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An infinite horizon model predictive control for stable and integrating processes

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Abstract

This paper deals with the linear model predictive control (MPC) with infinite prediction horizon (IHMPC) that is nominally stable. The study is focused on the output-tracking problem of systems with stable and integrating modes and unmeasured disturbances. To produce a bounded system response along the infinite prediction horizon, the effect of the integrating modes must be zeroed. The integrating mode zeroing constraint may turn the control problem infeasible, particularly when the system is affected by large disturbances. This work contributes in two ways to the problem of implementing IHMPC. The first contribution refers to the softening of some hard constraints associated with the integrating modes, while nominal stability is preserved. Another contribution is related to the strategy followed to deal with the infinite horizon and the removal of the matrix Lyapunov equation from the controller optimization problem. A real industrial example where the application of the controller has been studied is used to illustrate the advantages of the proposed strategy.

Keywords: Model Predictive Control; Stability; Infinite horizon; Constrained control; Integrating systems

1. Introduction

A desirable property of any control structure of a process plant is nominal stability, which means that the closed-loop system will be stable if the perfect model is used in the controller, independently of the controller tuning. In this case, the parameters of the controller will be selected considering solely the improvement on the performance of the system. In the MPC literature, the search for nominal stabilizing controllers has received plentiful attention in the last few years. Comprehensive reviews related to this issue can be found in the survey papers by Mayne, Rawlings, Rao, and Scokaert (2000) and Morari and Lee (1999). Several approaches have been developed to ensure nominal closed-loop stability to the linear constrained MPC. These methods can be classified in three main groups:

i) state terminal constraints;
ii) state terminal set constraint;
iii) infinite output prediction horizon.

The use of state terminal constraint was initially proposed by Keerthi and Gilbert (1988) for linear discrete-time systems and Mayne and Michalska (1990) for continuous-time nonlinear systems. This method uses an objective function with finite output prediction and control horizons subject to a terminal state equality constraint that forces the states to zero at the end of the output prediction horizon. Relevant contributions related to the second class of approaches are the papers by Gilbert and Tan (1991) and Michalska and Mayne (1993). In their work, Gilbert and Tan (1991) allowed a tolerance to the state at the end of output prediction horizon. They tried to remedy the drawback of the approaches belonging to group (i) that tend to have a narrow set of feasible initial states. Michalska and Mayne (1993) and Scokaert, Mayne, and Rawlings (1999) obtained stability results by addressing the problem of the nonlinear MPC with a dual mode controller. In the first mode, the system states are driven

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to a terminal set (some neighborhood of the origin) using an MPC where one searches for a control horizon that allows a feasible solution. Inside this terminal set, the system is controlled by a local linear stabilizing controller. Nonetheless, the evaluation of the length of the control horizon may result in a time-consuming optimization problem, and this approach does not seem to be easy to implement in practice.

The last framework of group (iii) for synthesizing a nominal stabilizing MPC makes use of an infinite prediction horizon. The infinite horizon MPC (IHMPC) was proposed by Rawlings and Muske (1993) for constrained linear systems. For stable systems, the authors replace the infinite horizon objective by a finite one after defining a penalty weight matrix (terminal cost) at the end of the input horizon. The terminal weight is obtained from the solution of a discrete-time Lyapunov equation. Stability is guaranteed as long as the related optimization problem is feasible. In the case of integrating systems, the authors included terminal equalities that zero the effect of integrating modes at the end of the control horizon. Terminal equalities constraints also need to be introduced in the infinite horizon controller when the system model is used in the incremental form, as it usually happens in MPC practice. In this case, the integrating modes arise naturally from the model formulation, which is necessary to apply the controller to other situations besides the regulator operation for which IHMPC was developed.

Genceli and Nikolaou (1993) showed that when the infinite horizon MPC problem is feasible, stability can be achieved with a finite horizon when sufficiently large input weights are adopted. Based on this idea, Zheng (1997) employed time-varying weights to guarantee stability of constrained MPC.

Another contribution that makes use of an infinite prediction horizon is due to Scokaert and Rawlings (1998), who proposed the constrained linear quadratic regulator. Their approach consists in determining a finite value of the input horizon and a control sequence that steers the states to an invariant region in the neighborhood of origin where the constraints in the inputs and states are satisfied. A similar result was presented by Chmielewski and Manousiouthakis (1996).

Chen and Allgöwer (1998) used a semi-infinite approach for continuous nonlinear systems and De Nicolao, Magni, and Scattolini (1998) presented a similar approach for discrete nonlinear systems. They presented methods to determine terminal costs and an invariant terminal region where a linear stabilizing control is used to establish a bound to the infinite horizon cost. There is no controller switching in these approaches.

Lee and Kouvaritakis (1999) and Lee, Kwon, and Choi (1998) present ways to determine a region where the system can be stabilized by properly evaluating terminal weighting matrices and using state feedback gains. Some related results can be found in Primbs and Nevistic (2000).

The available stability results require the feasibility of the MPC optimization problem where constraints associated to the terminal state are included. However, such constraints may be difficult to be implemented in the industrial controller since they may conflict with other constraints as the constraint on the maximum input move or maximum and minimum input values. This is particularly critical if the system is subject to large disturbances or set-point changes and a small control horizon is used to reduce the computer load.

Here, an infinite horizon MPC is studied from the point of view of the practical implementation. Also developed is a state-space model representation that is suitable for the infinite horizon strategy. Based on this model representation, alternatives are proposed to increase the feasibility region of the controller and to remove the solution of the matrix Lyapunov equation from the controller optimization problem. A linear time-invariant system is considered and it is assumed that the system contains only stable and integrating distinct poles, which can be considered the typical industrial case. The modeling approach considered in this study can be easily extended to systems with time delays, as shown by Rodrigues and Odloak (2000) that utilized a modified version of this model form to generate a compact formulation of the conventional MPC. The paper is organized as follows. In Section 2, the state-space model of the system with stable and integrated modes is presented. The model is formulated in such a way that the infinite horizon MPC can be developed under a new point of view. In Section 3, the IHMPC for the cases of output tracking and unknown disturbances is presented. In Section 4, the infinite horizon MPC is modified to enlarge the feasibility region of the controller. In Section 5, the application of the new approach to an industrial system is studied, and Section 6 gives concluding remarks.

2. Model formulation

We use a state-space model realization that is appropriate to deal with the infinite horizon cost function. To illustrate how the state-space model considered here can be assembled, we start with the example of a very simple system with transfer function

$$\frac{y(s)}{u(s)} = \frac{K}{(\tau s + 1)s} \tag{1}$$

whose corresponding step response is given by

$$S(t) = -K\tau + K\tau\,\mathrm{e}^{-\frac{t}{\tau}} + Kt$$

Observe that this function is of the form

$$S(t) = d^0 + d^d\ \mathrm{e}^{-\frac{t}{\tau}} + d^i t \tag{2}$$

where d^0, d^d and d^i are constants. The system is then discretized with a sampling period Δt following a particular procedure:

(1) Assume that at sampling step k, the system output can be represented by a function similar to the step response function represented in Eq. (2), more specifically, in the form

$$[y(t)]_k = [x^s]_k + [x^d]_k \mathrm{e}^{-\frac{t}{\tau}} + [x^i]_k t \tag{3}$$

where $[x^s]_k$, $[x^d]_k$, and $[x^i]_k$ are parameters of the system output trajectory.

(2) Update the output trajectory parameters with the control action Δu_k

$$[y'(t)]_k = [y(t)]_k + S(t)\ \Delta u_k$$

(3) Move the sampling step to $k+1$ and recompute the output trajectory using the new time origin

$$[y(t)]_{k+1} = [y'(t+\Delta t)]_k = [y(t+\Delta t)]_k + S(t+\Delta t)\Delta u_k \tag{4}$$

and substitute Eqs. (2) and (3) in Eq. (4) to obtain

$$[y(t)]_{k+1} = [x^s]_k + [x^d]_k \mathrm{e}^{-\frac{(t+\Delta t)}{\tau}} + [x^i]_k (t+\Delta t) + (d^0 + d^d \mathrm{e}^{-\frac{(t+\Delta t)}{\tau}} + d^i(t+\Delta t))\Delta u_k \tag{5}$$

(4) Write Eq. (5) as

$$[y(t)]_{k+1} = [x^s]_{k+1} + [x^d]_{k+1} \mathrm{e}^{-\frac{t}{\tau}} + [x^i]_{k+1} t$$

where

$$[x^s]_{k+1} = [x^s]_k + \Delta t [x^i]_k + (d^0 + \Delta t d^i)\Delta u_k \tag{6}$$

$$[x^d]_{k+1} = \mathrm{e}^{-\frac{\Delta t}{\tau}} [x^d]_k + d^d \mathrm{e}^{-\frac{\Delta t}{\tau}} \Delta u_k \tag{7}$$

$$[x^i]_{k+1} = [x^i]_k + d^i \Delta u_k \tag{8}$$

From Eq. (3), the system output at sampling instant k is given by

$$[y]_k = [y(0)]_k = [x^s]_k + [x^d]_k \tag{9}$$

Eqs. (6)–(9) can be consolidated in the conventional state-space model formulation

$$[x]_{k+1} = A[x]_k + B\Delta u_k \tag{10}$$

$$[y]_k = C[x]_k \tag{11}$$

where

$$[x]_k = \begin{bmatrix} x^s \\ x^d \\ x^i \end{bmatrix}_k, \quad A = \begin{bmatrix} 1 & 0 & \Delta t \\ 0 & \mathrm{e}^{-\Delta t/\tau} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} d^0 + d^i \Delta t \\ d^d \mathrm{e}^{-\Delta t/\tau} \\ d^i \end{bmatrix}, \quad C = [1 \quad 1 \quad 0]$$

This model realization has a very convenient property: at sampling step k, the entire output trajectory can be predicted by

$$[y(t)]_k = C(t)[x]_k$$

where $C(t) = [1 \quad \mathrm{e}^{-t/\tau} \quad t]$

In this state-space formulation, state x^s corresponds to the integrating pole that is related to the incremental form of the model. State x^d corresponds to the stable pole of the system and x^i corresponds to the true integrating pole of the system. These poles appear in the diagonal of matrix A.

