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Robust Output-Feedback Nonlinear Model Predictive Control Using High-Gain Observers

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Abstract

A robust output-feedback nonlinear model predictive control (NMPC) system is proposed by combining state-feedback NMPC with a certain type of high-gain observer. We show that robust stability of this formulation deteriorates from its state-feedback NMPC counterpart because of observation error. Then we demonstrate this idea through simulation studies of a CSTR example and offset free regulatory behavior is obtained because the estimated error is used to correct model-mismatch in the controller.
Keywords: Robust controller, Output-feedback system, High-gain observer.

1. Introduction

Nonlinear model predictive control (NMPC) has received considerable attention over the past years. Schemes to guarantee the stability of the closed-loop systems with state-feedback NMPC have been widely studied (Limon et al, 2008 & Mayne et al, 2000). However, the assumption that the plant is available for measurement usually does not hold in practice. Hence, the output-feedback NMPC controller is required by integrating a state-feedback NMPC controller with a state observer. For linear systems, the well-known separation principle guarantees that the closed-loop system is nominally stable if both the controller and the state observer are stable. For nonlinear systems, the issue of stability of the closed-loop systems is still open due to the lack of general valid separation principles. In order to analyze the nominal stability for certain nonlinear output-feedback systems, two possibilities have been explored: 1) developing a certain equivalent separation principle and designing observers accordingly (Atassi & Khalil, 1999, Findeisen, et al, 2003); 2) considering the observer error in the NMPC controller (Michalska & Mayne, 1995, Magni et al, 2004).

The above formulations focus on the nominal closed-loop stability or stability in the presence of vanishing perturbations. In this work, we propose a robust output-feedback NMPC based on a high-gain observer. We show the robust stability analysis for the proposed formulation with bounded observation errors. Section 2 starts with the introduction of the output-feedback system and stability analysis. Section 3 presents simulation examples of continuous stirred tank reactor (CSTR). We also demonstrate that the proposed formulation generates offset-free regulatory and servo behavior under moderate perturbations in model parameters. Section 4 concludes the paper.

2. Problem Formulation

2.1. State-feedback NMPC and stability

In this work, we consider the dynamic model of a plant with output,

$$x_{k+1} = f(x_k, u_k, \theta_k) \qquad y_k = Cx_k, \quad k \geq 0 \tag{1}$$

where $x_k \in \Re^{n_x}$, $u_k \in \Re^{n_u}$, $y_k \in \Re^{n_y}$ and $\theta_k \in \Re^{n_\theta}$ are the plant states, controls, outputs and uncertainty parameters respectively, defined at time steps t_k with integers $k \geq 0$. Without losing generality, we assume that the given plant (1) has an equilibrium point at the origin, that is $f(0,0,0)=0$.

Given $x(k)$, the current state value at time step t_k, the state-feedback NMPC formulation can be described in the following discretized form:

$$V(x_k) := \min \sum_{j=0}^{N-1} l(z_{k+j}, u_{k+j}) + F(z_{k+N}) \tag{2a}$$

$$\text{s.t} \qquad z_{k+j+1} = f(z_{k+j}, u_{k+j}, 0), \quad j = 0, \ldots, N-1 \tag{2b}$$

$$z_k = x_k, z_{k+j} \in \mathcal{X}, z_{k+N} \in \mathcal{X}_f, u_{k+j} \in \mathcal{U} \tag{2c}$$

where N is the finite time horizon, z_{k+j} is the sequence of the predicted state variables, and u_k is the calculated control action based on the plant state x_k at time step t_k. Note the uncertainty parameter is *0* in the controller, introducing the plant-model mismatch. The calculated state-feedback control law from (2) can be written as $u_k = h(x_k)$, and the plant state at the next time step t_{k+1} can be expressed as $f(x_k, h(x_k), \theta)$.

