbf{d}}$
abf	-1	$\overline{\mathbf{f}}$				$\overline{\mathbf{b}}$				$\overline{\mathbf{a}}$					
acf	-1		$\overline{\mathbf{f}}$			$\overline{\mathbf{c}}$						$\overline{\mathbf{a}}$			
adf	-1			$\overline{\mathbf{f}}$		$\overline{\mathbf{d}}$								$\overline{\mathbf{a}}$	
aef	-1				$\overline{\mathbf{f}}$	$\overline{\mathbf{e}}$									$\overline{\mathbf{a}}$
bcd	-1						$\overline{\mathbf{d}}$	$\overline{\mathbf{c}}$			$\overline{\mathbf{b}}$				
bde	-1							$\overline{\mathbf{e}}$	$\overline{\mathbf{d}}$				$\overline{\mathbf{b}}$		

.

If the points are in general position, there is no 3-stress:

(i) the points b, c, d, e will be affinely independent, and no coefficients for bcd, bde will add to 0 in the columns of bd $(\mathrm{mod}\{b, d\})$;

(ii) these guaranteed zero entries will pass, by similar arguments, to all rows.

We conclude that this set is independent for $\overline{\mathbf{p}}$ in general position. The remaining Betti numbers of the associated chain complex will be described below.

On the other hand, assume that we have a centrally symmetric configuration (Figure 16.1B) with $\overline{\mathbf{a}} + \overline{\mathbf{f}} = \overline{\mathbf{b}} + \overline{\mathbf{d}} = \overline{\mathbf{c}} + \overline{\mathbf{e}}$. The reader can check that the entries under ω are now a 3-stress, since we have such equations as:

column ab: $\overline{\mathbf{c}} + \overline{\mathbf{e}} - \overline{\mathbf{f}} = \overline{\mathbf{a}} \in \overline{\mathbf{a}, \mathbf{b}}$;
column ac: $\overline{\mathbf{b}} + \overline{\mathbf{d}} - \overline{\mathbf{f}} = \overline{\mathbf{a}} \in \overline{\mathbf{a}, \mathbf{b}}$;
column af: $\overline{\mathbf{b}} + \overline{\mathbf{d}} + \overline{\mathbf{c}} + \overline{\mathbf{d}} = 2(\overline{\mathbf{a}} + \overline{\mathbf{f}}) \in \overline{\mathbf{a}, \mathbf{f}}$;
column bc: $\overline{\mathbf{a}} + \overline{\mathbf{f}} - \overline{\mathbf{d}} = \overline{\mathbf{b}} \in \overline{\mathbf{b}, \mathbf{c}}$;
column bd: $\overline{\mathbf{c}} + \overline{\mathbf{e}} = (\overline{\mathbf{b}} + \overline{\mathbf{d}}) \in \overline{\mathbf{b}, \mathbf{d}}$.

This is a special position which has a 3-stress. Clearly this self-stress is unique, up to a single scalar multiple and $\Delta^{(2)}(\overline{\mathbf{p}})$ is a circuit of the matroid. (We can still obtain a 3-stress with a configuration which is special, but less symmetric: if the lines af, bd, c, e are concurrent then the same argument will apply, with appropriate scalars expressing the common point. We chose this symmetric configuration for the ease of verification.)

Once more, this basic matrix describes the 2-cycles (row dependencies) and 1-boundaries (rows) for a chain complex. For a simplicial complex Δ realized in affine 3-space, the 3-*skeletal chain complex* is

$$\mathcal{R}_{3,3}(\Delta; \overline{\mathbf{p}})\colon \mathbf{0} \to \oplus_{\sigma \in F} \mathbb{R} \xrightarrow{\partial_2} \oplus_{\rho \in E} V_4^{(1)} / \ker \rho \xrightarrow{\partial_1} \oplus_{v \in V} V_4^{(2)} / \ker v \xrightarrow{\partial_0} V_4^{(3)} \to \mathbf{0}.$$

We define the boundary maps as follows. For a 2-simplex $\sigma = \{a_0, a_1, a_2\}$:

$$\partial_i(1 \cdot [a_0, a_1, a_2]) = \overline{\mathbf{a}}_0 \cdot [a_1, a_2] + \overline{\mathbf{a}}_1 \cdot [a_0, a_2] + \overline{\mathbf{a}}_2 \cdot [a_0, a_1],$$

extended linearly to all 2-chains. For a general i-simplex τ, with coefficient P_τ:

$$\partial_i(P_\tau \cdot [a_0, \dots, a_i]) = \sum_{\{j:\ a_j \in \tau\}} P_\psi \vee \overline{\mathbf{a}}_j \cdot [a_0, \dots, \hat{a}_j, \dots, a_i]$$

where $\hat{a}_j$ indicates that this vertex is omitted. This is extended linearly to all i-chains. That $\partial_{i-1}\partial_i = 0$ follows from the basic antisymmetry of the operation $x \vee y$. If $\psi x, y = \sigma$ then the coefficient of $[\psi]$ in the double boundary of $c_\sigma \cdot [\sigma]$ is

$$\partial_{i-1}\partial_i c_\sigma \cdot [\sigma] = \partial_{i-1}\big(\overline{\mathbf{x}} \cdot [\psi, y] + \overline{\mathbf{y}} \cdot [\psi, x]\big) = \big(\overline{\mathbf{x}} \vee \overline{\mathbf{y}} + \overline{\mathbf{y}} \vee \overline{\mathbf{x}}\big) \cdot [\psi] = \mathbf{0} \cdot [\psi].$$

We define skeletal 3-independence as $\beta_2(\mathcal{R}_{d,r}(\langle\langle X\rangle\rangle; \overline{\mathbf{p}}|_X)) = 0$; skeletal 3-rigidity as $\beta_1(\mathcal{R}_{d,r}(\langle\langle X\rangle\rangle; \overline{\mathbf{p}}|_X)) = 0$; and skeletal 3-adequate complexes $\langle\langle Y\rangle\rangle$ for gluing as $\beta_i(\mathcal{R}_{d,r}(\langle\langle Y\rangle\rangle; \overline{\mathbf{p}}|_Y)) = 0$ for $i \leq 0$.

To calculate the Euler characteristic, we need the dimensions of the i-chains. The 2-chains $\oplus_{\sigma \in F} \mathbb{R}$ have dimension $|F|$. The 1-chains $\oplus_{\rho \in E} V_4^{(1)}/\ker\rho$ have dimension $2|E|$, the 0-chains $\oplus_{v \in V} V_4^{(2)}/\ker v$ have dimension $3|V|$ and the (-1)-chains $V_4^{(3)}$ have dimension 4. The chain complex on $\Delta^2(\overline{\mathbf{p}})$ has the Euler characteristic:

$$\chi(\mathcal{R}_{3,3}(\Delta; \overline{\mathbf{p}})) = |F| - 2|E| + 3|V| - 4.$$

The entire chain complex is isomorphic to the chain complex $\mathcal{K}_1^0(\Delta; \mathbf{p})$, provided $\mathbf{p}$ is the Euclidean coordinates of $\overline{\mathbf{p}}$. The same conditions worked out in §15.1 are required for the lower Betti numbers to be zero.

Proposition 16.1.2. *For a simplicial complex Δ^2 and configuration $\overline{\mathbf{p}}$ in affine 3-space,*

1. *$\beta_{-1}(\mathcal{R}_{3,3}(\Delta; \overline{\mathbf{p}})) = 0$ if and only if $\Delta^{(0)}(\overline{\mathbf{p}})$ contains two distinct points;*
2. *$\beta_0(\mathcal{R}_{3,3}(\Delta; \overline{\mathbf{p}})) = 0$ if and only if $\Delta^1(\overline{\mathbf{p}})$ has only trivial parallel drawings in 3-space.*

Example 16.1.3. Recall the simplicial complex and general position configuration $\overline{\mathbf{p}}$ of Figure 16.1A. The Euler characteristic is $\chi = 14 - 2 \times 14 + 18 - 4 = 0$. Since $\beta_2 = 0$, $\beta_0 = 0$, and $\beta_{-1} = 0$, we compute that $\beta_1 = 0$ and Δ is acyclic for the 3-rigidity matroid.

The centrally symmetric position (B), with $\beta_2 = 1$, $\beta_0 = 0$, and $\beta_{-1} = 0$, must have $\beta_1 = 1$.

A *projective transformation* T is expressed by a non-singular 4-by-4 matrix $[T]$ which is applied to the affine coordinates $[T]\overline{\mathbf{p}}_i = \lambda_i \overline{\mathbf{q}}_i$ ($\lambda_i \neq 0$). It is not difficult to show, even from our sketch of the chain complex, that such a transformation induces a non-singular linear transformation for all i-tensors, taking the i-cycles and i-boundaries of $\mathcal{R}_{3,3}(\Delta^2; \overline{\mathbf{p}})$ to the i-cycles and i-boundaries of $\mathcal{R}_{3,3}(\Delta^2; \overline{\mathbf{q}})$ for all i. We state, without proof, the basic projective invariance [**TWW3**].

Proposition 16.1.4. *For a 2-complex Δ^2 and a configuration $\overline{\mathbf{p}}$, any projective transformation T to an affine configuration $\overline{\mathbf{q}}$ induces an isomorphism of i-cycles, i-boundaries and i-homology from $\mathcal{R}_{3,3}(\Delta; \overline{\mathbf{p}})$ to $\mathcal{R}_{3,3}(\Delta; \overline{\mathbf{q}})$, for all i.*

Notice that the basic condition used to analyze Figure 16.1B: the statement "*af, bd, ce are concurrent*"; is projectively invariant. The theory of such special geometric positions is an extension of the 'pure conditions' of [**WW1**].

REMARK 16.1.5. This same projective invariance applies to all the skeletal rigidity chain complexes of this section and all the multivariate C_s^{s-1}-cofactor chain complexes of §15 [**ASW,TWW3,Wh10**].

All the results for $\mathcal{K}_1^0(\Delta;\mathbf{p})$ in §15 extend immediately to $\mathcal{R}_{3,3}(\Delta;\overline{\mathbf{p}})$. We will offer a brief overview of the corresponding skeletal cohomology in §16.3.

