# min_L2

December 19, 2019

[108]:
```python
from sympy import *
from sympy.plotting import plot_parametric
```

[4]:
```python
A = Matrix([[5, -3],
            [6, -4]])
A
```

[4]:
$$\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$$

[5]:
```python
B = Matrix([[0],
            [1]])
B
```

[5]:
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

[6]:
```python
x0 = Matrix([[0],
             [2]])
x0
```

[6]:
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

[7]:
```python
p = Matrix([[-3],
            [-1]])
p
```

[7]:
$$\begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

[8]:
```python
P = Matrix([[17, 12],
            [12, 10]])
P
```

[8]:
$$\begin{bmatrix} 17 & 12 \\ 12 & 10 \end{bmatrix}$$

[11]:
```
eigs = A.eigenvects()
pprint(eigs)
```

```
         1/2            1
 -1, 1,        ,  2, 1,
          1             1
```

[25]:
```
t, tau = symbols('t tau', real=True)
t0 = 0
t1 = log(2)
```

[26]:
```
Phi = (eigs[0][2][0] * exp(eigs[0][0] * t)).col_insert(1, eigs[1][2][0] *␣
 ↪exp(eigs[1][0] * t))
Phi
```

[26]:

$$\begin{bmatrix} \frac{e^{-t}}{2} & e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix}$$

[27]:
```
expA = Phi @ Phi.subs({t: 0}).inv()
expA
```

[27]:

$$\begin{bmatrix} 2e^{2t} - e^{-t} & -e^{2t} + e^{-t} \\ 2e^{2t} - 2e^{-t} & -e^{2t} + 2e^{-t} \end{bmatrix}$$

[24]:
```
l1, l2 = symbols('l1 l2', real=True)
l = Matrix([[l1],
            [l2]])
l
```

[24]:

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

[28]:
```
H = expA.subs({t: t1 - tau}) @ B
H
```

[28]:

$$\begin{bmatrix} \frac{e^{\tau}}{2} - 4e^{-2\tau} \\ e^{\tau} - 4e^{-2\tau} \end{bmatrix}$$

[73]:
```
norm_Hl = sqrt(integrate((H.T @ l) ** 2, (tau, t0, t1))[0])
norm_Hl
```

[73]:

$$\sqrt{\frac{17l_1^2}{8} + 3l_1l_2 + \frac{5l_2^2}{4}}$$

[75]:
```
simplify(norm_Hl * sqrt(8))
```

[75]:

$$\sqrt{17l_1^2 + 24l_1l_2 + 10l_2^2}$$

[53]:
```
simplify(l.dot(P @ l))
```

[53]:

$$17l_1^2 + 24l_1l_2 + 10l_2^2$$

[76]:
```
bnd = simplify(l.dot(p - expA.subs({t: t1 - t0}) @ x0) - sqrt(8))
bnd
```

[76]:

$$4l_1 + 5l_2 - 2\sqrt{2}$$

[77]:
```
constr = norm_Hl ** 2 - 1
constr
```

[77]:

$$\frac{17l_1^2}{8} + 3l_1l_2 + \frac{5l_2^2}{4} - 1$$

[78]:
```
lam = symbols('lambda', real=True)
lam
```

[78]:

$$\lambda$$

[79]:
```
L = bnd + lam * constr
L
```

[79]:

$$4l_1 + 5l_2 + \lambda\left(\frac{17l_1^2}{8} + 3l_1l_2 + \frac{5l_2^2}{4} - 1\right) - 2\sqrt{2}$$

[82]:
```
sol = solve([L.diff(l1), L.diff(l2), L.diff(lam)], (l1, l2, lam))
sol
```

[82]:
```
[(-8*sqrt(1365)/273, 74*sqrt(1365)/1365, -sqrt(1365)/13),
 (8*sqrt(1365)/273, -74*sqrt(1365)/1365, sqrt(1365)/13)]
```

[85]:
```
l1_opt0, l2_opt0, lam_opt0 = sol[0]
l1_opt1, l2_opt1, lam_opt1 = sol[1]
```

[87]:
```
bnd.subs({l1: l1_opt0, l2: l2_opt0})
```

[87]:

$$-2\sqrt{2} + \frac{2\sqrt{1365}}{13}$$

[88]:
```
bnd.subs({l1: l1_opt1, l2: l2_opt1})
```

[88]:

$$-\frac{2\sqrt{1365}}{13} - 2\sqrt{2}$$

[89]:
```
l1_max, l2_max = l1_opt0, l2_opt0
```

[90]:
```
mu = bnd.subs({l1: l1_opt0, l2: l2_opt0})
mu
```

[90]:

$$-2\sqrt{2} + \frac{2\sqrt{1365}}{13}$$

[93]:
```
mu.evalf()
```

[93]:

2.85555847584186

[99]:
```
u_opt = simplify(mu * H.T @ l.subs({l1: l1_max, l2: l2_max}) / norm_Hl.subs({l1:
 ↪ l1_max, l2: l2_max}))
u_opt
```

[99]:

$$\left[\frac{4\sqrt{1365}\left(-13\sqrt{2}+\sqrt{1365}\right)\left(27e^{3\tau}-68\right)e^{-2\tau}}{17745}\right]$$

[104]:
```
sqrt(integrate(u_opt ** 2, (tau, t0, t1))[0]).evalf()
```

[104]:

2.85555847584186

[106]:
```
x_opt = simplify(expA.subs({t: t - t0}) @ x0 + integrate(expA.subs({t: t -␣
 ↪tau}) @ B @ u_opt, (tau, t0, t)))
x_opt
```

