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Global tracking sliding mode control for a class of nonlinear systems via variable gain observer

Alessandro Jacoud Peixoto[1,*,†], Tiago Roux Oliveira[2], Liu Hsu[2], Fernando Lizarralde[2] and Ramon R. Costa[2]

[1] *Department of Electrical Engineering/CEFET-RJ, Federal Center of Technology, Rio de Janeiro, Brazil*
[2] *Department of Electrical Engineering/COPPE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil*

SUMMARY

A novel output-feedback sliding mode control strategy is proposed for a class of single-input single-output (SISO) uncertain time-varying nonlinear systems for which a norm state estimator can be implemented. Such a class encompasses minimum-phase systems with nonlinearities affinely norm bounded by unmeasured states with growth rate depending nonlinearly on the measured system output and on the internal states related with the zero-dynamics. The sliding surface is generated by using the state of a high gain observer (HGO) whereas a peaking free control amplitude is obtained via a norm observer. In contrast to the existing semi-global sliding mode control solutions available in the literature for the class of plants considered here, the proposed scheme is free of peaking and achieves global tracking with respect to a small residual set. The key idea is to design a time-varying HGO gain implementable from measurable signals. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: sliding mode control; uncertain nonlinear systems; output-feedback; high gain observer; norm observer; global practical tracking

1. INTRODUCTION

Several approaches to deal with the tracking problem by output-feedback sliding mode (OFSM) control for arbitrary relative degree uncertain systems have been proposed in the literature [1–5], where strategies using high gain observers (HGOs) [6, 7] represent a particular important design class. Exact output tracking can be achieved via higher order sliding mode control based on robust exact differentiators [8]. However, stability and/or convergence of the overall control system is guaranteed only locally. Most available OFSM designs achieve global results only under rather stringent assumptions, such as linearly or uniformly globally bounded vector fields [4, 5, 7].

More general nonlinear plants are dealt within [5, 6, 9, 10], but only semi-global tracking was achieved. This is not surprising since, as shown in [11], for systems with polynomial nonlinearities

*Correspondence to: Alessandro Jacoud Peixoto, Departamento de Engenharia Elétrica (DEPEL/CEFET-RJ), Maracanã 229, Bloco E, Andar 1, Maracanã, Rio de Janeiro, Brazil.
†E-mail: jacoud@coep.ufrj.br

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in the unmeasured states [6, 12], the global stabilization/tracking problem via continuous output-feedback may not be solvable.

Beyond the sliding mode control, several approaches to solve the global tracking problem by output feedback have been proposed based on backstepping-like designs, time-varying high gain techniques (HGO with variable gain) [13–17], homogeneity in the bi-limit [18, 19] and some kind of adaptation [20]. In contrast, no global tracking results are available for the class of systems dealt here in the domain of OFSM control, where robustness and good transient properties can be significant advantages.

We believe that the class of system considered in this note is in the state of the art of global output-feedback control framework commonly considered by other authors [13, 16, 18, 19, 21, 22]. We deal with time-varying minimum-phase nonlinear plants, affine in the control, transformable to a normal form and for which a norm state estimator can be implemented. Such a class encompasses the standard output feedback form, the parametric strict feedback form, lower triangular systems with linear growth condition in the unmeasured states and growth rate possibly depending on the inverse dynamics unmeasured state, on the system output and time. Strong polynomial nonlinearities in the inverse dynamics state and the output system are also allowed.

In the recent years Praly and several others have shown that, by using dynamic observer gain, global results can be achieved without invoking the global Lipschitz conditions or 'output-feedback' forms. Time-varying HGOs have also been used to cope with the effect of measurement noise and to establish the connections with the Extended Kalman Filter [17, 23].

In this paper, we extend the applicability of [24] to a wider class of nonlinear plants. The main result is to show that an OFSM control based on an HGO with dynamic observer gain can also be used for a state-of-the-art class of nonlinear systems to guarantee global practical tracking. Differently from most of the existing schemes, the HGO gain is not updated through a Riccati equation [13, 18, 22] but, instead, we use simple functions (e.g. polynomials) based on measurable signals and norm domination techniques [16, 18, 19]. To the best of our knowledge, this is the first *global* OFSM tracking scheme for the class of plants considered here.

A well-known drawback of HGO-based control strategies is the peaking phenomenon [25], which can degrade the system performance or even lead to instability. Peaking avoidance through control saturation has already been proposed by Oh and Khalil [6] and Esfandiari and Khalil [9], but such an approach leads only to semi-global results. Here, following [7], we circumvent control peaking by using measurable signals and estimates not based on high gain to generate the control law magnitude while the HGO is used in the proposed sliding mode scheme only to generate the sliding surface.

Global asymptotic stability with respect to a compact set and ultimate exponential convergence to a small residual set in the error space are obtained. Two academic examples illustrate the class of systems and the time-varying behaviour of the HGO gain.

2. PRELIMINARIES

The following notations and terminology are used:

- The 2-norm (Euclidean) of a vector $x=[x_1 \;\; x_2 \;\; \ldots \;\; x_n]^{\mathrm{T}}$ and the corresponding induced norm of a matrix A are denoted by $|x|$ and $|A|$, respectively. The symbol $\lambda[A]$ denotes the spectrum of A and $\lambda_m[A]=-\max_i\{Re\{\lambda[A]\}\}$.
- The $\mathscr{L}_{\infty e}$ norm of a signal $x(t)\in\mathbb{R}^n$ is defined as $\|x_t\|:=\sup_{0\leqslant\tau\leqslant t}|x(\tau)|$.
- Classes of $\mathscr{K}$, $\mathscr{K}_\infty$ functions are defined according to [26, p. 144]. ISS, OSS and IOSS mean Input-State-Stable (or Stability), Output-State-Stable (or Stability) and Input-Output-State-Stable, respectively [27].
- (i) α denotes class-$\mathscr{K}$ functions; (ii) β denotes class-$\mathscr{K}_\infty$ functions; (iii) π denotes class-$\mathscr{K}\mathscr{L}$ functions; (iv) Ψ denotes *known* class-$\mathscr{K}$ functions; and (v) φ, $\bar{\varphi}$ denotes *known* non-negative functions.

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Consider the single-input single-output (SISO) nonlinear systems of the form

$$\dot{x} = f(x,t) + g(x,t)u, \tag{1}$$

$$y = h(x,t), \tag{2}$$

where $u \in \mathbb{R}$ is the control input (discontinuous), $y \in \mathbb{R}$ is the measured output, x is the state and the uncertain functions $f(\cdot,\cdot), g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are smooth enough to ensure local existence and uniqueness of the solution through every initial condition (x_0, t_0). For each solution of (1) there exists a maximal time interval of definition given by $[0, t_M)$, where t_M may be finite or infinite. Thus, finite-time escape is not precluded, *a priori*. Filippov's definition of solution is adopted [28] and the extended equivalent control concept[‡] is used [4, Section 2.3] [29]. We denote the equivalent control signal (piecewise continuous) simply by $u(t)$.

Our output-feedback strategy relies on the implementation of a norm observer for the plant state x. In the following definition let: (i) u be the plant input, (ii) y be the plant output, (iii) γ_o be a smooth function and (iv) $\varphi_o(\cdot,\cdot,t)$ and $\bar{\varphi}_o(\cdot,t)$ be the non-negative functions, piecewise continuous and upper-bounded in t (as defined in [22]) and continuous in their other arguments.

Definition 1
A norm observer for system (1)–(2) is a m-order dynamic system of the form:

$$\tau_1 \dot{\omega}_1 = -\omega_1 + u, \tag{3}$$

$$\tau_2 \dot{\omega}_2 = \gamma_o(\omega_2) + \tau_2 \varphi_o(\omega_1, y, t), \tag{4}$$

with states $\omega_1 \in \mathbb{R}$, $\omega_2 \in \mathbb{R}^{m-1}$ and positive constants τ_1, τ_2 such that for $t \in [0, t_M)$: (i) if $|\varphi_o|$ is uniformly bounded by a constant $c_o > 0$, then $|\omega_2|$ can escape at most exponentially and there exists $\tau_2^*(c_o)$ such that the ω_2-dynamics is BIBS (Bounded-Input Bounded-State) stable w.r.t. φ_o for $\tau_2 \leqslant \tau_2^*$ and (ii) for each $x(0), \omega_1(0), \omega_2(0)$, there exists $\bar{\varphi}_o$ such that

$$|x(t)| \leqslant \bar{\varphi}_o(\omega(t), t) + \pi_o(t), \quad \omega := [\omega_1 \;\; \omega_2^{\mathrm{T}} \;\; y]^{\mathrm{T}}, \tag{5}$$

where $\pi_o := \beta_o(|\omega_1(0)| + |\omega_2(0)| + |x(0)|)e^{-\lambda_o t}$ with some $\beta_o \in \mathscr{K}_\infty$ and positive constant λ_o. $\square$

3. PROBLEM STATEMENT

We consider the global tracking problem of systems of the form (1)–(2) transformable into the normal form [26]:

$$\dot{\eta} = f_0(x,t), \tag{6}$$

$$\dot{\xi} = A_\rho \xi + b_\rho k_p(x,t)[u + d(x,t)], \quad y = c_\rho \xi, \tag{7}$$

where the transformed state is defined as

$$\bar{x} := [\eta^{\mathrm{T}} \;\; \xi^{\mathrm{T}}]^{\mathrm{T}} = T(x,t). \tag{8}$$

The η-subsystem represents the inverse dynamics with $\eta \in \mathbb{R}^{n-\rho}$ and the state of the external dynamics (ξ) is given by

$$\xi := [y \;\; \dot{y} \;\; \ldots \;\; y^{(\rho-1)}]^{\mathrm{T}}. \tag{9}$$

[‡]In general the equivalent control is defined only when the sliding surface is reached. The extended concept is valid also during the reaching phase.

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The pair (A_ρ, b_ρ) is in Brunovsky's canonical controllable form, i.e.

$$A_\rho = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}; \quad b_\rho = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

and $c_\rho = [1 \;\; 0 \cdots 0]$ and $d(x,t)$ is regarded as a nonlinear matched disturbance and $k_p(x,t)$ is the plant high frequency gain (HFG) assumed to be bounded away from zero. Note that, it is thus assumed that the plant (1)–(2) has a strong[§] relative degree ρ.

Remark 1 (*Normal form*)
For time invariant plants, the uniform relative degree assumption [26, 31] is a necessary and sufficient condition for the existence of a local change of coordinates (local diffeomorphism) which transforms (1)–(2) into (6)–(7). Here, we do not require mapping $T(x,t)$ (8) to be invertible, but it should be a global transformation. One sufficient condition to assure that the time-varying plant (1)–(2) is transformable to the normal form is given in Appendix A.1.

In the following assumption, we formulate the restrictions imposed on $T(x,t)$, $k_p(x,t)$ and $d(x,t)$, where the dependence on $y = h(x,t)$ is explicitly given to allow the implementation of less conservative upper bounds.

First of all, for $i = 1, 2, 3$, let: (a) $\varphi_i(|x|, y, t)$ are non-negative functions continuous and increasing in $|x|$, continuous in y, piecewise continuous and upper bounded in t; (b) $\bar{\varphi}_i(y,t)$ are non-negative functions continuous in y and piecewise continuous and upper bounded in t and (c) $\alpha_i(|x|)$ are locally Lipschitz class-$\mathscr{K}$ functions.

