A Globally Exponentially Stable Speed Observer for a Class of Mechanical Systems: Simulation Comparison with High-Gain and Sliding Mode Designs

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ABSTRACT
It is shown in the paper that the problem of speed observation for mechanical systems that are partially linearisable via coordinate changes admits a very simple and robust (exponentially stable) solution with a Luenberger-like observer. This result should be contrasted with the highly complicated observers based on immersion and invariance reported in the literature. A second contribution of the paper is to compare, via realistic simulations, the performance of the proposed observer with well-known high-gain and sliding mode observers. In particular, to show that—due to their high sensitivity to noise, that is unavoidable in mechanical systems applications—the performance of the two latter designs is well below par.

KEYWORDS
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1. Introduction

In this brief note we are interested in the problem of *speed observation* of mechanical systems that are partially linearisable via coordinate changes (PLvCC). This class, formally defined in (Venkatraman, Ortega, Sarras, & Van der Schaft, 2010), consists of mechanical systems whose dynamics becomes *linear in velocity* after a partial coordinate transformation, *e.g.*, a linear transformation of the velocities. PLvCC mechanical systems have been extensively studied (Bedrossian & Spong, 1995; Chang & McLenaghan, 2013; Romero & Ortega, 2015; Venkatraman et al., 2010) because, on one hand, observer design and controller synthesis are simplified for them while, on the other hand, there are many practical examples that satisfy this property. PLvCC mechanical systems have been characterized in (Venkatraman et al., 2010) via the solvability of a partial differential equation (PDE) defined by the inertia matrix. They contain as a particular case systems with zero Riemann symbols—also known as zero curvature systems—that are well known in analytical mechanics (Bedrossian & Spong, 1995; Spivak, 1999). Some verifiable conditions of PLvCC have been recently reported in (Chang & McLenaghan, 2013), where these systems are called quasi-linearizable.

In (Venkatraman et al., 2010) the problems of speed observation and position feedback stabilization of PLvCC systems are formulated and solved. The observer proposed in that paper is based on the immersion and invariance methodology proposed in (A. Astolfi, Karagiannis, & Ortega, 2007), this leads to a complicated high-order design that includes a dynamic scaling factor that injects high-gain during the transients. The first main contribution of this paper is to show that an *extremely simple* Luenberger-like observer yields a globally *exponentially* stable (GES) solution to the speed observation problem of PLvCC systems. Moreover, a standard, quadratic, *strict* Lyapunov function to prove GES is constructed with classical "addition of cross terms" techniques (Khalil, 2002; Malisoff & Mazenc, 2009). It should be underscored that the exponential qualifier is necessary to ensure that it can be combined—in a certainty equivalent way—with a full state-feedback controller to ensure a globally asymptotically stable (GAS) solution of the position feedback stabilization problem, see Proposition 9 of (Venkatraman et al., 2010). In this respect, using the observer proposed here yields a simpler solution to the stabilization problem than the one given in (Venkatraman et al., 2010), but the details are omitted for brevity. Furthermore, as is amply discussed in (Malisoff & Mazenc, 2009), disposing of a strict Lyapunov function is instrumental to carry out robustness analysis via total stability arguments.

As it is well known, it is undesirable to inject high-gain in a control loop. One of the deleterious effects of high-gain is the amplification of noise, which is unavoidable in any practical application, in particular, in mechanical systems. A second contribution of the paper is to show, via realistic simulations, that the high sensitivity to noise of high-gain (Esfandiari & Khalil, 1992; Khalil & Praly, 2014) and sliding mode observers (Davila, Fridman, & Levant, 2005) makes them less suitable for speed estimation of mechanical systems than the proposed one, which does not inject high-gain.

The remainder of the paper is structured as follows. In Section 2 we formulate the speed observation problem whose solution is given in Sections 3 and 4. Section 5 presents some simulation evidence of the proposed observer and two high-gain designs applied to the cart-and-pendulum system and the robotic leg. We wrap-up the paper with concluding remarks in Section 6.

Notation. I_n is the $n \times n$ identity matrix and $0_{n\times s}$ is an $n \times s$ matrix of zeros. For a vector $x \in \mathbb{R}^n$ and a square, symmetric matrix $A \in \mathbb{R}^{n\times n}$, we denote $|x|^2 := x^\top x$ and $\|A\|$, $\lambda_m\{A\}$ and $\lambda_M\{A\}$ the induced 2-norm and the minimum and maximum eigenvalue, respectively. All mappings are assumed smooth. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ we define $\nabla f := \left(\frac{\partial f}{\partial x}\right)^\top$.

2. Problem Formulation

2.1. *The class of mechanical systems*

We consider in the paper mechanical systems whose dynamics is described by the Hamiltonian equations of motion

$$\begin{bmatrix} \dot q \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -I_n & 0_{n\times n} \end{bmatrix} \nabla H(q,\mathbf{p}) + \begin{bmatrix} 0_{n\times m} \\ G(q) \end{bmatrix} u, \tag{1}$$

with total energy function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$H(q,\mathbf{p}) = \frac{1}{2}\mathbf{p}^\top M^{-1}(q)\mathbf{p} + V(q),$$

where $q, \mathbf{p} \in \mathbb{R}^n$ are the generalized positions and momenta, respectively, $u \in \mathbb{R}^m$ is the control input, $n \geq m$, the inertia matrix $M : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ verifies $M(q) > 0$, $V : \mathbb{R}^n \to \mathbb{R}$ is the potential energy function and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is the full–rank input matrix.

The definition below identifies the class of mechanical systems that we consider in the paper.

