Chapter – 01(Introduction)

Q.1 What is Numerical Method? Briefly describe the scope of numerical methods.

***Numerical Methods*:** Numerical Methods is the study of algorithms that use numerical for the problems of mathematical analysis. Numerical analysis involves the study of methods of computing numerical data. In many problems this implies producing a sequence of approximations by repeating the procedure again and again.

Numerical methods provide approximations to the problems in question that require Linear Algebra, whereas Differential Equations courses are sometimes bundled with Linear Algebra. No matter how accurate they are they do not, in most cases, provide the exact answer. In some instances working out the exact answer by a different approach may not be possible or may be too time consuming and it is in these cases where numerical methods are most often used.

Scope of Numerical Methods: Numerical methods can be a deep scope, but at their simplest level, they can help to solve many mathematical problems that appear for developing software. The area/scope of numerical methods are describe below:

- Numerical method is the key bridge between non-computer scientists (engineers, economists, medical researchers, mathematicians, scientists and engineers of all flavors) and computer science. It is crucial that computer scientists understand one of the most common styles, languages, and class of problems for which non-computer scientists use computers. Indeed numerical method has always been the leader in computer science.

- Numerical method extends, reviews, and applies in a wonderful way much crucial mathematics related to modeling of all kinds: multivariate calculus, linear algebra, and floating point arithmetic.

- Numerical Method is ultimately the most valuable CS course in opening up future job opportunities in that it provides specific skills that can be used in so many ways. Knowing how to program and knowing it is enough to be useful on many simulation projects.

- At an intellectual level, it is quite important to know something about numerical analysis since there are definite limits to what can be simulated on a computer. Similarly, complexity and algorithmic issues currently prevent the direct simulation of complex problems.

- In Software Engineering the students are writing a system to simulate the generation of large scale software packages - and this has them wondering about numerical integration, and finding zeros of functions given implicitly by other functions.

- To analysis some function/equation numerical method is used:

    - Linearization:
    - Finding Roots of Functions
    - Solving Systems of Equations
    - Numerical Integration and Differentiation

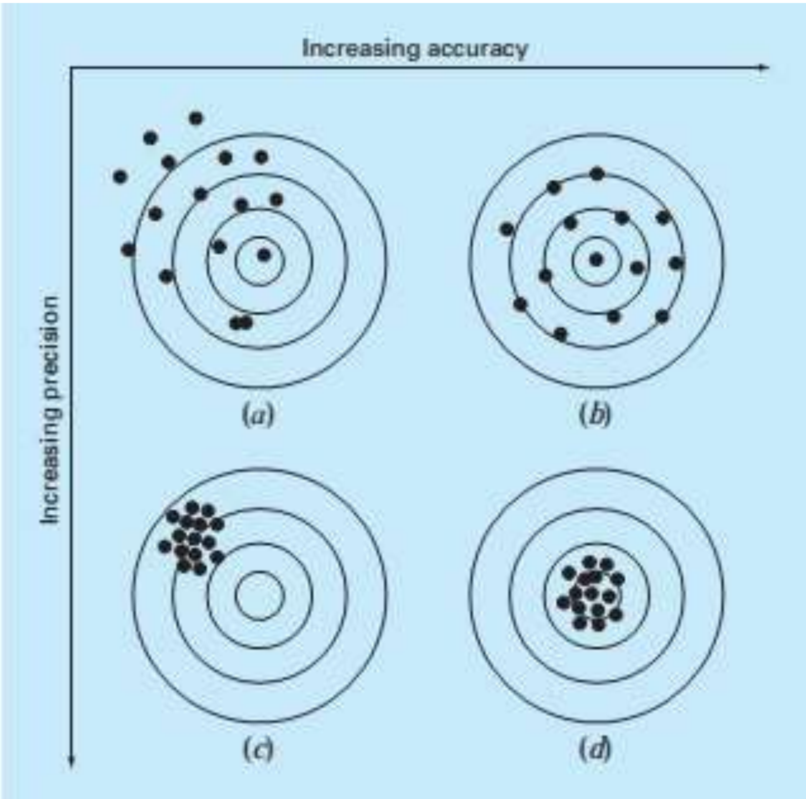
- Some algorithms/methods related to computer graphics and video games to make the AI take smart decisions.

Chapter – 03(Approximation & Round-off Errors)

Q.1 Define Accuracy and Precision. Describe the relation between them.

Accuracy: Accuracy refers to how closely a computed or measured value agrees with the true value. The systematic deviation from the truth is called Inaccuracy also called Bias. This concepts can be illustrated graphically from target practice. Here, the bullet holes on each target can be thought as a predictions of numerical technique, whereas the bull's-eye represents the truth. In the figure, the bullet holes of Fig b are more accurate than the Fig a and it agrees with the true.

