

Notes - A GP model for shear fields

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Our Gaussian process “model” for the projected lensing potential

$$\psi \sim N(0, \Sigma(\vec{x}, \vec{y})) \quad (1)$$

which is a scalar field evaluated at the positions $x_i = (\vec{x}_1, \vec{x}_2)_i$ or $y_i = (\vec{y}_1, \vec{y}_2)_i$ where we have data points. As usual, first column, x_1 or y_1 is the first spatial dimension, x_2 or y_2 is the second one, the i -th row correspond to spatial coordinates of the i -th data point.

\vec{x} and \vec{y} are the same but we call them different names for denoting their location in the covariance matrix
....

For inferring the convergence and shear, we need the 2nd spatial derivatives. The subscripts in these WL equations correspond to the spatial coordinates x_1, x_2 NOT the observation numbers i.e. $i, j = 1, 2, \dots, n$ observations

$$\begin{aligned} \kappa &= \frac{1}{2} \text{tr}(\psi_{,ij}) \\ &= \frac{1}{2} (\psi_{,11} + \psi_{,22}) \\ &= \frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} \right) \end{aligned}$$

$$\begin{aligned} \gamma_1 &= \frac{1}{2} (\psi_{,11} - \psi_{,22}) \\ &= \frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_2^2} \right) \end{aligned}$$

$$\begin{aligned} \gamma_2 &= \frac{1}{2} (\psi_{,12} + \psi_{,21}) \\ &= \frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_2 \partial x_1} \right) \end{aligned}$$

Covariances of the required functions

Note that ψ, κ and γ are scalar fields. However, we are evaluating them at the locations of the data points $(x_1, x_2)_i$, therefore, when we are writing down the shorthand for the m, n subscripts below, we mean, we first

take the spatial derivatives of those scalar field(s) with respect to x_1 or x_2 , then evaluate them at the m -th or n -th position $(x_1, x_2)_m$. The spatial derivatives are represented as follows:

$$\psi_{,1} = \frac{\partial \psi}{\partial x_1}$$

etc. with a comma in the subscript.

Also note expectation and derivative are both linear operators, so we can exchange their positions (and try not to let mathematicians read this and shoot us)

$$\begin{aligned} \text{Cov}_{m,n}(\kappa(\vec{x}), \kappa(\vec{y})) &= \mathbb{E}[(\kappa - \mathbb{E}[\kappa])|_m (\kappa - \mathbb{E}[\kappa])|_n] \\ &= \mathbb{E} \left[\left[\frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \psi - \mathbb{E} \left[\frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \psi \right] \right] \Big|_m \left[\frac{1}{2} \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \psi - \mathbb{E} \left[\frac{1}{2} \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \psi \right] \right] \Big|_n \right] \\ &= \frac{1}{4} \mathbb{E} \left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \Big|_m [\psi - \mathbb{E}[\psi]] \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \Big|_n [\psi - \mathbb{E}[\psi]] \right] \\ &= \frac{1}{4} \left(\left(\frac{\partial^2}{\partial x_1^2} \right) \Big|_m \left(\frac{\partial^2}{\partial y_1^2} \right) \Big|_n + \left(\frac{\partial^2}{\partial x_1^2} \right) \Big|_m \left(\frac{\partial^2}{\partial y_2^2} \right) \Big|_n + \left(\frac{\partial^2}{\partial x_2^2} \right) \Big|_m \left(\frac{\partial^2}{\partial y_1^2} \right) \Big|_n + \left(\frac{\partial^2}{\partial x_2^2} \right) \Big|_m \left(\frac{\partial^2}{\partial y_2^2} \right) \Big|_n \right) \Sigma_{mn} \\ &= \frac{1}{4} \left(\frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_2^2} \right) \Sigma_{mn} \end{aligned}$$

Now I have dropped the (m,n) subscripts and the following subscripts correspond to the spatial dimensions, the first two subscripts correspond to spatial derivatives w.r.t. x and evaluated for the m -th data points, the last two correspond to spatial derivatives w.r.t. y and evaluated for the n -th data points.

$$\text{Cov}(\kappa(\vec{x}), \kappa(\vec{y})) = \frac{1}{4} (\Sigma_{,1111} + \Sigma_{,1122} + \Sigma_{,2211} + \Sigma_{,2222}) \quad (2)$$