Definition 2.1. *(Venkatraman et al., 2010) The mechanical system* (1) *is said to be PLvCC if there exists a full rank mapping* $\Psi : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ *such that the (partial) change of coordinates*

$$(q, p) \mapsto (q, \Psi^\top(q)\mathbf{p}) \tag{2}$$

transforms (1) *into*

$$\begin{aligned} \dot{q} &= \mathcal{M}(q)p \\ \dot{p} &= -\Psi^\top(q)[\nabla V(q) - G(q)u], \end{aligned} \tag{3}$$

where, to simplify the notation, we have introduced the full-rank mapping $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ *as*

$$\mathcal{M}(q) := M^{-1}(q)\Psi^{-\top}(q).$$

Remark 1. As shown in (Venkatraman et al., 2010) the change of coordinates (2) transforms (1) into

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & \Psi(q) \\ -\Psi^\top(q) & J(q,p) \end{bmatrix} \nabla \bar{H}(q,p) + \begin{bmatrix} 0 \\ \Psi^\top(q)G(q) \end{bmatrix} u, \tag{4}$$

with new Hamiltonian $\bar{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\bar{H}(q,p) = \frac{1}{2} p^\top [\Psi^\top(q) M(q) \Psi(q)]^{-1} p + V(q),$$

and the jk–th element of the skew-symmetric matrix $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ given by

$$(J(q,p))_{jk} = -p^\top [(\Psi(q))_j, (\Psi(q))_k], \; j,k \in \bar{n}$$

with $[\cdot,\cdot]$ the standard Lie bracket (Spivak, 1999) and $(\cdot)_j$ the j-th column. In (Venkatraman et al., 2010) it is shown that (4) reduces to (3) *if and only if*

$$\Psi^\top(q) \frac{\partial}{\partial q} \left\{ \frac{1}{2} \mathbf{p}^\top M^{-1}(q) \mathbf{p} \right\} - \dot{\Psi}^\top(q)\mathbf{p} \equiv 0.$$

Interestingly, the latter is true if and only if a PDE in $\Psi(q)$, which is univocally defined by $M(q)$, admits a solution—see Assumption 1 in (Venkatraman et al., 2010). See also (Chang & McLenaghan, 2013) for a geometric characterisation of the PLvCC property.

Remark 2. The main feature of PLvCC systems is that, as seen from (3), their dynamics is *linear* in momenta. Adopting a Lagrangian description of the system this means that in PLvCC systems the quadratic terms in velocity—appearing in the

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Coriolis and centrifugal forces vector—vanishes when the dynamics is expressed in the new coordinates.

2.2. *Exponentially stable observer design problem*

Before presenting the observation problem that is formulated, and solved, in this paper we state the following assumption, which is standard in (open loop) observer design problems.

Assumption 2.2. $u \in \mathcal{L}_\infty$ and is such that $q, p \in \mathcal{L}_\infty$, with a *known* bound

$$\max\{\|q\|_\infty, \|p\|_\infty\} \leq K, \tag{5}$$

where $\|\cdot\|_\infty$ is the $\mathcal{L}_\infty$ norm.

Momenta observation problem. Consider the mechanical system (3). Find two mappings

$$F, H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,$$

such that (3) together with

$$\begin{aligned} \dot{\hat{q}} &= F(\hat{q}, \hat{p}, q, u) \\ \dot{\hat{p}} &= H(\hat{q}, \hat{p}, q, u), \end{aligned}$$

generates an error system

$$\dot{x} = B(x, t), \tag{6}$$

with

$$x := \mathrm{col}(\hat{q} - q, \hat{p} - p) \in \mathbb{R}^{2n}, \tag{7}$$

the error coordinates, for which there exists a quadratic function $W : \mathbb{R}_+ \times \mathbb{R}^{2n} \to \mathbb{R}_+$,

$$W(t, x) := \frac{1}{2} x^\top P(t) x \tag{8}$$

with $P : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ verifying

$$c_2 I_{2n} \leq P(t) \leq c_3 I_{2n} \tag{9}$$

and

$$\dot{W} \leq -c_4 |x|^2, \tag{10}$$

for some positive constants $c_i,\ i = 2, 3, 4$. As is well-known (Khalil, 2002) these properties ensure GES of the zero equilibrium of (6). In particular, we have

$$\left| \begin{bmatrix} \tilde{q}(t) \\ \tilde{p}(t) \end{bmatrix} \right|^2 = \frac{c_3}{c_2} e^{-\frac{2c_4}{c_3} t} \left| \begin{bmatrix} \tilde{q}(0) \\ \tilde{p}(0) \end{bmatrix} \right|^2,$$

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for all $(\tilde{q}(0), \tilde{p}(0)) \in \mathbb{R}^n \times \mathbb{R}^n$

Remark 3. In the problem formulation above we are aiming at an *exponentially* convergent momenta observer with a *strict* Lyapunov function. As discussed in the introduction the exponential requirement is necessary to, invoking Proposition 9 of (Venkatraman et al., 2010), be able to use the observer to give a GAS solution to the position feedback regulation problem. On the other hand, the importance for robustness analysis of disposing of a strict Lyapunov function can hardly be overestimated— (Malisoff & Mazenc, 2009, see) for a detailed discussion on this matter.

Remark 4. Notice that the system (3) does not verify the conditions for application of the classical "linearization up to an output injection" observers of (Krener & Respondek, 1985). On the other hand, the observer proposed in (Venkatraman et al., 2010)—besides being extremely complex for practical application—does not satisfy the strict Lyapunov requirement and only global (exp.) *convergence* is established.

3. Proposed Observer and Literature Review

3.1. *A Luenberger observer and resulting error equation*

The system (3) suggests the following standard Luenberger observer

$$\begin{aligned} \dot{\hat{q}} &= \mathcal{M}(q)\hat{p} - L(\hat{q} - q) \\ \dot{\hat{p}} &= -\Psi^\top(q)[\nabla V(q) - G(q)u] - \Gamma\mathcal{M}^\top(q)\,(\hat{q} - q)\,, \end{aligned} \tag{11}$$

where $L, \Gamma \in \mathbb{R}^{n\times n}, L, \Gamma > 0$. Defining the error signal $\tilde{q} := \hat{q} - q, \tilde{p} := \hat{p} - p$ the error equations take the familiar form,

$$\begin{aligned} \dot{\tilde{q}} &= -L\tilde{q} + \mathcal{M}(q)\tilde{p} \\ \dot{\tilde{p}} &= -\Gamma\mathcal{M}^\top(q)\tilde{q}. \end{aligned} \tag{12}$$

