Part II

CLOSED-CHAIN DYNAMICS FOR MULTIPLE ROBOT ARMS MOVING A COMMON OBJECT

ABSTRACT

The forward dynamics problem for a closed-chain system comprised of ℓ manipulators grasping a commonly held object is analyzed and solved. It is shown that this solution can be implemented either by using analytical expressions for the dynamics of each free arm, or by using any available open-chain forward dynamics algorithm which can be applied to each free arm. In particular, a recursive solution is presented based on an operator interpretation of the solution combined with an operator factorization and inversion of arm mass matrices. The resulting algorithm is shown to be but one of many possible recursive algorithms derivable from the operator approach applied to the analytical expression for the solution of the closed-chain forward dynamics problem. These algorithms require $O(n) + O(\ell^3)$ operations, where n is the total number of manipulator links in the closed chain system. The factorization approach for the closed-chain system can be built up modularly from the forward dynamics solution for each arm of the open-chain system. In the main body of the paper, the grasps are taken to be rigid attachments, in the sense that there are no contact degrees of freedom left unconstrained by the grasps. In an appendix, nonrigid grasps for which there are a prescribed number of unconstrained contact degrees of freedom are considered.

1. INTRODUCTION

The ability to simulate dynamical systems comprised of closed-chains of interconnected rigid links is important for understanding issues such as bipedal or multipedal motion, or multiple arm cooperative grasp and manipulation of objects. The ability to efficiently obtain joint accelerations from known driving moments — to solve the forward dynamics problem — is an important step toward making such simulations possible at a reasonable computational cost. The solution of the forward dynamics problem can also provide a thorough mathematical and physical understanding of the way multiple arms behave dynamically in response to applied moments. This understanding is essential for designing effective control algorithms and for searching for the sources of anomalous behavior when it occurs.

Several recursive approaches to the closed-chain forward dynamics problem have been proposed in [1-3] and [4-6]. Recursive approaches have the feature that they can be implemented having knowledge only of individual link mass and geometric properties, and of the nature of the link interconnections (joints). The need to have analytical expressions describing the complete manipulator dynamics is avoided. This makes it relatively easy to retarget the forward dynamics algorithms, and the corresponding software implementation, from any given arm to another.

In [1,2], a closed-chain forward dynamics algorithm is given based on the open-chain forward dynamics algorithms of [3], which utilize repeated applications of the recursive Newton-Euler algorithm [7,8]. The closed-chain algorithm of [1,2] is of complexity $O(\ell^3 N^3) = O(n^3)$, where ℓ gives the number of arms which grasp a commonly held object and the total number of manipulator links is $n = \ell N$, assuming that each arm has N links (N degrees of freedom). ℓ is also proportional to the number of independent closed loops of the system considered in [1,2]. (In fact, the number of such loops is just $\ell - 1$.) A strong point of the algorithm [1,2] is its conceptual simplicity.

In [4], a solution to the closed-chain forward dynamics problem is outlined. This solution is of complexity $O(n) + O(\ell^3)$, where n is the total number of links in a closed-chain mechanical system and ℓ is the total number of independent loops. The work of [7] builds on [4] to obtain an iterative algorithm of complexity no greater than $O(n) + O(\ell^2)$ and, for some special cases, at times as low as $O(n) + O(\ell \, log\ell)$. The algorithms in [4,5] are developed using the nonstandard Featherstone Spatial Calculus (which is based on Motor and Screw Calculus). The need to learn the Featherstone Spatial Calculus makes these algorithms less accessible than algorithms such as those in [1,2] which are based on the classical (Gibbsian) vector calculus.

In this paper, forward dynamics algorithms for a closed-chain system formed by several manipulators holding a common object are derived. An approach is taken which, before specializing to algorithms based on the operator approach and iterative techniques of [6,9-12], potentially allows any recursive open-chain forward dynamics algorithm to be used - e.g., those of [3] or [4]. In fact, the closed-chain forward dynamics problem is expressed in a form which allows for the use of closed-form analytical expressions for the dynamics of each arm, if and when these expressions are available.

The recursive algorithms given here are of complexity $O(n) + O(\ell^3)$, where ℓ is the number of arms and n is the total number of arm links in the system. These algorithms are based on the spatial dynamics of [6,10], and on the operator formulation of robot dynamics and the operator mass matrix factorization method of [9,11]. The development is done in the framework of classical mechanics, avoiding the need to work with the nonstandard spatial algebra of [4], and uses concepts of operator factorization and inversion which are well-known in the context of filtering and estimation theory [13,14].

As discussed in [9], the operator formulation and approach to robot dynamics is a powerful one with a wide range of applications. This arises from an ability to give an operator interpretation to many analytical expressions which can then be directly mapped to equivalent recursive algorithms. The operator interpretation also enhances the ability to manipulate abstract analytical expressions by the use of operator identities which are derived in [9]. Although these other applications are not directly addressed here, note, for example, the straightforward way that a recursive construction of $J\mathcal{M}^{-1}J^T$, with J a manipulator Jacobian and $\mathcal{M}$ a manipulator mass matrix, is obtained in Sec. 6. $[J\mathcal{M}^{-1}J^T]^{-1}$ is precisely the operational space manipulator mass matrix of [15], and in a similar way, recursions for computing other terms associated with the operational space formulation of [15] can be derived.

2. PROBLEM FORMULATION AND STATEMENT

The spatial notation and quantities of [9] are used throughout this paper, and [9] is therefore a fundamental reference for this paper. Results from [9] involving operator factorizations, inversions, and identities are used without proof. As in [9], the algorithms of this paper are given in coordinate free form. In actual applications, the algorithms can be projected onto fixed-link frames as is done in [3,7]. Table 2.1 provides definitions for many of the variables used in here.

The system of interest consists of ℓ rigid-link, possibly redundant, manipulators grasping a commonly held rigid object in Euclidean 3-space. This is the same setup considered in [2] and Fig. 2.1 represents this situation. The grasp contact points are at $O = col(O_1, \ldots, O_\ell)$, and (in the body of this paper) the grasps are taken to be completely rigid, in the sense that no contact degrees of freedom are left unconstrained [16]. The rigid-grasp assumption is made to reduce notational clutter and is relaxed in Appendix B, where a variety of possible grasp contacts is considered. The set of admissible contacts used in Appendix B is given in Appendix A. The effect of gravity loading is ignored with no loss of generality as gravity can be dealt with in the standard way [7,9] by giving an appropriate pseudo- spatial acceleration to the manipulator base (although care must be exercised to separate true spatial accelerations from pseudo-spatial accelerations). With the notation of [9], and the quantities defined in Table 2.1, the dynamics of a single N-link arm i, with $i = 1, \ldots, \ell$, is given by

$$\mathcal{M}_i\ddot{\theta}_i + \mathcal{C}_i + J_i^T f_i(0) = T_i \tag{2.1a}$$

$$X_i(0) = Q_i(\theta_i) \quad \text{is 6-dimensional} \tag{2.1b}$$

$$V_i(0) = J_i\dot{\theta}_i \in R^6 \tag{2.1c}$$

$$\alpha_i(0) = \dot{V}_i(0) = J_i\ddot{\theta}_i + \dot{J}_i\dot{\theta}_i \in R^6 \tag{2.1d}$$

$$\theta_i, \quad \dot{\theta}_i, \quad \ddot{\theta}_i \quad \text{are} \quad N\text{-dimensional.}$$

Note that every arm is taken to have the same N degrees of freedom when unconstrained by contact. This assumption is made only to simplify the notation and results in no real loss of generality. N itself can take on any value, subject to $N \geq 6$. $X_i(0)$ gives the spatial position (orientation and location) of the tip of arm i and evolves on a 6-dimensional manifold which is usually R^6 or $R^3 \times SO(3)$. $V_i(0)$ is the spatial velocity, and $\alpha_i(0) = \dot{V}_i(0)$ is the spatial acceleration of the tip of arm i. The contact force imparted to the held object is $f_i(0)$.

To succinctly describe the dynamics of all ℓ arms, define for $i = 1, \ldots, \ell$

$$\mathcal{M} = diag(\mathcal{M}_i), \quad J = diag(J_i), \quad \mathcal{C} = col(\mathcal{C}_i),$$

$$\theta = col(\theta_i), \quad T = col(T_i), \quad f(0) = col(f_i(0)), \quad Q = col(Q_i)$$

$$X(0) = col[X_i(0)], \quad V(0) = col[V_i(0)], \quad \alpha(0) = col[\alpha_i(0)],$$

and

$$n = \ell N. \tag{2.2}$$

n gives the total number of manipulator links in the system of Fig. 2.1. Note that θ is n-dimensional, $\mathcal{M} \in R^{n \times n}$, $V(0) \in R^{6\ell}$, $J \in R^{6\ell \times n}$, etc. With the above definitions, the aggregate arm dynamics are given by

$$\mathcal{M}\ddot{\theta} + \mathcal{C} + J^T f = T \tag{2.3a}$$

$$X(0) = Q(\theta) \tag{2.3b}$$

$$V(0) = J\dot{\theta} \tag{2.3c}$$

$$\alpha(0) = J\ddot{\theta} + \dot{J}\dot{\theta}. \tag{2.3d}$$

With C a point fixed with respect to the held object, let $M(C)$ be the held object's spatial mass referenced to the point C. $M(C)$ is given by

$$M(C) = \begin{pmatrix} I_C & m_C\tilde{p}_C \\ -m_C\tilde{p}_C & m_C I \end{pmatrix}$$

where m_C is the held object mass, I_C is the object rotational inertia tensor at C, and p_C is the 3- vector from C to the object mass center. Here, $\tilde{v}y \equiv v \times y$. In the spatial notation of [9], the object dynamics are given by

$$M(C)\alpha(c) + b(C) = f(C), \tag{2.4}$$

where

$$b(C) = \begin{pmatrix} \omega_C \times I_C \cdot \omega_C \\ m_C \omega_C \times (\omega_C \times p_C) \end{pmatrix} \quad \text{and} \quad f(C) = \begin{pmatrix} N_C \\ F_C \end{pmatrix}$$

are the bias spatial force and the resultant net spatial force at the point C respectively. Equation (2.4) is a succinct way of summarizing the coupled Newton and Euler equations which describe the rigid body dynamics of the held object [17].

