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A stable model predictive control for integrating processes

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Abstract

This paper proposes a stable model predictive control for systems with stable and integrating poles. The method presented here extends the method of [Rodrigues, M. A., & Odloak, D. (2003a). An infinite horizon model predictive control for stable and integrating processes. *Computers and Chemical Engineering, 27*, 1113–1128] to provide nominal stability for a set of process conditions, which is larger than in previous methods. The main effort is to eliminate the conflict between the constraints in the system inputs, which are usually included in the MPC, and the constraints created by zeroing the integrating modes of the system at the end of the control horizon. This problem has hindered the practical application of nominally stable infinite horizon MPC in industry. The improved controller is obtained through a modified control objective that includes additional decision variables to increase the set of feasible solutions to the control problem. The hard constraints associated with the integrating modes are softened and the resulting control problem is feasible to a much larger class of unknown disturbances and set point changes. Two methods are proposed to obtain stability: by imposing the contraction of the norm of the vector of slack variables associated with the integrating modes and by separating the control problem in two sub-problems. The methods are illustrated with the application of the proposed approach to two integrating examples of the literature.

Keywords: Model predictive control; Stability; Infinite horizon; Integrating system

1. Introduction

One of the main issues in the application of MPC to industrial processes is the stability of the closed loop system. In the MPC literature, we can find different approaches to obtain a controller, which is stable for any possible control structure and tuning parameters. When the state is perfectly known, nominal stability can be achieved by considering an infinite prediction horizon. Stability of infinite horizon MPC (IHMPC) was demonstrated by Rawlings and Muske (1993) for linear systems with constraints in the inputs and states. For the stable system, the strategy consists in reducing the infinite output horizon through the application of a terminal state weight, which is computed as the solution of a discrete-time Lyapunov equation. Rodrigues and Odloak (2003b) proposed a different approach in which the error in the output prediction appears as a continuous function of time and the squared output error is integrated along the infinite prediction horizon. In this approach, the Lyapunov equation is removed from the control optimization problem. Another method to obtain stability consists in explicitly including, in the control optimization problem, a terminal state constraint (Keerthi & Gilbert, 1988; Mayne & Michalska, 1990). This state constraint assumes that, at the end of the output prediction horizon, the state will lie at the origin. Although easy to implement, the terminal state constraint has shown little practical utility as far as systems subject to persistent output disturbances are concerned. For these systems, the optimization problem, which is solved by MPC, can become infeasible in normal operating conditions. To overcome the problem created by the terminal state constraint in the control problem, some authors (Michalska & Mayne, 1993; Scokaert, Mayne, & Rawlings, 1999) have proposed an alternative approach, in which the

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terminal state is expanded to a terminal set around the origin. In such approach, the control law is obtained through the solution of an optimization problem in which it is imposed that at the end of the control horizon the state reaches the terminal set. This terminal set is such that it is positive invariant in closed loop with the linear quadratic regulator. This last controller is obtained from the solution of the discrete Riccati equation. It is also assumed that when the state lies inside the terminal set the input constraints do not become active. Although, in this method, the region in which the controller is feasible is substantially increased in comparison to the terminal state or infinite horizon approaches, stability still depends on the control horizon and on the magnitude of the disturbances that enter the system. Recently, De Doná, Seron, Mayne, and Goodwin (2002) showed that a significant increase in the size of the terminal set could be obtained when we consider a non-linear controller based on the LQR, which allows saturation of the input. However, their approach can only be applied to the single input system and they give no clue on how to generalize the procedure to the multivariable system. For stable systems, Rodrigues and Odloak (2003b) proposed a stable MPC, which is based on the softening of the terminal state constraint. However, for systems with integrating modes, their approach still requires zeroing the effects of the integrating modes at the end of the control horizon. The consequence of including this condition in the control problem is that the controller may become infeasible, even when the equilibrium point of the system lies inside the set defined by its constraints. Here, the approach of Rodrigues and Odloak (2003b) was extended, by considering the modified cost function that was proposed by Cano and Odloak (2003) to control the pure integrating system. The result is a new stable infinite horizon MPC for systems with stable and integrating poles. The paper is organized as follows. In Section 2, it is detailed the state space model adopted in this work to represent the system with stable and integrating poles. This model is formulated in such a way that the Lyapunov equation, which defines the terminal state weight, has a straightforward solution. In Section 3, the conventional infinite horizon MPC is presented for the stable and integrating system in terms of the state space formulation adopted in this work. In Section 4, the infinite horizon MPC cost function defined by Cano and Odloak (2003) for the pure integrating system is developed for the case in which the system exhibits stable and integrating poles. It is also presented the main contribution of this work, which is the infinite horizon MPC with an extended set of feasible initial states. In Section 5, the new controller is applied to simulation studies of two typical systems of the literature, and Section 6 concludes the paper.

2. Model formulation

It is assumed that we have a multivariable system with nu inputs and ny outputs, and for each pair (y_i, u_j) there is a transfer function model:

$$G_{i,j}(z) = \frac{b_{i,j,1}z^{-1} + b_{i,j,2}z^{-2} + \cdots + b_{i,j,nb}z^{-nb}}{(1 + a_{i,j,1}z^{-1} + a_{i,j,2}z^{-2} + \cdots + a_{i,j,na}z^{-na})(1 - z^{-1})} \quad (1)$$

where $\{na, nb \in \mathbb{N}\}$. When the poles of the system are non-repeated, the kth coefficient of step response can be written as follows:

$$S_{i,j}(k) = d^0_{i,j} + \sum_{l=1}^{na} [d^d_{i,j,l}] r_l^k + d^i_{i,j} k\Delta t \quad (2)$$

where r_l, $l = 1, 2, \ldots, na$ are the non-integrating poles of the system, Δt is the sampling period and coefficients $d^0_{i,j}$, $d^d_{i,j,l}$, $d^i_{i,j}$ are obtained by partial fractions expansion of the transfer function $G_{i,j}$. An equivalent state space model, which produces an offset free MPC, can be written in the following form

$$x(k+1) = Ax(k) + B\Delta u(k) \quad (3)$$

$$y(k) = Cx(k) \quad (4)$$

where

$$[x] = \begin{bmatrix} x^s \\ x^d \\ x^i \end{bmatrix}, \quad x \in \mathbb{C}^{nx}, \quad nx = 2ny + nd,$$