In the x coordinates (7) the error equations (12) may be written as a linear time-varying (LTV) system of the form

$$\dot{x} = \begin{bmatrix} -L & \mathcal{M}(q(t)) \\ -\Gamma\mathcal{M}^\top(q(t)) & 0_{n\times n} \end{bmatrix} x =: A(t)x, \tag{13}$$

where, in view of Assumption 2.2, $A(t)$ and $\dot{A}(t)$ are bounded matrices. This kind of equations have been exhaustively studied in several contexts in the control literature, in particular, for adaptive systems (Anderson et al., 1986; Ioannou & Sun, 2012; Marino & Tomei, 1996; Sastry & Bodson, 2011). It is well-known that a *necessary and sufficient* condition for GES is that the matrix $\mathcal{M}(q(t))$—not necessarily square nor full-rank—verifies a persistency of excitation (PE) condition. Namely, that there exists positive constants T and ϵ such that

$$\int_t^{t+T} \mathcal{M}^\top(q(s))\mathcal{M}(q(s))ds \geq \epsilon I_n. \tag{14}$$

Notice that, in our case, this condition is clearly satisfied because $\mathcal{M}(q)$ is *square and*

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full-rank, therefore, $\mathcal{M}^\top(q)\mathcal{M}(q)$ is positive definite.

3.2. *Review of existing analysis results*

Unfortunately, as we discuss now, most proofs of GES of (12) available in the literature do not match the requirements of our problem formulation. In (Anderson et al., 1986; Ioannou & Sun, 2012; Sastry & Bodson, 2011) the equivalence between the PE condition (14) and GES of (13) is established without a strict Lyapunov function, but invoking instead properties of uniform complete observability of LTV systems—a feature that is ensured by the PE condition. The resulting proof is very long and technically involved and does not give much insight into the role of the various free parameters of the system, see (Loria, 2004; Panteley & Loria, 1998) for some discussion and (Maghenem & Loria, 2016) for recent related developments. Although the existence of a strict Lyapunov function can be established there is no explicit expression for it, instead it is is given in terms of the integral along trajectories of the fundamental matrix.

In Lemma B.2.3 of (Marino & Tomei, 1996) it is claimed that it is possible to prove that PE implies GES without invoking observability concepts. Unfortunately, the proof of this claim is wrong. Indeed, the derivation of the key inequality (B.43) relies on the following implication

$$\Omega \in PE \;\Rightarrow\; \int_t^{t+T} z^\top(s)\Omega(s)\Omega^\top(s)z(s)ds \geq \epsilon\delta^2,\; \forall |z| \geq \delta$$

where $\Omega, z \in \mathbb{R}^n$. The implication is clearly wrong because we cannot rule out the vector z orthogonal (or converging to orthogonal) to Ω. In Lemma A.3 of (R. Marino, 2010) a strict Lyapunov function is proposed but, similarly to the observability-based proofs mentioned above, requires an integration along trajectories for its construction.

Similarly, it is shown in Proposition 4 of (Barabanov & Ortega, 2017) that Theorem 2 of the highly cited paper (Morgan & Narendra, 1977) is also wrong, with a gap in the proof appearing in item d) of page 21.[1]

In view of the situation described above, in the next section we give an alternative proof of GES of (13) via the construction of a *strict Lyapunov function*, which follows closely (Malisoff & Mazenc, 2009). In spite of the simplicity of the construction, it seems that it has not been reported in the robotics literature.

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4. Global Exponential Stability Proof

To streamline the presentation of the result we find convenient to define the following positive constants

$$k_1 := \max_{t\geq 0} \|\mathcal{M}(q(t))\|,\; k_2 := \max_{t\geq 0} \|\dot{\mathcal{M}}(q(t))\|,\; k_3 := \|L\|,\; k_4 := \lambda_m\{L\},$$
$$k_5 := \|\Gamma\|,\; k_6 := \min\{1, \lambda_m\{\Gamma\}\},\; k_7 := \max\{1, \lambda_M\{\Gamma\}\}$$

[1]Theorem 2 of (Morgan & Narendra, 1977) claims GAS in the simpler case when $A(t)$ is a rank one matrix.

Notice that, under Assumption 2.2, these constants are well defined and can be *computed* from (5). We also introduce two positive constants

$$d_1 := k_1k_3 + k_2,\ d_2 := \frac{1}{2k_1^2}(k_1k_3 + k_2)^2 + k_1^2k_5 + \frac{k_4}{2}.$$

Proposition 4.1. The zero equilibrium of the error system (13) is GES with a strict Lyapunov function (8) verifying (9) and (10) where

$$P(t) := \begin{bmatrix} c_1 I_n & -\mathcal{M}(q(t)) \\ -\mathcal{M}^\top(q(t)) & c_1\Gamma^{-1} \end{bmatrix}, \tag{15}$$

and

$$c_1 := \frac{2}{k_4}d_2 + k_1k_7,\ c_2 := \frac{1}{k_7},\ c_3 := \frac{1}{k_6}\left(\frac{2}{k_4}d_2 + 2k_1k_7\right),\ c_4 := \min\left\{\frac{k_1^2}{2}, d_2 + k_1k_4k_7\right\}.$$

Proof. *The gist of the proof is to show that* (8), (15) *verifies* (9) *and* (10). *Towards this end, first, define the function* $E : \mathbb{R}^{2n} \to \mathbb{R}_+$

$$E(x) := \frac{1}{2}x^\top \begin{bmatrix} I_n & 0_{n\times n} \\ 0_{n\times n} & \Gamma^{-1} \end{bmatrix} x. \tag{16}$$

whose time derivative along (13) *verifies*

$$\dot{E} = -\tilde{q}^\top L\tilde{q} \le -k_4|\tilde{q}|^2. \tag{17}$$

Notice also that

$$k_6E(x) \le \frac{1}{2}|x|^2 \le k_7E(x), \tag{18}$$

where we have used the inequalities

$$\frac{1}{\lambda_M\{\Gamma\}} = \lambda_m\{\Gamma^{-1}\} \le \lambda_M\{\Gamma^{-1}\} = \frac{1}{\lambda_m\{\Gamma\}}.$$

