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Slides for Feb. 24, 1984:

VECTOR GEOMETRY AND ALGEBRA

IN A NOTATION SUITABLE FOR

COMPUTER PROGRAMMING

OF

CURVILINEAR GRAPHICS,

NAVIGATION,

COMPLICATED MECHANISMS,

ELABORATE BUILDINGS, and

ROBOTS' LIMBS,

illustrated by application to the
INTELLEDEX 605 ROBOT ARM.

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Some Pedantry:

POINTS are not VECTORS ;
what is the sum of two POINTS ?

VECTOR are DISPLACEMENTS between POINTS .

$\left.\begin{array}{l}\text{COLUMNS}\\ \text{ARRAYS}\\ \text{LISTS}\\ \vdots\end{array}\right\}$ are not VECTORS nor MATRICES ;

a Mailing List is an array of Names,
not a Matrix ;

a Railway Timetable is an array of Numbers,
not a Matrix ;
what is the sum of two Railway Timetables ?

A MATRIX is an array that represents a LINEAR OPERATOR in some (perhaps implicit) COORDINATE SYSTEMS.
A COLUMN-VECTOR is an array ... VECTOR ...

ROW-VECTOR ... VECTOR ... in DUAL-SPACE.

Notational Conventions:

Underlined names name geometrical objects

$\underset{\sim}{Q}$, $\underset{\sim}{P}$, ... are $\underset{\sim}{\text{POINTS}}$

$\underline{x}$, $\underline{y}$, ... are <u>VECTORS</u>

$\underline{A}$, $\underline{B}$, ... are <u>LINEAR OPERATORS</u>

<u>LINEAR TRANSFORMATIONS</u>

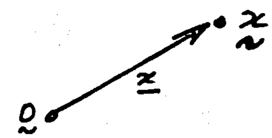
<u>COORDINATE FRAMES</u>

...

e.g. $\underline{C} = (\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}})$ is a COORDINATE FRAME; is is also a LINEAR OPERATOR from the space of COLUMN VECTORS like $x = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$ to the space spanned by the "Columns" of $\underline{C}$, consisting of vectors like $\underline{x} = \underline{C}\, x$.

> When $\underline{x} = \underline{C}\, x$, we say
> " x represents $\underline{x}$ in the coordinate frame $\underline{C}$ ".

Geometrical Objects & Coordinates



$\{\text{vectors}\} \not\equiv \{\text{points}\}$.

Origin $\underset{\sim}{o}$ ... a point. $\underline{x}$ ... a vector

Point $\underset{\sim}{x} = \underset{\sim}{o} + \underline{x}$ ... $\underline{x}$ is a DISPLACEMENT

Coordinate frame $\underline{C} = (\underline{\hat{\imath}}, \underline{\hat{\jmath}}, \underline{\hat{k}})$ is ORTHONORMAL

when $\underline{C}^T\underline{C} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, and then any vector

$$\underline{x} = \underline{C}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = \underline{\hat{\imath}}\,\xi + \underline{\hat{\jmath}}\,\eta + \underline{\hat{k}}\,\zeta = \underline{C}\,\underline{C}^T\underline{x}$$

<u>Distinguish</u> between geometrical vector $\underline{x} = \underline{C}\,x$

and its coordinate vector $x = \underline{C}^T\underline{x} = \begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}$ in the

orthonormal coordinate FRAME $\underline{C}$.

<u>Distinguish</u> between geometrical point $\underset{\sim}{x} = \underset{\sim}{o} + \underline{C}\,x$

and its coordinate vector $x = \underline{C}^T(\underset{\sim}{x} - \underset{\sim}{o})$ in the

orthonormal <u>coordinate SYSTEM</u> $\{\underset{\sim}{o}, \underline{C}\}$.

> $\underline{C}$ maps column vectors $x = \begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}$
>
> to geometrical vectors $\underline{x} = \underline{C}\,x = \underline{\hat{\imath}}\,\xi + \underline{\hat{\jmath}}\,\eta - \underline{\hat{k}}\,\zeta$
>
> just as a LINEAR OPERATOR maps vectors from one space to another.