With ℓ arms grasping the object at the ℓ distinct contact points O_j for $j = 1, \ldots, \ell$, there are $\ell - 1$ independent loops formed (the base is viewed as being common to all arms), and ℓ is also a measure of the number of closed kinematic chains for the sytem represented by Fig. 2.1. This can be seen by noting that cutting any $\ell - 1$ distinct arms will result in ℓ distinct and independent (i.e., kinematically decoupled) serial link systems, and that at least $\ell - 1$ cuts are needed to remove all loops. Assuming rigid arms, a rigid central body, and rigid grasps, the kinematic constraints imposed by the ℓ arms grasping the object at the contact points are holonomic [18] and consist of the requirement that the total change in position and orientation as one traverses around any loop of the system be zero. Differentiating these constraints leads to the following natural constraints on spatial velocities and accelerations for $i = 1, \ldots, \ell$.

$$V_i(0) = \phi^T(C, O_i) V(C) \tag{2.5a}$$

$$\alpha_i(0) = \phi^T(C, O_i)\alpha(C) + a_i(C) \tag{2.5b}$$

where

$$\phi(C, O_i) = \begin{pmatrix} I & \tilde{\ell}(C, O_i) \\ 0 & I \end{pmatrix}, \quad a_i(C) = \begin{pmatrix} 0 \\ \omega_C \times [\omega_C \times \ell(C, O_i)] \end{pmatrix}.$$

Here, $\ell(C, O_i)$ is the 3-vector from point C to point O_i. Since $\phi^{-1}(a, b) = \phi(b, a)$ and $\phi(a, b)\phi(b, c) = \phi(a, c)$ [6,9,10], (2.5a) can be rewritten as $V(C) = \phi^T(O_i, C)V_i(0)$ so that for every $i, j = 1, \ldots, \ell$

$$\phi^T(O_j, C)V_j(0) - \phi^T(O_i, C)V_i(0) = 0.$$

Equivalently,

$$V_j(0) - \phi^T(O_i, O_j)V_i(0) = 0.$$

This is just the requirement that the total change in velocity as a loop is traversed is zero, which is consistent with the original holonomic constraints. Equation (2.5b) follows from (2.5a) by direct differentiation, i.e. $\dot{V}_i(0) = \alpha_i(0)$.

Defining

$$A^T = [\phi(C, O_1), \cdots, \phi(C, O_\ell)] \in R^{6\times 6\ell} \tag{2.6}$$

and $a(C) = col[a_j(C)]$, Eqs. (2.5) become

$$V(0) = AV(C) \tag{2.7a}$$

$$\alpha(0) = A\alpha(C) + a(C). \tag{2.7b}$$

Because $\phi(C, O_i)$ is invertible, it can be shown that $A^T A = \sum_{i=1}^{\ell} \phi(C, O_i)\phi^T(C, O_i) \in R^{6\times 6}$ is invertible. This enables the determination of $V(C) \in R^6$ from $V(0) \in R^{6\ell}$ and the constraint relation (2.7a) by

$$V(C) = (A^T A)^{-1} A^T V(0) \tag{2.8a}$$

and $\alpha(C)$ from $\alpha(0)$ and (2.7b) by

$$\alpha(C) = (A^T A)^{-1} A^T [\alpha(0) - a(C)]. \tag{2.8b}$$

The resultant net spatial force at C due to the contact forces $f_i(0)$ is given by

$$f(C) = \sum_{i=1}^{\ell} \phi(C, O_i) f_i(0)$$

or

$$f(C) = A^T f(0). \tag{2.9}$$

The complete closed-chain dynamics are given by (2.3), (2.4), (2.7b), and (2.9) or

$$\mathcal{M}\ddot{\theta} + \mathcal{C} + J^T f(0) = T \tag{2.10a}$$

$$\alpha(0) = J\ddot{\theta} + \dot{J}\dot{\theta} = A\alpha(C) + a(C) \tag{2.10b}$$

$$M(C)\alpha(C) + b(C) = A^T f(0). \tag{2.10c}$$

When integrating the system (2.10), the holonomic loop constraints and the velocity constraints (2.7a) must be satisfied. Equations (2.10) can be written as

$$\begin{pmatrix} 0 & J & -A \\ J^T & \mathcal{M} & 0 \\ -A^T & 0 & M(C) \end{pmatrix} \begin{pmatrix} f(0) \\ \ddot{\theta} \\ \alpha(C) \end{pmatrix} = \begin{pmatrix} a(C) - \dot{J}\dot{\theta} \\ T - \mathcal{C} \\ -b(C) \end{pmatrix}. \tag{2.11}$$

From (2.10c), $\alpha(C) = M^{-1}(C)A^T f(0) - M^{-1}(C)b(C)$, which with (2.10b) yields

$$\alpha(0) = AM^{-1}(C)A^T f(0) - AM^{-1}(C)b(C) + a(C). \tag{2.12}$$

With (2.12) substituting for (2.10b) and (2.10c), Eqs. (2.10) can be simplified to

$$\mathcal{M}\ddot{\theta} + J^T f(0) = T - \mathcal{C} \tag{2.13a}$$

$$J\ddot{\theta} - AM^{-1}(C)A^T f(0) = a(C) - AM^{-1}(C)b(C) - \dot{J}\dot{\theta} \tag{2.13b}$$

or

$$\begin{pmatrix} \mathcal{M} & J^T \\ J & -AM^{-1}(C)A^T \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ f(0) \end{pmatrix} = \begin{pmatrix} T' \\ a'(C) \end{pmatrix} \tag{2.14}$$

where

$$T' = T - \mathcal{C} \tag{2.15a}$$

$$a'(C) = a(C) - AM^{-1}(C)b(C) - \dot{J}\dot{\theta}. \tag{2.15b}$$

We now state the

CLOSED-CHAIN FORWARD DYNAMICS PROBLEM (CCFDP):

Given joint data $(\theta, \dot{\theta})$ and input moments T, find the accelerations $\ddot{\theta}$ and the contact forces $f(0)$.

Note that from $(\theta, \dot{\theta})$, $V(0) = J\dot{\theta}$ and $V(C) = (A^T A)^{-1} AV(0)$ can be computed enabling the determination of $a(C)$ and $b(C)$ as well as the determination of the $(\theta, \dot{\theta})$-dependent terms $\mathcal{M}, J, \dot{J}\dot{\theta}, \mathcal{C}, M(C)$, and A. Thus knowledge of $(\theta, \dot{\theta})$ and T is sufficient for knowing the right-hand-sides of Eqs. (2.11) and (2.14) and the coefficient matrices of (2.11) and (2.14). The CCFDP is solved, then, by any $(\ddot{\theta}, f(0))$ for which (2.14) holds with $\mathcal{M}, J, A\mathcal{M}^{-1}(C)A^T, T'$, and $a'(C)$ determined by T and $(\theta, \dot{\theta})$. Similarly, $(\ddot{\theta}, f(0))$ and $\alpha(C)$ can be determined as the solution to (2.11). Note that having found $(\ddot{\theta}, f(0))$, $\alpha(0)$, we can compute $\alpha(C)$ and $f(C)$ from (2.3d), (2.8b), and (2.9) respectively.

Equation (2.11) is essentially [2; eq. 39] couched in the spatial notation of this paper and specialized to the case of several arms grasping a common object. (Reference [2] also deals with the more general case of multilegged locomotion). Note that the coefficient matrix of (2.11) is symmetric - a fact not readily evident in [2] due to the more general notation used. This matrix is certainly not positive definite due to the leading 0 on the block diagonal. The solution to (2.11) is seen to be equivalent to the simpler system

(2.14). In both cases, the nonsingularity of the coefficient matrices is not obvious. In [2], it is proposed that (2.11) be solved for $f(0)$, $\ddot{\theta}$, and $\alpha(C)$ by first determining the coefficient matrix and the "bias" terms by repeated application of the recursive Newton-Euler algorithm (in the manner first proposed in [3]), after which (2.11) is solved by Gauss-Jordan elimination or by the use of the Gauss-Seidel method. To solve for $(f(0), \ddot{\theta}, \alpha(C))$ in this manner requires $O(N^3\ell^3) = O(n^3)$ operations [2].

The same approach can be used to solve the simpler system (2.14), resulting in the same complexity of $O(n^3)$. Rather than do this, however, a solution for (2.14) is found for which the invertibility issue of the coefficient matrix is clearer.

3. AN ANALYTICAL SOLUTION TO THE CLOSED-CHAIN FORWARD DYNAMICS PROBLEM

Define $\Omega \in R^{6\ell \times 6\ell}$ by

$$\Omega \equiv J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T. \tag{3.1}$$

Premultiplication of both sides of (2.14) by

$$\begin{pmatrix} I & 0 \\ -J\mathcal{M}^{-1} & I \end{pmatrix}$$

results in

$$\begin{pmatrix} \mathcal{M} & J^T \\ 0 & -\Omega \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ f(0) \end{pmatrix} = \begin{pmatrix} I & 0 \\ -J\mathcal{M}^{-1} & I \end{pmatrix} \begin{pmatrix} T' \\ a'(C) \end{pmatrix}. \tag{3.2}$$

If Ω is nonsingular, Eq. (3.2) is certainly solvable. More generally, (3.2) is solvable if the right-hand side of (3.2) is in the image of the coefficient matrix. This more general case is not considered here, and Ω is assumed to be invertible. Further discussion on the relationship of the singularity of Ω to the solvability of the CCFDP is deferred to the next section.

