$\underline{A}^{sr}\underline{\boldsymbol{e}}^{(r)}$ is evaluated by the common multiplication rule of matrix algebra although one factor is composed of scalars and the other of vectors.

The scalar product $\boldsymbol{e}_\alpha^{(s)}\cdot\boldsymbol{e}_\beta^{(r)}$ of two base vectors belonging to different bases equals the cosine of the angle between the two vectors. Because of Eqs. (1.3) and (1.2) it is also equal to $A^{sr}_{\alpha\beta}$. This identity explains the name *direction cosine matrix* for $\underline{A}^{sr}$. Replace in Eq. (1.2) $\boldsymbol{e}_\alpha^{(s)}$ by Eq. (1.3) and $\boldsymbol{e}_\beta^{(s)}$ by the corresponding expression with β instead of α. The equation then states that the scalar product of the rows α and β of $\underline{A}^{sr}$ equals $\delta_{\alpha\beta}$. From this follows that $\underline{A}^{sr}(\underline{A}^{sr})^T$ is the unit matrix since every element of the product matrix represents the scalar product of two rows of $\underline{A}^{sr}$. From this identity two important properties of the matrix $\underline{A}^{sr}$ are deduced. First, the inverse of $\underline{A}^{sr}$ equals the transpose, $(\underline{A}^{sr})^{-1}=(\underline{A}^{sr})^T$. From this follows that the inverse of Eq. (1.4) reads

$$\underline{\boldsymbol{e}}^{(r)}=\underline{A}^{rs}\underline{\boldsymbol{e}}^{(s)}=(\underline{A}^{sr})^T\underline{\boldsymbol{e}}^{(s)}. \tag{1.5}$$

Second, the determinant of $\underline{A}^{sr}$ is either $+1$ or -1. The case -1 occurs only if one of the two bases $\underline{\boldsymbol{e}}^{(r)}$ and $\underline{\boldsymbol{e}}^{(s)}$ is a right hand system and the other a left hand system. This is not the case here so that

$$\det \underline{A}^{sr} = +1. \tag{1.6}$$

The right hand side of Eq. (1.1) can be given the form of a matrix product. For this purpose the column matrix $\underline{v}=[v_1\ \ v_2\ \ v_3]^T$ of the coordinates of $\boldsymbol{v}$ in the base $\underline{\boldsymbol{e}}$ is introduced (a shorter name for $\underline{v}$ is *coordinate matrix of* $\boldsymbol{v}$ *in* $\underline{\boldsymbol{e}}$). With this matrix Eq. (1.1) takes the form

$$\boldsymbol{v}=\underline{\boldsymbol{e}}^T\underline{v} \tag{1.7}$$

or

$$\boldsymbol{v}=\underline{v}^T\underline{\boldsymbol{e}}. \tag{1.8}$$

In two different bases $\underline{\boldsymbol{e}}^{(s)}$ and $\underline{\boldsymbol{e}}^{(r)}$ a vector $\boldsymbol{v}$ has different coordinate matrices. They are denoted $\underline{v}^{(s)}$ and $\underline{v}^{(r)}$, respectively. Eq. (1.7) establishes the identity

$$\underline{\boldsymbol{e}}^{(s)T}\underline{v}^{(s)}=\underline{\boldsymbol{e}}^{(r)T}\underline{v}^{(r)}.$$

On the right hand side Eq. (1.5) is substituted for $\underline{\boldsymbol{e}}^{(r)}$. This yields

$$\underline{\boldsymbol{e}}^{(s)T}\underline{v}^{(s)}=\underline{\boldsymbol{e}}^{(s)T}\underline{A}^{sr}\underline{v}^{(r)}$$

or

$$\underline{v}^{(s)}=\underline{A}^{sr}\underline{v}^{(r)}. \tag{1.9}$$

This equation represents the transformation rule for vector coordinates. It states that the *direction cosine matrix* is also the *coordinate transformation matrix*. Note the mnemonic position of the superscripts s and r.

The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as a matrix product. Let $\underline{a}^{(r)}$ and $\underline{b}^{(r)}$ be the coordinate matrices of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, in some vector base $\underline{\boldsymbol{e}}^{(r)}$. Then, $\boldsymbol{a}\cdot\boldsymbol{b}=\underline{a}^{(r)T}\underline{b}^{(r)}=\underline{b}^{(r)T}\underline{a}^{(r)}$. Often the coordinate matrices of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are known in two different bases, say $\underline{a}^{(r)}$ in $\underline{\boldsymbol{e}}^{(r)}$ and $\underline{b}^{(s)}$ in $\underline{\boldsymbol{e}}^{(s)}$. Then, $\boldsymbol{a}\cdot\boldsymbol{b}=\underline{a}^{(r)T}\underline{A}^{rs}\underline{b}^{(s)}$.

Consider, next, the eigenvalue problem $\underline{A}^{sr}\underline{u}^{(r)}=\lambda\underline{u}^{(r)}$ for a given direction cosine matrix $\underline{A}^{sr}$ relating two bases $\underline{\boldsymbol{e}}^{(r)}$ and $\underline{\boldsymbol{e}}^{(s)}$. This equation represents a special case of

Eq. (1.9) with $\underline{u}^{(s)} = \lambda \underline{u}^{(r)}$. Since the magnitude of a vector is the same in both bases the sums of squares of the elements of $\underline{u}^{(r)}$ and of $\lambda \underline{u}^{(r)}$ must be identical. From this follows that the absolute value of all (real or complex) eigenvalues λ is one. The characteristic polynomial $\det(\underline{A}^{rs} - \lambda \underline{E})$ with unit matrix $\underline{E}$ is of third order. Hence, there exists at least one real eigenvalue with $|\lambda| = 1$. That this eigenvalue is $\lambda = +1$ follows from the fact that the three eigenvalues λ_1, λ_2 and λ_3 satisfy the equation $\lambda_1 \lambda_2 \lambda_3 = \det \underline{A}^{rs} = +1$. The coordinates of the real eigenvector $\boldsymbol{u}$ which belongs to the eigenvalue $\lambda = +1$ are determined from the equation

$$\underline{A}^{rs} \underline{u}^{(r)} = \underline{u}^{(r)}. \tag{1.10}$$

This equation states that for any direction cosine matrix $\underline{A}^{rs}$ there exists (at least) one real vector $\boldsymbol{u}$ whose coordinates are the same in both bases $\underline{e}^{(r)}$ and $\underline{e}^{(s)}$. From this follows

Euler's Theorem *Two arbitrarily oriented bases $\underline{e}^{(r)}$ and $\underline{e}^{(s)}$ with common origin P can be made to coincide with one another by rotating one of them through a certain angle about an axis which is passing through P and which has the direction of the eigenvector $\boldsymbol{u}$ determined by Eq.* (1.10).

Besides vectors second-order tensors play an important role in rigid body dynamics. Such tensors are designated by boldface grotesque letters. In its most general form a tensor $\boldsymbol{D}$ is a sum of indeterminate products of two vectors each:

$$\boldsymbol{D} = \boldsymbol{a}_1 \boldsymbol{b}_1 + \boldsymbol{a}_2 \boldsymbol{b}_2 + \boldsymbol{a}_3 \boldsymbol{b}_3 + \cdots \tag{1.11}$$

A tensor is an operator. Its scalar product from the right with a vector $\boldsymbol{v}$ is defined as the vector

$$\begin{aligned} \boldsymbol{D} \cdot \boldsymbol{v} &= (\boldsymbol{a}_1 \boldsymbol{b}_1 + \boldsymbol{a}_2 \boldsymbol{b}_2 + \boldsymbol{a}_3 \boldsymbol{b}_3 + \cdots) \cdot \boldsymbol{v} \\ &= \boldsymbol{a}_1 \boldsymbol{b}_1 \cdot \boldsymbol{v} + \boldsymbol{a}_2 \boldsymbol{b}_2 \cdot \boldsymbol{v} + \boldsymbol{a}_3 \boldsymbol{b}_3 \cdot \boldsymbol{v} + \cdots . \end{aligned} \tag{1.12}$$

Similarly, the scalar product of $\boldsymbol{D}$ from the left with $\boldsymbol{v}$ is defined as

$$\boldsymbol{v} \cdot \boldsymbol{D} = \boldsymbol{v} \cdot \boldsymbol{a}_1 \boldsymbol{b}_1 + \boldsymbol{v} \cdot \boldsymbol{a}_2 \boldsymbol{b}_2 + \boldsymbol{v} \cdot \boldsymbol{a}_3 \boldsymbol{b}_3 + \cdots .$$

If in all indeterminate products of $\boldsymbol{D}$ the order of the factors is reversed a new tensor is obtained. It is called the conjugate of $\boldsymbol{D}$ and it is denoted by the symbol $\tilde{\boldsymbol{D}}$:

$$\begin{aligned} \boldsymbol{D} &= \boldsymbol{a}_1 \boldsymbol{b}_1 + \boldsymbol{a}_2 \boldsymbol{b}_2 + \boldsymbol{a}_3 \boldsymbol{b}_3 + \cdots \\ \tilde{\boldsymbol{D}} &= \boldsymbol{b}_1 \boldsymbol{a}_1 + \boldsymbol{b}_2 \boldsymbol{a}_2 + \boldsymbol{b}_3 \boldsymbol{a}_3 + \cdots . \end{aligned} \tag{1.13}$$

In vector algebra the distributive law is valid:

$$\boldsymbol{a} \boldsymbol{b}_1 \cdot \boldsymbol{v} + \boldsymbol{a} \boldsymbol{b}_2 \cdot \boldsymbol{v} = \boldsymbol{a}(\boldsymbol{b}_1 + \boldsymbol{b}_2) \cdot \boldsymbol{v}, \qquad \boldsymbol{a}_1 \boldsymbol{b} \cdot \boldsymbol{v} + \boldsymbol{a}_2 \boldsymbol{b} \cdot \boldsymbol{v} = (\boldsymbol{a}_1 + \boldsymbol{a}_2) \boldsymbol{b} \cdot \boldsymbol{v}.$$

Hence, the indeterminate products in a tensor are also distributive:

$$\boldsymbol{a} \boldsymbol{b}_1 + \boldsymbol{a} \boldsymbol{b}_2 = \boldsymbol{a}(\boldsymbol{b}_1 + \boldsymbol{b}_2), \qquad \boldsymbol{a}_1 \boldsymbol{b} + \boldsymbol{a}_2 \boldsymbol{b} = (\boldsymbol{a}_1 + \boldsymbol{a}_2) \boldsymbol{b}.$$

