

1 Scale factor as a function of time

From the Friedmann Equation we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}[\rho_m + \rho_k + \rho_r + \rho_\Lambda]$$

$$\frac{\dot{a}}{a} = \left(\frac{8\pi G}{3}\right)^{1/2}[\rho_m + \rho_k + \rho_r + \rho_\Lambda]^{1/2}$$

In homework 1 we worked out that the i th species $\rho_i = \rho_{0i}a^\alpha$ for alpha equal to -4, -3, -2, 0, 2 for relativistic, non-relativistic, lambda, and curvature matter respectively. If we take a flat model universes with only relativistic or non-relativistic matter, we can solve the Friedmann Equation in the following way

$$\frac{\dot{a}}{a} = \left(\frac{8\pi G}{3}\right)^{1/2}[\rho_i(a)]^{1/2}$$

$$\frac{\dot{a}}{a} = \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2}a^{\alpha/2}$$

$$\frac{da}{dt} \frac{1}{a} = \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2}a^{\alpha/2}$$

$$da \frac{1}{a} a^{-\alpha/2} = \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2} dt$$

$$da a^{-\alpha/2-1} = \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2} dt$$

$$a^{-\alpha/2} \frac{1}{-\alpha/2} = \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2} t + C$$

$$a(t) = \left[\frac{-t\alpha}{2} \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2} - C\alpha/2\right]^{-2/\alpha}$$

The initial condition $a(0) = 0$ requires that the integration constant is zero. The other condition $a_0 = a(t_0)$ can be used to show

$$a_0 = \left[\frac{-t_0\alpha}{2} \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2}\right]^{-2/\alpha}$$

$$a_0^{-\alpha/2} \frac{2}{-t_0\alpha} = \left(\frac{8\pi G\rho_{0i}}{3}\right)^{1/2}$$

Finally, for a relativistic matter only universe we have alpha=-4, which gives us the solution

$$a_r(t) = \left(\frac{t}{t_0}\right)^{1/2} a_0$$

For non-relativistic matter only universe we have $\alpha = -3$, which gives us the solution

$$a_m(t) = \left(\frac{t}{t_0}\right)^{2/3} a_0$$

The Friedmann equation for a ρ_Λ universe will lead us to

$$\begin{aligned} da \frac{1}{a} &= \left(\frac{8\pi G \rho_\Lambda}{3}\right)^{1/2} dt \\ \ln a &= \left(\frac{8\pi G \rho_\Lambda}{3}\right)^{1/2} t + C \\ a(t) &= \exp\left[\left(\frac{8\pi G \rho_\Lambda}{3}\right)^{1/2} t\right] C' \end{aligned}$$

Enforcing the condition $a(t_0) = a_0$ gives us

$$\begin{aligned} a_0 &= \exp\left[\left(\frac{8\pi G \rho_\Lambda}{3}\right)^{1/2} t_0\right] C' \\ \frac{\ln a_0 / C'}{t_0} &= \left(\frac{8\pi G \rho_\Lambda}{3}\right)^{1/2} \\ a(t) &= \exp[\ln(a_0 / C') t / t_0] C' \end{aligned}$$

2 Equation of state

If we parameterize the equation of state for dark energy with $w(a) = w_0 + w_a(1 - a)$, the dark energy density becomes

$$\begin{aligned} \omega_Q &= \omega_{Q0} a^{-3(1+w(a))} \\ \omega_Q &= \omega_{Q0} a^{-3(1+w_0+w_a(1-a))} \end{aligned}$$

3 Homogenous Scalar Field

3.1 Equation of motion

The equation of motion is $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$. We can rewrite H by using the Friedmann equation $\frac{8\pi G}{3}(\rho_m + \rho_\phi) = H^2$. Additionally, our potential $V(\phi) = V_0 e^{-\lambda\phi}$, which we use to evaluate $V'(\phi) = \frac{\partial V}{\partial \phi} = -\lambda V_0 e^{-\lambda\phi} = -\lambda V(\phi)$. Throwing all of this together, the equation of motion becomes

$$\ddot{\phi} + 3 \left[\frac{8\pi G}{3} \rho_{tot} \right]^{1/2} \dot{\phi} - \lambda V(\phi) = 0$$