BINDING-TIME QUESTIONS

Naming a vector $\underline{\underline{x}}$ does not really say which vector $\underline{\underline{x}}$ is.

$$\text{IS } \underline{\underline{x}} = \underline{\underline{y}} \text{ ?}$$

A vector $\underline{\underline{x}}$ is known completely when we know its representation x in some COORDINATE FRAME $\underline{\underline{C}}$;

e.g. if $\underline{\underline{C}} = (\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}})$ is given as an array of INDEPENDENT vectors, then specifying $x = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$, as a column of numbers, fully specifies the vector

$$\underline{\underline{x}} = \underline{\underline{C}}\, x = \hat{\underline{i}}\,\xi + \hat{\underline{j}}\,\eta + \hat{\underline{k}}\,\zeta$$

in terms of the "basis vectors" in the coordinate frame $\underline{\underline{C}}$.

VECTORS are almost always specified in terms of previously specified vectors; the earliest specifications must specify COORDINATE VECTORS.

Recall ORTHONORMAL FRAME $\underline{C} := (\underline{i}, \underline{j}, \underline{k})$ satisfied $\underline{C}^T \underline{C} = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$

where $\underline{C}^T := \begin{pmatrix} \underline{i}^T \\ \underline{j}^T \\ \underline{k}^T \end{pmatrix}$. Hence $\underline{C}^T = \underline{C}^{-1}$. ?

What does " $\underline{x}^T$ " mean? TRANSPOSE?

1. $\underline{x}^T$ is a LINEAR FUNCTIONAL, a linear operator from the space of vectors like $\underline{x}, \underline{y}, \dots$ to scalars;

$$\underline{x}^T(\alpha \cdot \underline{y} + \beta \cdot \underline{z}) = \alpha \cdot (\underline{x}^T \underline{y}) + \beta \cdot (\underline{x}^T \underline{z})$$

SCALARS

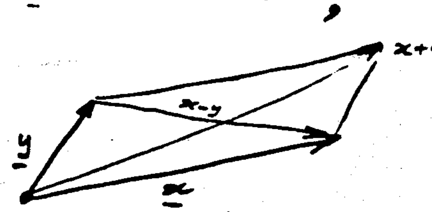
2. The map " T " from $\underline{x}$ to $\underline{x}^T$ is linear too;

$$(\alpha \cdot \underline{x} + \beta \cdot \underline{y})^T = \alpha \cdot \underline{x}^T + \beta\, \underline{y}^T .$$

cf. BILINEAR FORMS.

These 2 constraints do not yet specify " T " uniquely:

We assume that vectors $\underline{x}$ have LENGTH $\|\underline{x}\|$, and that this length satisfies the PARALLELOGRAM LAW:

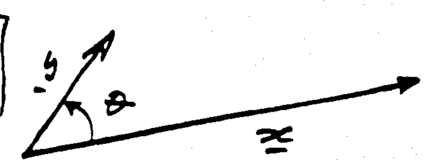


$$\|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2 = 2\|\underline{x}\|^2 + 2\|\underline{y}\|^2$$

VALID ONLY IN EUCLIDEAN SPACES.

THEN, set

$$\boxed{\underline{x}^T \underline{y} := \|\underline{x}\| \cdot \|\underline{y}\| \cdot \cos\theta}$$



This defines $\underline{x}^T$ uniquely as a function of $\underline{x}$.

Rigid Motion of a Point-set

PRESERVES DISTANCES BETWEEN POINTS

<u>Translation</u> moves $\underset{\sim}{p}$ to $\underline{d} + \underset{\sim}{p}$.

If $\underline{d} = \underline{C}d$ has coordinates $d = \underline{C}^T\underline{d}$ in the <u>same</u> orthonormal coordinate system as $\underset{\sim}{p}$'s coordinates $p = \underline{C}^T(\underset{\sim}{p} - \underset{\sim}{o})$

then this translation moves p to $d + p$.

