

# Notes - A GP model for shear fields

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December 19, 2014

## Our Gaussian process “model” for the projected lensing potential

$$\psi \sim N(0, \Sigma(\vec{x}, \vec{y})) \quad (1)$$

which is a scalar field evaluated at the positions  $x_i = (\vec{x}_1, \vec{x}_2)_i$  or  $y_i = (\vec{y}_1, \vec{y}_2)_i$  where we have data points. As usual, first column,  $x_1$  or  $y_1$  is the first spatial dimension,  $x_2$  or  $y_2$  is the second one, the  $i$ -th row correspond to spatial coordinates of the  $i$ -th data point.

$\vec{x}$  and  $\vec{y}$  are the same but we call them different names for denoting their location in the covariance matrix  
....

The convergence and shear,  $\kappa, \gamma_1, \gamma_2$  are the 2nd spatial derivatives of the lensing potential. The subscripts in these WL equations correspond to the spatial coordinates  $x_1, x_2$  NOT the observation numbers i.e.  $i, j = 1, 2, \dots, n$  observations

$$\begin{aligned} \kappa &= \frac{1}{2} \text{tr}(\psi_{,ij}) \\ &= \frac{1}{2} (\psi_{,11} + \psi_{,22}) \\ &= \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} \right) \end{aligned}$$

$$\begin{aligned} \gamma_1 &= \frac{1}{2} (\psi_{,11} - \psi_{,22}) \\ &= \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_2^2} \right) \end{aligned}$$

$$\begin{aligned} \gamma_2 &= \frac{1}{2} (\psi_{,12} + \psi_{,21}) \\ &= \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_2 \partial x_1} \right) \end{aligned}$$

## Covariances of the required functions

Note that  $\psi, \kappa$  and  $\gamma$  are scalar fields. However, we are evaluating them at the locations of the data points  $(x_1, x_2)_i$ , therefore, when we are writing down the shorthand for the  $m, n$  subscripts below, we mean, we first

take the spatial derivatives of those scalar field(s) with respect to  $x_1$  or  $x_2$ , then evaluate them at the  $m$ -th or  $n$ -th position  $(x_1, x_2)_m$ . The spatial derivatives are represented as follows:

$$\psi_{,1} = \frac{\partial \psi}{\partial x_1}$$

etc. with a comma in the subscript.

Also note expectation and derivative are both linear operators, so we can exchange their positions (and try not to let mathematicians read this and shoot us)

$$\begin{aligned} \text{Cov}_{m,n}(\kappa(\vec{x}), \kappa(\vec{y})) &= \mathbb{E}[(\kappa - \mathbb{E}[\kappa])|_m (\kappa - \mathbb{E}[\kappa])|_n] \\ &= \mathbb{E} \left[ \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \psi - \mathbb{E} \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \psi \right] \right] \Big|_m \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \psi - \mathbb{E} \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \psi \right] \right] \Big|_n \right] \\ &= \frac{1}{4} \mathbb{E} \left[ \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \Big|_m [\psi - \mathbb{E}[\psi]] \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \Big|_n [\psi - \mathbb{E}[\psi]] \right] \\ &= \frac{1}{4} \left( \left( \frac{\partial^2}{\partial x_1^2} \right) \Big|_m \left( \frac{\partial^2}{\partial y_1^2} \right) \Big|_n + \left( \frac{\partial^2}{\partial x_1^2} \right) \Big|_m \left( \frac{\partial^2}{\partial y_2^2} \right) \Big|_n + \left( \frac{\partial^2}{\partial x_2^2} \right) \Big|_m \left( \frac{\partial^2}{\partial y_1^2} \right) \Big|_n + \left( \frac{\partial^2}{\partial x_2^2} \right) \Big|_m \left( \frac{\partial^2}{\partial y_2^2} \right) \Big|_n \right) \Sigma_{mn} \\ &= \frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_2^2} \right) \Sigma_{mn} \end{aligned}$$

Now I have dropped the (m,n) subscripts and the following subscripts correspond to the spatial dimensions, the first two subscripts correspond to spatial derivatives w.r.t.  $x$  and evaluated for the  $m$ -th data points, the last two correspond to spatial derivatives w.r.t.  $y$  and evaluated for the  $n$ -th data points.

$$\text{Cov}(\kappa(\vec{x}), \kappa(\vec{y})) = \frac{1}{4} (\Sigma_{,1111} + \Sigma_{,1122} + \Sigma_{,2211} + \Sigma_{,2222}) \quad (2)$$