Assumption 1
There exist *known* functions $\varphi_i, \bar{\varphi}_i, \alpha_i$ and a *known* positive constant c_p such that the following inequalities hold $\forall x, y, \forall t \in [0, t_M)$:

$$\beta_T(|x|) + \gamma_T(y,t) \leqslant |T(x,t)| \leqslant \varphi_1(|x|, y, t),$$

$$0 < c_p \leqslant k_p(x,t) \leqslant \varphi_2(|x|, y, t),$$

$$|d(x,t)| \leqslant \varphi_3(|x|, y, t),$$

where φ_i satisfies $\varphi_i(|x|, y, t) \leqslant \alpha_i(|x|) + \bar{\varphi}_i(y,t)$, β_T is some class-$\mathscr{K}_\infty$ function and γ_T is some non-negative scalar function continuous in y and piecewise continuous and upper bounded in t.

The lower bound for $|T|$ assures boundedness of x from boundedness of $\bar{x}$ and the lower bound for k_p guarantees that it is positive (without lost of generality) and bounded away from zero.

On the other hand, the upper bounding functions for T, k_p and d are used to obtain implementable norm bounds for ξ, k_p and d from the plant state norm estimator vector ω (3)–(4).

In general, the upper bounds given in Assumption 1 impose significant restriction only w.r.t. the t-dependence, since $f(x,t), g(x,t)$ and $h(x,t)$ are sufficiently smooth (by assumption) so that T, k_p and d are continuous in x. We further assume that:

Assumption 2 (*Minimum phase*)
There exists a storage function $V(\eta)$ satisfying $\underline{\beta}(|\eta|) \leqslant V(\eta) \leqslant \bar{\beta}(|\eta|)$ with $\underline{\beta}, \bar{\beta} \in \mathscr{K}_\infty$, such that:

$$\frac{\partial V}{\partial \eta} f_0(x,t) \leqslant -\beta_0(|\eta|) + \varphi_0(|\xi|, t),$$

$\forall x, y, \forall t \in [0, t_M)$, for some non-negative scalar function $\varphi_0(|\xi|, t)$, continuous in $|\xi|$ and piecewise continuous and upper bounded in t and some $\beta_0 \in \mathscr{K}_\infty$.

[§]This terminology is used in [30] where the time dependence is considered in the so called 'modified Lie derivatives'.

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Assumption 2 assures that the inverse dynamics (6) has an ISS-like property with respect to an appropriate function of ξ and t. Hence, it corresponds to a generalization of the concept of minimum-phase plants and allows us to conclude boundedness of η from boundedness of ξ.

Assumption 3 (*Norm observability*)
The plant (1)–(2) admits a norm observer (Definition 1) for some *known* functions $\gamma_o, \varphi_o, \bar{\varphi}_o$ and positive constants τ_1, τ_2.

It is well known that, in the time-invariant case, if (1)–(2) is IOSS [32] then it admits a norm observer according to Assumption 3. In Section 8, we present a more general class of nonlinear time-varying plants which admit a norm observer as given by (3)–(4). Such class encompasses plants with linear growth condition in the unmeasured states and growth rate possibly depending on η, y and t. It should be stressed that strong polynomial nonlinearities in η and y are allowed.

3.1. Global practical tracking problem

The aim is to find an output feedback dynamic control law u to drive the *output tracking error*

$$e(t)=y(t)-y_m(t) \tag{11}$$

exponentially to zero or to some small neighborhood of zero (practical tracking), starting from any plant/controller initial conditions and maintaining uniform closed-loop signal boundedness, in spite of the uncertainties. The *desired trajectory* $y_m(t)$ is assumed to be generated by the following *reference model*:

$$\dot{\xi}_m=A_m\xi_m+b_\rho k_m r, \quad A_m=A_\rho+b_\rho K_m, \quad y_m=c_m^{\mathrm{T}}\xi_m, \tag{12}$$

where $\xi_m:=[y_m \ \ \dot{y}_m \ \ \ldots \ \ y_m^{(\rho-1)}]^{\mathrm{T}}$, $k_m>0$ is constant, $K_m\in\mathbb{R}^{1\times\rho}$ is such that A_m is Hurwitz and $r(t)$ is assumed piecewise continuous and uniformly bounded.

3.2. Reducing tracking to regulation

Subtracting (12) from (7) one has

$$\dot{\xi}_e=A_m\xi_e+b_\rho k_p[u+d_e], \quad e=c_m^{\mathrm{T}}\xi_e, \tag{13}$$

where $\xi_e:=\xi-\xi_m$ is the state tracking error, $c_m^{\mathrm{T}}=[1 \ \ 0 \ \ \cdots \ \ 0]$ (so $e=\xi_1-\xi_{m1}=y-y_m$) and the *error input disturbance* d_e is defined by

$$k_p d_e(x,\xi,t):=k_p d(x,t)-K_m\xi-k_m r. \tag{14}$$

Then, the tracking problem can be formulated as a regulation problem which consists in finding an OFSM control law u such that the output e is regulated to a neighborhood of zero, i.e. for all initial conditions $x(0), \omega_1(0), \omega_2(0)$: (i) the solutions of (3), (4), (6) and (7) are uniformly bounded and (ii) the output $e=\xi_1-\xi_{m1}$ of (13), i.e. the tracking error (11), tends to a neighbourhood of zero as $t\to\infty$.

3.3. Auxiliary upper bounds via norm observer

The following available upper bounds for ξ, k_p and d are obtained, *modulo* exponentially decaying term, by using the bounding functions given in Assumption 1 and the norm observer given in Definition 1 (for details, see Appendix A.3):

$$|\xi|\leqslant\psi_1(\omega,t)+\pi_1, \tag{15}$$

$$k_p(x,t)\leqslant\psi_2(\omega,t)+\pi_1, \tag{16}$$

$$|d(x,t)|\leqslant\psi_3(\omega,t)+\pi_1, \tag{17}$$

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where $\psi_i(\omega, t) := \varphi_i(2\bar{\varphi}_o, y, t) + \bar{\varphi}_i(y, t)$ $(i = 1, 2, 3)$ and $\pi_1 = \beta_1(|\omega(0)| + |x(0)|)e^{-\lambda_o t}$ with some $\beta_1 \in \mathscr{K}_\infty$ and λ_o in Definition 1.

Then, with c_p in Assumption 1 and from (14) one can verify that $|d_e| \leqslant |d| + (|K_m||\xi| + k_m|r|)/c_p$. Moreover, from (15) and (17) the following upper bound holds:

$$|d_e(x, \xi, t)| + \delta \leqslant \varrho(\omega, t) + \pi_2, \tag{18}$$

where δ is an arbitrary non-negative constant,

$$\varrho(\omega, t) := \psi_3 + (|K_m|\psi_1 + k_m|r|)/c_p + \delta, \tag{19}$$

and $\pi_2 := |K_m|\pi_1/c_p + \pi_1$.

4. OUTPUT-FEEDBACK SLIDING MODE CONTROL

When only y is available for feedback, we choose

$$\hat{\sigma} := S\hat{\xi}_e = 0, \quad \hat{\xi}_e := \hat{\xi} - \xi_m, \tag{20}$$

as the sliding surface, where S is such that (A_m, b_ρ, S) is strictly positive real and $\hat{\xi}$ is an estimate of ξ (9) provided by an HGO. The control law u is given by

$$u = -\varrho(\omega, t)\ \mathrm{sgn}(\hat{\sigma}(t)). \tag{21}$$

Then, defining the *estimation error* as

$$\tilde{\xi}_e := \xi_e - \hat{\xi}_e = \xi - \hat{\xi}, \tag{22}$$

the following lemma can be stated.

Lemma 1 (*ISS property from* $|\tilde{\xi}_e|$ *to* ξ_e)
Consider the ξ_e-dynamics (13) with output $\hat{\sigma} = S\xi_e - S\tilde{\xi}_e$, u given in (21), ϱ in (19) and d_e in (14). Then, (13) is ISS with respect to $\tilde{\xi}_e$ and the following inequality holds

$$|\xi_e(t)| \leqslant k_e|\tilde{\xi}_e(t)| + \pi_e,$$

where $\pi_e := \beta_e(|\omega(0)| + |x(0)| + |\xi_e(0)|)e^{-\lambda_e t}$, $\beta_e \in \mathscr{K}_\infty$, $0 < \lambda_e < \min\{\lambda_m[A_m], \lambda_o\}$, λ_o is given in Definition 1 and $k_e > 0$ is an appropriate constant.

Proof
See Appendix A.4. □

Our goal is to provide an estimate $\hat{\xi}$ by an HGO (with variable gain) such that the observer error norm $|\tilde{\xi}_e(t)|$ is an arbitrarily small, *modulo* exponentially decaying term, and use Lemma 1 to conclude global practical tracking.

As in [7], an eventual peaking [25] in $\hat{\sigma}$ is blocked by the sgn$(\cdot)$ function in (21) and the control signal u is peaking free since $\varrho(\omega, t)$ is implemented using only the well conditioned (without peaking) signals. The proposed scheme is depicted in Figure 1.

5. HIGH GAIN OBSERVER WITH VARIABLE GAIN

The HGO is given by

$$\dot{\hat{\xi}} = A_\rho\hat{\xi} + b_\rho u + H_\mu L_o(y - c_\rho\hat{\xi}), \tag{23}$$

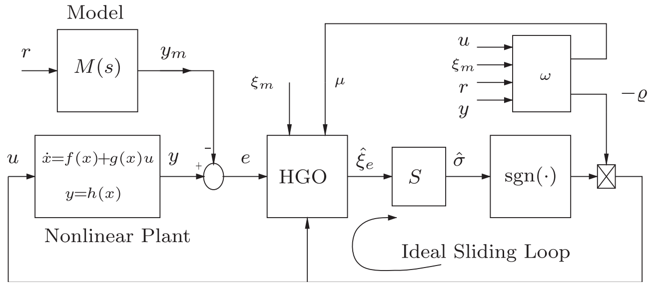

Figure 1. Global OFSM control using an HGO to generate $\hat{\sigma}(t)$.

where L_o and H_μ are given by

$$L_o := [l_1 \ \ \ldots \ \ l_\rho]^{\mathrm{T}} \quad \text{and} \quad H_\mu := \mathrm{diag}(\mu^{-1}, \ldots, \mu^{-\rho}). \tag{24}$$

The observer gain L_o is such that $s^\rho + l_1 s^{\rho-1} + \ldots + l_\rho$ is Hurwitz. In this paper, instead of using a constant μ, we introduce a *variable* parameter $\mu = \mu(t) \neq 0, \forall t \in [0, t_M)$, of the form

$$\mu(\omega, t) := \frac{\bar{\mu}}{1 + \psi_\mu(\omega, t)}, \tag{25}$$

where ψ_μ, named *domination function*, is a non-negative function (to be designed later on) continuous in its arguments and $\bar{\mu} > 0$ is a design constant.

For each system trajectory, μ is absolutely continuous and $\mu \leqslant \bar{\mu}$. Note that μ is bounded for t in any finite sub-interval of $[0, t_M)$. Therefore,

$$\mu(\omega, t) \in [\underline{\mu}, \bar{\mu}], \quad \forall t \in [t_*, t_M), \tag{26}$$

for some $t_* \in [0, t_M)$ and $\underline{\mu} \in (0, \bar{\mu})$.