$$nd = ny \cdot nu \cdot na, \quad x^s \in \mathbb{R}^{ny}, \quad x^d \in \mathbb{C}^{ny \cdot nu \cdot na}, \quad x^i \in \mathbb{R}^{ny}$$

$$A = \begin{bmatrix} I_{ny} & 0 & \Delta t I_{ny} \\ 0 & F & 0 \\ 0 & 0 & I_{ny} \end{bmatrix}, \quad A \in \mathbb{C}^{nx \cdot nx},$$

$$B = \begin{bmatrix} D^0 + \Delta t D^i \\ D^d FN \\ D^i \end{bmatrix}, \quad B \in \mathbb{R}^{nx \cdot nu} \quad (5)$$

$$[y] = \begin{bmatrix} y_1 \\ \vdots \\ y_{ny} \end{bmatrix}, \qquad C = [\, I_{ny} \quad \Psi \quad 0_{ny} \,]$$

$$[x^s] = [\, x_1^s \quad x_2^s \quad \cdots \quad x_{ny}^s \,]^{\mathrm{T}},$$

$$[x^i] = [\, x_1^i \quad x_2^i \quad \cdots \quad x_{ny}^i \,]^{\mathrm{T}},$$

$$[x^d] = \begin{bmatrix} x^d_{1,1,1} & \cdots & x^d_{1,1,na} & x^d_{1,2,1} & \cdots & x^d_{1,2,na} & \cdots \\ x^d_{2,1,1} & \cdots & x^d_{2,1,na} & \cdots & x^d_{ny,1,na} & \cdots & \\ x^d_{ny,nu,na} & & & & & & \end{bmatrix}^{\mathrm{T}}$$

$$D^0 = \begin{bmatrix} d^0_{1,1} & \cdots & d^0_{1,nu} \\ \vdots & \ddots & \vdots \\ d^0_{ny,1} & \cdots & d^0_{ny,nu} \end{bmatrix}, \quad D^0 \in \mathbb{R}^{ny \cdot nu},$$

$$D^i = \begin{bmatrix} d^i_{1,1} & \cdots & d^i_{1,nu} \\ \vdots & \ddots & \vdots \\ d^i_{ny,1} & \cdots & d^i_{ny,nu} \end{bmatrix}, \quad D^i \in \mathbb{R}^{ny \cdot nu}$$

$$F = \mathrm{diag}(r_{1,1,1} \cdots r_{1,1,na} \cdots r_{1,nu,1} \cdots r_{1,nu,na} \cdots r_{ny,1,1} \cdots r_{ny,1,na} \cdots r_{ny,nu,1} \cdots r_{ny,nu,na}) \quad F \in \mathbb{C}^{nd \cdot nd}$$

$$D^d = \mathrm{diag}(d^d_{1,1,1} \cdots d^d_{1,1,na} \cdots d^d_{1,nu,1} \cdots d^d_{1,nu,na} \cdots d^d_{ny,1,1} \cdots d^d_{ny,1,na} \cdots d^d_{ny,nu,1} \cdots d^d_{ny,nu,na}) \quad D^d \in \mathbb{C}^{nd \cdot nd}$$

$$N = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_{ny} \end{bmatrix}, \quad N \in \mathbb{R}^{nd \cdot nu}, \qquad J_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$J_i \in \mathbb{R}^{nu \cdot na \cdot nu} \quad i = 1, 2, \ldots, ny$

$$\Psi = \begin{bmatrix} \Phi & & 0 \\ & \ddots & \\ 0 & & \Phi \end{bmatrix}, \quad \Psi \in \mathbb{R}^{ny \cdot nd},$$

$$\Phi = [1 \quad \cdots \quad 1], \quad \Phi \in \mathbb{R}^{nu \cdot na}$$

In such model representation, x^s corresponds to the integrating states introduced by the incremental form of the input, x^d corresponds to the stable states and x^i corresponds to the true integrating states of the system. In the state matrix A, the system poles appear in its main diagonal. The first ny components of the main diagonal are the integrating poles created by the incremental form of the model. Matrix F is also diagonal and contains the stable poles of the system. Finally, the last ny components of the main diagonal correspond to the true integrating poles of the system. In the next section, we cast the infinite horizon MPC into the framework of the state space model presented above.

3. MPC with infinite prediction horizon for integrating systems

The cost function of the infinite horizon MPC for the discrete-time system can be written as follows:

$$V_{1,k} = \sum_{j=0}^{\infty} e(k+j)^{\mathrm{T}} Q e(k+j) + \sum_{j=0}^{m-1} \Delta u(k+j)^{\mathrm{T}} R \Delta u(k+j) \tag{6}$$

where $Q \in \mathbb{R}^{ny \cdot ny}$ is assumed positive definite and $R \in \mathbb{R}^{nu \cdot nu}$ is positive semi-definite, $e(k+j) = y(k+j) - y^r$ is the error of the predicted output at sampling step $k+j$ including the effect of future control actions, y^r is the desired output reference and m is the control horizon, where the input is allowed to move. Beyond the control horizon, the input move is assumed equal to zero. For the model formulation adopted in the previous section, in order to keep the control cost bounded, the following equality constraints should be obeyed by the control sequence, if the control objective (6) is to be kept bounded:

$$e^s(k) + (D^i_{2,m} - D^0_m)\Delta u_k = 0 \tag{7}$$

$$x^i(k) + D^i_{1m}\Delta u_k = 0 \tag{8}$$

where

$$e^s(k) = x^s(k) - y^r$$

$$D^0_m = [\overbrace{D^0 \quad D^0 \quad \cdots \quad D^0}^{m}], \qquad D^i_{1m} = [\overbrace{D^i \quad D^i \quad \cdots \quad D^i}^{m}],$$

$$D^i_{2m} = [0 \quad \Delta t D^i \quad \cdots \quad (m-1)\Delta t D^i],$$

$$\Delta u_k = \begin{bmatrix} \Delta u(k) \\ \vdots \\ \Delta u(k+m-1) \end{bmatrix},$$

We can find useful interpretations to the constraints represented by Eqs. (7) and (8). Eq. (8) corresponds to the rate balance constraint defined by Lee and Xiao (2000). This constraint means that the integrating modes of the system should be zeroed at the end of the control horizon. Analogously, the physical meaning to Eq. (7) is that the error on the system output at the steady state should be zeroed.