It is, therefore, possible to resolve all vectors on the right hand side of Eq. (1.11) in some vector base $\underline{e}$ and to regroup the resulting expression in the form

$$\boldsymbol{D}=\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} D_{\alpha\beta}\boldsymbol{e}_\alpha\boldsymbol{e}_\beta . \tag{1.14}$$

The nine scalars $D_{\alpha\beta}$ are called the coordinates of $\boldsymbol{D}$ in the base $\underline{e}$ (note that not this set of coordinates but only the quantity $\boldsymbol{D}$ is referred to as a tensor). They are combined in the (3×3) coordinate matrix $\underline{D}$. With this matrix the tensor becomes

$$\boldsymbol{D}=\underline{e}^{\mathrm{T}}\underline{D}\,\underline{e}. \tag{1.15}$$

It is a straightforward procedure to construct the matrix $\underline{D}$ from the coordinate matrices of the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ etc. Let these latter matrices be $\underline{a}_1$, $\underline{a}_2$, $\underline{b}_1$, $\underline{b}_2$ etc. Substitution of Eqs. (1.7) and (1.8) into Eq. (1.11) yields

$$\begin{aligned}\boldsymbol{D}&=\underline{e}^{\mathrm{T}}\underline{a}_1\underline{b}_1^{\mathrm{T}}\underline{e}+\underline{e}^{\mathrm{T}}\underline{a}_2\underline{b}_2^{\mathrm{T}}\underline{e}+\underline{e}^{\mathrm{T}}\underline{a}_3\underline{b}_3^{\mathrm{T}}\underline{e}+\cdots\\ &=\underline{e}^{\mathrm{T}}(\underline{a}_1\underline{b}_1^{\mathrm{T}}+\underline{a}_2\underline{b}_2^{\mathrm{T}}+\underline{a}_3\underline{b}_3^{\mathrm{T}}+\cdots)\underline{e}.\end{aligned}$$

Comparison with Eq. (1.15) shows that

$$\underline{D}=\underline{a}_1\underline{b}_1^{\mathrm{T}}+\underline{a}_2\underline{b}_2^{\mathrm{T}}+\underline{a}_3\underline{b}_3^{\mathrm{T}}+\cdots.$$

From this and from Eq. (1.13) follows that the coordinate matrix of the conjugate of $\boldsymbol{D}$ is the transpose of the coordinate matrix of $\boldsymbol{D}$. With Eqs. (1.14) and (1.1) the vector $\boldsymbol{D}\cdot\boldsymbol{v}$ is

$$\boldsymbol{D}\cdot\boldsymbol{v}=\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} D_{\alpha\beta}\boldsymbol{e}_\alpha\boldsymbol{e}_\beta\cdot\boldsymbol{v}=\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} D_{\alpha\beta}v_\beta\boldsymbol{e}_\alpha . \tag{1.16}$$

Its coordinate matrix in the base $\underline{e}$ is, therefore, the product $\underline{D}\,\underline{v}$ of the coordinate matrices of $\boldsymbol{D}$ and $\boldsymbol{v}$ in $\underline{e}$. The same result is obtained in a more formal way when Eqs. (1.15) and (1.7) are substituted for $\boldsymbol{D}$ and $\boldsymbol{v}$, respectively:

$$\boldsymbol{D}\cdot\boldsymbol{v}=\underline{e}^{\mathrm{T}}\underline{D}\,\underline{e}\cdot\underline{e}^{\mathrm{T}}\underline{v}. \tag{1.17}$$

In this expression a new type of matrix product appears, namely the scalar product $\underline{e}\cdot\underline{e}^{\mathrm{T}}$ of two *matrices whose elements are vectors.* Later, still another matrix product called cross product of two vectorial matrices will be met. These two products are defined as follows. Let $\underline{P}$ be an $(m\times r)$ matrix with vectors $\boldsymbol{P}_{ij}$ $(i=1\ldots m, j=1\ldots r)$ as elements and let $\underline{Q}$ be an $(r\times n)$ matrix with vectors $\boldsymbol{Q}_{ij}$ $(i=1\ldots r, j=1\ldots n)$ as elements. Then, the scalar product $\underline{P}\cdot\underline{Q}$ is a scalar $(m\times n)$ matrix with the elements

$$(\underline{P}\cdot\underline{Q})_{ij}=\sum_{k=1}^{r}\boldsymbol{P}_{ik}\cdot\boldsymbol{Q}_{kj}\qquad i=1\ldots m, j=1\ldots n,$$

and the cross product $\underline{P}\times\underline{Q}$ is a vectorial $(m\times n)$ matrix with the elements

$$(\underline{P}\times\underline{Q})_{ij}=\sum_{k=1}^{r}\boldsymbol{P}_{ik}\times\boldsymbol{Q}_{kj}\qquad i=1\ldots m, j=1\ldots n.$$

These definitions represent natural generalizations of the common matrix multiplication rule. Let us now return to Eq. (1.17). According to the definition just given the scalar product $\underline{e}\cdot\underline{e}^{\mathrm{T}}$ is the unit matrix so that $\boldsymbol{D}\cdot\boldsymbol{v}=\underline{e}^{\mathrm{T}}\underline{D}\,\underline{v}$ in agreement with Eq. (1.16).

Of particular interest is the tensor

$$E = e_1 e_1 + e_2 e_2 + e_3 e_3 = \underline{e}^T \underline{e} \tag{1.18}$$

whose coordinate matrix is the unit matrix. When this tensor is scalar multiplied with an arbitrary vector v the result is v itself: $E \cdot v = v$ and $v \cdot E = v$. For this reason E is called **unit tensor.**

With the help of Eq. (1.5) it is a simple matter to establish the law by which the coordinate matrix of a tensor is transformed when instead of a base $\underline{e}^{(r)}$ another base $\underline{e}^{(s)}$ is used for decomposition. Let $\underline{D}^{(r)}$ and $\underline{D}^{(s)}$ be the coordinate matrices of D in the two bases, respectively, so that by Eq. (1.15) the identity

$$\underline{e}^{(s)T} \underline{D}^{(s)} \underline{e}^{(s)} = \underline{e}^{(r)T} \underline{D}^{(r)} \underline{e}^{(r)}$$

holds. On the right hand side Eq. (1.5) is substituted for $\underline{e}^{(r)}$. This yields

$$\underline{e}^{(s)T} \underline{D}^{(s)} \underline{e}^{(s)} = \underline{e}^{(s)T} \underline{A}^{sr} \underline{D}^{(r)} \underline{A}^{rs} \underline{e}^{(s)}$$

or

$$\underline{D}^{(s)} = \underline{A}^{sr} \underline{D}^{(r)} \underline{A}^{rs}. \tag{1.19}$$

Note, here too, the mnemonic position of the superscripts.

In rigid body mechanics tensors with symmetric and with skew-symmetric coordinate matrices are met. The inertia tensor which will be defined in Chap. 3.1 and the unit tensor E have symmetric coordinate matrices. Tensors with unsymmetric coordinate matrices are found in connection with vector cross products. Consider, first, the double cross product $(a \times b) \times v$. It can be written in the form

$$(a \times b) \times v = b\, a \cdot v - a\, b \cdot v = (b\, a - a\, b) \cdot v \tag{1.20}$$

as scalar product of the tensor $(b\,a - a\,b)$ with v. If $\underline{a}$ and $\underline{b}$ are the coordinate matrices of a and b, respectively, in some vector base then the coordinate matrix of the tensor in this base is the skew-symmetric matrix

$$\underline{b}\underline{a}^T - \underline{a}\underline{b}^T = \begin{bmatrix} 0 & b_1 a_2 - b_2 a_1 & b_1 a_3 - b_3 a_1 \\ & 0 & b_2 a_3 - b_3 a_2 \\ \text{skew-symmetric} & & 0 \end{bmatrix}. \tag{1.21}$$

Also the single vector cross product $c \times v$ can be expressed as a scalar product of a tensor with v. For this purpose two vectors a and b are constructed which satisfy the equation $a \times b = c$. The tensor is then, again, $b\,a - a\,b$, and its coordinate matrix is given by Eq. (1.21). This matrix is seen to be identical with

$$\begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}$$

where c_1, c_2 and c_3 are the coordinates of c in the same base in which $\underline{a}$ and $\underline{b}$ are measured. For this matrix the symbol $\tilde{\underline{c}}$ (pronounced c tilde) is introduced so that the vector $c \times v$ has the coordinate matrix $\tilde{\underline{c}}\underline{v}$. This notation simplifies the transition from symbolic vector equations to scalar coordinate equations[1]. In order to be able to change formulations of coordinate matrix equations the following basic rules will be needed. If k is a scalar then

$$(\widetilde{k\underline{a}}) = k\tilde{\underline{a}}. \tag{1.22}$$

Furthermore,

$$(\widetilde{\underline{a}+\underline{b}}) = \tilde{\underline{a}} + \tilde{\underline{b}}. \tag{1.23}$$

$$\text{From } \tilde{\underline{a}} = \tilde{\underline{b}} \text{ follows } \underline{a} = \underline{b}. \tag{1.24}$$

The identity $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ yields

$$\tilde{\underline{a}}\underline{b} = -\tilde{\underline{b}}\underline{a} \tag{1.25}$$

and for the special case $\boldsymbol{a} = \boldsymbol{b}$

$$\tilde{\underline{a}}\underline{a} = \underline{0}. \tag{1.26}$$

With the help of the unit tensor $\boldsymbol{E}$ the double vector cross product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{v})$ can be written in the form

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{v}) = \boldsymbol{b}\,\boldsymbol{a}\cdot\boldsymbol{v} - \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{v} = (\boldsymbol{b}\,\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{E})\cdot\boldsymbol{v}. \tag{1.27}$$