The closed-loop stability with the state-feedback control law has been widely studied. Here we summarize the key definitions and analysis developed in Limon et al 2008. Here, $|\cdot|$ denotes the Euclidean vector norm in $\Re^n$ or the associated matrix norm. For a given sequence w_k, we denote $\|w\| \triangleq \sup_{k\geq 0}\{|w_k|\}$. For two $\mathcal{K}$ functions τ_1 and τ_2, we define $\tau_1 \circ \tau_2(s) \triangleq \tau_1(\tau_2(s))$.

Lemma 1: *If f(x,y) is a uniformly continuous function in both $x \in A$ and $y \in B$, then there exist $\mathcal{K}$ functions τ_1 and τ_2, such that $|f(x_1,y_1) - f(x_2,y_2)| \leq \tau_1(|x_1 - x_2|) + \tau_2(|y_1 - y_2|)$.*

Theorem 1: (Limon et al 2008) *Assume function $f(x, h(x), \theta)$ is uniformly continuous in θ, and the plant (1) is nominally asymptotically stable. If there exists a uniformly continuous Lyapunov function $V(x)$ for plant (1), then the closed-loop system with state-feedback controller is Input-to-State stable (ISS).*

2.2. Output-feedback NMPC and stability

Consider now a state observer for plant (1),

$$\hat{x}_{k+1} = g(\hat{x}_k, y_k, h(\hat{x}_k)), \quad k \geq 0 \tag{3}$$

where $\hat{x}_k$ is the state estimated from the outputs at time step t_k. Hence, the initial condition in the NMPC formulation is modified as $z_k = \hat{x}_k$, instead of $z_k = x_k$ in (2c). The calculated output-feedback control law is $u_k = h(\hat{x}_k)$, and the plant state at t_{k+1} can be modified as $f(x_k, h(\hat{x}_k), \theta)$.

In order to establish the robust stability of the output-feedback NMPC formulation, we make use the following assumptions.

Assumption 1 (Observer assumption)

a) the initial observer error is bounded by a positive constant e, i.e. $|\hat{x} - x| \leq e$;

b) the future observer error satisfies $|\hat{x}_k - x_k| \leq \rho|e|\eta^{-k} + \beta(\|\theta\|)$, where $\rho \geq 0, \eta > 1$ are constants, β is a $\mathcal{K}$ function.

c) the observer is a high-gain observer.

Lemma 2: *There exists a constant $\epsilon \geq 0$ such that $\rho|e|\eta^{-k} \leq \epsilon$. As a result $|\hat{x}_k - x_k| \leq \epsilon + \beta(\|\theta\|)$.*

Lemma 3: *If $|\hat{x}-x| \leq e$, then $|x|-e \leq |\hat{x}| \leq |x|+e$. For a $\mathcal{K}$ function $\alpha(\cdot)$, there are other $\mathcal{K}$ functions $\alpha_L(\cdot)$, $\alpha_U(\cdot)$ and positive constant c_1, c_2, such that $\alpha_L(|x|)-c_1 \leq \alpha(|x|-e) \leq \alpha(|\hat{x}|) \leq \alpha(|x|+e) \leq \alpha_U(|x|)+c_2$. Similarly if $|\hat{x}-x| \leq \epsilon+\beta(\|\theta\|)$, we can find other $\mathcal{K}$ functions $\hat{\alpha}_L(\cdot)$, $\hat{\alpha}_U(\cdot)$, $\beta_L(\cdot)$, $\beta_U(\cdot)$ and positive constants c_3, c_4, such that $\hat{\alpha}_L(|x|)-\beta_L(\|\theta\|)-c_3 \leq \alpha(|\hat{x}|) \leq \hat{\alpha}_U(|x|)+\beta_U(\|\theta\|)+c_4$.*

Robust stability of this output-feedback NMPC can be established by the following:

Theorem 2: *Assume function $f(x,h(\hat{x}),\theta)$ is uniformly continuous in θ and x, and system (1) is nominally asymptotically stable and the observer satisfies Assumption 1. If there exists a uniformly continuous Lyapunov function $V(x)$, then the closed-loop system with output-feedback controller is Input-to-state practical stable (ISpS).*

Proof: The analysis is similar as the proof for Theorem 2 in Limon et al 2008, but we need to consider the observer error. From the continuity of $V(x)$ and $f(x,h(\hat{x}),\theta)$, there exist $\mathcal{K}$ functions α, σ_V, σ_f and σ_θ such that $V(f(\hat{x},h(\hat{x}),0))-V(\hat{x}) \leq -\alpha(|\hat{x}|)$, $V(\hat{x})-V(x) \leq \sigma_V(|\hat{x}-x|)$, and $V(f(x,h(\hat{x}),\theta))-V(f(\hat{x},h(\hat{x}),0)) \leq \sigma_V(|f(x,h(\hat{x}),\theta)-f(\hat{x},h(\hat{x}),0)|) \leq \sigma_V \circ (\sigma_f(|\hat{x}-x|)+\sigma_\theta(|\theta|))$. As a result:

$$\begin{aligned} V(f(x,h(\hat{x}),\theta))-V(x) &= V(f(\hat{x},h(\hat{x}),0))-V(\hat{x})+V(\hat{x})-V(x) \\ &\quad +V(f(x,h(\hat{x}),\theta))-V(f(\hat{x},h(\hat{x}),0)) \\ &\leq -\alpha(|\hat{x}|)+\sigma_V(|\hat{x}-x|)+\sigma_V \circ (\sigma_f(|\hat{x}-x|)+\sigma_\theta(|\theta|)) \end{aligned} \tag{4}$$

Now, we need to ensure that the closed-loop system is robustly stable for both the initial stage of the observer when $|\hat{x}-x| \leq e$ and later stages when the observer error is corrupted by the uncertainties, i.e. $|\hat{x}-x| \leq \epsilon+\beta(\|\theta\|)$.

If $|\hat{x}-x| \leq e$, from Lemma 3, we have $-\alpha(|\hat{x}|) \leq -\alpha_L(|x|)+c_1$. In addition, we can find a constant $c_5 \geq 0$ such that $\sigma_V(e)+\sigma_V \circ \sigma_f(e)+c_1 \leq c_5$ and a $\mathcal{K}$ function σ_2 such that $\sigma_V \circ \sigma_\theta(|\theta|)) \leq \sigma_2(|\theta|)$. Consequently equation (4) leads to:

$$\begin{aligned} V(f(x,h(\hat{x}),\theta))-V(x) &\leq -\alpha(|\hat{x}|)+\sigma_V(|\hat{x}-x|)+\sigma_V \circ (\sigma_f(|\hat{x}-x|)+\sigma_\theta(|\theta|)) \\ &\leq -\alpha_L(|x|)+\sigma_2(|\theta|)+c_5 \end{aligned} \tag{5}$$

Then the closed-loop system is ISpS stable for the initial observer error.

If $|\hat{x}-x| \leq \epsilon+\beta(\|\theta\|)$, we have $-\alpha(|\hat{x}|) \leq -\hat{\alpha}_L(|x|)+\beta_L(\|\theta\|)+c_3$, then we can pick a constant $c_6 \geq 0$ and a $\mathcal{K}$ function σ_3 such that $\beta_L(\|\theta\|)+c_3+\sigma_V(|\hat{x}-x|)$ $+\sigma_V \circ \sigma_f(|\hat{x}-x|)+\sigma_V \circ \sigma_\theta(|\theta|) \leq \sigma_3(\|\theta\|)+c_6$. Then equation (4) leads to:

$$\begin{aligned} V(f(x,h(\hat{x}),\theta))-V(x) &\leq -\alpha(|\hat{x}|)+\sigma_V(|\hat{x}-x|)+\sigma_V \circ (\sigma_f(|\hat{x}-x|)+\sigma_\theta(|\theta|)) \\ &\leq -\hat{\alpha}_L(|x|)+\beta_L(\|\theta\|)+c_3+\sigma_V(|\hat{x}-x|)+\sigma_V \circ \sigma_f(|\hat{x}-x|)+\sigma_V \circ \sigma_\theta(|\theta|) \\ &\leq -\hat{\alpha}_L(|x|)+\sigma_3(\|\theta\|)+c_6 \end{aligned} \tag{6}$$