Substituting Eqs. (7) and (8) into Eq. (6), the control cost becomes

$$V_{1,k} = \sum_{j=1}^{m-1} e(k+j)^{\mathrm{T}} Q e(k+j) + x^d(k+m)^{\mathrm{T}} \bar{Q} x^d(k+m) + \sum_{j=0}^{m-1} \Delta u(k+j)^{\mathrm{T}} R \Delta u(k+j)$$

where $\bar{Q}$ is obtained from the solution of the following discrete Lyapunov equation

$$\bar{Q} - F^{\mathrm{T}} \bar{Q} F = F^{\mathrm{T}} \Psi^{\mathrm{T}} Q \Psi F$$

Writing model Eqs. (3) and (4) for future time instants, we can obtain

$$\bar{y} = \bar{L} x(k) + \bar{G} \Delta u_k$$

where

$$\bar{y} = [\, y(k+1)^{\mathrm{T}} \quad y(k+2)^{\mathrm{T}} \quad \cdots \quad y(k+m)^{\mathrm{T}} \,]^{\mathrm{T}}$$

$$\bar{L} = [\, L(1)^{\mathrm{T}} \quad L(2)^{\mathrm{T}} \quad \cdots \quad L(m)^{\mathrm{T}} \,]^{\mathrm{T}},$$

$$L(j) = [\, I_{ny} \quad \Psi F^j \quad j \Delta t I_{ny} \,]$$

$$U = \left\{ \Delta u(k+j) \,\middle|\, \begin{array}{c} -\Delta u^{\max} \le \Delta u(k+j) \le \Delta u^{\max} \\ \Delta u(k+j) = 0,\ j \ge m \\ u^{\min} \le u(k-1) + \sum_{i=0}^{j} \Delta u(k+i) \le u^{\max},\ j = 0, 1, \ldots, m-1 \end{array} \right\}$$

$$\bar{G} = \begin{bmatrix} G_{1,1} & 0 & \cdots & 0 \\ G_{2,1} & G_{2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{m,1} & G_{m,2} & \cdots & G_{m,m} \end{bmatrix},$$

$$G_{j,l} = D^0 + \Delta t D^i + \Psi F^{j-l} D^d F N + (j-l) \Delta t D^i$$

It can also be shown that

$$x^d(k+m) = F_x x(k) + F_u \Delta u_k$$

where

$$F_x = [\, 0_{nd \cdot ny} \quad F^m \quad 0_{nd \cdot ny} \,],$$

$$F_u = [\, D^d F^m N \quad D^d F^{m-1} N \quad \cdots \quad D^d F N \,]$$

Finally, the control cost can be written as follows:

$$V_{1,k} = \Delta u_k^{\mathrm{T}} H_1 \Delta u_k + 2 C_{f,1}^{\mathrm{T}} \Delta u_k + c \tag{9}$$

where

$$H_1 = \bar{G}^{\mathrm{T}} \tilde{Q} \bar{G} + F_u^{\mathrm{T}} \bar{Q} F_u + \tilde{R}$$

$$C_{f,1} = [(-\bar{I} y^r + \bar{L} x(k))^{\mathrm{T}} \tilde{Q} \bar{G} + x(k)^{\mathrm{T}} F_x \bar{Q} F_u]$$

$$\bar{I} = \begin{bmatrix} I_{ny} \\ \vdots \\ I_{ny} \end{bmatrix}, \qquad \bar{I} \in \mathbb{R}^{(m \cdot ny) ny}$$

$$c = [\bar{I} y^r - \bar{L} x(k)]^{\mathrm{T}} \tilde{Q} [\bar{I} y^r - \bar{L} x(k)] + x(k)^{\mathrm{T}} F_x^{\mathrm{T}} \bar{Q} F_x x(k)$$

$$\tilde{Q} = \operatorname{diag}[\overbrace{Q \quad \cdots \quad Q}^{m}], \qquad \tilde{R} = \operatorname{diag}[\overbrace{R \quad \cdots \quad R}^{m}]$$

Thus, the infinite horizon MPC for the integrating system can be written as the following optimization problem:

Problem 1.

$$\min_{\Delta u} [\Delta u_k^{\mathrm{T}} H_1 \Delta u_k + 2 c_{f,1}^{\mathrm{T}} \Delta u_k]$$

subject to

$$e^s(k) + (D_{2,m}^i - D_m^0) \Delta u_k = 0$$

$$x^i(k) + D_{1m}^i \Delta u_k = 0$$

$$\Delta u(k+j) \in U, \quad j \ge 0 \tag{10}$$

If at steady state, the input that corresponds to y^r lies inside the set U, the convergence of the system output in closed loop to the reference value is assured by the following theorem:

Theorem 1. *If Problem 1 is feasible at time step k, the application of the optimal solution to this problem to the undisturbed integrating system in closed loop drives the system output to its reference value.*

Proof. The proof of this theorem can be obtained by following the same steps followed by Rodrigues and Odloak (2003a) to prove the stability of the infinite horizon controller with continuous output horizon. □

Observe that, if the output converges to the desired value when $t \to \infty$, from Eq. (3) and the definition of the output matrix C, we have

$$y^{sp} = x^s(\infty) + \Psi x^d(\infty)$$

However, as x^d corresponds to the stable modes of the system, it tends to zero as time tends to infinite. Thus, the above equation becomes

$$y^{sp} = x^s(\infty)$$

and a physical meaning can be associated to the state component x^s: it corresponds to the predicted system output at steady state. The other state components x^d and x^i will rest at zero when the system reaches the steady state.

As long as, Problem 1 remains feasible, the above theorem guarantees the convergence of the closed loop system. However, Problem 1 may become infeasible quite easily in practice. This is so, because in practice, the controller tuning parameters may be such that the control horizon is short to reduce the computer effort and the maximum control move is small to produce a smooth operation of the system. Under such conditions, it may arise a conflict between constraints (7), (8) and (10). The consequence is that Problem 1 results infeasible. This will happen, for instance, when the controller is submitted to changes of considerable amplitude in the output reference, or, the system is affected by disturbances of significant magnitude. As shown by Rodrigues and Odloak (2003a), the feasibility range of the infinite horizon controller for the integrating system with an incremental state space model, can be enlarged by considering in the MPC a modified cost function as follows:

$$V_{2,k} = \sum_{j=0}^{\infty} (e(k+j) - \delta_k^s)^{\mathrm{T}} Q (e(k+j) - \delta_k^s) + \sum_{j=0}^{m-1} \Delta u(k+j)^{\mathrm{T}} R \Delta u(k+j) + \delta_k^{s\,\mathrm{T}} S_1 \delta_k^s \tag{11}$$

where $\delta_k^s \in \mathbb{R}^{ny}$ is a vector of additional decision variables of the controller optimization problem that provides extra degrees of freedom and widens substantially the set of operating conditions in which the controller is feasible. S_1 is a positive definite weighting matrix.

