Demostración de la identidad de Jacobi para los corchetes de Poisson

Ignacio Vallés

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$$\{A,B\} \;=\; \tfrac{\partial A}{\partial q}\tfrac{\partial B}{\partial p} \;-\; \tfrac{\partial A}{\partial p}\tfrac{\partial B}{\partial q} \quad\rightarrow\quad \{A,\{B.C\}\} \;+\; \{C,\{A,B\}\} \;+\; \{B,\{C,A\}\} \;=\; 0$$

$$\{A,\{B.C\}\} = \frac{\partial A}{\partial q}\frac{\partial}{\partial p}\{B,C\} - \frac{\partial A}{\partial p}\frac{\partial}{\partial q}\{B,C\} =$$

$$= \frac{\partial A}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial B}{\partial q}\frac{\partial C}{\partial p} \;-\; \frac{\partial B}{\partial p}\frac{\partial C}{\partial q}\right) - \frac{\partial A}{\partial p}\frac{\partial}{\partial q}\left(\frac{\partial B}{\partial q}\frac{\partial C}{\partial p} \;-\; \frac{\partial B}{\partial p}\frac{\partial C}{\partial q}\right) =$$

$$= \frac{\partial A}{\partial q}\left(\frac{\partial^2 B}{\partial q\partial p}\frac{\partial C}{\partial p} + \frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial p^2} - \frac{\partial^2 B}{\partial p^2}\frac{\partial C}{\partial q} - \frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q\partial p}\right) - \frac{\partial A}{\partial p}\left(\frac{\partial^2 B}{\partial q^2}\frac{\partial C}{\partial q} + \frac{\partial B}{\partial q}\frac{\partial^2 B}{\partial q^2} - \frac{\partial^2 B}{\partial q\partial p}\frac{\partial C}{\partial q} - \frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q^2}\right) =$$

$$= \quad +\frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial q\partial p}\frac{\partial C}{\partial p} + \frac{\partial A}{\partial q}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial p^2} - \frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial p^2}\frac{\partial C}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q\partial p}\;(\rightarrow)$$

$$(\rightarrow) - \frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q^2}\frac{\partial C}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial q\partial p} + \frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q\partial p}\frac{\partial C}{\partial q} + \frac{\partial A}{\partial p}\frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q^2} =$$

$$= \quad \cancel{+\frac{\partial A}{\partial q}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial p^2}} + \frac{\partial A}{\partial p}\frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q^2} \quad - \frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial p^2}\frac{\partial C}{\partial q} \quad \cancel{-\frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q^2}\frac{\partial C}{\partial p}}\;(\rightarrow)$$

$$(\rightarrow)\; + \frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial q\partial p}\frac{\partial C}{\partial p} + \frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q\partial p}\frac{\partial C}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q\partial p}\cancel{-\frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial q\partial p}} =$$

Analicemos el resultado obtenido y comparemos con los que se obtendrán al permutar cíclicamente los operadores $A,\ B$ y C

$$\{C,\{A,B\}\} \;=\; \qquad [A\rightarrow C;\;\; B\rightarrow A;\;\; C\rightarrow B]$$

$$= \quad +\frac{\partial C}{\partial q}\frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial p^2}\; \cancel{+\frac{\partial C}{\partial p}\frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q^2}} \quad - \frac{\partial C}{\partial q}\frac{\partial^2 A}{\partial p^2}\frac{\partial B}{\partial q} \quad - \frac{\partial C}{\partial p}\frac{\partial^2 A}{\partial q^2}\frac{\partial B}{\partial p}\;(\rightarrow)$$

$$(\rightarrow)\; + \frac{\partial C}{\partial q}\frac{\partial^2 A}{\partial q\partial p}\frac{\partial B}{\partial p} + \frac{\partial C}{\partial p}\frac{\partial^2 A}{\partial q\partial p}\frac{\partial B}{\partial q} - \frac{\partial C}{\partial q}\frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q\partial p} - \frac{\partial C}{\partial p}\frac{\partial A}{\partial q}\frac{\partial^2 B}{\partial q\partial p} =$$

$$\{B,\{C,A\}\} \; = \qquad [A\to B;\; B\to C;\; C\to A]$$

$$= \quad +\frac{\partial B}{\partial q}\frac{\partial C}{\partial q}\frac{\partial^2 A}{\partial p^2} + \frac{\partial B}{\partial p}\frac{\partial C}{\partial p}\frac{\partial^2 A}{\partial q^2} \; \cancel{-\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial p^2}\frac{\partial A}{\partial q}} \; - \frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q^2}\frac{\partial A}{\partial p} \quad (\rightarrow)$$

$$(\rightarrow) \; \cancel{+\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial q \partial p}\frac{\partial A}{\partial p}} + \frac{\partial B}{\partial p}\frac{\partial^2 C}{\partial q \partial p}\frac{\partial A}{\partial q} - \frac{\partial B}{\partial q}\frac{\partial C}{\partial p}\frac{\partial^2 A}{\partial q \partial p} - \frac{\partial B}{\partial p}\frac{\partial C}{\partial q}\frac{\partial^2 A}{\partial q \partial p} =$$

Sumando las tres expresiones, por cada término que aparece en las dos primeras, aparece su opuesto en las 4 segundas y todos se cancelan, como puede comprobar el lector.

