Spacecraft Dynamics + Control

2.4 EULERIAN ANGLES

For many practical applications, the relative orientation of two orthogonal reference frames is defined in terms of three angles. In many physical systems (e.g., polar telescope mounts, a zenith-elevation mounted radar antenna, gyroscope gimbals, etc.) a particular set of two or three angles are "built into" the gimbal axes of the particular hardware. For example, see the gyro assembly of Figure 2.4. In such a case, the most obvious choice of angles to orient the rigid body are those implicit in the gimbal design.

For the case of a reference frame imbedded in an unconstrained body (e.g., a space vehicle), however, an infinity of orientation coordinates is possible:

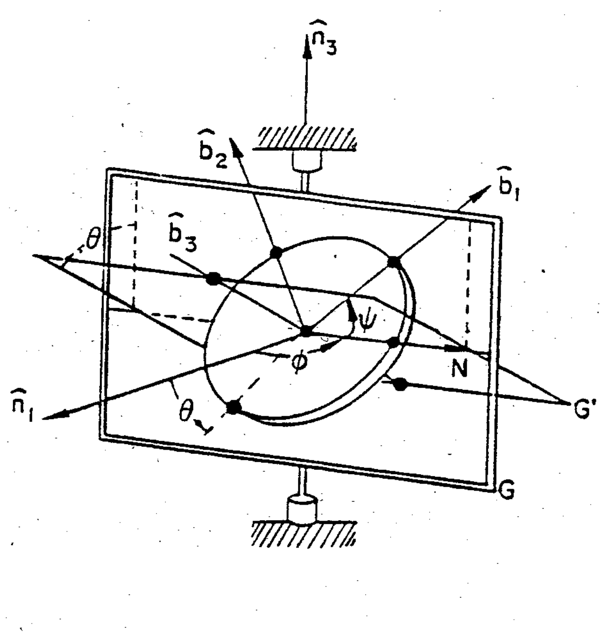

Figure 2.4 Two Gimbal Gyro with 3-1-3 Euler Angles

the particular choice of coordinates should be strongly influenced by the ease of motion visualization and perhaps more importantly, the absence of analytical or numerical singularities for a particular application or class of applications.

The most popular orientation coordinates in analytical dynamics generally and space vehicle dynamics in particular are a set of three Eulerian angles. The classical ("3-1-3") set of Euler angles are depicted in Figure 2.5. These were first used by astronomers to define the orientation of orbit planes of the planets relative to the earth's orbit plane (the eccliptic plane). In this context, $\theta_1 = \phi$ (the longitude of the ascending node), $\theta_2 = \theta$ (the inclination), $\theta_3 = \psi$ (the argument of perihelion). These "3-1-3" angles were first used in rigid body rotational dynamics by Euler during the early 1700's.

The successive rotational transformation property of direction cosines, as defined by Eq. 2.16, suggests using a set of three elementary rotations to parametrize (at any instant) the direction cosine matrix. If we restrict the three elementary rotations to be rigid right-handed rotations about fixed axes (as in Section 2.3), there still exists an infinity of three angle sets (owing to the infinity of available directions for fixing the three axes of rotation). If we restrict the axes of rotation to be colinear with one of the three orthogonal, right-handed body-fixed vectors $\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$, however, there are only twelve distinct cases. These are the classical Euler angles (although there is not universal conformity in adopting right-handed definitions for these angles). We introduce three indices α-β-γ to characterize these rotations

- α denotes the axis of the first rotation θ_1 about $\hat{b}_\alpha = \hat{b}'_\alpha$, which brings $\{\hat{b}\}$ into position $\{\hat{b}'\}$.
- β denotes the axis of the second rotation θ_2 about $\hat{b}'_\beta = \hat{b}''_\beta$, which brings $\{\hat{b}'\}$ into $\{\hat{b}''\}$

