Greedy Random Walk

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Abstract

A greedy random walk on a graph is the following discrete time self interacting random process. Each vertex maintains the set of adjacent edges touching it yet to be crossed. At each step, if the walk is in some vertex, it picks an edge uniformly at random among the adjacent edges of that vertex yet to be traversed. If all its adjacent edges have been crossed, then an adjacent edge is picked uniformly at random. After picking an edge the walker jump along it to a neighboring vertex. Several results are presented regarding edge cover-time and transience.

1   Introduction

Greedy random walk (GRW) on a graph $G = (V, E)$ is a discrete time random process, with a transition law defined as follows. Each vertex $v \in V$ maintains the set of adjacent edges touching it, that the walker hasn't crossed yet. At each step from vertex $v \in V$, the walker picks uniformly at random an unvisited edge, and if all the adjacent edges have been visited, an adjacent edge is picked uniformly at random. Then the walker jumps along the picked edge to a neighboring vertex.

Formally, a greedy random walk is a sequence $X_0, X_1, X_2, \ldots$ of random variables defined on V, with the following transition probabilities. For each time $t \geq 0$ define

$$H_t = \{\{X_s, X_{s+1}\} \in E : 0 \leq s < t\}$$

to be the set of all the edges traversed by the walk up to time t. For every vertex $v \in V$ and time $t \geq 0$ define

$$J_t(v) = \{e \in E : v \in e \text{ and } e \notin H_t\}$$

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to be the set of all the edges touching v that have not been traversed by the walk up to time t. The transition probabilities are given by:

$$\Pr[X_{t+1} = w | X_1, \ldots, X_t] = \begin{cases} \frac{1}{|J_t(X_t)|} & J_t(X_t) \neq \emptyset \text{ and } \{X_t, w\} \in J_t(X_t) \\ \frac{1}{|N_{X_t}|} & J_t(X_t) = \emptyset \text{ and } w \in N_{X_t} \\ 0 & \text{otherwise} \end{cases}$$

where N_v is the set of neighbors of v in G.

This variant of self interacting walk is related to bridge burning RW which is repelling rather than the attracting reinforced RW, rotor router RW, non-backtracking RW and excited RW, see [Pem07] for background, (see also [FS10], [ABLS06], [BW03])

One motivation for the study the GRW is arising from distributed computation in which on each vertex of the graph sits an agent, each agent knows and can communicates only with his neighbors. The goal of the agents is as fast as possible to let as many of the agents use some resource, while using no extra information regarding the graph and the vertices that were already visited. An agent has only a list of neighbors who communicated with him thus far. Each time the agent receives the resource, it can move it only to one neighbor. We will see that the GRW protocol can perform better than simple random walk (SRW) on some graphs.

The repelling nature of GRW suggests that the range of the walk will grow at least as fast as that of SRW and that it will mix at least as fast as SRW. We will study the cover time of finite graphs by the GRW, and establish faster GRW cover time for a few natural families of graph. E.g. the expected time it takes the GRW to go via all edges of the complete graph on n vertices is $O(n^2)$, which is asymptotically faster than $\Omega(n^2 \log(n))$, the edge cover time of SRW.

Recall that Feige [Fei95] showed that the expected cover time of a graph of size n by a simple random walk is at least $n \log n$, and recently in [BGM10] it was shown that for bounded degree graphs linear cover time is exponentially unlikely.

For many years probabilists are clueless regarding some basic conjectures about self interacting random walks. Occasionally some combinatorial observation, a coupling technique or other insights allow the analysis of such a process. We will see below that this is the case for the GRW, thus proving that GRW is transient on Z^d for $d > 2$. The current simple proof totally depends on the degrees of the graph being even and we don't know how to handle odd degrees at all. This suggests several conjectures.

1.1 Notation

We use standard notation of O, Ω and Θ. For two functions $f, g : \mathbb{N} \to \mathbb{R}_+$ we say $f = O(g)$, if there is a positive constant c_1 such that $f(n) < c_1 g(n)$ for all sufficiently large values of n. We say $f = \Omega(g)$, if there is a positive constant c_2

such that $f(n) < c_2 g(n)$ for all sufficiently large values of n. We say $f = \Theta(g)$, if both $f = O(g)$ and $f = \Omega(g)$.