16.2. Rigidity for r-skeleta in d-space. The constructions for 2-skeleta in 3-space extend to 2-skeleta in d-space in a natural way. For brevity, we pass immediately to general construction for k-skeleta in dimensions $d \geq k$ – with k-skeletal rigidity in k-space being homology.

For a simplicial complex Δ realized in projective d-space, the *skeletal r-rigidity chain complex* is [**TWW3**]:

$$\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}}):\ \mathbf{0} \to \oplus_{\rho\in\Delta^{(r-1)}} V_{d+1}^{(0)} \xrightarrow{\partial_{r-1}} \oplus_{\sigma\in\Delta^{(r-2)}} V_{d+1}^{(1)}/\ker\sigma \xrightarrow{\partial_{r-2}}$$
$$\cdots \xrightarrow{\partial_1} \oplus_{v\in\Delta^{(0)}} V_{d+1}^{(r-1)}/\ker v \xrightarrow{\partial_0} V_{d+1}^{(r)} \to \mathbf{0}.$$

For a general i-simplex σ, with coefficient c_σ (an $(r-i-1)$-tensor modulo $\overline{\sigma}$), the boundary operator is:

$$\partial_i c_\sigma \cdot [\sigma] = \sum_{x\in\sigma} c_\sigma \vee \overline{\mathbf{x}} \cdot [\sigma/x].$$

where $[\sigma/x]$ is the lexicographic order for ρ with $\pm[\rho x] = [\sigma]$. (This notation comes from indexing the simplices by square free monomials.) Again this is extended linearly to i-chains. Of course $\partial_{-1}(c\cdot\emptyset) = 0$.

The top $(r-1)$-cycles define the *skeletal r-rigidity matroid* on the $(r-1)$-simplices, $\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})$, for each configuration $\overline{\mathbf{p}}$, as well as the *generic r-rigidity matroid* $\mathcal{R}_{d,r}(\Delta)$, for algebraically independent coordinates (or alternately, using the maximal rank over all $\overline{\mathbf{p}}$). Some initial results for this are presented in [**TWW1**,**2**]. The rest of the homology plays the role of statics and kinematics to aid the analysis of this matroid [**TWW3**].

We define skeletal r-independence as $\beta_{r-1}(\mathcal{R}_{d,r}(\langle\langle X\rangle\rangle;\overline{\mathbf{p}}|_X)) = 0$; skeletal r-rigidity as $\beta_{r-2}(\mathcal{R}_{d,r}(\langle\langle X\rangle\rangle;\overline{\mathbf{p}}|_X)) = 0$; and skeletal r-acyclic complexes $\langle\langle Y\rangle\rangle$ for gluing as $\beta_i(\mathcal{R}_{d,r}(\langle\langle Y\rangle\rangle;\overline{\mathbf{p}}|_Y)) = 0$ for all i. Since the dimension of $V_{d+1}^{(r-i)}/\ker\rho$, $\rho\in\Delta^{(i)}$, is $\binom{d-i}{r-1-i}$, the Euler characteristic is:

$$\chi(\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})) = \sum_{i=-1}^{r-1}(-1)^i\binom{d-i}{r-i-1}f_i = \sum_{i=-1}^{r-1}(-1)^i\beta_i(\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}}).$$

The Mayer-Vietoris sequence yields results such as: gluing two complexes across a skeletal r-acyclic complex creates a complex with the sum of the Betti numbers of the pieces; and gluing two skeletal r-ample complexes across a skeletal r-adequate complex produces a new r-ample complex. Direct matrix analyses also verifies some simple 'Henneberg style' extension principles.

REMARK 16.2.1. If we look carefully at skeletal 2-rigidity on a graph G at a d-configuration $\overline{\mathbf{p}}$ in d-space, it is the first-order rigidity of $G(\mathbf{p})$ of Parts I and II of our survey. We have simply given an affine presentation of the 'vector' $\mathbf{p}_j - \mathbf{p}_i$ as the 'vector' $\overline{\mathbf{p}}_j \bmod \overline{\mathbf{p}}_i$. This gives an isomorphism between the 1-cycle matrix for 2-rigidity and the rigidity matrix for first-order rigidity.

In the Part II, we concentrated on β_1 (self-stresses) and β^0 (first-order motions), with the implicit understanding that $\beta_{-1} = 0$ for sufficiently large complexes: $|V| \geq d-1$ in dimension d. All of the difficulties with the counts such as $3|V|-6$ can be traced to the fact that a single bar has Euler characteristic -1 and $\beta_{-1} = 1$. *It is not skeletal* 2*-adequate* for $d > 2$. Therefore gluing across a single bar (Figure 9.4) produces a loss of 2-rigidity. (See Example 16.2.4 for the analogs in r-rigidity.)

REMARK 16.2.2. If we consider r-rigidity in $(r-1)$-space, then the i-coefficients have the form $c_\rho \in V_r^{(r-i)}/\ker\rho$, where $\rho \in \Delta^{(i)}$. Since this means $c_\rho \vee \overline{\rho}$ has step r in an r-dimensional vector space – these coefficients are just scalars. This r-rigidity has reduced to the simplicial r-cycle matroid and the chain complex is the ordinary chain complex of simplicial homology.

For r-rigidity in r-space, the i-coefficients have the form $c_\rho \in V_{r+1}^{(r-i)}/\ker\rho$, where $\rho \in \Delta^{(i)}$. For the top boundary operator the coefficients are r-extensors $c_\sigma\overline{\mathbf{v}} \vee \overline{\rho}$ in a space of dimension $r+1$ – the coefficients of the hyperplane $l_\sigma(\mathbf{p})$. The r-rigidity in r-space is the C_1^0-cofactor matroid in r-space in all dimensions.

Other results from $\mathcal{R}_{d,2}$ in Parts I and II extend immediately to these matroids. One example is the Coning Theorem.

CONING THEOREM 16.2.3 [**TWW3**]. *If* $\Pi_w\overline{\mathbf{p}}$ *in affine* d*-space is the projection of* $\overline{\mathbf{p}}$ *in* $(d+1)$*-space from the affine point* $\overline{\mathbf{p}}_w$*, and* $w * \Delta$ *is the cone of* Δ *then* $\beta_i(\mathcal{R}_{d+1,r}(w*\Delta; \overline{\mathbf{p}}_w, \overline{\mathbf{p}})) = \beta_i(\mathcal{R}_{d,r}(\Delta; \Pi_w\overline{\mathbf{p}}))$ *for all* i*.*

EXAMPLE 16.2.4. Coning explains the properties of the complete complexes K_n for skeletal r-rigidity in d-space [**TWW3**]. Since the complete complex K_n is the cone of K_{n-1}, we can project down from $\mathcal{R}_{d,r}(K_n)$ to $\mathcal{R}_{d-i,r}(K_{n-i})$ until one of two terminal events occurs:

1. $i = n$ and we have the empty complex K_0 in dimension $d-n \geq r-1$, which has the clear Betti numbers,
$$\beta_i(K_0) = \begin{cases} \binom{d-n-1}{r} & \text{for } i = -1 \\ 0 & \text{for } i \neq -1 \end{cases}$$
Therefore if $n \leq d+1-r$, then $\beta_{-1}(\mathcal{R}_{d,r}(K_n; \overline{\mathbf{p}})) = \binom{d-n+1}{r}$ in dimension d and all other Betti numbers are zero, provided $\overline{\mathbf{p}}$ is in general position.
2. $d-i = r-1$ and $n-i > 0$, we have the r-homology of §14, with the well-understood Betti numbers:
$$\beta_i(K_{n+r-d-1}) = \begin{cases} 0 & \text{for } i < r-1 \\ \binom{n+r-d-2}{r} & \text{for } i = r-1 \end{cases}$$
Therefore if $n > d-r+1$, then $\beta_i(\mathcal{R}_{d-n,r}(K_n; \overline{\mathbf{p}})) = 0$, $i < r-1$, for $\overline{\mathbf{p}}$ in general position and the complex is skeletal r-ample. (The value of $\beta_i(K_{n+r-d-1})$ is calculated from the Euler characteristic.) In particular, the complex is skeletal r-acyclic if and only if $d-r+2 \leq n \leq d+1$.

For $r = 2$, the standard first-order rigidity of frameworks, the complex K_n is 2-acyclic in dimension d if and only if $d \leq n \leq d+1$. In the plane, the 2-acyclic complexes for gluing are K_2, K_3. In 3-space, the 2-acyclic complexes are K_3, K_4.

For skeletal 3-rigidity the skeletal 3-acyclic complexes are K_n, $d-1 \leq n \leq d+1$. In 3-space, these are K_2, K_3, K_4. In general, the smallest skeletal 3-acyclic complex in dimension d is K_{d-1}, making explicit that $d-1$ is the minimum number of points in general position needed to make $\beta_{-1}(\mathcal{R}_{d,3}(\Delta; \overline{\mathbf{p}})) = 0$ in dimension d (something we had not previously derived). This also means gluing two 3-ample complexes across a single 2-simplex in d-space, $d > 4$, will cause the resulting complex to have $\beta_0 > 0$. This is an analog of gluing two frameworks across a single edge in 3-space.

More generally, we find that $d + 2 - r$ is the minimum number of points in general position needed to make $\beta_{-1}(\mathcal{R}_{d,r}(\Delta; \overline{\mathbf{p}})) = 0$ in dimension d. Gluing two skeletal r-ample complexes across a single $(r-1)$-simplex in d-space, $d > 2r - 2$, will create a complex with $\beta_0(\mathcal{R}_{d,r}(\Delta; \overline{\mathbf{p}})) > 0$.

Remark 16.2.5. As we described in Remark 14.2.14, shelling is a basic inductive construction for classes of simplicial complexes. In particular, all convex simplicial $(d+1)$-polytopes have a shelling. (This is given by the order in which the hyperplanes of the facets meet a line in general position relative to a convex realization in $(d+1)$-space.) All simplicial decompositions of a convex d-ball in d-space by a Delaunay (or Voronoi) decomposition are also shellable.

