Khrapchenko method for *k*-ary bases

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The method of formula complexity lower bounds – 1971 *sensitivity* of a boolean function *f*:

$$s(f) = \max_{N \subset f^{-1}(0),\, P \subset f^{-1}(1)} \frac{|R(N,P)|^2}{|N| \cdot |P|}$$

T. $B_0 = \{\vee, \wedge, \bar{\ }\}$: $\quad L_{B_0}(f) \geq s(f)$ (Khrapchenko'71)

formula: $(\overline{x_1} \vee x_2 x_3 \vee \overline{x_2} \cdot \overline{x_3})(x_1 \vee (x_2 \vee x_3)(\overline{x_2} \vee \overline{x_3}))$

$$l_n(x_1, \ldots, x_n) = x_1 \oplus \ldots \oplus x_n, \quad s(l_n) = n^2 \quad \Longrightarrow \quad L_{B_0}(l_n) \geq n^2$$

$$L_B(l_n) = \Omega(n^{\Gamma_B}), \quad \Gamma_B \text{ — shrinkage exponent of the basis } B$$

$\Gamma_{B_0} = 2$ (J. Håstad'98; A. Tal'14)

Bases of *k*-ary functions:

U_k – maximal basis *B*, where $L_B(l_n) \succ n$

U_k: all functions that are monotone/antimonotone in any variable

$\Gamma_{U_k} \geq 1 + \frac{1}{3k-4}$ (N.A. Peryazev'95)

$\Gamma_{U_3} \geq \frac{4}{3}$ (H. Chockler, U. Zwick'01) $U_3 \sim \{m_3(x,y,z), \bar{\ }\}$

χ_B — *Khrapchenko exponent*: maximal $\chi : \forall_f \; L_B(f) \geq s^\chi(f)$

$$L_B(l_n) = \Omega(n^{2\chi_B}) \qquad \chi_{B_0} = 1 \qquad \chi_{U_k} = ?$$

$G = (A, B, E)$ – bipartite graph with parts A and B

$$s(G) = \max_{X \subset A, Y \subset B} \frac{|E \cap (X \times Y)|^2}{|X| \cdot |Y|}$$

$f \to G_f$: $\quad A = f^{-1}(0),\ B = f^{-1}(1),\ E = R(A, B) \quad \Rightarrow \quad s(G_f) = s(f)$

$\{G_i = (A_i, B_i, E_i)\}$ — *covering* of a graph $G = (A, B, E)$, if

$$\forall i : \quad A_i \subset A, \quad B_i \subset B, \quad E_i = E \cap (A_i \times B_i), \text{ and } E = \bigcup E_i.$$

Covering is *monotone*, if $\forall I$: $A \setminus \bigcup_{i \in I} A_i = \emptyset$, or $B \setminus \bigcup_{i \notin I} B_i = \emptyset$.

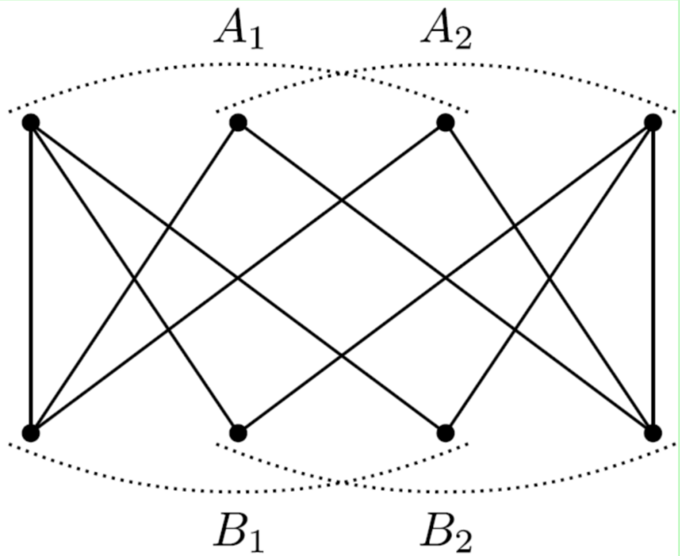
Complexity exponent χ_k (*monotone complexity exponent* χ_k^*) is the maximal χ: for any graph G, and for any its covering (monotone covering) $G_1, \ldots, G_k$:

$$s^\chi(G_1) + \ldots + s^\chi(G_k) \geq s^\chi(G).$$

T. $\chi_k \leqslant \chi_k^* \leqslant \chi_{U_k}$

T. $\chi_2^* = 1$ (Khrapchenko / A.E. Andreev, M.S. Paterson)

$\chi_2 < 0.95$



$$s(G) = 25/4$$
$$s(G_1) = s(G_2) = 3$$

T. $\chi_{U_k} \leqslant \log_{\lceil k/2 \rceil (\lfloor k/2 \rfloor + 1)} k$

consider the majority function m_k:

$$s(m_k) = \left\lceil \frac{k}{2} \right\rceil \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right)$$

T. $\chi_k \geq \frac{1}{2} + \frac{1}{10 \ln k}$

$$\frac{1}{2} + \frac{1}{10 \ln k} \leqslant \chi_k \leqslant \chi_k^* \leqslant \chi_{U_k} \leqslant \frac{1}{2} + \frac{1}{2 \log_2 (k/2)}$$

$$n^{1+\frac{1}{5 \ln k}} \preceq L_{U_k}(l_n) \preceq n^{1+\frac{1}{\log_2 \lfloor k/2 \rfloor}}$$

T. $0.769 < \chi_3^* \leq \chi_{U_3} \leq \log_4 3 \approx 0.792$

$$n^{1.53} \prec L_{U_3}(l_n) \prec n^{1.74}$$

(H. Chockler, U. Zwick'01)

reference:

1. Sergeev I. S. *Formula complexity of a linear function in a k-ary basis*. Mathematical Notes. 2021. V. 109, No. 3, 445-458.