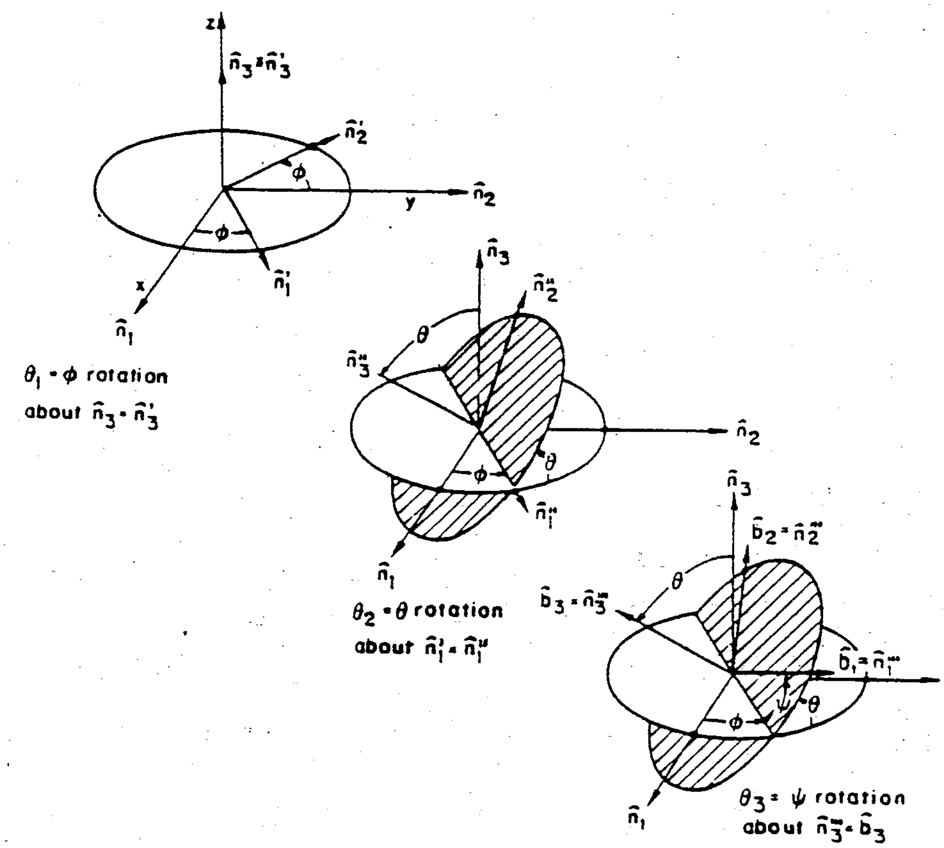

Figure 2.5 The 3-1-3 Euler Angles

γ denotes the axis of the third rotation θ_3 about $\hat{b}''_{\gamma} = \hat{b}'''_{\gamma}$, which brings $\{\hat{b}''\}$ into $\{\hat{b}'''\}$.

For example, the "3-1-3" description of the Euler angles of Figure 2.5 is clearly consistent with this designation. In this particular case, we observe rotation θ_1 about $\hat{n}_3 \equiv \hat{n}'_3$ results in the orthogonal projection

$$\begin{Bmatrix} \hat{n}'_1 \\ \hat{n}'_2 \\ \hat{n}'_3 \end{Bmatrix} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix} \tag{2.32a*}$$

Rotation θ_2 about $\hat{n}'_1 \equiv \hat{n}''_1$ results in

$$\begin{Bmatrix} \hat{n}''_1 \\ \hat{n}''_2 \\ \hat{n}''_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_2 & s\theta_2 \\ 0 & -s\theta_2 & c\theta_2 \end{bmatrix} \begin{Bmatrix} \hat{n}'_1 \\ \hat{n}'_2 \\ \hat{n}'_3 \end{Bmatrix} \tag{2.32b}$$