For a multivariable system with nu inputs and ny outputs, steps analogous to those of the previous example can be followed to build the state-space model of the system. Assume that for each output y_i and input u_j there is a transfer function model

$$G_{i,j}(s) = \frac{b_{i,j,0} + b_{i,j,1}s + b_{i,j,2}s^2 + \cdots + b_{i,j,nb}s^{nb}}{(1 + a_{i,j,1}s + a_{i,j,2}s^2 + \cdots + a_{i,j,na}s^{na})s} \tag{12}$$

where $\{na, nb \in \mathbb{N} | nb < na + 1\}$. If the poles of the system are distinct, the step response can be written as follows

$$S(t) = d^0_{i,j} + \sum_{l=1}^{na} [d^d_{i,j,l}]\mathrm{e}^{r_l t} + d^i_{i,j} t \tag{13}$$

where r_l, $l = 1, 2, \ldots, na$ are the non-integrating poles of the system and coefficients d^0, $d^d_i, \ldots, d^d_{na}, d^i$ can be obtained from the partial-fractions expansion of the transfer function $G_{i,j}$.

Combining the effects of all inputs, at sampling step k the predicted trajectory of output y_i can be represented by the continuous time function

$$[y_i(t)]_k = [x^s_i]_k + \sum_{j=1}^{nu} \sum_{l=1}^{na} [x^d_{i,j,l}]_k \mathrm{e}^{r_{i,j,l}t} + [x^i_i]_k t, \quad i = 1, \ldots, ny \tag{14}$$

For this system, the states can be defined as follows

$$[x^s_i]_{k+1} = [x^s_i]_k + \Delta t [x^i_i]_k + \sum_{j=1}^{nu} d^0_{ij}[\Delta u_j]_k + \Delta t \sum_{j=1}^{nu} d^i_{ij}[\Delta u_j]_k, \quad i = 1, \ldots, ny \tag{15}$$

$$[x^d_{ijl}]_{k+1} = \mathrm{e}^{r_{i,j,l}\Delta t}[x^d_{ijl}]_k + d^d_{ijl}\mathrm{e}^{r_{i,j,l}\Delta t}[\Delta u_j]_k \tag{16}$$

$i = 1, \ldots, ny$; $j = 1, \ldots, nu$; $l = 1, \ldots, na$

$$[x^i_i]_{k+1} = [x^i_i]_k + \sum_{j=1}^{nu} d^i_{ij}[\Delta u_j]_k, \quad i = 1, \ldots, ny \tag{17}$$

Or, in vector notation

$$[x^s]_{k+1} = [x^s]_k + \Delta t[x^i]_k + (D^0 + \Delta t D^i)\Delta u_k \tag{18}$$

$$[x^d]_{k+1} = F[x^d]_k + D^d F N \Delta u_k \tag{19}$$

$$[x^i]_{k+1} = [x^i]_k + D^i \Delta u_k \tag{20}$$

where

$$[x^s] = [x^s_1 \quad x^s_2 \quad \cdots \quad x^s_{ny}]^T, \quad [x^i] = [x^i_1 \quad x^i_2 \quad \cdots \quad x^i_{ny}]^T$$

$$[y_i]_k = [y_i(0)]_k = [x^s_i]_k + \sum_{j=1}^{nu} \sum_{l=1}^{na} [x^d_{i,j,l}]_k, \quad i = 1, \ldots, ny \tag{21}$$

Finally, Eqs. (18)–(21) can be written in the state-space form

$$[x]_{k+1} = A[x]_k + B\Delta u_k \tag{22}$$

$$[y]_k = C[x]_k \tag{23}$$

where

$$[x^d] = [x^d_{1,1,1} \quad \cdots \quad x^d_{1,1,na} \quad \cdots \quad x^d_{1,nu,1} \quad \cdots \quad x^d_{1,nu,na} \quad \cdots \quad x^d_{ny,1,1} \quad \cdots \quad x^d_{ny,1,na} \quad \cdots \quad x^d_{ny,nu,1} \quad \cdots \quad x^d_{ny,nu,na}]^T$$

$$D^0 = \begin{bmatrix} d^0_{1,1} & \cdots & d^0_{1,nu} \\ \vdots & \ddots & \vdots \\ d^0_{ny,1} & \cdots & d^0_{ny,nu} \end{bmatrix} \in \mathbb{R}^{ny \times nu};$$

$$D^i = \begin{bmatrix} d^i_{1,1} & \cdots & d^i_{1,nu} \\ \vdots & \ddots & \vdots \\ d^i_{ny,1} & \cdots & d^i_{ny,nu} \end{bmatrix} \in \mathbb{R}^{ny \times nu}$$

$$F = \mathrm{diag}(e^{r_{1,1,1}\Delta t} \cdots e^{r_{1,1,na}\Delta t} \cdots e^{r_{1,nu,1}\Delta t} \cdots e^{r_{1,nu,na}\Delta t} \cdots e^{r_{ny,1,1}\Delta t} \cdots e^{r_{ny,1,na}\Delta t} \cdots e^{r_{ny,nu,1}\Delta t} \cdots e^{r_{ny,nu,na}\Delta t}) \in \mathbb{C}^{nd \times nd}$$

$$D^d = \mathrm{diag}(d^d_{1,1,1} \quad \cdots \quad d^d_{1,1,na} \quad \cdots \quad d^d_{1,nu,1} \quad \cdots \quad d^d_{1,nu,na} \quad \cdots \quad d^d_{ny,1,1} \quad \cdots \quad d^d_{ny,1,na} \quad \cdots \quad d^d_{ny,nu,1} \quad \cdots \quad d^d_{ny,nu,na}) \in \mathbb{C}^{nd \times nd}$$

$$N = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_{ny} \end{bmatrix} \in \mathbb{R}^{nd \times nu};$$

$$J_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{nu.na \times nu}, \quad nd = ny.nu.na$$

Eq. (14) can be used to find an expression for the system output at sampling step k, i.e.

$$[y]_k = \begin{bmatrix} y_1 \\ \vdots \\ y_{ny} \end{bmatrix}_k, \quad [x]_k = \begin{bmatrix} x^s \\ x^d \\ x^i \end{bmatrix} \in \mathbb{C}^{nx},$$

$$A = \begin{bmatrix} I_{ny} & 0 & \Delta t I_{ny} \\ 0 & F & 0 \\ 0 & 0 & I_{ny} \end{bmatrix} \in \mathbb{C}^{nx \times nx}, \tag{24}$$

$$B = \begin{bmatrix} D^0 + \Delta t D^i \\ D^d F N \\ D^i \end{bmatrix} \in \mathbb{R}^{nx \times nu}$$

$$C = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}}^{ny} & \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{matrix}}^{nu.na} & \cdots & \overbrace{\begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{matrix}}^{nu.na} & \overbrace{\begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{matrix}}^{ny} \end{bmatrix},$$

$$nx = 2ny + nd$$

The output trajectory function (Eq. (14)) can also be written in the vector notation

$$[y(t)]_k = C(t)[x]_k \tag{25}$$

where

$$C(t) = [I \quad \Psi(t) \quad I^*(t)] \in \mathbb{C}^{ny \times nx} \tag{26}$$

$$\Psi(t) = \begin{bmatrix} \Phi_1(t) & 0 & \cdots & 0 \\ 0 & \Phi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_{ny}(t) \end{bmatrix} \in \mathbb{C}^{ny \times nd} \tag{27}$$

$$\Phi_i(t) = [e^{r_{i,1,1}t} \quad \cdots \quad e^{r_{i,1,na}t} \quad \cdots \quad e^{r_{i,nu,1}t} \quad \cdots \quad e^{r_{i,nu,na}t}] \in \mathbb{C}^{nd}, \tag{28}$$

$i = 1, 2 \ldots ny$

$$I^*(t) = \begin{bmatrix} f_1^*(t) & 0 & \cdots & 0 \\ 0 & f_2^*(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{ny}^*(t) \end{bmatrix} \in \mathbb{R}^{ny \times ny} \tag{29}$$

and where $f_i^*(t) = t$ if there is an integrating pole associated to output i, otherwise $f_i^*(t) = 0$.

Remark 1. In the state-space model represented in Eq. (22), the input appears in the incremental form. This model formulation is particularly suitable when, due to unmeasured disturbances or model uncertainties, the steady state of the system input is not known. In this case, the linear model represented by (22) and (23) may be a local approximation of a nonlinear model. If the true nonlinear model is not completely known, the exact value of the input steady state, which corresponds to a given set point for the output, can not be computed.