1.2 Organization of the Paper

In the next section we discuss edge cover time of finite graphs and show for some families of graphs the edge cover time of GRW is asymptotically smaller than that of SRW. In section 3 we study the behavior of GRW on $\mathbb{Z}^d$. In particular we show that the walk is transient on $\mathbb{Z}^d$ for $d \geq 2$ We end with a list of open problems.

2 Edge cover time on finite graphs

Let $G = (V, E)$ be a connected undirected finite graph on n vertices. We denote by T_G^E the number of steps it takes for the greedy random walk to traverse all edges of G. Note that since G is a finite graph, T_G^E is a.s. finite. Since the restriction on the random walk is with respect to the adjacent edges of the current vertex, it seems more natural to ask about the edge cover time, rather than vertex cover time. We hope that for many interesting families of graphs, the greedy walk covers the edges asymptotically faster than the simple random walk. We are able to show it on several natural families of graphs.

The basic idea is as following. Divide the GRW into two random parts:

1. The greedy parts - all steps that cover the yet uncovered edges.

2. The simple parts - all steps that traverse all the edges that have already been covered previously, i.e. the choice of edges is distributed according to the distribution of the simple random walk.

Roughly speaking, the GRW typically looks as following: It starts with a greedy part. The greedy part continues until reaching at time s_1 a vertex v_1 whose all edges have already been covered. We think in this situation that the walk got stuck. This means that the last step before getting stuck, covered the last edge touching v_1. Since all edges touching v_1 have been covered, the walk picks an edge at random among the already covered edges. In other words the walk is in a simple part. The simple part continues until the simple walk reaches at time t_1 a vertex u_1 that has an uncovered edge touching it. By definition, the next step will belong to a greedy part and will continue until reaching at time s_2 to some vertex v_2 whose all edges have already been covered, starting the second simple part. The second simple part will end when the walk reaches at time t_2 a vertex u_2 that has an uncovered edge touching it. The walk continues like this until all edges are covered.

Formally, define inductively for $i \geq 0$

$$
\begin{aligned}
t_0 &= 0 \\
s_{i+1} &= \inf\{t > t_i : J_t(X_t) = \emptyset\} \\
t_{i+1} &= \inf\{t > s_{i+1} : J_t(X_t) \neq \emptyset\}
\end{aligned}
$$

This gives us a random partition $(t_0 = 0, s_1, t_1, s_2, t_2, \ldots, s_k)$ of the time segment $[0, T_G^E]$. Note that $H_{s_k} = E$. We say the walk *got stuck* at time t if $t = s_i$ for some $i = 1, \ldots, k$. It should be clear from the description, that the vertices X_{s_i} must all be distinct, as X_{s_i} is exactly the i-th time that the walk got stuck and it is impossible to get stuck in the same vertex twice. In particular k is a random variable that takes values in $\{1, \ldots, n\}$. The distribution of k that depends only on the graph G and the starting vertex X_0.

Note that in any walk the total size of the greedy parts equals exactly to the number of edges $|E|$, implying the following simple formula:

$$
T_G^E = |E| + \sum_{i=1}^{k-1}(t_i - s_i) \tag{1}
$$

Thus, in order to bound $\mathbb{E}[T_G^E]$, we need to bound the expected size of each simple part $\mathbb{E}[t_i - s_i]$ for $i = 1, \ldots, k-1$.

We say a vertex v at time t is *occupied* if $J_t(v) = \emptyset$, i.e. all edges adjacent to v are covered by time t. Define B_i to be

$$
B_i = \{v \in V : J_{s_i}(v) = \emptyset\}
$$

the set of all occupied vertices by time s_i. By the definition of s_i and t_i we note that $B_i = \{v \in V : J_t(v) = \emptyset\}$ for every $t \in [s_i, t_i]$. Note that $B_j \subseteq B_i$ for all $j \leq i$ and $X_{s_j} \notin B_i$ for $j > i$. Thus $B_1 \subset B_2 \subset \ldots \subset B_k = V$, where all the containments are strict.

Having defined B_i, it is easy to see that the length of the time segment $[s_i, t_i]$ equals exactly the escape time of a simple random walk from B_i when started at X_{s_i}. That is $t_i - s_i = T(X_{s_i}, B_i)$, where

$$
T(v, B) = \min\{t : X_t \notin B | X_0 = v\}
$$

Since the behavior of the walk in the time segment $[s_i, t_i]$ is a simple random walk, we can apply known results regarding SRW to bound the expected escape time.

