Notes - A GP model for shear fields

Karen Y. Ng

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Our Gaussian process "model" for the projected lensing potential

$$\psi \sim N(0, \Sigma(\vec{x}, \vec{y})) \tag{1}$$

which is a scalar field evaluated at the positions $x_i = (\vec{x_1}, \vec{x_2})_i$ or $g_i = (\vec{y_1}, \vec{y_2})$ where we have data points. As usual, first column, x_1 or y_1 is the first spatial dimension, x_2 or y_2 is the second one, the i-th row correspond to spatial coordinates of the i-th data point.
$\vec{x}$ and $\vec{y}$ are the same but we call them different names for denoting their location in the covariance matrix ....

For inferring the convergence and shear, we need the 2nd spatial derivatives. The subscripts in these WL equations correspond to the spatial coordinates x_1, x_2 NOT the observation numbers i.e. i, j = 1, 2, ..., n observations

$$\begin{aligned}
\kappa &= \frac{1}{2} tr(\psi_{,ij}) \\
&= \frac{1}{2}(\psi_{,11} + \psi_{,22}) \\
&= \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right)
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}(\psi_{,11} - \psi_{,22}) \\
&= \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_2^2}\right)
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \frac{1}{2}(\psi_{,12} + \psi_{,21}) \\
&= \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_2 \partial x_1}\right)
\end{aligned}$$

Covariances of the required functions

Note that ψ, κ and γ are scalar fields. However, we are evaluating them at the locations of the data points $(x_1, x_2)_i$, therefore, when we are writing down the shorthand for the m, n subscripts below, we mean, we first take the spatial derivatives of those scalar field(s) with respect to x_1 or x_2, then evaluate them at the $m-th$ or $n-th$ position $(x_1, x_2)_m$. The spatial derivatives are represented as follows:

$$\psi_{,1} = \frac{\partial \psi}{\partial x_1}$$

etc. with a comma in the subscript.

Also note expectation and derivative are both linear operators, so we can exchange their positions (and try not to let mathematicians read this and shoot us)

$$\begin{aligned}
&\mathrm{Cov}_{m,n}(\kappa(\vec{x}), \kappa(\vec{y})) \\
&= \mathbb{E}\left[(\kappa - \mathbb{E}[\kappa])|_m (\kappa - \mathbb{E}[\kappa])|_n\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi\right]\right]\bigg|_m \left[\frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)\psi\right]\right]\bigg|_n\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\bigg|_m [\psi - \mathbb{E}[\psi]]|_m \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)\bigg|_n [\psi - \mathbb{E}[\psi]]|_n\right] \\
&= \frac{1}{4}\left(\left(\frac{\partial^2}{\partial x_1^2}\right)\bigg|_m \left(\frac{\partial^2}{\partial y_1^2}\right)\bigg|_n + \left(\frac{\partial^2}{\partial x_1^2}\right)\bigg|_m \left(\frac{\partial^2}{\partial y_2^2}\right)\bigg|_n + \left(\frac{\partial^2}{\partial x_2^2}\right)\bigg|_m \left(\frac{\partial^2}{\partial y_1^2}\right)\bigg|_n + \left(\frac{\partial^2}{\partial x_2^2}\right)\bigg|_m \left(\frac{\partial^2}{\partial y_2^2}\right)\bigg|_n\right)\Sigma_{mn} \\
&= \frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2}\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_1^2}\frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial x_2^2}\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2}\frac{\partial^2}{\partial y_2^2}\right)\Sigma_{mn}
\end{aligned}$$

Now I have dropped the (m,n) subscripts and the following subscripts correspond to the spatial dimensions

$$\mathrm{Cov}(\kappa(\vec{x}), \kappa(\vec{y})) = \frac{1}{4}\left(\Sigma_{,1111} + \Sigma_{,1122} + \Sigma_{,2211} + \Sigma_{,2222}\right) \tag{2}$$

Similarly,

$$\begin{aligned}
&\mathrm{Cov}_{mn}(\gamma_1(\vec{x}), \gamma_1(\vec{y})) \\
&= \mathbb{E}\left[(\gamma_1 - \mathbb{E}[\gamma_1])|_m (\gamma_1 - \mathbb{E}[\gamma_1])|_n\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\psi\right]\right]\bigg|_m \left[\frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}\right)\psi\right]\right]\bigg|_n\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\bigg|_i [\psi - \mathbb{E}[\psi]]|_m \left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}\right)\bigg|_j [\psi - \mathbb{E}[\psi]]|_n\right] \\
&= \frac{1}{4}\left(\frac{\partial^2}{\partial x_1^2}\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial x_1^2}\frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial x_2^2}\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2}\frac{\partial^2}{\partial y_2^2}\right)\Sigma_{mn}
\end{aligned}$$

$$\mathrm{Cov}(\gamma_1(\vec{x}), \gamma_1(\vec{y})) = \frac{1}{4}\left(\Sigma_{,1111} - \Sigma_{,1122} - \Sigma_{,2211} + \Sigma_{,2222}\right) \tag{3}$$

And,

$$
\begin{aligned}
\mathrm{Cov}_{mn}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) &= \mathbb{E}\left[(\gamma_2 - \mathbb{E}[\gamma_2])|_m(\gamma_2 - \mathbb{E}[\gamma_2])|_n\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1}\right)\psi\right]\right]\Big|_m \right. \\
&\qquad \left. \left[\frac{1}{2}\left(\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1}\right)\psi\right]\right]\Big|_n\right]
\end{aligned}
$$

$$
\mathrm{Cov}_{mn}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) = \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1}\right)\Big|_m [\psi - \mathbb{E}[\psi]]|_m \left(\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1}\right)\Big|_n [\psi - \mathbb{E}[\psi]]|_n\right]
$$

$$
\mathrm{Cov}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) = \frac{1}{4}\left(\Sigma_{,1212} + \Sigma_{,1221} + \Sigma_{,2112} + \Sigma_{,2121}\right) \tag{4}
$$

The squared exponential covariance function

$$
\Sigma(r^2; \lambda, \rho) = \lambda^{-1}\exp\left(-\frac{\beta}{2}r^2\right) \tag{5}
$$

where $\beta = -1/4 \ln \rho$, and $0 < \rho < 1$, note Σ is an N $\times$ N matrix and the covariance functions of the derivatives should have the same dimension.

The metric D

$$
r^2 = (t - t')^T D (t - t') \tag{6}
$$

Since we are working in projected (2D) space, D is a 2 $\times$ 2 matrix. More explicitly, in the GP model:

$$
\begin{aligned}
r_{ij}^2 &= (x_i - x_j, y_i - y_j)\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}\begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} \\
\Sigma_{ij} &= \lambda^{-1}\exp\left(-\frac{\beta}{2}r_{ij}^2\right)
\end{aligned}
$$

An example of r^2 with an Euclidean metric for a pair of data points, t_i, t_j would be:

$$
r^2 = (x_i - x_j)^2 + (y_i - y_j)^2
$$

Summary: basic derivatives with the preceeding coefficients

Comparison between parametrization of George and our parametrization