1 Scale factor as a function of time

From the Friedmann Equation we have

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}[\rho_m + \rho_k + \rho_r + \rho_\Lambda]$$

$$\frac{\dot{a}}{a} = (\frac{8\pi G}{3})^{1/2}[\rho_m + \rho_k + \rho_r + \rho_\Lambda]^{1/2}$$

In homework 1 we worked out that the ith species $\rho_i = \rho_{0i}a^\alpha$ for alpha equal to -4, -3, -2, 0, 2 for relativistic, non-relativistic, lambda, and curvature matter respectively. If we take a flat model universes with only relativistic or non-relativistic matter, we can solve the Friedmann Equation in the following way

$$\frac{\dot{a}}{a} = (\frac{8\pi G}{3})^{1/2}[\rho_i(a)]^{1/2}$$

$$\frac{\dot{a}}{a} = (\frac{8\pi G\rho_{0i}}{3})^{1/2}a^{\alpha/2}$$

$$\frac{da}{dt}\frac{1}{a} = (\frac{8\pi G\rho_{0i}}{3})^{1/2}a^{\alpha/2}$$

$$da\frac{1}{a}a^{-\alpha/2} = (\frac{8\pi G\rho_{0i}}{3})^{1/2}dt$$

$$daa^{-\alpha/2-1} = (\frac{8\pi G\rho_{0i}}{3})^{1/2}dt$$

$$a^{-\alpha/2}\frac{1}{-\alpha/2} = (\frac{8\pi G\rho_{0i}}{3})^{1/2}t + C$$

$$a(t) = [\frac{-t\alpha}{2}(\frac{8\pi G\rho_{0i}}{3})^{1/2} - C\alpha/2]^{-2/\alpha}$$

The initial condition $a(0) = 0$ requires that the integration constant is zero. The other condition $a_0 = a(t_0)$ can be used to show

$$a_0 = [\frac{-t_0\alpha}{2}(\frac{8\pi G\rho_{0i}}{3})^{1/2}]^{-2/\alpha}$$

$$a_0^{-\alpha/2}\frac{2}{-t_0\alpha} = (\frac{8\pi G\rho_{0i}}{3})^{1/2}$$

Finally, for a relativistic matter only universe we have alpha=-4, which gives us the solution

$$a_r(t) = (\frac{t}{t_0})^{1/2}a_0$$

For non-relativistic matter only universe we have alpha = -3, which gives us the solution

$$a_m(t) = (\frac{t}{t_0})^{2/3} a_0$$

The Friedmann equation for a ρ_Λ universe will lead us to

$$da\frac{1}{a} = (\frac{8\pi G\rho_\Lambda}{3})^{1/2} dt$$
$$\ln a = (\frac{8\pi G\rho_\Lambda}{3})^{1/2} t + C$$
$$a(t) = \exp[(\frac{8\pi G\rho_\Lambda}{3})^{1/2} t]C'$$

Enforcing the condition $a(t_0) = a_0$ gives us

$$a_0 = \exp[(\frac{8\pi G\rho_\Lambda}{3})^{1/2} t_0]C'$$

$$\frac{\ln a_0/C'}{t_0} = (\frac{8\pi G\rho_\Lambda}{3})^{1/2}$$

$$a(t) = \exp[\ln(a_0/C')t/t_0]C'$$

2 Equation of state

If we parameterize the equation of state for dark energy with $w(a) = w_0 + w_a(1-a)$, the dark energy density becomes

$$\omega_Q = \omega_{Q0} a^{-3(1+w(a))}$$
$$\omega_Q = \omega_{Q0} a^{-3(1+w_0+w_a(1-a))}$$