Then the closed-loop system is ISpS stable for the later stage when observer error is corrupted by the uncertainties. ?

Remark: From the analysis, we see that the ISS stability of state-feedback system deteriorates to ISpS stability of output-feedback system, because of the corruption of observer error. For the output-feedback system, all the past uncertainty signals affect the closed-loop stability, while only the current uncertainty signal plays a role in the robust stability for the state-feedback system. In addition, stronger assumption is required to guarantee the robust stability of the output-feedback system.

2.3. EKF as the High-gain observer stability

In this section we consider an extended Kalman filter (EKF) developed by Reif & Unbehauen (1999) as the state observer, satisfying Assumption 1.

$$x_{k+1}^{-} = f(\hat{x}_k, u_k, 0) \quad \hat{x}_k = x^{-} + K_k[y_k - Cx_k^{-}] \tag{7}$$

where x_k^- and $\hat{x}_k$ are called the prior and posterior estimate. K_k is the Kalman gain calculated from:

$$K_k = P_k^- C^T[CP_k^- C^T + R]^{-1} \quad P_{k+1}^- = \alpha^2 A_k P_k^+ A_k^T + Q \tag{8a}$$

$$P_k^+ = [I - K_k C]P_k^- \tag{8b}$$

where $A_k = \dfrac{\partial f}{\partial x}(\hat{x}_k, u_k, 0)$, $G_k = \dfrac{\partial f}{\partial \theta}(\hat{x}_k, u_k, 0)$ are the linearizations of the model. Q and R are symmetric positive definite matrices. $\alpha \geq 1$ is a exponential data weight to control the convergence rate of the EKF. Hence this EKF formulation is a high-gain observer. If $\alpha = 1$, this formulation reduces to the conventional EKF. Let $\varphi(\cdot,\cdot,\cdot)$ be the higher order term, the residual of the EKF can be defined by the following equations:

$$f(x_k, u_k, \theta) - f(\hat{x}_k, u_k, 0) = A_k[x_k - \hat{x}_k] + G_k\theta_k + \varphi(x_k, \hat{x}_k, u_k, \theta_k) \tag{9}$$

Defining estimation error as $\zeta_k = x_k - \hat{x}_k$, we have from (1), (7), (8) and (9) ,

$$\zeta_{k+1} = (I - K_{k+1}C)[A_k\zeta_k + r_k] \tag{10}$$

where $r_k = \varphi(x_k, \hat{x}_k, u_k, \theta_k) + G_k\theta_k$

Theorem 2: *Consider the EKF as stated by equations (7) and (8) and let the following assumptions hold:*

- *a) There are positive numbers $\overline{a}, \underline{p}, \overline{g}$ and $\overline{p}$ such that $|A_k| \leq \overline{a}$, $|G_k| \leq \overline{g}$, $\underline{p}I \leq P_k^- \leq \overline{p}I$ and $\underline{p}I \leq P_k^+ \leq \overline{p}I$, $\forall k \geq 0$.*
- *b) A_k is nonsingular for $k \geq 0$.*
- *c) $\theta_k \in \Omega_\theta \subset \mathfrak{R}^{n_\theta}$ where Ω_θ is a compact set.*
- *d) There are positive real numbers κ, γ_θ, $\mathcal{K}$ functions $\delta(\|\theta\|)$ and $\mu(\|\theta\|)$, such that $|\varphi(x_k, \hat{x}_k, u_k, \theta_k)| \leq \kappa|x - \hat{x}_k|^2 + \delta(\|\theta\|)$ whenever $|x - \hat{x}_k| \leq \mu(\|\theta\|) + \gamma_\theta \leq \bar{\varepsilon}$.*