For any shellable n-complex Δ realized in general position $\overline{\mathbf{p}}$ in $d \leq n$, this shelling can be retraced as a sequence of gluings for r-ample complexes:

(a) the initial simplex is r-ample;

(b) each $(n-1)$-ball from the boundary of an n-simplex is r-adequate and the added n-simplex is r-ample, so the result of this gluing is r-ample;

(c) the $(n-1)$-sphere is also r-ample, so the final complex Δ is r-ample.

Whether the result is skeletal r-acyclic (r-independent) will depend on the Euler characteristic (i.e. on the dimension d and the face numbers of the polytope). This can also be traced by the explicit form of the intersections $\sigma_{j+1} \cap \Delta_j$, for all j. An a analogous result holds for the multivariate cofactor matroids and yields some traditional heuristics for multivariate splines [**Al**].

Two simple examples covered by this shelling proof are the 2- and 3-rigidity of triangulated spheres in the plane, or the 3-rigidity of 4-polytopial complexes in 3-space. We do not obtain the 2-rigidity of 2-spheres in 3-space (Theorem of Cauchy and Dehn §9.3.10) or its analogs – see below.

The 'flavour of homology' in our definitions would suggest that the r-ample result extends to classes defined by homology and is not restricted to the inductive construction of shelling. For example, we *conjecture* that all simplicial homology n-spheres (all simplicial complexes with the homology of an n-sphere, some of which are not shellable) and all simplicial n-balls are r-ample for $d \leq n$. Some extended results in this direction will appear in [**TWW3**].

In polyhedral combinatorics, there are critical sequences of numbers defined from the face numbers of any n complex (in particular for any n-sphere):

1. the h-vector:

$$h_r(\Delta, n) = \sum_{j=0}^{r} (-1)^{j+r} \binom{n-j}{d-r} f_{j-1} = \sum_{j=-1}^{r-1} (-1)^{j+r+1} \binom{n-j-1}{n-r} f_j;$$

2. the g-vector:

$$g_r(\Delta, n) = h_r(\Delta, n) - h_{r-1}(\Delta, n) = \sum_{j=-1}^{r-1} (-1)^{j+r+1} \binom{n-j}{n-r+1} f_j.$$

These have a striking similarity to the Euler characteristics for skeletal rigidity defined above, in the special cases where $d = n-1$ for h_r and $d = n$ for g_r [**TWW3**]. These give:

$$\begin{aligned}
h_r(\Delta, n) &= \sum_{j=-1}^{r-1} (-1)^{j+r+1} \binom{n-j-1}{n-r} f_j = (-1)^{r+1} \chi(\mathcal{R}_{n-1,r}(\Delta; \overline{\mathbf{p}})) \\
&= \beta_{r-1}(\mathcal{R}_{n-1,r}(\Delta; \overline{\mathbf{p}})) + \sum_{i=-1}^{r-2} (-1)^{i+r+1} \beta_i(\mathcal{R}_{n-1,r}(\Delta; \overline{\mathbf{p}})) \\
g_r(\Delta, n) &= \sum_{j=-1}^{r-1} (-1)^{j+r+1} \binom{n-j}{n-r+1} f_j = (-1)^{r+1} \chi(\mathcal{R}_{n,r}(\Delta; \overline{\mathbf{p}})) \\
&= \beta_{r-1}(\mathcal{R}_{n,r}(\Delta; \overline{\mathbf{p}})) + \sum_{i=-1}^{r-2} (-1)^{i+r+1} \beta_i(\mathcal{R}_{n,r}(\Delta; \overline{\mathbf{p}})).
\end{aligned}$$

There are a number of fundamental results about these numbers, culminating in the g-theorem for the complexes of simplicial convex polytopes [**St1**,**2**,**Lee1**,**2**]. A critical aspect of this theorem is that the numbers given above are non-negative. If all the lower Betti numbers in these two expressions are zero, then, as the dimension of a vector space Z_{r-1}, $h_r(\Delta, n)$ and $g_r(\Delta, n)$ would be shown directly to be non-negative. The chain complex for skeletal rigidity was originally constructed for this purpose and significant partial results have been obtained [**TWW3**].

Homology Sphere Theorem 16.2.6 [**TWW3**]. *Let Δ be a simplicial homology n-sphere realized in d-space, with the vertices of each face in general position in the configuration $\overline{\mathbf{p}}$.*

1. *If $d = n$, then for all r, $\beta_i(\mathcal{R}_{d,r}(\Delta; \overline{\mathbf{p}})) = 0, i < r-1$;*
2. *If $d = n+1$, then for all r, $\beta_i(\mathcal{R}_{d,r}(\Delta; \overline{\mathbf{p}})) = 0, i < r-2$.*

Combinatorial theorems on convex n-polytopes Δ tell us that $h_i(\Delta) = h_{n-i}(\Delta)$, and $h_{i-1}(\Delta) < h_i(\Delta)$ for $i \leq \frac{n}{2}$ [**St2**,**Lee1**,**2**]. These results imply relationships on the spaces of dependencies in this sequence of skeletal k-cycle matroids on $\Delta(\overline{\mathbf{p}})$ in n-space. We conjecture that these relationships are geometric: there should be explicit isomorphisms and injections for the corresponding circuits in our matroids.

Geometric g-conjecture 16.2.7. *The r-skeleton of a simplicial (convex) $(2r-1)$-polytope is a basis for skeletal r-rigidity matroid in $(2r-1)$-space and is r-ample, for all strictly convex realizations $\overline{\mathbf{p}}$.*

The r-skeleton of a simplicial homology $(2r-2)$-sphere is a basis for generic skeletal r-rigidity and is generically r-ample in $(2r-1)$-space.

We note that the Theorem of Cauchy and Dehn 9.2.10 gives precisely this result for $r = 2$: the 2-rigidity of triangulated convex spheres in 3-space. There is some indication that recent geometric results of McMullen [**Mc**] will prove the first part of this conjecture. If the generic conjecture for homology spheres were also proven, this would extend known results.

16.3. r-skeletal cohomology. We have a companion cohomology which describes the 'skeletal kinematics' dual to the 'skeletal statics' of the previous subsections. For a simplicial $(r-1)$-complex Δ^{r-1} realized in affine d-space, the *r-skeletal* *cochain complex* is

$$\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}})\colon \mathbf{0} \leftarrow \oplus_{\rho\in\Delta^{(r-1)}} V_{d+1}^{(d+1-r)}/\ker\rho \xleftarrow{\delta_{r-1}} \oplus_{\pi\in\Delta^{(r-2)}} V_{d+1}^{(d+1-r)}/\ker\pi \xleftarrow{\delta_{r-2}} \cdots \xleftarrow{\delta_1} \oplus_{v\in\Delta^{(0)}} V_{d+1}^{(d+1-r)}/\ker v \xleftarrow{\delta_0} V_{d+1}^{(d+1-r)} \leftarrow \mathbf{0}.$$

For a general i-simplex σ, with coefficient c_σ, a $(d+1-r)$-extensor modulo $\overline{\sigma}$:

$$\delta_i(c_\sigma\cdot[\sigma]) = \sum_{\rho=\sigma x} Sign(\sigma,\rho)c_\sigma\cdot[\rho]$$

Again this is extended linearly to i-cochains. The maps are well-defined on the equivalence classes $[\sigma]$ and we have $\delta_i\delta_{i-1}=0$.

The elements of the kernel of δ_i are the *i-cocycles* $Z^i(\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}}))$, and the elements of the image of δ_{i-1} are the *i-coboundaries*, $B^i(\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}}))$. We have the usual cohomology spaces of the cochain complex:

$$\widetilde{H}^i(\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}})) = Z^i(\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}}))/B^i(\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}})).$$

and the corresponding *Betti numbers* $\beta^i(\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}})) = \dim H^i(\mathcal{R}^{d,r}(\Delta;\overline{\mathbf{p}}))$.

Given an i-chain $\mathbf{c}$, with $c_\pi \in V_{d+1}^{(r-i)}/\ker\pi$, and an i-cochain $\mathbf{d}$, with $d_\pi \in V_{d+1}^{(d+1-r)}/\ker\pi$, we have a bilinear form

$$\langle\ \mathbf{c},\mathbf{d}\ \rangle = \sum_{\pi\in\Delta^{(i)}} c_\pi \vee d_\pi \vee \overline{\pi}.$$

Notice that these products are $(r-i+d+1-r+i)=(d+1)$-tensors in V_{d+1} – or real numbers. With this inner product, the usual duality for an $(i+1)$-chain $\mathbf{c}$ and an i-cochain $\mathbf{d}$ holds:

$$\langle\ \partial_{i+1}\mathbf{c},\ \mathbf{d}\ \rangle = \langle\ \mathbf{c},\ \delta^i\mathbf{d}\ \rangle$$

and the resulting orthogonality and isomorphisms of homology and cohomology spaces and equality of i-Betti numbers, Euler characteristic etc. follow.

We are just at the initial stages of exploring the balance of 'skeletal r-static theory' (homology) and 'skeletal r-kinematic theory' (cohomology). We note that the $(r-2)$-cochains (the $(r-2)$-motions) have a direct geometric interpretation [**Lee1**,**2**]. Informally, these 'motions' are infinitesimal displacements of the $(r-2)$-faces which instantaneously preserve the $(r-1)$-volume of the $(r-1)$-faces (but may not preserve the global incidences on the $(r-3)$-faces). Indeed, this is the generalization of skeletal 2-motions: infinitesimal displacements of the vertices which preserve the length of the edges. It also gives an intuitive kinematic interpretation of ordinary homology for $d=r-1$.

Remark 16.3.1. For a simplicial complex in affine d-space, this cohomology is directly related to the Stanley-Reisner ring of the simplicial complex, an algebraic object used in algebraic combinatorics [**St2**,**Lee1**,**2**,**TWW1**,**2**,**3**].

The Stanley-Reisner ring does not exist for non-simplicial complexes. However we can define skeletal r-rigidity for non-simplicial cell complexes. Given an abstract cell complex, we need a d-configuration which makes each face 'flat' and ensures the vertices of an i face span an affine space of dimension i. With this assumption, we can complete extensions from §16 directly, using arbitrarily chosen $(i+1)$-extensors for the subspaces spanned by oriented i-faces and taking more care with the boundary operators. These matroids have not yet been studied for skeletal

r-rigidity beyond 3-rigidity in 3-space [**CrW4**]. Their significance for polyhedral combinatorics, extended h vectors etc. is unexplored.