3 Homogenius Scalar Field

3.1 Equation of motion

The equation of motion is $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$. We can rewrite H by using the Friedmann equation $\frac{8\pi G}{3}(\rho_m + \rho_\phi) = H^2$. Additionally, our potential $V(\phi) = V_0 e^{-\lambda\phi}$, which we use to evaluate $V'(\phi) = \frac{\partial V}{\partial \phi} = -\lambda V_0 e^{-\lambda\phi} = -\lambda V(\phi)$. Throwing all of this together, the equation of motion becomes

$$\ddot{\phi} + 3\Big[\frac{8\pi G}{3}\rho_{tot}\Big]^{1/2}\dot{\phi} - \lambda V(\phi) = 0$$

Now from the stress energy tensor, we have $\rho_\phi = \dot{\phi}^2 + V(\phi)$. If we require $V(\phi) = \frac{1}{2}\rho_\phi$, we can use the stress energy tensor relation to get the relationship $\rho_\phi = \dot{\phi}^2$. Now the equation of motion is

$$\ddot{\phi} + 3\Big[\frac{8\pi G}{3}\rho_{tot}\rho_\phi\Big]^{1/2} - \frac{\lambda}{2}\rho_\phi = 0$$

Keeping an eye towards what is to come in the next question, we divide through by ρ_ϕ so that the term ρ_{tot}/ρ_ϕ crops up. We then use $\rho_\phi = \dot{\phi}^2$ once more. Now we assert if the matter density is always a fixed fraction of the scalar field density, the fraction ρ_{tot}/ρ_ϕ is fixed for all times. Now we have an equation in terms of ϕ, time, and constants only.

$$\ddot{\phi}\frac{1}{\dot{\phi}^2} + 3\Big[\frac{8\pi G}{3}\frac{\rho_{tot}}{\rho_\phi}\Big]^{1/2} - \frac{\lambda}{2} = 0$$

For the sake of readabiltiy, let us define $3\Big[\frac{8\pi G}{3}\frac{\rho_{tot}}{\rho_\phi}\Big]^{1/2} - \frac{\lambda}{2} = \kappa$. To tackle the differential equation, let us re-write the equation of motion as a first order equation. We define $f = \dot{\phi}, \dot{f} = \ddot{\phi}$. Now we slide things around, separate variables, and integrate.

$$\frac{df}{dt} = -\kappa f^2$$

$$\frac{df}{f^2} = -\kappa dt$$

$$\frac{1}{f} = \kappa t + C_1$$

$$f = \frac{1}{\kappa t - C_1}$$

$$d\phi = \frac{1}{\kappa t - C_1}dt$$

$$\phi(t) = \frac{1}{\kappa}\ln(\kappa t - C_1) + C_2$$

Where C_1, C_2 are integration constants. Now we have shown a solution exists for the EOM given the assumptions in the problem.

3.2 ρ_ϕ/ρ_{tot} in terms of λ

We can eliminate the integration constants from our solution for the scalar field by using the relationship $\rho_\phi = 2V(\phi) = \dot{\phi}^2$

$$\frac{d\phi}{dt} = \sqrt{2V_0}e^{-\lambda\phi/2}$$

$$d\phi e^{\lambda\phi/2} = \sqrt{2V_0}dt$$

$$\frac{2}{\lambda}e^{\lambda\phi/2} = \sqrt{2V_0}t + C_3$$

$$e^{\lambda\phi/2} = \lambda\sqrt{V_0/2}t + \frac{\lambda}{2}C_3$$

$$\phi(t) = \frac{2}{\lambda}\ln\left(\lambda\sqrt{V_0/2}t + \frac{\lambda}{2}C_3\right)$$

Now at time = 0 and $\phi(0) = \phi_0$ we have

$$e^{\lambda\phi_0/2} = \frac{\lambda}{2}C_3$$

$$\frac{2}{\lambda}e^{\lambda\phi_0/2} = C_3$$

$$\phi(t) = \frac{2}{\lambda}\ln\left(\lambda\sqrt{V_0/2}t + e^{\lambda\phi_0/2}\right)$$