Rotation θ_3 about $\hat{n}''_3 = \hat{n}'''_3 \equiv \hat{b}_3$ yields

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{n}''_1 \\ \hat{n}''_2 \\ \hat{n}''_3 \end{Bmatrix} \tag{2.32c}$$

Substitution of Eq. 2.32a into Eq. 2.32b and the result into Eq. 2.32c, the direction cosine matrix has the 3-1-3 Euler angle parameterization

$$\{\hat{b}\} = [C(\theta_1,\theta_2,\theta_3)]\{\hat{n}\}$$

with

$$[C(\theta_1,\theta_2,\theta_3)] = \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_2 & s\theta_2 \\ 0 & -s\theta_2 & c\theta_2 \end{bmatrix} \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.33}$$

or, carrying out the implied matrix multiplications

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*Here we introduce the abbreviations $c \equiv \cos$, $s \equiv \sin$, which we will employ throughout this text to compact transcendental expressions.

$$[C(\theta_1,\theta_2,\theta_3)] = \begin{bmatrix} c\theta_3 c\theta_1 - s\theta_3 c\theta_2 s\theta_1 & c\theta_3 s\theta_1 + s\theta_3 c\theta_2 c\theta_1 & s\theta_3 s\theta_2 \\ -s\theta_3 c\theta_1 - c\theta_3 c\theta_2 s\theta_1 & -s\theta_3 s\theta_1 + c\theta_3 c\theta_2 c\theta_1 & c\theta_3 s\theta_2 \\ s\theta_2 s\theta_1 & -s\theta_2 c\theta_1 & c\theta_2 \end{bmatrix} \qquad (2.34)$$

From Eq. 2.34, we see that the angles can be calculated, given the direction cosines, from the inverse transformations

$$\theta_1 = \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right) \quad , \quad \theta_2 = \cos^{-1}(C_{33}) \quad , \quad \theta_3 = \tan^{-1}\left(\frac{C_{13}}{C_{23}}\right) \qquad (2.35a,b,c)$$

In general, the direction cosines can be formed from any of the twelve sets of Euler angles via multiplication of three elementary rotation matrices; for a general α-β-γ rotation sequence, the direction cosine matrix has the form

$$[C(\theta_1,\theta_2,\theta_3)] = [M_\gamma(\theta_3)][M_\beta(\theta_2)][M_\alpha(\theta_1)] \qquad (2.36)$$

where the three elementary rotation matrices are

$$[M_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \qquad (2.37a)$$

$$[M_2(\theta)] = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \qquad (2.37b)$$

$$[M_3(\theta)] = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.37c)$$

It is probably obvious, but we emphasize that the above discussion employs "sequential rotations" in the instantaneous geometric sense. By this we mean that the *instantaneous* position of $\{\hat{b}\}$ relative to $\{\hat{n}\}$ has direction cosines which can be calculated via Eq. 2.36. Clearly, we do not restrict our interpretation of Eq. 2.36 to the special case that the rotations are in fact α-β-γ sequential stop-start angular motions about fixed axes. An infinity of sequential motions *could have* led to any instantaneous values for [C], but

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the instantaneous [C] matrix, exclusive of certain singularities, can still be *described* as an instantaneous composite of three Eulerian rotations. This situation is quite analogous to the more familiar truth that one can choose rectangular or spherical coordinates to describe the same dynamical path of a particle. However, the issue is sometimes clouded by particular physical gimbaled devices which *do* execute specific Euler angle rotations, either sequentially or simultaneously.

In the rotational dynamics of Chapter 3 and subsequent developments, we will find that the differential equations for the three angles (or other parameters used to describe orientation) play a central role. Usually one encounters three or more kinematic equations of the functional form

$$\dot{\theta}_i = f_i(\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3, t, \ldots) \quad , \quad i = 1,2,3 \tag{2.38}$$

where

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{b}}_1 + \omega_2 \hat{\mathbf{b}}_2 + \omega_3 \hat{\mathbf{b}}_3 \tag{2.39}$$

is the angular velocity of $\{\hat{\mathbf{b}}\}$ relative to $\{\hat{\mathbf{n}}\}$.