In the multivariable system with distinct poles, the state matrix A has the system poles in its main diagonal, where the first ny elements correspond to the integrating poles introduced into the system by the incremental form of the model. The next $ny\, nu\, na$ elements of the diagonal of A correspond to the non-integrating poles of the system. Some of the diagonal elements will be equal to zero if one of the outputs is pure integrating with respect to one of the inputs. The last ny elements of the diagonal of A correspond to the integrating poles contained in the system. If an output is not integrating with respect to all the system inputs, the element of A corresponding to the integrating pole of this output will be equal to zero.

Remark 2. The multivariable continuous system step response can be represented by

$$S(t) = D^0 + \Psi(t)D^d N + I^*(t)D^i \tag{30}$$

3. MPC with infinite prediction horizon

The infinite-horizon MPC is characterized by the objective function

$$J_{k,\infty} = \sum_{n=1}^{\infty} [e(n\Delta t)]_k^T Q [e(n\Delta t)]_k + \sum_{j=0}^{m-1} \Delta u_{k+j}^T R \Delta u_{k+j} \tag{31}$$

The output error defined as $e(n\Delta t) = r - y(n\Delta t)$ is the predicted output error at sampling step n including the effect of the future control actions, and r is the output reference. In the output tracking case, the reference is supposed to be set, for example, by a plant optimizer, which lies in an upper layer of the control structure. In Eq. (31) $Q \in \mathbb{R}^{ny \times ny}$ is assumed positive definite, $R \in \mathbb{R}^{nu \times nu}$ is assumed positive semi definite, and m is the control horizon. The infinite horizon controller minimizes the objective in Eq. (31) subject to constraints in the inputs, outputs, and control moves. Rawlings and Muske (1993) showed that for the undisturbed regulator case, IHMPC would stabilize the ideal system independent of the controller tuning parameters. Muske and Rawlings (1993) showed how to extend the application of such controller to the problems of output tracking and disturbed system. To apply the IHMPC to a system that possesses stable and integrating poles, it is necessary to zero the integrating states at the end of the control horizon. This is necessary to ensure that $J_{k,\infty}$ will be bounded and that it can be used as a Lyapunov function for the controlled system. With the state-space model described in the previous section, the condition that the integrating states should be equal to zero at $k+m$ corresponds to

$$r - [x^s]_{k+m} = 0 \tag{32}$$

$$[x^i]_{k+m} = 0 \tag{33}$$

Observe that, even for the non-integrating system, Eq. (32) has to be included in the IHMPC optimization problem. This corresponds to zeroing the effect of the integrating modes created by the incremental form of the model. This model form is necessary to apply the controller to the output-tracking problem and to the disturbed system. From the model equations presented in the previous section, it is easy to show that Eq. (32) corresponds to

$$r - [x^s]_k - m\Delta t[x^i]_k - \tilde{D}\Delta u = 0 \tag{34}$$

where

$$\tilde{D} = [[D^0 + m\Delta t D^i] \quad [D^0 + (m-1)\Delta t D^i] \quad \cdots \quad [D^0 + \Delta t D^i]]$$

$$\Delta u = [\Delta u_k^T \quad \Delta u_{k+1}^T \quad \cdots \quad \Delta u_{k+m-1}^T]^T$$

Analogously, Eq. (33) can be written in the form

$$[x^i]_k + D_{1,m}^i \Delta u = 0 \tag{35}$$

where

$$D_{1,m}^i = [\overbrace{D^i \quad D^i \quad \cdots \quad D^i}^{m}]$$

Taking into account Eqs. (32) and (33), the infinite horizon objective becomes

$$J_{k,\infty} = \sum_{n=1}^{m-1} [e(n\Delta t)]_k^T Q [e(n\Delta t)]_k + [x^d]_{k+m}^T \tilde{Q} [x^d]_{k+m} + \sum_{j=0}^{m-1} \Delta u_{k+j}^T R \Delta u_{k+j}$$

where $\tilde{Q}$ is obtained from the solution of the following discrete Lyapunov equation

$$\tilde{Q} = \Psi(0)^T Q \Psi(0) + F^T \tilde{Q} F \tag{36}$$

The inclusion of the constraints corresponding to Eqs. (34) and (35) in the controller optimization problem may give rise to infeasibility. In some cases, this problem can be remedied with a large control horizon, which may be a computationally inefficient choice. Moreover, the system inputs may be required to assume values outside their operating range. This may happen when severe disturbances affect the system. In this case, the control problem will be infeasible independently of the control horizon. Besides this drawback, the conventional infinite horizon approach requires the solution of Eq. (36), which in some applications needs to be done on-line. This will happen, for instance, in the control strategies where the output is switched to the controlled variable status only when its predictions lie outside the corresponding operating range. The output status is selected by inserting ones or zeros in the corresponding position in the weighting matrix Q.

One could argue that the infeasibility problem related to constraints represented by Eqs. (34) and (35) could be alleviated by softening these constraints with the introduction of slack variables and minimizing the slacks in the objective function. However, it can be demonstrated that this approach would not preserve nominal stability, and hence the development of a more elaborated strategy is justified.

With the adopted model representation, Eq. (25), can used to describe the system output $y(t)$ as a continuous function of time. Such property can be explored to develop a new version of an infinite-horizon controller with the objective function

$$\tilde{J}_{k,\infty} = \sum_{n=1}^{m} \int_{(n-1)\Delta t}^{n\Delta t} [e(t)]_k^T Q [e(t)]_k dt + \int_{m\Delta t}^{\infty} [e(t)]_k^T Q [e(t)]_k dt + \sum_{j=0}^{m-1} \Delta u_{k+j}^T R \Delta u_{k+j} \tag{37}$$

where Q and R are the same weighting matrices used in the objective defined in Eq. (31). The error $e(t)$ is also a continuous function of time that can be obtained using the state-space model developed in the previous section. To develop the first term of the right-hand side of Eq. (37) we use Eq. (25) for the output prediction without future control actions, and Eq. (30) to include the effect of future control actions as follows:

$$[e(t)] = [e^s]_k - \Psi(t)[x^d]_k - I^*(t)[x^i]_k - (D_n^0 + \Psi(t) W_n Z + [t D_{1n}^i - D_{2n}^i]) \Delta u \tag{38}$$

where

$$[e^s]_k = r - [x^s]_k$$

$$(n-1)\Delta t \le t < n\Delta t, \quad 1 \le n \le m$$

$$D_n^0 = [\overbrace{D^0 \; D^0 \; \cdots \; D^0}^{n} \; 0 \; \cdots \; 0] \in \mathbb{R}^{ny \times m\ nu} \tag{39}$$

$$D_{1n}^i = [\overbrace{D^i \; D^i \; \cdots \; D^i}^{n} \; 0 \; \cdots \; 0] \in \mathbb{R}^{ny \times m\ nu} \tag{40}$$

$$D_{2n}^i = [0 \quad \Delta t D^i \quad \cdots \quad (n-1)\Delta t D^i \quad 0 \quad \cdots \quad 0] \in \mathbb{R}^{ny \times m\ nu} \tag{41}$$

$$W_n = [I \quad F^{-1} \quad \cdots \quad F^{-(n-1)} \quad 0 \quad \cdots \quad 0] \in \mathbb{C}^{nd \times nd\ m} \tag{42}$$

$$Z = \mathrm{diag}(D^d N, D^d N, \cdots, D^d N)$$

Consequently, the first term of Eq. (37) becomes

$$\int_{(n-1)\Delta t}^{n\Delta t} [e(t)]_k^T Q [e(t)]_k =$$

$$\begin{aligned}
&2\{-[e^s]_k^T Q [G_1(n) - G_1(n-1)] + [x^d]_k^T [G_2(n) - G_2(n-1)] \\
&\quad + [x^i]_k^T Q [G_3(n) - G_3(n-1)]\} W_n Z \Delta u \\
&+2\{-\Delta t [e^s]_k^T + [x^d]_k^T [G_1(n) - G_1(n-1)]^T \\
&\quad + (n - 1/2)\Delta t^2 [x^i]_k^T\} Q (D_n^0 - D_{2n}^i) \Delta u \\
&+2\{-(n-1/2)\Delta t^2 [e^s]_k^T + (n^3 - n + 1/3)\Delta t^3 [x^i]_k^T + [x^d]_k^T \\
&\quad \times [G_3(n) - G_3(n-1)]^T\} Q D_{1n}^i \Delta u \\
&+\Delta u^T \{Z^T W_n^T [G_2(n) - G_2(n-1)] W_n Z \\
&+2 Z^T W_n^T [[G_1(n) - G_1(n-1)]^T Q (D_n^0 - D_{2n}^i) \\
&\quad + [G_1(n) - G_1(n-1)]^T Q D_{1n}^i \\
&\quad + [G_3(n) - G_3(n-1)]^T Q D_{1n}^i] \\
&+\Delta t (D_n^0 - D_{2n}^i)^T Q (D_n^0 - D_{2n}^i) \\
&\quad + 2(n-1/2)\Delta t^2 (D_n^0 - D_{2n}^i)^T Q D_{1n}^i \\
&\quad + (n^3 - n + 1/3)\Delta t^3 (D_{1n}^i)^T Q D_{1n}^i\} \Delta u
\end{aligned} \tag{43}$$

where

$$G_2(n) = \int_0^{n\Delta t} \Psi(t)^T Q \Psi(t) dt, \quad G_2(n) = \int_0^{n\Delta t} \Psi(t)^T Q \Psi(t) dt,$$

$$\text{and } G_2(n) = \int_0^{n\Delta t} \Psi(t)^T Q \Psi(t) dt$$

The evaluation of these integrals is rather trivial since they involve only terms in e^{rt} and te^{rt}. Analytical expressions can be easily obtained and their calculation is not numerically expensive.