Thus, considering the SISO nonlinear plant (1)–(2) transformable into the normal form (6)–(7) under Assumptions 1–3, control law (21), with ϱ given by (19) and HGO (23) with μ be given by (25) and appropriate domination function ψ_μ. Then, for sufficiently small constants $\tau_2, \bar{\mu} > 0$, GAS of the error system with respect to the compact set and ultimate exponential convergence of error system state to a residual set of order $\bar{\mu}$ are guaranteed, with both sets being independent of the initial conditions. Moreover, all signals in the closed loop system are uniformly bounded. A detailed stability analysis and a formal statement of the main result (Theorem 1) will be presented later in Section 6.

5.1. *High gain observer error dynamics*

The transformation [6, 7]

$$\zeta := T_\mu \tilde{\xi}_e, \quad T_\mu := [\mu^\rho H_\mu]^{-1}, \tag{27}$$

is fundamental to represent the $\tilde{\xi}_e$-dynamics in convenient coordinates to allow us to show that $\tilde{\xi}_e$ is an arbitrarily small, *modulo* exponentially decaying term. First, from (10), (24) and (27), note that:

$$T_\mu (A_\rho - H_\mu L_o c_\rho) T_\mu^{-1} = \frac{1}{\mu} A_o, \tag{28}$$

$$T_\mu \, b_\rho = b_\rho, \tag{29}$$

$$\dot{T}_\mu T_\mu^{-1} = \frac{\dot{\mu}}{\mu} \Delta, \tag{30}$$

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where $A_o := A_\rho - L_o c_\rho$ and $\Delta := \mathrm{diag}(1-\rho, 2-\rho, \ldots, 0)$. Then, subtracting (23) from (7) and applying the above relationships, the dynamics of $\tilde{\xi}_e$ (22) in the new coordinates ζ (27) is given by:

$$\mu\dot{\zeta} = [A_o + \dot{\mu}(t)\Delta]\zeta + b_\rho[\mu v], \tag{31}$$

where

$$v := (k_p - 1)u + k_p d. \tag{32}$$

5.2. *Domination function design*

The HGO gain is inversely proportional to the small parameter μ which is time varying due to the domination function $\psi_\mu(\omega, t)$ in (25). In this section, our task is to establish properties for $\psi_\mu(\omega, t)$ so that $\mu|v|$ and $\dot{\mu}$ can be bounded by a constant of order $\mathcal{O}(\bar{\mu})$, at least after a finite time interval, where $\bar{\mu}$ in (25) is a design constant. With such properties, we can prove that the estimation error $\tilde{\xi}_e$ is ultimately small provided $\bar{\mu}$ is chosen sufficiently small. Then, by means of Lemma 1 we can conclude global practical tracking (Theorem 1).

5.2.1. Auxiliary upper bounds. Note that, from the definition of u (21), one has $|u(t)| \leqslant \varrho(\omega, t)$. Thus, from the upper bounds (16) and (17), the signal v (32) satisfies

$$|v| \leqslant \psi_v(\omega, t) + \pi_3, \tag{33}$$

where $\psi_v := \varrho\psi_2 + \varrho + \varrho^2 + \psi_2\psi_3 + \psi_2^2 + \psi_3^2$ is *known* and $\pi_3 := 3\pi_1^2$. Then, from (25) and (33), one can write

$$\mu|v| \leqslant \frac{\psi_v}{1+\psi_\mu}\bar{\mu} + \mu\pi_3. \tag{34}$$

In order to develop an upper bound for $|\dot{\mu}|$ we need an upper bound for $|\dot{\omega}|$. From (9), one has $|\dot{y}| \leqslant |\xi|$ and, from (15), one can verify that $|\dot{y}| \leqslant \psi_1(\omega, t) + \pi_1$. Moreover, from Definition 1 and (21), $\dot{\omega}_1$ and $\dot{\omega}_2$ satisfy $\tau_1|\dot{\omega}_1| \leqslant |\omega_1| + \varrho(\omega, t)$ and $\tau_2|\dot{\omega}_2| \leqslant |\gamma_o(\omega_2)| + \tau_2|\varphi_o|$, respectively. Then, one concludes that

$$|\dot{\omega}| \leqslant \psi_\omega(\omega, t) + \pi_1, \tag{35}$$

where $\psi_\omega(\omega, t) := \psi_1 + |\omega_1|/\tau_1 + \varrho/\tau_1 + |\gamma_o|/\tau_2 + |\varphi_o|$ is *known*. Then, multiplying (25) and (35), one gets

$$\mu|\dot{\omega}| \leqslant \frac{\psi_\omega}{1+\psi_\mu}\bar{\mu} + \mu\pi_1. \tag{36}$$

5.2.2. Domination function properties. We start by choosing the domination function ψ_μ in (25) so that the following property holds with ψ_v in (33) and ψ_ω in (35):

(P0) $\psi_v, \psi_\omega \leqslant c_{\mu 0}(1+\psi_\mu)$, $\forall t \in [0, t_M)$ where $c_{\mu 0} \geqslant 0$ is a *known* constant.

If ψ_μ satisfies (P0) then, from (34) and (36), $\mu|v|$ and $\mu|\dot{\omega}|$ can be bounded by

$$\mu|v| \leqslant \mathcal{O}(\bar{\mu}) + \mu\pi_3, \tag{37}$$

$$\mu|\dot{\omega}| \leqslant \mathcal{O}(\bar{\mu}) + \mu\pi_1. \tag{38}$$

In order to obtain a norm bound for $\dot{\mu}$, $\dot{\mu}$ can be calculated differentiating (25):

$$\dot{\mu}(t) = -\frac{\mu^2}{\bar{\mu}}\left[\frac{\partial\psi_\mu}{\partial\omega}\dot{\omega} + \frac{\partial\psi_\mu}{\partial t}\right] = -\frac{\dfrac{\partial\psi_\mu}{\partial\omega}}{1+\psi_\mu}\mu\dot{\omega} - \frac{\dfrac{\partial\psi_\mu}{\partial t}}{1+\psi_\mu}\mu. \tag{39}$$

Note that, $\dot{\mu}$ is a piecewise continuous time signal which can be upper bounded by

$$|\dot{\mu}(t)| \leqslant \frac{\left|\frac{\partial \psi_\mu}{\partial \omega}\right|}{1+\psi_\mu} \mu |\dot{\omega}| + \frac{\left|\frac{\partial \psi_\mu}{\partial t}\right|}{1+\psi_\mu} \mu. \tag{40}$$

Our strategy is to design $\psi_\mu(\omega, t)$ such that the following additional property holds:

(P1) $|\partial \psi_\mu / \partial \omega|, |\partial \psi_\mu / \partial t| \leqslant c_{\mu 1}(1+\psi_\mu)$, $\forall t \in [0, t_M)$ where $c_{\mu 1} \geqslant 0$ is a *known* constant.

This property is trivially satisfied by polynomial ψ_μ with positive coefficients (see Section 5.2.3).

Now, with ψ_μ satisfying (P1), one has that:

$$|\dot{\mu}(t)| \leqslant c_{\mu 1} \mu |\dot{\omega}| + c_{\mu 1} \mu. \tag{41}$$

Therefore, from (41), (37) and (38) the following holds:

$$|\dot{\mu}(t)|, \quad \mu |v| \leqslant \mathcal{O}(\bar{\mu}) + \mu \pi_4, \tag{42}$$

where $\pi_4 := c_{\mu 1} \pi_1 + \pi_3$. Note that, from (5) and Assumption 1, if any closed loop system signal escapes in some finite time, then ω also escapes not latter than that. Indeed, according to Assumption 3, the system possesses an unboundedness observability property [33]. We will use this fact to design $\psi_\mu(\omega, t)$ so that if ω escapes in some finite time then $\psi_\mu(\omega, t)$ also escapes not later than this time. From (25), this will ensure that the second term on the right-hand side of (42) will be of order $\mathcal{O}(\bar{\mu})$, before any eventual finite time escape.

To this end, we design ψ_μ to satisfy the property:

(P2) $\|\omega_t\| e^{-\lambda_\mu t} \leqslant \psi_\mu(\omega, t)$, $\forall \omega, \forall t \in [0, t_M)$ where λ_μ is a design positive constant.

The exponential term with rate λ_μ acts like a forgetting factor which allows a less conservative ψ_μ design. Reminding that π_4 can be written as $\pi_4 = \beta_4(|\omega(0)| + |x(0)|) e^{-\lambda_4 t}$, with some $\beta_4 \in \mathcal{K}_\infty$ and some positive constant λ_4, then if ψ_μ satisfies (P2), the following holds

$$\mu \pi_4 \leqslant \bar{\mu} \frac{\pi_4}{1+\psi_\mu} \leqslant \bar{\mu} \frac{\beta_4(|\omega(0)| + |x(0)|) e^{-\lambda_4 t}}{1 + \|\omega_t\| e^{-\lambda_\mu t}}, \tag{43}$$

$\forall t \in [0, t_M)$. We can show that (see Appendix A.3) the right-hand side of (43) is bounded by $\bar{\mu}$, at least after some finite time ($t_\mu \geqslant 0$). Finally, if ψ_μ is designed so that (P0)–(P2) hold, then from (42) and (43) one can verify that there exists a finite $t_\mu \in [0, t_M)$ such that:

$$|\omega|, |\zeta| \leqslant \beta_5(|\omega(0)| + |x(0)| + |\zeta(0)|) \quad \forall t \in [0, t_\mu) \tag{44}$$

$$|\dot{\mu}(t)|, \mu |v| \leqslant \mathcal{O}(\bar{\mu}) \quad \forall t \in [t_\mu, t_M), \tag{45}$$

with some $\beta_5 \in \mathcal{K}_\infty$. To see that (44) and (45) hold, refer to Appendix A.3.

5.2.3. One specific variable gain (μ) design. The following assumption is useful to determine at least one specific class of time-varying μ satisfying the aforementioned properties, at the expense of some conservatism:

Assumption 4

There exists a polynomial $\bar{p}_\mu(|\omega|)$ in $|\omega|$, with positive real coefficients, such that the functions $\varphi_o, \bar{\varphi}_o$ (Assumption 3) and the bounding functions $\varphi_i, \bar{\varphi}_i$ (Assumption 1) satisfy ($i = 1, 2, 3$):

$$|\gamma_o(\omega_2)|, |\varphi_o(\omega_1, y, t)| \leqslant \bar{p}_\mu(|\omega|),$$

$$\varphi_i(2\bar{\varphi}_o(\omega, t), y, t), \bar{\varphi}_i(y, t) \leqslant \bar{p}_\mu(|\omega|).$$

This assumption is not so restrictive since only polynomial growth condition is imposed on φ_o, $\bar{\varphi}_o$, γ_o, φ_i, $\bar{\varphi}_i$. Now, recall that $\psi_v(\omega,t)$ in (33) and $\psi_\omega(\omega,t)$ in (35) are given by $\psi_v(\omega,t)=\varrho\psi_2+\varrho+\varrho^2+\psi_2\psi_3+\psi_2^2+\psi_3^2$ and $\psi_\omega(\omega,t)=\psi_1+|\omega_1|/\tau_1+\varrho/\tau_1+|\gamma_o|/\tau_2+|\varphi_o|$, respectively, where $\psi_i=\varphi_i(2\bar{\varphi}_o,y,t)+\bar{\varphi}_i(y,t)$ $(i=1,2,3)$ in (15)–(17). Then, with Assumption 4, one can easily obtain a polynomial $p_\mu(|\omega|)$ in $|\omega|$, with positive real coefficients, such that:

$$\psi_v, \quad \psi_\omega \leqslant p_\mu(|\omega|). \tag{46}$$

We choose ψ_μ as:

$$\psi_\mu(\omega,t) := p_\mu(|\omega|)+\|\omega_t\|e^{-\lambda_\mu t}, \tag{47}$$

where $\lambda_\mu>0$ is a design constant. It is not difficult to verify that (47) satisfies (P0) and (P2). To see that (P1) also holds, see Appendix A.3.