Precision: Precision refers to how closely individual computed or measured values agree with each other. The magnitude of the scatter is called Imprecision also called uncertainty. This concepts can be illustrated graphically from target practice. Here, the bullet holes on each target can be thought as a predictions of numerical technique, whereas the bull's-eye represents the truth. In the figure, the bullet holes of Fig c are more precise than the Fig a.



Relation between Accuracy and Precision: The errors associated with both calculations and measurements can be characterized with regard to their accuracy and precision. Numerical methods should be sufficiently accurate or unbiased to meet the requirements of a particular engineering problem. They also should be precise enough for adequate engineering design.

In the figure, although the shots in Fig c are more tightly grouped than those in Fig a, the two cases are equally biased because they are both centered on the upper left quadrant of the target and Fig b and d are equally accurate. The example from marksmanship illustrating the concepts of accuracy and precision are;

(a) Inaccurate and imprecise
(b) accurate and imprecise
(c) inaccurate and precise
(d) accurate and precise.

Q.2 Define Round-off error and Truncation error.

Round-off Error: Round-off error occurs because of the computing device's inability to deal with certain numbers. When numbers having limited significant figures are used to represent exact numbers, this result happened. As an example of round-off error:

Consider the official value of the speed of light in a vacuum = 299,792,458 meters per second.
In scientific notation, that quantity is expressed as 2.99792458×10^8.
Rounding it to three decimal places, we have 2.998×10^8.

Now, Round-off error = the actual value - the rounded value
= $(2.998 - 2.99792458) \times 10^8$
= 0.00007542×10^8
= 7542×10^3, which equals 7542 in plain decimal notation.

Truncation Error: Truncation error refers to the error in a method, which occurs because some series (finite or infinite) is truncated to a fewer number of terms. Such errors are essentially algorithmic errors and when approximations are used to represent exact mathematical procedures, truncation errors occurs.

Q.3 Define Absolute and Relative Error.

Absolute Error: Absolute Error is the magnitude of the difference between the true value and the approximate value. Therefore,

$$\text{Absolute error } (\varepsilon_{abs}) = \text{True value} - \text{Approximate value}$$

Relative Error: The relative error of exact value is the absolute error relative to the exact value. The definition of the relative error is

$$\text{Relative error } (\varepsilon_{rel}) = \frac{\text{True value} - \text{Approximate value}}{\text{True value}}$$

Terms and Equations:

Approximate Error = Current approximation – Previous Approximation

$$\textit{Relative Approximate Error} = \frac{\text{Current approximation} - \text{Previous Approximation}}{\text{Current Approximation}} \times 100\ \%$$

True Error (E_t) = True value – Approximation

$$\textit{True fractional relative error} = \frac{\text{true error}}{\text{true value}}$$

$$\textit{The true percent relative error } (\varepsilon_t) = \frac{\text{true error}}{\text{true value}} \times 100\%$$

N.B: *The signs of **true error and relative approximate error** may be either positive or negative. If the approximation is greater than the true value (or the previous approximation is greater than the current approximation), the error is negative; if the approximation is less than the true value, the error is positive. Also, for **true present relative error and Relative Approximate Error** the denominator may be less than zero, which can also lead to a negative error.*

Q.4 Suppose that, you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute

(a) the true error and

(b) the true percent relative error for each case.

Solution: (a) The error for measuring the bridge and the rivet are;

for bridge, E_t = 10,000 – 9999 = 1 cm

for rivet, E_t = 10 – 9 = 1cm

(b) The percent relative error for the bridge and the rivet are;

for bridge, $\varepsilon_t = \frac{1}{10{,}000}$ x 100% = 0.01%

for bridge, $\varepsilon_t = \frac{1}{10}$ x 100% = 10%

Thus, although both measurements have an error of 1 cm, the relative error for the rivet is much greater.

Q.5 How to estimate error for iterative methods?

or

Q.6 Estimate the value of a Maclaurin series expansion given below.

$$\mathbf{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}}$$

Where true value is $e^{0.5}$ = 1.648721….

and the error criterion ε_s = 0.05%

Solution: Starting with the simplest version, $e^x = 1$ we add terms one at a time to estimate $e^{0.5}$. After each new term is added, we compute the true and approximate percent relative errors.

