Validation of the FV scheme in 2D

The computer code implementing the FV scheme is tested by performing a number of experiments. The test computations are performed for the equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f^{(x)}}{\partial x} + \frac{\partial f^{(y)}}{\partial y} = q, \quad (x, y) \in [x_1, x_2] \times [y_1, y_2], \tag{1}$$

where

$$f^{(x)} = a^{(x)} u - b^{(x)} \frac{\partial u}{\partial x}, \quad f^{(y)} = a^{(y)} u - b^{(y)} \frac{\partial u}{\partial y}. \tag{2}$$

In all cases, the periodic boundary conditions are used. The computational mesh consists of a few block elements. Each block element includes $m \times m$ finite elements (m elements per direction). Each finite element is discretized with a $n \times n$ tensor product grid of FV equidistant nodes.

The goal of these experiments is to test the key components of the FV scheme for different types of interelement contacts. Equation (1) is discretized using the FV scheme combined with the second-order total-variation-diminishing Runge-Kutta scheme [S. Gottlieb and C.-W. Shu, Mathematics of computation 67 (1998), 73-85].

Two problems are chosen for the experiments:

1 Problem 1

This formulation is based on the assumption that $a^{(x)} = a_x$, $a^{(y)} = a_y$, $b^{(x)} = b^{(y)} = b$, where $a_x = const$, $a_y = const$, $b = const$. The initial condition is given by $u(x, y) = \exp(-x^2 - y^2)$ at $t = 0$. The exact solution of this problem is given by

$$u(x, y) = \alpha \exp\left[-\alpha \left(x - a_x t\right)^2 - \alpha \left(y - a_y t\right)^2\right], \quad t > 0,$$

where $\alpha = (4b\tau)^{-1}$, with $\tau = (4b)^{-1} + t$. The following mesh configurations are considered:

Test 1.1.
The mesh includes a single block element $\Omega = [-4, 4] \times [-4, 4]$.
Test 1.2.
The mesh includes the block elements $\Omega_1 = [-4, 0] \times [-4, 0]$, $\Omega_2 = [0, 4] \times [-4, 0]$, $\Omega_3 = [-4, 4] \times [0, 8]$.
Test 1.3.
The mesh includes the block elements $\Omega_1 = [-4, 0] \times [-4, 0]$, $\Omega_2 = [-4, 0] \times [0, 4]$, $\Omega_3 = [0, 8] \times [-4, 4]$.

2 Problem 2

This formulation is designed using the method of manufactured solutions. The transport coefficients are defined as

$$a^{(x)} = 1 + \sin(x + y), \quad a^{(y)} = 1 + \cos(x + y), \quad b^{(x)} = 0.1 + 0.1 \cos(x + y), \quad b^{(y)} = 0.1 + 0.1 \sin(x + y).$$

The exact solution is assumed to be $u(x, y) = \sin(x + y - t)$. The source term q is derived analytically according to this assumption. The following mesh configurations are considered:

Test 2.1.
The mesh includes a single block element $\Omega = [-\pi, \pi] \times [-\pi, \pi]$.
Test 2.2.
The mesh includes the block elements $\Omega_1 = [-2\pi, 0] \times [-2\pi, 0]$, $\Omega_2 = [0, 2\pi] \times [-2\pi, 0]$, $\Omega_3 = [-2\pi, 0] \times [0, 2\pi]$, $\Omega_4 = [0, 2\pi] \times [0, 2\pi]$ and $\Omega_5 = [-2\pi, 2\pi] \times [-6\pi, -2\pi]$.
Test 2.3.
The mesh includes the block elements $\Omega_1 = [-2\pi, 0] \times [-2\pi, 0]$, $\Omega_2 = [0, 2\pi] \times [-2\pi, 0]$, $\Omega_3 = [-2\pi, 0] \times [0, 2\pi]$, $\Omega_4 = [0, 2\pi] \times [0, 2\pi]$ and $\Omega_5 = [2\pi, 6\pi] \times [-2\pi, 2\pi]$.