Then with Assumption 1.a, there exist $\rho \geq 0, \eta > 1$ and $c_e \geq 0$ are constants, β is a $\mathcal{K}$ function, such that the error sequence ζ_k defined in equation (10) behaves according to $|\zeta_k| \leq \rho|\zeta_0|\eta^{-k} + \beta(\|\theta\|)$.

The proof follows the similar line from the proof in Reif & Unbehauen (1999). With this theorem, we can see Assumption 1.b and 1.c are true for this EKF.

2.4. Output-feedback NMPC with EKF

To achieve offset free closed-loop behavior, we propose to carry out multi-step predictions by explicitly using observer errors for future predictions as follows:

$$z_{k+j+1} = f(z_{k+j}, u_{k+j}, 0) + K_k\beta_{k+j},\ y_{k+j} = Cz_{k+j} + \eta_{k+j},\ j = 0, \ldots, N-1 \tag{11a}$$

$$\beta_{k+j+1} = \beta_{k+j}, \eta_{k+j+1} = \eta_{k+j}, z_k = \hat{x}_k,\ \beta_k = y_k - Cx_k^-,\ \eta_k = y_k - C\hat{x}_k \tag{11b}$$

The predictive controller is then formulated as

$$\min_u \ \sum_{j=1}^{N-1}[E_{k+j}^T W_E E_{k+j}] + \sum_{i=0}^{q-1} \Delta u_{k+i}^T W_{\Delta u} u_{k+i} + E_{k+N}^T W_\infty E_{k+N}$$

$$s.t.\ z_{k+j} \in \mathcal{X}, z_{k+N} \in \mathcal{X}_f, u_{k+i} \in \mathcal{U}, j = 0, \ldots, N-1,\ i = 0, \ldots, q-1, \quad \text{(11a,b)}$$

$$u_{k+j} = u_{k+t_i} \text{ for } t_i \le j < t_{i+1},\ t_0 = 0 < t_1 < t_2 < \ldots < t_{q-1} = N-1. \quad (12)$$

where $E_{k+j} \triangleq (y_{k+j} - y_r)$, $W_E, W_{\Delta u}$ and W_∞ are positive semi-definite matrices.

3. Simulation Studies

We consider a simulated NMPC scenario with a nonlinear CSTR model represented by the following differential equations:

$$\frac{dz_c}{dt} = (z_c - 1)/u_1 + k_0 z_c \exp(-E_a/z_T) \quad \text{(13a)}$$

$$\frac{dz_T}{dt} = (z_T - z_T^f)/u_1 + k_0 z_c \exp(-E_a/z_T) + \nu u_2(z_T - z_T^{cw}) \quad \text{(13b)}$$

This system involves two states $z = [z_c, z_T]$ corresponding to dimensionless concentration and temperature, and two manipulated inputs, corresponding to the inverse of dilution rate (u_1) and cooling water flow rate (u_2). The model parameters are $z_T^{cw} = 0.38$, $z_T^f = 0.395$, $E_a = 5$, $\nu = 1.95 \times 10^4$ and k_0 is an uncertainty parameter in the plant with nominal value $k_0 = 300$ in the model. The system is operated at a stable steady state $z_c = 0.1247$ and $z_T = 0.74070$ corresponding to $u_1 = 20$ and $u_2 = 378$. The NMPC is formulated using $W_E = W_\infty = \text{diag}[1 \times 10^6 \ 1 \times 10^5]$, prediction horizon $N = 20$, control horizon $q = 5$, with input blocking and each block equals to 4 samples, and sampling time is 1. $W_{?u}$ is chosen as a null matrix. The EKF is tuned with $Q = \left[\frac{\partial f}{\partial u}|_{z_{ss}, u_{ss}}\right] \bar{Q} \left[\frac{\partial f}{\partial u}|_{z_{ss}, u_{ss}}\right]^T$, where $\bar{Q}$ is chosen to be the possible variations in the inputs $\text{diag}[6.25, 0.04]$, and $R = \text{diag}[1 \times 10^{-6}, 1 \times 10^{-6}]$ which is the possible covariance of the outputs. To ensure that EKF works as a high gain observer, α in equation (8a) is chosen equal to 2.5.