REMARK. This cohomology, and the cohomologies of §15, do not 'represent' the orthogonal matroid, even for lower faces of a larger complex. In the terms of §14.3, our chain complexes have only zero r-chains, because of the definitions of the coefficients. With this observation, the only time $\beta_{r-1}(\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})) = 0$ is when the entire set is independent and the orthogonal matroid is trivial!

However, for homology n-spheres in n-space, there is a critical Euclidean geometric construction, called the *reciprocal diagram*, which originated with Clerk Maxwell [**Max**] for plane statics on 2-spheres. In this geometric reciprocity, the original complex Δ at $\mathbf{p}$ and the dual complex Δ^* and $\overline{\mathbf{p}}^*$ are realized with the face $\sigma(\mathbf{p})$ orthogonal in n-space to the dual face $\sigma^*(\mathbf{p}^*)$. This reciprocal relationship induces a reciprocal pair of a skeletal i-cycle in $\Delta(\overline{\mathbf{p}})$ and a skeletal $(n-i)$-cycle in $\Delta^*(\overline{\mathbf{p}}^*)$, for each i. There are also several correspondences either proven or conjectured between skeletal i-cycles on $\Delta(\overline{\mathbf{p}})$ and skeletal $(n-i)$-cocycles of $\Delta^*(\overline{\mathbf{p}}^*)$ [**CrW2**,**3**,**4**].

There are further known and conjectured correspondences between i-cocycles (kinematics) on an n-sphere in $(n+1)$-space, $\Delta(\overline{\mathbf{p}})$, and $(n-i)$-cocycles on the same geometric complex [**CrW1**,**TWW3**]. These correspondences are directly related to the known combinatorial symmetry of the g-sequence and h-sequence of a convex spherical polytope [**Lee1**,**2**,**St2**].

16.4. Lower homologies and hypermatroids.

In §15.1 we saw that $\widetilde{H}_0(\mathcal{K}_1^0(\Delta^2;\mathbf{p}))$ matched the non-trivial parallel drawings of the underlying geometric graph $\Delta^1(\mathbf{p})$. This geometry gives a hypermatroid with independent sets defined by $2|E| \leq 3|V| - 4$ (§5.3). This is one small piece of a larger picture.

PROPOSITION 16.4.1. *For the chain complex $\mathcal{R}_{r,r}(\Delta;\overline{\mathbf{p}})$, β_0 is the Betti number for the parallel drawing matroid (and chain complex) of the geometric graph $\Delta^1(\overline{\mathbf{p}})$ in r-space.*

In particular, $\beta_0(\mathcal{R}_{r,r}(\Delta;\overline{\mathbf{p}})) = 0$ if and only if $\Delta^1(\overline{\mathbf{p}})$ has only trivial parallel drawings in r-space.

COROLLARY 16.4.2. *For the chain complex $\mathcal{R}_{r,r}(\Delta;\overline{\mathbf{p}})$, $\beta_0(\mathcal{R}_{r,r}(\Delta;\overline{\mathbf{p}})) = 0$ if the geometric graph $\Delta^1(\overline{\mathbf{p}})$ is first-order rigid in r-space.*

PROOF. For a geometric graph $\Delta^1(\overline{\mathbf{p}})$ in r-space, a non-trivial parallel drawing generates one (or more) non-trivial first-order motions [**Wh5**,**Wh15**].

These results apply the the precise cases where skeletal rigidity is known to be isomorphic to the C_1^0-cofactor matroids. We do not know to what depth the generic lower homologies and the corresponding matroids of these two sequences will actually coincide.

Each lower homology of $\mathcal{R}_{d,k}(\Delta;\mathbf{p})$ defines a corresponding hypermatroid on these lower faces Δ^i of the complex. Most of these hypermatroids have not been studied or even explicitly described. The 'off shoots' of Figures 13.1 and 1.1 give the rank estimates for some of the lowest of these families: counts obtained from the Euler characteristic of the truncated chain complex. We cannot demonstrate that each of these hypermatroids will have an independent geometric significance – but we suspect that all of them will.

The following tables summarize some connections which are confirmed and conjectured for the lower homologies of rigidity. The vertical arrows $\uparrow ?$ represent an conjectured injection of spaces under vertical projection in dimension, assuming the configurations have no faces 'vertical'. While we have not worked out all the details, there is strong evidence for this injection, building on the results and techniques for coning in [**TWW3**]. The horizontal arrows $\overset{?}{\hookleftarrow}$ represent conjectured injections of the homologies which would generalize the connection between parallel drawings and first-order rigidity in the $\mathbb{R}^3$ line of the upper table. It is likely that these can be proven directly, using similar techniques in the geometric homology [**Wh15**]. Similarly, some of the conjectures in the $\mathbb{R}^4$ line of the lower table are probably implicit in the connection between parallel drawing and first-order rigidity.

	1-skeleton		2-skeleton		3-skeleton		4-skeleton
$\mathbb{R}^6$	$\not{H}_0(\mathcal{R}_{6,2}(\Delta^1;\overline{\mathbf{p}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{6,3}(\Delta^2;\overline{\mathbf{p}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{6,4}(\Delta^3;\overline{\mathbf{p}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{6,5}(\Delta^4;\overline{\mathbf{p}}))$
$\Pi(\overline{\mathbf{p}})$	$\uparrow$		$\uparrow ?$		$\uparrow ?$		$\uparrow ?$
$\mathbb{R}^5$	$\not{H}_0(\mathcal{R}_{5,2}(\Delta^1;\overline{\mathbf{q}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{5,3}(\Delta^2;\overline{\mathbf{q}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{5,4}(\Delta^3;\overline{\mathbf{q}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{5,5}(\Delta^4;\overline{\mathbf{q}}))$
$\Pi(\overline{\mathbf{q}})$	$\uparrow$		$\uparrow ?$		$\uparrow ?$		$\uparrow ?$
$\mathbb{R}^4$	$\not{H}_0(\mathcal{R}_{4,2}(\Delta^1;\overline{\mathbf{r}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{4,3}(\Delta^2;\overline{\mathbf{r}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\mathcal{R}_{4,4}(\Delta^3;\overline{\mathbf{r}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\Delta)$
$\Pi(\overline{\mathbf{r}})$	$\uparrow$		$\uparrow ?$		$\uparrow ?$		
$\mathbb{R}^3$	$\not{H}_0(\mathcal{R}_{3,2}(\Delta^1;\overline{\mathbf{s}}))$	$\hookleftarrow$	$\not{H}_0(\mathcal{R}_{3,3}(\Delta^2;\overline{\mathbf{s}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\Delta)$		
$\Pi(\overline{\mathbf{s}})$	$\uparrow$		$\uparrow ?$				
$\mathbb{R}^2$	$\not{H}_0(\mathcal{R}_{2,2}(\Delta^1;\overline{\mathbf{t}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_0(\Delta)$				
$\Pi(\overline{\mathbf{t}})$	$\uparrow$						
$\mathbb{R}^1$	$\not{H}_0(\Delta)$						

	2-skeleton		3-skeleton		4-skeleton	
$\mathbb{R}^6$	$\not{H}_1(\mathcal{R}_{6,3}(\Delta^2;\overline{\mathbf{p}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_1(\mathcal{R}_{6,4}(\Delta^3;\overline{\mathbf{p}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_1(\mathcal{R}_{6,5}(\Delta^4;\overline{\mathbf{p}}))$	$\overset{?}{\hookleftarrow}$
$\Pi(\overline{\mathbf{p}})$	$\uparrow ?$		$\uparrow ?$		$\uparrow ?$	
$\mathbb{R}^5$	$\not{H}_1(\mathcal{R}_{5,3}(\Delta^2;\overline{\mathbf{q}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_1(\mathcal{R}_{5,4}(\Delta^3;\overline{\mathbf{q}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_1(\mathcal{R}_{5,5}(\Delta^4;\overline{\mathbf{q}}))$	$\overset{?}{\hookleftarrow}$
$\Pi(\overline{\mathbf{q}})$	$\uparrow ?$		$\uparrow ?$		$\uparrow ?$	
$\mathbb{R}^4$	$\not{H}_1(\mathcal{R}_{4,3}(\Delta^2;\overline{\mathbf{r}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_1(\mathcal{R}_{4,4}(\Delta^3;\overline{\mathbf{r}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_1(\Delta)$	
$\Pi(\overline{\mathbf{r}})$	$\uparrow ?$		$\uparrow ?$			
$\mathbb{R}^3$	$\not{H}_1(\mathcal{R}_{3,3}(\Delta^2;\overline{\mathbf{s}}))$	$\overset{?}{\hookleftarrow}$	$\not{H}_1(\Delta)$			
$\Pi(\overline{\mathbf{s}})$	$\uparrow ?$					
$\mathbb{R}^2$	$\not{H}_1(\Delta)$					

These conjectures are summarized in the following form, with (1) and (2) representing the horizontal rows and (3) and (4) representing the vertical columns. For β_0 and for the horizontal arrows, the existing evidence is stronger than for the vertical arrows on higher Betti numbers.

CONJECTURE 16.4.3.

1. *For the chain complexes* $\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})$, *and* $\mathcal{R}_{d,r+1}(\Delta;\overline{\mathbf{p}})$,
   $\beta_i(\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})) \geq \beta_i(\mathcal{R}_{d,r+1}(\Delta;\overline{\mathbf{p}}))$ *for* $i < r-1$.
2. *In particular,* $\beta_i(\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})) = 0$ *if* $\beta_i(\mathcal{R}_{d,r-k}(\Delta^i;\overline{\mathbf{p}})) = 0$ *for any* $k \leq r-1$ *and* $i < r-k-1$.
3. *For the chain complexes* $\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})$, *and* $\mathcal{R}_{d-1,r}(\Delta;\overline{\mathbf{q}})$ *with* $\overline{\mathbf{q}} = \Pi\overline{\mathbf{p}}$,
   $\beta_i(\mathcal{R}_{d,r}(\Delta;\overline{\mathbf{p}})) \geq \beta_i(\mathcal{R}_{d-1,r}(\Delta;\overline{\mathbf{q}}))$ *for* $i < r-1$.
4. *In particular,* $\beta_i(\mathcal{R}_{d,r}(\Delta;\Pi\overline{\mathbf{p}})) = 0$ *if* $\beta_i(\mathcal{R}_{d+k,r}(\Delta^i;\overline{\mathbf{p}})) = 0$ *for any* $i < r-1$.