<u>Rotation/Reflection</u> moves $\underset{\sim}{p}$ to $\underset{\sim}{o} + \underline{\underline{Q}}(\underset{\sim}{p} - \underset{\sim}{o})$

where $\underset{\sim}{o}$ is an origin and $\underline{\underline{Q}}$ a linear operator satisfying

$\underline{\underline{Q}}(\alpha\underline{x} + \beta\underline{y}) = \alpha\underline{\underline{Q}}\,\underline{x} + \beta\underline{\underline{Q}}\,\underline{y}$ (linearity)

$(\underline{\underline{Q}}\,\underline{x})^T(\underline{\underline{Q}}\,\underline{y}) = \underline{x}^T\underline{y} = \|\underline{x}\|\cdot\|\underline{y}\|\cdot\cos\angle(\underline{x},\underline{y})$ ...(Preserves length)

$\therefore$ Rotation/Reflection $\underline{\underline{Q}} = \underline{C}\,Q\,\underline{C}^T$ is represented by its matrix $Q = \underline{C}^T\underline{\underline{Q}}\,\underline{C}$ in coordinate system $\underline{C}$, and Q is orthogonal ... $Q^T = Q^{-1}$. cf. $\underline{C}^T = \underline{C}^{-1}$.

Moving $\underset{\sim}{p}$ to $\underline{d} + \underset{\sim}{o} + \underline{\underline{Q}}(\underset{\sim}{p} - \underset{\sim}{o})$

moves $p = \underline{C}^T(\underset{\sim}{p} - \underset{\sim}{o})$ to $d + Qp$.

i.e. Translation after Rotation/reflection

Moving $\underset{\sim}{p}$ to $\underset{\sim}{o} + \underline{\underline{Q}}(\underline{d} + \underset{\sim}{p} - \underset{\sim}{o})$

moves p to $Q(d + p)$

i.e. Rotation/reflection after translation.

The power of Mathematics is just the power we gain over Things by manipulating Symbols instead of the Things they represent.

Symbols are Things too, so ...

A good notation is one in which the rules for manipulating symbols suggest as directly as possible the corresponding ways in which the represented things may be manipulated.

The suggestion can be so direct that, at times, we fail to notice the distinctions between the Things and Symbols that represent them. A good notation is one which suffers little from this failure.

How good is our notation, in which we distinguish points, vectors, arrays ?

So, our notation seems to
permit us to write the
same sentences with
representations $p, d, Q, \ldots$
as with the things represented,
$\underset{\sim}{p}, \underline{d}, \underline{\underline{Q}}, \ldots$

<u>EXCEPT</u>

we omit the representation 0 of $\underset{\sim}{0}$.

Nothing special so far.

<u>BUT</u> we have used only ONE
COORDINATE SYSTEM.

what if we change coordinates?
That will change representations,
like changing everythings' names.

e.g. GROVE $\rightarrow$ MARTIN LUTHER KING JR.

Change of Orthonormal Coordinates.

Move $\underset{\sim}{o}$ to new origin $\underset{\sim}{n} = \underset{\sim}{o} + \underline{C}\, n$, and then

Rotate $\underline{C}$ to new frame $\underline{N} = \underline{\underline{Q}}\, \underline{C}$ $\quad \ldots \quad \underline{\underline{Q}} = \underline{C}\, Q\, \underline{C}^T$

$= \underline{C}\, Q$ $\quad$ where $Q^T = Q^{-1}$

So n is the coordinate of $\underset{\sim}{n}$ in old system $\{\underset{\sim}{o}, \underline{C}\}$,

Q's columns are coordinates of new coordinate frame in old.

If $\underset{\sim}{p} = \underset{\sim}{o} + \underline{C}\, x$ in old system $\{\underset{\sim}{o}, \underline{C}\}$

what are coordinates of $\underset{\sim}{p}$ in new system $\{\underset{\sim}{n}, \underline{N}\}$?

$$\underset{\sim}{p} = \underset{\sim}{n} + \underline{N}\, Q^T(x-n) \qquad \text{(SAME point } \underset{\sim}{p}\text{)}$$

has coordinates $Q^T\,(x-n)$

translation inverse to $\underset{\sim}{n} - \underset{\sim}{o}$

rotation inverse to $\underline{\underline{Q}}$.