Analogously, we can develop the second term of the right-hand side of Eq. (37). For this purpose, we need a prediction function for the output error at time instants beyond the control horizon. Eq. (38) can be extended to this case, but to help in the proof that the objective defined in Eq. (37) is a decreasing function of time, we add and subtract the term $m\Delta t\{[x^i]_k + D^i_{1m}\}$ to the output prediction function. The resulting expression is

$$[e(t)]_k = \{[e^s]_k - m\Delta t[x^i]_k - (D^0_m + D^i_{3m})\Delta u\} - \Psi(t) \times \{[x^d]_k + W_m Z\Delta u\} - (t - m\Delta t) \times \{[x^i]_k + D^i_{1m}\Delta u\} \tag{44}$$

where

$t \geq m\Delta t$

$D^i_{3m} = m\Delta t D^i_{1m} - D^i_{2m}$

Substituting Eq. (44) into the second term of the right-hand side of Eq. (37) yields

$$\int_{m\Delta t}^{\infty} [e(t)]_k^T Q[e(t)]_k dt =$$

$$\{[e^s]_k - m\Delta t[x^i]_k - (D^0_m + D^i_{3m})\Delta u\}^T Q\{[e^s]_k - m\Delta t[x^i]_k - (D^0_m + D^i_{3m})\Delta u\}\int_{m\Delta t}^{\infty} dt$$

$$-2\{[e^s]_k - m\Delta t[x^i]_k - (D^0_m + D^i_{3m})\Delta u\}^T Q[\int_{m\Delta t}^{\infty} \Psi^T(t)dt] \times \{[x^d]_k + W_m Z\Delta u\}$$

$$-2\{[e^s]_k - m\Delta t[x^i]_k - (D^0_m + D^i_{3m})\Delta u\}^T Q\{[x^i]_k + D^i_{1m}\Delta u\}\int_{m\Delta t}^{\infty} (t - m\Delta t)dt$$

$$+2\{[x^d]_k + W_m Z\Delta u\}^T \times [\int_{m\Delta t}^{\infty} \Psi^T(t)(t - m\Delta t)dt]Q\{[x^i]_k + D^i_{1m}\Delta u\}$$

$$+ \{[x^d]_k + W_m Z\Delta u\}^T[\int_{m\Delta t}^{\infty} \Psi^T(t)\ Q\ \Psi(t)dt] \times \{[x^d]_k + W_m Z\Delta u\}$$

$$+ \{[x^i]_k + D^i_{1m}\Delta u\}^T Q\{[x^i]_k + D^i_{1m}\Delta u\}\int_{m\Delta t}^{\infty} (t - m\Delta t)^2 dt \tag{45}$$

To prove nominal stability of the closed-loop system with this controller, it is necessary to show that the objective function is bounded, or equivalently that the integrals in Eq. (45) are bounded. However, the first term on the right side of Eq. (45) will be bounded only if the following equations hold:

$$[e^s]_k - m\Delta t[x^i]_k - (D^0_m + D^i_{3m})\Delta u = 0 \tag{46}$$

$$[x^i]_k + D^i_{1m}\Delta u = 0 \tag{47}$$

It is easy to show that $\tilde{D} = D^0_m + D^i_{3m}$, and that Eqs. (46) and (47) are equivalent to Eqs. (34) and (35) obtained in the conventional approach of the infinite horizon controller.

Substituting Eqs. (46) and (47) into Eq. (45) gives

$$\int_{m\Delta t}^{\infty} [e(t)]_k^T Q[e(t)]_k dt = \{[x^d]_k + W_m Z\Delta u\}^T[G_2(\infty) - G_2(m\Delta t)] \times \{[x^d]_k + W_m Z\Delta u\} = 2[x^d]_k^T[G_2(\infty) - G_2(m\Delta t)]W_m Z\Delta u + \Delta u^T Z^T W_m^T[G_2(\infty) - G_2(m\Delta t)]W_m Z\Delta u \tag{48}$$

Eqs. (48) and (45) can now be substituted into Eq. (37), and the following optimization scheme summarizes the infinite-horizon MPC problem:

Problem (P1).

$$\min_{\Delta u}[\Delta u^T H\Delta u + 2c_f^T\Delta u] \tag{49}$$

subject to Eqs. (46) and (47)

$$\Delta u_{k+j} \in \mathbb{U};\quad \mathbb{U} = \left\{ \begin{array}{l} \Delta u_{k+j} | -\Delta u^{\max} \leq \Delta u_{k+j} \leq \Delta u^{\max} \quad \text{and} \quad u^{\min} \leq u_{k-1} + \sum_{i=0}^{j-1} \Delta u_{k+i} \leq u^{\max};\quad j = 1, \cdots, m \\ \Delta u_{k+j} = 0;\quad j \geq m \end{array} \right\} \tag{50}$$

where

$$c_f^T = c_{f_m}^T + c_{f_\infty}^T$$

$$H_m = \sum_{n=1}^{m}\{Z^T W_n^T[G_2(n) - G_2(n-1)]W_n Z + 2Z^T W_n^T[(G_1(n) - G_1(n-1))^T Q(D^0_n - D^i_{2n}) + (G_3(n) - G_3(n-1))^T Q D^i_{1n} + (G_1(n) - G_1(n-1))^T Q D^i_{1n}]$$

$$+\Delta t(D_n^0 - D_{2n}^i)^T Q(D_n^0 - D_{2n}^i) + 2(n-1/2)\Delta t^2 (D_n^0 - D_{2n}^i)^T Q D_{1n}^i + (n^3 - n + 1/3)\Delta t^3 (D_{1n}^i)^T Q D_{1n}^i\}$$

$$H_\infty = Z^T W_m^T [G_2(\infty) - G_2(m)] W_m Z$$

$$c_{f_m}^T = \sum_{n=1}^{m} \{[-[e^s]_k^T Q(G_1(n) - G_1(n-1)) + [x^d]_k^T \times (G_2(n) - G_2(n-1)) + [x^i]_k^T Q(G_3(n) - G_3(n-1))] W_n Z$$
$$+[-\Delta t[e^s]_k^T + (n-1/2)\Delta t^2 [x^i]_k^T + [x^d]_k^T \times (G_1(n) - G_1(n-1))^T] Q(D_n^0 - D_{2n}^i)$$
$$+[-(n-1/2)\Delta t^2 [e^s]_k^T + (n^3 - n + 1/3)\Delta t^3 [x^i]_k^T + [x^d]_k^T \times (G_3(n) - G_3(n-1))^T] Q D_{1n}^i\}$$

$$c_{f_\infty}^T = [x^d]_k^T [G_2(\infty) - G_2(m)] W_m Z$$

$$R_1 = \text{diag}(R \quad \cdots \quad R)$$

The stability of the controller that results from the solution of Problem P1 is ensured by the following theorem.

Theorem 1. For a system with distinct stable and integrating poles, if Problem P1 has a feasible solution at sampling step k, then the closed-loop with the infinite horizon controller will be stable.

Proof 1. We follow the usual route to prove the stability of infinite horizon controllers, i.e. we prove that the objective function $\tilde{J}_{k,\infty}$ defined in Eq. (37) is positive and decreasing. Note that $\tilde{J}_{k,\infty}$ reduces to the objective function of Problem P1 when the constraints represented by Eqs. (46) and (47) are satisfied, and it is easy to show that the Hessian H satisfies $H > 0$. Then, if Problem P1 is feasible, there is a unique optimal solution, which obeys the constraints of Eqs. (46), (47) and (50). Suppose that this solution is represented by $\Delta u^* = [(\Delta u_k^*)^T \quad (\Delta u_{k+1}^*)^T \quad \cdots \quad (\Delta u_{k+m-1}^*)^T]^T$. The value of the objective function corresponding to this optimal solution is designated $\tilde{J}_{k,\infty}^*$. Next, suppose that the control action Δu_k^* is implemented in the true plant, and we move to sampling step $k+1$ where Problem P1 is solved again. Following the idea of Rawlings and Muske (1993), consider the value of the cost function corresponding to the control sequence $\Delta u' = [(\Delta u_{k+1}^*)^T \quad \cdots \quad (\Delta u_{k+m-1}^*)^T \quad 0]^T$, and designate this value as $\tilde{J}'_{k+1,\infty}$. It is easy to show that

$$\tilde{J}'_{k+1,\infty} = \tilde{J}_{k,\infty}^* - \int_0^T [e(t)]_k^T Q[e(t)]_k dt - \Delta u_k^{T*} R \Delta u_k^*$$

$$\tilde{J}'_{k+1,\infty} < \tilde{J}_{k,\infty}^*$$

Consequently $\tilde{J}_{k+1,\infty}^* < \tilde{J}_{k,\infty}^*$. It is also straightforward to show that $\Delta u'$ satisfies the constraints represented in Eq. (50), but it remains to prove that $\Delta u'$ also satisfies the equality constraints represented by Eqs. (46) and (47). Consider initially Eq. (46). Since Δu^* is a solution of Problem P1 at sampling step k, it obviously satisfies the equation

$$[e^s]_k - m\Delta t[x^i]_k - (D_m^0 + D_{3m}^i)\Delta u^* = 0$$

Let θ be the value of this constraint at step $k+1$ for the control sequence $\Delta u'$, i.e.

$$\theta = [e^s]_{k+1} - m\Delta t[x^i]_{k+1} - D_m^0 \Delta u' - D_{3m}^i \Delta u'$$