6. STABILITY ANALYSIS AND MAIN RESULTS

In order to account for all initial conditions involved in the *error system* (13) and (31), let:

$$z^{\mathrm{T}}(t) := [z^0(t), \ \ \xi_e^{\mathrm{T}}(t), \ \ \zeta^{\mathrm{T}}(t)], \quad z^0(t) := z^0(0)e^{-\lambda t}, \tag{48}$$

where $z^0(0) := [\eta^{\mathrm{T}}(0) \ \ \omega^{\mathrm{T}}(0)]$ and $\lambda>0$ is a generic constant. The stability analysis is carried out through the following steps:

Step 1. First, we demonstrate that $|z(t)|$ is uniformly bounded by a class-$\mathscr{K}_\infty$ function of $|z(0)|$, $\forall t\in[0,t_\mu)$.

Step 2. Then, for $t\in[t_\mu,t_M)$ we prove that the observer error norm is bounded by $|\tilde{\xi}_e(t)|\leqslant\beta_{z1}(|z(0)|)e^{-\lambda_{z1}t}+\mathcal{O}(\bar{\mu})$, where $\lambda_{z1}>0$ is a constant and $\beta_{z1}\in\mathscr{K}_\infty$, provided $\bar{\mu}$ is chosen sufficiently small and (P0)–(P1) hold.

Step 3. Applying Lemma 1, one can also verify that $|\xi_e|, |z(t)|\leqslant\beta_{z2}(|z(0)|)e^{-\lambda_{z2}t}+\mathcal{O}(\bar{\mu})$, where $\lambda_{z2}>0$ is a constant and $\beta_{z2}\in\mathscr{K}_\infty$. Moreover, $z(t)$ cannot escape in finite time.

Step 4. Finally, we verify that no closed loop signal can escape in finite time and, moreover, are uniformly bounded $\forall t$, provided τ_2 (in Definition 1) is chosen sufficiently small.

The following theorem summarizes the main result.

Theorem 1

Consider the SISO nonlinear plant (1)–(2) transformable into the normal form (6)–(7) under Assumptions 1–3. Let the control law be given by (21), with ϱ given by (19) and consider HGO (23) with μ be given by (25) and domination function ψ_μ designed such that the properties (P0)–(P2) hold. Then, for sufficiently small constants $\tau_2, \bar{\mu}>0$, there exist $\beta_z(\cdot)\in\mathscr{K}_\infty$ and positive constants a, b such that the complete error state z (48) satisfies

$$|z(t)|\leqslant[\beta_z(|z(0)|)+b]e^{-at}+\mathcal{O}(\bar{\mu}), \tag{49}$$

$\forall t\geqslant 0$ and $\forall z(0)$, i.e. GAS of the error system with respect to the compact set $\{z:|z|\leqslant b\}$ and ultimate exponential convergence of $z(t)$ to a residual set of order $\mathcal{O}(\bar{\mu})$ are guaranteed, with both sets being independent of the initial conditions. Moreover, all signals in the closed loop system are uniformly bounded.

Proof

See Appendix A.4. □

Finite frequency chattering is avoided and an ideal sliding mode is produced thanks to the *ideal sliding loop* (ISL) formed around the relay function (see Figure 1), according to the following corollary.

Corollary 1 (*Ideal sliding mode*)
Additionally to the assumptions of Theorem 1, if $\varrho \geqslant |K_m||\xi_m| + |k_m||r| + \delta$ with $\delta > 0$ then $\hat{\sigma} \equiv 0$ is reached in finite time.

Proof
See Appendix A.4. □

Remark 2
The importance of the existence of an ideal sliding mode has been discussed in several works, e.g. [34, p.210], [35, 36]. This is because, in the absence of noise, the frequency of chattering can be arbitrarily increased by reducing the sampling period in practical real time computer implementation of the control scheme. In many applications, such as in electrical drives or converters, sufficiently high frequency chattering is acceptable and allows the advantages of sliding mode control to be preserved. In contrast, when using linear causal differentiating filters to restore the states required in the switching function $\hat{\sigma}$, inevitable small lags are introduced in the high frequency loop and this usually leads to chattering with limited frequency, independently of the sampling period, thus deteriorating the sliding mode control performance.

Remark 3 (*Absence of peaking*)
In addition, one can conclude that ξ_e is peaking free by noting that (13) is ISS with respect to u and that the sgn$(\cdot)$ function in u (21) blocks the eventual peaking present in $\hat{\xi}$ to u.

7. CONTROLLER ALGORITHM

The complete controller is summarized in Table I. The design parameters can be obtained as follows.

First, we design a norm observer for the plant state x, according to Definition 1, and transform the original system to the normal form. From the bounding functions $\bar{\varphi}_o, \varphi_i, \bar{\varphi}_i$ $(i = 1, 2, 3)$, given in Assumptions 1–3, we obtain: the bounding functions ψ_i, the modulation function (19) ϱ and bounding functions ψ_ω and ψ_v.

Then, we design the domination function ψ_μ to satisfy the properties (P0)–(P2). The constant $\lambda_\mu \geqslant 0$ is arbitrary and L_o is such that $s^\rho + l_1 s^{\rho-1} + \cdots + l_\rho$ is Hurwitz. The HGO can be implemented from (23). Thus, the control law (21) is implemented with the sliding surface (20) chosen so that (A_m, b_ρ, S) is strictly positive real.

Table I. Proposed algorithm for achieving global tracking with a peaking free control signal.

Reference model (12)	$\dot{\xi}_m = A_m \xi_m + b_\rho k_m r, \quad \xi_m := [y_m \ \ \dot{y}_m \ \ \dots \ \ y_m^{(\rho-1)}]^{\mathrm{T}}$
Output error (11)	$e = y - y_m$
Norm observer (4)	$\tau_1 \dot{\omega}_1 = -\omega_1 + u$ and $\tau_2 \dot{\omega}_2 = \gamma_o(\omega_2) + \tau_2 \varphi_o(\omega_1, y, t)$ (see Definition 1)
Auxiliary upper bounds	$\psi_i(\omega, t) := \varphi_i(2\bar{\varphi}_o, y, t) + \bar{\varphi}_i(y, t)$ $(i = 1, 2, 3)$, with $\varphi_i, \bar{\varphi}_i$ in Assumption 1
Modulation function (19)	$\varrho(\omega, t) := \psi_3(\omega, t) + \lvert K_m \rvert \psi_1(\omega, t) + k_m \lvert r \rvert + \delta$.
Domination functions	$\psi_v(\omega, t) := \varrho\psi_2 + \varrho + \varrho^2 + \psi_2\psi_3 + \psi_2^2 + \psi_3^2$,
	$\psi_\omega(\omega, t) := \psi_1 + \lvert\omega_1\rvert/\tau_1 + \varrho/\tau_1 + \lvert\gamma_o\rvert/\tau_2 + \lvert\varphi_o\rvert$,
	$\psi_\mu(\omega, t)$ designed to satisfy (P0)(P1)(P2)
HGO (23)	$\dot{\hat{\xi}} = A_\rho \hat{\xi} + b_\rho u + H_\mu L_o (y - c_\rho \hat{\xi})$
	$L_o = [l_1 \ \ \dots \ \ l_\rho]^{\mathrm{T}}, H_\mu = \mathrm{diag}(\mu^{-1}, \dots, \mu^{-\rho})$,
(25)	$\mu(\omega, t) := \bar{\mu}/1 + \psi_\mu(\omega, t)$, where $\bar{\mu}$ is a design constant
Sliding surface (20)	$\hat{\sigma} := S(\hat{\xi} - \xi_m) = 0$ with (A_m, b_ρ, S) strictly positive real
Control law (21)	$u = -\varrho(\omega, t)\mathrm{sgn}(\hat{\sigma}(t))$

From the functions φ_o, γ_o and the constant τ_1 given in Assumption 3 we implement the norm observer. Finally, by simulation, we start with not so small values of $\bar{\mu}$, τ_2 and then decrease $\bar{\mu}$ until acceptable tracking error is obtained, which is guaranteed in the stability analysis. Then, we decrease τ_2 to assure ω_2 boundedness (see Definition 1).

8. AN ILLUSTRATIVE CLASS OF NONLINEAR PLANTS

We can tackle plants (1)–(2) of the form

$$\dot{\eta} = \phi_0(x, y, t), \tag{50}$$

$$\begin{aligned} \dot{v}_1 &= v_2 + \phi_1(x, y, t), \\ &\vdots \\ \dot{v}_{\rho-1} &= v_\rho + \phi_{\rho-1}(x, y, t), \\ \dot{v}_\rho &= k_u u + \phi_\rho(x, y, t), \\ y &= v_1, \end{aligned} \tag{51}$$

transformable to the normal form (see Remark 1) and satisfying Assumption 1. The state x is partitioned as $x^{\mathrm{T}} := [\eta^{\mathrm{T}} \;\; v^{\mathrm{T}}]$, with $v \in \mathbb{R}^\rho$, and $k_u > 0$ being a constant. Note that this system is neither in the triangular form nor time-invariant such as in [21].

Now, we formulate sufficient conditions on $\phi^{\mathrm{T}} = [\phi_1 \;\; \ldots \;\; \phi_\rho]$ such that (50)–(51) satisfies the minimum phase (Assumption 2) and the norm observer existence (Assumption 3).

First, as in [21, 37, 38], we consider that:

(**C0**) (Triangularity condition) For $i = 1, \ldots, \rho$:

$$|\phi_i| \leqslant \varphi_r(|\eta|, y, t)(|v_1| + \ldots + |v_i|) + \varphi_v(|\eta|, y, t),$$

$\forall t \in [0, t_M)$, where $\bar{\varphi}_r, \bar{\varphi}_v$ are *known* non-negative functions continuous in y and piecewise continuous and upper bounded in t satisfying $\varphi_r(|\eta|, y, t) \leqslant \Psi_r(|\eta|) + \bar{\varphi}_r(y, t)$ and $\varphi_v(|\eta|, y, t) \leqslant \Psi_v(|\eta|) + \bar{\varphi}_v(y, t)$ with *known* $\Psi_r, \Psi_v \in \mathscr{K}$ locally Lipschitz functions.

Then, for the η-subsystem, we assume that one can obtain a storage function $V(\eta)$ satisfying $\underline{\alpha}(|\eta|) \leqslant V(\eta) \leqslant \bar{\alpha}(|\eta|)$, with $\underline{\alpha}(\sigma) = \underline{\lambda}\sigma^2$, $\bar{\alpha}(\sigma) = \bar{\lambda}\sigma^2$ and $\underline{\lambda}, \bar{\lambda}$ *known* so that the following condition holds:

(**C1**) There exist a *known* non-negative function $\varphi_\eta(y, t)$, continuous in y, piecewise continuous and upper bounded in t and a *known* $\alpha \in \mathscr{K}$ such that $\forall t \in [0, t_M)$:

$$\frac{\partial V(\eta)}{\partial \eta} \phi_0 \leqslant -\alpha(|\eta|) + \varphi_\eta(y, t), \tag{52}$$

where the class-$\mathscr{K}$ function $\alpha \circ \bar{\alpha}^{-1}$ is stiffening[¶] in the interval $(0, \infty)$.