[106]:

$$\left[\begin{matrix}\frac{2\left(-3465e^{4t}+20\sqrt{2730}e^{4t}-81\sqrt{2730}e^{3t}+8505e^{3t}-15750e^{t}+163\sqrt{2730}e^{t}-102\sqrt{2730}+10710\right)e^{-2t}}{1365}\\ \frac{2\left(-3465e^{4t}+20\sqrt{2730}e^{4t}-108\sqrt{2730}e^{3t}+11340e^{3t}-31500e^{t}+326\sqrt{2730}e^{t}-238\sqrt{2730}+24990\right)e^{-2t}}{1365}\end{matrix}\right]$$

[115]:
```
z1, z2 = symbols('z1 z2', real=True)
z = Matrix([[z1],
            [z2]])
```

[119]:
```
X1 = (z - p).dot(P.inv() @ (z - p)) - 1
simplify(X1)
```

[119]:

$$\frac{5z_1^2}{13}-\frac{12z_1z_2}{13}+\frac{18z_1}{13}+\frac{17z_2^2}{26}-\frac{19z_2}{13}+\frac{9}{26}$$

[135]:
```
pprint(P.inv().eigenvects())
```

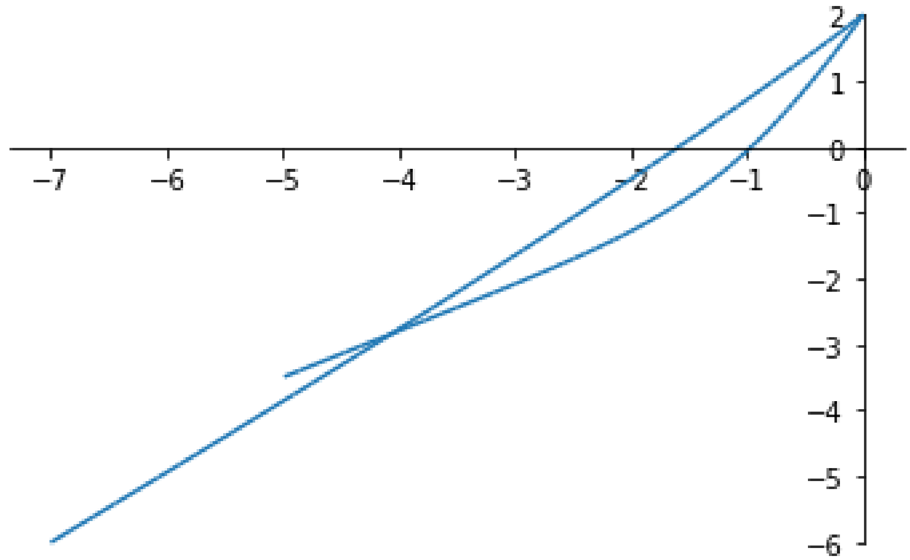
```
           4/3            -3/4
 1/26, 1,       ,  1, 1,
            1               1
```

[141]:
```
no_control = expA.subs({t: t - t0}) @ x0
no_control
```

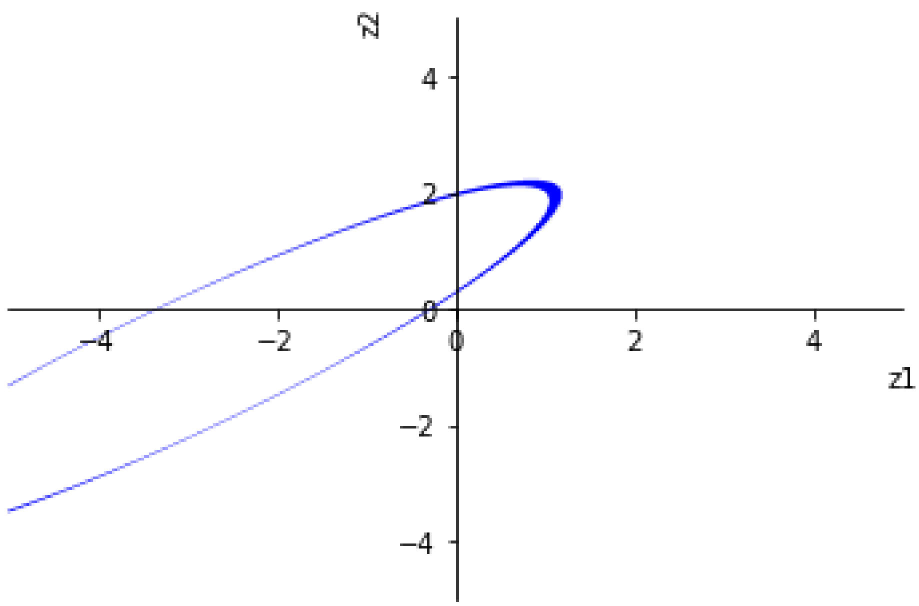
[141]:

$$\left[\begin{matrix}-2e^{2t}+2e^{-t}\\ -2e^{2t}+4e^{-t}\end{matrix}\right]$$

[145]:
```
p1 = plot_parametric((x_opt[0], x_opt[1], (t, t0, t1)), (no_control[0],␣
 ↪no_control[1], (t, t0, t1)))
```




[120]:
```
plot_implicit(X1)
```




[120]: <sympy.plotting.plot.Plot at 0x7fe5b2cafa10>

[122]:
```
x1 = x_opt.subs(t, t1).evalf()
X1.subs({z1: x1[0], z2: x1[1]})
```

[122]: $2.22044604925031 \cdot 10^{-16}$