Substituting D_m^0 and D_{3m}^i in terms of D^0 and D^i, and using the state-space model equations, the following expression is obtained:

$$\theta = [e^s]_k - \Delta t[x^i]_k - D^0 \Delta u_k^* - \Delta t D^i \Delta u_k^* - m\Delta t[x^i]_k - m\Delta t D^i \Delta u_k^* - D^0 \Delta u_{k+1}^* - D^0 \Delta u_{k+2}^* - \cdots$$
$$-D^0 \Delta u_{k+m-1}^* - m\Delta t D^i \Delta u_{k+1}^* - (m-1)\Delta t D^i \Delta u_{k+2}^* - (m-2)\Delta t D^i \Delta u_{k+3}^* - \cdots - 2\Delta t D^i \Delta u_{k+m-1}^*$$

$$\theta = [e^s]_k - m\Delta t[x^i]_k - D^0 \Delta u_k^* - D^0 \Delta u_{k+1}^* - D^0 \Delta u_{k+1}^* - \cdots - D^0 \Delta u_{k+m-1}^* - \Delta t[x^i]_k - \Delta t D^i \Delta u_k^*$$
$$-m\Delta t D^i \Delta u_k^* - m\Delta t D^i \Delta u_{k+1}^* - (m-1)\Delta t D^i \Delta u_{k+1}^* - (m-2)\Delta t D^i \Delta u_{k+3}^* - \cdots - 2\Delta t D^i \Delta u_{k+m-1}^*$$

Adding and subtracting the term $(\Delta t D^i \Delta u_{k+1}^* + \cdots + \Delta t D^i \Delta u_{k+m-1}^*)$ to the right-hand side of the equation above, we obtain

$$\theta = [e^s]_k - m\Delta t[x^i]_k - D_m^0 \Delta u^* - m\Delta t D^i \Delta u_k^* - (m-1)\Delta t D^i \Delta u_{k+1}^* - (m-2)\Delta t D^i \Delta u_{k+2}^* - \cdots - \Delta t D^i \Delta u_{k+m-1}^*$$
$$-\Delta t[x^i]_k - \Delta t D^i \Delta u_k^* - \Delta t D^i \Delta u_{k+1}^* - \cdots - \Delta t D^i \Delta u_{k+m-1}^*$$

$$\theta = [e^s]_k - m\Delta t[x^i]_k - D_m^0 \Delta u^* - D_{3m}^i \Delta u^* - \Delta t[x^i]_k - \Delta t D_{1m}^i \Delta u^*$$

$$\theta = \underbrace{[e^s]_k - m\Delta t[x^i]_k - (D_m^0 + D_{3m}^i)\Delta u^*}_{=0} - \Delta t \times \underbrace{\{[x^i]_k + D_{1m}^i \Delta u^*\}}_{=0}$$

Hence $\theta = 0$, and therefore $\Delta u'$ satisfies Eq. (46) at sampling step $k+1$. To show that $\Delta u'$ satisfies Eq. (47) at $k+1$, let β be the value of the left side of Eq. (47) at sampling step $k+1$ for the control sequence $\Delta u'$, i.e.

$$\beta = [x^i]_{k+1} + D^i_{1m}\Delta u'$$

or equivalently,

$$\beta = [x^i]_k + D^i \Delta u^*_k + D^i_{1m}\begin{bmatrix}\Delta u^*_{k+1}\\ \vdots \\ \Delta u^*_{k+m-1}\\ 0\end{bmatrix}$$

hence

$$\beta = [x^i]_k + D^i_{1m}\Delta u^* = 0$$

which completes the proof of the theorem □.

The infinite horizon MPC defined by Problem 1 does not need the solution of Eq. (36), which is required in the standard formulation of the infinite horizon controller. However, the constraints that appear in Problem 1 are the same as of the conventional version of the infinite horizon controller. The objective function considered here, although somewhat more involved than the conventional objective function, results in an optimization problem that is simpler than the conventional problem considered in the literature. It should be observed that no state constraints are included in the IHMPC formulation presented here. This is so, because in the model form considered in this work, the states represent the coefficients of the predicted output function and there is no clear scope in constraining those coefficients.

4. Overcoming some infeasibilities of IHMPC

In the previous section, it was shown that a new version of IHMPC could be generated by defining an objective function that considers the integral of the error, instead of the sum of the errors at the sampling instants. Although the Lyapunov equation was removed from the controller formulation, the equality constraints, which are the main cause of infeasibility, remained in the proposed controller. To overcome this drawback, we present a method to soften the equality constraints corresponding to Eq. (46). The controller objective function is modified as follows:

$$\bar{J}_{k,\infty} = \sum_{n=1}^{m}\int_{(n-1)\Delta t}^{n\Delta t}\{[e(t)]_k + \delta_k\}^T Q\{[e(t)]_k + v\}dt + \int_{m\Delta t}^{\infty}\{[e(t)]_k + \delta_k\}^T Q\{[e(t)]_k + \delta_k\}dt + \sum_{j=0}^{m-1}\Delta u^T_{k+j}R\Delta u_{k+j} + [\delta_k]^T S\delta_k \quad (51)$$

where $\delta_k \in \mathbb{R}^{ny}$ is a vector of slack variables, and $S \in \mathbb{R}^{ny\times ny}$ is a positive-definite weighting matrix. Using this objective and following the same steps of the previous section, the control problem reduces to solving the following optimization problem:

Problem (P2).

$$\min_{\Delta u,\delta_k}[\Delta u^T \quad (\delta_k)^T]\begin{bmatrix}H_{11} & H_{21}^T\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\Delta u\\ \delta_k\end{bmatrix} + 2[c^T_{f_{11}} \quad c^T_{f_{12}}] \times \begin{bmatrix}\Delta u\\ \delta_k\end{bmatrix} \quad (52)$$

subject to

$$[e^s]_k + \delta_k - m\Delta t[x^i]_k - (D^0_m + D^i_{3m})\Delta u = 0 \quad (53)$$

$$[x^i]_k + D^i_{1m}\Delta u = 0$$

$$\Delta u_{k+j} \in \mathbb{U}$$

where

$$H_{11} = \sum_{n=1}^{m}\{Z^T W_n^T[G_2(n) - G_2(n-1)]W_n Z$$
$$+2Z^T W_n^T[(G_1(n) - G_1(n-1))^T Q(D^0_n - D^i_{2n}) + (G_3(n) - G_3(n-1))^T QD^i_{1n} + (G_1(n) - G_1(n-1))^T QD^i_{1n}]$$
$$+\Delta t(D^0_n - D^i_{2n})^T Q(D^0_n - D^i_{2n}) + 2(n-1/2)\Delta t^2(D^0_n - D^i_{2n})^T QD^i_{1n}$$
$$+(n^3 - n - 1/3)\Delta t^3(D^i_{1n})^T QD^i_{1n}\} + Z^T W_m^T[G_2(\infty) - G_2(m)]W_m Z + R_1$$

$$H_{21} = \sum_{n=1}^{m}\{-Q[G_1(n) - G_1(n-1)]W_n Z - \Delta tQ(D^0_n - D^i_{2n}) \text{ - } (n-1/2)\Delta t^2 QD^i_{1n}\}$$

$$H_{22} = \sum_{n=1}^{m}\{\Delta tQ\} + S$$

$$c^T_{f11} = \sum_{n=1}^{m}\{[-[e^s]^T_k Q(G_1(n) - G_1(n-1)) + [x^d]^T_k \times (G_2(n) - G_2(n-1)) + [x^i]^T_k Q(G_3(n) - G_3(n-1))]W_n Z$$
$$+[-\Delta t[e^s]^T_k + (n-1/2)\Delta t^2[x^i]^T_k + [x^d]^T_k \times (G_1(n) - G_1(n-1))^T]Q(D^0_n - D^i_{2n})$$
$$+[-(n-1/2)\Delta t^2[e^s]^T_k + (n^3 - n - 1/3)\Delta t^3[x^i]^T_k + [x^d]^T_k \times (G_3(n) - G_3(n-1))^T]QD^i_{1n}\}$$
$$+[x^d]^T_k[G_2(\infty) - G_2(m)]W_m Z$$

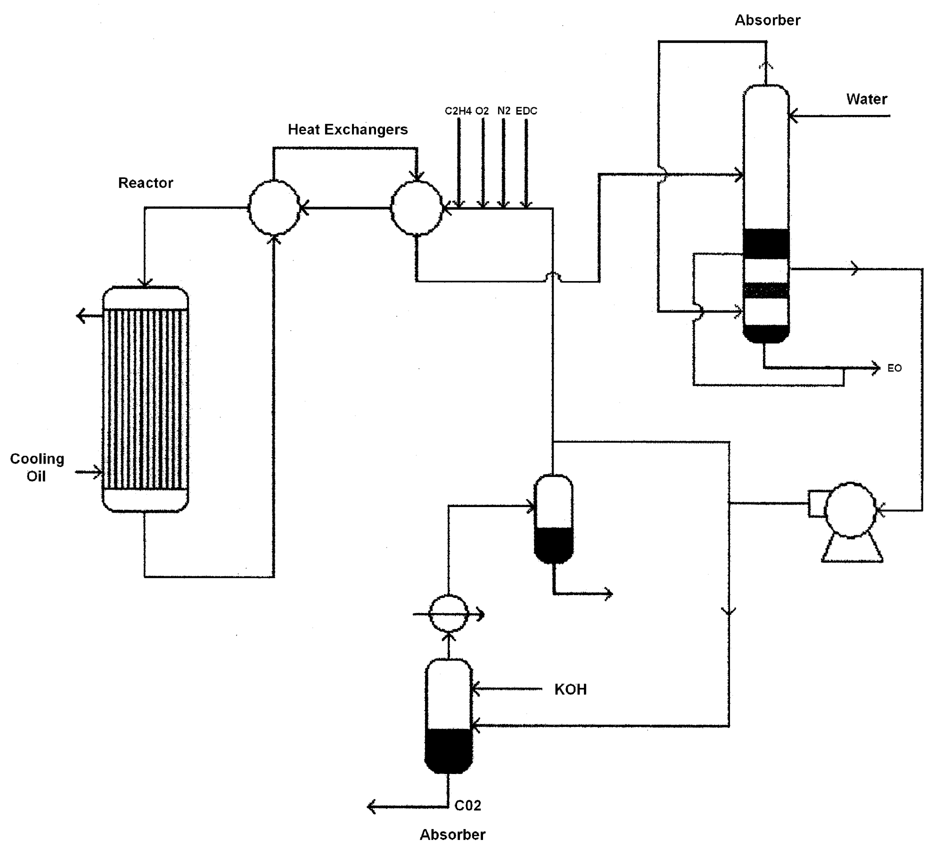

Fig. 1. Schematic representation of an industrial ethylene-oxide plant.