The corresponding coordinate equation reads $\tilde{\underline{a}}\tilde{\underline{b}}\underline{v} = (\underline{b}\underline{a}^{\mathrm{T}} - \underline{a}^{\mathrm{T}}\underline{b}\,\underline{E})\underline{v}$ with the unit matrix $\underline{E}$. Since this equation holds for every $\underline{v}$ the identity

$$\tilde{\underline{a}}\tilde{\underline{b}} = \underline{b}\underline{a}^{\mathrm{T}} - \underline{a}^{\mathrm{T}}\underline{b}\,\underline{E} \tag{1.28}$$

is valid. According to Eq. (1.20) the coordinate matrix of $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{v}$ is $(\underline{b}\underline{a}^{\mathrm{T}} - \underline{a}\underline{b}^{\mathrm{T}})\underline{v}$. It can also be written in the form $(\widetilde{\tilde{\underline{a}}\underline{b}})\underline{v}$. Since both forms are identical for every $\underline{v}$ the identity

$$(\widetilde{\tilde{\underline{a}}\underline{b}}) = \underline{b}\underline{a}^{\mathrm{T}} - \underline{a}\underline{b}^{\mathrm{T}} \tag{1.29}$$

holds. Finally, the transformation rule for tensor coordinates (Eq. (1.19)) states that

$$\tilde{\underline{a}}^{(s)} = (\widetilde{\underline{A}^{sr}\underline{a}^{(r)}}) = \underline{A}^{sr}\tilde{\underline{a}}^{(r)}\underline{A}^{rs}.$$

Systems of linear vector equations can be written in a very compact form if, in addition to matrices with vectorial elements, *matrices with tensors as elements* are used. Such matrices are designated by underlined boldface grotesque letters. They have the general form

$$\underline{\boldsymbol{D}} = \begin{bmatrix} \boldsymbol{D}_{11} \dots \boldsymbol{D}_{1r} \\ \vdots \\ \boldsymbol{D}_{m1} \quad \boldsymbol{D}_{mr} \end{bmatrix}$$

with arbitrary numbers of rows and columns. The scalar product $\underline{\boldsymbol{D}}\cdot\underline{\boldsymbol{b}}$ of the $(m \times r)$ matrix $\underline{\boldsymbol{D}}$ from the right with a vectorial matrix $\underline{\boldsymbol{b}}$ whose elements are

[1] The notation $\tilde{\underline{c}}\underline{v}$ for the coordinates of $\boldsymbol{c} \times \boldsymbol{v}$ is equivalent to the notation $\varepsilon_{\alpha\beta\gamma}c_\beta v_\gamma$ $(\alpha = 1,2,3)$ which is commonly used in tensor algebra.

b_{ij} $(i=1\ldots r, j=1\ldots n)$ is defined as an $(m\times n)$ matrix with the elements

$$\sum_{k=1}^{r} \boldsymbol{D}_{ik}\cdot \boldsymbol{b}_{kj} \qquad i=1\ldots m, j=1\ldots n\,.$$

A similar definition holds for the scalar product of $\underline{\boldsymbol{D}}$ from the left with an $(n\times m)$ matrix with elements $\boldsymbol{b}_{ij}$ $(i=1\ldots n, j=1\ldots m)$. The following example illustrates the practical use of these notations. Suppose it is desired to write the scalar

$$c=\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{a}_i\cdot \boldsymbol{D}_{ij}\cdot \boldsymbol{b}_j$$

as a matrix product. This can be done in symbolic form, $c=\underline{\boldsymbol{a}}^{\mathrm{T}}\cdot\underline{\boldsymbol{D}}\cdot\underline{\boldsymbol{b}}$, with the factors

$$\underline{\boldsymbol{a}}=\begin{bmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_n\end{bmatrix},\qquad \underline{\boldsymbol{D}}=\begin{bmatrix}\boldsymbol{D}_{11}\ldots \boldsymbol{D}_{1n}\\ \vdots\\ \boldsymbol{D}_{n1}\quad \boldsymbol{D}_{nn}\end{bmatrix},\qquad \underline{\boldsymbol{b}}=\begin{bmatrix}\boldsymbol{b}_1\\ \vdots\\ \boldsymbol{b}_n\end{bmatrix}.$$

When it is desired to calculate c numerically the following expression in terms of coordinate matrices is more convenient. Let $\underline{a}_i, \underline{b}_i$ and $\underline{D}_{ij}$ $(i,j=1\ldots n)$ be the coordinate matrices of $\boldsymbol{a}_i$, $\boldsymbol{b}_i$ and $\boldsymbol{D}_{ij}$, respectively, in some common vector base. Then,

$$c=\sum_{i=1}^{n}\sum_{j=1}^{n} \underline{a}_i^{\mathrm{T}}\,\underline{D}_{ij}\underline{b}_j.$$

This can, in turn, be written as the matrix product $c=\underline{a}^{\mathrm{T}}\underline{D}\,\underline{b}$ where

$$\underline{a}=\begin{bmatrix}\underline{a}_1\\ \vdots\\ \underline{a}_n\end{bmatrix},\qquad \underline{D}=\begin{bmatrix}\underline{D}_{11}\ldots \underline{D}_{1n}\\ \vdots\\ \underline{D}_{n1}\quad \underline{D}_{nn}\end{bmatrix},\qquad \underline{b}=\begin{bmatrix}\underline{b}_1\\ \vdots\\ \underline{b}_n\end{bmatrix}$$

are partitioned scalar matrices with the submatrices $\underline{a}_i, \underline{D}_{ij}$ and $\underline{b}_i$, respectively.

Problems

1.1 Given is the direction cosine matrix $\underline{A}^{rs}$ relating the vector bases $\underline{\boldsymbol{e}}^{(r)}$ and $\underline{\boldsymbol{e}}^{(s)}$. Express the matrix products $\underline{\boldsymbol{e}}^{(r)}\cdot\underline{\boldsymbol{e}}^{(r)\mathrm{T}}$, $\underline{\boldsymbol{e}}^{(r)\mathrm{T}}\cdot\underline{\boldsymbol{e}}^{(r)}$, $\underline{\boldsymbol{e}}^{(r)}\times\underline{\boldsymbol{e}}^{(r)\mathrm{T}}$, $\underline{\boldsymbol{e}}^{(r)\mathrm{T}}\times\underline{\boldsymbol{e}}^{(r)}$, $\underline{\boldsymbol{e}}^{(s)}\cdot\underline{\boldsymbol{e}}^{(r)\mathrm{T}}$, $\underline{\boldsymbol{e}}^{(r)}\cdot\underline{\boldsymbol{e}}^{(s)\mathrm{T}}$ and $\underline{\boldsymbol{e}}^{(s)\mathrm{T}}\cdot\underline{\boldsymbol{e}}^{(r)}$ in terms of $\underline{A}^{rs}$ or of elements of $\underline{A}^{rs}$.

1.2 Let $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ be vectorial matrices and let $\underline{c}$ be a scalar matrix of such dimensions that the products $\underline{\boldsymbol{a}}\cdot\underline{c}\,\underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{a}}\times\underline{c}\,\underline{\boldsymbol{b}}$ exist. Show that the former product is identical with $\underline{\boldsymbol{a}}\,\underline{c}\cdot\underline{\boldsymbol{b}}$ and the latter with $\underline{\boldsymbol{a}}\,\underline{c}\times\underline{\boldsymbol{b}}$.

1.3 $\underline{\boldsymbol{e}}^{(r)}$ and $\underline{\boldsymbol{e}}^{(s)}=\underline{A}^{sr}\underline{\boldsymbol{e}}^{(r)}$ are two vector bases, and $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors whose coordinate matrices $\underline{a}^{(r)}$ and $\underline{b}^{(r)}$ in $\underline{\boldsymbol{e}}^{(r)}$ and $\underline{c}^{(s)}$ in $\underline{\boldsymbol{e}}^{(s)}$, respectively, are given. Furthermore, $\boldsymbol{D}$ is a tensor with the coordinate matrix $\underline{D}^{(s)}$ in $\underline{\boldsymbol{e}}^{(s)}$. Formulate in terms of $\underline{A}^{sr}$ and of the given coordinate matrices the scalars $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{c}\cdot\boldsymbol{D}\cdot\boldsymbol{a}$ and $\boldsymbol{c}\cdot\boldsymbol{b}\times\boldsymbol{D}\cdot\boldsymbol{c}$ as well as the coordinate matrices in $\underline{\boldsymbol{e}}^{(r)}$ of the vectors $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{a}\times\boldsymbol{c}$, $\boldsymbol{a}\times(\boldsymbol{c}\times\boldsymbol{b})$, $\boldsymbol{c}\times\boldsymbol{D}\cdot\boldsymbol{a}$ and $\boldsymbol{a}\times[(\boldsymbol{D}\cdot\boldsymbol{b})\times\boldsymbol{c}]$.

1.4 Rewrite the vector equations

$$\boldsymbol{a}_1=\boldsymbol{b}\times(\boldsymbol{v}_1\times\boldsymbol{b}+\boldsymbol{v}_2\times\boldsymbol{c})+\boldsymbol{d}\times\boldsymbol{v}_2$$
$$\boldsymbol{a}_2=\boldsymbol{c}\times(\boldsymbol{v}_1\times\boldsymbol{b}+\boldsymbol{v}_2\times\boldsymbol{c})-\boldsymbol{d}\times\boldsymbol{v}_1$$

in the form

$$\begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$$

with explicit expressions for the tensors $\boldsymbol{D}_{ij}$ $(i,j=1,2)$. How are $\boldsymbol{D}_{12}$ and $\boldsymbol{D}_{21}$ related to one another? In some vector base the vectors in the original equations have the coordinate matrices $\underline{a}_1, \underline{a}_2, \underline{v}_1, \underline{v}_2, \underline{b}, \underline{c}$ and $\underline{d}$, respectively. Write down the coordinate matrix equation

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \end{bmatrix} = \begin{bmatrix} \underline{D}_{11} & \underline{D}_{12} \\ \underline{D}_{21} & \underline{D}_{22} \end{bmatrix} \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{bmatrix}$$

giving explicit expressions for the (3×3) submatrices $\underline{D}_{ij}$ $(i,j=1,2)$. What can be said about the (6×6) matrix on the right hand side?