Similarly,

$$\begin{aligned} \text{Cov}_{mn}(\gamma_1(\vec{x}), \gamma_1(\vec{y})) &= \mathbb{E}[(\gamma_1 - \mathbb{E}[\gamma_1])|_m (\gamma_1 - \mathbb{E}[\gamma_1])|_n] \\ &= \mathbb{E} \left[ \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \psi - \mathbb{E} \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \psi \right] \right] \Big|_m \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} \right) \psi - \mathbb{E} \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} \right) \psi \right] \right] \Big|_n \right] \\ &= \frac{1}{4} \mathbb{E} \left[ \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \Big|_i [\psi - \mathbb{E}[\psi]] \left( \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} \right) \Big|_j [\psi - \mathbb{E}[\psi]] \right] \\ &= \frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} \frac{\partial^2}{\partial y_2^2} \right) \Sigma_{mn} \end{aligned}$$

$$\text{Cov}(\gamma_1(\vec{x}), \gamma_1(\vec{y})) = \frac{1}{4} (\Sigma_{,1111} - \Sigma_{,1122} - \Sigma_{,2211} + \Sigma_{,2222}) \quad (3)$$

And,

$$\begin{aligned}
\text{Cov}_{mn}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) &= \mathbb{E}[(\gamma_2 - \mathbb{E}[\gamma_2])|_m (\gamma_2 - \mathbb{E}[\gamma_2])|_n] \\
&= \mathbb{E} \left[ \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1} \right) \psi - \mathbb{E} \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1} \right) \psi \right] \right] \Big|_m \right. \\
&\quad \left. \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1} \right) \psi - \mathbb{E} \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1} \right) \psi \right] \right] \Big|_n \right]
\end{aligned}$$

$$\text{Cov}_{mn}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) = \frac{1}{4} \mathbb{E} \left[ \left( \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_1} \right) \Big|_m [\psi - \mathbb{E}[\psi]] \left( \frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2 \partial y_1} \right) \Big|_n [\psi - \mathbb{E}[\psi]] \right]$$

$$\text{Cov}(\gamma_2(\vec{x}), \gamma_2(\vec{y})) = \frac{1}{4} (\Sigma_{,1212} + \Sigma_{,1221} + \Sigma_{,2112} + \Sigma_{,2121}) \quad (4)$$

$$\text{Cov}(\kappa(\vec{x}), \gamma_1(\vec{y})) = \frac{1}{4} (\Sigma_{,1111} + \Sigma_{,2211} - \Sigma_{,1122} - \Sigma_{,2222}) \quad (5)$$

$$\text{Cov}(\kappa(\vec{x}), \gamma_2(\vec{y})) = \frac{1}{4} (\Sigma_{,1112} + \Sigma_{,2212} + \Sigma_{,1121} + \Sigma_{,2221}) \quad (6)$$

$$\text{Cov}(\gamma_1(\vec{x}), \gamma_2(\vec{y})) = \frac{1}{4} (\Sigma_{,1112} + \Sigma_{,1121} - \Sigma_{,2212} - \Sigma_{,2221}) \quad (7)$$

## The squared exponential covariance function

$$\Sigma(r^2; \lambda, \rho) = \lambda^{-1} \exp \left( -\frac{\beta}{2} r^2 \right) \quad (8)$$

where  $\beta = -1/4 \ln \rho$ , and  $0 < \rho < 1$ , note  $\Sigma$  is an  $N \times N$  matrix and the covariance functions of the derivatives should have the same dimension.

## The metric D

$$r^2 = (\vec{x} - \vec{y})^T D (\vec{x} - \vec{y}) \quad (9)$$

Since we are working in projected (2D) space, D is a  $2 \times 2$  matrix. More explicitly, I will use i,j,h,k as subscripts for the spatial dimensions and m, n for the observation number in the GP model:

$$\begin{aligned}
r_{mn}^2 &= (x_{m1} - y_{n1}, x_{m2} - y_{n2}) \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} x_{m1} - y_{n1} \\ x_{m2} - y_{n2} \end{pmatrix} \\
\Sigma_{mn} &= \lambda^{-1} \exp \left( -\frac{\beta}{2} r_{mn}^2 \right)
\end{aligned}$$

An example of  $r^2$  with an Euclidean metric for a pair of data points,  $\vec{x}_i$  and  $\vec{y}_j$  would be:

$$r_{mn}^2 = D_{11}(x_{m1} - y_{n1})^2 + D_{22}(x_{m2} - y_{n2})^2 \quad (10)$$

assuming diagonal metric.