The following table summarizes results for the top Betti numbers which follow from the coning theorems (if the cone point is then dropped).

	1-skeleton	2-skeleton	3-skeleton	4-skeleton
$\mathbb{R}^5$	$\not H_1(\mathcal{R}_{5,2}(\Delta^1;\overline{\mathbf{q}}))$	$\not H_2(\mathcal{R}_{5,3}(\Delta^2;\overline{\mathbf{q}}))$	$\not H_3(\mathcal{R}_{5,4}(\Delta^3;\overline{\mathbf{q}}))$	$\not H_4(\mathcal{R}_{5,5}(\Delta^4;\overline{\mathbf{q}}))$
$\Pi(\overline{\mathbf{q}})$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathbb{R}^4$	$\not H_1(\mathcal{R}_{4,2}(\Delta^1;\overline{\mathbf{r}}))$	$\not H_2(\mathcal{R}_{4,3}(\Delta^2;\overline{\mathbf{r}}))$	$\not H_3(\mathcal{R}_{4,4}(\Delta^3;\overline{\mathbf{r}}))$	$\not H_4(\Delta^4)$
$\Pi(\overline{\mathbf{r}})$	$\downarrow$	$\downarrow$	$\downarrow$	
$\mathbb{R}^3$	$\not H_1(\mathcal{R}_{3,2}(\Delta^1;\overline{\mathbf{s}}))$	$\not H_2(\mathcal{R}_{3,3}(\Delta^2;\overline{\mathbf{s}}))$	$\not H_3(\Delta^3)$	
$\Pi(\overline{\mathbf{s}})$	$\downarrow$	$\downarrow$		
$\mathbb{R}^2$	$\not H_1(\mathcal{R}_{2,2}(\Delta^1;\overline{\mathbf{t}}))$	$\not H_2(\Delta^2)$		
$\Pi(\overline{\mathbf{t}})$	$\downarrow$			
$\mathbb{R}^1$	$\not H_1(\Delta^1)$			

We have very little evidence about analogous relationships for $K_s^{s-1}(\Delta^m)$. We can only suggest that the possibility of similar connections should be investigated.

16.5. The analogy between skeletal matroids and cofactor matroids. A recurrent theme for §15 and §16 has been the fundamental analogy between the generic matroids $\mathcal{K}_s^{s-1}(\Delta^m)$ and the generic matroids $\mathcal{R}_{d,r}(\Delta^{r-1})$, where $r = m+1$ and $d = m+s$. Certain specific correspondences have been observed:

(a) the i-chains of the two complexes have identical dimensions and the resulting Euler characteristics are equal;

(b) $\mathcal{K}_1^0(\Delta^m;\mathbf{p})$ and $\mathcal{R}_{r,r}(\Delta^{r-1};\overline{\mathbf{p}})$ are isomorphic and the three chain complexes $\mathcal{K}_0^{-1}(\Delta^m)$, $\mathcal{R}_{r-1,r}(\Delta)$, and the simplicial $(r-1)$-homology of Δ^{r-1} are isomorphic.

Some additional correspondences have been conjectured:

(c) *Conjecture*: the matroids $\mathcal{K}_2^1(\Delta^1)$ and $\mathcal{R}_{3,2}(\Delta)$ are isomorphic (§10.3);

(d) *Conjecture*: all sets independent in $\mathcal{R}_{k,2}(\Delta)$ are independent $\mathcal{K}_k^{k-1}(\Delta^1)$ (§11.5);

(e) *Conjecture*: the geometric map from the cone $\mathcal{R}_{d+1,r}(w * \Delta, \overline{\mathbf{p}}_w;\overline{\mathbf{p}})$ to the projection $\mathcal{R}_{d,r}(\Delta; Pi(\overline{\mathbf{p}}))$ in skeletal r-rigidity transfers to a geometric correspondence from $\mathcal{K}_{s+1}^s((w*\Delta)^r;\mathbf{q}_w,\mathbf{q})$ to $\mathcal{K}_s^{s-1}(\Delta^r;\mathbf{q})$ (§15.2)

To these we add several new conjectures:

(f) *Conjecture*: all the results of Example 16.2.4 for K_n extend to the corresponding $\mathcal{K}_s^{s-1}(K_n)$. For example, K_n^m is $\mathcal{K}_s^{s-1}$-acyclic in dimension m if and only if $s+1 \le n \le m+s+1$.

(g) *Conjecture*: for each r, there is a generic isomorphism between $\mathcal{K}_r^{r-1}(\Delta^{r-1})$ and $\mathcal{R}_{r+1,r}(\Delta^{r-1})$.

(h) *Conjecture*: For each k, m, all sets independent in $\mathcal{R}_{k+m,r}(\Delta^m)$ are independent in $\mathcal{K}_k^{k-1}(\Delta^m)$.

(i) The Homology Sphere Theorem 16.2.6 and Geometric g-Conjecture 16.2.7 transfer to the cofactor matroids.

Remark 16.5.1. For applications back to multivariate splines it will be important to develop the appropriate theory of $\mathcal{K}_k^{k-1}(\Delta^r,\Delta';\overline{\mathbf{p}})$ – the homology of the corresponding chain complexes relative to the 'boundary' of the simplicial complex. We conjecture that the same relationships conjectured for the original homologies also apply to these relative homologies, using 'pinned skeletal rigidity' for $\mathcal{R}_{r,d}(\Delta^r,\Delta';\overline{\mathbf{p}})$. Only a few pieces of these theories are now evident –

one of which is a valuable geometric 'coning' result giving an isomorphism from $\mathcal{K}_k^{k-1}(w * \Delta^r, (\Delta^r \cup w * \Delta'); \overline{\mathbf{p}}_0, \overline{\mathbf{p}})$ to $\mathcal{K}_k^{k-1}(\Delta^r, \Delta'; \Pi(\overline{\mathbf{p}}))$ [**ASW**].

The interested reader can certainly make a longer list of conjectures and problems to be investigated for each of these structures. So little work has been completed that it remains unclear which directions will be most fruitful and which techniques implicitly require graphs and are not open to this generality.

17. Summary of Themes

We have displayed a wide range of matroids arising from work in discrete geometry. In presenting each of these examples in Figures 1.1 and 13.1 these matroids have been displayed in families, organized:

1. in vertical strips: by dimension of the space of realization (for skeletal rigidity matroids); or algebraic powers (for cofactor matroids) for fixed topological dimension of the abstract complex;
2. by homological (lower homologies) and geometric relationships;
3. in layers by dimension of the underlying abstract objects: edges, 2-faces, $\ldots$, r-faces, $\ldots$;
4. by 'off shoots' such as parallel drawings – and other unexplored generalizations which complete each strip to a full quadrant of interrelated matroids.

There are a number of common themes in the analysis of these matroids. Some of these have been explicit, but some of the patterns were left implicit in the flow of our presentation. Examples include:

5. homology and cohomology for augmented chain complexes over the reals, leading to a representing matrix for the matroid, a 'static theory' of cycles, a 'kinematic theory' of cocycles and an accounting of when lower homologies are zero;
6. inductive techniques, including 'gluing' of complexes (based implicitly or explicitly on the Mayer-Vietoris sequence of homology) and shellings for complexes;
7. combinatorial estimates of the rank of the matroid, based on the numbers of faces f_i of each dimension i;
8. combinatorial / topological conditions which guarantee these rank estimates are generically correct (based on showing lower homologies are geometrically or generically zero);
9. geometric conditions for a change from this generic rank (implicitly or explicitly projective geometry) and some geometric conditions for the generic rank to be achieved geometrically.

There are numerous connections among the matroids in neighboring cells of this display. Some of these have been explicitly pointed out, but many more have been left to remarks or omitted entirely. Examples include connections:

10. by the analogy between skeletal rigidity matroids and cofactor matroids of the same rank on complete (ample) complexes – displayed as parallel strips which coincide for the first two (conjectured three) cells in each strip;
11. by coning up and projection down between cells of skeletal k-rigidity (conjectured to also hold for cofactor matroids);
12. by lower homologies, such as parallel drawings of edges for skeletal 3-rigidity in 3-space, and numerous similar connections we could not present;

13. by coning and sectioning between matroids on objects of adjacent dimensions;

The significance of these matroids comes from several sources. The original matroids for first-order rigidity, for scene analysis, and for cofactors, arose directly from applied problems. Of course we have generalized beyond the questions arising in these applications, in part to grasp the underlying structure, in part to find solutions to residual unsolved problems, and in part to enjoy the rich interplay of patterns.

The 'secondary' matroids, such as parallel drawings and skeletal rigidity, correspond precisely to existing geometrical problems (such as Minkowski decomposability and the g-theorem) but arose from our geometric play and from the continuing drive to generalize patterns which gave such insight to the geometry of polytopes.

The matroids make explicit the essential role of homology (and k-cycle matroids) and the need for substantial combinatorial and algorithmic theory for the homologies of larger complexes, both simplicial and non-simplicial, extending the base case of graph theory (1-cycle homology). We anticipate an expanding role for homology-based matroids and geometric homology ([**Cr1**,**3**,**4**,**CR2**,**Wh 15**]) and corresponding techniques within matroid theory and its applications.

A central example – generic rigidity in 3-space – highlights essential unsolved problems in matroid theory / graph theory / algorithms. We have no effective combinatorial characterization (other than take a matrix with variable entries and crush it); we have the 'maximal matroid' conjecture for $3|V| - 6$ highlighting the issue of submodular functions negative on singletons which still appear to describe a matroid; and we have an explicit example of a graph theoretic property which is random polynomial, but may not be polynomial in complexity.