$e^x = 1$

$e^{0.5} = 1$

For this term, $\varepsilon_t = \frac{true\ value - aproximate\ value}{true\ value}$ x 100%

$\varepsilon_t = \frac{1.648721 - 1}{1.648721}$ x 100%

ε_t = 39.3 %

And, $\varepsilon_a = \frac{Current\ approximation - Previous\ Approximation}{Current\ Approximation}$ x 100 %

ε_a = none

Again; $e^x = 1 + x$

$e^{0.5} = 1 + 0.5 = 1.5$

For this term, $\varepsilon_t = \frac{true\ value - aproximate\ value}{true\ value}$ x 100%

$\varepsilon_t = \frac{1.648721 - 1.5}{1.648721}$ x 100%

ε_t = 9.02 %

And, $\varepsilon_a = \frac{Current\ approximation - Previous\ Approximation}{Current\ Approximation}$ x *100 %*

$\varepsilon_a = \frac{1.5-1}{1.5}$ x *100 %*

ε_a *= 33.3%*

Similarly, we get the following result by adding one term in that series The process is continued until $\varepsilon_a < \varepsilon_s$. The entire computation can be summarized as

Term	Result	ε_t %	ε_a %
1	1	39.3	
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833333	0.175	1.27
5	1.648437500	0.0172	0.0158
6	1.648697917	0.00142	0.0158

Q.7 Evaluate e^{-5} using the following approaches;

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

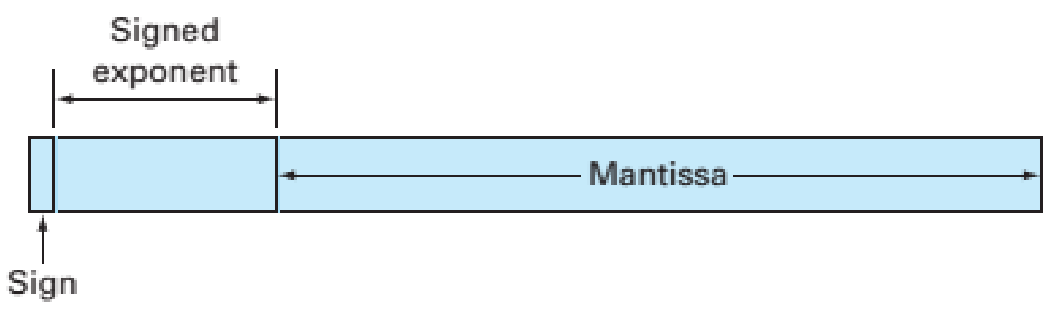
and compare with the true value of 6.737947×10^{-3}. Use 20 terms to evaluate each series and compute true and approximate relative errors as terms are added.

Q.8 Short Note: Floating point representation.

Floating-Point Representation: Fractional quantities are typically represented in computers using floating-point form. In this approach, the number is expressed as a fractional part, called a mantissa or significand and an integer part, called an exponent or characteristic. And the expression is;

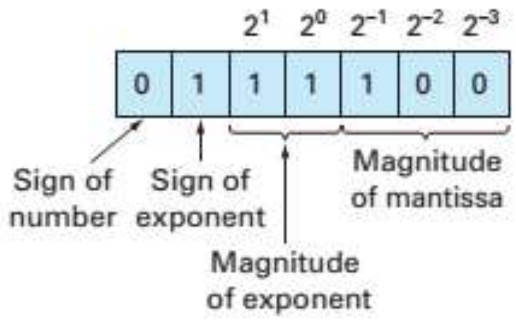
$m.b^e$

where, m = the mantissa, b = the base of the number system being used, and e = the exponent. For instance, the number 156.78 could be represented as 0.15678×10^3 in a floating-point base-10 system. The Figure shows one way that a floating-point number could be stored in a word. The first bit is reserved for the sign, the next series of bits for the signed exponent, and the last bits for the mantissa.



Q.9 Create a hypothetical floating-point number set for a machine that stores information using 7-bit words. Employ the first bit for the sign of the number, the next three for the sign and the magnitude of the exponent, and the last three for the magnitude of the mantissa

Solution: The smallest possible positive number is depicted in Figure. The initial 0 indicates that the quantity is positive. The 1 in the second place designates that the exponent has a negative sign. The 1's in the third and fourth places give a maximum value of exponent.