In the following derivations, it is NOT important to keep the  $m, n$  subscripts. We are taking the derivatives w.r.t to the spatial dimensions, so I will drop the  $m, n$  subscripts. But keep in mind each forth derivative of  $\Sigma$  with the indices written out should be a scalar, each is an element in the big  $m \times n \Sigma_{,x_i x_j y_h y_k}$  matrix.

## Summary: basic derivatives of components, assuming diagonal D

Derivation of the derivatives for a non-symmetric / non-diagonal D would give more terms ...

$$\begin{aligned} \frac{\partial r^2}{\partial x_i} &= \frac{\partial(x_q - y_q)D_{qr}(x_r - y_r)}{\partial x_i} \\ &= \delta_{iq}D_{qr}(x_r - y_r) + (x_q - y_q)\delta_{iq}D_{qr} \\ &= 2D_{ii}(x_i - y_i) \\ &= 2[D(\vec{x} - \vec{y})]_i \end{aligned}$$

The third line in the above derivation is only true for diagonal D or else there are other summation terms.

$$r_{,x_i} = 2[D(\vec{x} - \vec{y})]_i \equiv 2X_i \quad (11)$$

Similarly,

$$\begin{aligned} \frac{\partial r^2}{\partial y_h} &= \frac{\partial(x_q - y_q)D_{qr}(x_r - y_r)}{\partial y_h} \\ &= -\delta_{iq}D_{qr}(x_r - y_r) - (x_q - y_q)\delta_{iq}D_{qr} \\ &= -2[D(\vec{x} - \vec{y})]_h \end{aligned}$$

$$r_{,y_h} = -2[D(\vec{x} - \vec{y})]_h \equiv -2X_h \quad (12)$$

where  $i = 1, 2$

## Second derivatives of $r^2$

$$r_{,x_i x_j}^2 = 2\delta_{ij}D_{ij} \quad (13)$$

$$r_{,y_h y_k}^2 = 2\delta_{hk}D_{hk} \quad (14)$$

$$r_{,x_i y_h}^2 = -2\delta_{ih}D_{ih} \quad (15)$$

$$X_i = [D(\vec{x} - \vec{y})]_i \quad (16)$$

$$X_{i,x_j} = D\delta_{ij} \quad (17)$$

$$X_{i,y_h} = -D\delta_{ih} \quad (18)$$

## Derivatives of the kernel

$$\Sigma = \lambda^{-1}k$$

$$k = \exp\left(\frac{-\beta}{2}r^2\right) \quad (19)$$

$$k_{,x_i} = \frac{-\beta}{2}kr_{,x_i}^2 = -\beta kX_i \quad (20)$$

$$k_{,y_h} = \beta kX_h \quad (21)$$

$$k_{,x_i x_j} = \frac{-\beta}{2}(k_{,x_j}r_{,x_i}^2 + kr_{,x_i x_j}^2) \quad (22)$$

$$k_{,x_i x_j y_h} = \frac{-\beta}{2}(k_{,x_j y_h}r_{,x_i}^2 + k_{,x_j}r_{,x_i y_h}^2 + k_{,y_h}r_{,x_i x_j}^2) \quad (23)$$

$$k_{,x_i x_j y_h y_k} = \frac{-\beta}{2}(k_{,x_j y_h y_k}r_{,x_i}^2 + k_{,x_j y_h}r_{,x_i y_k}^2 + k_{,x_j y_k}r_{,x_i y_h}^2 + k_{,y_h y_k}r_{,x_i x_j}^2) \quad (24)$$

Just work on terms that are parts of the 4th kernel derivative

$$\begin{aligned} k_{,x_i x_j} &= \frac{\partial}{\partial x_j}(-\beta kX_i) \\ &= -\beta(k_{,x_j}X_i + kX_{i,x_j}) \\ &= -\beta(-\beta kX_jX_i + k\delta_{ij}D_{ij}) \\ &= (\beta^2 X_jX_i - \beta\delta_{ij}D_{ij})k \end{aligned}$$

$$\begin{aligned} k_{,x_i y_h} &= \frac{\partial}{\partial y_h}(-\beta kX_i) \\ &= -\beta(k_{,y_h}X_i + kX_{i,y_h}) \\ &= -\beta(\beta kX_hX_i - k\delta_{ih}D_{ih}) \\ &= -(\beta^2 X_hX_i - \beta\delta_{ih}D_{ih})k \end{aligned}$$

