factors that depend on the numerical integration scheme being used as follows:

Simple summation:

$$\overline{A}_N \doteq \frac{\triangle\tau}{m\,\omega}\Big[y_0 + y_1 + y_2 + \cdots + y_{N-1}\Big] \qquad N = 1, 2, 3, \cdots \tag{6-12a}$$

Trapezoidal rule:

$$\overline{A}_N \doteq \frac{\triangle\tau}{2\,m\,\omega}\Big[y_0 + 2\,y_1 + 2\,y_2 + \cdots + 2\,y_{N-1} + y_N\Big] \qquad N = 1, 2, 3, \cdots \tag{6-12b}$$

Simpson's rule:

$$\overline{A}_N \doteq \frac{\triangle\tau}{3\,m\,\omega}\Big[y_0 + 4\,y_1 + 2\,y_2 + \cdots + 4\,y_{N-1} + y_N\Big] \qquad N = 2, 4, 6, \cdots \tag{6-12c}$$

Using any one of these equations, $\overline{A}_N$ can be obtained directly for any specific value of N indicated. However, usually the entire time-history of response is required so that one must evaluate $\overline{A}_N$ for successive values of N until the desired time-history of response is obtained. For this purpose, it is more efficient to use these equations in their recursive forms:

Simple summation:

$$\overline{A}_N \doteq \overline{A}_{N-1} + \frac{\triangle\tau}{m\,\omega}\Big[y_{N-1}\Big] \qquad N = 1, 2, 3, \cdots \tag{6-13a}$$

Trapezoidal rule:

$$\overline{A}_N \doteq \overline{A}_{N-1} + \frac{\triangle\tau}{2\,m\,\omega}\Big[y_{N-1} + y_N\Big] \qquad N = 1, 2, 3, \cdots \tag{6-13b}$$

Simpson's rule:

$$\overline{A}_N \doteq \overline{A}_{N-2} + \frac{\triangle\tau}{3\,m\,\omega}\Big[y_{N-2} + 4\,y_{N-1} + y_N\Big] \qquad N = 2, 4, 6, \cdots \tag{6-13c}$$

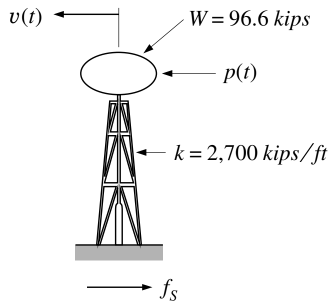
such that $\overline{A}_0 = 0$.

Evaluation of $\overline{B}(t)$ in Eq. (6-10) can be carried out in the same manner, leading to expressions for $\overline{B}_N$ having exactly the same forms shown by Eqs. (6-13); however, in doing so, the definition of $y(\tau)$ must be changed to $y(\tau) \equiv p(\tau)\ \sin\omega\tau$ consistent with the second of Eqs. (6-11). Having calculated the values of $\overline{A}_N$ and $\overline{B}_N$ for successive values of N, the corresponding values of response $v_N \equiv v\ (t = N\,\triangle\tau)$ are obtained using

$$v_N = \overline{A}_N\ \sin\omega t_N - \overline{B}_N\ \cos\omega t_N \tag{6-14}$$

Example E6-1. The dynamic response of a water tower subjected to a blast loading will now be presented to illustrate the above numerical procedure for obtaining undamped response through the time domain in accordance with Eq. (6-14). The idealizations of the structure and blast loading are shown in Fig. E6-1. For this system, the vibration frequency and period are

$$\omega = \sqrt{\frac{kg}{W}} = \sqrt{\frac{2,700\ (32.2)}{96.6}} = 30\ rad/sec \qquad T = \frac{2\,\pi}{\omega} = 0.209\ sec$$



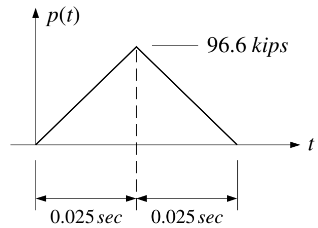


FIGURE E6-1
Water tower subjected to blast load.

The time increment used in the numerical integration is $\triangle\tau = 0.005\ sec$, which corresponds to an angular increment in free vibrations of $\omega\,\triangle\tau = 0.15\ rad$ (probably a somewhat longer increment would give equally satisfactory results). In this analysis, Simpson's-rule summation as given by Eq. (6-13c) is used.

An evaluation of response over the first 10 time steps is presented in a convenient tabular format in Table E6-1. The operations performed in each column are generally apparent from the labels at the top; however, a few brief comments may be helpful as follows: (a) Columns (*4*) through (*10*) are used to evaluate $\overline{A}_N/F$ (where $F \equiv \triangle t/3m\,\omega$) in accordance with Eq. (6-13c) using $y_N \equiv p_N \cos\omega t_N$. (b) Columns (*11*) through (*17*) are used to evaluate $\overline{B}_N/F$ in accordance with its equivalence of Eq. (6-13c). (c) Columns (*18*) through (*21*) are used to evaluate v_N in accordance with Eq. (6-14). (d) The last column is used to evaluate the spring force $f_{S_N} = k\,v_N$. (e) The multiplication factor $M_2 = 1$ need not be shown in Table E6-1; however, it is entered for later comparison with $M_2 \neq 1$ as required in the damped-response solution.

