Table 2.1 Continued

α-β-γ ROTATION SEQUENCE	B	B^{-1}	ANGLES AS FUNCTIONS OF DIRECTION COSINES	SINGULAR AT θ_2 =
2-1-3	$\begin{bmatrix} c_2s_3 & c_3 & 0 \\ c_2c_3 & -s_3 & 0 \\ -s_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} s_3 & c_3 & 0 \\ c_2c_3 & -c_2s_3 & 0 \\ s_2s_3 & s_2c_3 & c_2 \end{bmatrix}$	$\theta_1=t^{-1}(C_{31}/C_{33})$ $\theta_2=s^{-1}(-C_{32})$ $\theta_3=t^{-1}(C_{12}/C_{22})$	$\pm\pi/2$
2-3-1	$\begin{bmatrix} s_2 & 0 & 1 \\ c_2c_3 & s_3 & 0 \\ -c_2s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} 0 & c_3 & -s_3 \\ 0 & c_2s_3 & c_2c_3 \\ c_2 & -s_2c_3 & s_2s_3 \end{bmatrix}$	$\theta_1=t^{-1}(-C_{13}/C_{11})$ $\theta_2=s^{-1}(C_{12})$ $\theta_3=t^{-1}(-C_{32}/C_{22})$	$\pm\pi/2$
2-3-2	$\begin{bmatrix} s_2c_3 & -s_3 & 0 \\ c_2 & 0 & 1 \\ s_2s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} c_3 & 0 & s_3 \\ -s_2s_3 & 0 & s_2c_3 \\ -c_2c_3 & s_2 & -c_2s_3 \end{bmatrix}$	$\theta_1=t^{-1}(C_{23}/-C_{21})$ $\theta_2=c^{-1}(C_{22})$ $\theta_3=t^{-1}(C_{32}/C_{12})$	$0,\pm\pi$
3-1-2	$\begin{bmatrix} -c_2s_3 & c_3 & 0 \\ s_2 & 0 & 1 \\ c_2c_3 & s_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} -s_3 & 0 & c_3 \\ c_2c_3 & 0 & c_2s_3 \\ s_2s_3 & c_2 & -s_2c_3 \end{bmatrix}$	$\theta_1=t^{-1}(-C_{21}/C_{22})$ $\theta_2=s^{-1}(C_{23})$ $\theta_3=t^{-1}(-C_{13}/C_{33})$	$\pm\pi/2$
3-1-3	$\begin{bmatrix} s_2s_3 & c_3 & 0 \\ s_2c_3 & -s_3 & 0 \\ c_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} s_3 & c_3 & 0 \\ c_3s_2 & -s_3s_2 & 0 \\ -s_3c_2 & -c_3c_2 & s_2 \end{bmatrix}$	$\theta_1=t^{-1}(C_{31}/-C_{32})$ $\theta_2=c^{-1}(C_{33})$ $\theta_3=t^{-1}(C_{13}/C_{23})$	$0,\pm\pi$
3-2-1	$\begin{bmatrix} -s_2 & 0 & 1 \\ c_2s_3 & c_3 & 0 \\ c_2c_3 & -s_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} 0 & s_3 & c_3 \\ 0 & c_2c_3 & -c_2s_3 \\ c_2 & s_2s_3 & s_2c_3 \end{bmatrix}$	$\theta_1=t^{-1}(C_{12}/C_{11})$ $\theta_2=s^{-1}(-C_{13})$ $\theta_3=t^{-1}(C_{23}/C_{33})$	$\pm\pi/2$
3-2-3	$\begin{bmatrix} -s_2c_3 & s_3 & 0 \\ s_2s_3 & c_3 & 0 \\ c_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} -c_3 & s_3 & 0 \\ s_2s_3 & s_2c_3 & 0 \\ c_2c_3 & -c_2s_3 & s_2 \end{bmatrix}$	$\theta_1=t^{-1}(C_{32}/C_{31})$ $\theta_2=c^{-1}(C_{33})$ $\theta_3=t^{-1}(C_{23}/-C_{13})$	$0,\pm\pi$

singularity. A very familiar device in space vehicle dynamics is to define a moving frame which is the "nominal" or desired motion, then introduce three small Euler angles to describe departure motion. A casual inspection of Table 2.1 reveals any of the six α-β-γ Euler angle sets which *have non-repeated indices* (i.e., 1-2-3, 3-2-1, 2-3-1, 3-1-2, 2-1-3, 1-3-2) are well-suited for describing small departure motions. The most common ("yaw, pitch, roll") set of Euler angles for aircraft and spacecraft applications is the 3-2-1 sequence, taken as typically small displacements from a moving "local vertical" set of reference axes. So long as these angles remain small, of course, no analytical or numerical difficulty associated with the geometric singularities will occur.