$$c_{f12}^T = \sum_{n=1}^{m} \{[\Delta t[e^s]_k^T Q - (n-1/2)\Delta t^2[x^i]_k^T Q - [x^d]_k^T \times (G_1(n) - G_1(n-1))^T Q]\}$$

The stability of the control law generated by the solution of Problem P2 is guaranteed by the following theorem.

Theorem 2. For a system with stable and integrating poles if Problem P2 has a feasible solution at sampling step k the following condition is satisfied

$$R < [m\Delta t D_{1m}^i - D_m^0 - D_{3m}^i]^T S[m\Delta t D_{1m}^i - D_m^0 - D_{3m}^i] \quad (54)$$

then, the closed-loop with the infinite horizon controller will be asymptotically stable.

Proof 2. The proof follows the same steps of Theorem 1. Consider the control sequence $\Delta u'$ at sampling step $k+1$ and assume that $\delta_{k+1} = \delta_k$. The corresponding value of the objective function is designated $\bar{J}'_{k+1,\infty}$, and it can be calculated as follows:

$$\bar{J}'_{k+1,\infty} = \bar{J}^*_{k,\infty} - \int_0^{\Delta t} \{[e(t)]_k + \delta_k\}^T Q \{[e(t)]_k + \delta_k\} dt - \Delta u_k^{T*} R \Delta u_k^*$$

Observe that the second term of the right-hand side of the above equation can be made equal to zero if

$$[e^s]_k = \delta_k, \quad [x^d]_k = 0 \quad \text{and} \quad [x^i]_k = 0$$

In this case, if $\delta_k \neq 0$ with $\Delta u_k^* = 0$, the objective function would be non-increasing but $e(t)$ would different from zero. However, the condition defined in Eq. (54) assures that this situation will not be optimal, and the states and the slack variables tend to zero when k tends to infinite. Consequently, $\bar{J}'_{k+1,\infty} < \bar{J}^*_{k,\infty}$, and $\bar{J}^*_{k+1,\infty} < \bar{J}^*_{k,\infty}$.

The proof that $(\Delta u', \ \delta_k)$ satisfies Eqs. (53) and (47) at sampling step $k+1$ is similar to the proof that $\Delta u'$ satisfies Eqs. (46) and (47) in Theorem 1. This completes the proof □.

5. Example: the ethylene-oxide industrial system

Fig. 1 shows the schematic representation of an industrial system where ethylene-oxide (EO) is produced by the exothermic reaction

$$C_2H_4 + \frac{1}{2}O_2 \overset{Ag_2O}{\rightarrow} C_2H_4O$$

The complete combustion of ethylene occurs through the undesirable secondary reaction

$$C_2H_4 + 3O_2 \rightarrow 2CO_2 + 2H_2O$$

The gaseous mixture that leaves the reactor is sent to an absorber where ethylene-oxide is absorbed from the gas by means of water and leaves the system through the bottom stream. The remaining gas is compressed and directed to a second absorber where carbon dioxide is absorbed by means of an aqueous solution of potassium hydroxide (KOH). The residual gas is recycled to the reactor inlet where it is mixed with the feed stream. The integrating character of the system can be observed when one of the feed components, oxygen or ethylene, is introduced in excess into the system. The concentration of this component in the recycle gas tends to increase continuously since the excess amount of the component cannot leave the system in the product streams.

A stable MPC controller is sought to manipulate the following inputs ($nu = 6$):

u_1—the pure oxygen feed flow rate
u_2—the ethylene feed flow rate
u_3—the flow rate of nitrogen, which is introduced into the system and acts as ballast to moderate the reaction and helps to remove the reaction heat
u_4—the temperature of cooling oil used in the reactor
u_5—the ethylene dichloride (EDC) flow rate (valve opening), a combustion inhibitor, introduced into the system to improve the reaction selectivity
u_6—the flow rate of potassium hydroxide, which is introduced in the CO_2 absorber

The MPC controller needs to control the following outputs ($ny = 5$):
y_1—the molar fraction of oxygen in the gas stream at the entrance of the reactor
y_2—the temperature of reaction products in the reactor
y_3—the reaction selectivity, which is defined as the percentage of the reacted ethylene that is converted to ethylene-oxide.
y_4—the system pressure
y_5—the molar fraction of ethylene in the gas at the entrance of the reactor

Table 1
Controller bounds for the inputs

Input	u^{max}	u^{min}	Δu^{max}	Unit
u_1	0.10	−0.10	0.02	ton/h
u_2	0.15	−0.20	0.02	ton/h
u_3	0.05	−0.05	0.02	ton/h
u_4	1.00	−2.00	0.50	°C
u_5	60.0	−5.00	1.00	%
u_6	15.0	−10.0	1.00	ton/h

All the system inputs and outputs are available, with low measurement noise at a sampling rate of 1 min^{-1}. Plant tests were performed and the system model was identified. The transfer function model

$$G(s) = \begin{bmatrix} \frac{6.43\times10^{-2}}{s} & \frac{-3.67\times10^{-2}}{s} & \frac{-2.50\times10^{-2}}{s} & \frac{-9.80\times10^{-3}}{s} & \frac{2.61\times10^{-4}}{s} & \frac{-8.34\times10^{-5}}{s} \\ \frac{4.3}{(15.1s+1)} & \frac{-2.1}{(19.5s+1)} & \frac{-7.3}{(199.5s+1)} & \frac{1.2}{(7.8s+1)} & \frac{-2.37\times10^{-2}}{(49.5s+1)} & \frac{2.97\times10^{-2}}{(55.1s+1)} \\ \frac{-1.7}{(19.5s+1)} & \frac{-0.767}{(31.8s+1)} & \frac{-2.5}{(17.7s+1)} & \frac{-6.0}{(199.5s+1)} & \frac{4.45\times10^{-2}}{(49.5s+1)} & \frac{2.40\times10^{-3}}{(10.9s+1)} \\ \frac{-4.33\times10^{-2}}{s} & \frac{5.66\times10^{-2}}{s} & \frac{6.32\times10^{-2}}{s} & \frac{6.33\times10^{-2}}{(11.4s+1)} & \frac{5.36\times10^{-2}}{(142.4s+1)} & \frac{-1.59\times10^{-2}}{(36.5s+1)} \\ \frac{-0.190}{s} & \frac{0.235}{s} & \frac{-5.96}{(98.5s+1)} & \frac{-0.250}{(12.5s+1)} & \frac{0.140}{(55.1s+1)} & 0 \end{bmatrix} \quad (57)$$

can be considered as the ideal model corresponding at a given time instant in the operating cycle of the system. This model is expected to represent the behavior of the system for a period of a few months, inside the operation cycle of the ethylene oxide system. Practical experience shows that, as the catalyst deactivates, the temperature profile along the reactor bed is gradually increased and the linear model that represents the system locally tends to change slowly. Thus, a more

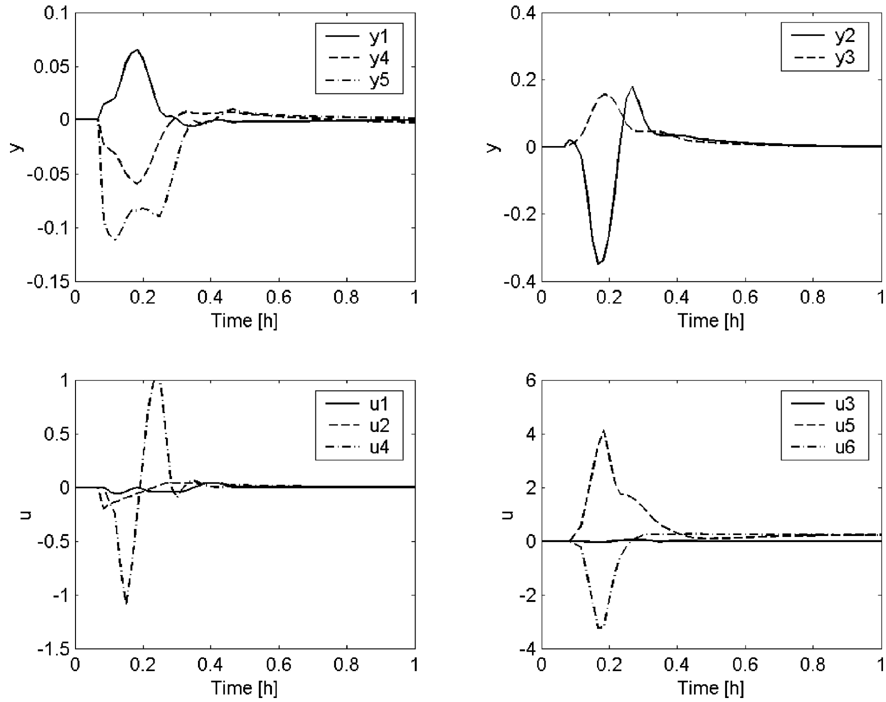

Fig. 2. Inputs and outputs responses for the proposed controller in case 1.