Following (Malisoff & Mazenc, 2009) we propose a Lyapunov function candidate

$$W(t,x) = c_1E(x) + U(t,x)$$

where $V(x)$ *is defined in* (16) *and we defined the cross term function* $U : \mathbb{R}_+ \times \mathbb{R}^{2n} \to \mathbb{R}$

$$U(t,x) := -\tilde{q}^\top \mathcal{M}(q(t))\tilde{p}.$$

First, we compute the bounds (9). *Towards this end, we have that*

$$|U(t,x)| \le k_1|\tilde{q}||\tilde{p}| \le k_1k_7E(x), \tag{19}$$

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consequently

$$k_1 k_7 E(x) + U(t,x) \geq 0.$$

From (19) *we get*

$$W(t,x) = \left(\frac{2}{k_4} d_2 + k_1 k_7\right) E(x) + U(t,x) \geq \frac{2}{k_4} d_2 E(x) \geq E(x) \geq \frac{1}{2k_7} |x|^2,$$

where, to obtain the second inequality, we used the fact that $\frac{2}{k_4} d_2 > 1$ *and the last one follows from* (18)*—proving the lower bound of* (9)*. From* (19) *the following upper bound for* $W(t,x)$ *can be established:*

$$W(t,x) \leq \left(\frac{2}{k_4} d_2 + k_1 k_7\right) E(x) + k_1 k_7 E(x) = c_3 k_6 E(x) \leq \frac{c_3}{2} |x|^2.$$

To complete the proof we establish now the upper bound (10)*. The time derivative of* $U(t,x)$ *is given by*

$$\dot{U} = -\tilde{p}^\top \mathcal{M}^\top(q) \left[-L\tilde{q} + \mathcal{M}(q)\tilde{p}\right] - \tilde{p}^\top \dot{\mathcal{M}}^\top(q)\tilde{q} + \tilde{q}^\top \mathcal{M}(q)\Gamma\mathcal{M}^\top(q)\tilde{q},$$

whose terms can be bounded as follows:

$$\begin{aligned} \tilde{q}^\top \mathcal{M}(q)\Gamma\mathcal{M}^\top(q)\tilde{q} &\leq k_1^2 k_5 |\tilde{q}|^2, \\ \tilde{p}^\top \left(\mathcal{M}^\top(q)L - \dot{\mathcal{M}}^\top(q)\right)\tilde{q} &\leq d_1 |\tilde{q}||\tilde{p}|, \\ -\tilde{p}^\top \mathcal{M}^\top(q)\mathcal{M}(q)\tilde{p} &\leq -k_1^2 |\tilde{p}|^2. \end{aligned}$$

This leads to

$$\dot{U} \leq -k_1^2 |\tilde{p}|^2 + k_1^2 k_5 |\tilde{q}|^2 + d_1 |\tilde{q}||\tilde{p}|.$$

Using the triangle inequality

$$d_1 |\tilde{q}||\tilde{p}| \leq \frac{1}{2k_1^2} d_1^2 |\tilde{q}|^2 + \frac{k_1^2}{2} |\tilde{p}|^2$$

and the definition of d_2 *one obtains*

$$\dot{U} \leq -\frac{k_1^2}{2} |\tilde{p}|^2 + d_2 |\tilde{q}|^2.$$

Combining the latter bound with (17) *we get*

$$\begin{aligned} \dot{W} &\leq -\frac{k_1^2}{2} |\tilde{p}|^2 + d_2 |\tilde{q}|^2 - k_4 \left(\frac{2}{k_4} d_2 + k_1 k_7\right) |\tilde{q}|^2 \\ &\leq -\frac{k_1^2}{2} |\tilde{p}|^2 - (d_2 + k_1 k_4 k_7) |\tilde{q}|^2 \leq -c_4 |x|^2, \end{aligned}$$

which completes the proof. □

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Remark 5. In the proposed observer (11) it has been assumed that the matrix L is positive definite. This assumption has been made to simplify the proof, from which it is clear that any *Hurwitz* matrix L will ensure the GAS property—replacing the first term of the Lyapunov function (16) by $\frac{1}{2}\tilde{q}^\top P_0 \tilde{q}$, with $P_0 \in \mathbb{R}^{2n\times 2n}$ the positive definite solution of the Lyapunov matrix equation $P_0 L + L^\top P_0 < 0$.

Remark 6. The derivations above show that any system of the form

$$\begin{aligned} \dot{x}_1 &= B(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, u) \\ y &= x_1, \end{aligned}$$

with $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, u \in \mathbb{R}^m$ and $B : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1 \times n_2}, f_2 : \mathbb{R}^{n_1 \times m} \to \mathbb{R}^{n_2}$ smooth mappings, admit a Luenberger-like observer like the one proposed here. GAS is guaranteed for all values of $n_1, n_2 > 0$, but the proof of GES imposes the constraints $n_1 \geq n_2$ and $B(x_1)$ full rank. The latter condition ensures the existence of the positive constants k_1 and c_4. Although GAS observers for systems which are linear in the unmeasurable states, like the one above, are available in the literature (A. Astolfi et al., 2007), the proposed observer is particularly attractive due to its simplicity and the fact that it ensures GES.

5. Two Simulation Examples

In this section we present simulations of three speed observers applied to two PLvCC mechanical systems in the presence of, practically unavoidable, measurement noise. Besides the proposed Luenberger observer—called in the sequel GES observer (GESO)—we simulate a high-gain observer (HGO) and a sliding mode observer (SMO).

The first considered example is the cart-pendulum system, which is a well-known 2-dof mechanical example. For this example we simulate the HGO reported in (Lee, Mukherjee, & Khalil, 2015), which is designed exactly for the cart-pendulum system, and the SMO reported in (Davila et al., 2005), which is designed for a pendulum and is modified in a straightforward manner to fit the cart-pendulum example.

The second example is the robotic leg, which is a 3-dof mechanical system. For this example we simulate the HGO and SMO reported in (Khalil & Praly, 2014) and (Cruz-Zavala, Moreno, & Fridman, 2010), respectively. These references present rather general theory of high-gain and sliding mode observers design and can be applied for the considered mechanical system.

Our interest in this section is twofold: first, to show the excellent behavior of the proposed GESO in spite of the presence of the noise. Second, to prove that—due to their high sensitivity to noise—the performances of the HGO and the SMO are well below par.