Premultiplication of (3.2) by

$$\begin{pmatrix} I & J^T\Omega^{-1} \\ 0 & I \end{pmatrix}$$

yields

$$\begin{pmatrix} \mathcal{M} & 0 \\ 0 & -\Omega \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ f(0) \end{pmatrix} = \begin{pmatrix} I - J^T\Omega^{-1}J\mathcal{M}^{-1} & J^T\Omega^{-1} \\ -J\mathcal{M}^{-1} & I \end{pmatrix} \begin{pmatrix} T' \\ a'(C) \end{pmatrix} \tag{3.3}$$

or

$$\begin{pmatrix} \ddot{\theta} \\ f(0) \end{pmatrix} = \begin{pmatrix} \mathcal{M}^{-1} - \mathcal{M}^{-1}J^T\Omega^{-1}J\mathcal{M}^{-1} & \mathcal{M}^{-1}J^T\Omega^{-1} \\ \Omega^{-1}J\mathcal{M}^{-1} & -\Omega^{-1} \end{pmatrix} \begin{pmatrix} T' \\ a'(C) \end{pmatrix}. \tag{3.4}$$

In other words,

$$\begin{pmatrix} \mathcal{M} & J^T \\ J & -A\mathcal{M}^{-1}(C)A^T \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{M}^{-1} - \mathcal{M}^{-1}J^T\Omega^{-1}J\mathcal{M}^{-1} & \mathcal{M}^{-1}J^T\Omega^{-1} \\ \Omega^{-1}J\mathcal{M}^{-1} & -\Omega^{-1} \end{pmatrix} \tag{3.5a}$$

$$= \begin{pmatrix} \mathcal{M}^{-1} & 0 \\ 0 & \Omega^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{M} - J^T\Omega^{-1}J & J^T \\ J & -\Omega \end{pmatrix} \begin{pmatrix} \mathcal{M}^{-1} & 0 \\ 0 & \Omega^{-1} \end{pmatrix}. \tag{3.5b}$$

Eq. (3.4) is the analytical solution to the CCFDP given by (2.4), which in component form is

$$\ddot{\theta} = (\mathcal{M}^{-1} - \mathcal{M}^{-1}J^T\Omega^{-1}J\mathcal{M}^{-1})T' + \mathcal{M}^{-1}J^T\Omega^{-1}a'(C) \tag{3.6a}$$

$$f(0) = \Omega^{-1}JM^{-1}T' - \Omega^{-1}a'(C) \tag{3.6b}$$

where T' and a' are given by (2.15). In this paper, Eq. (3.6b) is viewed as equivalent to the more general $\Omega f(0) = J\mathcal{M}^{-1}T' - a'(C)$ found from (3.2).

The solution (3.6) is just the solution that arises from (2.13) by substituting $\ddot{\theta}$ determined by (2.13a) into (2.13b) and then solving for the constraint forces $f(0)$. This gives (3.6b). After this step, the solution for $f(0)$ is substituted into (2.13a) resulting in an equation involving only $\ddot{\theta}$ which can be solved to give (3.6a). This approach of first solving for the constraint forces and then obtaining an equation purely in terms of $\ddot{\theta}$ is precisely that suggested in [19]. Reference [19] states this procedure abstractly, after which it is applied to the constrained equations arising from bipedal motion. This technique is also used in [20] to obtain the equations describing various robotics applications, including the case of two robot arms carrying a common object for which equations similar to (3.6) are derived. For now we will defer the analysis of (3.6) as a means of obtaining recursive closed-chain forward dynamics algorithms until Section 6. The following Section 4 gives a general algorithm for solving the CCFDP based on an alternative derivation to (3.6), after which the relationship of this general algorithm to (3.6) will be discussed.

4. A GENERAL ALGORITHM FOR SOLVING THE CLOSED-CHAIN FORWARD DYNAMICS PROBLEM

For given $(\theta, \dot{\theta})$ and T, the "free dynamics" are defined by

$$\mathcal{M}\ddot{\theta}_f + \mathcal{C} = T \tag{4.1a}$$

$$\alpha_f(0) = J\ddot{\theta}_f + \dot{J}\dot{\theta}. \tag{4.1b}$$

Eqs. (4.1) should be compared to Eqs. (2.10). $\ddot{\theta}_f$ and $\alpha_f(0)$ are the joint accelerations and arm tip spatial velocities that would occur if the arms were not constrained by contact with the object. Also, define

$$\Delta\ddot{\theta} = \ddot{\theta} - \ddot{\theta}_f \tag{4.2a}$$

$$\Delta\alpha(0) = \alpha(0) - \alpha_f(0) \tag{4.2b}$$

so that $\ddot{\theta} = \ddot{\theta}_f + \Delta\ddot{\theta}$ and $\alpha(0) = \alpha_f(0) + \Delta\alpha(0)$.

Equations (2.13a), (4.1a), and (4.2a) yield

$$\mathcal{M}\Delta\ddot{\theta} = -J^T f(0) \tag{4.3}$$

or

$$\Delta\ddot{\theta} = -\mathcal{M}^{-1}J^T f(0). \tag{4.4}$$

Equations (2.12), (2.13b), (4.1b) and (4.2b) yield

$$J\Delta\ddot{\theta} = \Delta\alpha(0) \tag{4.5}$$

with

$$\Delta\alpha(0) = AM^{-1}(C)A^T f(0) - AM^{-1}(C)b(C) + a(C) - \alpha_f(0). \tag{4.6}$$

Again the matrix $\Omega \in R^{6\ell\times 6\ell}$ is defined by

$$\Omega = J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T. \tag{4.7}$$

Eqs. (4.4)-(4.7) combine to give

$$\Omega f(0) = \alpha_f(0) + AM^{-1}(C)b(C) - a(C) \tag{4.8}$$

Equations (4.1) and (4.8) completely determine the tip contact forces acting on the held object.

Equations (4.1)–(4.8) together result in the following general 4-step closed-chain forward dynamics (CCFD) algorithm.

GENERAL CCFD ALGORITHM.

1) Find the free accelerations $(\ddot{\theta}_f, \alpha_f(0))$.

$$\mathcal{M}\ddot{\theta}_f + \mathcal{C} = T \tag{4.9a}$$

$$\alpha_f(0) = J\ddot{\theta}_f + \dot{J}\dot{\theta} \tag{4.9b}$$

2) Find the constraint contact forces, $f(0)$.

$$\Omega = J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T \tag{4.9c}$$

$$\Omega f(0) = \alpha_f(0) + AM^{-1}(C)b(C) - a(C). \tag{4.9d}$$

3) Find the correction accelerations, $(\Delta\ddot{\theta}, \Delta\alpha(0))$.

$$\mathcal{M}\Delta\ddot{\theta} + J^T f(0) = 0 \tag{4.9e}$$

$$\Delta\alpha(0) = J\Delta\ddot{\theta}. \tag{4.9f}$$

4) Find the true accelerations, $(\ddot{\theta}, \alpha(0))$.

$$\ddot{\theta} = \ddot{\theta}_f + \Delta\ddot{\theta} \tag{4.9g}$$

$$\Delta\alpha(0) = \alpha_f(0) + \Delta\alpha(0). \tag{4.9h}$$

It is evident that the key step is the ability to solve (4.9d) for the constraint forces $f(0)$. The solvability of (2.11) or (2.14) is evidently equivalent to the solvability of (4.9d), which is equivalent to the requirement that the right-hand side of (4.9d) be in the image of Ω. This most general condition for solvability will not be investigated, except to say that it is conjectured that for any trajectory of the closed-chain system which satisfies the kinematic constraints, the right-hand side of (4.9d) will start and remain in the image of Ω. From (4.9c), it is seen that a sufficient condition for the invertibility of Ω is that all arms be kinematically nonsingular, so that J is of full rank and $J\mathcal{M}^{-1}J^T$ is therefore invertible. The matrix Ω^{-1} is a generalization of the Operational Space mass matrix of [15] and represents the mass of the system reflected to the contact points.

Note that any available forward dynamics algorithm can be used to solve (4.9a), (4.9b), (4.9e), and (4.9f). For example, these equations can be solved using analytical expressions for arm dynamics when available; or, they can be solved by repeated applications of the recursive Newton-Euler algorithm, as in [1- 3] (to determine J, $\mathcal{M}$, $\mathcal{C}$, $\dot{J}\dot{\theta}$, $\ddot{\theta}_f$, and $\alpha_f(0)$, after which Ω is inverted to obtain $f(0)$, $\Delta\ddot{\theta}$, and $\Delta\alpha(0)$); or finally these equations can be solved using recursive algorithms which directly yield $\ddot{\theta}_f$, $\Delta\ddot{\theta}$, $\alpha_f(0)$ and $\Delta\alpha(0)$ such as those in [4] or in [9].

To show how the general CCFD algorithm (4.9) can be obtained from (3.6) with T' and a' given by (2.15), write (3.6b) as

$$\Omega f(0) = J\mathcal{M}^{-1}T' - a'(C)$$
$$= J\ddot{\theta}_f - a(C) + AM^{-1}(C)b(C) + \dot{J}\dot{\theta}.$$
$$= \alpha_f(0) + AM^{-1}(C)b(C) - a(C).$$

Additionally, (3.6a) can be written as

$$\ddot{\theta} = \mathcal{M}^{-1}T' - \mathcal{M}^{-1}J^T\Omega^{-1}[J\mathcal{M}^{-1}T' - a'(C)]$$
$$= \ddot{\theta}_f - \mathcal{M}^{-1}J^T f(0) \quad \text{by (3.6b)}$$

or

$$\mathcal{M}\Delta\ddot{\theta} + J^T f(0) = 0.$$

The point to be drawn here is that the algorithm (4.9) is only one of several possible algorithms obtainable from (3.6), depending on how terms are grouped and interpreted. This is particularly pertinent since, due to the work of [6,9-12], it is known that the coefficient terms in (3.6) can be viewed as operators which can be combined in a variety of ways using the operator factorizations, inversions, and identities to be found for example in [9]. Although it is true that algorithm (4.9) has a particularly nice form and interpretation, as we shall see it is by no means the only way to implement the solution given by (3.6).

5. AN OVERVIEW OF OPERATOR METHODS FOR SOLVING THE FREE FORWARD DYNAMICS PROBLEM

Consider the dynamics of a single arm given by (2.1). In [9], it is shown that (2.1) has an interpretation as an operator- formulated dynamics. From this perspective M_i and J_i are viewed as operators of the form

$$\mathcal{M}_i = H_i\phi_i M_i \phi_i^T H_i^T$$

and

$$J_i = B_i^T \phi_i^T H_i^T$$

where the specific definitions of the component operators H_i, B_i, ϕ_i, and M_i can be found in [9]. $\mathcal{C}_i$ and $\dot{J}_i\dot{\theta}_i$ also have the forms

$$\mathcal{C}_i = H_i\phi_i(M_i\phi_i^T a_i + b_i)$$

and

$$\dot{J}_i\dot{\theta}_i = B_i^T \phi_i^T a_i + a_i(0)$$

where $[a_i, a_i(0)]$ are the "bias spatial accelerations" and b_i are the "bias spatial forces" associated with the joints of arm i. C_i and $\dot{J}_i\dot{\theta}_i$ are seen to arise from the action of certain operators acting on the biases a_i and b_i.