Since the blast loading terminates at the end of the first 10 time steps, the values of $\overline{A}$ and $\overline{B}$ remain constant after time $t = 0.050$. If these constant values are designated $\overline{A}^*$ and $\overline{B}^*$, the free vibrations which follow the blast loading are given by

$$v(t) = \overline{A}^* \ \sin\omega t - \overline{B}^* \ \cos\omega t$$

in accordance with Eq. (6-10). The amplitude of this motion is

$$v_{\text{max}} = \left[(\overline{A}^*)^2 + (\overline{B}^*)^2\right]^{1/2}$$

In the above example, $\overline{A}^* = 1026\,F = 0.0190\ ft\ [0.579\ cm]$ and $\overline{B}^* = 956\,F = 0.0177\ ft\ [0.539\ cm]$ [see Columns (*10*) and (*17*) for $N = 10$] so that $v_{\text{max}} = 0.0260\ ft\ [0.792\ cm]$ and $f_{S_{\text{max}}} = 70.2\ kips\ [31,840\ kg]$.

TABLE E6-1
Numerical Duhamel integral analysis without damping

N	t_N sec	p_N kips (1)	$\sin 30t_N$ (2)	$\cos 30t_N$ (3)	y_N (1)×(3) kips (4)	y_{N-1} (5)	y_{N-2} (6)	$M_1 \times$ (5) (7)	$M_2 \times$ [(6)+(9)] (8)	$\frac{\bar{A}_{N-2}}{F}$ (9)	$\frac{\bar{A}_N}{F}$ (4)+(7)+(8) (10)	y_N (1)×(2) kips (11)	y_{N-1} (12)	y_{N-2} (13)	$M_1 \times$ (12) (14)	$M_2 \times$ [(13)+(16)] (15)	$\frac{\bar{B}_{N-2}}{F}$ (16)	$\frac{\bar{B}_N}{F}$ (11)+(14)+(15) (17)	(10)×(2) (18)	(17)×(3) (19)	(18)−(19) (20)	v_N F×(20) ft (21)	f_{S_N} k×(21) kips (22)
0	0.000	0	0	1.000	0	—	—	—	—	—	0	0	—	—	—	—	—	0	0	0	0	0	0
1	0.005	19.32	0.149	0.989	19.1	0	—	—	—	—	—	2.88	0	—	—	—	—	—	—	—	—	—	—
2	0.010	38.64	0.296	0.955	36.9	19.1	0	76.4	0	0	113.3	11.4	2.88	0	11.5	0	0	22.9	33.5	21.9	11.6	0.0002	0.54
3	0.015	57.96	0.435	0.900	52.2	36.9	19.1	—	—	—	—	25.2	11.4	2.88	—	—	—	—	—	—	—	—	—
4	0.020	77.28	0.565	0.825	63.8	52.2	36.9	208.8	150.2	113.3	422.8	43.7	25.2	11.4	100.8	34.3	22.9	178.8	239	148	91	0.0017	4.60
5	0.025	96.60	0.682	0.732	70.7	63.8	52.2	—	—	—	—	65.9	43.7	25.2	—	—	—	—	—	—	—	—	—
6	0.030	77.28	0.783	0.622	48.1	70.7	63.8	282.8	486.6	422.8	817.5	60.5	65.9	43.7	263.6	222.5	178.8	546.6	640	340	300	0.0056	15.1
7	0.035	57.96	0.867	0.498	28.9	48.1	70.7	—	—	—	—	50.3	60.5	65.9	—	—	—	—	—	—	—	—	—
8	0.040	38.64	0.932	0.362	14.0	28.9	48.1	115.6	865.6	817.5	995.2	36.0	50.3	60.5	201.2	607.1	546.6	844.3	928	306	622	0.015	31.0
9	0.045	19.32	0.976	0.219	4.23	14.0	28.9	—	—	—	—	18.9	36.0	50.3	—	—	—	—	—	—	—	—	—
10	0.050	0	0.997	0.0707	0	4.23	14.0	16.9	1009	995.2	1026	0	18.9	36.0	75.6	880.3	844.3	955.9	1023	67.6	955	0.0177	47.8

$\omega = \sqrt{\frac{kg}{W}} = 30\ rad/sec$ $\quad \Delta\tau = 0.0005\ sec$ $\quad M_1 = 4$ $\quad M_2 = 1$ $\quad F \equiv \frac{\Delta\tau}{3m\omega} = 1.852 \times 10^{-5}\ ft/kip$ $\quad k = 2700\ kips/ft$