5.1. *Cart-pendulum system*

The cart-pendulum system is a well-known example of a 2-dof, underactuation degree one, mechanical system depicted in Fig. 1. Its dynamics, in a normalized form

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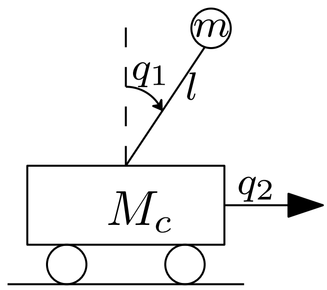

Figure 1. The inverted pendulum on a cart system

(Venkatraman et al., 2010), is described by (1) with

$$
\begin{aligned}
M(q) &= \begin{bmatrix} 1 & b\cos(q_1) \\ b\cos(q_1) & m \end{bmatrix}, \\
V(q) &= a\cos(q_1),\ G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned} \tag{20}
$$

As shown in (Venkatraman et al., 2010) this system is PLvCC with $M^{-1}(q) \equiv \Psi(q)\Psi^\top(q)$, therefore, it can be represented as (3) where

$$
\mathcal{M}(q) = \Psi(q) := \begin{bmatrix} \frac{\sqrt{m}}{\sqrt{m-b^2\cos^2(q_1)}} & 0 \\ \frac{-b\cos(q_1)}{\sqrt{m}\sqrt{m-b^2\cos^2(q_1)}} & \frac{1}{\sqrt{m}} \end{bmatrix}.
$$

The behaviour of the proposed GESO (11) is compared in simulations with the HGO and SMO proposed in (Lee et al., 2015) and (Davila et al., 2005), respectively. To design both observers we need the Coriolis matrix $C : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$, defined via the Christoffel symbols of the inertia matrix, which is given by

$$
C(q,\dot{q}) = \begin{bmatrix} 0 & 0 \\ -b\dot{q}_1\sin(q_1) & 0 \end{bmatrix}.
$$

The HGO has the form

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \left(\frac{h_1}{\varepsilon_h}\right)\tilde{x}_1 \\
\dot{\hat{x}}_2 &= M^{-1}(x_1)\left[Gu - \nabla V(x_1) - C(x_1,\hat{x}_2)\hat{x}_2\right] + \left(\frac{h_2}{\varepsilon_h^2}\right)\tilde{x}_1
\end{aligned} \tag{21}
$$

where $x_1 = q$, $x_2 = \dot{q}$, its estimates are $\hat{x}_1$ and $\hat{x}_2$, respectively, the observation errors are defined as

$$
\tilde{x}_1 = x_1 - \hat{x}_1 =: \mathrm{col}(\tilde{x}_{1_1}, \tilde{x}_{1_2})
$$

and h_1, h_2, ε_h are positive tuning parameters. The SMO is defined by the following equations

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \Phi_1(\tilde{x}_1) \\
\dot{\hat{x}}_2 &= M^{-1}(x_1)\left(Gu - \nabla V(x_1) - C(x_1, \hat{x}_2)\hat{x}_2\right) + \Phi_2(\tilde{x}_1)
\end{aligned} \tag{22}
$$

where the expressions for Φ_1 and Φ_2 are given in (Davila et al., 2005) as

$$
\Phi_1(\tilde{x}_1) = \begin{bmatrix} 1.5\sqrt{\mu_1}|\tilde{x}_{1_1}|^{\frac{1}{2}}\mathrm{sign}(\tilde{x}_{1_1}) \\ 1.5\sqrt{\mu_2}|\tilde{x}_{1_2}|^{\frac{1}{2}}\mathrm{sign}(\tilde{x}_{1_2}) \end{bmatrix},
$$
$$
\Phi_2(\tilde{x}_1) = \begin{bmatrix} 1.1\mu_1\mathrm{sign}(\tilde{x}_{1_1}) \\ 1.1\mu_2\mathrm{sign}(\tilde{x}_{1_2}) \end{bmatrix},
$$

with positive gains μ_1 and μ_2.

For the numerical simulations we chose the system parameters $a = 1$, $b = 0.1$ and $m = 1$. The simulation scenario for the three observers is $u(t) \equiv 0$, $q(0) = [\frac{\pi}{2} - 0.2,\ -0.1]^\top$, $\dot{q}(0) = [0.4,\ 0.35]^\top$ and $\hat{q}(0) = q(0)$, $\hat{\dot{q}}(0) = \hat{p}(0) = [0,\ 0]^\top$. The HGO (21) and the SMO (22) are tuned according to the recommendations given in (Lee et al., 2015) and (Davila et al., 2005), respectively, and are given as $h_1 = 3 \cdot 10^{-2}$, $h_2 = 2 \cdot 10^{-4}$, $\epsilon_h = 0.01$ for the HGO, and $\mu_1 = 2.2$, $\mu_2 = 4$ for the SMO.

The gains of the proposed Luenberger observer (11) are chosen to have approximately the same transient time as SMO and HGO, yielding, $L = 10I_2$ and $\Gamma = 70I_2$. Note that the GESO estimates the generalized momenta vector $\hat{p}$, and the SMO and HGO estimate the velocity $\hat{\dot{q}}$, thus conversion of momenta to velocity is performed for comparison.

In an ideal case when both q_1 and q_2 are measured without any distortion, the three observers perform well, and simulation results for such a case are not of interest. Next, we consider the following realistic scenario of noisy measurements.

- Both q_1 and q_2 are perturbed by an additive normally (Gaussian) distributed random measurement noise with zero mean and a small variance 10^{-4}.
- The noisy measurements are sampled with an analogue-to-digital converter with the quantization intervals $\frac{2\pi}{256}$ for q_1 and $\frac{1}{500}$ for q_2.
- The sampling frequency is fixed as 1 kHz.