M_i and H_i are block diagonal and are therefore said to be "memoryless". ϕ_i is a lower block-triangular matrix and is therefore said to be "causal". A causal operator is equivalent to a recursive algorithm which processes link data in an iterative sweep which goes from the arm tip to the arm base. ϕ_i^T is upper block triangular and therefore "anticausal". The action of the anticausal ϕ_i^T on link quantities is equivalent to a base-to-tip iteration. As in [9], a link numbering scheme is used which increases from the tip (on link 1) to the base (link $N+1$). For this reason a base-to-tip anticausal recursion iterates from $k = N+1$ to $k = 1$.

A statement like $V_i(O) = B_i^T \phi_i^T H_i^T \dot{\theta}_i$ says that the tip velocity $V_i(0)$ is equivalent to an anticausal (base-to-tip) processing of the joint rates $\dot{\theta}_i$. The action of B_i^T is to pick out the spatial velocity associated with link 1 at joint 1 and propagate its effect to the tip. The equivalent recursive formulation is

$$V_i(N+1) = 0;$$

for $k = N, \ldots, 1$ loop

$$V_i(k) = \phi_i^T(k+1,k)V_i(k+1) + H_i^T(k)\dot{\theta}_i(k)$$

end loop

$$V_i(0) = \phi_i^T(1,0)V_i(1)$$

where $H_i^T(k)$ gives joint axis k of arm i and $\phi_i^T(k+1,k)$ is the interlink Jacobian for link $k+1$ of arm i.

Since the product $H_i\phi_i$ is lower block triangular, or causal, $\mathcal{M}_i = H_i\phi_i M_i \phi_i^T H_i^T$ is seen to be an operator with a causal-memoryless-anticausal factorization.

Now let $H = diag(H_i)$, $\phi = diag(\phi_i)$, $M = diag(M_i)$, $B = diag(B_i)$, $a = col(a_i)$, $a(0) = col[a_i(0)]$, and $b = col(b_i)$. The aggregate dynamics (2.10) will be given an operator interpretation by taking

$$\mathcal{M} = H\phi M \phi^T H^T \tag{5.1a}$$

$$J = B^T \phi^T H^T \tag{5.1b}$$

$$\mathcal{C} = H\phi(M\phi^T a + b) \tag{5.1c}$$

$$\dot{J}\dot{\theta} = B^T\phi^T a + a(0). \tag{5.1d}$$

Note that the aggregate forms (5.1) have precisely the same structure as the single arm case. This is always the case for the operator forms extended from the single arm case studied in [9] to the multiarm case considered in this paper. Thus all operator factorizations, inversions, and identities developed in [9] hold here, with the understanding that an operator expression used here stands for the parallel application of this operator to all arms. As an example,

$$V(0) = J\dot{\theta} = B^T H^T \phi^T \dot{\theta}$$

is equivalent to

$$V_i(0) = J_i\dot{\theta}_i = B_i^T H_i^T \phi_i^T \dot{\theta}_i \quad \text{for} \quad i = 1, \ldots, \ell.$$

A more detailed and formal, but somewhat different, extension from single arms to multiple arms may be found in [6].

In [9], it is shown that $\mathcal{M}$ has an alternative causal-memoryless-anticausal factorization given by

$$\mathcal{M} = (I + H\Phi L)D(I + H\Phi L)^T \tag{5.2a}$$

where Φ_i, L_i, and D_i correspond to arm i, as defined in [9], and $\Phi = diag(\Phi_i)$, $L = diag(L_i)$, and $D = diag(D_i)$. Eq. (5.2) is just the aggregate statement that for $i = 1, \ldots, \ell$

$$\mathcal{M}_i = (I + H_i\Phi_i L_i)D_i(I + H_i\Phi_i L_i)^T \tag{5.2b}$$

and it is actually (5.2b) that can be found in [9]. Φ is a strictly causal operator which is related to ϕ. L and D are memoryless and are specified by a causal tip-to-base recursion given in [9]. It is shown in [9] that D is invertible and that

$$(I + H\Phi L)^{-1} = (I - H\Psi L) \tag{5.3}$$

where Ψ is a strictly causal (strictly lower block triangular) operator which is determined from ϕ, L, and H. Equation (5.3) is a statement about two causal operators, $(I + H\Phi L)$ and $(I - H\Psi L)$, being causal inverses of each other.

Equations (5.2) and (5.3) result in the important operator factorization of the inverse mass operator,

$$\mathcal{M}^{-1} = (I - H\Psi L)^T D^{-1}(I - H\Psi L). \tag{5.4}$$

The action of $\mathcal{M}^{-1}$ is equivalent to a causal tip-to-base filtering recursion given by $D^{-1}(\mathcal{I} - H\Psi L)$, followed by an anticausal base-to-tip smoothing recursion given by $(\mathcal{I} - H\Psi L)^T$. For each arm all of these actions are $O(N)$, and thus the action of $\mathcal{M}^{-1}$ is $O(n) = O(\ell N)$.

With (5.1), the closed-chain dynamics (2.13)–(2.15) are given by

$$T' = T - H\phi(M\phi^T a + b) \tag{5.5a}$$

$$a'(C) = a(C) - AM^{-1}(C)b(C) - B^T\phi^T a - a(0) \tag{5.5b}$$

$$H\phi M\phi^T H^T\ddot{\theta} + H\phi Bf(0) = T' \tag{5.5c}$$

$$B^T\phi^T H^T\ddot{\theta} - AM^{-1}(C)A^T f(0) = a'(C). \tag{5.5d}$$

For brevity, usually the expressions (2.13)–(2.15) are written, with the understanding that they have the interpretation of (5.5). This is the sense in which (2.13) is said to provide an "operator-formulated" closed-chain dynamics.

The "free dynamics" (4.1) can be written as

$$H\phi M\phi^T H^T\ddot{\theta}_f = T - H\phi(M\phi^T a + b) \tag{5.6a}$$

$$\alpha_f(0) = B^T\phi^T H^T\ddot{\theta}_f + B^T\phi^T a + a(0). \tag{5.6b}$$

Equation (5.6a) can be equivalently written as

$$T' = T - H\phi(M\phi^T a + b) \tag{5.7a}$$

$$H\phi M\phi^T H^T\ddot{\theta}_f = T'. \tag{5.7b}$$

Whether one chooses to solve (5.6a) or (5.7b) results in a different algorithmic implementation. To solve (5.7b), the "bias" moments $\mathcal{C} = H\phi(M\phi^T a + b)$ are first removed by a 2-sweep (a base-to-tip sweep followed by a tip-to-base sweep) Newton-Euler recursion [9]. This approach is now standard and is used in [1-3] and [4,5]. With (5.6a), different algorithmic options are available as discussed in [9].

With (5.4), (5.7) has the solution

$$T' = T - H\phi(M\phi^T a + b) \tag{5.8a}$$

$$\ddot{\theta}_f = (I - H\Psi L)^T D^{-1}(I - H\Psi L)T'. \tag{5.8b}$$

Equation (5.8b) can be written as

$$\epsilon = (I - H\Psi L)T' \tag{5.9a}$$

$$\nu = D^{-1}\epsilon \tag{5.9b}$$

$$\ddot{\theta}_f = (I - H\Psi L)^T \nu. \tag{5.9c}$$

For the i^{th} arm (5.9) is equivalent to the following recursions [9]

$$\hat{z}_i(0) = 0, \quad T_i'(0) = 0 \tag{5.10a}$$

for $k = 1, \ldots, N$ loop;

$$\hat{z}_i(k) = \psi_i(k, k-1)\hat{z}_i(k-1) + L_i(k-1)T_i(k-1) \tag{5.10b}$$

$$\epsilon_i(k) = T_i(k) - H_i(k)\hat{z}_i(k) \tag{5.10c}$$

$$\nu_i(k) = D_i^{-1}(k)\epsilon_i(k) \tag{5.10d}$$

end loop;

$$\lambda_i(N+1) = 0 \tag{5.10e}$$

for $k = 1, \ldots, N$ loop

$$\lambda_i(k) = \psi_i^T(k+1, k)\lambda_i(k+1) + H_i^T(k)\nu_i(k) \tag{5.10f}$$

$$\ddot{\theta}_{i_f}(k) = \nu_i(k) - L_i^T(k)\lambda_i(k+1) \tag{5.10g}$$

end loop;

Here, the matrix $\psi_i(k+1, k)$ is defined as

$$\psi_i(k+1, k) = \phi_i(k+1, k) - L_i(k)H_i(k).$$

Eq. (5.10b) is a Kalman filter and $L_i(k)$ are the Kalman filter gains. Eq. (5.10c) computes the white-noise residuals $\epsilon_i(k)$. Eq. (5.10b) and (5.10c) show a Kalman filter being used as a causal whitening filter. Eq. (5.10d) computes the weighted residual $\nu_i(k)$. Eqs. (5.10e)–(5.10g) show an anticausal filtering, or smoothing, action which processes the weighted residuals to obtain $\ddot{\theta}_{i_f}$.

Such filtering and smoothing ideas underlie the results of this section, and are an extension of filtering and smoothing concepts from linear estimation theory [13,14] to mechanical multibody systems [6,9-12]. The key concept which is exploited to make this extension in [6,9-12] is the linear relationship which exists between accelerations and forces for mechanical systems.

For notational simplicity, what has not been made explicit in (5.8) is the need to perform the recursions given by $V = B^T \phi^T H^T \dot{\theta}$ in order to compute the velocity-dependent biases a and b, as well as the need for recursions to compute D, L, and Ψ. As discussed in [9], the necessary operations can be performed simultaneously with the explicitly stated recursions.