Now from the stress energy tensor, we have $\rho_\phi = \dot{\phi}^2 + V(\phi)$. If we require $V(\phi) = \frac{1}{2}\rho_\phi$, we can use the stress energy tensor relation to get the relationship $\rho_\phi = \dot{\phi}^2$. Now the equation of motion is

$$\ddot{\phi} + 3 \left[\frac{8\pi G}{3} \rho_{tot} \rho_\phi \right]^{1/2} - \frac{\lambda}{2} \rho_\phi = 0$$

Keeping an eye towards what is to come in the next question, we divide through by ρ_ϕ so that the term ρ_{tot}/ρ_ϕ crops up. We then use $\rho_\phi = \dot{\phi}^2$ once more. Now we assert if the matter density is always a fixed fraction of the scalar field density, the fraction ρ_{tot}/ρ_ϕ is fixed for all times. Now we have an equation in terms of ϕ , time, and constants only.

$$\ddot{\phi} \frac{1}{\dot{\phi}^2} + 3 \left[\frac{8\pi G}{3} \frac{\rho_{tot}}{\rho_\phi} \right]^{1/2} - \frac{\lambda}{2} = 0$$

For the sake of readability, let us define $3 \left[\frac{8\pi G}{3} \frac{\rho_{tot}}{\rho_\phi} \right]^{1/2} - \frac{\lambda}{2} = \kappa$. To tackle the differential equation, let us re-write the equation of motion as a first order equation. We define $f = \dot{\phi}$, $\dot{f} = \ddot{\phi}$. Now we slide things around, separate variables, and integrate.

$$\frac{df}{dt} = -\kappa f^2$$

$$\frac{df}{f^2} = -\kappa dt$$

$$\frac{1}{f} = \kappa t + C_1$$

$$f = \frac{1}{\kappa t - C_1}$$

$$d\phi = \frac{1}{\kappa t - C_1} dt$$

$$\phi(t) = \frac{1}{\kappa} \ln(\kappa t - C_1) + C_2$$

Where C_1, C_2 are integration constants. Now we have shown a solution exists for the EOM given the assumptions in the problem.

3.2 ρ_ϕ/ρ_{tot} in terms of λ

We can eliminate the integration constants from our solution for the scalar field by using the relationship $\rho_\phi = 2V(\phi) = \dot{\phi}^2$

$$\begin{aligned}\frac{d\phi}{dt} &= \sqrt{2V_0}e^{-\lambda\phi/2} \\ d\phi e^{\lambda\phi/2} &= \sqrt{2V_0}dt \\ \frac{2}{\lambda}e^{\lambda\phi/2} &= \sqrt{2V_0}t + C_3 \\ e^{\lambda\phi/2} &= \lambda\sqrt{V_0/2}t + \frac{\lambda}{2}C_3 \\ \phi(t) &= \frac{2}{\lambda} \ln \left(\lambda\sqrt{V_0/2}t + \frac{\lambda}{2}C_3 \right)\end{aligned}$$

Now at time = 0 and $\phi(0) = \phi_0$ we have

$$\begin{aligned}e^{\lambda\phi_0/2} &= \frac{\lambda}{2}C_3 \\ \frac{2}{\lambda}e^{\lambda\phi_0/2} &= C_3 \\ \phi(t) &= \frac{2}{\lambda} \ln \left(\lambda\sqrt{V_0/2}t + e^{\lambda\phi_0/2} \right)\end{aligned}$$

Comparing terms of this solution to our previous solution of ϕ which solved the EOM, we must conclude $\frac{2}{\lambda} = \frac{1}{\kappa}$

$$\begin{aligned}\frac{\lambda}{2} &= 3 \left[\frac{8\pi G}{3} \frac{\rho_{tot}}{\rho_\phi} \right]^{1/2} - \frac{\lambda}{2} \\ \lambda^2 &= 24\pi G \frac{\rho_{tot}}{\rho_\phi} \\ \frac{24\pi G}{\lambda^2} &= \frac{\rho_\phi}{\rho_{tot}}\end{aligned}$$

3.3 limits on λ

In order to keep the appropriate form of the potential, λ must be non negative and non zero. Additionally, if this model universe contains matter in addition to the scalar field, the ratio $\frac{\rho_\phi}{\rho_{tot}} < 1$. All this together tells us that

$$\lambda > \sqrt{24\pi G}$$