Note that, (C1) implies Assumption 2. Moreover, (C0) and (C1) allow us to implement the following three-order norm observer for x:

$$\tau_1 \dot{\omega}_1 = -\omega_1 + u, \tag{53}$$

$$\dot{\omega}_{21} = -c_0 \omega_{21} + \varphi_1(y, t), \tag{54}$$

$$\tau_2 \dot{\omega}_{22} = -(1 - e^{-\omega_{22}}) + \tau_2 \varphi_2(\omega_{21}) + \tau_2 \varphi_3(y, t), \tag{55}$$

[¶]As in [39], we say that $\alpha_1(\sigma)$ is stiffening if for every $\lambda > 0$, there exists $\varepsilon > 0$ such that $\sigma \geqslant \varepsilon \Rightarrow \alpha_1(\sigma) \geqslant \lambda\sigma$.

which is in agreement with Definition 1. The ω_{21}-dynamics provides a norm bound for η whereas the ω_{22}-dynamics provides a norm bound for v, so that:

$$|x| \leqslant \varphi_4(\omega_1, \omega_{21}, y, t, \tau_2) + c_1 e^{c_2|\omega_{22}|} + \pi, \tag{56}$$

where c_0, c_1, c_2 are non-negative constants, $\pi := \beta_0(|\omega(0)| + |x(0)|)e^{-\lambda_o t}$ and $\tau_1, \tau_2, \lambda_o$ are positive design constants. In Appendix A.2 we give the steps needed to obtain the norm observer functions φ_1, φ_2, φ_3 and φ_4. Now, we illustrate the class of system by the following nontrivial academic example.

Example 1 (*Class of systems*)
Consider the four-order nonlinear plant with $\rho = 3$:

$$\dot{\eta} = -\eta^5 - |y|\eta^2 + y\theta(t), \tag{57}$$

$$\dot{v}_1 = v_2 + \eta y^2,$$

$$\dot{v}_2 = v_3 + \frac{v_3^2}{4+4v_3^2}\sin(v_2) + \eta^2 y v_2,$$

$$\dot{v}_3 = u + \eta^2 y \quad v_2^{2/3} v_3^{1/3},$$

$$y = v_1, \tag{58}$$

where $\theta(t)$ is a uniformly bounded time-varying function. The nonlinear terms in the v-dynamics satisfy (C0) with $\varphi_r = \eta^2|y|$ and $\varphi_v = |\eta|y^2 + 0.25$ and the inverse dynamics, adapted from [40, Ex. 1], satisfies (C1) with $V(\eta) = \eta^2/2$, $\alpha = |\eta|^6/4$ and $\varphi_\eta = y^2[1+\theta^2]/2 + 0.5^{1/3}$. The cross term $v_2^{2/3} v_3^{1/3}$ was inspired from [37, 38, Ex. 2.4]. Note that the system is non-triangular but transformable to the normal form (6)–(7). Moreover, by computing the time derivatives $\dot{y}$, $\ddot{y}$, $\dddot{y}$ one can obtain $T(x,t), k_p(x,t)$ and $d(x,t)$ satisfying Assumption 1. Assumption 3 also holds and the steps to construct the norm observer (53)–(55) are given in Appendix A.2. □

In the next example, we focus only on the time varying behavior of $\mu(t)$.

Example 2 (*Simulation results*)
We consider the simple academic case, with no zeroes dynamics and relative degree 2 ($\rho = 2$), where (50)–(51) is reduced to:

$$\dot{v}_1 = v_2,$$

$$\dot{v}_2 = k_u u - \delta_1 v_2 + \delta_2 y^2 + \delta_3 \sin(2\pi\delta_4 t),$$

$$y = v_1.$$

The plant is already in the normal form (6)–(7) (with $T = I$), where $\xi = x$, $k_p = k_u$ and $k_p d = -\delta_1 \xi_2 + \delta_2 y^2 + \delta_3 \sin(2\pi\delta_4 t)$. Assumption 1 is satisfied with: $\varphi_1 = \gamma_T = \bar{\varphi}_1 = 1$, $\alpha_1 = \beta_T = 0$, $c_p = 1$, $\varphi_2 = \bar{\varphi}_2 = 2$, $\alpha_2 = 0$, $\alpha_3 = 3|x|$, $\bar{\varphi}_3 = 3y^2 + 2$ and $\varphi_3 = \alpha_3 + \bar{\varphi}_3$.

The uncertain parameters are: $1 \leqslant k_u \leqslant 2$, $1 \leqslant \delta_1, \delta_2 < 3, 0.5 \leqslant \delta_3 < 2$ and $8 \leqslant \delta_4 \leqslant 10$. The *actual* plant parameters, assumed unknown, are $k_u = 2$, $\delta_1 = 2$, $\delta_2 = 1$, $\delta_3 = 0.7$ and $\delta_4 = 10$.

Of course, Assumption 2 (and (C1)) is absent. Moreover, since $\delta_1 > 0$, it is not difficult to verify that Assumption 3 holds with the (two-order) norm observer in Table I, where: $\tau_1 = \tau_2 = 1$, $\gamma_o = -\omega_2$, $\varphi_o = 8|\omega_1| + 3y^2 + 2$ and $\bar{\varphi}_o = 2|\omega_1| + |\omega_2| + |y|$. Note that, since $v_1 = y$ is measured, only a norm bound for $v_2 = \dot{y}$ is needed.

In addition, in Table I, the pair (A_ρ, b_ρ) is in Brunovsky's canonical controllable form (with $\rho = 2$) and the desired trajectory y_m is generated with $k_m = 4$, $A_m = A_\rho + b_\rho K_m$, $K_m = [-4 \;\; -2]$ and $r = \text{sgn}(\sin(0.5\pi t))$. The modulation and domination functions are given by $\varrho = 15|\omega_1| + 7.4|\omega_2| +$

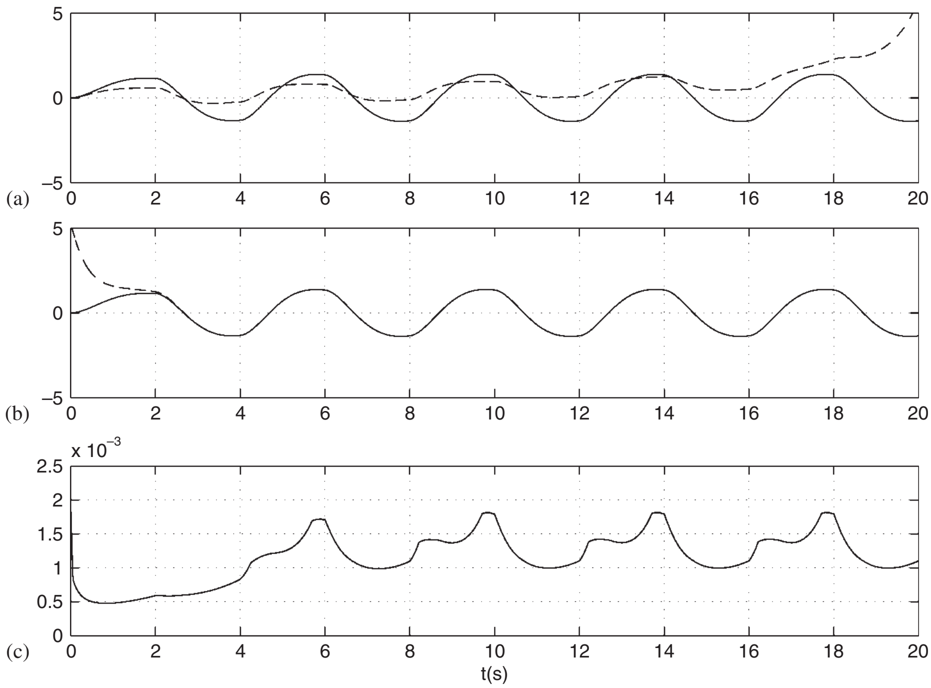

Figure 2. Simulation result: (a) (dashed line) y, (solid line) y_m when μ is held constant at $\mu=1$, with $y(0)=0$ and $\dot{y}(0)=0$; (b) (dashed line) y, (solid line) y_m when $\mu(t)$ is time varying according to (25) with $\bar{\mu}=1$, $y(0)=5$ and $\dot{y}(0)=0$; and (c) the time varying $\mu(t)$.

$4.4|y|+3y^2+4|r|+2.1$ and $\psi_\mu=56|\omega_1|+28|\omega_2|+13|y|+15y^2+22$, respectively. Moreover, the HGO and the sliding surface are implemented with $l_1=2$, $l_2=1$ and $S=[2\ \ 1]$.

For $y(0)=0$ and $\dot{y}(0)=0$ and with a constant and large value of $\mu(t)=\bar{\mu}=1$ an apparent degradation in the closed loop tracking accuracy (y does not even converge to y_m) is observed in Figure 2(a). Moreover, for $y(0)=5$ and $\dot{y}(0)=0$, the plant output escapes at $t\approx 1.79$ (not shown). On the other hand, when the time varying $\mu(t)$ is implemented with the same large value for $\bar{\mu}=1$, the plant output converges to the desired trajectory from $y(0)=5$, as shown in Figure 2(b). In this case, the time evolution of $\mu(t)$ is shown in Figure 2(c), from which one can verify that a constant $\mu=\bar{\mu}=0.0005$ could be used. However, this value is not known *a priori*. Moreover, care must be taken in reducing $\bar{\mu}$, since there exist a tradeoff between measurement noise reduction and tracking accuracy. In Figure 3, the estimated state $\hat{\xi}$ transient is presented. Figure 3(a) shows the fast convergence of $\hat{\xi}_1$ to plant output y. Figure 3(b) shows the peaking present in $\hat{\xi}_2$ which converges to $\dot{y}$ after a fast transient. □

9. CONCLUSIONS

The global tracking problem of SISO uncertain time-varying nonlinear systems has been solved by using OFSM control. We have considered a rather general class of plants which includes nonlinearities affinely norm bounded by unmeasured states with growth rate depending nonlinearly on the internal states and measured system output. This note shows that it is possible to apply domination techniques and to design a dynamic gain HGO in order to obtain global practical tracking by OFSM control free of peaking. An illustrative simulation example was presented. We believe that such result is new in the context of sliding mode control of nonlinear uncertain systems.