It can be shown that with the modified control cost function defined in (11), the condition represented in Eq. (7) results

$$e^s(k) + \delta_k^s + (D_{2,m}^i - D_m^0)\Delta u_k = 0 \tag{12}$$

Since δ_k^s is not bounded, the constraint represented in (12) can be satisfied by any control sequence that also satisfies (10), or at any operating condition defined by $e^s(k)$.

Including the new decision variable δ_k^s, the control cost defined in (11) can be written as follows:

$$V_{2,k} = [\,\Delta u_k^{\mathrm{T}} \quad \delta_k^{s\,\mathrm{T}}\,] H_2 \begin{bmatrix} \Delta u_k \\ \delta_k^s \end{bmatrix} + 2C_{f,2}^{\mathrm{T}} \begin{bmatrix} \Delta u_k \\ \delta_k^s \end{bmatrix} + c \tag{13}$$

where

$$H_2 = \begin{bmatrix} \bar{G}^{\mathrm{T}} \tilde{Q} \bar{G} + F_u^{\mathrm{T}} \bar{Q} F_u + \bar{R} & \bar{G}^{\mathrm{T}} \tilde{Q} \bar{I} \\ \bar{I}^{\mathrm{T}} \tilde{Q} \bar{G} & \bar{I}^{\mathrm{T}} \tilde{Q} \bar{I} + S_1 \end{bmatrix} \tag{14}$$

$$C_{f,2} = [[(-\bar{I} y^r + \bar{L} x(k))^{\mathrm{T}} \tilde{Q} \bar{G} + x(k)^{\mathrm{T}} F_x \bar{Q} F_u]^{\mathrm{T}} \times [-\bar{I} y^r + \bar{L} x(k)]^{\mathrm{T}} \tilde{Q} \bar{I}]$$

$$c = [\bar{I} y^r - \bar{L} x(k)]^{\mathrm{T}} \tilde{Q} [\bar{I} y^r - \bar{L} x(k)] + x(k)^{\mathrm{T}} F_x^{\mathrm{T}} \bar{Q} F_x x(k)$$

$$x(k) = [\,x^s(k)^{\mathrm{T}} \quad x^d(k)^{\mathrm{T}} \quad x^i(k)^{\mathrm{T}}\,]^{\mathrm{T}}$$

With the proposed cost function, the controller with infinite output horizon is now defined by the following optimization problem:

Problem 2.

$$\min_{\Delta u, \delta_k^s} [\,\Delta u_k^{\mathrm{T}} \quad \delta_k^{s\,\mathrm{T}}\,]\; H_2 \begin{bmatrix} \Delta u_k \\ \delta_k^s \end{bmatrix} + 2C_{f,2}^{\mathrm{T}} \begin{bmatrix} \Delta u_k \\ \delta_k^s \end{bmatrix} \tag{15}$$

subject to

$$x^i(k) + D_{1m}^i \Delta u_k = 0$$

$$\Delta u(k+j) \in U, \quad j \geq 0$$

$$e^s(k) + \delta_k^s + (D_{2,m}^i - D_m^0)\Delta u_k = 0$$

The convergence of the system output to the reference value in closed-loop system with the controller defined by Problem 2 is guaranteed by the following theorem:

Theorem 2. *For the undisturbed system with stable and integrating poles and an input weight R sufficiently small, if Problem 2 is feasible at sampling step k, then it will be feasible at time steps $k+1$, $k+2$, …, and the output of the closed-loop system with the control law defined by Problem 2 will converge to the reference value.*

Proof. The proof of this theorem can also be found in Rodrigues and Odloak (2003a) for the case in which the state space model is built considering a continuous output prediction horizon. A similar proof can be developed for the discretized time state space model considered here. □

4. Extended infinite horizon MPC for integrating systems

For the regulator operation case, the controller defined by Problem 2 will be feasible for a larger set of initial states and/or unknown disturbances than the controller defined by Problem 1. Similarly, for the set point tracking case, the new controller will remain feasible for larger changes in y^r than the controller defined by Problem 1. However, depending on the size of the disturbance or set point change, it may still arise a conflict between constraints (8) and (10). In such case, Problem 2 becomes infeasible and the infinite horizon controller can no longer be implemented. To circumvent such limitation, we propose a new extension of the infinite horizon controller applied to systems with integrating poles. The extended proposed controller is based on the following cost function (Cano & Odloak, 2003)

$$V_{3,k} = \sum_{j=0}^{\infty} (e(k+j) + \delta_k^s + j\Delta t \delta_k^i)^{\mathrm{T}} Q (e(k+j) + \delta_k^s + j\Delta t \delta_k^i) + \sum_{j=0}^{m-1} \Delta u(k+j)^{\mathrm{T}} R \Delta u(k+j) + \delta_k^{s\,\mathrm{T}} S_1 \delta_k^s + \delta_k^{i\,\mathrm{T}} S_2 \delta_k^i \tag{16}$$

where $S_2 \in \mathbb{R}^{ny \cdot ny}$ is positive definite and δ_k^i is a vector of additional decision variables of the control optimization problem. As it will be shown in the sequel, these new slack variables provide the necessary degrees of freedom in order to the control problem to be feasible to a larger set of initial states and external disturbances. In order to show the benefits of considering the cost defined in (16), we observe the conditions necessary to a bounded cost. We can show that, for the new cost function to be bounded, constraint (12) has to be satisfied as in Problem 2, but Eq. (8) is substituted by the following equation

$$x^i(k) + \delta_k^i + D_{1m}^i \Delta u_k = 0 \tag{17}$$

From Eqs. (17) and (8), it can be seen that a physical meaning to δ_k^i is that this variable corresponds to a slack to the rate balance constraint. If this slack is made equal to zero by the future control actions, then, observing Eqs. (7) and (12), it becomes clear that δ_k^s corresponds to a slack to the predicted output error at steady state.