Max value of exponent, $1\times2^1 + 1\times2^0 = 3$ Therefore, the exponent will be −3.
Finally, the mantissa is specified by the 100 in the last three places, which conforms to

$$1\times2^{-1} + 0\times2^{-2} + 0\times2^{-3} = 0.5$$

This possible positive number for this system is $+ 0.5\times2^{-3}$ which is equal to $(0.0625)_{10}$. The next highest numbers are developed by increasing the mantissa, as in

$$0111101 = (1\times2^{-1} + 0\times2^{-2} + 1\times2^{-3}) \times 2^{-3} = (0.078125)_{10}$$
$$0111110 = (1\times2^{-1} + 1\times2^{-2} + 0\times2^{-3}) \times 2^{-3} = (0.093750)_{10}$$
$$0111111 = (1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3}) \times 2^{-3} = (0.109375)_{10}$$

At this point, to continue increasing, we must decrease the exponent to 10, which gives a value of
$1\times2^1 + 0\times2^0 = 2$ Therefore, the exponent will be −2

Therefore, the next number is;

$$0101100 = (1\times2^{-1} + 0\times2^{-2} + 0\times2^{-3}) \times 2^{-2} = (0.125000)_{10}$$
$$0101101 = (1\times2^{-1} + 0\times2^{-2} + 1\times2^{-3}) \times 2^{-2} = (0.156250)_{10}$$
$$0101110 = (1\times2^{-1} + 1\times2^{-2} + 0\times2^{-3}) \times 2^{-2} = (0.187500)_{10}$$
$$0101111 = (1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3}) \times 2^{-2} = (0.218750)_{10}$$

At this point, to increasing mantissa and decreasing exponent we should change the second bit to 0 for positive sign for exponent to 10. This pattern is repeated as each larger quantity is formulated until a maximum number is reached, and the maximum number is;

$$0011111 = (1\times2^{-1} + 1\times2^{-2} + 1\times2^{-3}) \times 2^{3} = (7)_{10}$$

Chapter – 04(Truncation Errors and the Taylors Series)

Q.1 Define Taylor series. Expressed the Remainder basis on the integral mean-value theorem.

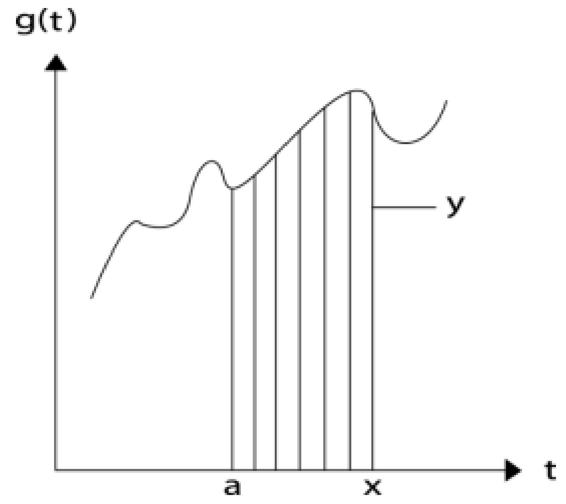
Taylor series: If the function f and its first n+1 derivatives are continuous on an interval containing a and x, then the value of the function at x is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n$$

where the remainder R_n is defined by, $R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$ … … … (i)

Here t = a dummy variable. Equation (i) is but one way, called the integral form, by which the remainder can be expressed. An alternative formulation can be derived on the basis of the integral mean-value theorem.

First Theorem of Mean for Integrals: If the function g is continuous and integrable on an interval containing a and x, then there exists a point ξ between a and x such that

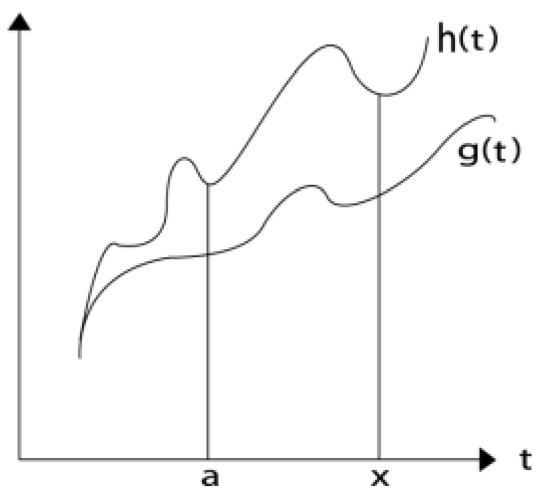


$$\int_a^x g(t)dt = g(\xi)(x-a)$$

This theorem states that the integral can be represented by an average value for the function g(ξ) times the interval length x - a. Because the average must occur between the minimum and maximum values for the interval, there is a point x = ξ at which the function takes on the average value.