As I mentioned in the introduction, I am a geometer. I view areas like graph theory, matroid theory, homology theory and related combinatorics as (abstract) layers in the rich hierarchy of geometries – an essential ground for further studies, but only one portion of a much larger study. The geometric patterns underlying these matroids was essential and delightful. The generic combinatorial patterns are sometimes accessible through combinatorial techniques, sometimes through geometric techniques. It seems very appropriate that geometric and topological techniques should appear in this study of 'combinatorial geometries'.

Appendix: Matroids from Counts on Graphs and Hypergraphs.

A.1. The basic counts. A number of the 'rank counts' in this paper directly defined matroids on graphs (and hypergraphs). Some years ago, a joint project with Neil White [**WW2**] derived some basic results for these 'count matroids' which we will present (and extend) here. Independently, and earlier, the basic constructions of this section (and the connectivity results of the next section) were also described by Lorea [**Lor**]. An examination of these constructions will highlight the line which separates the well-understood matroids, such as generic 2-rigidity and parallel drawings, from the matroids with fundamental unsolved problems, such as generic 3-rigidity and homology.

Throughout this appendix we will work with general hypergraphs $H = (V, E)$, with multiple edges and loops (edges of cardinality 1) allowed.

COUNTS TO MATROIDS PROPOSITION A.1.1 [**Lor**,**WW2**]. *For integers $m \geq 0$ and k, the following definition gives the independent sets of a matroid $\mathcal{M}_{m,k}(E)$ on* *the edges of a hypergraph (in particular, of the complete hypergraph on a finite set of vertices V):*

A set of edges E is independent if and only if for all non-empty subsets E' on vertices V': $|E'| \le m|V'| + k$.

Proof. We verify the circuit exchange axiom. A *circuit* in this independence structure $\mathcal{M}_{m,k}$ is a minimal dependent set defined by:

a non-empty set E of edges in the hypergraph is a circuit if and only if
(i) $|E| = m|V| + k + 1$ *and,*
(ii) *for all proper subsets E' with vertices $V(E')$: $|E'| \le m|V(E')| + k$.*

Given circuits $|E_1| = m|V_1| + k + 1$, $|E_2| = m|V_2| + k + 1$, sharing an edge e, we show that $|E_1 \cup E_2| - \{e\}$ contains a circuit. First, we have:

$$\begin{aligned} |E_1 \cup E_2| &= |E_1| + |E_2| - |E_1 \cap E_2| \\ |V(E_1)| + |V(E_2)| &= |V(E_1 \cap E_2)| + |V(E_1) \cap V(E_2)| \\ &\le |V(E_1 \cap E_2)| + |V(E_1 \cap E_2)| \end{aligned}$$

Since $|E_1 \cap E_2| \le m|V(E_1) \cap V(E_2)| + k$, substitution gives:

$$\begin{aligned} |E_1 \cup E_2| &\ge m|V(E_1)| + k + 1 + m|V(E_2)| + k + 1 - [m|V_1 \cap V_2| + k] \\ &\ge m\big(|V(E_1)| + |V(E_2)| - |V(E_1 \cap E_2)|\big) + 2k - k + 2 \\ &\ge m\big(|V(E_1)| + |V(E_2)| - |V(E_1) \cap V(E_2)|\big) + k + 2 \\ &\ge m|V(E_1 \cup E_2)| + k + 2 \end{aligned}$$

In passing from $-m|V(E_1 \cap E_2)|$ to $-m|V(E_1) \cap V(E_2)|$ we used the assumption that $m \ge 0$. We now delete e from the left hand side and 1 from the right:

$$|(E_1 \cup E_2) - \{e\}| \ge m|V(E_1 \cup E_2)| + k + 1 \ge m|V((E_1 \cup E_2) - \{e\})| + k + 1.$$

This set must contain a circuit. For example, a minimal non-empty subset E_* of the set $E_1 \cup E_2 - \{e\}$ with $|E_*| = m|V(E_*)| + k + 1$ will give the required circuit.

Special cases of this matroid are abundant in matroid theory [**Lor**]:
(a) $\mathcal{M}_{1,-1}(E)$ is the cycle matroid of the graph $(V(E), E)$;
(b) $\mathcal{M}_{1,0}(E)$ is the bicycle matroid (circuits are two connected polygons);
(c) $\mathcal{M}_{1,0}(H)$ is the transversal matroid of a general hypergraph;
(d) for any positive integer m, $M_{m,0}(H)$ is the matroid union of m copies of the transversal matroid of a general hypergraph;
(e) $\mathcal{M}_{1,-2}(H)$ is Crapo's geometry of triples, if H is a hypergraph with all edges of cardinality 3.
(f) $\mathcal{M}_{2,-3}(E)$ is the generic 2-rigidity matroid on ordinary graphs.

Remark A.1.2. If we try $k \le -2m$, the matroid will have all ordinary graph edges (pairs) dependent. Each singleton $\{e\}$ will satisfy $|\{e\}| = 1 > 2m - 2m = m|V'| + k$, so it is dependent. Therefore, for ordinary graphs we should assume that $-k < 2m$.

In general, we need $k > -nm$ to make the matroid non-trivial on n-tuple edges. We also need $k > -m$ to make loops (single vertices) into independent elements. If $k \ge -mn + t$ then t copies of n-tuple edges will be independent.

If we try non-integers $m > 0$, we do not get a matroid. For example, the count $|E'| \leq .5|V'|$ on non-empty subsets of E, defines an interesting structure: the matchings on a graph (Figure A.1). This is not a matroid: the circuits of Figure A.1.B 'exchange' on their shared edge to the independent set of Figure A.1.C.

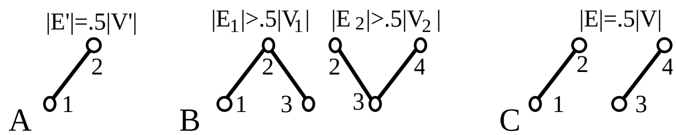

Fig. A.1. *A counterexample to $|E| \leq .5|V|$ defining independence in a matroid on graphs.*

If we try to adapt the proof above, and replace the condition for dependence $|E| > m|V| + k$ by $|E| = \lfloor m|V| + k \rfloor + 1$, the problem is that the floor function need not be submodular on non-integers.

For rational m, these counts do produce 'polymatroids' or 'hypermatroids' on appropriate multigraphs (see §A.3).

Remark A.1.3. Most of the results in this appendix can be viewed as applications of a general result defining a matroid on a set A from a submodular function which is non-negative on non-empty subsets of the set A [**Ed**,**ER**]. Recall that an real valued function $f : 2^A \to \mathbb{R}$ is *submodular* if
for all subsets B, C, $f(B) + f(C) \geq f(B \cup C) + f(B \cap C)$.

Submodular Theorem A.1.4 [**We**]. *Given a submodular function μ on 2^S, non-negative on all non-empty subsets of S, the collection:*

$$\mathcal{I}(\mu) = \{A \mid |A \cap Y| \leq \mu(Y), Y \in 2^S - \emptyset\}$$

are the independent sets of a matroid $\mathcal{M}(\mu)$ on A.
Moreover, the rank function ρ_μ is given for $A \subseteq S$ by

$$\rho_\mu(A) = \min\Big(|A|, \sum_{i=1}^{k} \mu(X_i) + |A \cap (S - \cup X_i)|\Big).$$

Lemma A.1.5. *$f(E) = m|V(E)| + k$ is a submodular function on 2^E, provided $m \geq 0$ and $m|V(e)| + k \geq 0$ on singletons.*

Proof. The core of our proof above amounted to:

$$f(E_1) + f(E_2) \geq f(E_1 \cup E_2) + f(E_1 \cap E_2).$$

To keep $f(E_1) \geq 0$ on non-empty subsets of large sets E, we need $m \geq 0$. Finally to make the matroid positive on singletons, we need $m|V(e)| + k \geq 1$.

One sign of trouble for the generic 3-rigidity count is that $f(\{e\}) = 3|V(e)| - 6 = 0$. The count of this matroid would define all edges of a graph as dependent.

A.2. Some structure results from counts. We let $\mathcal{M}_{m,k}(n)$ denote the m, k counting matroid on the complete hypergraph on n vertices (i.e. $E = 2^V$) and let $\mathcal{M}_{m,k}(d, n)$ denote the matroid on the d-complete hypergraph on n vertices which contains d copies of each edge.

Recall that the *matroid union* of two matroids $\mathcal{M}_1$, and $\mathcal{M}_2$ is the matroid $\mathcal{M}_1 \vee \mathcal{M}_2$ defined by:

A set I is independent in the union if and only if it can be partitioned into $I_1 \cup I_2$ with I_1 independent in $\mathcal{M}_1$ and I_2 independent in $\mathcal{M}_2$.

The following general result connects matroid unions with sums of submodular functions.

Proposition A.2.1 [**PP**]. *If μ_1, μ_2 are non-decreasing, integer valued, non-negative submodular functions on 2^S then $\mathcal{M}(\mu_1) \vee \mathcal{M}(\mu_2) = \mathcal{M}(\mu_1 + \mu_2)$.*

Applied to our matroids, this gives:

Proposition A.2.2. *If $m = m_1 + m_2$ and $k = k_1 + k_2$, with $m_1, m_2 \geq 0$ and $m_1|V(e)| + k_1 \geq 0, m_2|V(e)| + k_2 \geq 0$ for all edges $e \in E$, then $\mathcal{M}_{m,k}(E)$ is the matroid union $\mathcal{M}_{m_1,k_1}(E) \vee \mathcal{M}_{m_2,k_2}(E)$.*

Example A.2.3. The restriction in this proposition is essential. For example

$$\mathcal{M}_{2,-3}(E) \neq \mathcal{M}_{1,-1}(E) \vee \mathcal{M}_{1,-2}(E),$$

since $\mathcal{M}_{1,-2}$ is trivial on graphs (making all doubleton edges dependent). This matroid union is correct for a hypergraph with all edges of valence at least 3.