Comparing terms of this solution to our previous solution of ϕ which solved the EOM, we must conclude $\frac{2}{\lambda} = \frac{1}{\kappa}$

$$\frac{\lambda}{2} = 3\left[\frac{8\pi G}{3}\frac{\rho_{tot}}{\rho_\phi}\right]^{1/2} - \frac{\lambda}{2}$$

$$\lambda^2 = 24\pi G\frac{\rho_{tot}}{\rho_\phi}$$

$$\frac{24\pi G}{\lambda^2} = \frac{\rho_\phi}{\rho_{tot}}$$

3.3 limits on λ

In order to keep the appropriate form of the potential, λ must be non negative and non zero. Additionally, if this model universe contains matter in addition to the scalar field, the ratio $\frac{\rho_\phi}{\rho_{tot}} < 1$. All this together tells us that

$$\lambda > \sqrt{24\pi G}$$

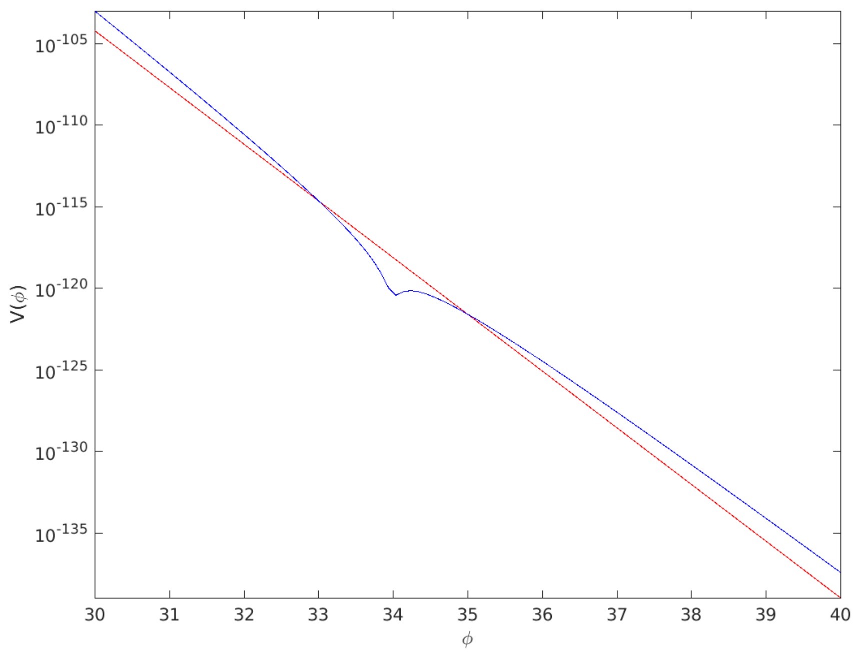

Figure 1: the exponential potential and exponential prefactor as a function of ϕ. The parameters are set to $\lambda = 8$, $\beta = 34$, $V_0 = 1$, $\delta = 0.005$, $\chi = 1$, $\phi_0 = \beta 1\mathrm{e}-3$

4 Numerical solutions

4.1 Exponential prefactor Potential

Please see Figure 1

4.2 Approximate the analytic solution

First we examine the exponential prefactor potential.

$$V(\phi) = V_0(\chi(\phi - \beta)^2 + \delta)e^{-\lambda\phi}$$

For $\phi \ll \beta$ we can expand ϕ/β to first order.

$$V(\phi) = V_0(\chi\phi^2 + \chi\beta^2 - 2\chi\phi\beta) + \delta)e^{-\lambda\phi}$$

$$V(\phi) = V_0\chi\beta^2((\frac{\phi}{\beta})^2 + 1 - 2\chi\frac{\phi}{\beta} + \frac{\delta}{\chi\beta^2})e^{-\lambda\phi}$$