Rigid Motion of Coordinates
= <u>inverse</u> of Rigid Motion of Coordinate System

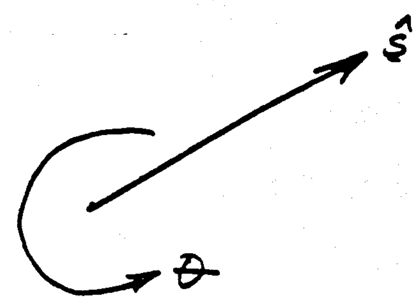
ROTATIONS IN 3-SPACE

ARE VERY SPECIAL.

Every rotation $\underline{\underline{Q}}$ in 3-space

has an axis $\underline{\underline{\hat{s}}}$, a unit vector,

and an angle θ:



Can the relation between $\underline{\underline{Q}}$ and $(\underline{\underline{\hat{s}}}, \theta)$ be expressed conveniently in COORDINATE-FREE terms?

YES: $\underline{\underline{Q}} = \exp(\theta \, \underline{\underline{\hat{s}}} \times)$ ...

In Matrix terms, $Q = \exp(\theta \, \hat{s} \times)$ ...

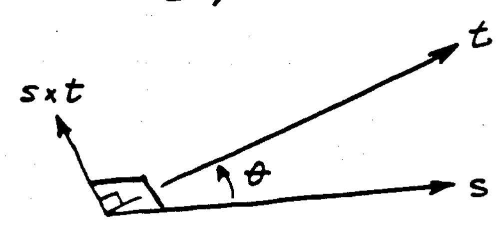
EASILY COMPUTED!

Vector $s = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}$, $\quad s^T s = \lambda^2 + \mu^2 + \nu^2$, $\quad \|s\| = \sqrt{s^T s}$, $\quad s^T = (\lambda \ \ \mu \ \ \nu)$

Skew Matrix $\quad s\times = S = \begin{pmatrix} 0 & -\nu & \mu \\ \nu & 0 & -\lambda \\ -\mu & \lambda & 0 \end{pmatrix} = -S^T$.

$s \times t = -t \times s$

so $\quad s \times s = 0$..



$\|s\times t\|^2 = \|s\|^2\|t\|^2 - (s^T t)^2 = \|s\|^2\|t\|^2 \sin^2\theta$.

$$(s\times t)\times = ts^T - st^T \neq s\times(t\times) = ts^T - s^T t$$

$$(s\times)^2 = s\times(s\times) = ss^T - s^T s \quad \text{so} \quad (s\times)^3 = -s^T s\,(s\times)$$

$$\text{cf. } 0^3 = -0, \quad i^3 = -i.$$

$$(s\times)^T = -s\times, \quad (s\times t)^T = s^T(t\times) = -t^T(s\times).$$

Orthogonal $\quad Q: \quad Q^T = Q^{-1} \iff \|Qs\| = \|s\|$.

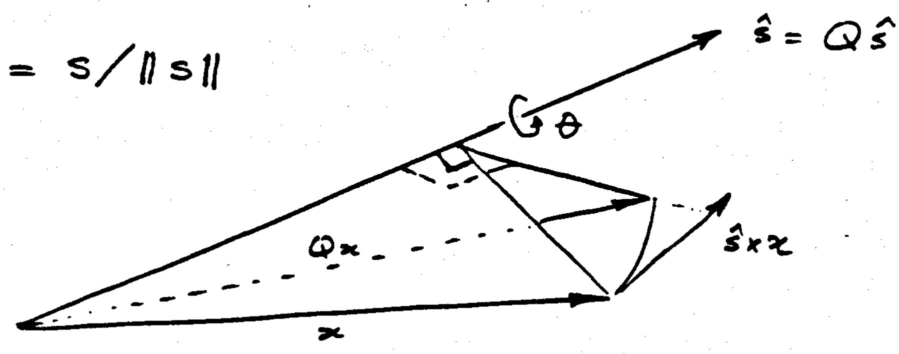
$$(Qs)\times Qt = Q(s\times t)$$

i.e. $\quad (Qs)\times = Q(s\times)Q^T$.

cf. change of coordinates.