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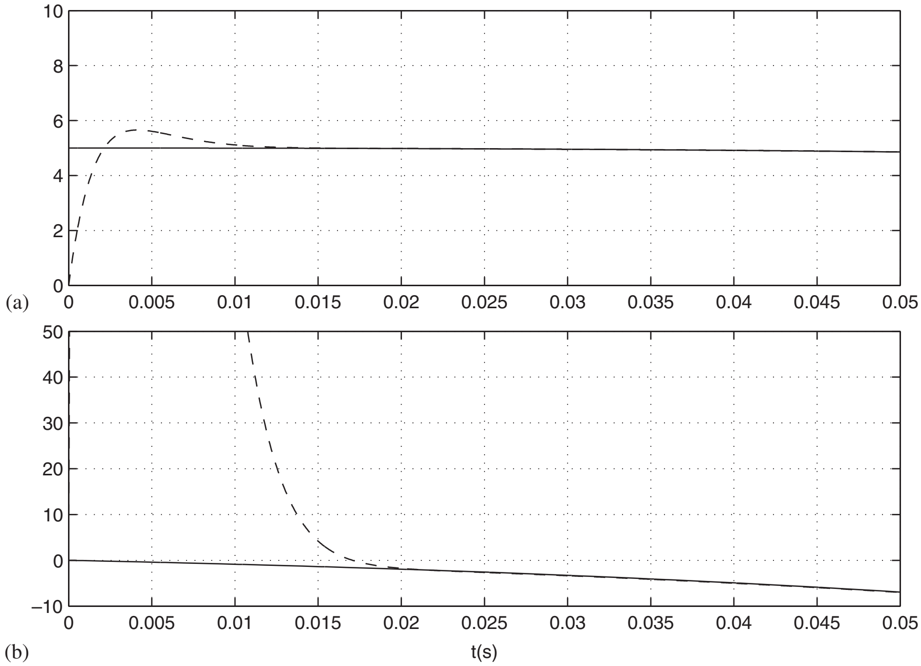

Figure 3. Simulation result. HGO estimated states transient: (a) (solid line) output y, (dashed line) $\hat{\xi}_1$ and (b) (solid line) $\dot{y}$, (dashed line) $\hat{\xi}_2$.

APPENDIX A

A.1. Geometric conditions for normal form

In order to consider explicitly the time dependence of $f(x,t)$ in (1), let: $\beta_k := L_f\beta_{k-1} + \partial\beta_{k-1}/\partial t + \partial[L_f^{k-1}h]/\partial t$, for $k \in \{1,\ldots,\rho\}$ and $\beta_0 := 0$. A sufficient condition to assure that the *time-varying* plant (1)–(2) is transformable to the normal form is given by: $L_g[L_f^k h + \beta_k] \equiv 0$ $(k \in \{0,\ldots,\rho-2\})$, where Lie derivative of a function h along a vector field f is denoted by $L_f h$, as in [26, pp. 510]. In this case, the transformation $T(x,t) = [\eta^{\mathrm{T}}\ T_\xi^{\mathrm{T}}(x,t)]$ is such that $T_\xi := [L_f^0 h + \beta_0, L_f h + \beta_1, \ldots L_f^{\rho-1}h + \beta_{\rho-1}]^{\mathrm{T}}$. In addition, the plant HFG $k_p(x,t) = L_g[L_f^{\rho-1}h + \beta_{\rho-1}]$, the input disturbance $d(x,t) = (L_f^\rho h + \beta_\rho)/k_p$ and T must satisfy Assumption 1.

A.2. Norm observer

In this section, we consider systems in the form (50)–(51) satisfying (C0) and (C1) in Section 8. In what follows, we give the steps to obtain the norm observer (53)–(55), according to Definition 1.

Norm bound for η: obtaining c_0 and φ_1 in (54): From (C1), the function α_1 is stiffening. It guarantees that $\alpha_1(\sigma) > \lambda\sigma$, $\forall\sigma > \varepsilon$, for any $\varepsilon > 0$ and $0 < \lambda < \alpha_1(\varepsilon)/\varepsilon$. Moreover, from (52), one can write $\dot{V} \leqslant -\alpha_1(V) + \varphi_\eta(y,t)$ or, equivalently, $\dot{V} \leqslant -\lambda V + [\lambda V - \alpha_1(V)] + \varphi_\eta(y,t)$. Now, given any V, either $V \leqslant \varepsilon$ or $V > \varepsilon$. Hence, either $\dot{V} \leqslant -\lambda V + [\lambda\varepsilon + \alpha_1(\varepsilon)] + \varphi_\eta$ or $\dot{V} \leqslant -\lambda V + \varphi_\eta$, leading to the conclusion that $\dot{V} \leqslant -\lambda V + [\lambda\varepsilon + \alpha_1(\varepsilon)] + \varphi_\eta$. Therefore, by using comparison theorems [26], one has

$$V \leqslant e^{-c_0 t} * \varphi_1(y,t) + V(\eta(0))e^{-c_0 t},$$

where $\varphi_1 = \varphi_\eta + c_0\varepsilon + \alpha_1(\varepsilon)$, $c_0 = \lambda$ are *known* and the operator $*$ denotes pure convolution. Reminding that $\underline{\lambda}|\eta|^2 \leqslant V$, then one can obtain $|\eta| \leqslant \sqrt{|\omega_{21}|/\underline{\lambda}} + \pi_0$, with ω_{21} in (54) and π_0 is an exponentially decaying term depending on $|\eta(0)|$ and $|\omega_{21}(0)|$.

Norm bound for v: obtaining φ_2 and φ_3 in (55): It is useful to rewrite (51) in the compact form

$$\dot{v} = A_\rho v + b_\rho k_u u + \phi(x,t), \tag{A1}$$

where (A_ρ, b_ρ) is the Brunovsky's canonical pair and apply the change of variable $\bar{v} = v - b_\rho k_u \tau_1 \omega_1$ to obtain:

$$\dot{\bar{v}} = A_\rho \bar{v} + b_\rho k_u \omega_1 + \phi.$$

By observability of the pair (A_ρ, c_ρ), where $c_\rho = [1 \ 0 \ \ldots \ 0]$, there exist a matrix $P = P^{\mathrm{T}} > 0$ and an arbitrary column vector L satisfying $A_L^{\mathrm{T}} P + P A_L = -I$, where $A_L = A_\rho - L c_\rho$ is a Hurwitz matrix.

Now, with $T := \mathrm{diag}(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{\rho-1})$ and any given constant $\varepsilon > 0$, the following properties can be checked: (i) $T A_\rho T^{-1} = \varepsilon^{-1} A_\rho$, (ii) $c_\rho T^{-1} = c_\rho$ and (iii) $T b_\rho = b_\rho \varepsilon^{\rho-1}$. Then, adding and subtracting the term $(\varepsilon T)^{-1} L c_\rho \bar{v}$ to the $\bar{v}$-dynamics, one can write $\dot{\bar{v}} = [A_\rho - (\varepsilon T)^{-1} L c_\rho]\bar{v} + b_\rho k_u \omega_1 + \phi + (\varepsilon T)^{-1} L y$. Moreover, applying the transformation $\vartheta = T\bar{v}$ and the above properties (i)–(iii), one can also write

$$\dot{\vartheta} = \varepsilon^{-1} A_L \vartheta + b_\rho \varepsilon^{\rho-1} k_u \omega_1 + \varepsilon^{-1} L y + T\phi.$$

The key step is to note that, due to the triangularity condition (C0):

$$|T\phi| \leqslant k_\vartheta \varphi_r |\vartheta| + \varphi_\vartheta,$$

where k_ϑ is ε independent. Then, by using the Dini derivative[‖] and the bounding function Ψ_v given in (C0), the time derivative of $V := (\vartheta^{\mathrm{T}} P \vartheta)^{1/2}$ along the solution of the ϑ-dynamics satisfies

$$\dot{V} \leqslant -\frac{c_1}{\varepsilon} V + c_2 \varphi_r V + \bar{\varphi}_1(\omega_{21}, \omega_1, y, t, \varepsilon) + \pi_1,$$

where π_1 is a exponentially decaying term and the non-negative function $\bar{\varphi}_1$ and the non-negative constants c_1, c_2 are all *known* and satisfy $c_1 \leqslant 1/(2\lambda_M[P])$, $c_2 \geqslant |P| k_\vartheta / \lambda_m[P]$ and $[|b_\rho \varepsilon^{\rho-1} k_u \omega_1 + \varepsilon^{-1} L y| + \varphi_\vartheta] c_3 \leqslant \bar{\varphi}_1 + \pi_1$, with $c_3 \geqslant |P| / \sqrt{\lambda_m[P]}$.

Therefore, given any V, either

$$V \leqslant \bar{\varphi}_1 \quad \text{or} \quad \dot{V} \leqslant -\frac{c_1}{\varepsilon} V + c_2 \varphi_r V + V + \pi_1. \tag{A2}$$

Now, let

$$\bar{\varphi}_4(\omega_{21}, y, t) := \varphi_2(\omega_{21}) + \varphi_3(y, t), \tag{A3}$$

with the non-negative functions φ_2, φ_3 in (55) to be determined. Then, (55) can be written as

$$\dot{\omega}_{22} = -\frac{1}{\tau_2} \gamma(\omega_{22}) + \bar{\varphi}_4, \tag{A4}$$

with $\gamma(\sigma) := 1 - e^{-\sigma}$. Hence, by using the bounding function Ψ_r, given in (C0), we must choose $\bar{\varphi}_4$ in (A3) (and the functions φ_2, φ_3) in order to satisfy:

$$c_2 \varphi_r + 1 \leqslant \bar{\varphi}_4(\omega_{21}, y, t).$$

Norm bound for v: The norm bound for the v-subsystem can be obtained considering two cases for the growth rate $\varphi_r(|\eta|, y, t)$: $\varphi_r > k_r$ and $\varphi_r \leqslant k_r$, where $k_r = 3/(c_2 \tau_2)$ and τ_2 is the positive design constant in (A4).

Case 1: In this case, one has $3/\tau_2 \leqslant c_2 k_r + 1 \leqslant c_2 \varphi_r + 1 \leqslant \bar{\varphi}_4$. Thus, one can verify that

$$\gamma(\sigma) \leqslant 2 \leqslant \tau_2 \bar{\varphi}_4 - 1, \quad \forall \sigma. \tag{A5}$$

[‖] To avoid the Dini derivative we could have used the relationship $ab \leqslant a^2 + b^2$, valid $\forall a, b > 0$, at the expense of some conservatism.

Now, let $W := \ln(V+1)$ [22]. Then, $\dot{W} = \dot{V}/(V+1)$ and, from (A2), one can write

$$V \leqslant \bar{\varphi}_3 \quad \text{or} \quad \dot{W} \leqslant -\frac{1}{\tau_2}\gamma(W) + \bar{\varphi}_4 + \pi_1, \tag{A6}$$

with $\varepsilon = c_1\tau_2$, $\bar{\varphi}_3 := \bar{\varphi}_1|_{\varepsilon = c_1\tau_2}$ and noting that $V/(V+1)$, $1/(V+1) \leqslant 1$.

Now, given any W, we have two possibilities: $W < \omega_{22}$ or $W \geqslant \omega_{22}$. Considering the latter case, one can write $-\gamma(\omega_{22}) \geqslant -\gamma(W)$, since γ is a increasing function. Therefore, from (A6) and (A4), one has $\dot{\omega}_{22} \geqslant \dot{W} - \pi_1$. In addition, from (A5), $\dot{\omega}_{22}$ also satisfies $\dot{\omega}_{22} \geqslant 1/\tau_2$. Consequently, adding the last two inequalities one has

$$\dot{W} - 2\dot{\omega}_{22} \leqslant -\frac{1}{\tau_2} + \pi_1.$$

Now, recall that $\pi_1 = \beta_1 e^{-\lambda_1 t}$ and let $\bar{W} = W + \pi_1/\lambda_1$, for some positive constant λ_1 and some $\beta_1 \in \mathscr{K}_\infty$. Then, one has $\dot{\bar{W}} - 2\dot{\omega}_{22} \leqslant -\frac{1}{\tau_2}$, from which one can conclude that, $\bar{W} \leqslant 2\omega_{22} - t/\tau_2 + |\bar{W}(0) - 2\omega_{22}(0)|$. Note that, it is always possible to find an exponential decaying term which is an upper bound for the above affine time function, i.e. $-t/\tau_2 + |\bar{W}(0) - 2\omega_{22}(0)| \leqslant \pi_2$, where $\pi_2 := \beta_2(|\bar{W}(0)| + |\omega_{22}(0)|)e^{-\lambda_2 t}$, with $\beta_2 \in \mathscr{K}_\infty$ and some constant $\lambda_2 > 0$. Finally, given W, one can conclude that $W \leqslant 2|\omega_{22}| + \pi_2 + \pi_1/\lambda_1$ and, by using comparison theorems [26] and recalling that $V = e^W - 1$ one can write

$$V \leqslant e^{2|\omega_{22}|} + \pi_3, \tag{A7}$$

where π_3 is an exponential decaying term.