We apply this idea to several families of graphs, showing that the expected edges cover time of the GRW is asymptotically faster than that of SRW.

2.1 Clique

Consider $G = K_n$, the complete graph of n vertices. A simple observation here is as following: Fix $k \in \{1, \ldots, n\}$ and let $B_1 \subset B_2 \subset \ldots \subset B_k = V$. The expected

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escape time from B_i (conditioned on k and B_j's) is distributed geometrically

PROVE why this is true!!!
Note that conditioning on k and B_j's might change the escape time.
Also might be that not every fixing is possible, i.e. has zero probability!!!

$$t_i - s_i \sim G(1 - \frac{|B_i| - 1}{n - 1})$$

Therefore

$$\mathbb{E}[t_i - s_i] = \frac{n - 1}{n - |B_i|}$$

and hence

$$\sum_{i=1}^{k-1} \mathbb{E}[t_i - s_i] = \sum_{i=1}^{k-1} \frac{n - 1}{n - |B_i|} \leq \sum_{i=1}^{n-1} \frac{n - 1}{n - i}$$

where the inequality uses the strict containment of B_i's and the fact that $k \leq n$. Since the bound holds for any fixed k and B_i's, we conclude that the same bound holds when taking the expectation over k and B_i's. This gives the following result for cliques.

Proposition 2.1. *Let $G = K_n$. Then the expected edge cover time of GRW on K_n has*

$$\mathbb{E}[T_{K_n}^E] \leq |E| + n \log(n) = (1 + o(1))|E|$$

$\square$

This is an improvement over the $\Omega(n^2 \log(n))$ time of the SRW.

We mention a result of Omer Angel and Yariv Yaari who recently proved that the expected number of unvisited edges in the n-complete graph until the first time the walk got stuck (i.e. up to time s_1) is linear in n, for odd values of n [AY].

2.2 Expanders

We apply the same techniques on (n, d, λ)-expander graphs. Recall a graph $G = (V, E)$ is a (n, d, λ)-expander, if it is a d-regular graph on n vertices and the second largest eigenvalue of its normalized adjacency matrix is at most λ (for more see e.g. [HLW06]).

We are able to show that for $d = \log(n)$, the greedy edge cover time is linear in the number of edges. This is faster than a simple random walk, which covers the edges in $\Omega(|E| \log |E|)$ steps.

Theorem 2.2. *Let G be any (n, d, λ)-expander. Then $\mathbb{E}[T_G^E] \leq |E| + O\left(\frac{n \log(n)}{1 - \lambda}\right)$. In particular for an expander with $d = \log(n)$ the expected greedy edge cover time is linear in the number of edges.*

Proof. We upper bound the quantity $t_i - s_i$ using the lemma due to Broder and Karlin.

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Lemma 2.3 ([BK89] Lemma 3)**.** *Let G be an (n, d, λ)-graph and let $B \subseteq V$ be a set of vertices. Consider a random walk $X_0, X_1 \ldots$ from some $v \in B$. Recall that $T(v, B)$ is the escape time of the walk from B when started from v. Then*

$$\mathbb{E}[T(v, B)] \leq \left(\frac{O(\log(n)) + \frac{n}{n - |B|}}{1 - \lambda} \right) (1 + o(1)) \qquad \square$$

Plugging into the lemma, we have

$$\mathbb{E}[t_i - s_i] \leq \left(\frac{O(\log(n)) + \frac{n}{n - |B_i|}}{1 - \lambda} \right) (1 + o(1))$$

Summing over all i's we get

$$\sum_{i=1}^{k-1} \mathbb{E}[t_i - s_i] \leq \sum_{i=1}^{k-1} \left(\frac{O(\log(n)) + \frac{n}{n - |B_i|}}{1 - \lambda} \right) (1 + o(1)) \leq O \left(\frac{n \log(n)}{1 - \lambda} \right)$$

which completes the proof. $\qquad \square$

We do not know if for a bounded degree expander the greedy edge cover time is $o(n \log n)$.