elaborated study, which is not the scope of this work, has to be performed concerning the robustness of the proposed control strategy for model uncertainties. It can be seen that the system has several integrating poles, and that fast and slow elements appear together. This model is used to simulate the system and to compare the performance of the proposed controller to a standard IHMPC. All variables are represented in deviation form, i.e. as differences between their true values and their initial steady states. From the controlled outputs, the oxygen molar fraction (y_1), is particularly important since it strongly affects the process productivity. This variable has also a constraint associated to the maximum oxygen concentration in the reactor inlet before

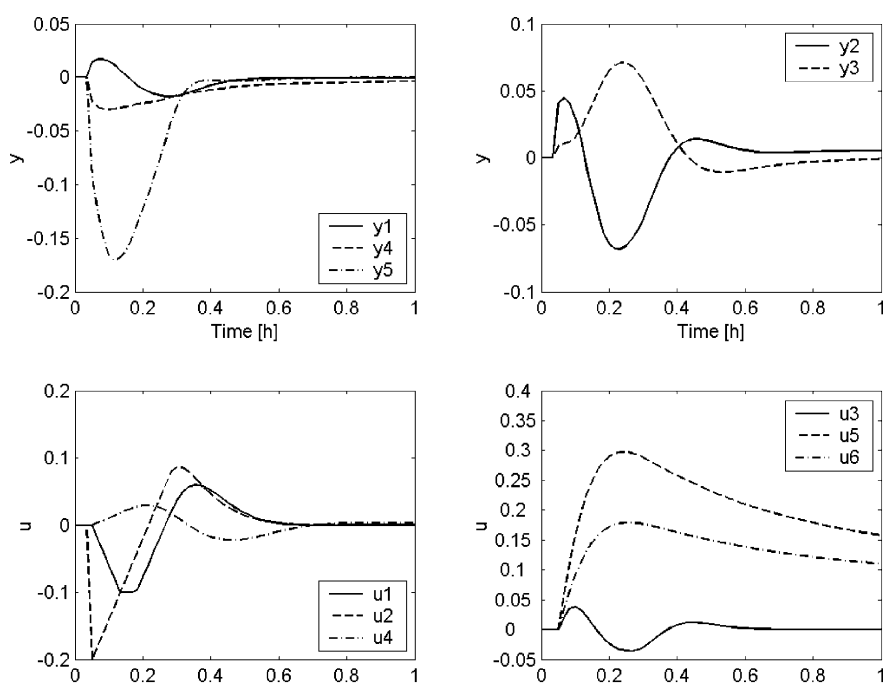

Fig. 3. Inputs and outputs responses for the standard IHMPC in case 1.

the reaction becomes explosive. Reaction selectivity (y_3) must be kept between close bounds since it directly affects the economic performance, as the secondary reaction is the complete combustion of ethylene. The control of the reactor temperature (y_2) is also critical since there is a trade off between conversion and catalyst deactivation.

Experienced plant operators provided the bounds to the system variables, which they deem adequate for the smooth and safe operation of the system. Furthermore, the operators want to be allowed to redefine, plant equipment permitting, new values for these bounds whenever they find this is necessary to accommodate new optimal operating conditions along the operating cycle of the system. Table 1 shows the suggested bounds for the inputs and input increments. These bounds were considered in all the cases simulated in this study. Moreover, a sampling period equal to 1 min ($\Delta t = 1$) was adopted.

The infinite-horizon MPC based on Problem P1 was compared to the controller that solves Problem P2. The controller that solves Problem P1 can be considered as a standard IHMPC. From the countless disturbances that can affect the system, we selected three cases to illustrate the differences between the two approaches. In each case, the best tuning parameters for each controller were considered.

Case 1 *(Disturbance in the ethylene feed supply).* This kind of disturbance is expected to happen quite often since the ethylene-oxide plant is connected to a neighbor ethylene plant with no intermediary tank. To simulate this event, we assume a step decrease in the ethylene feed flow rate. The operators identified the typical size of this step as 0.200 ton/h. The tuning parameters corresponding to the optimal performance of the proposed controller were the following:

$$Q = I_{ny}, \quad R = 0.01 I_{nu}, \quad m = 3$$
$$\text{and } S = 400\,\mathrm{diag}([1 \quad 1 \quad 1 \quad 1 \quad 0.20])$$

It can be verified that the adopted values of R and S satisfy Eq. (54). Fig. 2 shows the responses of the system outputs and inputs for the proposed MPC. A short control horizon was used ($m = 3$) to demonstrate that, for the disturbance considered, the proposed controller is feasible with any control horizon. If the inputs connected to an integrating output become saturated, the controller will become infeasible. Once the operator realizes that this is happening, he has to enlarge the ranges of the saturated inputs. The performance of the controller can be considered acceptable, and the ethylene concentration (y_5) and the reactor temperature (y_2) are the variables most affected by this disturbance.

The same disturbance was considered for the ethylene-oxide system controlled by the standard IHMPC with the following tuning parameters:

$$Q = I_{ny}, \quad R = 0.10 I_{nu}, \quad m = 10$$

The control horizon is the minimum necessary such that a feasible solution to Problem P1 exists. The

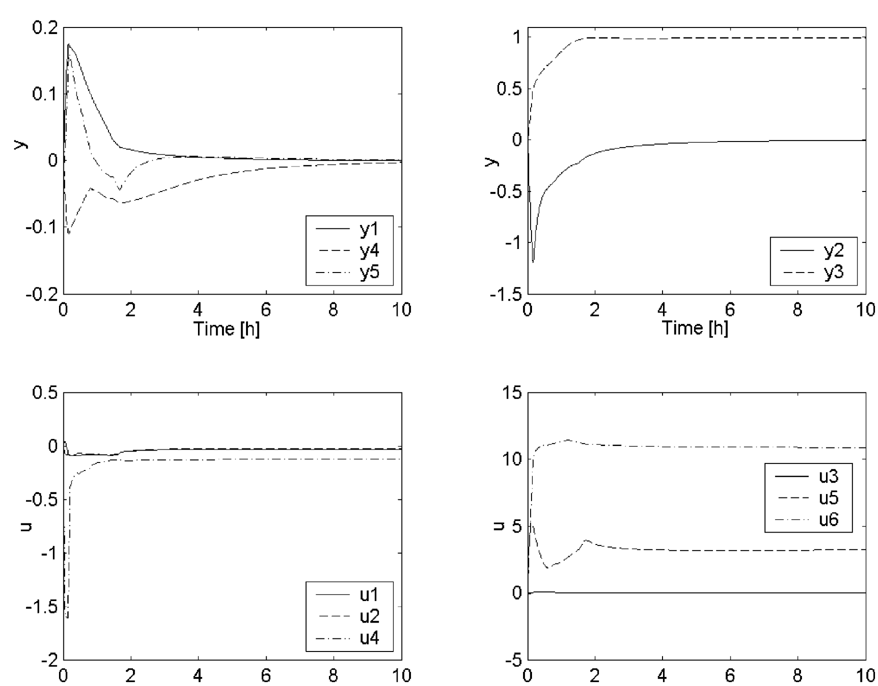

Fig. 4. Inputs and outputs responses for the proposed IHMPC in case 2.

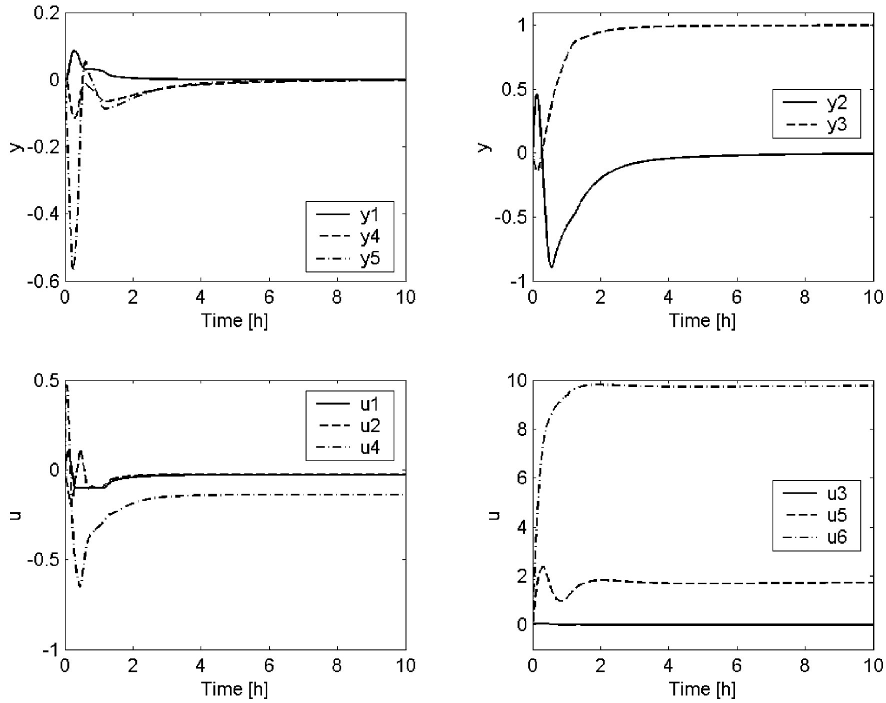

Fig. 5. Inputs and outputs responses for the standard IHMPC in case 2.

corresponding responses of the system are represented in Fig. 3. There is not a clear difference between the performance of the two controllers. For the reactor temperature (y_2), the conventional IHMPC seems to perform better, while for the ethylene molar fraction (y_5), the proposed controller gives a better response. Additional simulation results not shown here indicate that the new controller can be made to approximate the conventional IHMPC by increasing its control horizon to intermediary values.

