Notes - A GP model for shear fields

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Our Gaussian process "model" for the projected lensing potential

$$\psi(\vec{t}) \sim N(0, \Sigma(\vec{t}, \vec{t}')) \tag{1}$$

which is a scalar field evaluated at the positions $t_i = (\vec{x}, \vec{y})_i$ where we have data points. As usual, first column, x is the first spatial dimension, y is the second one, the i-th row correspond to spatial coordinates of the i-th data point.

For inferring the convergence and shear, we need the 2nd spatial derivatives. The subscripts in these WL equations correspond to the spatial coordinates x, y instead of the observation numbers i.e. i, j = 1, 2, ..., n observations

$$\begin{aligned}\kappa &= \frac{1}{2} tr(\psi_{,ij}) \\ &= \frac{1}{2}(\psi_{,11} + \psi_{,22}) \\ &= \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)\end{aligned}$$

$$\begin{aligned}\gamma_1 &= \frac{1}{2}(\psi_{,11} - \psi_{,22}) \\ &= \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right)\end{aligned}$$

$$\begin{aligned}\gamma_2 &= \frac{1}{2}(\psi_{,12} + \psi_{,21}) \\ &= \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x}\right)\end{aligned}$$

Covariances of the required functions

Note that ψ, κ and γ are scalar fields. However, we are evaluating them at the locations of the data points (x_i, y_i), therefore, when we are writing down the shorthand for the i, j subscripts below, we mean, we first take the spatial derivatives of those scalar field(s) with respect to x or y, then evaluate them at (x_i, y_i). The spatial derivatives are represented as follows:

$$\psi_{,1} = \frac{\partial \psi}{\partial x}$$

etc. with a comma in the subscript.

Also note expectation and derivative are both linear operators, so we can exchange their positions (and try not to let mathemticians read this and shoot us)

$$\begin{aligned}
\mathrm{Cov}_{ij}(\kappa) &= \mathbb{E}\left[(\kappa - \mathbb{E}[\kappa])|_i(\kappa - \mathbb{E}[\kappa])|_j\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\bigg|_i \left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\bigg|_j\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_i \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_j\right] \\
&= \frac{1}{4}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial y^4}\right)\Sigma_{ij}
\end{aligned}$$

$$\mathrm{Cov}(\kappa) = \frac{1}{4}\left(\Sigma_{,1111} + \Sigma_{,1122} + \Sigma_{,2211} + \Sigma_{,2222}\right) \tag{2}$$

Similarly,

$$\begin{aligned}
\mathrm{Cov}_{ij}(\gamma_1) &= \mathbb{E}\left[(\gamma_1 - \mathbb{E}[\gamma_1])|_i(\gamma_1 - \mathbb{E}[\gamma_1])|_j\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\bigg|_i \left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\bigg|_j\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_i \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_j\right] \\
&= \frac{1}{4}\left(\frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial y^4}\right)\Sigma_{ij}
\end{aligned}$$

$$\mathrm{Cov}(\gamma_1) = \frac{1}{4}\left(\Sigma_{,1111} - \Sigma_{,1122} - \Sigma_{,2211} + \Sigma_{,2222}\right) \tag{3}$$

And,

$$\begin{aligned}
&\mathrm{Cov}_{ij}(\gamma_2) \\
&= \mathbb{E}\left[(\gamma_2 - \mathbb{E}[\gamma_2])|_i(\gamma_2 - \mathbb{E}[\gamma_2])|_j\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi\right]\right]\bigg|_i \left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi\right]\right]\bigg|_j\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)[\psi - \mathbb{E}[\psi]]|_i \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)[\psi - \mathbb{E}[\psi]]|_j\right]
\end{aligned}$$

$$\mathrm{Cov}(\gamma_2) = \frac{1}{4}\left(\Sigma_{,1212} + \Sigma_{,1221} + \Sigma_{,2112} + \Sigma_{,2121}\right) \tag{4}$$

The squared exponential covariance function

$$\Sigma(r^2; \lambda, \rho) = \lambda^{-1} \exp\left(-\frac{\beta}{2} r^2\right) \tag{5}$$

where $\beta = -1/4 \ln \rho$, and $0 < \rho < 1$, note Σ is an N $\times$ N matrix and the covariance functions of the derivatives should have the same dimension.

The metric D

Since we are working in projected (2D) space, D is a 2 $\times$ 2 matrix.

$$r^2 = (t - t')^T D (t - t') \tag{6}$$

More explicitly, in the GP model:

$$\begin{aligned}
r_{ij}^2 &= (x_i - x_j, y_i - y_j) \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} \\
\Sigma_{ij} &= \lambda^{-1} \exp\left(-\frac{\beta}{2} r_{ij}^2\right)
\end{aligned}$$

Summary: basic derivatives with the preceeding coefficients

$$\frac{\partial r^2}{\partial x_i} = [(D + D^T)(\vec{x} - \vec{y})]_i \tag{7}$$

$$\frac{\partial r^2}{\partial y_i} = -[(D + D^T)(\vec{x} - \vec{y})]_i \tag{8}$$

$$\text{Hess}(r^2(\vec{x}, \vec{y})) = (D + D^T) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{9}$$

Comparison between parametrization of George and our parametrization