In Fig. 2 the trajectories $q_1(t)$ and $q_2(t)$ are given, note the relatively small distortion of the measurements. Estimates of the velocities $\dot{q}_1$ and $\dot{q}_2$ are given in Fig. 3, which clearly shows the superior performance of GESO. To quantify this fact a numerical comparison of the observers is given in Table 1, where the metrics

$$
\begin{aligned}
ME_j &:= \frac{1}{N}\sum_{i=1}^{N} |\hat{\dot{q}}_j(t_i) - \dot{q}_j(t_i)|, \\
MSE_j &:= \frac{1}{N}\sum_{i=1}^{N} (\hat{\dot{q}}_j(t_i) - \dot{q}_j(t_i))^2,
\end{aligned} \tag{23}
$$

are computed with t_i, $i = 1, \ldots, N$, being the sampling instants, and $j = 1, 2$ for q_1 and q_2, respectively. To distinguish the effect of the noise with respect to the errors due to the mismatched initial conditions the metrics were computed omitting the transients. The experiment duration is 15 seconds, and dropping away the first 1.5 seconds of

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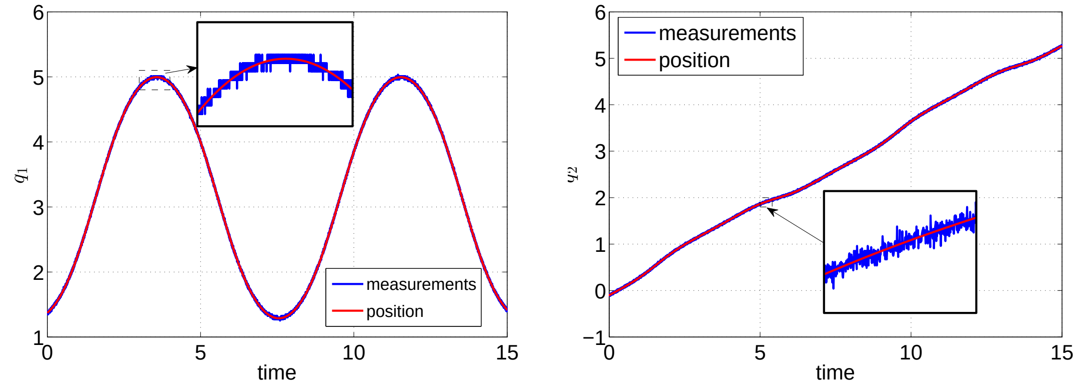
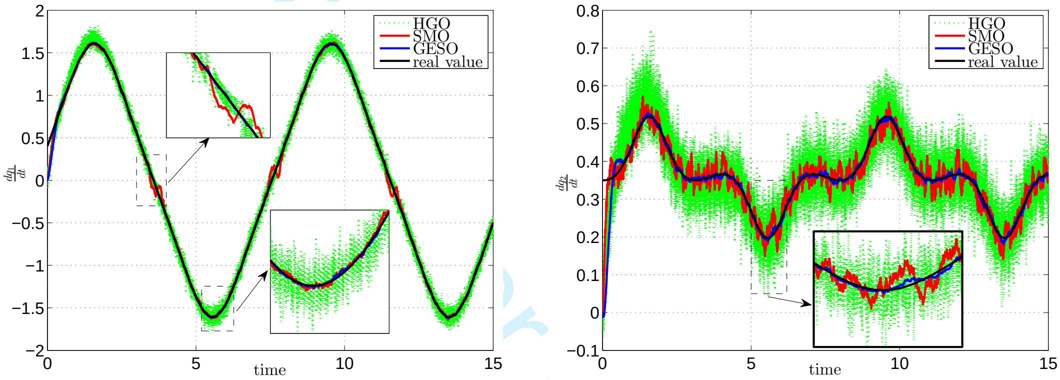


(a) Trajectory $q_1(t)$ (b) Trajectory $q_2(t)$

Figure 2. Trajectories $q_1(t)$ and $q_2(t)$ in the considered operation scenario, real position and noisy measurements



(a) Estimates of the velocity $\dot{q}_1$ (b) Estimates of the velocity $\dot{q}_2$

Figure 3. Estimates of the velocities $\dot{q}_1$ and $\dot{q}_2$ with different observers

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transients, the number of samples N equals to 13500. It can be seen that the proposed GESO significantly outperforms the HGO and the SMO.

5.2. *Robotic leg*

This is an 3-dof, underactuation degree one, mechanical system depicted in Fig. 4. Its dynamics is described by (1) with $V(q) = 0$,

$$M(q) = \mathrm{diag}\{m_1,\, m_1 q_1^2,\, m_2\},\ G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

with $m_1 > 0$, $m_2 > 0$, $q_1 \geq \epsilon > 0$. As shown in (Venkatraman et al., 2010) the system is PLvCC with the matrix $\Psi(q)$ given as

$$\Psi(q) := \begin{bmatrix} \sin(q_2) & \sin(q_2) & 0 \\ \frac{1}{q_1}\cos(q_2) + \kappa & \frac{1}{q_1}\cos(q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \kappa \neq 0,$$

Table 1. Numerical comparison of the observers.

	$\dot{q}_1$		$\dot{q}_2$	
	$ME_1 \cdot 10^2$	$MSE_1 \cdot 10^4$	$ME_2 \cdot 10^2$	$MSE_2 \cdot 10^4$
HGO	3.1	15.5	2.6	10.5
SMO	2.0	12.6	1.8	4.9
GESO	0.5	0.3	0.4	0.2

which is well-defined and full-rank for all $q_2 \neq i\pi$, $i \in \mathcal{Z}_+$. This yields

$$\mathcal{M}(q) = M(q)^{-1}\Psi^{-\top}(q) = \begin{bmatrix} -\frac{\cot(q_2)}{\kappa m_1 q_1} & \frac{\cos(q_2)+\kappa q_1}{\kappa m_1 \sin(q2) q_1} & 0 \\ \frac{1}{\kappa m_1 q_1^2} & -\frac{1}{\kappa m_1 q_1^2} & 0 \\ 0 & 0 & \frac{1}{m_2} \end{bmatrix}.$$

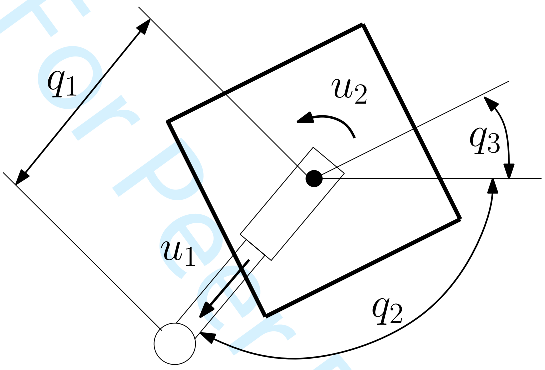

Figure 4. Robotic-leg system

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As in the previous example we compared in simulations the behaviour of the GESO (11) and the HGO and the SMO proposed in (Khalil & Praly, 2014) and (Cruz-Zavala et al., 2010), respectively, which require the Coriolis matrix $C : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ given by