It is a remarkable truth that the corresponding kinematic relationships for the four Euler (quaternion) parameters of Section 2.6 are *rigorously*, and *universally* linearly related to the angular velocity components, for all orientations. These orientation parameters are closely related to Euler's principal rotation theorem developed in the following section.

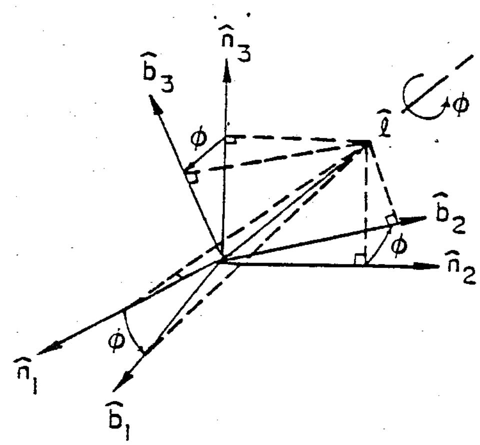

Figure 2.6 Euler's Principal Rotation

2.5 EULER'S PRINCIPAL ROTATION THEOREM

Euler (refs. 5, 6, 7) is generally credited with being responsible for the Principal Rotation Theorem:

> A rigid body can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rotation of the body through a *principal angle* (ϕ) about a *principal line* ($\hat{\ell}$); the principal line being a judicious axis <u>fixed in the body and fixed in space</u>.

Letting (see Figure 2.6) the body fixed axes $\{\hat{b}\}$ be "initially" coincident with fixed axes $\{\hat{n}\}$, we can use Euler's principal rotation theorem to develop several elegant parameterizations of the direction cosine matrix $[C]$ defining $\{\hat{b}\}$'s instantaneous angular position in the sense

$$\{\hat{b}\} = [C]\{\hat{n}\}.$$

Let us denote the $\{\hat{b}\}$ and $\{\hat{n}\}$ components of $\hat{\ell}$ as

$$\hat{\ell} = \ell_{b1}\hat{b}_1 + \ell_{b2}\hat{b}_2 + \ell_{b3}\hat{b}_3 \tag{2.46a}$$

and

$$\hat{\ell} = \ell_{n1}\hat{n}_1 + \ell_{n2}\hat{n}_2 + \ell_{n3}\hat{n}_3 \tag{2.46b}$$

As a direct consequence of the fact that ℓ_{bi} and ℓ_{ni} are constants (during a rotation ϕ about a fixed $\hat{\ell}$) and $\hat{b}_i$ are "initially" coincident with $\hat{n}_i$, we see that $\ell_{ni} \equiv \ell_{bi} \equiv \ell_i$, for $i = 1,2,3$. According to Eq. 2.15, we have

$$\begin{Bmatrix} \ell_{b1} \\ \ell_{b2} \\ \ell_{b3} \end{Bmatrix} = [C] \begin{Bmatrix} \ell_{n1} \\ \ell_{n2} \\ \ell_{n3} \end{Bmatrix} \tag{2.47}$$

but since $\ell_{bi} = \ell_{ni} = \ell_i$, we have

$$\begin{Bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{Bmatrix} = [C] \begin{Bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{Bmatrix} \tag{2.48}$$

Careful inspection of Eq. 2.48 reveals the truth that $\hat{\ell}$ exists if and only if

[C] has an eigenvalue of +1; in which case $\hat{\ell}$ is the corresponding unit eigenvector of [C]. Goldstein (ref. 3) proves that all "proper" direction cosine matrices (those corresponding to displacement of right-handed axes embedded in a rigid body) do in fact have an eigenvalue of +1; the eigenvalue and corresponding eigenvector are unique (to within a sign on $\hat{\ell}$ and ϕ) except for the case of zero angular displacement. The lack of sign uniqueness, as will be evident, does not cause a practical difficulty.