$$\begin{aligned} k_{,y_h y_k} &= \frac{\partial}{\partial y_h}(\beta kX_h) \\ &= \beta(k_{,y_k}X_h + kX_{h,y_k}) \\ &= \beta(\beta kX_kX_h - k\delta_{hk}D_{hk}) \\ &= (\beta^2 X_hX_k - \beta\delta_{hk}D_{hk})k \end{aligned}$$

$$k_{,x_i x_j} = (\beta^2 X_jX_i - \beta\delta_{ij}D_{ij})k \quad (25)$$

$$k_{,x_i y_h} = -(\beta^2 X_hX_i - \beta\delta_{ih}D_{ih})k \quad (26)$$

$$k_{,y_h y_k} = (\beta^2 X_hX_k - \beta\delta_{hk}D_{hk})k \quad (27)$$

### Term 1 of the 4th derivative in eqn (24)

$$\begin{aligned}
k_{,x_j y_h y_k} &= \frac{\partial}{\partial y_k} k_{,x_j y_h} \\
&= -\frac{\partial}{\partial y_k} (\beta^2 X_h X_j - \beta \delta_{jh} D_{jh}) k \\
&= (\beta^2 D_{hk} \delta_{hk} X_j + \beta^2 X_h D_{jk} \delta_{jk}) k - (\beta^2 X_h X_j - \beta D_{jh} \delta_{jh}) \beta X_k k
\end{aligned}$$

$$\begin{aligned}
&k_{,x_j y_h y_k} r_{,x_i}^2 \\
&= 2[\beta^2 X_j D_{hk} \delta_{hk} + \beta^2 X_h D_{jk} \delta_{jk} + \beta^2 X_k D_{jh} \delta_{jh} - \beta^3 X_h X_j X_k] X_i k \\
&= \boxed{2\beta^2 [X_j X_i D_{hk} \delta_{hk} + X_h X_i D_{jk} \delta_{jk} + X_k X_i D_{jh} \delta_{jh}] k - 2\beta^3 X_h X_j X_k X_i k}
\end{aligned}$$

### Term 2 of the 4th derivative

$$\begin{aligned}
k_{,x_j y_h} r_{,x_i y_k}^2 &= -(\beta^2 X_h X_j - \beta D_{jh} \delta_{jh}) (-2D_{ik} \delta_{ik} k) \\
&= \boxed{(2\beta^2 X_h X_j D_{ik} \delta_{ik} - 2\beta D_{jh} D_{ik} \delta_{jh} \delta_{ik}) k}
\end{aligned}$$

### Term 3 of the 4th derivative

This is completely analogous to term 2 except the subscripts are slightly different

$$k_{,x_j y_k} r_{,x_i y_h}^2 = \boxed{(2\beta^2 X_k X_j D_{ih} \delta_{ih} - 2\beta D_{jk} D_{ih} \delta_{jk} \delta_{ih}) k}$$

### Term 4 of the 4th derivative

$$\begin{aligned}
k_{,y_h y_k} r_{,x_i x_j}^2 &= (\beta^2 X_k X_h - \beta D_{hk} \delta_{hk}) k 2D_{ij} \delta_{ij} \\
&= \boxed{(2\beta^2 X_k X_h D_{ij} \delta_{ij} - 2\beta D_{ij} D_{hk} \delta_{hk} \delta_{ij}) k}
\end{aligned}$$

### Collect terms of $\Sigma_{,x_i x_j y_h y_k}$ by plugging them in eqn 20

All the relevant terms are boxed above,

$$\nu_{,x_i x_j y_h y_k} = (\beta^4 X_h X_j X_k X_i - \beta^3 (X_j X_i D_{hk} \delta_{hk} + 5\text{perm.}) + \beta^2 (D_{jh} D_{ik} \delta_{jh} \delta_{ik} + 2\text{perm.})) \nu \quad (28)$$

$$= \gamma \nu \quad (29)$$

Where  $\nu$  is an entry in the matrix  $\Sigma$

$$\Sigma = \begin{pmatrix} \nu_{11} & \cdots & \nu_{1n} \\ \vdots & \ddots & \vdots \\ \nu_{n1} & \cdots & \nu_{nn} \end{pmatrix} \quad (30)$$

Note that when we evaluate the terms in the parenthesis, they come out to be a  $n \times n$  matrix, and we should multiply those terms to  $\Sigma$  using a [Schur product](#).

Each spatial derivative result in an extra factor of inverse length in terms of the units. Therefore, the covariance function of the 4th spatial derivative has units of (inverse length)<sup>4</sup>.