Now since $\phi \ll \beta$, $\delta \ll \beta^2$

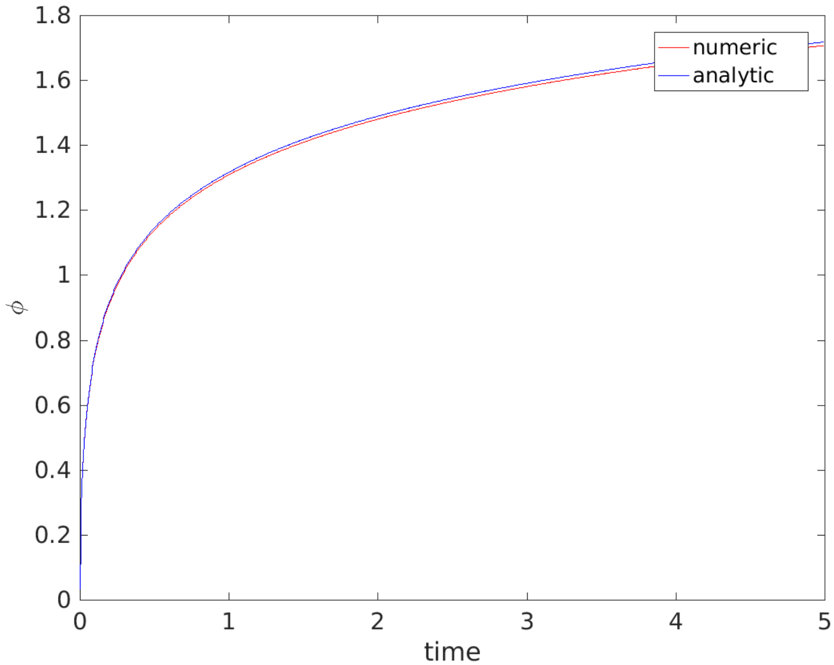

Figure 2: analytic and numerical solutions of ϕ, calculated using the exponential and exponential prefactor solutions, respectively. The parameters are set to $\lambda = 8$, $\beta = 34$, $V_0 = 1$, $\delta = 0.005$, $\chi = 1$, $\phi_0 = \beta 1\mathrm{e}{-3}$. In the small ϕ limit, the solutions are approximately equal

$$V(\phi) \approx V_0 \chi \beta^2 (1 - 2\chi \frac{\phi}{\beta}) e^{-\lambda\phi}$$

$$V(\phi) \approx V_0 \chi \beta^2 e^{-\lambda\phi} - 2V_0 \chi \phi \beta e^{-\lambda\phi}$$

By inspection, in this limit the potential looks like the exponential potential from question 3 with a first order correction term subtracted off. Only now we have to substitute $V_0 \to V_0 \chi \beta^2$.

Now we will compare the numerical solution of $\phi(t)$ to the analytical solution. When calculating The analytical solution, we will use $V(\phi) \approx V_0 \chi \beta^2 e^{-\lambda\phi}$.