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -m_1 q_1 \dot{q}_2 & 0 \\ m_1 q_1 \dot{q}_2 & m_1 q_1 \dot{q}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The HGO is similar to (21), but now there are 3-dimensional vectors

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + H_1 \tilde{x}_1, \\ \dot{\hat{x}}_2 &= M^{-1}(x_1)\left(Gu - C(x_1, \hat{x}_2)\hat{x}_2\right) + H_2 \tilde{x}_1, \end{aligned}$$

where H_1 and H_2 are the constant matrices. The SMO also has the form (22) and is given by

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1 \Phi_3(\tilde{x}_1) \\ \dot{\hat{x}}_2 &= M^{-1}(x_1)\left(Gu - C(x_1, \hat{x}_2)\hat{x}_2\right) + k_2 \Phi_4(\tilde{x}_1) \end{aligned}$$

where

$$\Phi_3(\tilde{x}_1) = \begin{bmatrix} \mu_{1_1}|\tilde{x}_{1_1}|^{\frac{1}{2}}\mathrm{sign}(\tilde{x}_{1_1}) + \mu_{2_1}|\tilde{x}_{1_1}|^{\frac{3}{2}}\mathrm{sign}(\tilde{x}_{1_1}) \\ \mu_{1_2}|\tilde{x}_{1_2}|^{\frac{1}{2}}\mathrm{sign}(\tilde{x}_{1_2}) + \mu_{2_2}|\tilde{x}_{1_2}|^{\frac{3}{2}}\mathrm{sign}(\tilde{x}_{1_2}) \\ \mu_{1_3}|\tilde{x}_{1_3}|^{\frac{1}{2}}\mathrm{sign}(\tilde{x}_{1_3}) + \mu_{2_3}|\tilde{x}_{1_3}|^{\frac{3}{2}}\mathrm{sign}(\tilde{x}_{1_3}) \end{bmatrix},$$

$$\Phi_4(\tilde{x}_1) = \begin{bmatrix} \frac{\mu_{1_1}^2}{2}\mathrm{sign}(\tilde{x}_{1_1}) + 2\mu_{2_1}\mu_{1_1}\tilde{x}_{1_1} + \frac{3}{2}\mu_{2_1}^2|\tilde{x}_{1_1}|^2\mathrm{sign}(x_{1_1}) \\ \frac{\mu_{1_2}^2}{2}\mathrm{sign}(\tilde{x}_{1_2}) + 2\mu_{2_2}\mu_{1_2}\tilde{x}_{1_2} + \frac{3}{2}\mu_{2_1}^2|\tilde{x}_{1_2}|^2\mathrm{sign}(x_{1_2}) \\ \frac{\mu_{1_3}^2}{2}\mathrm{sign}(\tilde{x}_{1_3}) + 2\mu_{2_3}\mu_{1_3}\tilde{x}_{1_3} + \frac{3}{2}\mu_{2_1}^2|\tilde{x}_{1_3}|^2\mathrm{sign}(x_{1_3}) \end{bmatrix},$$

with free positive constant parameters μ_{ij} for $i, j = \{1, 2, 3\}$.

The system parameters were taken as $m_1 = 4.5$, $m_2 = 1.7$ and the change of coordinates gain set to $\kappa = 5$. The simulation scenario is $u_1(t) = 0.0535\cos(10t)$ and $u_2(t) = 0.067\sin(10t)$, $q(0) = [2.2,\ 1.8,\ 0.4]^\top$ and $\dot{q}(0) = p(0) = [0,\ 0,\ 0]^\top$ and initial conditions for the three observers $\hat{q}(0) = [0.1,\ 0.2,\ 0.4]$ and $\hat{\dot{q}}(0) = \hat{p}(0) = [0,\ 0,\ 0]^\top$. The SMO and HGO have been designed and tuned following the procedure suggested in (Cruz-Zavala et al., 2010) and (Khalil & Praly, 2014), respectively, and are given as $H_1 = \mathrm{diag}\{1.2,\ 2,\ 2\}$, $H_2 = \mathrm{diag}\{3.9,\ 2.5,\ 1.3\}$ and $k_1 = \mathrm{diag}\{1,\ 0.52,\ 0.71\}$, $k_2 = \mathrm{diag}\{1,\ 1.2,\ 1.17\}$; and $\mu_{1j} := 1.86$ and $\mu_{2j} := 0.303$.

We have chosen the gains of the GESO as $L = \mathrm{diag}\{0.61,\ 0.9,\ 1.9\}$ and $\Gamma = \mathrm{diag}\{1.74,\ 4.9,\ 0.71\}$ to approximately match the convergence rate of the other observers.

If the position signals q are measured without distortion, then all the observers have a good performance. Thus we adopted the realistic simulation scenario, which is similar to the scenario used in Subsection 5.1:

- The signals are affected by an additive normally (Gaussian) distributed random measurement noise with zero mean and variance of 0.1.
- The noisy measurements are sampled with an analogue-to-digital converter with the quantization intervals $\frac{1}{500}$ for q_1 and $\frac{2\pi}{256}$ for q_2 and q_3.
- The sampling frequency is fixed as 1 kHz.

The transient behaviors of the error signals $\dot{\tilde{q}}$ for the three observers are shown in Fig.5–7. As seen from the figures, the GESO has a somehow slower convergence rate for the first coordinate, but with a much smaller overshoot. On the other hand, it significantly outperforms the SMO and HGO for the other two coordinates. The excellent behaviour of the third speed observation error of the GESO stems from the fact that its dynamics is described by the linear homogeneous equation

$$\ddot{\tilde{q}}_3 + L_3\dot{\tilde{q}}_3 + \frac{\Gamma_3}{m_2^2}\tilde{q}_3 = 0.$$

Making use of the metrics (23) a numerical comparison of the observers is presented in Table 2 where—as in the previous example—the transients have been omitted. It can be seen that in all coordinates and metrics the proposed GESO largely outperforms the HGO and the SMO.