Example A.2.4. Examples of this result include:

(i) $\mathcal{M}_{n,-n}(G) = \bigvee_{i=1}^{n} \mathcal{M}_{1,-1}(G)$. Equivalently, for $n|V| - n = n(|V| - 1)$ the matroid is the matroid union of n copies of the cycle matroid for an ordinary graph (recall §12.1);

(ii) for $m \geq n \geq 0$, $\mathcal{M}_{m,-n}(G) = \bigvee_{i=1}^{n} \mathcal{M}_{1,-1}(G) \vee \bigvee_{j=1}^{m-n} \mathcal{M}_{1,0}$ or the matroid for $m|V| - n$, $m > n \geq 0$, is the matroid union of n copies of the cycle matroid for a ordinary graph and $m - n$ copies of the bicycle matroid;

(iii) $\mathcal{M}_{2,0}(G) = \mathcal{M}_{1,1}(G) \vee \mathcal{M}_{1,-1}(G) = \mathcal{M}_{1,0}(G) \vee \mathcal{M}_{1,0}(G)$, so the decomposition is *not unique*.

Many of these matroids actually appear as the first-order rigidity matroids of generic frameworks on alternate surfaces [**Wh7**]:

(i) $\mathcal{M}_{2,-2}$ appears as the first-order rigidity matroid of generic frameworks on the flat torus or the cylinder (which have a 2-space of global congruences);

(ii) $\mathcal{M}_{2,-1}$ appears as the first-order rigidity matroid of generic frameworks on the surface of a cone (which has a 1-dimensional space of global congruences);

(iii) $\mathcal{M}_{2,0}$ appears as the first-order rigidity matroid of generic frameworks on the surface of a convex polyhedron (which has a 0-dimensional space of global congruences).

Similar connections can be found for matroids on graphs of the form $\mathcal{M}_{m,k}(G)$ for $m > 2$, $0 \geq k \geq -\binom{m}{2}$ on surfaces of dimension m.

We next show that all count matroids $\mathcal{M}_{m,k}(n)$ (and therefore $\mathcal{M}_{m,k}(E)$ for all subsets E) are coordinatizable over $\mathbb{R}$ or over any infinite field.

Coordinatization Theorem A.2.5 [**WW2**]. *All count matroids $\mathcal{M}_{m,k}(n)$ are coordinatizable over each infinite field F.*

Proof. We begin with a coordinatization of $\mathcal{M}_{m,k}(n)$ for $k \leq 0$.

Lemma A.2.6. *For $k \leq 0$, $\mathcal{M}_{m,k}(n)$ can be constructed in rank $mn + k$ over any infinite field.*

PROOF. We choose n flats of co-rank m in general position in F^{mn+k}, labeled by the vertices V. For each edge $W \subseteq V$, we place a point corresponding to W in general position on the intersection of the flats labeled by the elements of $V \setminus W$. We must verify that this intersection is non-empty, provided W is not a loop (dependent edge) of $\mathcal{M}_{m,k}(n)$. If $j = |W|$, then the intersection of flats corresponding to $V \setminus W$ has co-rank $m(n-j)$ and this is nonempty if and only if $m(n-j) < mn + k$ or equivalently if and only if $mj + k > 0$. This is the condition that W is not a loop.

If we have multiple copies of edges, and therefore several sets W, W' corresponding to the same intersection, we pick distinct points (in general position to one another) in the appropriate intersection.

These points give the desired representation. For each set $V' \subseteq V$, the rank of the flat of its edges $2^{V'}$ is $m|V'| + k$ as required by the counts.

Our next step is to convert a representation of $\mathcal{M}_{m,0}(n)$ to a representation of $\mathcal{M}_{m,k}(n)$, $k > 0$.

LEMMA A.2.7 ([**WW2**]). *$\mathcal{M}_{m,k+1}(n)$ is the Higgs lift of $\mathcal{M}_{m,k}(n)$.*

PROOF. For E' contained in E, E' is independent in the Higgs lift of $\mathcal{M}_{m,k}(n)$ if and only if E' has nullity at most one in $\mathcal{M}_{m,k}(n)$ [**Wht1**, p. 160], which is true if and only if

$|E'| \leq m|V'| + k + 1$ and

$|E''| \leq m|V''| + k + 1$ for every $E'' \subset E'$.

This is the definition of $\mathcal{M}_{m,k+1}(n)$ as a Higgs lift.

To complete the proof for $k > 0$, the Higgs lift of a coordinatized matroid is coordinatizable for all suitably large fields [**Wht1**, p. 161].

REMARK A.2.8. The representation of Lemma A.2.5 is *not* the representation one is likely to find in the applications. For example, the usual representation of the cycle matroid $\mathcal{M}_{1,-1}(n)$ lies in F^n not in F^{n-1}.

An important problem in applying these matroids to represented matroids arising in applications is to 'recognize' such alternate representations which arise from the relevant systems of equations. For example, the proof that generic plane rigidity is the matroid $\mathcal{M}_{2,-3}(n)$ amounts to recognizing the representation of the matroid union of two copies of the cycle matroid (by its shape of zero and non-zero entries) and the changes due to the Dilworth truncation represented by the additional -1 [**Lov**,**LY**,**Wh7**].

There is much ingenuity that goes into the demonstration that alternate matrices, such as the rigidity matrix, indeed represent one of the matroids $\mathcal{M}_{m,k}(n)$. Such problems are the essence of several of the conjectures in this chapter.

Finally, we observe some connectivity results for circuits and bases which follow from the counts. Recall that a hypergraph $H = (V, E)$ is *j-vertex-connected* (or *j-connected*, for short) if there do not exist $j-1$ vertices whose removal disconnects H, that is, there does not exist a cover V_1, V_2 of the vertices, $|V_1 \cap V_2| \leq j-1$ such that every edge lies in V_1 or in V_2. A hypergraph $H = (V, E)$ is *j-edge-connected* if for every partition $\emptyset \neq V_1, V_2 \subset V$, there exist at least j edges intersecting both V_1 and V_2. When we talk about the vertex-connectivity (or edge-connectivity) of a set of edges E, we are really talking about the vertex-connectivity (or edge-connectivity) of the induced hypergraph $H = (V(E), E)$.

A matroid $\mathcal{M}_{m,k}(E)$ is called *full* if it has rank $m|V(E)|+k$, that is, if it contains at least some bases of the complete matroid $\mathcal{M}_{m,k}(|V(E)|)$. For example, all circuits in $\mathcal{M}_{m,k}(E)$ are full (corresponding to the observation that circuits in the plane rigidity matroid are rigid).

PROPOSITION A.2.9. *If $k \leq -(j-1)m$, then the circuits of $\mathcal{M}_{m,k}(E)$ are j-connected. If $\mathcal{M}_{m,k}(E)$ is full and $k < -(j-1)m$, then the bases are j-connected.*

PROOF. Suppose that E' is a circuit of $\mathcal{M}_{m,k}(E)$ which is not j-connected. Then there exist V_1, V_2 containing all the edges with $|V_1 \cap V_2| \leq j-1$. Let E_1, E_2 be the corresponding partition of the edges. Then

$$\begin{aligned}|E'| = |E_1| + |E_2| &\leq m|V_1| + k + m|V_2| + k \leq m|V(E')| + m|V_1 \cap V_2| + 2k \\ &\leq m|V(E')| + k + [m(j-1)+k] \leq m|V(E')| + k.\end{aligned}$$

However, this violates the claim that E' is a circuit of $\mathcal{M}_{m,k}(E)$.

The proof for a basis is similar, except that the last inequality is strict, giving $|E'| < m|V(E')| + k$, which contradicts the claim that $\mathcal{M}_{m,k}(E)$ is full.

PROPOSITION A.2.10. *If $k \leq 0$, then the circuits of $\mathcal{M}_{m,k}(E)$ are $(1-k)$-edge-connected. If $\mathcal{M}_{m,k}(E)$ is full, then the bases are $(-k)$-edge-connected.*

PROOF. Suppose that E' is a circuit of $\mathcal{M}_{m,k}(E)$ which is not $(1-k)$-edge-connected, by virtue of the partition V_1, V_2 with at most $-k$ edges intersecting both. We partition the remaining edges of E' into E_1, E_2. Then

$$|E'| \leq |E_1| + |E_2| + k \leq m|V_1| + k + m|V_2| + k - k \leq m|V(E')| + k.$$

Again we have contradicted the definition of a circuit. The obvious modifications complete the proof for bases.

PROPOSITION A.2.11. *If $k > 0$, then the circuits of $\mathcal{M}_{m,k}(E)$ have at most $k+1$ components.*

PROOF. Suppose that E' is a circuit of $\mathcal{M}_{m,k}(E)$ with j components, given by a partition of $V(E')$ into nonempty subsets $V_1, \dots V_j$. Then $|E'| = m|V(E')|+k+1$ and $|E_i| \leq m|V_i| + k$ for all i. If for some h, $|E_h| \leq m|V_h|$ then we can remove this set to give

$$|E' \setminus E_h| = m(|V(E')| - |V(E_h)|) + k + 1 > m|V(E' - E_h)| + k + 1.$$

This contradicts the minimality of the dependence of the circuit. Therefore, for each i, $|E_i| \geq m|V_i| + 1$. Now,

$$m|V(E')| + k + 1 = |E'| \geq \sum_{i=1}^{j} (m|V_i| + 1) = m|V(E')| + j$$

Therefore $k + 1 \geq j$.

A.3. Hypermatroids from counts. In §8.2 and §12.2, we encountered counts to bound the rank of the matroid of the form $(d-1)|E| \leq d|V|-(d+1)$. Put in raw submodular terms, these correspond to rational valued submodular functions such as $f(E)=\frac{3}{2}|V|-2$ and $f(|E|)=\frac{6}{5}(|B|-1)$. A simple extension of the previous results describes how such submodular functions define hypermatroids.

Given a submodular function f, we define

$$r_f(A) = \min\{\ |A|,\ \sum_i f(H_i) + |A - \bigcup_i H_i|\ \}$$

minimum over all collections $\{H_i\}$ of non-empty, disjoint subsets of A.

Proposition A.3.1. *For a real valued, increasing, submodular function, $f : 2^S \to \mathbb{R}$, nonnegative on all $A \neq \emptyset$, the following are equivalent:*

1. *the set A is f-independent: $|A| \leq f(A)$ and for all nonempty subsets B, $|B| \leq f(B)$;*
2. *the set A is r_f-independent: for all $B \subseteq A$, $|B| \leq r_f(B)$;*
3. $r_f(A) = |A|$.