2 Rigid Body Kinematics

2.1 Generalized coordinates for the angular orientation of a rigid body

In order to specify the angular orientation of a rigid body in a vector base $\underline{e}^{(1)}$ it is sufficient to specify the angular orientation of a vector base $\underline{e}^{(2)}$ which is rigidly attached to the body. This can be done, for instance, by means of the direction cosine matrix:

$$\underline{e}^{(2)} = \underline{A}^{21}\,\underline{e}^{(1)}.$$

The nine elements of this matrix are generalized coordinates which describe the angular orientation of the body in the base $\underline{e}^{(1)}$. Between these coordinates there exist six constraint equations of the form

$$\sum_{\lambda=1}^{3} A^{21}_{\alpha\lambda} A^{21}_{\beta\lambda} = \delta_{\alpha\beta} \qquad \alpha, \beta = 1, 2, 3. \tag{2.1}$$

It is often inconvenient to work with nine coordinates and six constraint equations. There are several useful systems of three coordinates without constraint equations and of four coordinates with one constraint equation which can be used as alternatives to direction cosines. In the following subsections generalized coordinates known as Euler angles, Bryant angles and Euler parameters will be discussed.

2.1.1 Euler angles Z - X - Z

The angular orientation of the body-fixed base $\underline{e}^{(2)}$ is thought to be the result of three successive rotations. Before the first rotation the base $\underline{e}^{(2)}$ coincides with the base $\underline{e}^{(1)}$. The first rotation is carried out about the axis $e_3^{(1)}$ through an angle ψ. It carries the base from its original orientation to an orientation denoted $\underline{e}^{(2)''}$ (Fig. 2.1). The second

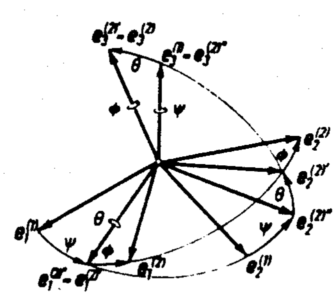

Fig. 2.1
Euler angles ψ, θ, ϕ

rotation through the angle θ about the axis $e_1^{(2)''}$ results in the orientation denoted $\underline{e}^{(2)'}$. The third rotation through the angle ϕ about the axis $e_3^{(2)'}$ produces the final orientation of the base. It is denoted $\underline{e}^{(2)}$ in Fig. 2.1. A characteristic property of Euler angles is that each rotation is carried out about a base vector of the body-fixed base in a position which is the result of all previous rotations. A further characteristic is the sequence (3,1,3) of indices of rotation axes. The desired presentation of the transformation matrix $\underline{A}^{21}$ in terms of ψ, θ and ϕ is found from the transformation equations for the individual rotations which are according to Fig. 2.1

$$\underline{e}^{(2)''} = \underline{A}^{\psi}\underline{e}^{(1)}, \qquad \underline{e}^{(2)'} = \underline{A}^{\theta}\underline{e}^{(2)''}, \qquad \underline{e}^{(2)} = \underline{A}^{\phi}\underline{e}^{(2)'}$$

with

$$\underline{A}^{\psi} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{A}^{\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad \underline{A}^{\phi} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that $\underline{A}^{21} = \underline{A}^{\phi}\underline{A}^{\theta}\underline{A}^{\psi}$ or explicitly with the abbreviations c_ψ, c_θ and c_ϕ for $\cos\psi$, $\cos\theta$ and $\cos\phi$ and s_ψ, s_θ and s_ϕ for $\sin\psi$, $\sin\theta$ and $\sin\phi$, respectively,

$$\underline{A}^{21} = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & s_\psi c_\phi + c_\psi c_\theta s_\phi & s_\theta s_\phi \\ -c_\psi s_\phi - s_\psi c_\theta c_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & s_\theta c_\phi \\ s_\psi s_\theta & -c_\psi s_\theta & c_\theta \end{bmatrix}. \tag{2.2}$$

The advantage of having only three coordinates without any constraint equation is paid for with the disadvantage that the direction cosines are complicated circular functions. There is still another problem. Fig. 2.1 shows that in the case $\theta = n\pi$ $(n = 0, 1, \ldots)$ the axes of the first and third rotation coincide so that ψ and ϕ cannot be distinguished. Euler angles can be illustrated by means of a rigid body in a two-gimbal cardanic suspension. The bases $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ are attached to the material base and to the

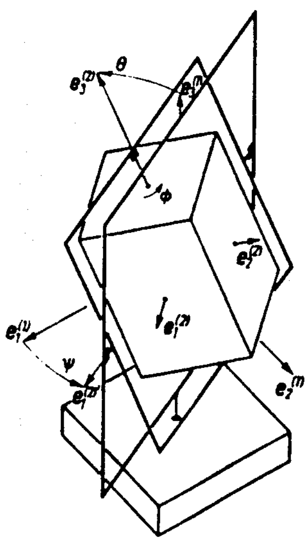

Fig. 2.2
Euler angles in a cardan suspension

suspended body, respectively, as shown in Fig. 2.2. The angles ψ, θ and ϕ are in this order the rotation angle of the outer gimbal relative to the material base, of the inner gimbal relative to the outer gimbal and of the body relative to the inner gimbal. For $\theta = n\pi$ $(n=0,1,\ldots)$ the two gimbals coincide (gimbal lock). With this device all three angles can be adjusted independently since the auxiliary vector bases $\underline{e}^{(2)''}$ and $\underline{e}^{(2)'}$ are materially realized by the gimbals. Euler angles owe their practical importance to the fact that there are many technical systems in which a rigid body is moving in such a way that θ is exactly or approximately constant and that, both, ψ and ϕ are exactly or approximately proportional to time, i.e. $\dot{\psi} \approx$ const and $\dot{\phi} \approx$ const. The use of Euler angles is advantageous also whenever there exist two axes of particular physical significance, one fixed in the base $\underline{e}^{(1)}$ and the other fixed on the body. In such cases the base vectors $e_3^{(1)}$ and $e_3^{(2)}$ are given these directions so that θ is the angle between the two axes (an example of this kind will be treated in Sec. 4.1.4). The use of Euler angles as generalized coordinates is not limited to such cases, however.

It is sometimes necessary to calculate Euler angles which correspond to a given matrix $\underline{A}^{21}$. For this purpose the following formulas are deduced from Eq. (2.2).

$$\begin{aligned} \cos\theta &= A_{33}^{21}, & \sin\theta &= \varepsilon\sqrt{1-\cos^2\theta}, \quad \varepsilon = +1 \text{ or } -1 \\ \cos\psi &= -\frac{A_{32}^{21}}{\sin\theta}, & \sin\psi &= \frac{A_{31}^{21}}{\sin\theta} \\ \cos\phi &= \frac{A_{23}^{21}}{\sin\theta}, & \sin\phi &= \frac{A_{13}^{21}}{\sin\theta}. \end{aligned} \tag{2.3}$$

The formulas show that numerical difficulties are to be expected for values of θ which are close to the critical values $n\pi$ $(n=0,1,\ldots)$.

2.1.2 Bryant angles X-Y-Z

These angles are also referred to as Cardan angles. The angular orientation of the body-fixed base $\underline{e}^{(2)}$ is, again, represented as the result of a sequence of three rotations at the beginning of which the base coincides with the reference base $\underline{e}^{(1)}$. The first rotation through an angle ϕ_1 is carried out about the axis $e_1^{(1)}$. It results in the auxiliary base $\underline{e}^{(2)''}$ (Fig. 2.3). The second rotation through an angle ϕ_2 about the axis $e_2^{(2)''}$

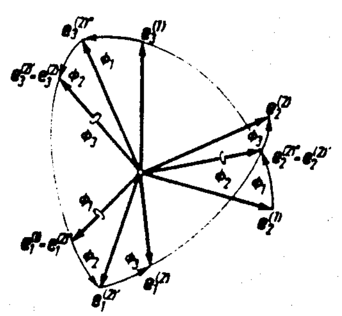

Fig. 2.3 Bryant angles ϕ_1, ϕ_2, ϕ_3

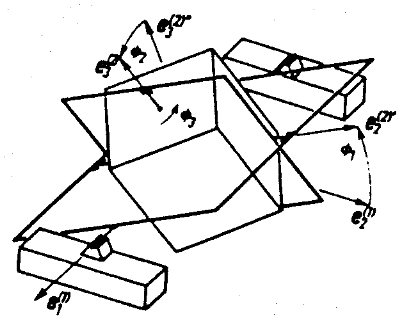

Fig. 2.4
Bryant angles in a cardan suspension

produces the base $\underline{e}^{(2)'}$. The third rotation through an angle ϕ_3 about the axis $e_3^{(2)'}$ gives the body-fixed base its final orientation denoted $\underline{e}^{(2)}$ in Fig. 2.3. The transformation equations for the individual rotations are

$$\underline{e}^{(2)''} = \underline{A}^1 \underline{e}^{(1)}, \qquad \underline{e}^{(2)'} = \underline{A}^2 \underline{e}^{(2)''}, \qquad \underline{e}^{(2)} = \underline{A}^3 \underline{e}^{(2)'}$$

with
$$\underline{A}^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{bmatrix}, \quad \underline{A}^2 = \begin{bmatrix} \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 1 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 \end{bmatrix},$$

$$\underline{A}^3 = \begin{bmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.4}$$

The direction cosine matrix relating $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ is the product $\underline{A}^{21} = \underline{A}^3 \underline{A}^2 \underline{A}^1$ or explicitly with the abbreviations $c_\alpha = \cos\phi_\alpha$, $s_\alpha = \sin\phi_\alpha$ $(\alpha = 1, 2, 3)$

$$\underline{A}^{21} = \begin{bmatrix} c_2 c_3 & c_1 s_3 + s_1 s_2 c_3 & s_1 s_3 - c_1 s_2 c_3 \\ -c_2 s_3 & c_1 c_3 - s_1 s_2 s_3 & s_1 c_3 + c_1 s_2 s_3 \\ s_2 & -s_1 c_2 & c_1 c_2 \end{bmatrix}. \tag{2.5}$$