Figure 1 presents the variation of controlled output and state estimation errors generated in response to a sequence of +/- 40 % step changes in model parameter k_0 (see Figure 2) for two different scenarios, 1) without output noises, 2) outputs are corrupted with white noise with standard deviation of $[1 \times 10^{-3}, 1 \times 10^{-3}]^T$. The corresponding variation of manipulated inputs is presented in Figure 2. It is clear from Figure 1 that, each time after a step change is introduced in k_0, the state estimator errors generated by the high gain observer settle to a constant value in a few samples. Moreover, the high gain observer's performance is hardly influenced by the tuning of the controller. It may be noted that the proposed output-feedback NMPC formulation eliminates the offset in both the cases.

4. Concluding Remarks

The purpose of this paper is to point out that although there are no general separation principles for nonlinear systems with plant-model mismatches, the closed-loop stability can still be ensured by using certain types of observers, and the controller performance can be improved by considering the observer error and tuning the observer. In future, this technique will be implemented on large scale applications and other types of observers will also be studied.

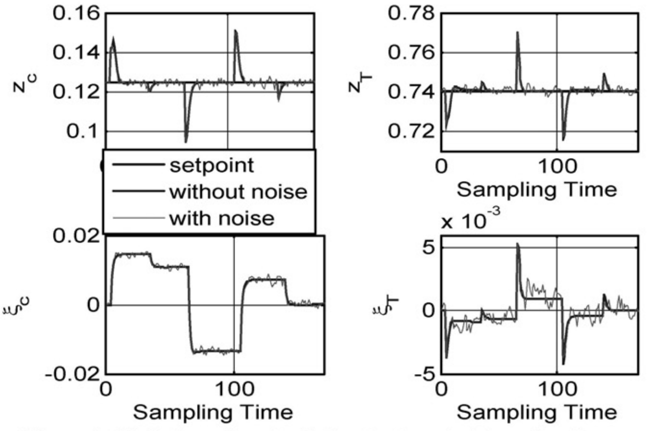

Figure 1: Variation of controlled outputs and state estimation errors

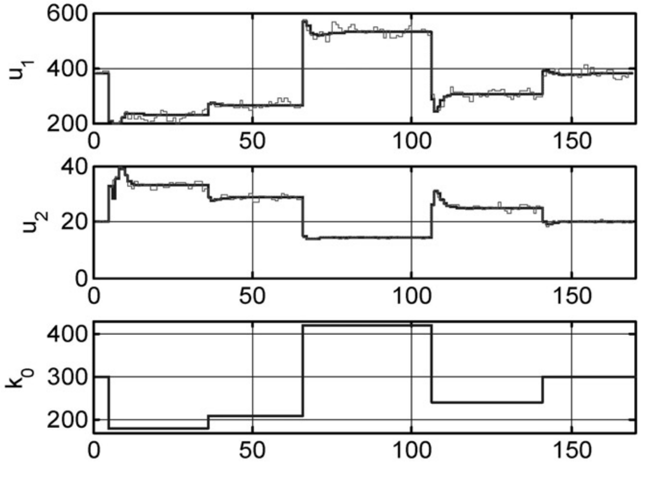

Figure 2: Variation of manipulated input and parameter disturbance

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