Since Euler's theorem reduces the general angular displacement to a single rotation about a fixed line, we can make immediate use of the developments in Section 2.3 to parameterize the direction cosine matrix in terms of $\hat{\ell}$ and ϕ. Specifically, if we take $r = \hat{n}_i$ and $r' = \hat{n}_i' \equiv \hat{b}_i$ in the general Eq. 2.29, we obtain

$$\hat{b}_i = (1 - \cos\phi)(\hat{\ell} \cdot \hat{n}_i)\hat{\ell} + \cos\phi\hat{n}_i + \sin\phi(\hat{\ell} \times \hat{n}_i) \ , \quad i = 1,2,3 \qquad (2.49)$$

If we substitute $\hat{\ell} = \ell_1\hat{n}_1 + \ell_2\hat{n}_2 + \ell_3\hat{n}_3$ and carry out the implied algebra in Eq. 2.49, we immediately obtain Eq. 2.11 with

$$[C] = \begin{bmatrix} \ell_1^2(1-c\phi)+c\phi & \ell_1\ell_2(1-c\phi)+\ell_3 s\phi & \ell_1\ell_3(1-c\phi)-\ell_2 s\phi \\ \ell_2\ell_1(1-c\phi)-\ell_3 s\phi & \ell_2^2(1-c\phi)+c\phi & \ell_2\ell_3(1-c\phi)+\ell_1 s\phi \\ \ell_3\ell_1(1-c\phi)+\ell_2 s\phi & \ell_3\ell_2(1-c\phi)-\ell_1 s\phi & \ell_3^2(1-c\phi)+c\phi \end{bmatrix} \qquad (2.50)$$

Since $\ell_1^2 + \ell_2^2 + \ell_3^2 = 1$, we have $[C(\ell_1,\ell_2,\ell_3,\phi)]$, but only three degrees of freedom, as expected. Notice we can verify immediately (by summing the trace of Eq. 2.50) that

$$\cos\phi = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \qquad (2.51)$$

and, by differencing the symmetric elements, we see that

$$2\ell_3\sin\phi = C_{12} - C_{21}$$
$$2\ell_2\sin\phi = C_{31} - C_{13}$$
$$2\ell_1\sin\phi = C_{23} - C_{32}$$

Multiplying the above three equations by ℓ_3, ℓ_2, ℓ_1, respectively (and making use of $\sum_{i=1}^{3} \ell_i^2 = 1$), we have

$$\sin\phi = \frac{1}{2}\left[\ell_3(C_{12} - C_{21}) + \ell_2(C_{31} - C_{13}) + \ell_1(C_{23} - C_{32})\right] \tag{2.52}$$

Thus, given the direction cosines $[C]$, we can solve for $\hat{\boldsymbol{\ell}}$ from Eq. 2.48, normalized so that $\hat{\boldsymbol{\ell}}$ is a unit vector, and ϕ from Eqs. 2.51 and 2.52. Notice that the sign choice on ℓ_i correctly affects the quadrant of ϕ in Eq. 2.52. Clearly a positive rotation about $\pm\hat{\boldsymbol{\ell}}$ is equivalent to a negative rotation about $\mp\hat{\boldsymbol{\ell}}$.

2.6 EULER PARAMETERS

Following Euler, we define the four *Euler Parameters* in terms of the *Principal Line* ($\hat{\boldsymbol{\ell}}$) and *Principal Angle* (ϕ) as

$$\begin{aligned}
\beta_0 &= \cos\frac{\phi}{2}\\
\beta_1 &= \ell_1\sin\frac{\phi}{2}\\
\beta_2 &= \ell_2\sin\frac{\phi}{2}\\
\beta_3 &= \ell_3\sin\frac{\phi}{2}
\end{aligned} \tag{2.53}$$

It is obvious that the four β_i's satisfy the constraint equation

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \tag{2.54}$$

If we make use of the half-angle identities

$$\begin{aligned}
\sin\phi &= 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\\
\cos\phi &= 2\cos^2\frac{\phi}{2} - 1
\end{aligned}$$

and Eq. 2.54, the direction cosine matrix of Eq. 2.50 can be parameterized as a function of the Euler Parameters:

$$[C(\beta)] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2)\\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1)\\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix} \tag{2.55}$$

The inverse relationships (β's in terms of the elements of $[C]$) can be deduced from Eq. 2.55:

$$\begin{aligned}
\beta_0 &= \pm\frac{1}{2}(C_{11} + C_{22} + C_{33} + 1)^{1/2}\\
\beta_1 &= (C_{23} - C_{32})/4\beta_0
\end{aligned}$$

$$\beta_2 = (C_{31} - C_{13})/4\beta_0$$
$$\beta_3 = (C_{12} - C_{21})/4\beta_0 \tag{2.56}$$