2.3 Hypercube $\{0, 1\}^d$

As a corollary from the previous result we obtain a bound on the $\{0, 1\}^d$ hypercube. The bound is useful (improves over the SRW) only when we connect two vertices if the hamming distance between them is ℓ, for some $\ell \geq 2$ (and not 1 as usual). We do not know whether for the usual hypercube graph $\mathbb{E}[T_G^E]$ is smaller that $O(n \log^2(n))$.

Proposition 2.4. *. Let $G = (V, E)$ be the d dimensional hypercube graph, where $(x, y) \in E$ iff $d(x, y) = \ell$. That is $n = |V| = 2^d$ and $|E| = \Theta(\binom{d}{\ell} \cdot 2^d)$. Then $\mathbb{E}[T_G^E] \leq |E| + O(n \log^2(n))$.*
The assumption $d(x, y) = \ell$ can be replaced by $d(x, y) \leq \ell$.

Proof. The proof is just by plugging the second eigenvalues associated with G, $\lambda = 1 - \Theta_\ell(\log^{-1}(n))$ in Theorem 2.2, implying $\mathbb{E}[T_G^E] \leq |E| + O(n \log^2(n))$. $\quad \square$

2.4 d-trees

So far we considered graphs with superconstant degree. We are also able to upper bound the edge cover time of GRW on trees.

Theorem 2.5. *Let $G = (V, E)$ be any tree rooted at r such that $\deg(r) \geq 2$. For any $v \in V$ denote by T_v the subtree rooted at v and let $|T_v|$ denote the number of edges in T_v. Then the GRW covers all the edges in expected time*

$$\mathbb{E}[T_G^E] = \Theta \left(\sum_{v \in G \setminus \{r\}} |T_v| \right)$$

In particular for a d-tree, the expected edge cover time is $\Theta(n \log_d(n))$.

This improves over the $\Theta(n \log_d^2(n))$ time of the SRW [Ald91].

Proof. Consider a GRW on a tree starting at the root r. We observe that the vertices $X_{s_1}, X_{s_2}, \ldots X_{s_k}$ define some postorder traversal sequence (first the children and then the root) on the tree, except that it stops after visiting all the leaves and doesn't necessarily return to the root.

Then $t_i - s_i$ equals the time it takes for the simple random walk starting at X_{s_i} to visit its parent. It is well known (see e.g. [AKL$^+$79], Lemma 1) that $\mathbb{E}[t_i - s_i] = 2|T_{X_{s_i}}| + 1$ (remember that $|T_v|$ is the number of edges in T_v, the subtree rooted at $v \in V$). This proves the upper bound of the theorem.

The lower bound follows by the fact that the subtrees of r are explored completely one after another (the order of the subtrees is random), and the walk will return to the origin from all but the last subtree. Thus in order to cover all edges, the walk must return to the origin $\deg(r) - 1$ times. Hence a lower bound on $\mathbb{E}[T_G^E]$ is

$$\mathbb{E}[T_G^E] \geq \frac{\deg(r) - 1}{\deg(r)} \left(\sum_{v \in G \setminus \{r\}} |T_v| \right) = \Omega \left(\sum_{v \in G \setminus \{r\}} |T_v| \right)$$

$\square$

3   Greedy random walk on $\mathbb{Z}^d$

In this section we study the behavior of GRW on infinite graphs. Specifically we ask whether the walk is recurrent or transient in different graphs.

Obviously the greedy random walk on $\mathbb{Z}$ visits every vertex at most once. We show that for $d \geq 3$ the greedy random walk on $\mathbb{Z}^d$ is transient.

Theorem 3.1. *Let $G = (V, E)$ be an infinite graph where all its vertices are of even degree. If the simple random walk on G is transient, then so is the greedy random walk.*

In particular for $d \geq 3$, the greedy random walk on $\mathbb{Z}^d$ returns to the origin only finitely many times a.s..

Proof. As in the case of finite graphs in Section 2, we have a partition of the walk into two types of parts: the greedy parts and the simple parts. The difference here is that the times can have the value $+\infty$. Let $(t_0 = 0, s_1, t_1, s_2, t_2, ...)$ be a sequence of times defined as in Section 2. Let X_{s_i}, X_{t_i} be the locations of the walk at these times.