Proof. 1.⇒3. Since $f(H_i) \geq |H_i|$ for all H_i, for all partitions $\{H_i\}$,

$$\sum_i f(H_i) + |A - \bigcup_i H_i| \geq \sum_i |H_i| + |A - \bigcup_i H_i| = |A|.$$

The minimum for $r_f(A)$ is $|A|$.

3.⇒2. We proceed by contraction. Suppose $r_f(B) \leq |B|$ for some B, with the minimum based on the partition $\{H_i\}$. Then

$$r_f(A) \leq \sum_i f(H_i) + |A - \bigcup_i H_i| < |A|$$

2.⇒1. This is obvious since $r_f(B) \leq f(B)$ for each $B \neq \emptyset$.

If f is rational valued, then for $d = \gcd(\dots, f(A), \dots)$ the function $g = df$ is integer valued, as is the corresponding rank function:

$$r_{df}(A) = \min\{\ d|A|,\ \sum_i df(H_i) + d|A - \bigcup_i H_i|\ \}.$$

Let dB denote the multiset of d copies of each edge in B. If $C \subseteq dE$, let C_0 be the ordinary set in which each edge in C is taken with multiplicity 1. We further define $f^d(C) = df(C_0)$ for subsets $C \subseteq dA$. In particular, $f^d(C_0) = f^d(dC_0) = df(C_0)$.

Theorem A.3.2. *For any rational-valued, nonnegative, non-decreasing submodular function f on E, and d the* gcd *for $\{f(B) \mid B \subseteq E\}$, the following equalities hold for each $A \subseteq E$,*

$$r_{df}(A) = d(r_f(A)) = r_{f^d}(dA).$$

In general r_f gives a hypermatroid. Since f^d is integer valued on dE, r_{f^d} is a matroid rank function on dE. [**Lor**] calls $r_{df} = d(r_f(A))$ a hypermatroid rank function.

Example A.3.3. For body-and-hinge frameworks in three space (§12.2), the rational function is $f(H) = \frac{6}{5}|B(H)| - \frac{6}{5}$ and $df(H) = 5f(H) = 6|B(H)| - 6$.

We interpreted $5|E| \leq 6|B(H)|-6$ via a matrix which gave 5 equations for each hinge and 6 variables for each body. The count $5|E| \leq 6|B(H)|-6 = 6(|B(H)|-1)$, directly implies the fact that a basis in the matroid on $5E$ is formed by 6 edge-disjoint trees in the multigraph with five distinct copies for each edge §11.2.

A.4. Counts for partitioned vertex sets. The counting which characterized independent sets for the generic parallel drawings of hyperplanes in d-space indicates a mild generalization of the basic Counts to Matroid Proposition A.1.1. In this setting, we partition the vertices of the hypergraph into different 'types', $H = (V^1, \ldots V^p; E)$, and these classes are assigned different coefficients. (If we worked with subsets V^i of V which was not a partition, this could be easily changed to a partition using $\{V^1 \cap \ldots \cap V^p, \ldots, \overline{V^1} \cap \ldots \cap \overline{V^p}\}$, where $\overline{V^p}$ represents the complement of V^p.)

We present one case of a more general *reduction* result of [**Su2**]. This result applies to the counts in the Parallel d-Scene Theorem of §8.

Proposition A.4.1. *For a hypergraph $H = (V; E)$ with partitioned vertices $\{V^1, \ldots V^p\}$ and non-negative integers $m_1, \ldots, m_p$, and integer k, the following definition gives the independent sets of a matroid $\mathcal{M}_{m_1,\ldots,m_p;k}(H)$ on the edges E of the hypergraph (in particular a complete graph on a finite set of vertices V):*

A set of edges E' is independent if and only if for all non-empty subsets $E_* \subseteq E'$ on the vertices $V_*^1, \ldots, V_*^p$, $|E_*| \leq m_1|V_*^1| + \ldots + m_p|V_*^p| + k$.

Proof. To clarify the underlying structure, we verify that

$$f(E) = \sum_{1 \leq i \leq p} m_i |V^i(E)| + k$$

is submodular and then apply the Submodular Theorem A.1.4.

$$\begin{aligned} f(E_1) + f(E_2) &= \sum_{1\leq i\leq p} m_i|V^i(E_1)| + k + \sum_{1\leq i\leq p} m_i|V^i(E_2)| + k \\ &= \sum_{1\leq i\leq p} m_i|V^i(E_1) \cup V^i(E_2)| + k + \sum_{1\leq i\leq p} m_i|V^i(E_1) \cap V^i(E_2)| + k \\ &\geq \sum_{1\leq i\leq p} m_i|V^i(E_1 \cup E_2)| + k + \sum_{1\leq i\leq p} m_i|V^i(E_1 \cap E_2)| + k \\ &= f(E_1 \cup E_2) + f(E_1 \cap E_2). \end{aligned}$$

Notice that the last step used $V^i(E_1 \cap E_2) \subseteq V^i(E_1) \cap V^i(E_2)$ and $m_i \geq 0$. Since f is an appropriate submodular function, we have the desired matroid.

We note that in this matroid, the independent sets A are defined, as usual, by the condition $|B| \leq \sum_{1\leq i\leq p} m_i|V^i(B)| + k$ for all non-empty subsets $B \subseteq A$.

Remark A.4.2. All of the results of §A.2 regarding matroid unions, coordinatizability, and connectivity extend to these matroids. Specifically, for all k, the proof of the Coordinatizability Theorem A.2.4 carries over immediately. The connectivity results also carry over – using the minimum m_i for the guaranteed connectivity.

A.5. Variable counts on edge sets. The 'counting' properties listed in §6 for mixed directions and distances and for spherical incidences, lengths and angles, point to another generalization of the basic counting matroids. We have a set of integers $k(A)$, indexed by selected subsets of an overall set E. These negative integers represent 'minus the degree of freedom' of different sets.

Sugihara [**Su2**] analyzes some related patterns of 'counts' arising from the variables and 'degrees of freedom' in systems of equations. He offers certain conditions which guarantee that these counts generate matroids or hypermatroids. Significant work remains to be carried out in this area.

We present a different pattern which also generates a matroid and applies to our examples. We follow with some specific application to counts on graphs, multigraphs and hypergraphs.

PROPOSITION A.5.1. *Given a set of edges E on vertices V, let Φ be a lattice of subsets of V (including the full set V) with intersection as meet and union as join, and a submodular, integer valued function $k : \Phi \to \mathbb{Z}$. Define $k(B) = \min\{k(W) \mid V(B) \subseteq W \in \Phi\}$. Then the function: $f(B) = m|V(B)| + k(B)$ is submodular.*

PROOF. Let W_B be the subset generating the minimum for $k(B)$.

$$\begin{aligned}
f(B)+f(C) &= m|V(B)| + k(B) + m|V(C)| + k(C)\\
&= m(|V(B)| + |V(C)|) + k(W_B) + k(W_C)\\
&\geq m(|V(B) \cup V(C)| + |V(B) \cap V(C)|) + k(W_B \cup W_C) + k(W_B \cap W_C)\\
&\geq m|V(B \cup C)| + k(B \cup C) + m|V(B \cap C)| + k(B \cap C)\\
&= f(B \cup C) + f(B \cap C).
\end{aligned}$$

REMARK A.5.2. It is natural to consider that case where the constant m also varies with the lattice Φ, in a submodular way. To keep the submodular function non-decreasing over reasonable sets, we would need $m(W)$ to be non-decreasing. In addition, to guarantee that the overall function is submodular, the proof also requires $m(W_B \cup W_C) \leq m(W_B), m(W_C)$. The net conclusion is that this proof requires $m(W)$ to be a constant. Sugihara [**Su2**] considers alternate submodular functions not directly defined by counts which do vary over a lattice.

EXAMPLE A.5.3. For the matroid on directions and lengths in §6.2 and [**SW**], we employ the simple lattice on the full set of multiple edges: $D \cup E$, the two designated sets of all edges of the graph D and E, and the empty set. With $k(D \cup E) = -2$ and $k(D) = -3, k(E) = -3$, and $k(\emptyset) = -4$, we have: $k(D)+k(E) = 6 = k(D \cup E) + k(D \cap E)$.

For frameworks in 3-space, the true degree of freedom is $k(\{v\}) = -3$ for singletons, $k(\{v_1, v_2\}) = -5$ on all pairs and $k(B) = -6$ on all subsets with $|B| \geq 3$. Unfortunately, with two larger sets B, C intersecting on a pair:

$$k(B) + k(C) = -6 - 6 < -6 - 5 = k(B \cup C) + k(B \cap C).$$

The appropriate degrees of freedom do not generate a matroid.

EXAMPLE A.5.4. One of the basic difficulties in characterizing matroids such as the simplicial homology matroid by 'counts' is our inability to give a simple formula for $k(B)$ for arbitrary simplicial complexes. While we have estimates such as $k(B) = -|E| + |V| - 1$ for sets of triangles, this is not correct for complexes for which the edges are not 'full enough' on the vertices (e.g. are not a connected graph). In fact, this estimate may be either too high or too low! Working out the rank carefully requires an understanding of each of the lower homologies of the complex.

This same problem arises, often in more complicated form, for all of the matroids and counts of Part III. To understand these estimates and the bases of these matroids one must understand the sequence of matroids corresponding to all the lower homologies.

With a correct calculation of the degrees of freedom, it is unlikely that the true degrees of freedom $(-k(B))$ will be submodular on arbitrary subsets. The combinatorics of these matroids lies beyond simple counting.

REMARK A.5.5. Consider a general set of counting conditions as described in this Appendix and set of 'edges' E to be tested for independence. The test appears to be exponential: "for all nonempty subsets $A \subseteq E$". Sugihara and Imai [**Su2**,**Im**] have developed general polynomial algorithms, typically $O(|E|^2)$, for all of these counting conditions as well as other more general (non-matroidal and non-hypermatroidal) counts. For the special case of graphs, with the counts $m|V| + k$, $-2m < k \leq -m$, there are also specific matroidal algorithms for detecting the appropriate tree coverings [**Cr5**,**Ha**]. These algorithms also appear to be $O(|E|^2)$.

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