The first of these equations has an apparent sign ambiguity. However, there is no loss of generality in adopting the positive sign, since it is evident that changing all four signs of the β_i's in Eq. 2.55 *does not* change the direction cosine matrix. This is another reflection of the equivalence of rotating by a positive angle Φ about $\pm\hat{\ell}$ to a $-\Phi$ rotation about $\mp\hat{\ell}$. It is evident that Eqs. 2.56 contain a 0/0 indeterminacy whenever β_0 goes through zero; a computationally superior algorithm has been developed by Stanley (ref. 8):

$$\beta_0^2 = \frac{1}{4}(1 + 2C_{00} - T)$$
$$\beta_1^2 = \frac{1}{4}(1 + 2C_{11} - T)$$
$$\beta_2^2 = \frac{1}{4}(1 + 2C_{22} - T) \tag{2.57}$$
$$\beta_3^2 = \frac{1}{4}(1 + 2C_{33} - T)$$

where Stanley defines

$$T = \text{Trace}\,[C] = C_{11} + C_{22} + C_{33} \tag{2.58a}$$
$$C_{00} = T \tag{2.58b}$$

and selects (for division by) the β_i computed from Eq. 2.57 which has the largest absolute value assuming it to be positive; the other three β_j's can be obtained by dividing β_i into the appropriate three of the following six equations (obtained by differencing and summing the symmetric elements of [C]):

$$\beta_0\beta_1 = (C_{23} - C_{32})/4$$
$$\beta_0\beta_2 = (C_{31} - C_{13})/4$$
$$\beta_0\beta_3 = (C_{12} - C_{21})/4 \tag{2.59}$$
$$\beta_2\beta_3 = (C_{23} + C_{32})/4$$
$$\beta_3\beta_1 = (C_{31} + C_{13})/4$$
$$\beta_1\beta_2 = (C_{12} + C_{21})/4$$

The Euler parameters have many elegant properties; we shall develop the more important relationships in the following discussion.

One important property is the simple fashion in which Euler parameters of sequential rotations parameterize an equivalent single rotation. To develop these results, let $\{\hat{b}\}$, $\{\hat{b}'\}$, and $\{\hat{b}''\}$ be three arbitrary positions of a triad of unit vectors; the direction cosine matrix $[C]$ defining the relative orientations of $\{\hat{b}\}$, $\{\hat{b}'\}$, and $\{\hat{b}''\}$ can be parameterized in terms of three sets of Euler parameters as

$$\{\hat{b}'\} = [C(\beta')]\{\hat{b}\} \tag{2.60a}$$

$$\{\hat{b}''\} = [C(\beta'')]\{\hat{b}'\} \tag{2.60b}$$

$$\{\hat{b}''\} = [C(\beta)]\{\hat{b}\} \tag{2.60c}$$

We seek a relationship of the form

$$\beta_i = \text{function}\,(\beta_0',\beta_1',\beta_2',\beta_3';\beta_0'',\beta_1'',\beta_2'',\beta_3'') \quad , \quad i = 0,1,2,3 \tag{2.61}$$

relating the three sets of Euler parameters. From Eq. 2.60, it is obvious that

$$[C(\beta)] = [C(\beta'')][C(\beta')] \tag{2.62}$$

Direct substitution of Eq. 2.55 into Eq. 2.62 and equating the corresponding elements leads to the most elegant result

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \begin{bmatrix} \beta_0'' & -\beta_1'' & -\beta_2'' & -\beta_3'' \\ \beta_1'' & \beta_0'' & \beta_3'' & -\beta_2'' \\ \beta_2'' & -\beta_3'' & \beta_0'' & \beta_1'' \\ \beta_3'' & \beta_2'' & -\beta_1'' & \beta_0'' \end{bmatrix} \begin{Bmatrix} \beta_0' \\ \beta_1' \\ \beta_2' \\ \beta_3' \end{Bmatrix} \tag{2.63}$$

or, by *transmutation* of Eq. 2.63

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \begin{bmatrix} \beta_0' & -\beta_1' & -\beta_2' & -\beta_3' \\ \beta_1' & \beta_0' & -\beta_3' & \beta_2' \\ \beta_2' & \beta_3' & \beta_0' & -\beta_1' \\ \beta_3' & -\beta_2' & \beta_1' & \beta_0' \end{bmatrix} \begin{Bmatrix} \beta_0'' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{Bmatrix} \tag{2.64}$$