The only significant difference as compared with Euler angles is the sequence (1, 2, 3) of indices of rotation axes. Bryant angles can also be illustrated by means of a rigid body in a two-gimbal cardanic suspension. The bases $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ are attached to the material base and to the body, respectively, as shown in Fig. 2.4. The angles ϕ_1, ϕ_2 and ϕ_3 are in this order the rotation angle of the outer gimbal relative to the material base, of the inner gimbal relative to the outer gimbal and of the body relative to the inner gimbal. For $\phi_2 = 0$ the three rotation axes are mutually orthogonal. As with Euler angles there exists a critical case, namely the case $\phi_2 = \pi/2 + n\pi$ $(n = 0, 1 \ldots)$ in which the planes of the two gimbals coincide so that the rotation axes of ϕ_1 and ϕ_3 become identical. In contrast to Euler angles no mathematical difficulties arise if all three angles ϕ_1, ϕ_2 and ϕ_3 are close to zero. For this reason Bryant angles are particularly useful in cases where a body is moving in such a way that the body-fixed base $\underline{e}^{(2)}$ deviates only little from $\underline{e}^{(1)}$. For sufficiently small angles the linear approximation $\sin\phi_\alpha \approx \phi_\alpha$, $\cos\phi_\alpha \approx 1$ $(\alpha = 1, 2, 3)$ yields

$$\underline{A}^{21} \approx \begin{bmatrix} 1 & \phi_3 & -\phi_2 \\ -\phi_3 & 1 & \phi_1 \\ \phi_2 & -\phi_1 & 1 \end{bmatrix}. \tag{2.6}$$

This expression suggests defining a vector ϕ with coordinates ϕ_1, ϕ_2 and ϕ_3 and writing $\underline{A}^{21} \approx \underline{E} - \tilde{\underline{\phi}}$. Note that it makes no difference whether ϕ_1, ϕ_2 and ϕ_3 are interpreted as coordinates of ϕ in the base $\underline{e}^{(1)}$ or in the base $\underline{e}^{(2)}$ or along the axes $e_1^{(1)}$, $e_2^{(2)''}$ and $e_3^{(2)'}$, respectively. This can be shown as follows. If $\underline{\phi} = [\phi_1 \;\; \phi_2 \;\; \phi_3]^T$ designates the coordinate matrix in $\underline{e}^{(2)}$ then the coordinate matrix in the base $\underline{e}^{(1)}$ is in the linear approximation $\underline{A}^{21}\underline{\phi} \approx (\underline{E} - \tilde{\underline{\phi}})\underline{\phi}$. Because of the identity $\tilde{\underline{\phi}}\underline{\phi} = \underline{0}$ (cf. Eq. (1.26)) this is identical with $\underline{\phi}$. The result of these considerations is that within linear approximations small rotation angles can be added like vectors.

It is sometimes necessary to calculate Bryant angles which correspond to a given direction cosine matrix $\underline{A}^{21}$. This can be done with the help of the formulas derived from Eq. (2.5)

$$\begin{aligned}
\sin\phi_2 &= A_{31}^{21}\,, & \cos\phi_2 &= \varepsilon\sqrt{1-\sin^2\phi_2}\,, \quad \varepsilon = +1 \text{ or } -1\\
\sin\phi_1 &= -\frac{A_{32}^{21}}{\cos\phi_2}\,, & \cos\phi_1 &= \frac{A_{33}^{21}}{\cos\phi_2}\\
\sin\phi_3 &= -\frac{A_{21}^{21}}{\cos\phi_2}\,, & \cos\phi_3 &= \frac{A_{11}^{21}}{\cos\phi_2}\,.
\end{aligned} \tag{2.7}$$

2.1.3 Euler parameters

The angular orientation of the body-fixed base $\underline{e}^{(2)}$ is considered to be the result of a single rotation at the beginning of which the base coincides with the reference base $\underline{e}^{(1)}$. The rotation is carried out clockwise through an angle χ about an axis which is defined by a unit vector $\boldsymbol{u}$. Euler's theorem (see Chap. 1) states that for any angular orientation of the base $\underline{e}^{(2)}$ to be described a real angle χ and a unit vector $\boldsymbol{u}$ with real coordinates exist. These coordinates are the same in both bases $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$. The following considerations are based upon the assumption that χ as well as $\boldsymbol{u}$ are given and that the direction cosine matrix $\underline{A}^{21}$ relating $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ is to be expressed as a function of these quantities. The function is found from a comparison of the coordinate matrices in the bases $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ of a body-fixed vector. In Fig. 2.5 this vector is shown in its positions $\boldsymbol{r}^*$ and $\boldsymbol{r}$ before and after the rotation, respectively. In both positions the vector lies on a circular cone whose axis is defined by the unit vector $\boldsymbol{u}$. Let $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$ be the coordinate matrices of $\boldsymbol{r}$ in $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$, respectively. Then, $\underline{r}^{(1)} = \underline{A}^{12}\underline{r}^{(2)}$ where $\underline{A}^{12}$ is the transpose of the matrix $\underline{A}^{21}$ under consideration. The coordinate matrix $\underline{r}^{(2)}$ is identical with the coordinate matrix $\underline{r}^{*(1)}$ of $\boldsymbol{r}^*$ in the base $\underline{e}^{(1)}$ since before the rotation $\underline{e}^{(2)}$ coincides with $\underline{e}^{(1)}$ and the body-fixed vector coincides with $\boldsymbol{r}^*$. Therefore,

$$\underline{r}^{(1)} = \underline{A}^{12}\underline{r}^{*(1)}. \tag{2.8}$$

According to Fig. 2.5 $\boldsymbol{r}$ and $\boldsymbol{r}^*$ are related by the equation $\boldsymbol{r} = \boldsymbol{r}^* + (1-\cos\chi)\boldsymbol{b} + \sin\chi\,\boldsymbol{a}$ or recognizing that $\boldsymbol{a}$ equals $\boldsymbol{u}\times\boldsymbol{r}^*$ and that $\boldsymbol{b}$ equals $\boldsymbol{u}\times\boldsymbol{a}$

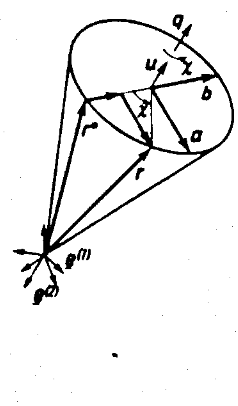

Fig. 2.5
Rotation of a body-fixed vector $\boldsymbol{r}$ about an axial unit vector $\boldsymbol{u}$

$$\boldsymbol{r} = \boldsymbol{r}^* + (1-\cos\chi)\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{r}^*) + \sin\chi\, \boldsymbol{u} \times \boldsymbol{r}^* . \tag{2.9}$$

By means of the relationships

$$1-\cos\chi = 2\sin^2\frac{\chi}{2}, \qquad \sin\chi = 2\sin\frac{\chi}{2}\cos\frac{\chi}{2} \tag{2.10}$$

the semi rotation angle is introduced. Now the new quantities

$$q_0 = \cos\frac{\chi}{2} \quad \text{and} \quad \boldsymbol{q} = \boldsymbol{u}\sin\frac{\chi}{2} \tag{2.11}$$

are defined. The vector $\boldsymbol{q}$ has equal coordinates in both bases $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ since $\boldsymbol{u}$ has this property. The coordinates of $\boldsymbol{q}$ are denoted q_1, q_2 and q_3. It is the four scalars q_0, q_1, q_2 and q_3 which are called **Euler parameters**. They satisfy the constraint equation

$$q_0^2 + \boldsymbol{q}\cdot\boldsymbol{q} = \cos^2\frac{\chi}{2} + \boldsymbol{u}\cdot\boldsymbol{u}\sin^2\frac{\chi}{2} = 1 .$$

Alternative formulations are

$$\left.\begin{aligned} &q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \\ \text{or}\quad &q_0^2 + \underline{q}^{\mathrm{T}}\underline{q} = 1 . \end{aligned}\right. \tag{2.12}$$

Mathematically speaking Euler parameters represent normalized quaternions. Together with Eqs. (2.11) and (2.10) Eq. (2.9) becomes

$$\boldsymbol{r} = \boldsymbol{r}^* + 2\boldsymbol{q} \times (\boldsymbol{q} \times \boldsymbol{r}^*) + 2q_0\boldsymbol{q} \times \boldsymbol{r}^* .$$

In the base $\underline{e}^{(1)}$ this yields the coordinate equation

$$\underline{r}^{(1)} = (\underline{E} + 2\tilde{\underline{q}}\tilde{\underline{q}} + 2q_0\tilde{\underline{q}})\underline{r}^{*(1)} .$$

Comparison with Eq. (2.8) shows that the term in brackets is the desired expression for the matrix $\underline{A}^{12}$. Applying Eq. (1.28) to the product $\tilde{\underline{q}}\tilde{\underline{q}}$ and using Eq. (2.12) the transpose $\underline{A}^{21}$ becomes

$$\underline{A}^{21} = (2q_0^2 - 1)\underline{E} + 2(\underline{q}\,\underline{q}^{\mathrm{T}} - q_0\tilde{\underline{q}}) \tag{2.13}$$

or explicitly

$$\underline{A}^{21} = \begin{bmatrix} 2(q_0^2+q_1^2)-1 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & 2(q_0^2+q_2^2)-1 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & 2(q_0^2+q_3^2)-1 \end{bmatrix} . \tag{2.14}$$

It is a simple matter to derive from this expression explicit formulas for the inverse problem in which the matrix $\underline{A}^{21}$ is given and the corresponding Euler parameters are to be determined. For the trace of $\underline{A}^{21}$ the relationship is found

$$\frac{\operatorname{tr}\underline{A}^{21} + 1}{2} = q_0^2 + q_1^2 + q_2^2 + q_3^2 + 2q_0^2 - 1 = 2q_0^2$$

and, therefore,

$$q_0^2 = \frac{\operatorname{tr} \underline{A}^{21} + 1}{4} . \tag{2.15}$$

Substitution of this expression into the formula for the diagonal elements, $A_{ii}^{21} = 2(q_0^2 + q_i^2) - 1$, results in

$$q_i^2 = \frac{A_{ii}^{21}}{2} - \frac{\operatorname{tr} \underline{A}^{21} - 1}{4} \qquad i = 1, 2, 3 . \tag{2.16}$$

In contrast to Euler and Bryant angles (and to any other set of three generalized coordinates) there is no critical case in which the right hand sides of these inverse formulas are singular.

Problems

2.1 The angular orientation of a body is described in terms of Euler angles. It is desired to convert ψ, θ and ϕ into equivalent Euler parameters. How can this be done?