Second Theorem of Mean for Integrals: If the functions g and hare continuous and integrable on an interval containing a and x, and h does not change sign in the interval, then there exists a point ξ between a and x such that,

$$\int_a^x g(t)h(t)dt = g(\xi)\int_a^x h(t)dt$$



Thus, equation of first mean value theorem is equivalent to this equation with h(t) = 1. The second theorem can be applied to equation (i) with

$$g(t) = f^{(n+1)}(t)$$

$$h(t) = \frac{(x-t)^n}{n!}$$

As t varies from a to x, h(t) is continuous and does not change sign. Therefore, if $f^{(n+1)}(t)$ is continuous, then the integral mean-value theorem holds and

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x-a)^{n+1}$$

This equation is referred to as the derivative or Lagrange form of the remainder.

Q.2 Use zero-through fourth order Taylor series expansion to approximate the function:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

Or

Q.3 Use zero-through fourth order Taylor series expansion to approximate the function:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

from $x_1 = 0$ with h = 1. That is predict the function value at $x_{i+1} = 1$. Compute E_t and $\mathcal{E}_t$ for each approximation.

Solution: we can compute values for f(x) between $x_i = 0$ and $x_{i+1} = 1$. That means the function starts at f(0) = 1.2 and then curves downward to f(1) = 0.2. Thus, the true value that we are trying to predict is 0.2.

The Taylor series approximation with n = 0 is,

$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$

$f(0) = 1.2$

$f(x_{i+1}) \approx 1.2$

$E_t = 0.2 - 1.2 = -1.0$

$|\mathcal{E}_t| = \frac{0.2-1.2}{0.2} \times 100\% = 500\%$

The Taylor series approximation with n = 1 is,

$f'(x) = -0.4x^3 - 0.45x^2 - 1.0x^1 - 0.25$

$f'(0) = -0.25$

$f(x_{i+1}) \approx 1.2 - 0.25h$

$f(x_{i+1}) \approx 0.95$ [as h = 1]

$E_t = 0.2 - 0.95 = -0.75$

$|\mathcal{E}_t| = 375\%$

The Taylor series approximation with n = 2 is,

$f''(x) = -1.2x^2 - 0.90x^1 - 1.0$

$f''(0) = -1.0$

$\frac{f''(0)}{2!} = -0.5$

$f(x_{i+1}) \approx 1.2 - 0.25h - 0.5h^2$

$f(x_{i+1}) \approx 0.45$ [as h = 1]

$E_t = 0.2 - 0.45 = -0.25$

$|\mathcal{E}_t| = 125\%$

Additional terms would improve the approximation even more. In fact, the inclusion of the third and the fourth derivatives results in exactly the same equation we started with:

$$f(x) = 1.2 - 0.25h - 0.5h^2 - 0.15h^3 - 0.1h^4$$

And $R_4 = \frac{f^5(\xi)}{5!} h^5 = 0$

Because the fifth derivative of a fourth-order polynomial is zero. Consequently, the Taylor series expansion to the fourth derivative yields an exact estimate at $x_{i+1} = 1$:

$$f(1) = 1.2 - 0.25(1) - 0.5(1)^2 - 0.15(1)^3 - 0.1(1)^4$$

$$f(1) = 0.2$$

Q.3 Use zero-through three order Taylor series expansion to predict f(2) for:

$$f(x) = 25x^3 - 6x^2 + 7x - 88$$

using a base point at x = 1. Compute the true present error ε_t for each approximation.

Solution: we can compute values for f(x) between $x_i = 1$ and $x_{i+1} = 2$. That means the function starts at f(1) = - 62 and then curves downward to f(2) = 102. Thus, the true value that we are trying to predict is 102.
The Taylor series approximation with n = 0 is,

$$f(x) = 25x^3 - 6x^2 + 7x - 88$$
$$f(1) = -62$$
$$f(x_{i+1}) \approx -62$$
$$E_t = 102 + 62 = 164$$
$$|\varepsilon_t| = 160.8\%$$

The Taylor series approximation with n = 1 is,

$$f'(x) = 75x^2 - 12x^1 + 7$$
$$f'(1) = 70$$
$$f(x_{i+1}) \approx -62 + 70h$$
$$f(x_{i+1}) \approx 8 \qquad [h = 1;\ h = (x_{i+1} - x_i)]$$
$$E_t = 102 - 8 = 96$$
$$|\varepsilon_t| = 92.1\%$$

The Taylor series approximation with n = 2 is,

$$f''(x) = 150x^1 - 12$$
$$f''(1) = 138$$
$$\frac{f''(1)}{2!} = 69$$
$$f(x_{i+1}) \approx -62 + 70h + 69h^2$$
$$f(x_{i+1}) \approx 77 \qquad [h = 1]$$
$$E_t = 102 - 77 = 5$$
$$|\varepsilon_t| = 24.5\%$$

The Taylor series approximation with n = 3 is,

$$f''(x) = 150$$
$$f''(1) = 150$$
$$\frac{f''(0)}{2!} = 25$$
$$f(x_{i+1}) \approx -62 + 70h + 69h^2 + 25h^3$$
$$f(x_{i+1}) \approx 102 \qquad [h = 1;\ \text{why?}]$$
$$E_t = 102 - 102 = 0$$
$$|\varepsilon_t| = 0\%$$

And $\quad R_3 = \frac{f^4(\xi)}{4!}h^4 = 0$

Because the fifth derivative of a fourth-order polynomial is zero.