It is obvious by inspection that the coefficient matrices in Eqs. 2.63 and 2.64 are orthogonal; thus, any set of β's can be solved universally as a simple, nonsingular, bilinear combination of the other two. The successive rotation properties of the Euler parameters are thus quite elegant compared to any alternative parameterizations of $[C]$. For example, the analogous transformations for 3-1-3 (ϕ,θ,ψ) Euler angles can be verified to be

$$\begin{aligned} \theta &= \cos^{-1}(C_{33}) \\ \phi &= \tan^{-1}(C_{31}/-C_{32}) \\ \psi &= \tan^{-1}(C_{13}/C_{23}) \end{aligned} \tag{2.65}$$

where the C_{ij} are the lengthy transcendental functions obtained by carrying out the matrix multiplication

$$[C(\phi,\theta,\psi)] = [C(\phi'',\theta'',\psi'')][C(\phi',\theta',\psi')] \tag{2.66}$$

using Eq. 2.34 to parameterize the two matrices on the right side of Eq. 2.66. These nine equations can be inverted for (ϕ,θ,ψ) as in Eq. 2.65, where the C_{ij} are the functions of $(\phi',\theta',\psi',\phi'',\theta'',\psi'')$ obtained from Eq. 2.66. It is a trivial observation that Eqs. 2.63 and 2.64 are vastly more attractive that Eqs. 2.65 and 2.66!

In applications, we often require a transformation from a set of Euler angles to the corresponding Euler parameters, and the inverse transformation. The transformation from Euler parameters into Euler angles is entirely straightforward, we simply calculate the direction cosine matrix from Eq. 2.55, then employ the inverse trigonometric relationships in Table 2.1, corresponding to the particular Euler angle rotation sequence. The transformation from Euler angles to Euler parameters can proceed by calculating numerical values for the direction cosines from Eq. 2.36, then using Eqs. 2.56, 2.57, or 2.59 to calculate the β's. However it is possible to derive very compact analytical expressions that are more efficient and avoid the branching logic of the path through the direction cosines.

To develop the direct transformation from Euler angles to Euler parameters, we first note from Eq. 2.53 that the elementary Euler angle rotations have the corresponding Euler parameter values

Rotation about a "1" axis:
$$\beta_0 = \cos\frac{1}{2}\alpha$$
$$\beta_1 = \sin\frac{1}{2}\alpha$$
$$\beta_2 = \beta_3 = 0$$

Rotation about a "2" axis:
$$\beta_0 = \cos\frac{1}{2}\beta$$
$$\beta_2 = \sin\frac{1}{2}\beta$$
$$\beta_1 = \beta_3 = 0$$

Rotation about a "3" axis:
$$\beta_0 = \cos\frac{1}{2}\gamma$$
$$\beta_3 = \sin\frac{1}{2}\gamma$$
$$\beta_1 = \beta_2 = 0$$

If we let $c_i = \cos\frac{1}{2}\theta_i$ and $s_i = \sin\frac{1}{2}\theta_i$, for the 3-1-3 Euler angles $(\theta_1, \theta_2, \theta_3)$, we can write from Eq. 2.63 the equivalent Euler parameters (replacing the above three elementary rotations) as

$$\begin{Bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{Bmatrix} = \begin{bmatrix} c_3 & 0 & 0 & -s_3\\ 0 & c_3 & s_3 & 0\\ 0 & -s_3 & c_3 & 0\\ s_3 & 0 & 0 & c_3\end{bmatrix}\begin{bmatrix} c_2 & -s_2 & 0 & 0\\ s_2 & c_2 & 0 & 0\\ 0 & 0 & c_2 & s_2\\ 0 & 0 & -s_2 & c_2\end{bmatrix}\begin{Bmatrix} c_1\\ 0\\ 0\\ s_1\end{Bmatrix}$$

Upon carrying out the matrix multiplications we obtain

$$\beta_0 = \cos\frac{1}{2}\theta_3\cos\frac{1}{2}\theta_2\cos\frac{1}{2}\theta_1 - \sin\frac{1}{2}\theta_3\cos\frac{1}{2}\theta_2\sin\frac{1}{2}\theta_1$$
$$\beta_1 = \cos\frac{1}{2}\theta_3\sin\frac{1}{2}\theta_2\cos\frac{1}{2}\theta_1 + \sin\frac{1}{2}\theta_3\sin\frac{1}{2}\theta_2\sin\frac{1}{2}\theta_1$$
$$\beta_2 = -\sin\frac{1}{2}\theta_3\sin\frac{1}{2}\theta_2\cos\frac{1}{2}\theta_1 + \cos\frac{1}{2}\theta_3\sin\frac{1}{2}\theta_2\sin\frac{1}{2}\theta_1$$
$$\beta_3 = \sin\frac{1}{2}\theta_3\cos\frac{1}{2}\theta_2\cos\frac{1}{2}\theta_1 + \cos\frac{1}{2}\theta_3\cos\frac{1}{2}\theta_2\sin\frac{1}{2}\theta_1$$