If (5.6a) is directly solved by an application of (5.4), operator identities found in [9] and various groupings of terms can be used to place the solution in a number of alternative forms, each having a different algorithm implementation. For example, with (5.4), (5.6a) becomes

$$\ddot{\theta}_f = (I - H\Psi L)^T D^{-1} (I - H\Psi L)\{T - H\phi(M\phi^T a + b)\}. \tag{5.11}$$

If the terms in the curly brackets are kept distinct and computed before the action of $(I - H\Psi L)$, then the solution of (5.8)–(5.10) is applicable. On the other hand the identities

$$\Psi = \psi S \tag{5.12a}$$

$$(I - H\Psi L)H\phi = H\psi, \tag{5.12b}$$

where ψ and S are operators defined in [9], enables (5.11) to be written as

$$\ddot{\theta}_f = (I - H\Psi L)^{-1} D^{-1} \{T - H\psi(SLT + M\phi^T a + b)\}. \tag{5.13}$$

Eq. (5.13) is equivalent to

$$\varsigma = \phi^T a \tag{5.14a}$$

$$\epsilon = T - H\psi(SLT + M\varsigma + b) \tag{5.14b}$$

$$\nu = D^{-1}\epsilon \tag{5.14c}$$

$$\ddot{\theta}_f = (I - H\Psi L)^T \nu. \tag{5.14d}$$

The algorithm represented by (5.14) is significantly different from that of (5.8a) and (5.9). In particular, the algorithm (5.8a) and (5.9) requires four base-to-tip or tip-to-base sweeps (two to compute T' in (5.8a) and two to compute $\ddot{\theta}_f$ via (5.9)), while the algorithm of (5.14) requires only three sweeps.

The identities of [9] can also be used to obtain $\alpha_f(0)$ of (5.6b) simultaneously with the computation of $\ddot{\theta}_f$ from (5.9) or (5.14), resulting in additional computational savings. However, this possibility is not exploited in this paper. The details may be found in [9].

For additional simplicity, in the remainder of this paper, the focus will be on the algorithmic possibilities obtained from removing the bias moments in the manner (5.5a), and working with bias-free moments T', and the bias-free dynamics given by (5.5c) or (5.7b). Even with this (unnecessary) restriction, it will be seen that there remain a variety of algorithmic possibilities to solving the CCFDP.

6. RECURSIVE CLOSED-CHAIN FORWARD DYNAMICS BY OPERATOR METHODS

With the background of Sec. 5, first consider the general CCFD algorithm (4.9). Step 1 of (4.9) can be recursively performed by (5.8a), (5.8b), and (5.6b), resulting in

$$T' = T - H\phi(M\phi^T a + b) \tag{6.1a}$$

$$\ddot{\theta}_f = (I - H\Psi L)^T D^{-1}(I - H\Psi L)T' \tag{6.1b}$$

$$\alpha_f(0) = B^T\phi^T H^T\ddot{\theta}_f + B^T\phi^T a + a(0). \tag{6.1c}$$

Eqs. (6.1) represent ℓ recursive algorithms, each with a cost which is $O(N)$. See Eqs. (5.8)−(5.10). This results in a total cost of $O(\ell N) = O(n)$ computations. As was noted in the last section, (6.1) can be simplified by the application of the techniques in [9]. In particular, (6.1b) and (6.1c) can be combined to generate a single recursive algorithm for obtaining $\ddot{\theta}_f$ and $\alpha_f(0)$.

Step 3 of (4.9) involves solving (4.9e). Using (5.1b) and (5.4), (4.9e) becomes

$$\Delta\ddot{\theta} = -(I - H\Psi L)^T D^{-1}(I - H\Psi L)H\phi Bf(0). \tag{6.2}$$

With the identity (5.12b), (6.2) becomes

$$\Delta\ddot{\theta} = -(I - H\Psi L)^T D^{-1} H\psi Bf(0). \tag{6.3}$$

Using the techniques of [9], (6.3) can be shown to be equivalent to a causal tip-to-base recursion followed by an anticausal base-to-tip recursion. For each arm i, (6.3) is equivalent to the $O(N)$ recursion

$$\hat{z}_i(1) = \phi_i(1,0)f_i(0) \tag{6.4a}$$

For $k = 2, \ldots, N$ loop

$$\hat{z}_i(k) = \psi_i(k, k-1)\hat{z}_i(k-1); \tag{6.4b}$$

$$\nu_i(k) = D_i^{-1}(k)H_i(k)\hat{z}_i(k); \tag{6.4c}$$

end loop;

$$\lambda_i(N+1) = 0; \tag{6.4d}$$

For $k = N, \ldots, 1$ loop;

$$\lambda_i(k) = \psi_i^T(k+1, k)\lambda_i(k+1) + H_i^T(k)\nu_i(k) \tag{6.4e}$$

$$\Delta\ddot{\theta}_i = \nu_i(k) - L_i^T(k)\lambda_i(k+1) \tag{6.4f}$$

end loop;

Equations (5.10e)–(5.10g) and (6.4d)–(6.4f) are equivalent as they both represent the action of the operator $(I - H\Psi L)^T$. Compare (5.8b) and (6.3).

Step 3 of (4.9) is solved then by the $O(\ell N) = O(n)$ recursions implied by

$$\Delta\ddot{\theta} = -(I - H\Psi L)^T D^{-1}\psi B f(0) \tag{6.5a}$$

$$\Delta\alpha = B^T\phi^T H^T \Delta\ddot{\theta}. \tag{6.5b}$$

Again, (6.5a) and (6.5b) could be combined into a single algorithm using the techniques of [9].

So far Steps 1 and 3 of (4.9) involve $O(n)$ operations. The next task is to show that $J\mathcal{M}^{-1}J^T$, which is needed to obtain Ω of (4.9c) can be obtained by an $O(n)$ recursion. Note that $J\mathcal{M}^{-1}J^T$ has $J_i\mathcal{M}_i^{-1}J_i^T$ on the block diagonal where $(J_i\mathcal{M}_i^{-1}J_i^T)^{-1}$ is precisely the operational space effective mass of [15] for arm i. Use of (5.1b) and (5.4) leads to

$$J\mathcal{M}^{-1}J^T = B^T\phi^T H^T(I - H\Psi L)^T D^{-1}(I - H\Psi L)H\phi B. \tag{6.6}$$

With the identity (5.12b), (6.6) becomes

$$J\mathcal{M}^{-1}J^T = B^T\psi^T H^T D^{-1}H\psi B. \tag{6.7}$$

With the definitions of B, ψ, H, and D to be found in [9], (6.7) can be shown to be equivalent to

$$J\mathcal{M}^{-1}J^T = \Lambda(0) \tag{6.8a}$$

$$\Lambda(0) = diag[\Lambda_i(0)] \tag{6.8b}$$

where $\Lambda_i(0) = J_i \mathcal{M}_i^{-1} J_i^T$ for each arm $i = 1, \ldots, \ell$ is given by

$$\Lambda_i(0) = \phi_i^T(1,0)\Lambda_i(1)\phi_i(1,0) \tag{6.9a}$$

$$\Lambda_i(1) = \sum_{k=1}^{N} \psi_i^T(k,1) H_i^T(k) D_i^{-1}(k) H_i^T(k) \psi_i(k,1) \tag{6.9b}$$

which is equivalent to the $O(N)$ recursion

$$\Lambda_i(N+1) = 0 \tag{6.10a}$$

for $k = N, \ldots, 1$ loop

$$\Lambda_i(k) = \psi_i^T(k+1,k)\Lambda_i(k+1)\psi_i(k+1,k) + H_i^T(k) D_i^{-1} H_i(k) \tag{6.10b}$$

end loop;

$$\Lambda_i(0) = \phi_i^T(1,0)\Lambda_i(1)\phi_i(1,0). \tag{6.10c}$$

With ℓ arms, the computation of $J\mathcal{M}^{-1}J^T$ via (6.8) and (6.10) is $O(\ell N) = O(n)$.

With $J\mathcal{M}^{-1}J^T$ available, then $\Omega = J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T$, and $f(0)$ can be solved from (4.9d). Since Ω is $6\ell \times 6\ell$, this is an $O(\ell^3)$ operation.

Together the above yield the following $O(n) + O(\ell^3) = O(\ell N) + O(\ell^3)$ recursive form of (4.9):

ALGORITHM RCCFD 1.

1) $$T' = T - H\phi(M\phi^T a + b) \tag{6.11a}$$

$$\ddot{\theta}_f = (I - H\Psi L)^T D^{-1}(I - H\Psi L)T' \tag{6.11b}$$

$$\alpha_f(0) = B^T\phi^T H^T \ddot{\theta}_f + B^T\phi^T a + a(0) \tag{6.11c}$$

2) $$J\mathcal{M}^{-1}J^T = B^T\psi^T H^T D^{-1} H\psi B \tag{6.11d}$$

$$\Omega = J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T \tag{6.11e}$$

$$f(0): \quad \Omega f(0) = \alpha_f(0) + AM^{-1}(C)b(C) - a(C) \tag{6.11f}$$

3) $$\Delta\ddot{\theta} = (I - H\Psi L)^T D^{-1}\psi B f(0) \tag{6.11g}$$

$$\Delta\alpha = B^T\phi^T H^T \Delta\ddot{\theta} \tag{6.11h}$$

4) $$\ddot{\theta} = \ddot{\theta}_f + \Delta\ddot{\theta} \tag{6.11i}$$

$$\Delta\alpha(0) = \alpha_f(0) + \Delta\alpha(0) \tag{6.11j}$$

Now turn to the solution (3.6) viewed as operator equations. As discussed in Sec. 4, the algorithm (4.9), and hence its recursive form (6.11), is just one way that the general solution (3.6) can be interpreted. Alternative groupings of terms in (3.6), and the use of different operator identities from [9], will result in many different $O(n) + O(\ell^3)$ RCCFD algorithms, where the $O(\ell^3)$ complexity arises from the need to solve for $f(0)$, after Ω has been obtained by an $O(n)$ recursion. It may well be that solving for $f(0)$ can be found to require less than $O(\ell^3)$ operations by exploiting the special structure of Ω. But for now, the inversion of Ω, or the solution of the linear system of equations (6.11f), will be assumed to be done using traditional $O(\ell^3)$ methods.