To illustrate the general process for establishing these differential equations for the α-β-γ Euler angles, we consider the 3-1-3 case in detail.

From Figure 2.5, it is apparent that the angular velocity can be written as

$$\boldsymbol{\omega} = \dot{\theta}_1 \hat{\mathbf{n}}_3 + \dot{\theta}_2 \hat{\mathbf{n}}_1'' + \dot{\theta}_3 \hat{\mathbf{b}}_3 \tag{2.40}$$

From Eqs. 2.32c and 2.34, it follows that

$$\hat{\mathbf{n}}_1'' = \cos\theta_3 \hat{\mathbf{b}}_1 - \sin\theta_3 \hat{\mathbf{b}}_2 \tag{2.41a}$$

$$\hat{\mathbf{n}}_3 = \sin\theta_3 \sin\theta_2 \hat{\mathbf{b}}_1 + \cos\theta_3 \sin\theta_2 \hat{\mathbf{b}}_2 + \cos\theta_2 \hat{\mathbf{b}}_3 \tag{2.42b}$$

which upon substituting into Eq. 2.40 and equating the result to Eq. 2.39 yields the kinematic equation

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} s\theta_3 s\theta_2 & c\theta_3 & 0 \\ c\theta_3 s\theta_2 & -s\theta_3 & 0 \\ c\theta_2 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} ; \tag{2.43}$$

the inverse of Eq. 2.43 is the kinematic differential equation for 3-1-3 Euler angles

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \frac{1}{s\theta_2} \begin{bmatrix} s\theta_3 & c\theta_3 & 0 \\ c\theta_3 s\theta_2 & -s\theta_3 s\theta_2 & 0 \\ -s\theta_3 c\theta_2 & -c\theta_3 c\theta_2 & s\theta_2 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{2.44}$$

which has an obvious singularity for $\theta_2 = 0, \pi$ (in which $\dot{\theta}_1$ and $\dot{\theta}_3$ are undefined, regardless of the behavior of the $\omega_i(t)$). This singularity is also evident in the inverse transformations of Eq. 2.35 for θ_1 and θ_3. Referring to Figure 2.5, the geometric interpretation of this singularity is that the $\theta_2 = 0, \pi$ conditions correspond to the vanishing of the line of nodes (the $(\hat{n}_1, \hat{n}_2)$ and $(\hat{b}_1, \hat{b}_2)$ planes are coincident). In general, when two of three Euler angles are measured in the same plane, their values are not uniquely determined and a singularity occurs. The kinematic relationships Eqs. 2.43 and 2.44 can be written compactly as

$$\{\omega\} = [B(\theta_2, \theta_3)]\{\dot{\theta}\} \tag{2.45a}$$

$$\{\dot{\theta}\} = [B(\theta_2, \theta_3)]^{-1}\{\omega\} \tag{2.45b}$$

The $[B]$ and $[B]^{-1}$ matrices are summarized in Table 2.1 for all 12 sets of α-β-γ Euler angles. Table 2.1 also summarizes the inverse transformations from direction cosines of Eq. 2.36 to the corresponding Euler angle parameterization of $[C]$.

In many applications, it is possible to select a judicious set of the Euler angles which avoids, for all practical purposes, the singularity at $\theta_2 = 0, \pm\pi$ or $\theta_2 = \pm\pi/2$ (see Table 2.1). In a significant subset of applications, it is desirable to linearize the kinematic relationships of Table 2.1. In this situation, it is extremely important that an Euler angle set be chosen so that the anticipated small motions are "far away" from the singularity (preferably 90° away). Regardless of the "smallness" of the physical angular motion, linearizations of the results in Table 2.1 are likely to be invalid near a ation for 3-1-3 Euler