Finally, using trigonometric identities to simplify to the final transformation

$$\begin{aligned}
\beta_0 &= \cos \tfrac{1}{2}\theta_2 \cos \tfrac{1}{2}(\theta_1 + \theta_3) \\
\beta_1 &= \sin \tfrac{1}{2}\theta_2 \cos \tfrac{1}{2}(\theta_1 - \theta_3) \\
\beta_2 &= \sin \tfrac{1}{2}\theta_2 \sin \tfrac{1}{2}(\theta_1 - \theta_3) \\
\beta_3 &= \cos \tfrac{1}{2}\theta_2 \sin \tfrac{1}{2}(\theta_1 + \theta_3)
\end{aligned} \tag{2.67}$$

The above can be paralleled for the other 11 sets of Euler angle sequences; the transformations for all 12 sets of Euler angles are summarized in Table 2.2. For ease in computer programming, we can employ the most useful universal transformation

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = [R_{\alpha\beta\gamma}] \cdot \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.68}$$ *

where

$$\begin{aligned}
[R_{\alpha\beta\gamma}] = {} & [\cos \tfrac{1}{2}\theta_3 R_0 + \sin \tfrac{1}{2}\theta_3 R_\gamma] \cdot \\
& [\cos \tfrac{1}{2}\theta_2 R_0 + \sin \tfrac{1}{2}\theta_2 R_\beta] \cdot \\
& [\cos \tfrac{1}{2}\theta_1 R_0 + \sin \tfrac{1}{2}\theta_1 R_\alpha] \cdot
\end{aligned} \tag{2.69}$$

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

*Eq. 2.68 was developed by H. S. Morton, Jr. of the University of Virginia, Dec. 1973 (ref. 4).

TABLE 2.2

TRANSFORMATION FROM THE TWELVE SETS OF EULER ANGLES TO EULER PARAMETERS*

α-β-γ	β_0	β_1	β_2	β_3
1-2-1	$c\frac{\theta_2}{2}\, c(\frac{\theta_1+\theta_3}{2})$	$c\frac{\theta_2}{2}\, s(\frac{\theta_1+\theta_3}{2})$	$s\frac{\theta_2}{2}\, c(\frac{\theta_1-\theta_3}{2})$	$s\frac{\theta_2}{2}\, s(\frac{\theta_1-\theta_3}{2})$
2-3-2	$c\frac{\theta_2}{2}\, c(\frac{\theta_1+\theta_3}{2})$	$s\frac{\theta_2}{2}\, s(\frac{\theta_1-\theta_3}{2})$	$c\frac{\theta_2}{2}\, s(\frac{\theta_1+\theta_3}{2})$	$s\frac{\theta_2}{2}\, c(\frac{\theta_1-\theta_3}{2})$
3-1-3	$c\frac{\theta_2}{2}\, c(\frac{\theta_1+\theta_3}{2})$	$s\frac{\theta_2}{2}\, c(\frac{\theta_1-\theta_3}{2})$	$s\frac{\theta_2}{2}\, s(\frac{\theta_1-\theta_3}{2})$	$c\frac{\theta_2}{2}\, s(\frac{\theta_1+\theta_3}{2})$
1-3-1	$c\frac{\theta_2}{2}\, c(\frac{\theta_3+\theta_1}{2})$	$c\frac{\theta_2}{2}\, s(\frac{\theta_3+\theta_1}{2})$	$s\frac{\theta_2}{2}\, s(\frac{\theta_3-\theta_1}{2})$	$s\frac{\theta_2}{2}\, c(\frac{\theta_3-\theta_1}{2})$
2-1-2	$c\frac{\theta_2}{2}\, c(\frac{\theta_3+\theta_1}{2})$	$s\frac{\theta_2}{2}\, c(\frac{\theta_3-\theta_1}{2})$	$c\frac{\theta_2}{2}\, s(\frac{\theta_3+\theta_1}{2})$	$s\frac{\theta_2}{2}\, s(\frac{\theta_3-\theta_1}{2})$
3-2-3	$c\frac{\theta_2}{2}\, c(\frac{\theta_3+\theta_1}{2})$	$s\frac{\theta_2}{2}\, s(\frac{\theta_3-\theta_1}{2})$	$s\frac{\theta_2}{2}\, c(\frac{\theta_3-\theta_1}{2})$	$c\frac{\theta_2}{2}\, s(\frac{\theta_3+\theta_1}{2})$
1-2-3	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$
	$-s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$+c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$+s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$
2-3-1	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$
	$-s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$+s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$+c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$
3-1-2	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$
	$-s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$+s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$+c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$
1-3-2	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$
	$+s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$+s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$
2-1-3	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$
	$+s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$+s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$
3-2-1	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$
	$+s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-s\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, c\frac{\theta_3}{2}$	$+s\frac{\theta_1}{2}\, c\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$	$-c\frac{\theta_1}{2}\, s\frac{\theta_2}{2}\, s\frac{\theta_3}{2}$

*c = cos, s = sin

Equation 2.68 captures all 12 transformations of Table 2.2 in a general form suitable for a universal computer algorithm.