Cross Products

$\hat{s} = s/\|s\|$



$$Q = \exp(\theta\,\hat{s}\times) \quad \ldots \text{ is orthogonal } (Q^T = Q^{-1})$$

$$= I + (\sin\theta)\,s\times + (1-\cos\theta)\,(s\times)^2$$

$$= (\cos\theta)\,I + (\sin\theta)\,s\times + (1-\cos\theta)\,s\,s^T$$

Q is a PROPER ROTATION ($\det Q = 1$).

$$dQ/d\theta = \hat{s}\times Q = Q\,s\times .$$

Given an orthogonal Q, can we find θ and $\hat{s}$ so that $Q = \pm\exp(\theta\,\hat{s}\times)$? Yes:

Say $\det Q = +1$, else replace Q by $-Q$;

Solve $(Q-I)\,s \doteq 0$ by Gaussian Elimination

i.e. $\|(Q-I)\,s\| \ll \|Q-I\|\cdot\|s\|$. $\quad \hat{s} \doteq s/\|s\|$.

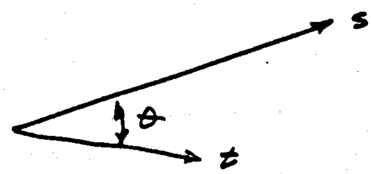
$\theta := 2\tan^{-1}\left(2\,\mathrm{trace}(Q-I)\,/\,\mathrm{trace}(\hat{s}\times(Q-Q^T))\right)$

accurately even when $|\sin\theta|$ is tiny.

ROTATION ABOUT AN AXIS $\hat{s}$, USING THE CROSS PRODUCT $\hat{s}\times$.

Finding Angles

Problem 1:

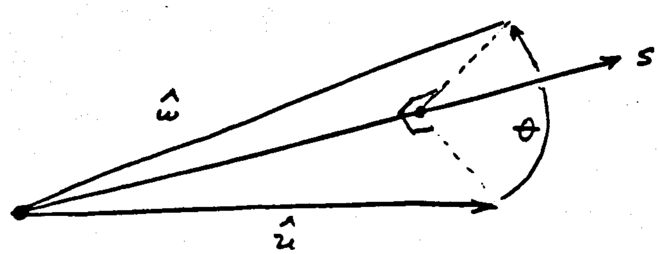


$|\theta| = \cos^{-1}\left(s^T t / (\|s\| \cdot \|t\|) \right)$ inaccurately

$= 2 \arctan\left(\|\hat{s} - \hat{t}\| / \|\hat{s} + \hat{t}\| \right)$ accurately

(where $\hat{s} = s/\|s\|$ etc.)

Problem 2:



Solve $\exp(\theta \hat{s}\times)\, \hat{u} = \hat{w}$ for θ.

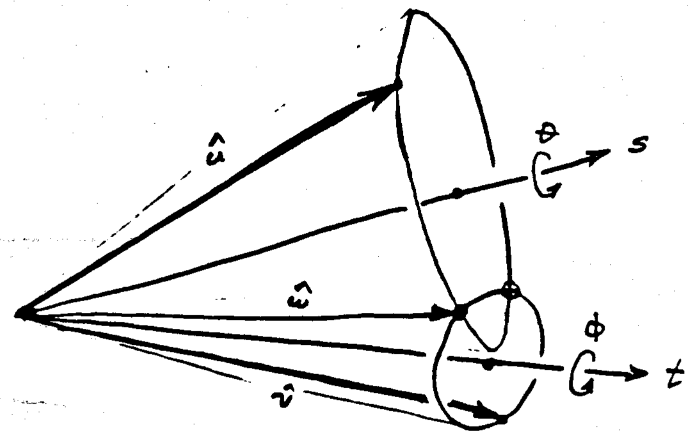
If & Only If $\hat{s}^T \hat{u} = \hat{s}^T \hat{w}$,

$|\theta| = 2 \arctan\left(\|\hat{s} \times (\hat{u} - \hat{w})\| / \|\hat{s} \times (\hat{u} + \hat{w})\| \right)$

and $\mathrm{sign}(\theta) = \mathrm{sign}\left(\hat{w}^T \left(\hat{s} \times \left(\hat{u} \pm \hat{w}\right)\right) \right)$

whichever is the smaller.