2.2 Show that the rotation angle χ and the coordinates of the unit vector $\boldsymbol{u}$ of Fig. 2.5 are determined by the equations

$$\cos\chi = \frac{\operatorname{tr} \underline{A}^{21} - 1}{2} , \qquad u_i^2 = \frac{A_{ii}^{21} - \cos\chi}{1 - \cos\chi} \qquad i = 1, 2, 3 .$$

2.3 A reference base $\underline{e}^{(1)}$ and a body-fixed base $\underline{e}^{(2)}$ are initially coincident. The base $\underline{e}^{(2)}$ is then subject to a sequence of three rotations. It is rotated, first, through an angle ϕ_1 about the axis $e_1^{(1)}$, then through an angle ϕ_2 about $e_2^{(1)}$ and, finally, through an angle ϕ_3 about $e_3^{(1)}$. Note that in contrast to Bryant angles all three rotations are carried out about base vectors of the reference base $\underline{e}^{(1)}$. The matrix $\underline{A}^{21}$ relating the final orientation of $\underline{e}^{(2)}$ to $\underline{e}^{(1)}$ is a function of ϕ_1, ϕ_2 and ϕ_3. Find this function and evaluate it numerically for the three sets of angles $(\pi/2, \pi/2, \pi/2)$, $(0, \pi/2, 0)$ and (π, π, π). Check the results experimentally.

2.2 The notion of angular velocity

Let $\underline{e}^{(1)}$ be some arbitrarily moving base. Relative to this base a rigid body is in arbitrary motion. Fixed on this body there is a base $\underline{e}^{(2)}$ with origin P. Furthermore, a point Q is considered which is moving relative to the body. With the notations of Fig. 2.6

$$\boldsymbol{z} = \boldsymbol{z}_P + \boldsymbol{r} . \tag{2.17}$$

The goal of the present investigation is an expression for the velocity of Q relative to the base $\underline{e}^{(1)}$ in terms of the velocity of Q relative to the body, of the velocity of P relative to

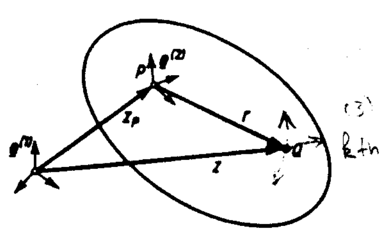

Fig. 2.6
Radius vectors of two points P (body-fixed) and Q (not body-fixed)

$$\dot{z} = \dot{z}_P + \omega \times r$$
$$\dot{z}_P = \dot{z} - \omega \times r$$

$\underline{e}^{(1)}$ and of some as yet unknown quantity which accounts for changes of the body angular orientation in the base $\underline{e}^{(1)}$. Velocities are represented as time derivatives of radius vectors. Since a point of a body has, in general, different velocities relative to different vector bases it must be specified in which base a radius vector is differentiated with respect to time. The same is true also for vectors which are not radius vectors. One frequently used notation for the time derivative of a vector c in a base $\underline{e}^{(n)}$ is ${}^{(n)}\mathrm{d}c/\mathrm{d}t$. This derivative is calculated from the equation

$$\frac{{}^{(n)}\mathrm{d}}{\mathrm{d}t}c = \sum_{\alpha=1}^{3}\frac{\mathrm{d}}{\mathrm{d}t}c_\alpha^{(n)}e_\alpha^{(n)}. \tag{2.18}$$

In words: The coordinates of ${}^{(n)}\mathrm{d}c/\mathrm{d}t$ in the base $\underline{e}^{(n)}$ are found by calculating the coordinates $c_\alpha^{(n)}$ ($\alpha=1,2,3$) of c in $e^{(n)}$ and by differentiating them with respect to time. The time derivative $\mathrm{d}c_\alpha^{(n)}/\mathrm{d}t$ of the scalar $c_\alpha^{(n)}$ is unambiguous.

After these preparatory remarks Eq. (2.17) is considered, again. The vector r is expressed in terms of its coordinates in $\underline{e}^{(2)}$, $r = \sum_{\alpha=1}^{3} r_\alpha^{(2)}e_\alpha^{(2)}$. This is substituted into Eq. (2.17), and then the entire equation is differentiated with respect to time in the base $\underline{e}^{(1)}$:

$$\frac{{}^{(1)}\mathrm{d}}{\mathrm{d}t}z = \frac{{}^{(1)}\mathrm{d}}{\mathrm{d}t}z_P + \sum_{\alpha=1}^{3}\frac{\mathrm{d}}{\mathrm{d}t}r_\alpha^{(2)}e_\alpha^{(2)} + \sum_{\alpha=1}^{3}r_\alpha^{(2)}\frac{{}^{(1)}\mathrm{d}}{\mathrm{d}t}e_\alpha^{(2)}. \tag{2.19}$$

The derivative on the left hand side is the velocity of Q relative to $\underline{e}^{(1)}$. It is called v. Similarly, the first term on the right hand side is the velocity v_P of the point P relative to $\underline{e}^{(1)}$. The second term on the right represents (according to Eq. (2.18)) the velocity ${}^{(2)}\mathrm{d}r/\mathrm{d}t$ of the point Q relative to $\underline{e}^{(2)}$. It is abbreviated v_{rel}. In the last term the derivatives ${}^{(1)}\mathrm{d}e_\alpha^{(2)}/\mathrm{d}t$ can be calculated from Eq. (2.18). Since the coordinates of $e_\alpha^{(2)}$ in the base $\underline{e}^{(1)}$ are the direction cosines $A_{\alpha\beta}^{21}$ ($\beta=1,2,3$) we get

$$\frac{{}^{(1)}\mathrm{d}}{\mathrm{d}t}e_\alpha^{(2)} = \sum_{\beta=1}^{3}\frac{\mathrm{d}}{\mathrm{d}t}A_{\alpha\beta}^{21}\,e_\beta^{(1)} \qquad \alpha=1,2,3. \tag{2.20}$$

This expression leads to very complicated formulas. Therefore, another approach is chosen. The derivatives can also be represented as linear combinations of base vectors of $\underline{e}^{(2)}$ with as yet unknown coefficients $c_{\alpha\beta}$:

$$\frac{{}^{(1)}\mathrm{d}}{\mathrm{d}t}e_\alpha^{(2)} = \sum_{\beta=1}^{3}c_{\alpha\beta}e_\beta^{(2)} \qquad \alpha=1,2,3. \tag{2.21}$$

The base vectors satisfy the relationship $e_\alpha^{(2)}\cdot e_\gamma^{(2)}=\delta_{\alpha\gamma}$ ($\alpha,\gamma=1,2,3$). Differentiation of this relationship with respect to time in the base $e^{(1)}$ leads to

$$\frac{{}^{(1)}\mathrm{d}}{\mathrm{d}t}e_\alpha^{(2)}\cdot e_\gamma^{(2)} + e_\alpha^{(2)}\cdot\frac{{}^{(1)}\mathrm{d}}{\mathrm{d}t}e_\gamma^{(2)} = 0 \qquad \alpha,\gamma=1,2,3.$$

When Eq. (2.21) is substituted for the derivatives one obtains $c_{\alpha\gamma}+c_{\gamma\alpha}=0$. This means that the matrix formed by the coefficients $c_{\alpha\beta}$ ($\alpha,\beta=1,2,3$) is skew-symmetric. Its three nonzero elements are given the new names $\omega_1=c_{23}=-c_{32}$, $\omega_2=-c_{13}=c_{31}$ and $\omega_3=c_{12}=-c_{21}$. These three quantities are interpreted as coordinates of a vector ω in the body-fixed base $\underline{e}^{(2)}$. Eq. (2.21) then becomes

$$\begin{aligned}
\frac{^{(1)}\mathrm{d}}{\mathrm{d}t}\,e_1^{(2)} &= \quad \omega_3 e_2^{(2)} - \omega_2 e_3^{(2)} = \omega \times e_1^{(2)} \\
\frac{^{(1)}\mathrm{d}}{\mathrm{d}t}\,e_2^{(2)} &= -\omega_3 e_1^{(2)} + \omega_1 e_3^{(2)} = \omega \times e_2^{(2)} \\
\frac{^{(1)}\mathrm{d}}{\mathrm{d}t}\,e_3^{(2)} &= \quad \omega_2 e_1^{(2)} - \omega_1 e_2^{(2)} = \omega \times e_3^{(2)} .
\end{aligned}$$

With this the third term in Eq. (2.19) takes the simple form $\omega \times r$, and the basic relationship is obtained

$$v = v_P + v_{\mathrm{rel}} + \omega \times r . \tag{2.22}$$

The vector ω is referred to as angular velocity of the body relative to the base $\underline{e}^{(1)}$. It has some important properties two of which will now be discussed. From the identity of the right hand side expressions of Eqs. (2.20) and (2.21) follows that the coefficients $c_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ and, hence, also ω depend on direction cosines and on time derivatives of direction cosines only. This means that ω (in contrast to v_P) is independent of the choice of the body-fixed point P in Fig. 2.6 because the direction cosine matrix $\underline{A}^{21}$ is independent of it.