An alternative algorithmic possibility for obtaining the solution (3.6) is now demonstrated. From (5.1b), (5.4), and (5.12b)

$$J\mathcal{M}^{-1} = B^T \psi^T H^T D^{-1}(I - H\Psi L). \tag{6.12}$$

With (6.12), (3.6b) becomes

$$\begin{aligned} f(0) &= \Omega^{-1} B^T \psi^T H^T D^{-1}(I - H\Psi L)T' - \Omega^{-1} a'(C) \\ &= \Omega^{-1}\{B^T \psi^T H^T D^{-1}(I - H\Psi L)T' - a'(C)\}. \end{aligned} \tag{6.13}$$

Eq. (6.13) is a useful form of (3.6b) which shows how the tip contact forces imparted to the held object can be obtained by: (1) A causal $O(n)$ filtering of the bias-free torques T' to produce $\nu = D^{-1}(I - H\Psi L)$, followed by; (2) An anticausal $O(n)$ filtering of ν, given by the operation $B^T \psi^T H^T \nu$, where during this operation the $O(n)$ recursions necessary to produce Ω can be done, and (3) an $O(\ell^3)$ inversion of Ω to produce $f(0)$. With (3.6b), (3.6a) can be written as

$$\ddot{\theta} = \mathcal{M}^{-1}[T' - J^T f(0)] \tag{6.14}$$

which, of course, can be found directly from (2.13a) and (2.15a). With (5.12a) and $\mathcal{M}^{-1}$ given by (5.4) note that

$$\begin{aligned} \mathcal{M}^{-1} T' &= (I - H\Psi L)^T D^{-1}(I - H\psi S L)T' \\ &= (I - H\Psi L)^T D^{-1}\{T' - H\psi S L T'\}. \end{aligned} \tag{6.15}$$

Also, from (6.12) note that

$$\mathcal{M}^{-1} J^T f(0) = (I - H\Psi L)^T D^{-1} H\psi B f(0). \tag{6.16}$$

Eqs. (6.14)–(6.16) result in

$$\ddot{\theta} = (I - H\Psi L)^T D^{-1}\{T' - H\psi(SLT' + Bf(0)\}. \tag{6.17}$$

Eq. (6.17) shows that a causal filtering of T' and $f(0)$, followed by an anticausal filtering results in $\ddot{\theta}$. Eqs. (6.1a), (6.7), (6.13) and (6.17) yield the following algorithm.

ALGORITHM RCCFD 2.

$$T' = T - H\phi(M\phi^T a + b) \tag{6.18a}$$

$$J\mathcal{M}^{-1}J^T = B^T\psi^T H^T D^{-1} H\psi B \tag{6.18b}$$

$$\Omega = J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T \tag{6.18c}$$

$$f(0) = \Omega^{-1}\{B^T\psi^T H^T D^{-1}(I - H\Psi L)T' - a'(C)\} \tag{6.18d}$$

$$\ddot{\theta} = (I - H\Psi L)D^{-1}\{T' - H\psi(SLT' + Bf(0)\}. \tag{6.18e}$$

A third recursive algorithm can be obtained by taking $\ddot{\theta} = \ddot{\theta}_1 + \ddot{\theta}_2$ in (3.6a) where

$$\ddot{\theta}_1 = (\mathcal{M}^{-1} - \mathcal{M}^{-1}J^T\Omega^{-1}J\mathcal{M}^{-1})T' \tag{6.19a}$$

$$\ddot{\theta}_2 = \mathcal{M}^{-1}J^T\Omega^{-1}a'(C). \tag{6.19b}$$

With (6.12), (6.19b) becomes

$$\ddot{\theta}_2 = (I - H\Psi L)^T D^{-1} H\psi B\Omega^{-1}a'(C), \tag{6.20}$$

while with (5.4) and (6.12), it is possible to write

$$\ddot{\theta}_1 = (I - H\Psi L)^T\{D^{-1} - D^{-1}H\psi B\Omega^{-1}B^T\psi^T H^T D^{-1}\}(I - H\Psi L)T' \tag{6.21}$$

Eqs. (6.1a), (6.7), (6.20), and (6.21) result in

ALGORITHM RCCFD3.

$$T' = T - H\phi(M\phi^T a + b) \tag{6.22a}$$

$$J\mathcal{M}^{-1}J^T = B^T\psi^T H^T D^{-1} H\psi B \tag{6.22b}$$

$$\Omega = J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T \tag{6.22c}$$

$$\varsigma(C) = \Omega^{-1}a'(C) \tag{6.22d}$$

$$f(0) = \Omega^{-1}B^T\psi^T H^T D^{-1}(I - H\Psi L)T' - \varsigma(C) \tag{6.22e}$$

$$\ddot{\theta}_1 = (I - H\Psi L)^T\{D^{-1} - D^{-1}H\psi B\Omega^{-1}B^T\psi^T H^T D^{-1}\}(I - H\Psi L)T' \tag{6.22f}$$

$$\ddot{\theta}_2 = (I - H\Psi L)^T D^{-1}H\psi B\varsigma(C). \tag{6.22g}$$

$$\ddot{\theta} = \ddot{\theta}_1 + \ddot{\theta}_2. \tag{6.22h}$$

Note that (6.22f) requires 4 causal or anticausal sweeps to produce $\ddot{\theta}_1$ from T'. Although algorithm RCCFD3 may be more computationally intensive than RCCFD1 or RCCFD2, it is an implementable $O(n)+O(\ell^3)$ recursive forward dynamics algorithm. The differences between the forward dynamics algorithms RCCFD1-RCCFD3 show dramatically the algorithmic variety and possibilities available from the tools provided by the operator perspective developed in [9] and here. Again, note that additional variations are possible which relax the need for the *a priori* determination of the bias-free moments T', and which allow the determination of the tip acceleration in the same iteration which produces the joint accelerations $\ddot{\theta}$. The details may be found in [9].

7. CONCLUSIONS

The exact analytical solution for the closed-chain forward dynamics problem of several arms grasping a commonly held object has been derived and is given by Eqs. (3.6). It is shown that the key step in obtaining the solution is the ability to solve for the tip forces $f(0)$, from Eqs. (3.2) or, equivalently from (4.9d). The solvability question is related to the nature of Ω given by (3.1), and a sufficient condition for solvability is that $J\mathcal{M}^{-1}J^T$ be full rank, which is true if no arm is at a kinematic singularity. A general algorithm, given by Eqs. (4.9), can be obtained from (3.6) which can be implemented using any available open-chain forward dynamics solver, analytical or recursive. In particular, in this paper this algorithm is implemented recursively using the linear operator formulation of robot dynamics and the operator factorization and inversion techniques of [9]. More generally, the theory of [9] can be applied to the general analytical solution (3.6) to produce a variety of different recursive $O(n) + O(\ell^3)$ closed-chain forward dynamics algorithms where ℓ is the total number of grasping arms and $n = \ell N$ is the total number of manipulator links, each arm having N links, where $N \geq 6$.

The primary motivation for using the operator approach is to allow an understanding of closed-chain dynamics at a higher level of abstraction than is possible with other methods. By use of the operator methods patterns and mathematical structures begin to emerge that are otherwise not detectable. An example of this is the ability to express dynamics problems in terms of the spatial operator algebra of [9] and of this paper. The benefit is not only conceptual however, as the operator equations can be made specific to produce recursive algorithms that are readily implemented using the more detailed state space notation.

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APPENDIXES

APPENDIX A: CONTACT MODELS FOR NON-FULLY-CONSTRAINED GRASPS

Only contacts are considered for which the contact point does not slip on the object surface. As will be discussed, this still leaves several types of contact. The nonslippage assumption is related to the "force closure" assumption [16], which is the assumption that all contact forces are in an appropriate friction cone, so that friction forces are active which prevent slipping. During a simulation this assumption must be continually verified.

As in the body of this paper, here, and in Appendix B, a "stacked notation" is continued to be used in which a vector $V(0)$ is a composite vector $V(0) = col[V_i(0)]$ for $i = 1, \ldots, \ell$. A useful conceptual feature of this notation is that it allows viewing $V(0)$ as representing either all arm tip velocities or the tip velocity of any single arm (viewed generically) so that the distinction between $V(0)$ and $V_i(0)$ tends to disappear. For this reason, the aim will be to avoid the subscripted notation. It is to be understood that the expressions written below and in Appendix B pertain to all arms as well as to each arm.

Since the contact points are assumed fixed relative to the held object, only 1, 2, and 3 degree of freedom rotational motions of the arm tips relative to the object at the contact point are allowed. Equivalently, only 1, 2, or 3 degree of freedom rotational motions of the object at the contact point relative to the arm tips are allowed. At the contact point, then, the linear and angular velocities obey

$$v^-(0) = v^+(0)$$

$$\omega^-(0) = \omega^+(0) + \Delta\omega^+, \qquad (a.1)$$

where: $\omega^+(0)$ gives the arm tip angular velocities ("+" being the manipulator side of the contact); $\omega^-(0)$ gives the object angular velocities at the contact points ("$-$" being on the side of the contact away from the manipulators); and $\Delta\omega^+$ gives the relative angular velocity of the object (on the "$-$" side of the contact) with respect to the arm tips (on the "+" side of the contacts). Equation (a.1) can be recast as

$$V^-(0) = V^+(0) + \Delta V^+$$

$$\Delta V^+ = col[\Delta\omega^+, 0]. \qquad (a.2)$$

As the contact is crossed in an outward direction from the arm tip to the task object, imagine that the frame on the arm tip side (the "+" side) is left behind in order to enter a frame on the object side ("$-$"). The "+" and "$-$" contact frames differ only by a rotation, $R^+ = diag[R_i^+]$ which is configuration dependent. The contact configuration parameters are given by $\Theta(0) = col[\Theta_i(0)]$.

At a given contact i, $\Theta_i(0) \in R^{m_i}$, where in general m_i is different for each arm. However, for notational simplicity, take $m_i = m$ for all i, and $\Theta(0) \in R^{\ell m}$. This results in no loss of generality, as the relevant equations written below can be taken as statements on an individual arm for implementation

size of paramatrized space

purposes. For nonrigid contacts $m \geq 1$, and, more generally, $m \geq n_g$, where n_g is the number of degrees of freedom of the contact which is assumed the same for each contact. In general, m can be larger than the number of grasp degrees of freedom, if we are to have a parameterization which gives a globally nonsingular representation of the relative attitude R^+ [21,22].