## Actual Kernel used

It is customary for people to add a white-noise term to the kernel in the form of:

$$K = \Sigma + \sigma_{noise}^2 I \quad (31)$$

## Gradient function for optimizing hyperparameters

With  $\Gamma$  being the matrix containing all the derivative coefficients in eqn (29), the gradient function can be thought of as

$$g(r^2) = \frac{\partial}{\partial r^2} \Sigma_{,hijk} \quad (32)$$

$$= \Gamma \frac{\partial \Sigma}{\partial r^2} + \frac{\partial \Gamma}{\partial r^2} \Sigma \quad (33)$$

$$= -\frac{\beta}{2} \Gamma \Sigma \quad (34)$$

This is due to equation (11) showing how

$$\frac{\partial X_i}{\partial r^2} = 0.$$

## Conditional distribution to learn from $\gamma_1$ or $\kappa$

Our entire covariance matrix with second derivatives give:

$$\Sigma_{,hijk} = \begin{pmatrix} \kappa\kappa & \kappa\gamma_1 & \kappa\gamma_2 \\ \gamma_1\kappa & \gamma_1\gamma_1 & \gamma_1\gamma_2 \\ \gamma_2\kappa & \gamma_2\gamma_1 & \gamma_2\gamma_2 \end{pmatrix} \quad (35)$$

with a data vector:

$$\vec{d} = \begin{pmatrix} \vec{x}_\kappa \\ \vec{x}_{\gamma_1} \\ \vec{x}_{\gamma_2} \end{pmatrix} \quad (36)$$

$$N(\mu_s, \Sigma_s) = N(\mu_{\kappa\kappa}, \Sigma_{\kappa\kappa} | \vec{d}_{\gamma_1}, \vec{d}_\kappa) \quad (37)$$

$$\Sigma_s = \Sigma_{\kappa\kappa} - \Sigma_{\kappa\gamma_1} \Sigma_{\gamma_1\gamma_1}^{-1} \Sigma_{\gamma_1\kappa} \quad (38)$$

## Implementation details

### Hard coded member variables that should have at most ONE member copy

- `__ix_list__` = actual subscripts on the R.H.S. of eqn. (2 - 7),  $4 \times 4$  in dimension
- `__term_signs__` = signs of the terms on the R.H.S. of (2 - 7),  $4 \times 1$  in dimension
- `__comb_B_ix__` = actual permutation of each of the 4 rows (variations) of `__ix_list__` after taking the order represented by `__pair__of_B_indices__` into account,  $6 \times 4$  in dimension (we have 6 terms of type B, each term has 4 subscripts). In conclusion, `__comb_B_ix__` is going to be  $6 \times 4$  by 4, i.e. 24 by 4.
- `__comb_C_ix__` = actual permutation of each of the 4 rows (variation) of `__ix_list__` after taking the order represented by `__pair__of_C_indices__` into account,  $3 \times 4$  in dimension (we have 3 terms of type C, each term has 4 subscripts) In conclusion, `__comb_C_ix__` is going to be  $3 \times 4$  by 4, i.e. 12 by 4.

### Miscellaneous

- the “distance matrix”  $X_1$  and  $X_2$  should be precomputed / distributed a priori and called when needed.

**Within the virtual class `DerivativeExpSquaredKernel`** The following should only have one copy (per instance)

- hyperparameter  $\beta$
- hyperparameter  $\lambda$
- `__pairs_of_B_indices__` = order of permutations of subscripts order of the second term on the RHS of eqn. (28),  $6 \times 4$  in dimension
- `__pairs_of_C_indices__` = order of permutations of subscripts order of the third term on the RHS of eqn. (28)  $3 \times 4$  in dimension

### Notes

- $\gamma_2$ , unlike  $\kappa$  and  $\gamma_1$  does not have any pair of repeated indices, e.g. 1122, nor 2211 nor 1111 etc., so for small angular separation, only  $\kappa$  and  $\gamma_1$  has increased covariances on the diagonal compared to  $\psi_s$

### Parameters

The variable that the `ExpSquaredKernel` and the `DerivativeKernel` uses is  $l^2 = 1/\beta$ .

### Thoughts on implementation

- The metric object should incorporate the  $\delta_{ij}$  condition for diagonal D, which will kill a lot of terms (sorry for being pedantic about including  $\delta$  since I don't want myself to forget about it)

### Comparison between parametrization of George and our parametrization

#### Test 1:

Let's check that our general expression of the 4th derivative of  $\Sigma$  is correct by working out an example