The above properties are significant, attractive features of the Euler parameters; however, the most impressive property is the kinematical differential equation which we now develop. We seek equations of the form $\dot\beta_i =$ function$_i$ $(\beta_0, \beta_1, \beta_2, \beta_3, \omega_1, \omega_2, \omega_3)$. These can be obtained by differentiating Eq. 2.56. For example, consider the derivation of the equation for $\dot\beta_0$, by differentiating the first term of Eq. 2.56, we have

$$\dot\beta_0 = \frac{(\dot C_{11} + \dot C_{22} + \dot C_{33})}{8\beta_0} \tag{2.70}$$

From Eq. 2.22, substitute for the $\dot C_{ij}$'s

$$\begin{aligned} \dot C_{11} &= \omega_3 C_{21} - \omega_2 C_{31} \\ \dot C_{22} &= -\omega_3 C_{12} + \omega_1 C_{32} \\ \dot C_{33} &= \omega_2 C_{13} - \omega_1 C_{23} \end{aligned} \tag{2.71}$$

so that Eq. 2.70 becomes

$$\dot\beta_0 = \frac{(C_{32} - C_{23})\omega_1 + (C_{13} - C_{31})\omega_2 + (C_{21} - C_{12})\omega_3}{8\beta_0} \tag{2.72}$$

and eliminating the C_{ij}'s in favor of the β_i's using the first three terms of Eq. 2.59, we have the desired equation for $\dot\beta_0$.

$$\dot\beta_0 = -\frac{1}{2}(\beta_1 \omega_1 + \beta_2 \omega_2 + \beta_3 \omega_3) \tag{2.73}$$

Similarly, we can derive equations for $\dot\beta_1$, $\dot\beta_2$, $\dot\beta_3$; the resulting four equations can be written in matrix form

$$\begin{Bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} \tag{2.74}$$

or, by *transmutation* of Eq. 2.74

$$\begin{Bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{2.75}$$

Eq. 2.74 or, alternatively, Eq. 2.75 are indeed useful results. Observe that the transformation matrix in Eq. 2.75 relating $\dot\beta_i$'s and ω_j's is orthogonal; therefore this relationship is universally nonsingular (whereas the corresponding kinematical equations for any three angle set is transcendental, nonlinear, and contains a 0/0 singularity). Further, note Eq. 2.74, for $\omega_i(t)$ measured or integrated a priori, is a *rigorously linear* differential equation

$$\{\dot\beta\} = [\omega(t)]\{\beta\} \tag{2.76}$$

Since $[\omega(t)]$ is skew symmetric (compare Eqs. 2.74 and 2.76), we can show that $\sum_i \beta_i^2 = 1$ is a rigorous integral of the solution. To see this, let

$$\|\beta\| = \sum_{i=0}^{3} \beta_i^2 = \{\beta\}^T\{\beta\} \tag{2.77}$$

then

$$\frac{d}{dt}\|\beta\| = \{\dot\beta\}^T\{\beta\} + \{\beta\}^T\{\dot\beta\} \tag{2.78}$$

substitution of Eq. 2.76 into Eq. 2.78 yields

$$\frac{d}{dt}\|\beta\| = \{\beta\}^T[[\omega]^T + [\omega]]\{\beta\} \tag{2.79}$$

and since $[\omega]^T = -[\omega]$, we see from Eq. 2.79 that $\frac{d}{dt}\|\beta\| = 0$. Since $\|\beta(t_0)\| = 1$ will be established by any valid choice of initial Euler parameters, we can see that any (accurate) solution of Eqs. 2.74, 2.75, or 2.76 will guarantee $\|\beta(t)\| = 1$. In fact, this is a standard (necessary) condition used to test numerical solutions of Eq. 2.75, to, for example, control step size.

The Euler parameters represent a fascinating example of *regularization* in mechanics. Through a judicious choice of coordinates, we are able to eliminate the singularities usually present. It is also a beautiful truth that all possible rotational motions correspond to a path on the surface of a four dimensional unit sphere. For certain cases, analytical solution for the Euler parameters are possible, see refs. 4 and 11.

2.7 OTHER ORIENTATION PARAMETERS

Aside from Euler angles, Euler parameters, and the direction cosines, there are an infinity of less commonly adopted possible descriptions of orientation. We summarize here the most prominent members of this large family of possibilities.