The "contact configuration velocity" associated with $\Theta(0)$ is given by

$$W(0) = col[W_i(0)] \in R^{\ell n_g} \tag{a.3}$$

with $W_i(0) \in R^{n_g}$. Here,

$$\dot{\Theta}(0) = F[\Theta(0)]W(0) \tag{a.4}$$

where $F[\Theta(0)] \in R^{(m \times n_g)\ell}$, and $W(0)$ is related to $\Delta\omega^+$ by

$$\Delta\omega^+ = h(0)W(0) \tag{a.5}$$

with $h[0, \Theta(0)] \in R^{3\ell \times n_g \ell}$. Equations (a.2) and (a.5) result in

$$V^-(0) = V^+(0) + H^T(0)W(0) \tag{a.6}$$

with $H^T(0) = col[h(0), 0] \in R^{6\ell \times n_g \ell}$. $h(0)$, and hence $H^T(0)$, is assumed to be full rank.

In order to make the above clearer, consider the following three types of contact.

<u>Case 1</u>: 1 dof rotational contact. A contact which forms a revolute lower kinematic pair has one rotational degree of freedom and is two-sided in the sense that the contact cannot be broken [23]. A kinematically equivalent contact is formed by an edge or line contact with friction, which is one-sided in the sense that the contact will be broken if appropriate contact forces normal to the object surface are not maintained [16]. Let $h(0) = diag[h_i(0)] \in R^{3\ell \times \ell}$ give the ℓ unit 3-vectors about which the contact rotations can occur. Then, the configuration parameters are the angles $\Theta(0) = \theta(0) \in R^{\ell}$. This gives $W(0) = \dot{\Theta}(0)$ and $\Delta\omega^+ = h(0)\dot{\theta}(0)$. Note that $m = n_g = 1$.

<u>Case 2</u>: 3 dof rotational contact. A contact which forms a spherical (ball-in-socket) lower kinematic pair has a full three rotational degrees of freedom and is two-sided in the sense that the contact cannot be broken [23].

A kinematically equivalent contact is formed by a point contact with friction, which is one sided since the contact can be broken if proper contact forces are not maintained [16]. There are many possible configuration parameters for this type of contact, corresponding to different representations of the rotation group $SO(3)$. Minimal 3-parameter representations, such as Euler angles, have the drawback that they become singular for certain contact orientations. Moreover, this singularity is purely a mathematical problem having nothing to do with the possible contact motions which are completely rotationally unconstrained. A commonly used 4-parameter representation which is globally nonsingular on $SO(3)$ is given by the Euler 4-parameters (the unit quaternions) [21,22]. With the Euler parameters, global control of a 3-dof rotational contact is possible with no control breakdown due to representation singularity [24,25]. In either case, $n_g = 3$ and we take $W(0) = \Delta\omega^+$, $h(0) = I \in R^{3\ell \times 3\ell}$. If the Euler parameters are used, $m = 4$ and $\Theta(0) = q(0) \in R^{4\ell}$, while if the Euler angles are used instead, $m = 3$ and $\Theta(0) = \beta \in R^{3\ell}$. The appropriate relationship (a.4) may be found in [21,22] or in [24].

Case 3. 2 dof rotational contact. The "soft finger contact" case is dealt with last because of its somewhat peculiar nature. This a one-sided point contact with friction such that a moment friction force exists about the normal to the object tangent plane at the contact point, prohibiting rotational motion about that normal [16]. All allowable motions are rotations about any axis in the contact tangent plane. Since this contact is one-sided, it can be broken if an appropriate contact normal force is not maintained. There exists no equivalent 2-dof rotational lower kinematic pair (and hence two-sided) contact [23]. The soft finger contact has the nice modeling feature that it is a one-sided point contact for which a two-finger grasp (i.e. two soft finger point contacts) of the held object is statically stable [16]. This is in contrast to the point contact with friction considered in Case 2, for which at least three contact points are needed for a stable grasp. Let $h_i^1(0) \in R^3$ be the unit 3-vector normal to the contact tangent plane at contact point O_i. Let $h_i^2(0), h_i^3(0) \in R^3$ be mutually orthogonal 3-vectors which are orthogonal to $h_i^1(0)$ and lie in the contact tangent plane. Admissible relative angular velocities at O_i are given by

$$\begin{aligned}\Delta\omega_i^+ &= h_i^2(0)\Delta\omega_i^2 + h_i^3(0)\Delta\omega_i^3 \\ &= [h_i^2(0), h_i^3(0)]\begin{pmatrix}\Delta\omega_i^2 \\ \Delta\omega_i^3\end{pmatrix}.\end{aligned}$$

Define $W_i(0) = col[\Delta\omega_i^2, \Delta\omega_i^3] \in R^2$, $W(0) = col[W_i(0)]$, $h_i(0) = [h_i^2(0), h_i^3(0)] \in R^{3\times 2}$, and $h(0) = diag[h_i(0)]$. For the configuration parameters take, as in Case 2, $\Theta(0) = q(0) \in R^{4\ell}$ (Euler parameters, $m = 4$) or $\Theta(0) = \beta(0) \in R^{3\ell}$ (Euler angles, $m = 3$), i.e., take a representation for the entire rotation group. This may seem surprising since $n_g = 2$, but it can be shown that even though rotations about the contact normal are not allowed, every possible contact attitude is attainable [26]. In [21,22] or [24], the relationship $\dot{\Theta}(0) = \bar{F}[\Theta(0)]\Delta\omega$ is given in detail, which with $\Delta\omega^+ = h(0)W(0)$ gives $\dot{\Theta}(0) = \bar{F}[\Theta(0)]h(0)W(0)$. Thus F of (a.4) is given by $F = \bar{F}h(0)$.

Now that (a.2)-(a.6) has been explained in some detail, consider the force and acceleration conditions which must hold at the contact points. Since ΔV^+ represents a motion at the contact point which is freely allowed, then

$$f^T(0)\Delta V^+ = 0 = f^T(0)H^T(0)W(0) \quad \text{for all} \quad W(0).$$

Equivalently,

$$H(0)f(0) = 0. \tag{a.7}$$

(In the terminology of [16], $f(0)$ and ΔV^+ are "reciprocal").

Differentiation of (a.6) leads to

$$\alpha^-(0) = \alpha^+(0) + H^T(0)\dot{W}(0) + \dot{H}^T(0)W(0) \tag{a.8}$$

Furthermore, it must be true that

$$\alpha^+(0) = J\ddot{\theta} + \dot{J}\dot{\theta} \tag{a.9}$$

$$\alpha^-(0) = A\alpha(C) + a(C). \tag{a.10}$$

APPENDIX B: THE CLOSED-CHAIN FORWARD DYNAMICS PROBLEM FOR NON-FULLY-CONSTRAINED GRASPS

In this appendix, the forward dynamics equations for a closed-chain system with nonrigid, nonslipping contacts of the type given in Appendix A are derived. The notation of Appendix A is used freely here without additional explanation.

Together with the manipulator dynamics $\mathcal{M}\ddot{\theta}+\mathcal{C}+J^T f(0)=T$, and the held object dynamics $M(C)\alpha(C)+b(C)=A^T f(0)$, (a.7)-(a.10) give the nonrigid grasp closed-chain dynamics:

$$\mathcal{M}\ddot{\theta}+\mathcal{C}+J^T f(0)=T \tag{b.1a}$$

$$M(C)\alpha(C)+b(C)=A^T f(0) \tag{b.1b}$$

$$H(0)f(0)=0 \tag{b.1c}$$

$$\alpha^+(0)=J\ddot{\theta}+\dot{J}\dot{\theta} \qquad v^+(0)=J\dot{\theta} \tag{b.1d}$$

$$\alpha^-(0)=A\alpha(C)+a(C) \qquad v^-(0)=Av(C) \tag{b.1e}$$

$$\alpha^-(0)=\alpha^+(0)+H^T(0)\dot{W}(0)+\dot{H}^T(0)W(0). \tag{b.1f}$$

$v^-(0)=v^+(0)+H^T(0)W(0)$

With rigid grasps $H(0)=\dot{H}(0)=0$ and (b.1) reduces to (2.10).

The forward dynamics problem for (b.1) is to produce $[\ddot{\theta},\dot{W}(0),f(0)]$ from $[\theta,\dot{\theta},\Theta(0),W(0),T]$. Note that having $f(0)$ enables $\ddot{\theta}$ to be computed from (b.1a) via the techniques developed in the main body of this paper, and thus the issues of obtaining $f(0)$ and $\dot{W}(0)$ are the novel ones.

From (b.1a), $\ddot{\theta}=\mathcal{M}^{-1}[T-J^T f(0)-\mathcal{C}]$, which with (b.1d) gives

$$\alpha^+(0)=J\mathcal{M}^{-1}[T-J^T f(0)-\mathcal{C}]+\dot{J}\dot{\theta} \tag{b.2}$$

From (b.1b), $\alpha(C)=M^{-1}(C)[A^T f(0)-b(C)]$, which with (b.1e) gives

$$\alpha^-(0)=AM^{-1}(C)A^T f(0)-AM^{-1}(C)b(C)+a(C). \tag{b.3}$$

Eqs. (b.1f), (b.2), and (b.3) result in

$$\Omega=J\mathcal{M}^{-1}J^T+AM^{-1}(C)A^T \tag{b.4a}$$

$$\Omega f(0)=J\mathcal{M}^{-1}T+H^T(0)\dot{W}(0)+G_1 \tag{b.4b}$$

$$G_1 = AM^{-1}(C)b(C) + \dot{J}\dot{\theta} + \dot{H}^T(0)W(0) - a(C) - J\mathcal{M}^{-1}\mathcal{C} \tag{b.4c}$$

Note that with $H(0) = \dot{H}(0) = 0$, then (b.4) becomes (3.6b). Under the assumption that Ω is invertible

$$f(0) = \Omega^{-1}(J\mathcal{M}^{-1}T + G_1) + \Omega^{-1}H^T\dot{W}(0),$$

which with (b.1c) gives

$$[H(0)\Omega^{-1}H^T(0)]\dot{W}(0) = -H(0)\Omega^{-1}J\mathcal{M}^{-1}T - H(0)\Omega^{-1}G_1, \tag{b.5a}$$

or

$$\dot{W}(0) = -[H(0)\Omega^{-1}H^T(0)]^{-1}\{H(0)\Omega^{-1}J\mathcal{M}^{-1}T + H(0)\Omega^{-1}G_1\}. \tag{b.5b}$$

(b.1), (b.4), and (b.5) combine to give the following solution to the closed-chain forward dynamics problem.

$$\dot{W}(0) = -[H(0)\Omega^{-1}H^T(0)]^{-1}[H\Omega^{-1}J\mathcal{M}^{-1}T + H(0)\Omega^{-1}G_1] \tag{b.6a}$$

$$f(0) = \Omega^{-1}[J\mathcal{M}^{-1}T + H^T(0)\dot{W}(0) + G_1] \tag{b.6b}$$

$$\mathcal{M}\ddot{\theta} + \mathcal{C} + J^T f(0) = T. \tag{b.6c}$$

Inversion of $H(0)\Omega^{-1}H^T(0) \in R^{n_g\ell \times n_g\ell}$ is an $O(\ell^3)$ operation, and finding the nonrigid grasp closed-chain forward dynamics via the recursive techniques of this paper remains $O(n) + O(\ell^3)$.