Case 2: Assume now that $\varphi_r \leqslant k_r$ and set $\varepsilon = c_1/(c_2 k_r + 2)$ in (A2). Then, one can write:

$$V \leqslant \bar{\varphi}_2 \quad \text{or} \quad \dot{V} \leqslant -V + \pi_1, \tag{A8}$$

where $\bar{\varphi}_2 = \bar{\varphi}_1|_{\varepsilon = c_1/(c_2 k_r + 2)}$. In this case, adding the two upper bounds obtained from (A8) one can write

$$V \leqslant \bar{\varphi}_2 + \pi_4, \tag{A9}$$

where π_4 is an exponential decaying term. Then, from (A7) and (A9) one has

$$V \leqslant e^{2|\omega_{22}|} + \bar{\varphi}_1(\omega_{21}, \omega_1, y, t, \varepsilon) + \pi_5, \tag{A10}$$

with $\varepsilon = c_1/(3/\tau_2 + 2)$ and using the Rayleigh's inequality one can obtain an upper bound for v.

Finally, putting together the norm bounds for v and η we obtain the non-negative function φ_4 and the non-negative constants in (56).

A.3. Auxiliary proofs

Proof of Inequalities (15), (16) *and* (17): From the plant state estimator, one has $|x| \leqslant \bar{\varphi}_o(\omega, t) + \pi_o$. Note that, for any increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, one can write $\psi(a+b) \leqslant \psi(2a) + \psi(2b)$, $\forall a, b \in \mathbb{R}^+$. Since φ_i $(i = 1, 2, 3)$ are non-negative and increasing functions in their first arguments, then follows that $\varphi_i(|x|, y, t) \leqslant \varphi_i(2\bar{\varphi}_o, y, t) + \varphi_i(2\pi_o, y, t)$. Moreover, one can further conclude from Assumption 1 that $\varphi_i(2\pi_o, y, t) \leqslant \alpha_i(2\pi_o) + \bar{\varphi}_i(y, t)$ and, since $\alpha_i \in \mathscr{K}$ are locally Lipschitz functions, $\alpha_i(2\pi_o) \leqslant \pi_1$, where $\pi_1 = \beta_1(|\omega(0)| + |x(0)|)e^{-\lambda_o t}$ with some $\beta_1 \in \mathscr{K}_\infty$. Therefore, one can write $\varphi_i(|x|, y, t) \leqslant \psi_i(\omega, t) + \pi_1$, where $\psi_i(\omega, t) := \varphi_i(2\bar{\varphi}_o, y, t) + \bar{\varphi}_i(y, t)$. Recalling that $[\eta^{\mathrm{T}} \ \xi^{\mathrm{T}}]^{\mathrm{T}} = T(x, t)$, then $|\xi| \leqslant |T(x, t)|$. Hence, from Assumption 1, one has $|\xi| \leqslant \varphi_1(|x|, y, t)$. Therefore, ξ, k_p and d satisfy (15), (16) and (17).

Proof of Inequalities (44) *and* (45): If $\beta_4(|\omega(0)| + |x(0)|) \leqslant 1$ or t_M is infinite it is trivial due to the vanishing exponential $e^{-\lambda_4 t}$. Now, consider that $\beta_4(|\omega(0)| + |x(0)|) > 1$ and t_M is finite. Then, one has: (i) $e^{-\lambda_\mu t} \geqslant e^{-\lambda_\mu t_M}$, $\forall t \in [0, t_M)$ and (ii) $\exists t_1 \in [0, t_M)$ such that $\|\omega_t\| \geqslant \delta$, $\forall t \in [t_1, t_M)$, where δ is an arbitrary constant. Hence, from (i) and (ii) and taking $\delta \geqslant (\beta_4 - 1)e^{\lambda_\mu t_M}$, one also has that the right-hand side of (43) is bounded by $\bar{\mu}$. In addition, during the interval $[0, t_\mu)$, by definition of

t_μ, one has that $|\omega(t)| \leqslant \beta_5(|\omega(0)|+|x(0)|)$. By noting that (i) $e^{\lambda_\mu t_\mu}$ can be bounded by a class-$\mathscr{K}$ function of $|\omega(0)|+|x(0)|$ and (ii) ζ (31) escapes at most exponentially, one can conclude that $|\zeta|$ and $|\omega|$ can be bounded by some class-$\mathscr{K}$ function of $|\omega(0)|+|x(0)|+|\zeta(0)|$.

Proving that (47) *Satisfies (P1)*: First note that for any absolutely continuous function $g(t)$, $\|g_t\| = |g(t)|$ or $\|g_t\|$ is a positive constant. Thus, $|\mathrm{d}\|g_t\|/\mathrm{d}|g|| \leqslant 1$, almost everywhere, consequently, $|\partial\psi_\mu/\partial|\omega|| \leqslant |\mathrm{d}p_\mu(|\omega|)/\mathrm{d}|\omega|| + e^{-\lambda_\mu t}$. Moreover, since $\mathrm{d}p(a)/\mathrm{d}a \leqslant k_1 p(a)$, where $p(a)$ is any polynomial in a with positive real coefficients and k_1 is an appropriate constant, one can also write $|\partial\psi_\mu/\partial|\omega|| \leqslant k_1 p_\mu(|\omega|) + e^{-\lambda_\mu t}$. In addition, since $|\partial|\omega|/\partial\omega| \leqslant 1$, then $|\partial\psi_\mu/\partial\omega| \leqslant |\partial\psi_\mu/\partial|\omega||$ and one can conclude that (P1) holds, since $\partial\psi_\mu/\partial t = -\lambda_\mu\|\omega_t\|e^{-\lambda_\mu t}$.

A.4. Main proofs

Proof of Lemma 1

First, applying the coordinate transformation $\xi_{en} = T_n\xi_e$, where $T_n := [I \;\; S^{\mathrm{T}}]^{\mathrm{T}}$, system (13) can be rewriting into the normal form and one can conclude that (13) is OSS w.r.t. the output $S\xi_e$, i.e. ξ_e satisfies

$$|\xi_e| \leqslant k_1|S\xi_e| + \pi_1,$$

where k_1 is a positive constant and $\pi_1 = \beta_1(|\xi_e(0)|)e^{-\lambda_m t}$, with some $\beta_1 \in \mathscr{K}_\infty$ and $0 < \lambda_m < \lambda_m[A_m]$. Given any $\tilde{\xi}_e$, either $|S\xi_e| \leqslant |S\tilde{\xi}_e|$ or $|S\xi_e| > |S\tilde{\xi}_e|$. Hence, either $|S\xi_e| \leqslant |S\tilde{\xi}_e|$ or $\mathrm{sgn}(\hat{\sigma}) = \mathrm{sgn}(S\xi_e)$. Consider the latter case. Then, by using the storage function $V = \xi_e^{\mathrm{T}} P\xi_e$, where $P = P^{\mathrm{T}} > 0$ is the solution of $A_m^{\mathrm{T}}P + PA_m = -I$, one can conclude that the time derivative of V along the solutions of (13) satisfies

$$\dot{V} \leqslant -|\xi_e|^2 - 2k_p|S\xi_e|[\varrho - |d_e|].$$

Thus, since ϱ in (19) satisfies (18), i.e. $\varrho > |d_e|$, then one has $\dot{V} \leqslant -|\xi_e|^2$, which leads to the conclusion that $|S\xi_e| \leqslant |S\tilde{\xi}_e| + \pi_2$ and, consequently, the ξ_e-dynamics is ISS w.r.t. $\tilde{\xi}_e$. □

Proof of Theorem 1

- [*STEP1*]: From Definition 1, Assumption 1 and (44), one can verify that $|z(t)| \leqslant \beta_1(|z(0)|) + k_1, \forall t \in [0, t_\mu]$, where $\beta_1 \in \mathscr{K}_\infty$ and $k_1 \geqslant 0$ is a constant.
- [*STEP2*]: Consider the ζ-dynamics (31) and the storage $V = \zeta^{\mathrm{T}}P\zeta$, where $P = P^{\mathrm{T}} > 0$ is the solution of $A_o^{\mathrm{T}}P + PA_o = -I$. Then, the time derivative of V along the solutions of (31) satisfies $\mu\dot{V} = -|\zeta|^2 + (\dot{\mu})[2\zeta^{\mathrm{T}}P\Delta\zeta] + (\mu v)[2\zeta^{\mathrm{T}}Pb_\rho]$. Now, designing μ to satisfy (P0)–(P2), (45) holds and the following inequality is valid $\forall t \in [t_\mu, t_M)$: $\mu\dot{V} \leqslant -|\zeta|^2 + \mathcal{O}(\bar{\mu})k_1|\zeta|^2 + \mathcal{O}(\bar{\mu})k_2|\zeta|$, where $k_1 := 2|P||\Delta|$ and $k_2 := 2|P||b_\rho|$. Moreover, since $ab < a^2 + b^2$, for any positive real numbers a, b, then

  $$\mu\dot{V} \leqslant -[1 - \mathcal{O}(\bar{\mu})k_1 - \mathcal{O}(\bar{\mu})]|\zeta|^2 + \mathcal{O}(\bar{\mu}),$$

  from which one can conclude that $\mu\dot{V} \leqslant -\lambda_1 V + \mathcal{O}(\bar{\mu})$, with an appropriate constant $\lambda_1 > 0$. Now, either $V \leqslant 2\mathcal{O}(\bar{\mu})/\lambda_1$ or $\mu\dot{V} \leqslant -\lambda_1 V/2$. Consider the latter case. Since $\mu < \bar{\mu}$, then one has $\dot{V} \leqslant -\lambda_1 V/(2\bar{\mu})$. Hence, one can conclude that $|\zeta|, |\tilde{\xi}_e| \leqslant \beta_2(|\zeta(0)|)e^{-\lambda_2 t} + \mathcal{O}(\bar{\mu})$, $\forall t \in [t_\mu, t_M)$, with an appropriate constant $\lambda_2 > 0$ and some $\beta_2 \in \mathscr{K}_\infty$. In the last inequality, the norm bound for $\tilde{\xi}_e$ was obtained by noting that $\tilde{\xi}_e = T_\mu^{-1}\zeta$ implies $|\tilde{\xi}_e| \leqslant |\zeta|$, since $|T_\mu^{-1}| \leqslant 1$ for $\mu < 1$.
- [*STEP3*]: Applying Lemma 1, there exists an ISS Property from $|\tilde{\xi}_e|$ to ξ_e and, considering the norm bound given in STEP-1, one can further concluded that $|\xi_e|, |z(t)| \leqslant [\beta_3(|z(0)|) + k_3]e^{-\lambda_3 t} + \mathcal{O}(\bar{\mu})$, $\forall t \in [0, t_M)$, with an appropriate constants $\lambda_3 > 0$, $k_3 \geqslant 0$ and some $\beta_3 \in \mathscr{K}_\infty$. Thus, $|z(t)|$ cannot escape in finite time and it is uniformly bounded in $\mathscr{I} := [0, t_M)$ (UB$\mathscr{I}$).