Unfortunately, minimizing the cost defined in (16) subject to constraint (10), (12) and (17), does not produce a control law that forces the cost function to decrease. This is so, because at sampling step $k+1$, $\Delta u_{k+1} = [\,\Delta u^*(k+1)^{\mathrm{T}} \quad \cdots \quad \Delta u^*(k+m-1)^{\mathrm{T}} \quad 0\,]^{\mathrm{T}}$ and $\delta_{k+1}^i = \delta_k^{i*}$ does not satisfy (17) and so, it is not a feasible solution to the controller optimization problem that would minimize (16) subject to constraints (10), (12) and (17). Thus, a different strategy needs to be followed to produce a stable controller that is based on the minimization of the cost defined in (16). For this purpose, define a secondary positive cost $V_{l,k} = \delta_k^{i\mathrm{T}} S_2 \delta_k^i$ and consider the following constraint to be included in the control optimization problem:

$$V_{l,k} < \tilde{V}_{l,k-1} \tag{18}$$

where

$$\tilde{V}_{l,k-1} = \tilde{\delta}_{k-1}^{i\mathrm{T}} S_2 \tilde{\delta}_{k-1}^i$$

$\tilde{V}_{l,k-1}$ is computed with $\tilde{\delta}_{k-1}^i$, which is obtained as follows:

$$\tilde{\delta}_{k-1}^i = -\tilde{x}^i(k-1) - D_{1m}^i \Delta u_{k-1} \tag{19}$$

$$\tilde{x}^i(k-1) = x^i(k) - D^i \Delta u(k-1) \tag{20}$$

Observe that for the undisturbed system $\tilde{x}^i(k-1) = x^i(k-1)$ and $\tilde{\delta}_{k-1}^i = \delta_{k-1}^i$. Consequently, as long as the system input at the equilibrium point lies inside U, inequality (18) will be satisfied for any $\Delta u_{\max} \neq 0$ and $V_{l,k}$ will decrease asymptotically. For the disturbed system, using Eq. (20) to the computation of the state at $(k-1)$ makes (18) always feasible. Thus, the infinite horizon MPC, which is stable for systems containing stable and integrating poles, can be formulated as the solution of the following optimization problem:

Problem 3.

$$\min_{\Delta u_k, \delta_k^s, \delta_k^i} [\,\Delta u^{\mathrm{T}} \quad \delta_k^{s\mathrm{T}} \quad \delta_k^{i\mathrm{T}}\,] H_3 \begin{bmatrix} \Delta u \\ \delta_k^s \\ \delta_k^i \end{bmatrix} + 2C_{f,3}^{\mathrm{T}} \begin{bmatrix} \Delta u \\ \delta_k^s \\ \delta_k^i \end{bmatrix}$$

subject to

$$\Delta u(k+j) \in U, \quad j \geq 0$$

$$e^s(k) + \delta_k^s + (D_{2,m}^i - D_m^0)\Delta u_k = 0$$

$$x^i(k) + \delta_k^i + D_{1m}^i \Delta u_k = 0$$

$$V_{l,k} < \tilde{V}_{l,k-1}$$

where

$$H_3 = \begin{bmatrix} \bar{G}^{\mathrm{T}} \tilde{Q} \bar{G} + F_u^{\mathrm{T}} \bar{Q} F_u + \bar{R} & \bar{G}^{\mathrm{T}} \tilde{Q} \bar{I} & \bar{G}^{\mathrm{T}} \tilde{Q} T \\ \bar{I}^{\mathrm{T}} \tilde{Q} \bar{G} & \bar{I}^{\mathrm{T}} \tilde{Q} \bar{I} + S_1 & \bar{I}^{\mathrm{T}} \tilde{Q} T \\ T^{\mathrm{T}} \tilde{Q} \bar{G} & T^{\mathrm{T}} \tilde{Q} \bar{I} & T^{\mathrm{T}} \tilde{Q} T + S_2 \end{bmatrix},$$

$$T = \begin{bmatrix} \Delta t I_{ny} \\ 2\Delta t I_{ny} \\ \vdots \\ m\Delta t I_{ny} \end{bmatrix},$$

$$C_{f,3} = \begin{bmatrix} [(-\bar{I} y^r + \bar{L} x(k))^{\mathrm{T}} \tilde{Q} \bar{G} + x(k)^{\mathrm{T}} F_x \bar{Q} F_u]^{\mathrm{T}} \\ [-\bar{I} y^r + \bar{L} x(k)]^{\mathrm{T}} \tilde{Q} \bar{I} \\ [-\bar{I} y^r + \bar{L} x(k)]^{\mathrm{T}} \tilde{Q} T \end{bmatrix}$$

The convergence of the extended infinite horizon MPC resulting from Problem 3 can be summarized in the following theorem:

Theorem 3. *For systems with stable and integrating poles whose input at steady state lies inside U, the control sequence produced by the successive solution of* Problem 3 *drives the system output asymptotically to the reference value.*

Proof. The condition that the system input lies inside the definition set U means that the input will not become saturated at steady state. Thus, the integrating modes are stabilizable at the desired steady state. Consider now the undisturbed regulator case or the output tracking case. In both cases, if the system input corresponding to steady state lies inside U, Problem 3 is feasible at k and it will remain feasible at the subsequent time steps. Consequently, function $V_{l,k}$ will converge to zero. Once this function has converged, constraint (8) becomes feasible and it will remain feasible for all the subsequent time instants. Then, Problem 3 reduces to Problem 2 and, convergence of the system output to the reference, is ensured by Theorem 2. □

The practical implementation of controllers defined by Problems 2 and 3 could be achieved through the following steps: at time k, check if there is a feasible solution to the problem defined by constraint (8) and (10). If yes, solve Problem 2 and implement the resulting control action. If not, solve Problem 3 and implement the corresponding control action.

Lee and Xiao (2000) propose a two-step approach to solve the problem of including the steady state economic optimization in the conventional MPC of stable and integrating systems represented by step response models. A similar approach can be used here to produce a stable MPC. At each sampling step k, the controller is obtained as the solution of the two following problems:

Problem 4a.

$$\min_{\delta_k^i, \Delta u_{a,k}} V_{4a,k} = \delta_k^{i^{\mathrm{T}}} S_2 \delta_k^i + \Delta u_{a,k}^{\mathrm{T}} \bar{R} \Delta u_{a,k}$$

subject to

$$\Delta u_a(k+j) \in U, \quad j \geq 0$$

$$x^i(k) + \delta_k^i + D_{1m}^i \Delta u_{a,k} = 0$$

where
$\Delta u_{a,k} = [\, \Delta u_a(k)^{\mathrm{T}} \;\; \Delta u_a(k+1)^{\mathrm{T}} \;\; \cdots \;\; \Delta u_a(k+m-1)^{\mathrm{T}} \,]^{\mathrm{T}}$
and $\bar{R}$ is positive definite.