Similarly,

$$\begin{aligned} \text{Cov}_{mn}(\gamma_1(\vec{x}), \gamma_1(\vec{y})) &= \mathbb{E}[(\gamma_1 - \mathbb{E}[\gamma_1])|_m (\gamma_1 - \mathbb{E}[\gamma_1])|_n] \\ &= \mathbb{E} \left[\left[\frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \psi - \mathbb{E} \left[\frac{1}{2} \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \psi \right] \right] \Big|_m \left[\frac{1}{2} \left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} \right) \psi - \mathbb{E} \left[\frac{1}{2} \left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} \right) \psi \right] \right] \Big|_n \right] \\ &= \frac{1}{4} \mathbb{E} \left[\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \Big|_i [\psi - \mathbb{E}[\psi]] \left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} \right) \Big|_j [\psi - \mathbb{E}[\psi]] \right] \\ &= \frac{1}{4} \left(\frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_2^2} \right) \Sigma_{mn} \end{aligned}$$

$$\text{Cov}(\gamma_1(\vec{x}), \gamma_1(\vec{y})) = \frac{1}{4} (\Sigma_{,1111} - \Sigma_{,1122} - \Sigma_{,2211} + \Sigma_{,2222}) \quad (3)$$

And,

$$\begin{aligned}
\text{Cov}_{mn}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) &= \mathbb{E}[(\gamma_2 - \mathbb{E}[\gamma_2])|_m (\gamma_2 - \mathbb{E}[\gamma_2])|_n] \\
&= \mathbb{E} \left[\left[\frac{1}{2} \left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1} \right) \psi - \mathbb{E} \left[\frac{1}{2} \left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1} \right) \psi \right] \right] \Big|_m \right. \\
&\quad \left. \left[\frac{1}{2} \left(\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1} \right) \psi - \mathbb{E} \left[\frac{1}{2} \left(\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1} \right) \psi \right] \right] \Big|_n \right] \\
\text{Cov}_{mn}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) &= \frac{1}{4} \mathbb{E} \left[\left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1} \right) \Big|_m [\psi - \mathbb{E}[\psi]] \left(\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1} \right) \Big|_n [\psi - \mathbb{E}[\psi]] \right] \\
\text{Cov}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) &= \frac{1}{4} (\Sigma_{,1212} + \Sigma_{,1221} + \Sigma_{,2112} + \Sigma_{,2121}) \tag{4}
\end{aligned}$$

The squared exponential covariance function

$$\Sigma(r^2; \lambda, \rho) = \lambda^{-1} \exp \left(-\frac{\beta}{2} r^2 \right) \tag{5}$$

where $\beta = -1/4 \ln \rho$, and $0 < \rho < 1$, note Σ is an $N \times N$ matrix and the covariance functions of the derivatives should have the same dimension.

The metric D

$$r^2 = (\vec{x} - \vec{y})^T D (\vec{x} - \vec{y}) \tag{6}$$

Since we are working in projected (2D) space, D is a 2×2 matrix. More explicitly, I will use i,j,h,k as subscripts for the spatial dimensions and m, n for the observation number in the GP model:

$$\begin{aligned}
r_{mn}^2 &= (x_{m1} - y_{m1}, x_{n2} - y_{n2}) \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x_{m1} - y_{n1} \\ x_{m2} - y_{n2} \end{pmatrix} \\
\Sigma_{mn} &= \lambda^{-1} \exp \left(-\frac{\beta}{2} r_{mn}^2 \right)
\end{aligned}$$

An example of r^2 with an Euclidean metric for a pair of data points, \vec{x}_i and \vec{y}_j would be:

$$r_{mn}^2 = (x_{m1} - y_{n1})^2 + (x_{m2} - y_{n2})^2$$

In the following derivations, it is NOT important to keep the m, n subscripts. We are taking the derivatives w.r.t to the spatial dimensions, so I will drop the m, n subscripts. But keep in mind each forth derivative of Σ with the indices written out should be a scalar, each is an element in the big $m \times n$ $\Sigma_{,x_i x_j y_h y_k}$ matrix.