Together, Equations (b.6a) and (b.6b) give

$$f(0) = \Pi(J\mathcal{M}^{-1}T + G_1) \tag{b.7a}$$

$$\Pi = \Omega^{-1} - \Omega^{-1}H^T(0)[H(0)\Omega^{-1}H^T(0)]^{-1}H(0)\Omega^{-1} \tag{b.7b}$$

which can be compared to (3.6b). With (b.7), (b.6c) is

$$\ddot{\theta} = (\mathcal{M}^{-1} - \mathcal{M}^{-1}J^T\Pi J\mathcal{M}^{-1})T - \mathcal{M}^{-1}J^T\Pi G_1 - \mathcal{M}^{-1}\mathcal{C}. \tag{b.8}$$

The combination of (b.4), (b.6a), (b.7) and (b.8) gives

$$G_1 = AM^{-1}(C)b(C) + \dot{J}\dot{\theta} + \dot{H}^T(0)W(0) - a(C) - J\mathcal{M}^{-1}\mathcal{C} \tag{b.9a}$$

$$\Omega = J\mathcal{M}^{-1}J^T + AM^{-1}(C)A^T \tag{b.9b}$$

$$\Pi = \Omega^{-1} - \Omega^{-1}H^T(0)[H(0)\Omega^{-1}H^T(0)]^{-1}H(0)\Omega^{-1} \tag{b.9c}$$

$$f(0) = \Pi(J\mathcal{M}^{-1}T + G_1) \tag{b.9d}$$

$$\ddot{\theta} = (\mathcal{M}^{-1} - \mathcal{M}^{-1}J^T\Pi J\mathcal{M}^{-1})T - \mathcal{M}^{-1}J^T\Pi G_1 - \mathcal{M}^{-1}\mathcal{C} \tag{b.9e}$$

$$\dot{W}(0) = -[H(0)\Omega^{-1}H^T(0)]^{-1}[H(0)\Omega^{-1}J\mathcal{M}^{-1}T + H(0)\Omega^{-1}G_1] \tag{b.9f}$$

Eqs. (b.9) give the complete analytical solution to the closed-chain forward dynamics problem of several rigid link manipulators holding a rigid object with nonrigid grasps.

From (b.3), (b.1d) and (b.1f) follows

$$\alpha^-(0) - \alpha^+(0) = AM^{-1}(C)A^Tf(0) - AM^{-1}(C)b(C) + a(C) - J\ddot{\theta} - \dot{J}\dot{\theta}$$

$$= H^T(0)\dot{W}(0) + \dot{H}^T(0)W(0)$$

which can be rewritten as

$$H^T(0)\dot{W}(0) + J\ddot{\theta} = AM^{-1}(C)A^Tf(0) - G_1 + J\mathcal{M}^{-1}\mathcal{C}.$$

Premultiplying by $H(0)\Omega^{-1}$,

$$H(0)\Omega^{-1}H^T(0)\dot{W}(0) + H(0)\Omega^{-1}J\ddot{\theta} = H(0)\Omega^{-1}AM^{-1}(C)A^Tf(0) + H(0)\Omega^{-1}(J\mathcal{M}^{-1}\mathcal{C} - G_1).$$

With (b.9d), this becomes

$$\begin{aligned} &H(0)\Omega^{-1}H^T(0)\dot{W}(0) + H(0)\Omega^{-1}J\ddot{\theta} \\ &= H(0)\Omega^{-1}AM^{-1}(C)A^T\Pi J\mathcal{M}^{-1}T + H(0)\Omega^{-1}AM^{-1}(C)A^T\Pi G_1 \\ &\quad + H(0)\Omega^{-1}(J\mathcal{M}^{-1}\mathcal{C} - G_1) \\ &= H(0)\Omega^{-1}AM^{-1}(C)A^T\Pi J\mathcal{M}^{-1}T + G_2 \end{aligned} \tag{b.10}$$

where

$$G_2 = H(0)\Omega^{-1}\{(AM^{-1}(C)A^T\Pi - I)G_1 + J\mathcal{M}^{-1}\mathcal{C}\}. \tag{b.11}$$

With (b.6b), (b.6c) can be written as

$$\mathcal{M}\ddot{\theta} + J^T\Omega^{-1}H^T(0)\dot{W}(0) = (I - J^T\Omega^{-1}J\mathcal{M}^{-1})T - J^T\Omega^{-1}G_1 - \mathcal{C}$$

$$= (I - J^T\Omega^{-1}J\mathcal{M}^{-1})T + G_3, \qquad (b.12)$$

where

$$G_3 = -J^T\Omega^{-1}G_1 - \mathcal{C}. \qquad (b.13)$$

Equations (b.10) and (b.12) combine to give

$$\begin{pmatrix} H(0)\Omega^{-1}H^T(0) & H(0)\Omega^{-1}J \\ J^T\Omega^{-1}H^T(0) & \mathcal{M} \end{pmatrix} \begin{pmatrix} \dot{W}(0) \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} H(0)\Omega^{-1}AM^{-1}(C)A^T\Pi J\mathcal{M}^{-1} \\ I - J^T\Omega^{-1}J\mathcal{M}^{-1} \end{pmatrix} T + \begin{pmatrix} G_2 \\ G_3 \end{pmatrix} \qquad (b.14)$$

Equation (b.14) describes the dynamical behavior of $(\theta, \dot{\theta})$ and $[\Theta(0), W(0)]$. The forward dynamics solution to (b.14) is given by (b.9e) and (b.9f).

TABLE 2.1 DEFINITION OF SYMBOLS

$\mathcal{M}_i \equiv$ mass operator of arm i

$\mathcal{C}_i \equiv$ coriolis and centrifugal terms for arm i

$J_i \equiv$ Jacobian operator of arm i

$\theta_i \equiv$ vector of joint variables of arm i

$T_i \equiv$ generalized joint forces or moments of arm i

$N \equiv$ number of joints (links) of arm i

$\ell \equiv$ number of arms (grasp points)

$n \equiv \ell N \equiv$ total number of manipulator links and joints

$X_i(0) \equiv$ orientation and location of arm i tip. Usually $X_i(0) \in R^6$ or $X_i(0) \in R^3 \times SO(3)$

$V_i(0) = col[\omega_i(0), v_i(0)] \in R^6 \equiv$ spatial velocity of arm i tip

$\alpha_i(0) = col[\dot{\omega}_i(0), \dot{v}_i(0)] \in R^6 \equiv$ spatial accelerations of arm i tip

$f_i(0) = col[N_i(0), F_i(0)] \in R^6 \equiv$ spatial contact force imparted to object at contact point O_i by tip of arm i

$O_i \equiv$ contact point of arm i tip with held object

$C \equiv$ point fixed with respect to held object

$M(C) \equiv$ spatial mass of held object referenced to C

$b(C) \equiv$ "bias spatial force" acting on held object at C

$f(C) = col[N(C), F(C)] \in R^6 \equiv$ resultant spatial force on held object at C due to grasp contact forces

$X(C) \equiv$ orientation and location of held object at C

$V(C) = col[\omega(C), v(C)] \in R^6 \equiv$ spatial velocity of object at C

$\alpha(C) = col[\dot{\omega}(C), \dot{v}(C)] \in R^6 \equiv$ spatial acceleration of object at C

$\ell(C, O_i) \equiv$ 3-vector from point C to point O_i

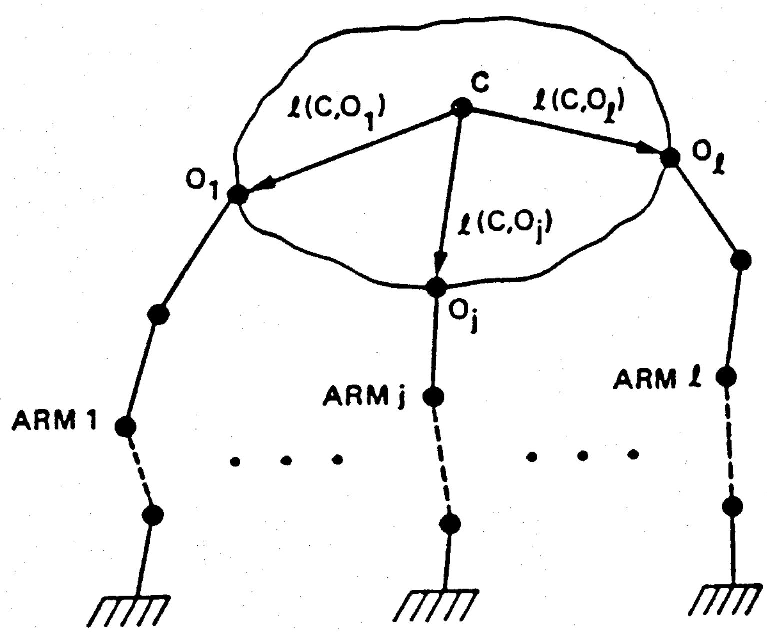

Fig. 2.1 ℓ Manipulators grasping a common object