In order to find another important property of ω it is necessary to investigate, first, the relationship which exists between the time derivatives of one and the same vector in two different vector bases. Let $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$ be two vector bases which move relative to each other in an arbitrary way and let c be some vector (not necessarily a radius vector). In the base $\underline{e}^{(2)}$ the vector c has coordinates $c_\alpha^{(2)}$ $(\alpha = 1, 2, 3)$ so that $c = \sum_{\alpha=1}^{3} c_\alpha^{(2)} e_\alpha^{(2)}$. This equation is differentiated with respect to time in the base $\underline{e}^{(1)}$:

$$\frac{^{(1)}\mathrm{d}}{\mathrm{d}t}\,c = \sum_{\alpha=1}^{3} \frac{\mathrm{d}}{\mathrm{d}t}\,c_\alpha^{(2)} e_\alpha^{(2)} + \sum_{\alpha=1}^{3} c_\alpha^{(2)} \frac{^{(1)}\mathrm{d}}{\mathrm{d}t}\,e_\alpha^{(2)} .$$

The terms on the right hand side have the same form as the last two terms in Eq. (2.19). Using the same arguments as before one gets

$$\frac{^{(1)}\mathrm{d}}{\mathrm{d}t}\,c = \frac{^{(2)}\mathrm{d}}{\mathrm{d}t}\,c + \omega \times c . \tag{2.23}$$

This is the desired general relationship between the time derivatives of an arbitrary vector in two different bases. The vector ω represents the angular velocity of the base $\underline{e}^{(2)}$ relative to $\underline{e}^{(1)}$. For the vector ω itself the equation has the special form

$$\frac{^{(1)}\mathrm{d}}{\mathrm{d}t}\,\omega = \frac{^{(2)}\mathrm{d}}{\mathrm{d}t}\,\omega . \tag{2.24}$$

This is the second important property of the angular velocity which is mentioned here. We now return to Eq. (2.22). If only body-fixed points Q are considered the equation takes the special form

$$\boldsymbol{v} = \boldsymbol{v}_P + \boldsymbol{\omega} \times \boldsymbol{r}\,. \tag{2.25}$$

It describes the **velocity distribution** of a rigid body. All points along the straight line parallel to $\boldsymbol{\omega}$ and passing through P have the same velocity $\boldsymbol{v}_P$ since for these points $\boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{0}$. The velocity distribution can, therefore, be interpreted as the result of a superposition of two separate motions. One is a pure translation with the velocity $\boldsymbol{v}_P$ of the point P, and the other is a pure rotation with the angular velocity $\boldsymbol{\omega}$ about an axis which has the direction of $\boldsymbol{\omega}$ and which passes through P. This interpretation holds for any arbitrarily chosen body-fixed point P. It becomes particularly simple if a point P^* is chosen whose velocity $\boldsymbol{v}_P^*$ has the same direction as $\boldsymbol{\omega}$. The pure translation then has the direction of the axis of the pure rotation. Thus, the velocity distribution of the body is the same as that of a **screw**. It remains to be shown that in the case $\boldsymbol{\omega} \neq \boldsymbol{0}$ there exists a unique **screw axis** (in the trivial case $\boldsymbol{\omega} = \boldsymbol{0}$ the motion is a pure translation). It is assumed that the velocity $\boldsymbol{v}_P$ is known for some point P of the body. Let $\boldsymbol{\varrho}$ be the vector from P to a point on the screw axis. Then, $\boldsymbol{v}_P^* = \boldsymbol{v}_P + \boldsymbol{\omega} \times \boldsymbol{\varrho}$ is the velocity of this point and by definition of the screw axis it is parallel to $\boldsymbol{\omega}$ so that cross multiplication with $\boldsymbol{\omega}$ yields $\boldsymbol{0} = \boldsymbol{\omega} \times \boldsymbol{v}_P + \boldsymbol{\omega} \cdot \boldsymbol{\varrho}\,\boldsymbol{\omega} - \omega^2 \boldsymbol{\varrho}$. Since the direction of the screw axis is known it suffices to determine a single point on the axis. We choose the particular point P^* for which $\boldsymbol{\omega} \cdot \boldsymbol{\varrho}^*$ is zero. The radius vector $\boldsymbol{\varrho}^*$ for this point is $\boldsymbol{\varrho}^* = \boldsymbol{\omega} \times \boldsymbol{v}_P / \omega^2$. It is, indeed, uniquely defined if $\boldsymbol{\omega}$ is different from zero.

In general, the location of the screw axis in the body varies with time. Of particular interest is the case in which one point of the body is fixed in the reference base $\underline{e}^{(1)}$. The screw axis then degenerates to an **instantaneous axis of rotation** which is always passing through the fixed point. In the course of time this axis with the direction of $\boldsymbol{\omega}(t)$ is sweeping out two cones one of which is fixed in the body and the other in the base $\underline{e}^{(1)}$ (Fig. 2.7). At time t the two cones share the instantaneous axis as a

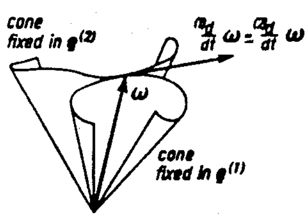

Fig. 2.7
The cones generated by $\boldsymbol{\omega}$ in two bases $\underline{e}^{(1)}$ and $\underline{e}^{(2)}$, $\boldsymbol{\omega}$ being the angular velocity of $\underline{e}^{(2)}$ relative to $\underline{e}^{(1)}$

common generating line. That the cones also have a common tangential plane at time t is a consequence of Eq. (2.24) which states that $\boldsymbol{\omega}(t)$ is sweeping out both cones with equal rates of change. Summarizing all these facts *the general motion of a rigid body relative to a base $\underline{e}^{(1)}$ with a point fixed in this base can be interpreted as rolling motion without slipping of a body-fixed cone on a cone fixed in the base $\underline{e}^{(1)}$.*

Problems

2.4 A body has an angular velocity $\boldsymbol{\omega} \neq \boldsymbol{0}$, and a point P of the body has a velocity $\boldsymbol{v}_P \neq \boldsymbol{0}$, both measured relative to the same reference base. What is the condition for the existence of body points with zero velocity and where are these points located?

2.5 On a rigid body three non-collinear points are defined by their radius vectors $\boldsymbol{r}_1, \boldsymbol{r}_2$ and $\boldsymbol{r}_3$. The velocities $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ of these points relative to a base $\underline{e}^{(1)}$ are known. Show that the angular velocity of the body relative to the same base is

$$\boldsymbol{\omega} = 2\frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2 + \boldsymbol{v}_2 \times \boldsymbol{v}_3 + \boldsymbol{v}_3 \times \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_3) + \boldsymbol{v}_2 \cdot (\boldsymbol{r}_3 - \boldsymbol{r}_1) + \boldsymbol{v}_3 \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2)}.$$

2.3 Relationships between the angular velocity of a body and generalized coordinates describing the angular orientation of the body

The angular velocity of a body cannot, in general, be represented as the time derivative of another vector (this is possible only in the trivial case where the direction of $\boldsymbol{\omega}$ in the body is constant). The coordinates ω_1, ω_2 and ω_3 of $\boldsymbol{\omega}$ in a body-fixed vector base do not, therefore, represent generalized velocities in the sense of analytic mechanics. From this follows that generalized coordinates for the angular orientation of a body cannot be determined from $\omega_\alpha(t)$ $(\alpha = 1,2,3)$ by simple integration. Instead, differential equations must be solved in which $\omega_\alpha(t)$ $(\alpha = 1,2,3)$ appear as variable coefficients. These equations will now be formulated for direction cosines, Euler angles, Bryant angles and Euler parameters as generalized coordinates.

2.3.1 Direction cosines

Let $\boldsymbol{\omega}$ be the angular velocity of a body-fixed base $\underline{e}^{(2)}$ relative to another base $\underline{e}^{(1)}$ and let $\boldsymbol{r}$ be a body-fixed vector with a constant coordinate matrix $\underline{r}^{(2)}$ in $\underline{e}^{(2)}$. The time varying coordinate matrix of $\boldsymbol{r}$ in $\underline{e}^{(1)}$ is then $\underline{r}^{(1)}(t) = \underline{A}^{12}(t)\underline{r}^{(2)}$ where $\underline{A}^{12}(t)$ is the direction cosine matrix relating the two bases. The time derivative of $\underline{r}^{(1)}(t)$ is

$$\dot{\underline{r}}^{(1)} = \dot{\underline{A}}^{12}\underline{r}^{(2)} \tag{2.26}$$

(here and in the remainder of this chapter time derivatives of scalars are designated by a dot). The same quantity is obtained by decomposing the vector ${}^{(1)}\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{\omega} \times \boldsymbol{r}$ in the base $\underline{e}^{(1)}$. This gives $\dot{\underline{r}}^{(1)} = \underline{A}^{12}\tilde{\underline{\omega}}^{(2)}\underline{r}^{(2)}$. Comparison with Eq. (2.26) yields $\dot{\underline{A}}^{12}\underline{r}^{(2)} = \underline{A}^{12}\tilde{\underline{\omega}}^{(2)}\underline{r}^{(2)}$. Since this holds for any coordinate matrix $\underline{r}^{(2)}$ the factors in front must be identical. Omitting the superscript of $\underline{\omega}^{(2)}$ and taking the transpose of either side one obtains

$$\dot{\underline{A}}^{21} = -\tilde{\underline{\omega}}\underline{A}^{21}. \tag{2.27}$$

These are, in matrix form, the desired differential equations for the nine direction cosines. They are known as Poisson's equations. For the individual elements of $\underline{A}^{21}$ they read

$$\dot{A}^{21}_{11} = \omega_3 A^{21}_{21} - \omega_2 A^{21}_{31} \quad \text{etc.}$$

Because of the six constraint equations (2.1) only three differential equations need be integrated.

2.3.2 Euler angles

From Fig. 2.1 the angular velocity ω of the base $\underline{e}^{(2)}$ relative to $\underline{e}^{(1)}$ is seen to be

$$\omega = \dot{\psi}\, e_3^{(1)} + \dot{\theta}\, e_1^{(2)\prime} + \dot{\phi}\, e_3^{(2)}.$$

Decomposition in $\underline{e}^{(2)}$ yields the coordinate equations

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \sin\theta \sin\phi & \cos\phi & 0 \\ \sin\theta \cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}. \tag{2.28}$$

Their explicit solutions for $\dot{\psi}, \dot{\theta}$ and $\dot{\phi}$ read

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\phi}{\sin\theta} & \dfrac{\cos\phi}{\sin\theta} & 0 \\ \cos\phi & -\sin\phi & 0 \\ -\sin\phi \,\mathrm{ctan}\,\theta & -\cos\phi \,\mathrm{ctan}\,\theta & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{2.29}$$

These are the desired kinematic differential equations. They show, again, that numerical problems will arise if θ is close to the critical values $n\pi$ $(n=0,1,\ldots)$.

