# Об арифметической сложности вычисления некоторых линейных преобразований

# (On the arithmetic complexity of some linear mappings)

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(Gashkov S. B., Sergeev I. S.)

# Definitions

Linear map

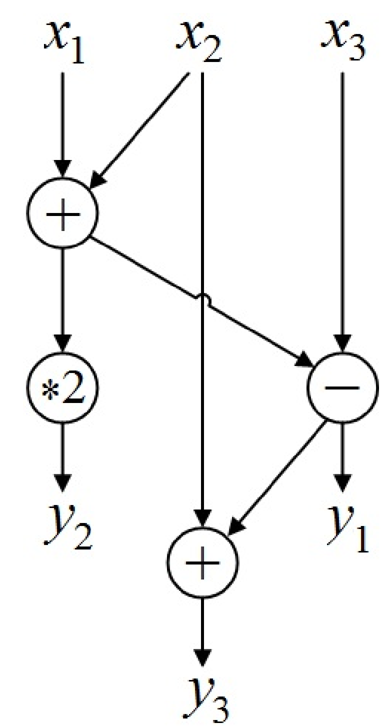
$$y = A \cdot x$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

basis $B = \{x + y, x - y, 2x\}$

complexity $L_B(A) = 4$

Arithmetic circuit

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Examples of linear bases:

- addition basis $\{x + y\}$
- full linear basis $B_\infty = \{ax + by \mid a, b \in \mathbb{R}\}$

# Simple properties of the complexity

1. $$L_B(A_1 \times A_2) \le L_B(A_1) + L_B(A_2)$$

2. $$L_B(A) \ge L_B(\text{any submatrix of } A)$$

3. $$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \Rightarrow \quad L_B(A) \le L_B(A_1) + L_B(A_2)$$

   for a monotone $B$: $L_B(A) = L_B(A_1) + L_B(A_2)$

4. Transposition principle. For $m \times n$ matrix $A$:

$$\left| L_B(A) - L_B(A^T) \right| = O(m+n)$$

# Determinant lower bound

Theorem.

$$B_C = \{x \pm y\} \cup \{ax \mid |a| \leq C\}$$

$$L_{B_C}(A) \geq \log_{\max\{2,C\}} |\det A|$$

J. Morgenstern ’1973

# Sylvester-Hadamard matrices

$$H_1 = [1], \quad H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad H_{2n} = \begin{bmatrix} H_n & -H_n \\ H_n & H_n \end{bmatrix}$$

**Fact.** $\det H_n = n^{n/2}$

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$$\Rightarrow \quad L_{\{x \pm y\}}(H_n) \sim L_{B_2}(H_n) \sim \frac{1}{2} n \log_2 n$$

# Pascal (binomial) matrix. I

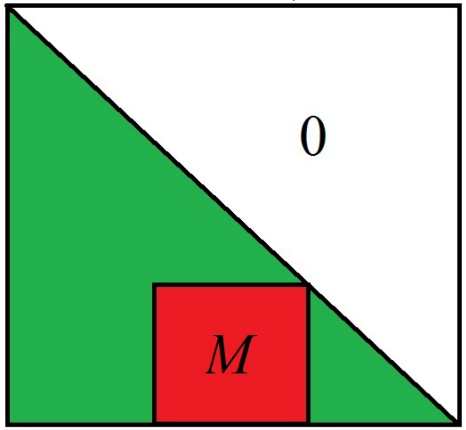
$$C_n = \begin{bmatrix} C_0^0 & 0 & \cdots & 0 \\ C_1^0 & C_1^1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ C_{n-1}^0 & C_{n-1}^1 & \cdots & C_{n-1}^{n-1} \end{bmatrix}$$

Pascal's rule: $C_{n+1}^{k+1} = C_n^{k+1} + C_n^k$

$$\Rightarrow \quad L_{\{x+y\}}(C_n) \le n^2/2$$

# Pascal (binomial) matrix. II

**Fact 1.** Matrix $C_n$ has a submatrix $M$ with determinant of order $c^{n^2}$ for a constant $c>1$.




$$\Rightarrow \quad L_{B_2}(C_n) = \Theta(n^2)$$

**Fact 2.**

$$C_n = \Delta \times \begin{bmatrix} \frac{1}{0!} & 0 & \cdots & 0 \\ \frac{1}{1!} & \frac{1}{0!} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \frac{1}{0!} \end{bmatrix} \times \Delta^{-1}$$

$$\Delta = \mathrm{diag}(0!, 1!, \ldots, (n-1)!)$$

$$\Rightarrow \quad L_{B_\infty}(C_n) = O(n \log n)$$ Gashkov '2014

# Mod2 binomial matrix ≡ Disjointness matrix ≡ Sierpiński matrix

$$D_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad D_{2n} = \begin{bmatrix} D_n & 0 \\ D_n & D_n \end{bmatrix}$$

$$L_{\{x+y\}}(D_n) \sim L_{B_2}(D_n) \sim \frac{1}{2} n \log_2 n$$

S.N. Selezneva; J. Boyar, M.G. Find ‘2012

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$$\Rightarrow \quad L_{B_{>0}}(C_n) = \Omega(n \log n),$$