Q.4 Use Taylor series expansions with n = 0 to 6 to approximate f(x) = cos x at $x_{i+1} = \pi/3$ on the basis of the value of f(x) and its derivatives at xi = $\pi/4$. [Example of 4.4]

Q.5 Write shortly about Numerical difference (Forward, Backward and Centered).

Forward Difference: Generally the equation of finite divided difference is expressed by:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1}) - (x_i)} + O(x_{i+1} - x_i)$$

Or, $$f'(x_i) = \frac{\Delta f_i}{h} + O(h)$$

where Δf_i is referred to as the ***first forward difference*** and h is called the step size, that is, the length of the interval over which the approximation is made. It is termed a *"forward"* difference because it utilizes data at I and i+1 to estimate the derivative (Figure a).

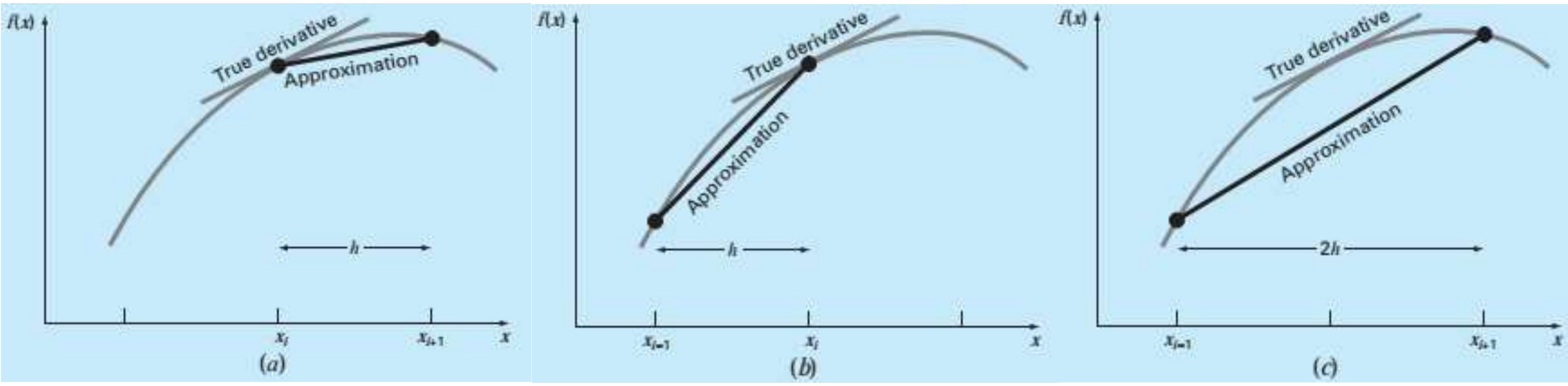
Backward difference: The Taylor series can be expanded backward to calculate a previous value on the basis of a present value as in,

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \cdots$$

Truncating this equation after the first derivative and rearranging yields,

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{(x_i) - (x_{i-1})} = \frac{\nabla f_i}{h}$$

where the error is O(h), and ∇f_i is referred to as the ***first backward difference*** (Figure b).



Centered Difference: A third way to approximate the first derivative is to subtract backward Taylor series expansion from the forward Taylor series expansion. We have,

$$f(x_{i-1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \cdots \qquad \text{[Forward]}$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \cdots \qquad \text{[Backward]}$$

Subtracting backward Taylor series expansion from the forward Taylor series expansion,

$$f(x_{i-1}) = f(x_{i-1}) + 2f'(x_i)h + \frac{2f^{(3)}(x_i)}{3!}h^3 + \cdots$$

Which can be solved, $$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-i})}{2h} - \frac{2f^{(3)}(x_i)}{3!}h^3 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-i})}{2h} \; O(h^2)$$

This is a centered difference representation of the first derivative (Figure c). The centered difference is a more accurate representation of the derivative. If we halve the step size using a forward or backward difference, we would approximately halve the truncation error, whereas for the central difference, the error would be quartered.