2.3.3 Bryant angles

Fig. 2.3 yields

$$\omega = \dot{\phi}_1 e_1^{(1)} + \dot{\phi}_2 e_2^{(2)\prime} + \dot{\phi}_3 e_3^{(2)}. \tag{2.30}$$

Decomposition in $\underline{e}^{(2)}$ leads to the coordinate equations

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \cos\phi_2 \cos\phi_3 & \sin\phi_3 & 0 \\ -\cos\phi_2 \sin\phi_3 & \cos\phi_3 & 0 \\ \sin\phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix}. \tag{2.31}$$

Their solutions for $\dot{\phi}_1, \dot{\phi}_2$ and $\dot{\phi}_3$ read

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\cos\phi_3}{\cos\phi_2} & -\dfrac{\sin\phi_3}{\cos\phi_2} & 0 \\ \sin\phi_3 & \cos\phi_3 & 0 \\ -\cos\phi_3 \tan\phi_2 & \sin\phi_3 \tan\phi_2 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{2.32}$$

These are the kinematic differential equations for Bryant angles. They fail numerically in the vicinity of the critical values $\phi_2 = \pi/2 + n\pi$ $(n=0,1,\ldots)$. In connection with Eq. (2.6) it has been shown that for small angles ϕ_1, ϕ_2 and ϕ_3 the orientation of the body in the base $\underline{e}^{(1)}$ can, within linear approximations, be characterized by a rotation vector $\boldsymbol{\phi}$ whose coordinates are ϕ_1, ϕ_2 and ϕ_3. From Eq. (2.31) follow the linear approximations $\omega_1 \approx \dot{\phi}_1$, $\omega_2 \approx \dot{\phi}_2$ and $\omega_3 \approx \dot{\phi}_3$. The angular orientation of the body is, therefore, found by simple integration:

$$\phi_\alpha(t) \approx \int \omega_\alpha(t)\,\mathrm{d}t \qquad \alpha = 1, 2, 3\ (\phi_1, \phi_2, \phi_3 \text{ small}).$$

These approximation formulas for small angular displacements are often used in technical problems.

2.3.4 Euler parameters

In Poisson's equations (2.27) which can be written in the form $\underline{\tilde{\omega}} = -\underline{\dot{A}}^{21}\underline{A}^{12}$ the matrices $\underline{A}^{12}$ and $\underline{\dot{A}}^{21}$ are replaced by the transpose of the expression in Eq. (2.13) and by its time derivative, respectively:

$$\underline{A}^{12} = (2q_0^2 - 1)\underline{E} + 2(\underline{q}\underline{q}^{\mathrm{T}} + q_0\underline{\tilde{q}})$$
$$\underline{\dot{A}}^{21} = 2(2q_0\dot{q}_0\underline{E} + \underline{\dot{q}}\underline{q}^{\mathrm{T}} + \underline{q}\underline{\dot{q}}^{\mathrm{T}} - \dot{q}_0\underline{\tilde{q}} - q_0\underline{\dot{\tilde{q}}}).$$

This yields

$$-\frac{\underline{\tilde{\omega}}}{2} = 2q_0\dot{q}_0(2q_0^2 - 1)\underline{E} + 4q_0\dot{q}_0(\underline{q}\underline{q}^{\mathrm{T}} + q_0\underline{\tilde{q}}) + \\ + (2q_0^2 - 1)(\underline{\dot{q}}\underline{q}^{\mathrm{T}} + \underline{q}\underline{\dot{q}}^{\mathrm{T}} - \dot{q}_0\underline{\tilde{q}} - q_0\underline{\dot{\tilde{q}}}) + \\ + 2(\underline{\dot{q}}\underline{q}^{\mathrm{T}} + \underline{q}\underline{\dot{q}}^{\mathrm{T}} - \dot{q}_0\underline{\tilde{q}} - q_0\underline{\dot{\tilde{q}}})(\underline{q}\underline{q}^{\mathrm{T}} + q_0\underline{\tilde{q}}). \tag{2.33}$$

In simplifying this expression the constraint equation (2.12) and its time derivative will be used:

$$\underline{q}^{\mathrm{T}}\underline{q} = 1 - q_0^2, \qquad \underline{\dot{q}}^{\mathrm{T}}\underline{q} = \underline{q}^{\mathrm{T}}\underline{\dot{q}} = -q_0\dot{q}_0.$$

Also Eqs. (1.22) to (1.29) will be applied. The product of the last two expressions in brackets in Eq. (2.33) contains, among others, the following terms

$$\begin{aligned} &\underline{\dot{q}}\underline{q}^{\mathrm{T}}\underline{q}\underline{q}^{\mathrm{T}} = (1 - q_0^2)\underline{\dot{q}}\underline{q}^{\mathrm{T}}, &&\underline{\dot{q}}\underline{q}^{\mathrm{T}}q_0\underline{\tilde{q}} = \underline{0} \\ &\underline{q}\underline{\dot{q}}^{\mathrm{T}}\underline{q}\underline{q}^{\mathrm{T}} = -q_0\dot{q}_0\underline{q}\underline{q}^{\mathrm{T}}, &&\dot{q}_0\underline{\tilde{q}}\underline{q}\underline{q}^{\mathrm{T}} = \underline{0} \\ &\dot{q}_0\underline{\tilde{q}}q_0\underline{\tilde{q}} = \dot{q}_0q_0[\underline{q}\underline{q}^{\mathrm{T}} - (1 - q_0^2)\underline{E}], &&q_0^2\underline{\dot{\tilde{q}}}\underline{\tilde{q}} = q_0^2(\underline{q}\underline{\dot{q}}^{\mathrm{T}} + q_0\dot{q}_0\underline{E}). \end{aligned}$$

With them Eq. (2.33) can be rewritten in the form

$$-\frac{\underline{\tilde{\omega}}}{2} = \underline{\dot{q}}\underline{q}^{\mathrm{T}} - \underline{q}\underline{\dot{q}}^{\mathrm{T}} + \dot{q}_0\underline{\tilde{q}} + q_0\underline{\dot{\tilde{q}}} + 2q_0[(q_0\dot{q}_0\underline{E} + \underline{q}\underline{\dot{q}}^{\mathrm{T}})\underline{\tilde{q}}\underline{\dot{\tilde{q}}}(q_0^2\underline{E} + \underline{q}\underline{q}^{\mathrm{T}})].$$

In this equation the identities

$$q_0\dot{q}_0\underline{E} + \underline{q}\underline{\dot{q}}^{\mathrm{T}} = \underline{\dot{\tilde{q}}}\underline{\tilde{q}}, \qquad q_0^2\underline{E} + \underline{q}\underline{q}^{\mathrm{T}} = \underline{E} + \underline{\tilde{q}}\underline{\tilde{q}}$$

are used. They reduce the expression in square brackets to $-\underline{\dot{\tilde{q}}}$. This yields

$$-\frac{\underline{\tilde{\omega}}}{2} = \underline{\dot{q}}\underline{q}^{\mathrm{T}} - \underline{q}\underline{\dot{q}}^{\mathrm{T}} + \dot{q}_0\underline{\tilde{q}} - q_0\underline{\dot{\tilde{q}}}$$

or with Eq. (1.29)

$$\underline{\tilde{\omega}} = -2[(\widetilde{\underline{\tilde{q}}\underline{\dot{q}}}) + \dot{q}_0\underline{\tilde{q}} - q_0\underline{\dot{\tilde{q}}}]$$

so that $\quad \underline{\omega} = -2(\underline{\tilde{q}}\underline{\dot{q}} + \dot{q}_0\underline{q} - q_0\underline{\dot{q}}).$

Thus, the coordinates of $\boldsymbol{\omega}$ in the base $\underline{e}^{(2)}$ are linear combinations of q_i as well as of

$\dot{q}_i\,(i=0\dots3)$. In explicit form the equations represent the last three rows of the matrix equation

$$\begin{bmatrix}0\\ \omega_1\\ \omega_2\\ \omega_3\end{bmatrix}=2\begin{bmatrix}q_0 & q_1 & q_2 & q_3\\ -q_1 & q_0 & q_3 & -q_2\\ -q_2 & -q_3 & q_0 & q_1\\ -q_3 & q_2 & -q_1 & q_0\end{bmatrix}\begin{bmatrix}\dot{q}_0\\ \dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix}.$$

The first row is the time derivative of the constraint equation. It is added in order to produce a square coefficient matrix. Because of the constraint equation this matrix is orthogonal and normalized to unity. Its inverse equals, therefore, its transpose. Solving for $\dot{q}_i\,(i=0\dots3)$ and rearranging the right hand side results in

$$\begin{bmatrix}\dot{q}_0\\ \dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix}=\frac{1}{2}\left[\begin{array}{c|c}0 & -\boldsymbol{\omega}^{\mathrm{T}}\\ \hline \boldsymbol{\omega} & -\tilde{\boldsymbol{\omega}}\end{array}\right]\begin{bmatrix}q_0\\ q_1\\ q_2\\ q_3\end{bmatrix}. \tag{2.34}$$

These are the desired kinematic differential equations for Euler parameters. In numerical calculations all four equations are integrated. The constraint equation is used for correcting round-off errors. When after a few integration steps values for $q_i\,(i=0\dots3)$ are obtained which do not strictly satisfy the constraint equation then the calculation is continued not with q_i but with the corrected values

$$q_i^*=q_i\left(\sum_{j=0}^{3}q_j^2\right)^{-1/2}\qquad(i=0\dots3).$$

Problems

2.6 Develop the above correction formula for $q_i^*\,(i=0\dots3)$ from the condition that the sum of squares of the corrections, i.e. the sum $\sum_{i=0}^{3}(q_i^*-q_i)^2$, is a minimum.

2.7 A rigid body is suspended in two gimbals as shown in Fig. 2.8. In the outer gimbal the two axes are offset from 90° by an angle α and in the inner gimbal by an angle β. Let ϕ_1, ϕ_2 and ϕ_3 be defined like Bryant angles, i.e. as rotation angles of the outer gimbal about $\boldsymbol{e}_1^{(1)}$, of the inner gimbal relative to the outer gimbal and of the body relative to the inner gimbal, respectively. For $\phi_1=\phi_2=\phi_3=0$ the planes of the gimbals are perpendicular to one another and, furthermore, the base vectors $\boldsymbol{e}_1^{(1)}$ and $\boldsymbol{e}_2^{(1)}$ of the reference base as well as the body-fixed base vector $\boldsymbol{e}_1^{(2)}$ lie in the plane of the outer gimbal. Develop an expression for the direction cosine matrix $\underline{A}^{21}$ and kinematic differential equations similar to Eq. (2.32).

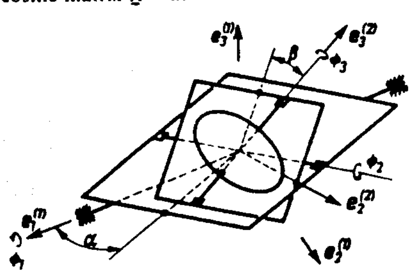

Fig. 2.8
Two-gimbal suspension with non-orthogonal gimbal axes