$$B_{>0} = \{ax + by \mid a, b \geq 0\}$$

# Stirling matrices

$$s_n = \| s_m^k \|_{0 \le k,m<n}, \qquad S_n = \| S_m^k \|_{0 \le k,m<n}$$

$s_m^k$ - kind I Stirling numbers

$S_m^k$ - kind II Stirling numbers

$$s_m^k = s_{m-1}^{k-1} - (k-1)s_m^{k-1}, \qquad S_m^k = S_{m-1}^{k-1} + mS_m^{k-1},$$

$$s_0^0 = S_0^0 = 1, \qquad s_0^k = s_k^0 = S_0^k = S_k^0 = 0, \quad k > 0$$

**Facts.** $S_n = (s_n)^{-1}$

$$\{1, (x)_1, \ldots, (x)_{n-1}\} \xrightarrow{s_n} \{1, x, \ldots, x^{n-1}\} \xleftarrow{|s_n|} \{1, (x)^1, \ldots, (x)^{n-1}\}$$

$$(x)_k = x(x-1) \cdot \ldots \cdot (x-k+1),$$

$$(x)^k = x(x+1) \cdot \ldots \cdot (x+k-1)$$

# Evaluation, interpolation matrices

**Fact 1.** Matrices $s_n$ and $|s_n|$ have submatrices with determinants of order $2^{\Theta(n^2 \log n)}$.

Gashkov '2014

$$\Rightarrow \quad L_{B_2}(s_n) \asymp L_{\{x \pm y\}}(s_n) = \Theta(n^2 \log n),$$

$$L_{B_2}(|s_n|) \asymp L_{\{x+y\}}(|s_n|) = \Theta(n^2 \log n)$$

Vandermonde matrix $V_n$: $V_n = ||k^m||_{0 \le k,m < n}$

**Fact 2.** $\det V_n = \prod_{k=1}^{n-1} k! = 2^{\Theta(n^2 \log n)}$

**Fact 3.** $V_n = C_n \times \Delta \times S_n^T$

$$\Rightarrow \quad L_{B_2}(V_n) \asymp L_{\{x+y\}}(V_n) = \Theta(n^2 \log n),$$

Gashkov '2014

$$L_{B_\infty}(V_n),\; L_{B_\infty}(S_n),\; L_{B_\infty}(s_n) = O(n \log^2 n)$$

# N.T. Auxiliary matrices

$$D = \mathrm{diag}(1, \ldots, n), \quad f(D) = \mathrm{diag}(f(1), \ldots, f(n))$$

Division matrix $E$: $E[i,k] = (k \mid i)$

Möbius matrix $M$: $M[i,k] = \begin{cases} \mu\left(\frac{i}{k}\right), & k \mid i \\ 0, & k \nmid i \end{cases}$

Möbius inversion formula: $M = E^{-1}$

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## Facts:

$$L_{B_2}(f(D)) \sim L_{\{x+y\}}(f(D)) \sim \log_2 \prod_{i=1}^{n} f(i) + O(n)$$

A. Brauer ‘1929

$$L_{\{x+y\}}(E) = O(n \log\log n), \qquad L_{\{x\pm y\}}(M) = o\left(n\sqrt{\log n}\right)$$

# GCD matrix

$$\mathrm{GCD} = \| \gcd(i,k) \|$$

**Fact.** $\mathrm{GCD} = E \times \phi(D) \times E^T$

$\phi(x)$ - Euler totient function

H. Smith '1875

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$$\Rightarrow \quad \log_2 \det \mathrm{GCD} \sim n \log_2 n$$

**Theorem.** Gashkov, Sergeev '2015

$$L_{B_2}(\mathrm{GCD}) \sim L_{\{x+y\}}(\mathrm{GCD}) \sim n \log_2 n$$

$$\left| L_{\{x\pm y\}}(\mathrm{GCD}) - L_{\{x\pm y\}}(\phi(D)) \right| = o\left( n\sqrt{\log n} \right)$$

# LCM matrix

$$\mathrm{LCM} = \| \ \mathrm{lcm}(i,k) \ \|$$

$$\gcd(i,k) \cdot \mathrm{lcm}(i,k) = ik$$

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$$\Rightarrow \quad \mathrm{LCM} = D \times E \times J_{-1}(D) \times E^T \times D$$

$$J_r(k) = k^r \prod_{p \in \mathbb{P},\, p|k} (1 - p^{-r})$$ - Jordan function

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$$\Rightarrow \quad \log_2 \det \mathrm{LCM} \sim 2n \log_2 n$$

**Theorem.** Gashkov, Sergeev '2015

$$L_{B_2}(\mathrm{LCM}) \sim L_{\{\pm\}}(\mathrm{LCM}) \sim 2n \log_2 n$$

$$\mathrm{LCM} = E \times \phi(\mathrm{core}(D)) \times \left\| \phi\left(\frac{i}{k}\right) \cdot I\{\mathrm{core}(i) = \mathrm{core}(k)\} \right\| \times U \times \mu^*(D) \times E^T \times D$$

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