A modo de ejemplo buscamos tres de ellos, $+\frac{\partial A}{\partial q}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial p^2}$, $-\frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\frac{\partial^2 C}{\partial q \partial p}$ y $-\frac{\partial A}{\partial p}\frac{\partial^2 B}{\partial q^2}\frac{\partial C}{\partial p}$

Finalmente, $\{A,\{B.C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} \; = \; 0$ $\square$

Notación SIMPLÉCTICA :

En 1-dim: $z = \begin{pmatrix} q \\ p \end{pmatrix}$, en n-dim: $z = \begin{pmatrix} q_1 & q_2 & \cdots & q_n & p_1 & p_2 & \cdots & p_n \end{pmatrix}^T$, con una transformación canónica, $\mathbb{Z} = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_n & P_1 & P_2 & \cdots & P_n \end{pmatrix}^T$.

En 1-dim: $\dot{z} = \begin{pmatrix} \dot{q} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} = J \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = J \frac{\partial H}{\partial z}$ con $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

En n-dim: $J = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}$ Para 2-dim, $J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

Corchete de Poisson en notación simpléctica, para 1-dim

$$\{A,B\} = \begin{pmatrix} \frac{\partial A}{\partial q} & \frac{\partial A}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial B}{\partial p} \\ -\frac{\partial B}{\partial q} \end{pmatrix} = \begin{pmatrix} \frac{\partial A}{\partial q} & \frac{\partial A}{\partial p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial B}{\partial q} \\ \frac{\partial B}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial A}{\partial q} & \frac{\partial A}{\partial p} \end{pmatrix} J \begin{pmatrix} \frac{\partial B}{\partial q} \\ \frac{\partial B}{\partial p} \end{pmatrix}$$

$$\boldsymbol{\{A,B\} \; = \; \left(\frac{\partial A}{\partial z}\right)^T J \left(\frac{\partial B}{\partial z}\right)}$$

Identidad de Jacobi $\{A,\{B.C\}\} \;+\; \{C,\{A,B\}\} \;+\; \{B,\{C,A\}\} \;=\; 0$

Corchete de Poisson $\{A,B\} \;=\; \partial A^T \; J \; \partial B$

Para mayor comodidad en la demostración, escribiremos el corchete de Poisson en la forma:

$$(\partial A)^T J(\partial B) \quad = \quad \left(\frac{\partial A}{\partial z}\right)^T J \left(\frac{\partial B}{\partial z}\right)$$

$\{A,\{B,C\}\} \; = \{A, \partial B^T J \partial C\} = \partial A^T J \partial(\partial B^T J \partial C) = \; (\partial J = 0) \; = \partial A^T J[\partial\partial B^T J\partial C + \partial B^T J \partial\partial C] =$
$\partial A^T J \partial\partial B^T J \partial C + \partial A^T J \partial B^T J \partial\partial C = \{A, \partial B^T\} J \partial C + \partial A^T J \{B, \partial C\}$

Análogamente, alternando cíclicamente los valores de A, B y C, tenemos:

$\{C,\{A,B\}\} = \{C, \partial A^T\} J \partial B + \partial C^T J \{A, \partial B\}$ y $\{B\{C,A\}\} \; = \{B, \partial C^T\} J \partial A + \partial B^T J \{C, \partial A\} =$

Sumando estas tres expresiones,

$= \{A,\{B.C\}\} \;+\; \{C,\{A,B\}\} \;+\; \{B,\{C,A\}\} \; = \{A, \partial B^T\} J \partial C + \partial A^T J \{B, \partial C\} \;+\; \{C, \partial A^T\} J \partial B +$
$\partial C^T J \{A, \partial B\} \; + \{B, \partial C^T\} J \partial A + \partial B^T J \{C, \partial A\} =$

Reorganizando,

$$= \{A, \partial B^T\} J \partial C + \partial C^T J \{A, \partial B\} + \{B, \partial C^T\} J \partial A + \partial A^T J \{B, \partial C\} + \{C, \partial A^T\} J \partial B + \partial B^T J \{C, \partial A\} \overset{\divideontimes}{=}$$

Como $\partial C^T J \{A, \partial C\}$ es un número real, coincide con su traspuesto:

$\partial C^T J \{A, \partial C\} = [\partial C^T J \{A, \partial C\}]^T = \{A, \partial B\}^T J^T (\partial C^T)^T = (\rightarrow)$

Como $\{A, \partial B\}^T = \{A, \partial B\}$ por ser un número; $(\partial C^T)^T = \partial C$; $J^T = -J$, tendremos

$(\rightarrow) = \{A, \partial B\}(-J)\partial C = -\{A, \partial B\} J \partial C$ y análogamente para los términos pares de la expresión $\divideontimes$ con lo que podremos escribir,

$$\overset{\divideontimes}{=} \{A, \partial B^T\} J \partial C - \; \{A, \partial B\} J \partial C \;+\; \{B, \partial C^T\} J \partial A \;-\; \{B, \partial C\} J \partial A \;+\; \{C, \partial A^T\} J \partial B \;+\; \{C, \partial A\} J \partial B =$$

$$[\{A, \partial B^T\} - \{A, \partial B\}] J \partial C + [\{B, \partial C^T\} - \{B, \partial C\}] J \partial A + [\{C, \partial A^T\} - \{C, \partial A\}] J \partial B \underset{\divideontimes}{\overset{\divideontimes}{=}}$$

Como B es un escalar, $B = B^T \;\rightarrow\; \partial B = \partial B^T$ y $\{A, \partial B\} - \{A, \partial B^T\} = 0$ y análogamente para los otros dos cocientes, entonces,

$$\underset{\divideontimes}{\overset{\divideontimes}{=}} 0 \; J \partial C \;+\; 0 \; J \partial B \;+\; 0 \; J \partial B \;=\; 0 \qquad \square$$