Let the optimal solution to Problem 4a be designated $(\delta_k^{i^*}, \Delta u_{a,k}^*)$ and consider the corresponding input increment

$$u^*(k+m-1) - u(k-1) = \sum_{j=0}^{m-1} \Delta u_a^*(k+j)$$

This optimal input increment is passed to a second problem, which is solved within the same time step:

Problem 4b.

$$\min_{\Delta u_{b,k}, \delta_k^s} V_{4b,k} = [\, \Delta u_{b,k}^{\mathrm{T}} \quad \delta_k^{s^{\mathrm{T}}} \,] H_2 \begin{bmatrix} \Delta u_{b,k} \\ \delta_k^s \end{bmatrix} + 2 C_{f,2}^{\mathrm{T}} \begin{bmatrix} \Delta u_{b,k} \\ \delta_k^s \end{bmatrix} + c$$

subject to

$$\Delta u_b(k+j) \in U, \quad j \geq 0$$

$$e^s(k) + \delta_k^s + (D_{2,m}^i - D_m^0) \Delta u_{b,k} = 0$$

$$u^*(k+m-1) - u(k-1) = \sum_{j=0}^{m-1} \Delta u_b(k+j) \tag{21}$$

where
$\Delta u_{b,k} = [\, \Delta u_b(k)^{\mathrm{T}} \;\; \Delta u_b(k+1)^{\mathrm{T}} \;\; \cdots \;\; \Delta u_b(k+m-1)^{\mathrm{T}} \,]^{\mathrm{T}}$.

The control law obtained through the sequential solution of Problems 4a and 4b above leads to the convergence of the system output to the reference value as summarized in the following theorem:

Theorem 4. *For a system with stable and integrating poles, if at sampling step k,* Problem 4a *is feasible then* Problem 4b *is also feasible. Also, the control law, which results from the successive solution of these two problems, drives the output of the closed loop system to the reference value.*

Proof. Consider the undisturbed system and assume that at time step k Problem 4a is feasible. Let us designate the optimal value of the objective of Problem 4a as $V_{4a,k}^*$. The total increment of the control input is passed to Problem 4b as equality constraint (21). As far as Problem 4b is concerned, the decrease of its cost function cannot be guaranteed at this stage. However, the use of the slack variable δ_k^s assures that there is a feasible solution, which satisfies constraints (10), (12) and (19). Let this optimal solution to Problem 4b be designated $(\Delta u_{b,k}^*, \delta_k^{s*})$. The first control move $\Delta u_b^*(k)$ is injected into the true process and we move to time step $k+1$. At this time step, it can be easily shown that $\Delta u_{a,k+1} = [\, \Delta u_b^*(k+1)^{\mathrm{T}} \quad \cdots \quad \Delta u_b^*(k+m-1)^{\mathrm{T}} \quad 0 \,]^{\mathrm{T}}$ and $\delta_{k+1}^i = \delta_k^{i*}$ is a feasible solution to Problem 4a. Now, considering that $\bar{R}$ is negligible in comparison to S_2, the corresponding value of the objective function of Problem 2 is still $V_{a,k}^*$. Consequently, the optimal solution of Problem 4a at time step $k+1$ will be $V_{4a,k+1}^* \leq V_{4a,k}^*$. Since we have selected $\bar{R} \ll S_2$, the objective function of Problem 4a will converge to zero, which corresponds to $x^i(k) = \delta_k^i = 0$. The consequence of the convergence of the objective of Problem 4a is that Problem 4b has been made equivalent to Problem 2, which is now feasible. After convergence, the solution of the Problem 4a reduces to the trivial solution $\delta_k^i = 0$, and Theorem 2 assures the convergence of the system output to the reference through the successive application of the control sequence produced by the solution of Problems 4a and 4b. □

5. Examples

In this section, we illustrate the application of the proposed stable MPC strategy to two systems of the control literature. The first example considers the control of a system, which is based on the ethylene oxide reactor system presented by Rodrigues and Odloak (2003a). This is a typical example of the chemical process industry that exhibits stable and integrating poles. We use this system to compare the performances of the controllers defined by Problems 3, 4a and 4b. The second example shows that the attraction region of the controller defined by Problem 3 is larger than the attraction region of the stable MPC proposed by De Doná et al. (2002).

