Saarland University, Department of Computer Science
Neural Network Assignment 2

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Exercise 2.1

2.1.a

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$
$$|A - \lambda I| = 0$$
$$(4-\lambda)(3-\lambda) - 2 = 0$$
$$\lambda^2 - 7\lambda + 10 = 0$$
$$\lambda(\lambda-2) - 5(\lambda-2) = 0$$
$$\lambda_1 = 5; \lambda_2 = 2$$
Case for $\lambda_1 = 5$;
$$4x_1 + 2x_2 = \lambda x_1 \Rightarrow 4x_1 + 2x_2 = 5x_1$$
$$x_1 + 3x_2 = \lambda x_2 \Rightarrow x_1 + 3x_2 = 5x_2$$
$$\Rightarrow -x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_1$$
$$x_1 - 2x_2 = 0$$
$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
Case for $\lambda_2 = 2'$
$$4x_1 + 2x_2 = 2x_1$$
$$x_1 + 3x_2 = 2x_2$$
$$2x_1 + 2x_2 = 0$$
$$x_1 + x_2 = 0$$
$$x_1 = -x_2$$
$$V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2.1.b

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$$

$$|A^{-1} - \lambda I| = 0$$

$$|\begin{bmatrix} \frac{3}{10} & -\frac{1}{5} \\ \frac{-1}{10} & -\frac{2}{5} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}| = 0$$

$$\left(\frac{3}{10} - \lambda\right)\left(\frac{2}{5} - \lambda\right) - \left(\frac{-1}{5}x - \frac{1}{10}\right) = 0$$

$$\frac{3}{25} - \frac{3}{10}\lambda - \frac{2}{5}\lambda + \lambda^2 - \frac{1}{50} = 0$$

$$\frac{1}{10} - \frac{7}{10}\lambda + \lambda^2 = 0 \tag{1}$$

$$\lambda^2 - \frac{7}{10}\lambda + \frac{1}{10} = 0$$

$$\lambda^2 - \frac{1}{5}\lambda - \frac{1}{2}\left(\lambda - \frac{1}{5}\right) = 0$$

$$\lambda\left(\lambda - \frac{1}{5}\right) - \frac{1}{2}\left(\lambda - \frac{1}{5}\right) = 0$$

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{5}$$

The relationship between eigenvalues of the inverse matrix; A^{-1} are equal to the inverse of the eigenvalues of the original matrix A: Eigenvalues of A; 5 and 2 and A^{-1}; $\frac{1}{2}$ and $\frac{1}{5}$

2.1.c

$$A \in R^{n \times n}, B \in R^{n \times n}, \text{ and we know } \quad A^{-1} \cdot A = I, \quad B^{-1} \cdot B = I$$

If λ is an eigen value of AB;
for AB; $\det(AB - \lambda I) = 0$
$\det(A^{-1}) \det(AB - \lambda I) \det(B^{-1}) = 0$
Rearranging we have;

$$\det\left(A^{-1}(AB - \lambda I)B^{-1}\right) = 0$$

But we know $A \in R^{n \times n}$ and $A^{-1}A = I$

$$\det\left(\left(B - \lambda A^{-1} I\right) B^{-1}\right) = 0$$

$$= \det\left(I - \lambda A^{-1} B^{-1}\right) = 0$$

for BA; $\det(BA - \lambda I) = 0$

$$\det\left(B^{-1}\right) \det(BA - \lambda I) \det\left(A^{-1}\right) = 0$$

Rearranging we have; $\det(B^{-1}(BA - \lambda I)A^{-1}) = 0$
$\det((A - \lambda B^{-1} I) A^{-1}) = 0$

$$\det\left(I - \lambda A^{-1} B^{-1}\right) = 0$$

And we see this is the same as the eigenvalue of $AB = BA$

We can also proof this analytically and using these examples $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}, \quad BA = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$$

Finding eigen values of AB and BA will both yield the following characteristics equation

$$\begin{aligned}
&\lambda^2 - 15 + 25 = 0 \\
&\lambda_1 = 1.91 \text{ and } \lambda_2 = 13.09
\end{aligned}$$

Exercise 2.2

2.2 a

(a) Using first-derivative formulae

$$\begin{aligned}
&f'(x) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&f(x) = \omega^\top x; f(x+h) = \omega^\top (x+h) \\
&f'(x) = \lim_{h\to 0} \frac{\omega^\top (x+h) - \omega^\top x}{h} \\
&F'(x) = \lim_{h\to 0} \frac{w^\top x + w \cdot h - w^\top x}{h} \\
&= \lim_{h\to 0} \frac{w \cdot h}{h} \\
&\text{As } h \to 0; \quad f'(x) = w \text{ as } \lim_{h\to 0}; h = 0
\end{aligned}$$

2.2 b

$$\begin{aligned}
&f(x) = x^\top Ax; f(x+h) = (x+h)^\top A(x+h) \\
&f'(x) = \lim_{h\to 0} \frac{\left(x^\top + h\right)^\top A(x+h) - x^\top Ax}{h} \\
&f'(x) = \lim_{h\to 0} \frac{x^\top Ax + x^\top Ah + h^\top Ax + h^\top Ah - x^\top Ax}{h} \\
&f'(x) = \lim_{h\to 0} \frac{x^\top Ah + h^\top Ax + h^\top Ah}{h} \\
&f'(x) = \lim_{h\to 0} x^\top A + Ax + h^\top A \\
&F'(x) = Ax + A^\top x, \text{ as } \lim_{h\to 0}; h = 0
\end{aligned} \tag{2}$$

2.2 c

$$\begin{aligned}
&f(x) = (Bx)^2, f(x+h) = (B(x+h))^2 \\
&f'(x) = \lim_{h\to 0} \frac{B(x+h) \cdot B(x+h) - (Bx)^2}{h} \\
&f'(x) = \lim_{h\to 0} \frac{(Bx)^2 + 2B^\top Bxh + (Bh)^2 - (Bx)^2}{h} \\
&f'(x) = \lim_{h\to 0} \frac{2B^\top Bxh + (Bh)^2}{h} \\
&f'(x) = \lim_{h\to 0} 2B^\top Bx + B^2 \cdot h \\
&f'(x) = 2B^\top Bx, \text{ as } \lim_{h\to 0}; h = 0
\end{aligned}$$

2.2 d

$f(x) = \|Bx - c\|_2^2$, then $\nabla_x f(x) = 2B^\top(Bx - c)$

$f(x) = \|Bx - c\|_2^2$,

$f(x+h) = \|B(x+h) - c\|_2^2$

$= (Bx + Bh - c)(Bx + Bh - c)$

$= Bx(Bx + Bh - c) + Bh(Bx + Bh - c) - c(Bx + Bh - c)$

$= B^\top Bx^2 + B^\top Bhx - cBx + B^\top Bhx - cBh - cBx - cBh + c^2$

Also $f(x) = \|Bx - c\|_2^2$

$= B^\top Bx^2 - 2Bcx - Bcx + c^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{B^\top Bx^2 + B^\top Bhx - cBx + B^\top Bhx - cBh - cBx - cBh + c^2 - (\|Bx - c\|_2^2)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2B^\top Bhx - 2B^\top ch}{h}$$

$f'(x) = 2B^\top Bx - 2B^\top c$

$\Rightarrow 2B^\top(Bx - c)$, as $\lim_{h \to 0}; h = 0$