Q.6 Use forward and backward difference approximations of O(h) and a centered difference approximation of O(h^2) to estimate the first derivative of

$$\mathbf{f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2}$$

at x = 0.5 using a step size h = 0.5. Repeat the computation using h = 0.25.

Solution: The derivative can be calculated directly from the given equation:

$$f'(x) = -0.4x^3 - 0.45x^2 - 1.0x - 0.25$$

And the true value is, $f'(0.5) = -0.9125$

For h = 0.5, the function can be employed to determine,

$x_{i-1} = 0$	and	$f(x_{i-1}) = 1.2$	[$x_{i-1} = x - h$]
$x_i = 0.5$	and	$f(x_i) = 0.925$	[$x_i = x$ fixed]
$x_{i+1} = 1$	and	$f(x_{i+1}) = 0.2$	[$x_{i+1} = x + h$]

For forward divided difference:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1}) - (x_i)}$$

$$f'(0.5) = -1.45$$

And $|\varepsilon_t| = 58.9\%$

For backward divided difference:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{(x_i) - (x_{i-1})}$$

$$f'(0.5) = -0.55$$

And $|\varepsilon_t| = 39.7\%$

For centered divided difference:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$f'(0.5) = -1.0$$

And $|\varepsilon_t| = 9.6\%$

Here, the centered difference approximation is more accurate than forward or backward differences. Also, as predicted by the Taylor series analysis, halving the step size approximately halves the error of the backward and forward differences and quarters the error of the centered difference.

Q.7 Use a centered difference approximation of O(h^2) to estimate the second derivative of the function:

$$\mathbf{f(x) = 25x^3 - 6x^2 + 7x - 88}$$

Perform the evaluation at x = 2 using step sizes of h = 0.25 and 0.125. Compare your estimates with the true value of the second Derivative.

Chapter – 05(Bracketing Methods)

Q.1 What do you mean by roots of equation?

To solve a quadratic equation we can use the quadratic formula;

$$f(x) = ax^2 + bx + c = 0 \qquad \dots \quad \dots \quad \dots \quad \text{(i)}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \dots \quad \dots \quad \dots \quad \text{(ii)}$$

The value calculated with equation (ii) are called the 'roots' of the equation (i). They represents the value of x that makes equation (i) equal to zero. Thus we can define the root of an equation as the value of x that makes f(x) = 0. For this reason, roots are sometimes called the zeros of the equation.

Q.2. What is Bracketing Method?

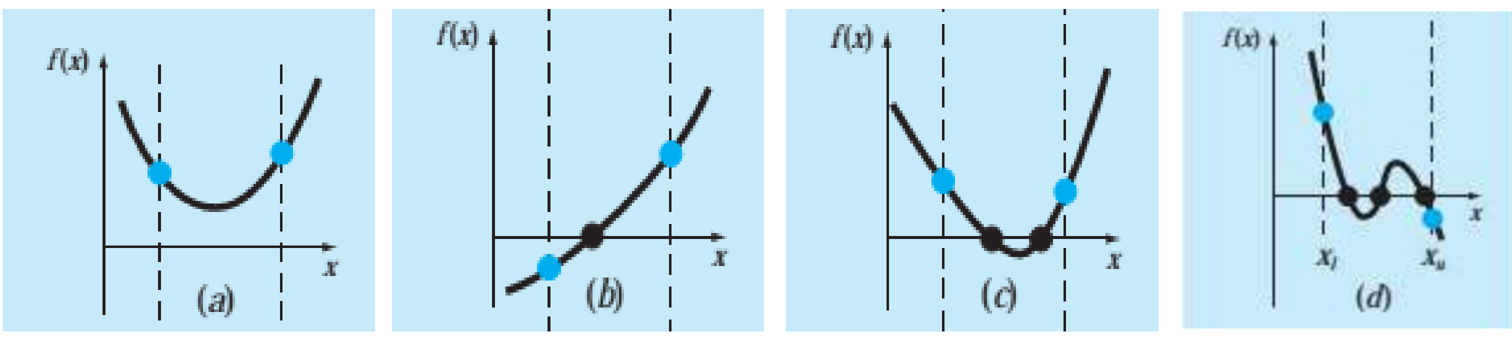
***Bracketing Method*:** Bracketing methods are based on making two initial guesses that "bracket" the root - that is, are on either side of the root. Brackets are formed by finding two guesses x_l and x_u where the sign of the function changes. In general if f(x) is a real and continuous in the interval from x_l and x_u and $f(x_l)$ and $f(x_u)$ have opposites signs, that is;

$$f(x_l) \,.\, f(x_u) < 0$$

then there at least one real root between x_l and x_u.