Assume that the event that one of the times s_i, t_i equals $+\infty$ has a positive probability. Conditioning on this event, if t_i is the first such time, the walk is in a simple part from s_i onwards. Since the walk is transient, conditioning on this event, the walk will return to **0** finitely many times a.s.. If s_i is the first

such time, the walk is in a greedy part from t_{i-1} onwards. Hence, as the degree of $\mathbf{0}$ is finite, the maximal number of of returns to zero is at most $\frac{\deg(\mathbf{0})}{2}$.

Assume now that the event that $s_i, t_i < +\infty$ for all $i = 1, 2, ...$ has a positive probability. Note that since all vertices have even degrees, the first time the walk gets stuck will be exactly when it visits the origin for the $\left(\frac{\deg(\mathbf{0})}{2}\right)$-th time. In other words $X_{s_1} = \mathbf{0}$.

Remember the discussion in the beginning of Section 2. When the walk gets stuck, it performs a SRW on the covered vertices until reaching for the first time a vertex X_{t_1} that touches an uncovered edge. The walk then performs a greedy walk until gets stuck again. It gets stuck in the same vertex X_{t_1}, that it started from, i.e. $X_{s_2} = X_{t_1}$, again because all degrees are even. Then performs an SRW starting from X_{s_2} until reaching a vertex X_{t_2}. The walk continues in this manner satisfying the property $X_{s_i} = X_{t_{i-1}}$. In such case we see that the walk in time segments $[s_{i-1}, t_{i-1}]$ and $[s_i, t_i]$ can be concatenated. Hence the walk restricted to $\bigcup_i [s_i, t_i]$ is distributed exactly as an SRW on G, which by transience, returns to $\mathbf{0}$ finitely many times a.s.. Since in the overall greedy parts visits $\mathbf{0}$ at most $\frac{\deg(\mathbf{0})}{2}$ times, it returns to $\mathbf{0}$ finitely many times a.s.. $\quad\square$

Note that we strongly used the fact that all vertices in our graph have even degree. In fact we can prove the same result on a slightly noisy version of $\mathbb{Z}^3$.

Proposition 3.2. *Let $G = (V, E)$ be a graph obtained from $\mathbb{Z}^3$ by removing at most $r^{1-\epsilon}$ edges from any box of radius r around the origin for some $\epsilon > 0$. Then the greedy random walk on G is transient.*

Proof. We start with a time partition $(t_0 = 0, s_1, t_1, s_2, t_2, ...)$ as previously. Unlike the proof of Theorem 3.1, in this graph it is possible that the simple parts cannot be concatenated into one walk. We say that a vertex v is a *new start* if $v = X_{s_i}$ for some i and $X_{t_j} \neq v$ for all $j < i$. Note that every new start vertex v must be either the origin or of odd degree.

Consider the walk restricted to the segments $[s_i, t_i]$. The indices $i \geq 1$ can be partitioned into classes $C_1, C_2, \dots$ such that in each class C_j the segments $[s_i, t_i]$, $i \in C_j$, can be concatenated into one walk that starts with a new start vertex. Namely, if $C_j = \{i_1 < i_2 < i_3 < \dots\}$, then $X_{s_{i_1}}$ is a new start and $X_{s_{i_2}} = X_{t_{i_1}}, X_{s_{i_3}} = X_{t_{i_2}}, \dots$ Denoting by $m_j = \min C_j$ we have $X_{s_{m_j}}$ is necessarily a new start and therefore is either the origin or a vertex of odd degree. Moreover all $\left\{s_{m_j}\right\}_j$ are distinct.

For each C_j, restricting the walk to times $\bigcup \{[s_i, t_i] : i \in C_j\}$ gives us a simple random walk (possibly finite) starting from $X_{s_{m_j}}$. Therefore we have at most $O(r^{1-\epsilon})$ simple random walks, starting from a box of radius r around the origin. Using the fact that a random walk in $\mathbb{Z}^3$ starting from distance r hits the origin with probability $O(1/r)$, we conclude that the sum of probabilities of hitting zero when summing over all random walks converges.