Case 2 *(Step increase in the reference value of selectivity).* In this case, the controller is requested to increase the reaction selectivity (y_3) by a small

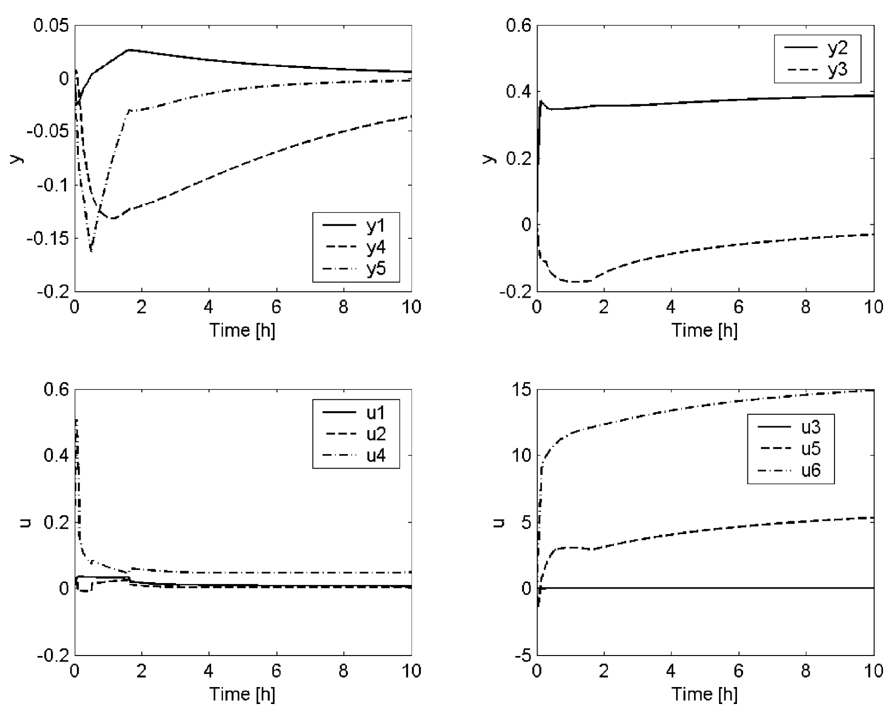

Fig. 6. Inputs and outputs responses for the proposed IHMPC in case 3

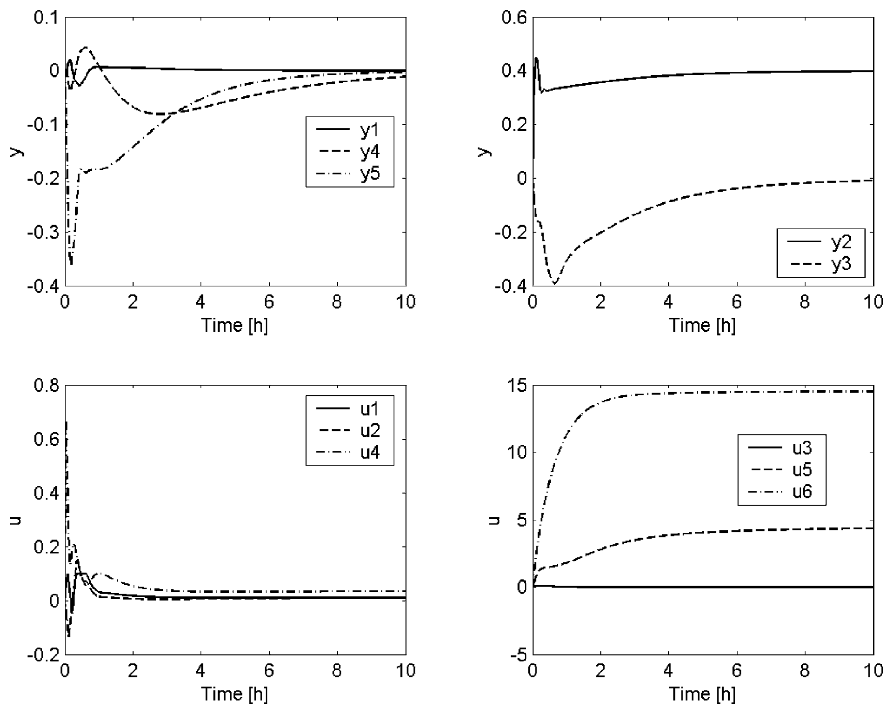

Fig. 7. Inputs and outputs responses for the standard IHMPC in case 3.

Table 2
Execution time for the IHMPC

	Time (s)		
	Case 1	Case 2	Case 3
IHMPC (proposed)	0.271	0.471	0.261
IHMPC (standard)	3.565	104.17	275.32

percentage. In normal operation, the increase in selectivity is always accompanied by the increase in the reactor temperature. However, the higher the reactor temperature the faster the catalyst deactivates, and the plant optimizer may seek to increase selectivity without increasing the temperature. The following parameters were selected for the proposed controller:

$$Q = \mathrm{diag}([1 \quad 1 \quad 2 \quad 1 \quad 1]), \quad R = 0.01 I_{nu}, \quad m = 4,$$
$$S = 400\mathrm{diag}([1 \quad 1 \quad 1 \quad 1 \quad 0.20]),$$
$$r = [0 \quad 0 \quad 1 \quad 0 \quad 0]$$

The responses of the closed-loop with the proposed controller are plotted in Fig. 4. The increase of 1% in the selectivity was also simulated with the standard IHMPC with the following set of parameters, which optimizes the controller performance

$$Q = \mathrm{diag}([1 \quad 1 \quad 2 \quad 1 \quad 1]), \quad R = 0.01 I_{nu}, \quad m = 26,$$
$$r = [0 \quad 0 \quad 1 \quad 0 \quad 0]$$

As in Case 1, the control horizon used in the standard controller is the minimum necessary to ensure a feasible solution to Problem P1. The responses of the system with this controller are shown in Fig. 5. With small input weights, the standard IHMPC seems to have a more aggressive response with larger overshoots in the controlled outputs. As in Case 1, there is not a clear superiority of any of the controllers.

Case 3 *(Step increase in the reference value of the reactor temperature).* It is usually necessary to increase the reactor temperature (y_2) as the catalyst deactivates. In this case, an increase of 0.4 °C in the reference value of the reactor temperature was simulated. The proposed controller has the same parameters considered in Case 2, except for the output error matrix weight, which was modified to $Q = \mathrm{diag}([1 \quad 10 \quad 1 \quad 1 \quad 1])$, and the control horizon that which was selected as $m = 2$. The weight for the reactor temperature was increased from 1 to 10 in order to force a faster response. The vector of reference values was also modified to represent the step in the temperature. The resulting input and output responses are shown in Fig. 6.

In the standard IHMPC for this case, the same values for Q and R are used, and the smallest control horizon necessary to allow a feasible solution to Problem P1 is $m = 42$. The corresponding results are represented in Fig. 7.

Remark 3 *(Comparison of the computation time required by the two controllers).* The computer execution time required by the infinite horizon approaches considered in this paper is presented in Table 2. The computer time shown in Table 2 is the maximum time required to compute the control sequence in any of the simulated sampling steps. The simulations were performed with Matlab 6.0 for Windows NT 4.0 in a Pentium III, 550 MHz computer.

From Table 2, we observe that the practical implementation of the standard IHMPC may not be possible because of the large execution time requirements. Solution of Problem P1 with the minimum control horizon to allow a feasible controller is substantially more computer demanding than the solution of Problem P2, which is feasible for any control horizon. The ratio between the computer times spent by the two approaches ranges from 13.16 in Case 1 to 1055 in Case 3, in favor of the new approach. The proposed controller can be configured to run with a computer effort comparable to the usual MPC control packages already implemented in industry. On the contrary, the conventional IHMPC needs very large control horizons to have a feasibility range comparable to the feasibility range of the proposed controller. Therefore, unless a powerful computer system is available in the industrial site, the standard IHMPC cannot be implemented. This is the case of the existing control system of the ethylene-oxide plant studied here, which justifies the development of the proposed approach.

6. Conclusion

In this paper, we have proposed a nominally stable linear MPC, which extends existing approaches for practical implementation. The stable MPC approaches existing in the control literature ensure stability by including hard constraints in the control optimization problem. A consequence of those strategies is that the feasible range of the controller may result quite limited for small to moderate control horizons. We have presented a method to enlarge the feasible range of the controller without substantially deteriorating the performance. The developed controller was compared to the standard IHMPC for an industrial multivariable system with several integrating poles.

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