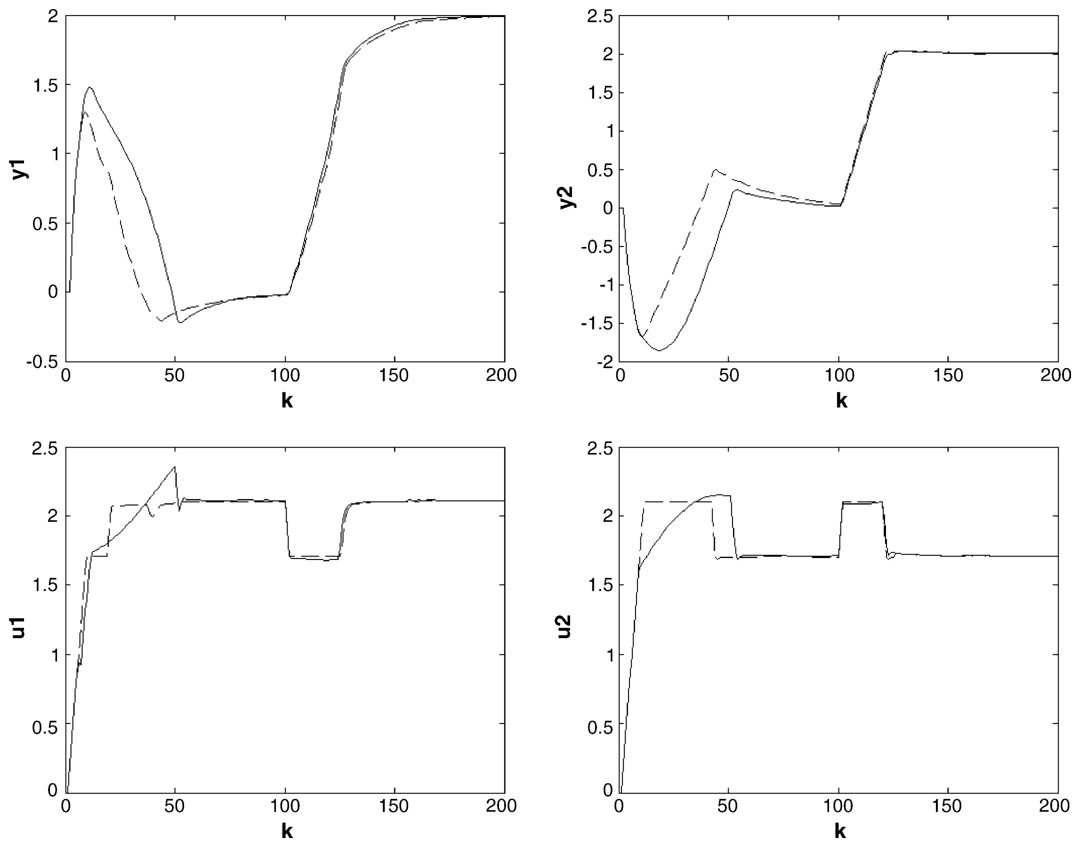

Fig. 1. System responses for example 1. Controller I (—) and Controller II (— —).

Example 1. The system is represented by the following transfer matrix

$$G(s)=\begin{bmatrix} \dfrac{-0.19}{s} & \dfrac{-1.7}{19.5s+1} \\ \dfrac{-0.763}{31.8s+1} & \dfrac{0.235}{s} \end{bmatrix}$$

Using a sampling period $T=1$, this model can be translated into the model formulation represented in (3) and (4) as

Table 1
Tuning parameters of Controllers I and II in Example 1

m	3
Q	diag(1, 1)
R	diag(10^{-2}, 10^{-2})
S_1	diag(10, 10)
S_2	diag(10^3, 10^3)
$\Delta u_{\max}$	[0.2, 0.2]

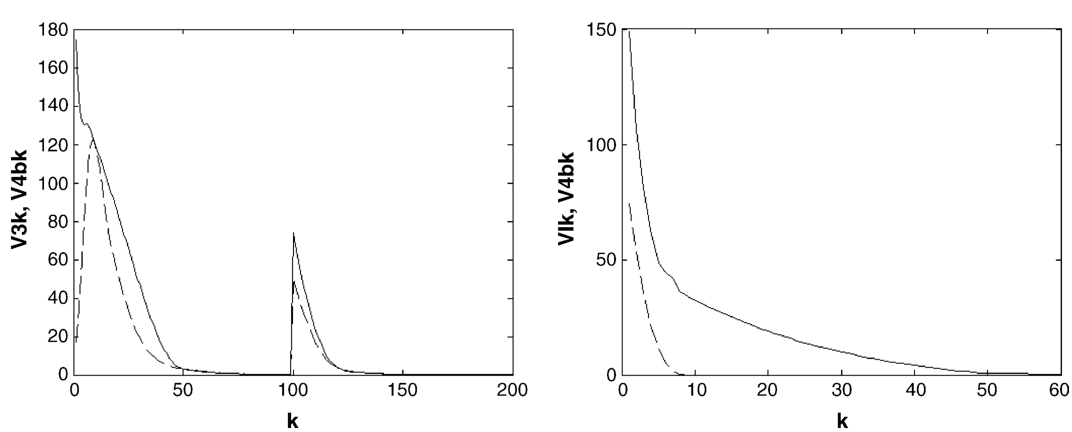

Fig. 2. Objective functions for example 1. Controller I (—), Controller II (— —).

follows:

$$\begin{bmatrix} x_1^s(k+1) \\ x_2^s(k+1) \\ x_1^d(k+1) \\ x_2^d(k+1) \\ x_1^i(k+1) \\ x_2^i(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9690 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1^s(k) \\ x_2^s(k) \\ x_1^d(k) \\ x_2^d(k) \\ x_1^i(k) \\ x_2^i(k) \end{bmatrix} + \begin{bmatrix} -0.1900 & -1.7000 \\ -0.7630 & 0.2350 \\ 0 & 1.6150 \\ 0.7394 & 0 \\ -0.1900 & 0 \\ 0 & 0.2350 \end{bmatrix} \Delta u(k),$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^s(k) \\ x_2^s(k) \\ x_1^d(k) \\ x_2^d(k) \\ x_1^i(k) \\ x_2^i(k) \end{bmatrix}$$

This system is part of the ethylene oxide reactor system studied by Rodrigues and Odloak (2003a). Each of the controlled outputs (y_1, y_2) is integrating with respect to one of the inputs (u_1, u_2) and stable with respect to the other input. This system is simulated with Controller I, which is produced by the solution of Problem 3 and with Controller II, which corresponds to the sequential solution of Problems 4a and 4b. The tuning parameters for both controllers are represented in Table 1.

For the comparison of Controllers I and II, the following scenario is simulated: The initial state is $x^s = [0 \quad 0]^{\mathrm{T}}$; $x^d = [0 \quad 0]^{\mathrm{T}}$; $x^i = [0.4 \quad -0.4]^{\mathrm{T}}$ and the output reference is $y^r = [0 \quad 0]^{\mathrm{T}}$. At time step $k = 100$, the reference is changed to $y^r = [2 \quad 2]^{\mathrm{T}}$. Fig. 1 shows the system responses for the proposed Controllers I and II. There is no clear superiority of any of the two controllers. For the regulator operation (k between 0 and 100), Controller II has output responses that are slightly better than Controller I. For the output tracking operation ($k > 100$), the two controllers are equivalent.

Fig. 2 shows cost $V_{3,k}$ that is minimized in Problem 3 solved by Controller I. It is compared to cost $V_{4b,k}$ that is minimized in Problem 4b of Controller II. In the same figure, it is compared the time responses of $V_{I,k}$, which is forced to contract in Controller I, and $V_{4a,k}$, which is minimized in Problem 4a. We observe that cost $V_{4b,k}$ does not decrease at the beginning of the regulator operation with Controller II. This is so because, while $V_{4a,k} \neq 0$ in Problem 4a, this means that Problem 2 would not be feasible and, consequently, there is no guarantee that the controller cost will decrease. As soon as, $V_{4a,k}$ converges to zero, Problems 4b and 2 become equivalent and cost $V_{4b,k}$ is forced to decrease. Since Problem 2 corresponds to the controller proposed by Rodrigues and Odloak (2003a), it is clear that their controller could not be applied to the regulator part of this example. In the output tracking part of this example, $V_{I,k}$ remains equal to zero and Controllers I and II reduce to the controller proposed by Rodrigues and Odloak (2003a) that would also be feasible. Thus, in comparison with the previous control strategy, the main advantage of the control strategy proposed in this work is the enlargement of the range of disturbances that can be tolerated by the stable controller. The next example confirms this property of the proposed controller when compared to the dual MPC–LQR strategy.