More precisely, let P_v be the probability that a SRW starting at v reaches

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the origin. And let ODD be the set of all vertices of odd degree. Then

$$\sum_{v \in ODD} P_v = \sum_{n=1}^{\infty} \sum_{\substack{v \in ODD \\ 2^{n-1} \leq \|v\| < 2^n}} P_v \leq \sum_n (2^n)^{1-\epsilon} \cdot O(\frac{1}{2^n}) = O\left(\sum_{n=1}^{\infty} 2^{-\epsilon n}\right) < \infty$$

Thus, by the Borel-Cantelli lemma, almost surely only finitely many of the walks will reach the origin, implying that the GRW on this graph is transient. $\square$

Using the concatenation argument above, we show that for any graph with even degrees the expected number of edges covered by the GRW in t steps cannot be asymptotically smaller than that of the SRW.

Proposition 3.3. *Let $G = (V, E)$ be a vertex-transitive graph with vertices of even degree. Denote by $range_W(t)$ the expected number of edges covered by the walk W in t steps. Then $range_{GRW}(t) = \Omega(range_{SRW}(t))$.*

Proof. Let s be the number of steps that GRW spent in the simple parts and $(t-s)$ to be the number of steps spent in the greedy parts. Then $\text{range}_{\text{GRW}}(t) \geq \max(t - s, \text{range}_{\text{SRW}}(s))$.

If $s < t/2$, then $\text{range}_{\text{GRW}}(t) \geq t/2 \geq \text{range}_{\text{SRW}}(t/2)$. Otherwise, we have $\text{range}_{\text{GRW}}(t) \geq \text{range}_{\text{SRW}}(t/2)$. In both cases $\text{range}_{\text{GRW}}(t) = \Omega(\text{range}_{\text{SRW}}(t))$, since by the Markov property of SRW holds $\text{range}_{\text{SRW}}(t) \leq 2 \cdot \text{range}_{\text{SRW}}(t/2)$. for any vertex-transitive graph. $\square$

Remark   Adapting the proof one gets an analogous result also for the expected number of vertices covered by the GRW in t steps.

4   Simulations

We present some simulations comparing between GRW and SRW. For the first simulation, see Figures 1 and 2. It is aimed to show the speed of covering in a 500×500 torus. The green part in Figures 1 and 2 consists of the vertices that were visited by the GRW and SRW, respectively, up to the first time that GRW has covered 0.8 of the total area of the torus. For the second simulation, see Figures 3 and 4. It is aimed to compare the hitting time of the boundary of a 500×500 box in $\mathbb{Z}^2$. The green part in Figures 3 and 4 consists of the vertices that were visited by GRW and SRW, respectively, starting at the center of a 500×500 box in $\mathbb{Z}^2$, up to the hitting time in the boundary of the box.
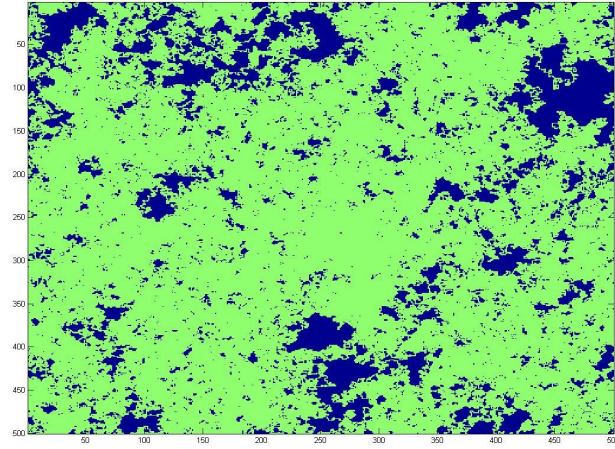
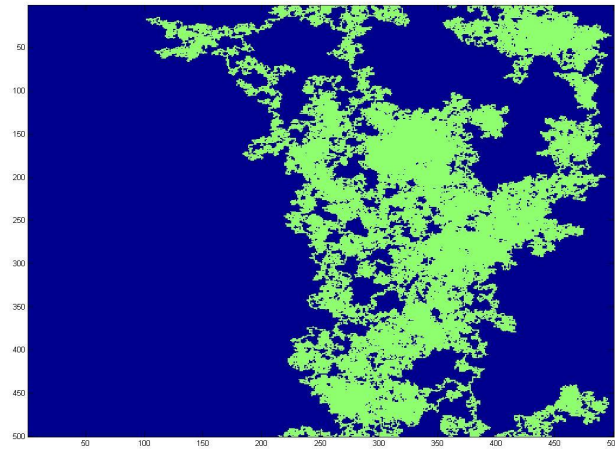Figure 1: GRW up to the time it visits 0.8 of the area of the torus