Summary: basic derivatives of components, assuming diagonal D

Derivation of the derivatives for a non-symmetric / non-diagonal D would give more terms ...

$$\begin{aligned}
\frac{\partial r^2}{\partial x_i} &= \frac{\partial(x_q - y_q)D_{qr}(x_r - y_r)}{\partial x_i} \\
&= \delta_{iq}D_{qr}(x_r - y_r) + (x_q - y_q)\delta_{iq}D_{qr} \\
&= 2D_{ii}(x_i - y_i) \\
&= 2[D(\vec{x} - \vec{y})]_i
\end{aligned}$$

The third line in the above derivation is only true for diagonal D or else there are other summation terms.

$$r_{,x_i} = 2[D(\vec{x} - \vec{y})]_i \equiv 2X_i \quad (7)$$

Similarly,

$$\begin{aligned}
\frac{\partial r^2}{\partial y_h} &= \frac{\partial(x_q - y_q)D_{qr}(x_r - y_r)}{\partial y_h} \\
&= -\delta_{iq}D_{qr}(x_r - y_r) - (x_q - y_q)\delta_{iq}D_{qr} \\
&= -2[D(\vec{x} - \vec{y})]_h
\end{aligned}$$

$$r_{,y_h} = -2[D(\vec{x} - \vec{y})]_h \equiv -2X_h \quad (8)$$

where $i = 1, 2$

Second derivatives of r^2

$$r_{,x_i x_j}^2 = 2\delta_{ij}D_{ij} \quad (9)$$

$$r_{,y_h y_k}^2 = 2\delta_{hk}D_{hk} \quad (10)$$

$$r_{,x_i y_h}^2 = 2\delta_{ih}D_{ih} \quad (11)$$

derivatives of the kernel

$$\Sigma = \lambda^{-1}k$$

$$k = \exp\left(\frac{-\beta}{2}r^2\right) \quad (12)$$

$$k_{,x_i} = \frac{-\beta}{2}kr_{,x_i}^2 = -\beta kX_i \quad (13)$$

$$k_{,y_h} = \beta kX_h \quad (14)$$

$$X_i = [D(\vec{x} - \vec{y})]_i \quad (15)$$

$$X_i, x_j = D\delta_{ij} \quad (16)$$

$$X_i, y_h = -D\delta_{ih} \quad (17)$$

$$X_h, y_k = D\delta_{ih} \quad (18)$$

second derivatives

$$\begin{aligned}
k_{,x_i x_j} &= \frac{\partial}{\partial x_j}(-\beta k X_i) \\
&= -\beta(k_{,x_j X_i} + k X_{i,x_j}) \\
&= -\beta(-\beta k X_j X_i + k \delta_{ij} D_{ij}) \\
&= (\beta^2 X_j X_i - \beta \delta_{ij} D_{ij})k
\end{aligned}$$

$$\begin{aligned}
k_{,x_i y_h} &= \frac{\partial}{\partial y_h}(-\beta k X_i) \\
&= -\beta(k_{,y_h X_i} + k X_{i,y_h}) \\
&= -\beta(\beta k X_h X_i - k \delta_{ih} D_{ih}) \\
&= -(\beta^2 X_h X_i - \beta \delta_{ih} D_{ih})k
\end{aligned}$$

$$\begin{aligned}
k_{,y_h y_k} &= \frac{\partial}{\partial y_h}(\beta k X_h) \\
&= \beta(k_{,y_k X_h} + k X_{h,y_k}) \\
&= \beta(\beta k X_k X_h - k \delta_{hk} D_{hk}) \\
&= (\beta^2 X_h X_k - \beta \delta_{hk} D_{hk})k
\end{aligned}$$

$$k_{,x_i x_j} = (\beta^2 X_j X_i - \beta \delta_{ij} D_{ij})k \quad (19)$$

$$k_{,x_i y_h} = -(\beta^2 X_h X_i - \beta \delta_{ih} D_{ih})k \quad (20)$$

$$k_{,y_h y_k} = (\beta^2 X_h X_k - \beta \delta_{hk} D_{hk})k \quad (21)$$

want to work out general $\Sigma_{,x_i x_j y_h y_k}$

Comparison between parametrization of George and our parametrization

Coefficients in front of Σ that we need for the kernel wrapper

Test 1:

Let's check that our general expression of the 4th derivative of Σ is correct