Example 2. This example was borrowed from De Doná et al. (2002) who studied the integrating system represented by the following state space model

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.08 \end{bmatrix} \Delta u(k)$$

$$y(k) = x_2(k)$$

De Doná et al. (2002) used this example to show that their non-linear dual controller has the largest attraction region among other existing stable constrained MPC controllers. In their approach a constrained MPC is used to bring the state to a terminal set and inside the terminal set they propose a non-linear control law $u = -\mathrm{sat}(Kx)$, where K is the optimal gain for the unconstrained LQR. This gain is computed from the solution of the algebraic Riccati equation. Fig. 3 shows in dashed lines the attraction region of the stable receding horizon controller proposed by De Doná et al. (2002) with parameters listed in Table 2. In this table, it is also shown the parameters of Controller I, which was applied to this example. Observe that a shorter control horizon was used in Controller I. In Fig. 3, it is also represented the trajectory followed by the states when the system is controlled by Controller I. The

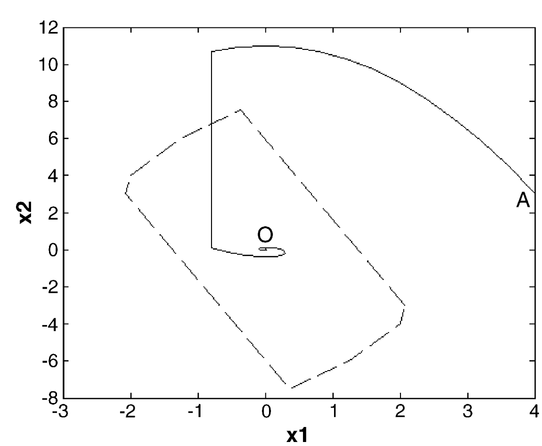

Fig. 3. States trajectory for Controller I in Example 2.

Table 2
Tuning parameters of the MPC controller for Example 2

Controller	m	Q	R	S_1	S_2	Δu_{max}
De Dona et al.	10	diag(1, 1)	0.25	–	–	1
I	3	diag(1, 1)	0.25	diag(10, 10)	diag(10, 10)	1

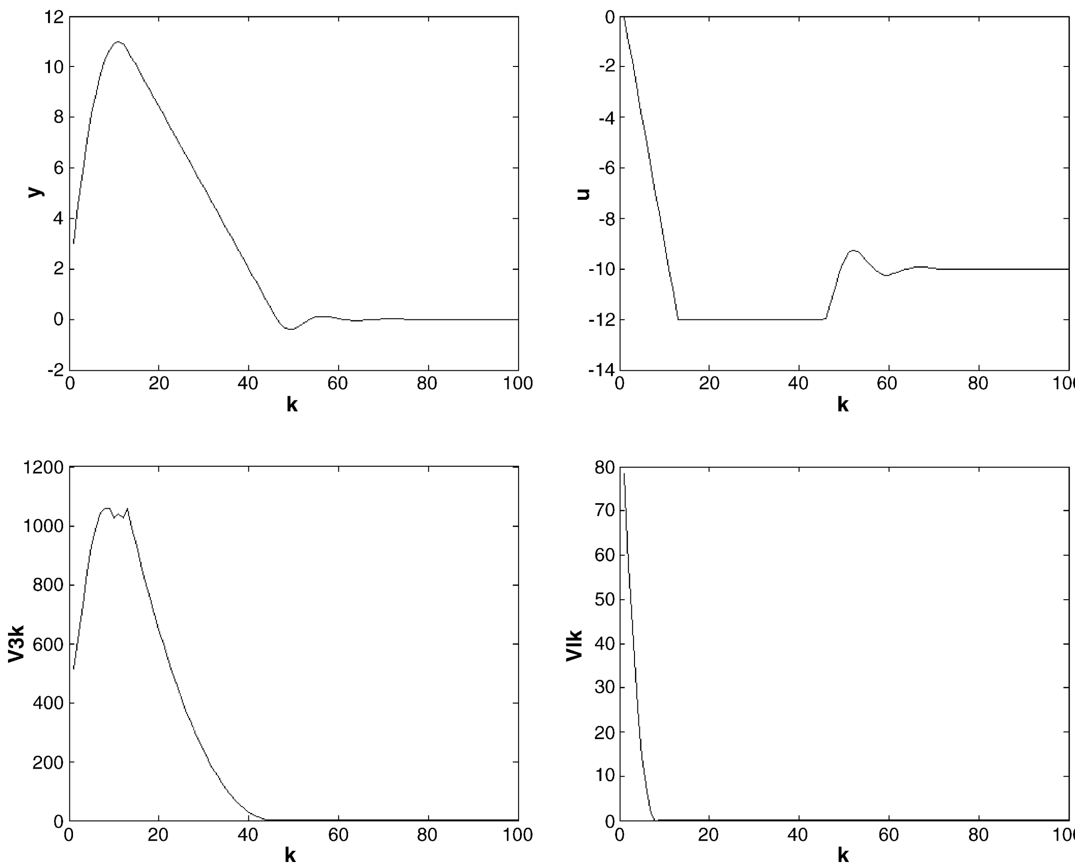

Fig. 4. Responses of Controller I in Example 2.

system starts at the state corresponding to point $A = (4, 3)$, which is beyond the attraction region of the controller of De Doná et al. (2002). A similar trajectory could be obtained with Controller II, which solves Problems 4a and 4b. Fig. 4 shows the system input and output dynamic responses corresponding to this case. It is also shown the cost function $V_{3,k}$ and the contracting function $V_{l,k}$.

6. Conclusions

In this paper, we have presented an extension of a previous approach to the nominal stable infinite horizon MPC. The extension concerns systems with stable and integrating modes. With the approach presented here, the limitations related to infeasibilities generated by the presence of unknown disturbances and input constraints are completely eliminated. The only condition that remains to be attended by the system in order to guarantee stability of the closed loop with MPC is that the terminal steady state for the system input lies inside the set Ω of allowable inputs. With the approach presented here, practical implementation of the stable MPC becomes as simple as the implementation of conventional MPC.

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