Figure 2: SRW up to the time that GRW visits at 0.8 of the area of the torus
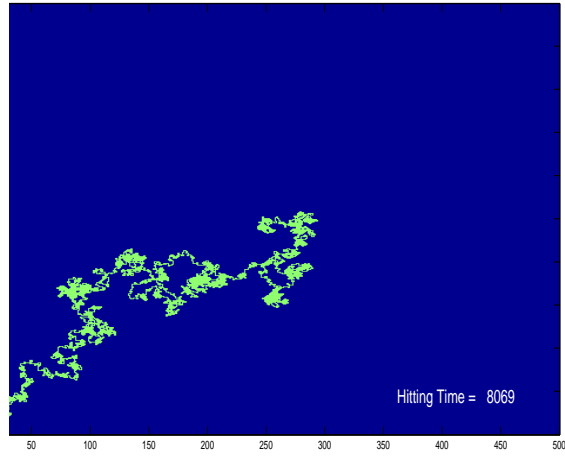
10

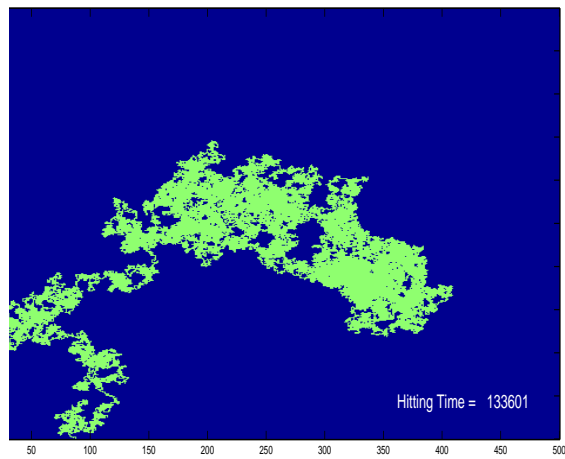Figure 3: GRW up to the first hitting time of the boundary of the box

Figure 4: SRW up to the first hitting time of the boundary of the box

5 Remarks and Open Questions

One may consider a variant of the greedy random walk defined as follows. At each step from a vertex v, the random walk picks uniformly at random a non visited neighbor and moves to it. If all neighbors were visited, it picks one of them uniformly at random and moves to it. This walk may be beneficial to lower the expected *vertex* cover time, for finite graphs of high degree. For example, it is obvious that in the clique K_n, the vertex cover time is n.

However if we change the rule a little, then for any graph G on n vertices the walk will cover the vertices in time at most $2n$. The rule will cause the walk to perform DFS. explain how this can be turned into a DFS walk

We conclude with a list several open questions.

1. Show upper bounds on T_G^E for other families of graphs. Interesting graph to look at could be a d-dimensional torus and a hypercube.

2. It seems also interesting to analyze the GRW on graphs with power-law degree distribution. As on such graphs there are hubs of very large degrees and when visiting these the GRW will be very efficient.

3. Show that the expected edge cover time of the GRW cannot be asymptotically larger than that of the SRW for any finite graph (This is true for graphs of even degree). Note that for $G = P_n$, a path of length n, when starting from the midpoint, the expected cover time of GRW and that of SRW or of the same order ($\Theta(n^2)$).

4. Give bounds on the expected vertex cover time of the GRW for finite graphs.

5. Define GRW mixing time and show that GRW mixing time is as fast as that of SRW (here [ABLS06] is relevant, and also [MW05] may be found useful).

6. Give bounds on the expected hitting time of GRW for different graphs.

   The coming problems are regarding recurrence of GRW on infinite graphs.

7. Show that the GRW on the ladder $\mathbb{Z} \times \mathbb{Z}_2$ is recurrent.

8. Is GRW on $\mathbb{Z}^2$ recurrent? Is GRW diffusive on $\mathbb{Z}^d$, for all $d > 1$?

9. Prove that any graph rough isometric to the $\mathbb{Z}^3$ square lattice is transient. In particular any odd degree lattice.

10. Show that if an infinite graph G is nonamenable, then GRW on G is transient.

6 Acknowledgements

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