A GP model for shear fields

Michael D. Schneider

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Basic premise

We model the 2D lens potential ψ as a Gaussian Process (GP) model,

$$\psi \sim \mathrm{GP}(0, \Sigma),$$

where the GP covariance $\Sigma(\vec{x}, \vec{y})$ is a function of two 2D position vectors to denote the covariance of the realizations of the lens potential at sky positions $\vec{x}$ and $\vec{y}$.

Observable lensing quantities

- The lensing convergence and shear are 2nd derivatives of the lens potential.
- **Because the derivative of a GP is also a GP,** the vector of $\mathbf{s} \equiv (\kappa, \gamma_1, \gamma_2)$ is GP distributed.
- The covariance of $\mathbf{s}$ is given by 4th derivatives of the covariance of Σ of ψ. That is,

  $$\mathrm{Cov}(\psi_{,ij}(\vec{x}), \psi_{,k\ell}(\vec{y})) = \Sigma_{,x_i x_j y_k y_\ell}(\vec{x}, \vec{y}).$$

  where $i, j, k, l = 1, 2$ denoting the two spatial coordinates

GP covariance parameterizations

Generally, we will write the GP covariance as the product of a precision parameter λ and a 'kernel' k,

$$\Sigma(\vec{x}, \vec{y}) \equiv \lambda^{-1} k(r^2(\vec{x}, \vec{y}, D)),$$

where,

$$r^2 \equiv (\vec{x} - \vec{y})^T D (\vec{x} - \vec{y})$$

for some metric D^{-1}.

GP kernels in george

The george python package has a number of GP kernels already implemented. Two common interesting ones are:

1. Squared exponential kernel,

   $$k(r^2) = \exp(-\frac{1}{2} r^2)$$

2. Matern kernels with parameter ν (only $\nu = (3/2, 5/2)$ implemented in george),

   $$k_\nu(r^2) = \frac{1}{\Gamma(\nu) 2^{\nu-1}} (2\nu r^2)^{\nu/2} K_\nu(\sqrt{2\nu r^2}),$$

   where K_ν is the modified Bessel function of the 2nd kind.

GP covariance derivatives

General relations (1): derivatives of r^2

From before,

$$r^2 \equiv (\vec{x} - \vec{y})^T D(\vec{x} - \vec{y}).$$

Then,

$$\frac{\partial r^2}{\partial x_i} = [D(\vec{x} - \vec{y})]_i \equiv X_i$$

$$\frac{\partial r^2}{\partial y_i} = -X_i$$

$$\mathrm{Hess}_{ij}(r^2) = D_{ij} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

General relations (2a): derivatives of Σ

The 1st, 2nd, and 4th derivatives of Σ that we need are,

$$\Sigma_{,x_i} = \lambda^{-1} k^{'} r^2_{,x_i}$$

$$\Sigma_{,x_i x_j} = \lambda^{-1} \left[k^{''} r^2_{,x_j} r^2_{,x_i} + k^{'} r^2_{,x_i x_j} \right]$$

$$\Sigma_{,x_i x_j y_k y_\ell} = \lambda^{-1} [k^{''''} r^2_{,x_i} r^2_{,x_j} r^2_{,y_k} r^2_{,y_\ell} + k^{'''} \left(r^2_{,x_i x_j} r^2_{,y_k} r^2_{,y_\ell} + 5\text{perm.} \right) + k^{''} \left(r^2_{,x_i x_j} r^2_{,y_k y_\ell} + 2\text{perm.} \right)]$$

where primes on k denote derivatives with respect to r^2.

General relations (2b): derivatives of Σ (simplified)

$$\Sigma_{,(x,y)_i} = \pm \lambda^{-1} k^{'} X_i$$

$$\Sigma_{,x_i x_j} = \lambda^{-1} \left[k^{''} X_i X_j + k^{'} D_{ij} \right]$$

$$\Sigma_{,x_i x_j y_k y_\ell} = \lambda^{-1} [k^{''''} X_i X_j X_k X_\ell + k^{'''} \left(D_{ij} X_k X_\ell + 5\text{perm.} \right) + k^{''} \left(D_{ij} D_{k\ell} + 2\text{perm.} \right)]$$

Observations about GP covariances of convergence and shear

- The derivatives of Σ have terms that depend on $X_i \equiv [D(\vec{x} - \vec{y})]_i$, which *increases* in magnitude with increasing separation on the sky. **So, distant points can be highly correlated.**
- In our nominal model, $D_{ij} \propto -\ln(\rho)$ where ρ is the correlation parameter for the GP.
- At small sky separations $X_i \approx 0$ and the covariance of the convergence or shear $\sim D^2_{ij} \Sigma$. **So, the variance of the convergence and shear is enhanced by a factor $[\ln(\rho)]^2$ over that of the lens potential.**

Derivatives of specific GP kernels

Kernel derivatives: Squared exponential

$$k(r^2) \equiv \exp(-\frac{1}{2}r^2)$$

Because there's just the exponential term, the nth derivatives are trivial,

$$k^{(n)} = \left(-\frac{1}{2}\right)^n k$$

Final GP covariance derivatives: Squared exponential

$$\Sigma_{,(x,y)_i} = \mp\frac{1}{2}X_i\Sigma$$

$$\Sigma_{,x_ix_j} = \left[\frac{1}{4}X_iX_j - \frac{1}{2}D_{ij}\right]\Sigma$$

$$\Sigma_{,x_ix_jy_ky_\ell} = [\frac{1}{16}X_iX_jX_kX_\ell - \frac{1}{8}\left(D_{ij}X_kX_\ell + 5\text{perm.}\right) + \frac{1}{4}\left(D_{ij}D_{k\ell} + 2\text{perm.}\right)]\Sigma$$

Kernel derivatives: Matern (1)

$$k_\nu(r^2) = \frac{1}{\Gamma(\nu)2^{\nu-1}}(2\nu r^2)^{\nu/2}K_\nu(\sqrt{2\nu r^2}),$$

The nth derivatives of the modified Bessel function of the 2nd kind can be represented by the Meijer G function. But, george only implements the cases $\nu = 3/2, 5/2$ and the kernel has a simpler form for these cases,

$$k_\nu(r^2) = p_\nu(r)\exp\left(-\sqrt{2\nu r^2}\right)$$

where p_ν is a polynomial in $\sqrt{r^2}$.

We'll consider the 1st-4th derivatives of this expression next.

Lensing parameters

Convergence and shear from lens potential

Note the subscripts correspond to derivatives w.r.t. spatial coords

$$\kappa = \frac{1}{2}\left(\psi_{,11} + \psi_{,22}\right)$$

$$\gamma_1 = \frac{1}{2}\left(\psi_{,11} - \psi_{,22}\right)$$

$$\gamma_2 = \frac{1}{2}\left(\psi_{,12} + \psi_{,21}\right)$$

Convergence and shear GP covariances

$$\mathrm{Cov}(\kappa) = \frac{1}{4}\left(\Sigma_{,1111} + \Sigma_{,1122} + \Sigma_{,2211} + \Sigma_{,2222}\right)$$

$$\mathrm{Cov}(\gamma_1) = \frac{1}{4}\left(\Sigma_{,1111} - \Sigma_{,1122} - \Sigma_{,2211} + \Sigma_{,2222}\right)$$

$$\mathrm{Cov}(\gamma_2) = \frac{1}{4}\left(\Sigma_{,1212} + \Sigma_{,1221} + \Sigma_{,2112} + \Sigma_{,2121}\right)$$