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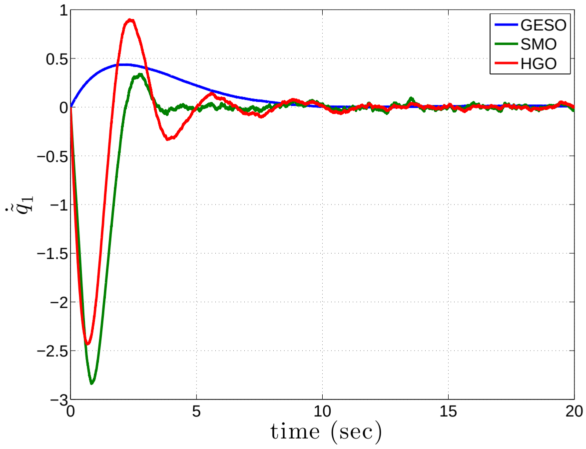

Figure 5. Transient behavior of $\dot{\tilde{q}}_1(t)$

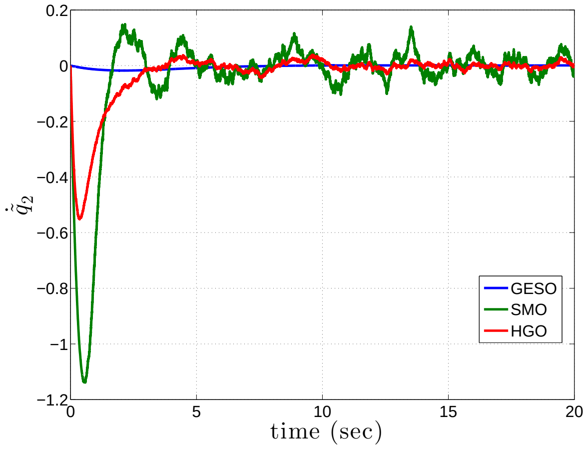

Figure 6. Transient behavior of $\dot{\tilde{q}}_2(t)$

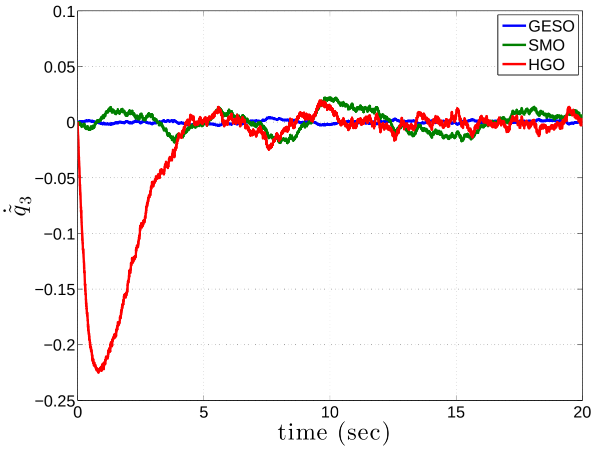

Figure 7. Transient behavior of $\dot{\tilde{q}}_3(t)$

Table 2. Numerical comparison of the observers.

	$\dot{q}_1$		$\dot{q}_2$		$\dot{q}_3$	
	$ME_1 \cdot 10^2$	$MSE_1 \cdot 10^4$	$ME_2 \cdot 10^2$	$MSE_2 \cdot 10^4$	$ME_3 \cdot 10^2$	$MSE_3 \cdot 10^4$
HGO	2.04	7.56	1.04	1.79	0.52	0.47
SMO	2.14	7.25	2.20	7.76	0.86	0.98
GESO	1.00	1.80	0.14	0.06	0.10	0.02

6. Concluding Remarks

It has been shown in the paper that the problem of speed (or momenta) observation of PLvCC mechanical systems admits a very simple—even trivial—Luenberger-like solution. In spite of its remarkable simplicity this fact does not seem to have been reported in the robotics literature, where only far more complicated observers are available. The observer ensures the very strong property of GES of the error system without any excitation requirement. The only assumption on the input is the usual boundedness of trajectories condition. The GES nature of the proposed observer makes it a suitable candidate to—combined with a full-state GAS controller *à la* (Venkatraman et al., 2010)—yield a GAS solution to the position feedback stabilization problem, but the details are omitted for brevity.

It should be noted that, although the assumption of PLvCC may seem restrictive, the class contains a very long list of benchmark examples, including the pendulum on a cart, the mass and beam system, the spherical pendulum on a puck, the 3-link underactuated planar manipulator and the planar redundant manipulator with one elastic degree of freedom. In a recent paper (Chang, Song, & Kim, 2016) this class has been enlarged adding to the change of coordinates a position feedback term. New sufficient conditions for quasilinearizability are given and are shown to be satisfied by the Acrobot example. It is interesting to see how the presence of this new term affects the observation problem.

Another contribution of the paper is to exhibit, via realistic simulations, the high sensitivity to noise of HGO and SMO in mechanical systems. It should be underscored that a lot of effort is under way to palliate this problem for high gain observers, in particular, to reduce the peaking phenomenon when they are applied to *high order* systems, *e.g.*, (D. Astolfi & Marconi, 2015; Teel, 2016). However, these modifications are not applicable to *second* order mechanical equations.

At a more philosophical level, we quote below Slavoj Zizek (Žižek, 1989) and ask ourselves if all these fixes are merely a Ptolemization of an intrinsically fragile design—that contradicts a basic premise of control theory: "high-gain is bad for feedback systems!"

> "When a discipline is in crisis, attempts are made to change or supplement its theses within the terms of its basic framework - a procedure one might call 'Ptolemization' (since when data poured in which clashed with Ptolemy's earth-centred astronomy, his partisans introduced additional complications to account for the anomalies). But the true 'Copernican' revolution takes place when, instead of just adding complications and changing minor premises, the basic framework itself undergoes a transformation. So, when we are dealing with a self-professed 'scientific revolution', the question to ask is always: is this truly a Copernican revolution, of the old paradigm?"
>
> S. Zizek

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