4.3 ρ_ϕ and ρ_λ

lol

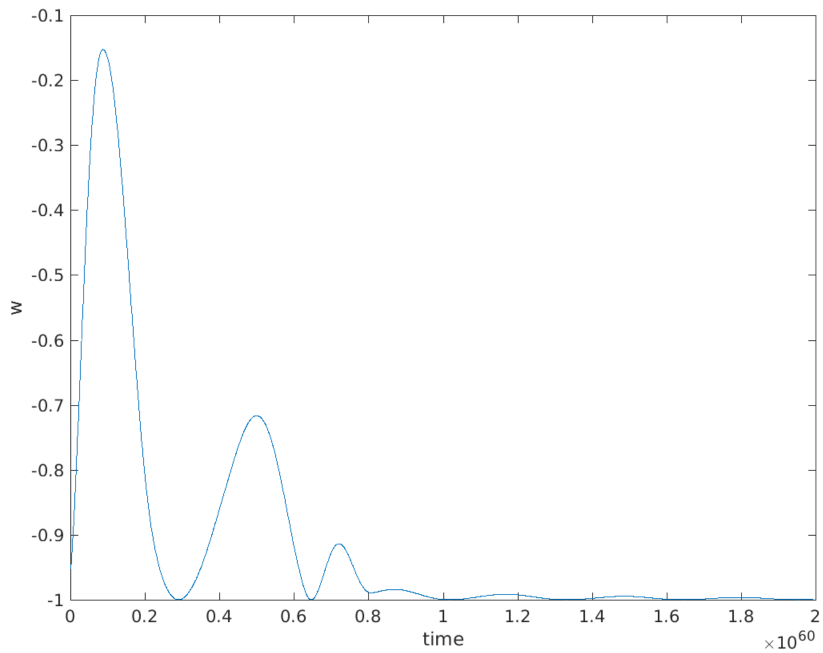

Figure 3: The equation of state for the scalar field ϕ. We use the initial condition $\phi_0 = 33.95$ and $\dot{\phi} =$ 2e-61. The equation of state oscillates but dampens out to -1. For such a scenario, $\dot{\phi} \to 0$

4.4 Equation of state

w_ϕ can be calculated by recalling the relations from the stress energy tensor for the field scalar field ϕ. Namely, $w = p_\phi/\rho_\phi$, $p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$ and $\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)$. Our ODE solver gives us $\dot{\phi}$, and we know the form of the potential, so we can evaluate the equation of state for all time steps. Please see figure 3 for solution.

4.5 Ω_i for different scenarios

Please see figure 4 for the solution using the results of 4.2, and figure 5 for the results of 4.3

4.6 ϕ and $V(\phi)$ vs time

Please see figure 6 below

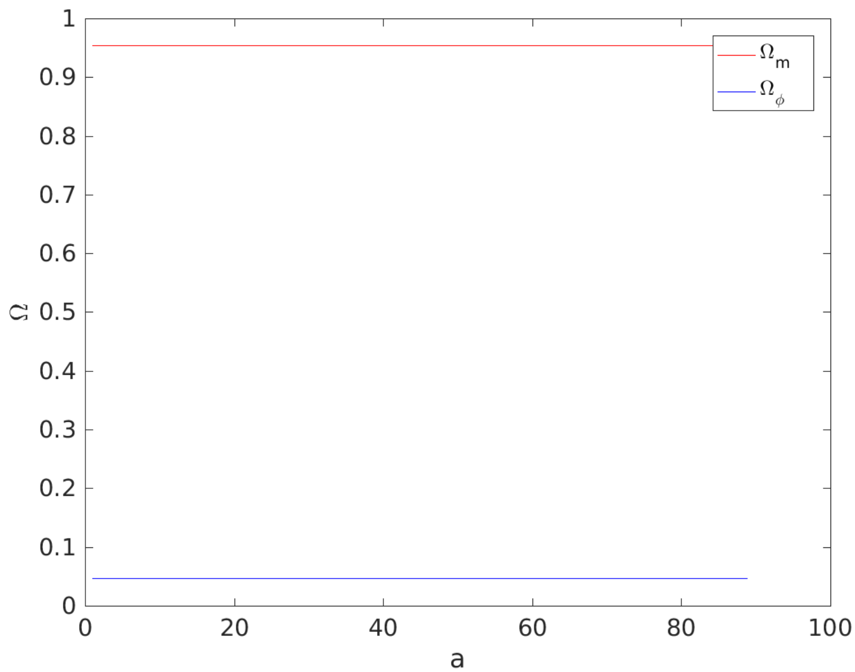

Figure 4:

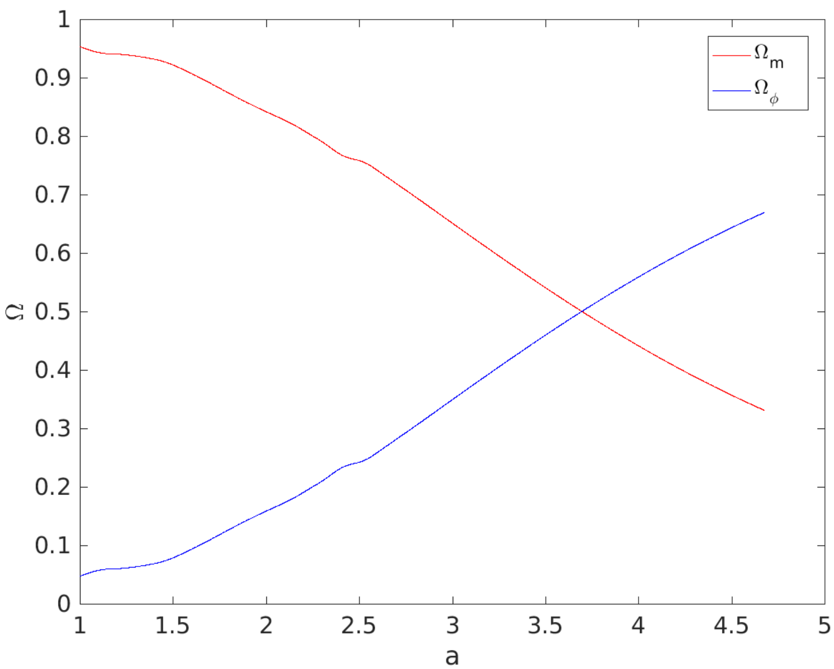

Figure 5: This universe starts out matter dominated, but as time elapses, switches over to a ϕ dominated universe.

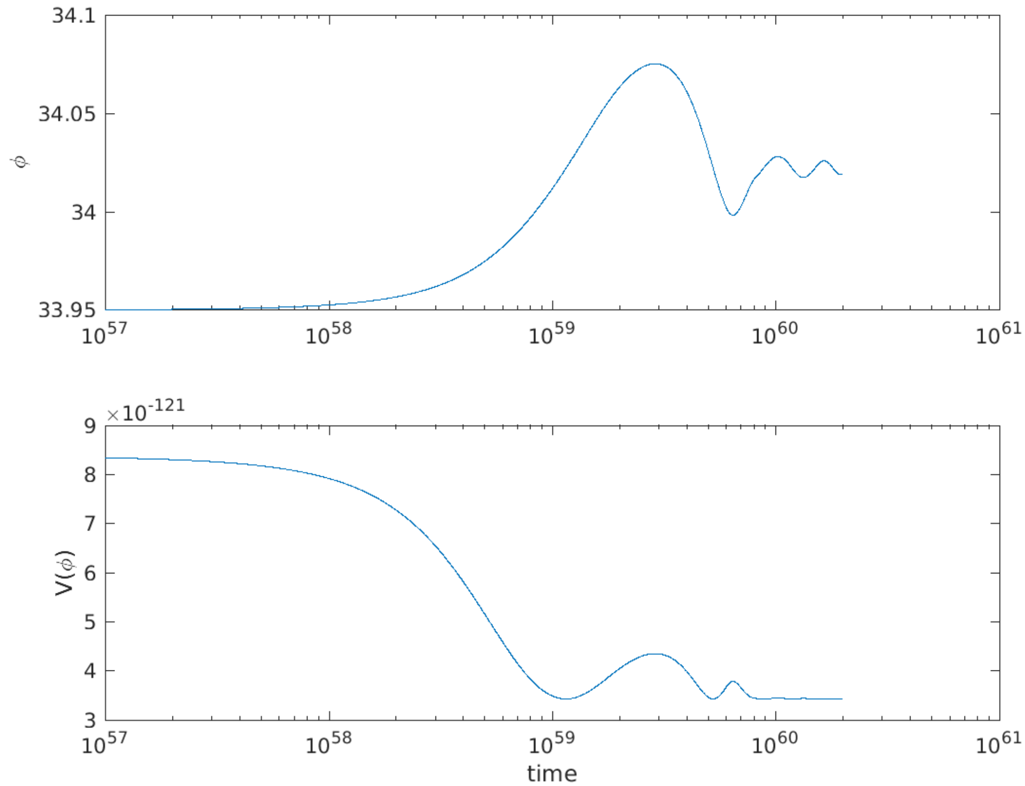

Figure 6: The scalar field drops through the minima of the potential, then oscillates about it. The motion mimics that of a damped harmonic oscillator