Problem 3 :



solve

$$\exp(\theta\,\hat{s}\times)\,\hat{u} = \exp(\phi\,\hat{t}\times)\,\hat{v} \quad \text{for } \theta \text{ and } \phi .$$

Skip trivial case $\hat{s} = \pm\hat{t}$.

No QUARTICS TO FACTOR !

Tactic : find common vector(s)

$$\exp(\theta\,\hat{s}\times)\,\hat{u} = \hat{\omega} = \exp(\phi\,\hat{t}\times)\,\hat{v} .$$

$$\hat{\omega} = \left(\hat{s}(\hat{s}^T\hat{u} - \hat{s}^T\hat{t}\cdot\hat{t}^T\hat{v}) + \hat{t}(\hat{t}^T\hat{v} - \hat{t}^T\hat{s}\cdot\hat{s}^T\hat{u})\right)/\|\hat{s}\times t\|^2$$
$$\pm\ \lambda \cdot \hat{s}\times\hat{t}/\|\hat{s}\times\hat{t}\|$$

two choices

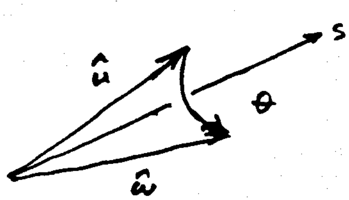
where

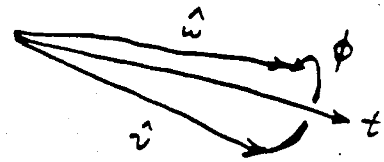
$$\lambda^2 = 1 - (\hat{s}^T\hat{u} \pm \hat{t}^T\hat{v})^2/\|\hat{s}\times\hat{t}\|^2 + 2\,\hat{s}^T\hat{u}\,\hat{t}^T\hat{v}/(-\hat{s}^T\hat{t} \pm 1)$$

Match these signs to avoid cancellation here

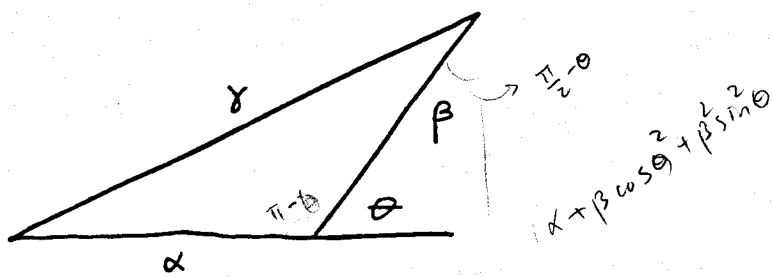
(If $\lambda^2 < 0$ then no θ and ϕ exist.)

Then invoke Problem 2 once for θ, once for ϕ.





Problem 4: Angle in a Triangle with Given Sides.



$\alpha \geq 0$
$\beta \geq 0$
$\gamma \geq 0$
$\alpha+\beta \geq \gamma$
$\beta+\gamma \geq \alpha$
$\gamma+\alpha \geq \beta$

Solve $\alpha^2+\beta^2+2\alpha\beta\cos\theta = \gamma^2$ for

$|\theta| = \cos^{-1}((\gamma^2-\alpha^2-\beta^2)/(2\alpha\beta))$.

INACCURATE in Near-degenerate cases.

$|\theta|(\alpha, \beta; \gamma)$:

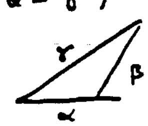
If $\alpha < \beta$ then swap(α, β) to ensure $\alpha \geq \beta$.

$\lambda := (\alpha-\beta)+\gamma$; $\mu := (\alpha-\gamma)+\beta$;

$\sigma := (\alpha+\gamma)+\beta$; ... DON'T DROP PARENTHESES!

If $\beta \geq \gamma \geq 0$ then $\nu := \gamma-(\alpha-\beta)$

else if $\gamma > \beta \geq 0$ then $\nu := \beta-(\alpha-\gamma)$

else NO SUCH TRANGLE AS  EXISTS ;

return $|\theta| := 2\arctan(\sqrt{(\mu\sigma)}/\sqrt{(\nu\lambda)})$

unless $\sqrt{(\text{Negative})} \Rightarrow$ NO SUCH TRIANGLE.