Notes - A GP model for shear fields

Karen Y. Ng

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Our Gaussian process "model" for the projected lensing potential

$$\psi(\vec{x}, \vec{y}) \sim N(0, \Sigma(\vec{x}, \vec{y})) \tag{1}$$

which is a scalar field evaluated at the positions $(\vec{x}, \vec{y})$ where we have data points

For inferring the convergence and shear, we need the 2nd spatial derivatives. The subscripts in these WL equations correspond to the spatial coordinates x, y instead of the observation numbers i.e. i, j = 1, 2, . . . , n observations

$$\begin{aligned}
\kappa &= \frac{1}{2} tr(\psi_{,ij}) \\
&= \frac{1}{2}(\psi_{,11} + \psi_{,22}) \\
&= \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}(\psi_{,11} - \psi_{,22}) \\
&= \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \frac{1}{2}(\psi_{,12} + \psi_{,21}) \\
&= \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x \partial y} + \frac{\partial^2\psi}{\partial y \partial x}\right)
\end{aligned}$$

Covariances of the required functions

Note that ψ, κ and γ are scalar fields. However, we are evaluating them at the locations of the data points (x_i, y_i), therefore, when we are writing down the shorthand for the i, j subscripts below, we mean, we first take the spatial derivatives of those scalar field(s) with respect to x or y, then evaluate them at (x_i, y_i). The spatial derivatives are represented as follows:

$$\psi_{,1} = \frac{\partial \psi}{\partial x}$$

etc. with a comma in the subscript.

Also note expectation and derivative are both linear operators, so we can exchange their positions (and try not to let mathemticians read this and shoot us)

$$
\begin{aligned}
Cov_{ij}(\kappa) &= \mathbb{E}\left[(\kappa - \mathbb{E}[\kappa])|_i(\kappa - \mathbb{E}[\kappa])|_j\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\Big|_i \left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\Big|_j\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_i \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_j\right] \\
&= \frac{1}{4}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial y^4}\right)\Sigma_{ij}
\end{aligned}
$$

$$
Cov(\kappa) = \frac{1}{4}\left(\Sigma_{,1111} + \Sigma_{,1122} + \Sigma_{,2211} + \Sigma_{,2222}\right) \tag{2}
$$

Similarly,

$$
\begin{aligned}
Cov_{ij}(\gamma_1) &= \mathbb{E}\left[(\gamma_1 - \mathbb{E}[\gamma_1])|_i(\gamma_1 - \mathbb{E}[\gamma_1])|_j\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\Big|_i \left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi\right]\right]\Big|_j\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_i \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)[\psi - \mathbb{E}[\psi]]|_j\right] \\
&= \frac{1}{4}\left(\frac{\partial^4}{\partial x^4} - \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial y^4}\right)\Sigma_{ij}
\end{aligned}
$$

$$
Cov(\gamma_1) = \frac{1}{4}\left(\Sigma_{,1111} - \Sigma_{,1122} - \Sigma_{,2211} + \Sigma_{,2222}\right) \tag{3}
$$

And,

$$
\begin{aligned}
&Cov_{ij}(\gamma_2) \\
&= \mathbb{E}\left[(\gamma_2 - \mathbb{E}[\gamma_2])|_i(\gamma_2 - \mathbb{E}[\gamma_2])|_j\right] \\
&= \mathbb{E}\left[\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi\right]\right]\Big|_i \left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi - \mathbb{E}\left[\frac{1}{2}\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)\psi\right]\right]\Big|_j\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)[\psi - \mathbb{E}[\psi]]|_i \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)[\psi - \mathbb{E}[\psi]]|_j\right]
\end{aligned}
$$

$$
Cov(\gamma_2) = \frac{1}{4}\left(\Sigma_{,1212} + \Sigma_{,1221} + \Sigma_{,2112} + \Sigma_{,2121}\right) \tag{4}
$$

The squared exponential covariance function

$$\Sigma(r^2; \lambda, \rho) = \lambda^{-1} \exp\left(-\frac{\beta}{2} r^2\right) \tag{5}$$

where $\beta = -1/4 \ln \rho$, and $0 < \rho < 1$, note Σ is an N $\times$ N matrix and the covariance functions of the derivatives should have the same dimension.

Summary: basic derivatives with the preceeding coefficients

$$\frac{\partial r^2}{\partial x_i} = [(D + D^T)(\vec{x} - \vec{y})]_i \tag{6}$$

$$\frac{\partial r^2}{\partial y_i} = -[(D + D^T)(\vec{x} - \vec{y})]_i \tag{7}$$

$$\mathrm{Hess}(r^2(\vec{x}, \vec{y})) = (D + D^T) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{8}$$

Comparison between parametrization of George and our parametrization