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TABLE 2.1 EULER ANGLE GEOMETRIC AND KINEMATIC FORMULA SUMMARY

Direction Cosine Parameterization: $[C(\theta_1,\theta_2,\theta_3)] = [M_\gamma(\theta_3)][M_\beta(\theta_2)][M_\alpha(\theta_1)]$

Angular Velocity/Angular Rate Transformation: $\{\omega\} = [B(\theta_2,\theta_3)]\{\dot{\theta}\}$

Abbreviation $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$, $c^{-1}(\ )$ = arc cosine( ), $s^{-1}(\ )$ = arc sine ( ), $t^{-1}(\ )$ = arc tangent( )

α-β-γ ROTATION SEQUENCE	B	B^{-1}	ANGLES AS FUNCTIONS OF DIRECTION COSINES	SINGULAR AT θ_2 =
1-2-1	$\begin{bmatrix} c_2 & 0 & 1 \\ s_2 s_3 & c_3 & 0 \\ s_2 c_3 & -s_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} 0 & s_3 & c_3 \\ 0 & s_2 c_3 & -s_2 c_3 \\ s_2 & -c_2 s_3 & -c_2 c_3 \end{bmatrix}$	$\theta_1 = t^{-1}(C_{12}/-C_{13})$ $\theta_2 = c^{-1}(C_{11})$ $\theta_3 = t^{-1}(C_{21}/C_{31})$	$0, \pm\pi$
1-2-3	$\begin{bmatrix} c_2 c_3 & s_3 & 0 \\ -c_2 s_3 & c_3 & 0 \\ s_2 & 0 & 1 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} c_3 & -s_3 & 0 \\ c_2 s_3 & c_2 c_3 & 0 \\ -s_2 c_3 & s_2 s_3 & c_2 \end{bmatrix}$	$\theta_1 = t^{-1}(-C_{32}/C_{33})$ $\theta_2 = s^{-1}(C_{31})$ $\theta_3 = t^{-1}(-C_{21}/C_{11})$	$\pm\pi/2$
1-3-1	$\begin{bmatrix} c_2 & 0 & 1 \\ -s_2 c_3 & s_3 & 0 \\ s_2 s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} 0 & -c_3 & s_3 \\ 0 & s_2 s_3 & s_2 c_3 \\ s_2 & c_2 c_3 & -c_2 s_3 \end{bmatrix}$	$\theta_1 = t^{-1}(C_{13}/C_{12})$ $\theta_2 = c^{-1}(C_{11})$ $\theta_3 = t^{-1}(C_{31}/-C_{21})$	$0, \pm\pi$
1-3-2	$\begin{bmatrix} c_2 c_3 & -s_3 & 0 \\ -s_2 & 0 & 1 \\ c_2 s_3 & c_3 & 0 \end{bmatrix}$	$\frac{1}{c_2}\begin{bmatrix} c_3 & 0 & s_3 \\ -c_2 s_3 & 0 & c_2 c_3 \\ s_2 c_3 & c_2 & s_2 s_3 \end{bmatrix}$	$\theta_1 = t^{-1}(C_{23}/C_{22})$ $\theta_2 = s^{-1}(-C_{21})$ $\theta_3 = t^{-1}(C_{31}/C_{11})$	$\pm\pi/2$
2-1-2	$\begin{bmatrix} s_2 s_3 & c_3 & 0 \\ c_2 & 0 & 1 \\ -s_2 c_3 & s_3 & 0 \end{bmatrix}$	$\frac{1}{s_2}\begin{bmatrix} s_3 & 0 & -c_3 \\ s_2 c_3 & 0 & s_2 s_3 \\ -c_2 s_3 & s_2 & c_2 c_3 \end{bmatrix}$	$\theta_1 = t^{-1}(C_{21}/C_{23})$ $\theta_2 = c^{-1}(C_{22})$ $\theta_3 = t^{-1}(C_{12}/-C_{32})$	$0, \pm\pi$