Rodriquez Parameters $(\lambda_1, \lambda_2, \lambda_3)$

These parameters (ref. 6) are intimately related to the principal rotation and, therefore, to the Euler parameters. In fact, the λ_i's are simply

$$\begin{aligned} \lambda_1 &= \beta_1/\beta_0 = \ell_1 \tan \tfrac{1}{2}\Phi \\ \lambda_2 &= \beta_2/\beta_0 = \ell_2 \tan \tfrac{1}{2}\Phi \\ \lambda_3 &= \beta_3/\beta_0 = \ell_3 \tan \tfrac{1}{2}\Phi \end{aligned} \tag{2.80}$$

Clearly, these parameters have an unbounded behavior near $\Phi = (2n+1)\pi$ where n is integer and therefore are less attractive than the β's themselves for most purposes. The geometric and kinematic relationships governing the λ_i's can be immediately derived from the corresponding Euler parameter equations.

Cayley-Klein Parameters

These four complex parameters (ref. 7) are the combinations of the Euler parameters

$$\begin{aligned} \alpha &= \beta_0 + i\beta_3 \quad , \quad i^2 = -1 \\ \beta &= -\beta_2 + i\beta_1 \\ \gamma &= \beta_2 + i\beta_1 \\ \delta &= \beta_0 - i\beta_3 \end{aligned} \tag{2.81}$$

with the inverse relations being

$$\begin{aligned}
\beta_0 &= (\alpha + \delta)/2 \\
\beta_1 &= -i(\beta + \gamma)/2 \\
\beta_2 &= -(\beta - \gamma)/2 \\
\beta_3 &= -i(\alpha - \delta)/2
\end{aligned} \tag{2.82}$$

and the parameterization of the direction cosine matrix being

$$[C(\alpha,\beta,\gamma,\delta)] = \begin{bmatrix} (\alpha^2-\beta^2-\gamma^2+\delta^2)/2 & i(-\alpha^2+\beta^2-\gamma^2+\delta^2)/2 & (\beta\delta-\alpha\gamma) \\ i(\alpha^2+\beta^2-\gamma^2-\delta^2)/2 & (\alpha^2+\beta^2+\gamma^2+\delta^2)/2 & -i(\alpha\gamma+\beta\delta) \\ (\gamma\delta-\alpha\beta) & i(\alpha\beta+\gamma\delta) & (\alpha\delta+\beta\gamma) \end{bmatrix} \tag{2.83}$$

An alternate set of four complex Cayley-Klein-like parameters (having advantages for certain applications) is discussed in Ref. [11].

Quaternions

Hamilton (ref. 7) developed quaternion algebra due to the intimate connection with Euler Parameters and rotation of vectors. Quaternions are four dimensional entities having one real and three complex parts,

$$q = q_0 + iq_1 + jq_2 + kq_3 \tag{2.84}$$

where

$$i\cdot i = -1,\ j\cdot j = -1,\ k\cdot k = -1$$

$$i\cdot j = k,\ j\cdot k = i,\ -k\cdot j = i, \text{ etc.}$$

For a unit quaternion $\sum_{i=0}^{3} q_i^2 = 1$. From Ames and Murnahan's (ref. 10) development of quaternion algebra, it can be shown that the quaternion multiplication $q = q''q'$ yields a new quaternion (q) whose components relate to those of q'' and q' as

$$\begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} q_0'' & -q_1'' & -q_2'' & -q_3'' \\ q_1'' & q_0'' & -q_3'' & q_2'' \\ q_2'' & q_3'' & q_0'' & -q_1'' \\ q_3'' & -q_2'' & q_1'' & q_0'' \end{bmatrix} \begin{Bmatrix} q_0' \\ q_1' \\ q_2' \\ q_3' \end{Bmatrix} \tag{2.84}$$

Comparison of Eq. 2.84 with Eq. 2.63 allows us to identify

$$
\begin{aligned}
q_0 &= \beta_0 \\
q_i &= -\beta_i \quad , \quad i = 1,2,3
\end{aligned}
\tag{2.85}
$$

Or, alternatively, we can consider the Euler parameters to be the elements of the quaternion

$$q = \beta_0 - i\beta_1 - j\beta_2 - k\beta_3. \tag{2.86}$$

Historically, quaternions played a "stepping stone" role en route to modern vector algebra and quaternion algebra is not presently widely used. Unfortunately, Hamilton's association of Euler parameters with quaternions served to make them more obscure when quaternion algebra fell largely into a "museum-piece" status. As we have seen in the foregoing, the basic kinematic and geometric relationships governing the Euler parameters can be easily developed without relying upon interpreting them as elements of a quaternion.

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CHAPTER 3

BASIC PR...