[*STEP4*]: Since $z(t)$ is UB$\mathscr{I}$, then $\xi_e, \sigma = S\xi_e, \zeta$ and $\xi = \xi_e + \xi_m$ are UB$\mathscr{I}$ and, from Assumption 2, $\eta, \bar{x}$ are also UB$\mathscr{I}$. In addition, according to the lower bound for $|T(x,t)|$ given in Assumption 1 one has that x UB$\mathscr{I}$. Thus, the bounding functions given in Assumption 1 guarantee that d, k_p, d_e are also UB$\mathscr{I}$. Now, rewriting (13) into the normal form one can write $\dot{\sigma} = -\lambda_4\sigma + k_4(u + d_e)$, for some constants $\lambda_4, k_4 > 0$. Moreover, by linearity of the solution of the last equation, one can further write $\sigma = \sigma_1 + \sigma_2$, where $\dot{\sigma}_1 = -\lambda_4\sigma_1 + k_4 u$ and $\dot{\sigma}_2 = -\lambda_4\sigma_2 + k_4 d_e$, with appropriate initial conditions. Thus, since σ and d_e are UB$\mathscr{I}$ so are σ_2 and σ_1. Then, any signal satisfying $\dot{\sigma}_3 = -\lambda_5\sigma_3 + k_5 u$, where $\lambda_5, k_5 > 0$ are constants, is also UB$\mathscr{I}$, in particular, ω_1 defined in (3). Since y, ω_1 is UB$\mathscr{I}$ and φ_o is piecewise continuous in its arguments then the ω_2-dynamics, in Definition 1, cannot escape in finite time. Finally, one can conclude that all system signals cannot escape in finite time, i.e. $t_M \to \infty$. Now, from STEP 3, one can directly verify that the error system is GAS with respect to the compact set $\{z : |z| \leqslant b\}$ and ultimate exponential convergence of $z(t)$ to a residual set of order $\mathcal{O}(\bar{\mu})$.

Closed loop signals boundedness: One can further conclude, subsequently, that $|\xi|, y, |\eta|, |x|, \sigma_1$ and ω_1 converge to a set of order $\mathcal{O}(|r| + k_5)$ after some finite time, where k_5 is a positive constant depending on the time-varying disturbances. Then, there exists τ_2 sufficiently small and independent of the initial conditions, which assures that ω_2 is bounded after some finite time. Finally, one can conclude that all system signals are UB $\forall t$. □

Proof of Corollary 1

Recalling that $A_\rho = A_m - b_\rho K_m$, $\hat{\xi} = \hat{\xi}_e + \xi_m$, $\hat{\xi} = \xi_e + \xi_m - \tilde{\xi}_e$, $\hat{\xi}_e = \xi_e - \tilde{\xi}_e$ and $\tilde{\xi}_e = T_\mu^{-1}\zeta$, then from (23) one can write $\dot{\hat{\xi}}_e = A_m\hat{\xi}_e + b_\rho u + \varsigma_m + \varsigma_e$, where $\varsigma_m = -b_\rho(K_m\xi_m + k_m r)$ and $\varsigma_e = (b_\rho K_m + H_\mu L_o c_\rho)(\tilde{\xi}_e - \xi_e) + H_\mu L_o e$. Note that, from Theorem 1, all system signals are uniformly bounded and $z(t) \to \mathcal{O}(\bar{\mu})$. Then, there exists a finite time $T_1 > 0$ such that $|\varsigma_e| \leqslant \delta_1, \forall t \geqslant T_1$, for any $\delta_1 > 0$. Now, consider the storage function $V = \hat{\xi}_e^{\mathrm{T}} P \hat{\xi}_e$, where $P = P^{\mathrm{T}} > 0$ is the solution of $A_m^{\mathrm{T}} P + P A_m = -Q$, where $Q = Q^{\mathrm{T}} > 0$ and $P b_\rho = S^{\mathrm{T}}$ (recall that (A_m, b_ρ, S) is strictly positive real). Then, computing $\dot{V}$ along the solutions of the $\hat{\xi}_e$-dynamics, one can verify that the condition for the existence of sliding mode $\hat{\sigma}\dot{\hat{\sigma}} < 0$ is verified for some finite time $T_2 \geqslant T_1$ provided that $\varrho \geqslant \varsigma_m + \delta$, where $\delta > 0$ is an arbitrary constant. □

REFERENCES

1. Yu X, Xu J-X (eds). *Variable Structure Systems*: *Towards the 21st Century*. Springer: Berlin, 2002.
2. Edwards C, Colet EF, Fridman L (eds). *Advances in Variable Structure and Sliding Mode Control*. Springer: Berlin, 2006.
3. Sabanovic A, Fridman L, Spurgeon SK (eds). *Variable Structure Systems*: *From Principles to Implementation* (*IEE Control Engineering*). Academic Press: New York, 2004.
4. Hsu L, Cunha JPVS, Costa RR, Lizarralde F. Multivariable output-feedback sliding mode control. In *Variable Structure Systems*: *Towards the 21st Century*, Yu X, Xu J-X (eds). Springer: Berlin, 2002; 283–313.
5. Hsu L, Peixoto AJ, Cunha JPVS, Costa RR, Lizarralde F. Output feedback sliding mode control for a class of uncertain multivariable systems with unmatched nonlinear disturbances. *Advances in Variable Structure and Sliding Mode Control*. Springer: Berlin, 2006; 195–225.
6. Oh S, Khalil HK. Nonlinear output-feedback tracking using high-gain observer and variable structure control. *Automatica* 1997; **33**(10):1845–1856.
7. Cunha JPVS, Hsu L, Costa RR, Lizarralde F. Sliding mode control of uncertain linear systems based on a high gain observer free of peaking. *Preprints of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005.
8. Levant A. Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control* 2003; **76**(9):924–941.
9. Esfandiari F, Khalil HK. Output feedback stabilization of fully linearizable systems. *International Journal of Control* 1992; **56**:1007–1037.
10. Oliveira TR, Peixoto AJ, Nunes EVL, Hsu L. Control of uncertain nonlinear systems with arbitrary relative degree and unknown control direction using sliding modes. *International Journal of Adaptive Control and Signal Processing* 2007; **21**:692–707.
11. Mazenc F, Praly L, Dayawansa W. Global stabilization by output feedback: examples and counterexamples. *Systems and Control Letters* 1994; **23**:119–125.

DOI: 10.1002/rnc

12. Teel A, Praly L. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization* 1995; **33**(5):1443–1488.
13. Praly L. Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate. *Proceedings of the IEEE Conference on Decision and Control*, Orlando, FL, U.S.A., 2001; 3808–3813.
14. Krishnamurthy P, Khorrami F, Jiang ZP. Global output feedback tracking for nonlinear systems in generalized output-feedback canonical form. *IEEE Transactions on Automatic Control* 2002; **47**(5):814–819.
15. Krishnamurthy P, Khorrami F, Chandra RS. Global high-gain-based observer and backstepping controller for generalized output-feedback canonical form. *IEEE Transactions on Automatic Control* 2003; **48**(12):2277–2284.
16. Lei H, Lin W. Universal output feedback control of nonlinear systems with unknown growth rate. *Preprints of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005.
17. Ahrens JH, Khalil HK. Closed-loop behavior of a class of nonlinear systems under EKF-based control. *IEEE Transactions on Automatic Control* 2007; **52**(3):536–540.
18. Andrieu V, Praly L, Astolfi A. *Asymptotic Tracking of a State Trajectory by Output-feedback for a Class of Non Linear Systems*. CDC: New Orleans, LA, 2007; 5228–5233.
19. Andrieu V, Praly L, Astolfi A. High gain observers with updated gain and homogeneous correction terms. *Automatica* 2009; **45**(2):422–428.
20. Marino R, Tomei P. *Nonlinear Control Design*. Prentice-Hall: Englewood Cliffs, NJ, 1995.
21. Praly L, Jiang Z. Linear output feedback with dynamic high gain for nonlinear systems. *Systems and Control Letters* 2004; **53**:107–116.
22. Kaliora G, Astolfi A, Praly L. Norm estimators and global output feedback stabilization of nonlinear systems with iss inverse dynamics. *IEEE Transactions on Automatic Control* 2006; **51**(3):493–498.
23. Ahrens JH, Khalil HK. High-gain observers in the presence of measurement noise: a switched-gain approach. *Automatica* 2009; DOI: 10.1016/j.automatica.2008.11.012.
24. Peixoto AJ, Hsu L, Costa RR, Lizarralde F. Global tracking sliding mode control for uncertain nonlinear systems based on variable high gain observer. *Proceedings of the IEEE Conference on Decision and Control*, New Orleans, LA, U.S.A., 2007; 2041–2046.
25. Sussmann HJ, Kokotović PV. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Transactions on Automatic Control* 1991; **36**(4):424–440.
26. Khalil HK. *Nonlinear Systems* (3rd edn). Prentice-Hall: Englewood Cliffs, NJ, 2002.
27. Sontag ED, Wang Y. On characterizations of the input-to-state stability property. *Systems and Control Letters* 1995; **24**:351–359.
28. Filippov AF. Differential equations with discontinuous right-hand side. *American Mathematical Society Translations* 1964; **42**(2):199–231.
29. Utkin VI. *Sliding Modes and their Application in Variable Structure Systems*. MIR: Moscow, 1978.
30. Diao Y, Passino KM. Adaptive control for a class of nonlinear time-varying systems. *Proceedings of the American Control Conference*, Arlington, June 2001; 4161–4166.
31. Isidori A. *Nonlinear Control Systems*. Springer: Berlin, 1995.
32. Sontag ED, Wang Y. Output-to-state stability and detectability of nonlinear systems. *Systems and Control Letters* 1997; **29**:279–290.
33. Angeli D, Sontag E. Forward completeness unboundedness observability, and their Lyapunov characterizations. *Systems and Control Letters* 1999; **38**(5):209–217.
34. Utkin VI. *Sliding Modes in Control and Optimization*. Springer: Berlin, 1992.
35. Bondarev AG, Bondarev SA, Kostyleva NE, Utkin VI. Sliding modes in systems with asymptotic state observers. *Automatic Remote Control* 1985; **46**(6):679–684.
36. Hsu L. Smooth sliding control of uncertain systems based on a prediction error. *International Journal of Robust and Nonlinear Control* 1997; **7**:353–372.
37. Qian C, Lin W. Output feedback control of a class of nonlinear systems a nonseparation principle paradigm. *IEEE Transactions on Automatic Control* 2002; **47**(10):1710–1715.
38. Choi H-L, Lim J-T. Global exponential stabilization of a class of nonlinear systems by output feedback. *IEEE Transactions on Automatic Control* 2005; **50**(2):255–257.
39. Arcak M, Larsen M, Kokotović P. Boundedness without absolute stability in systems with stiffening nonlinearities. *European Journal of Control* 2002; **8**(3):243–250.
40. Jiang ZP, Mareels I, Hill DJ, Huang J. A unifying framework for global regulation nonlinear output feedback: from ISS to iISS. *IEEE Transactions on Automatic Control* 2004; **49**(4):549–562.