Q.3 How graphical method can be useful to obtain rough estimates of roots?

A simple method for obtaining an estimate of the root of the equation f(x) = 0 is to make a plot of the function and observe where it crosses the x axis. This point, which represents the x value for which f(x) = 0, provides a rough approximation of the root. Graphing the function can also indicate where roots may be and where some root-finding methods may fail.



Example of Graphical Method:
(a) same sign, No roots (b) Different sign, One root (c) Same sign, Two roots (d) Different sign, Three roots

The estimate of graphical methods (*a rough estimate)* can be employed as starting guesses for other numerical methods. Therefore, graphical interpretation of the problem are useful for understanding the properties of the functions and anticipating the pitfalls of the numerical methods.

Q.4 Determine the drag coefficient c needed for a parachutist of mass m = 68.1 kg to have a velocity of 40 m/s after free-falling for time t = 10 s. Note: The acceleration due to gravity is 9.8 m/s^2. Using the following methods

a) Graphically

b) Bisection methods using the initial guesses of x_l = 12 and x_u = 16 until the approximate error falls below a stopping criterion of ε_s =0.5%.

Solution: a) This problem can be solved by determining the root of equation given below and using the parameters t = 10, g = 9.8, v = 40, and m = 68.1.

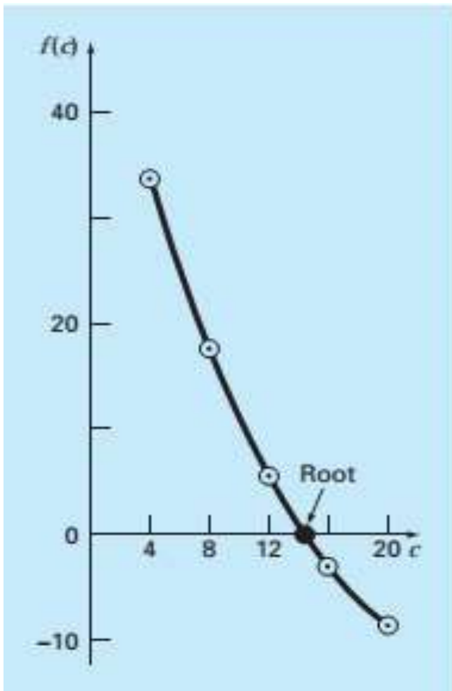
$$f(c) = \frac{gm}{c}(1 - e^{-(c/m)t}) - v$$

$$f(c) = \frac{9.8(68.1)}{c}(1 - e^{-(c/68.1)10}) - 40$$

$$f(c) = \frac{667.38}{c}(1 - e^{-0.146843c}) - 40 \qquad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

For various value of c we have the following calculation. From this calculation the figure if plotted. The resulting curve crosses the c axis between 12 and16. Visual inspection of the plot provides a rough estimate of the root 14.75.

C	f(c)
4	34.115
8	17.653
12	6.067
16	-2.269
20	-8.401



The validity of the graphical estimate can be checked by substituting it into the equation (i) to yield that is very close to zero.

$$f(14.75) = \frac{667.38}{14.75}(1 - e^{-0.146843(4.75)}) - 40 = 0.059$$

b) We have the equation $f(c) = \frac{667.38}{c}(1 - e^{-0.146843c}) - 40$. We can see that the function changes sign between values of 12 and 16. Therefore, the initial estimate of the root x_r lies at the midpoint of the interval,

$$x_r = \frac{x_l + x_u}{2} = \frac{12+16}{2} = 14$$

Now, x_l = 12, x_u = 16 and x_r = 14.

From the equation we have,

$f(12) = 6.067 \qquad x_l = 12$

$f(14) = 1.569 \qquad x_r = 14.$

$f(16) = -2.269 \qquad x_u = 16$

and $f(x_r).f(x_u) < 0$

So new $x_l = 14$ and $x_u = 16$

$$\boldsymbol{\varepsilon_a} = \frac{x_r new - x_r old}{x_r new} \text{ x } 100\% = 0 \text{ (there is no new } x_r\text{) or may be } 100\%$$

If true value (x_t) is known, then we can easily determine the percent of true error ($\boldsymbol{\varepsilon_t}$). Here the true value is 14.7802 that will be given. And the approximate value is $x_a = x_r$. So

$$\boldsymbol{\varepsilon_t} = \frac{x_t - x_a}{x_t} \text{ x } 100\% = 5.279\ \%$$

Continue this process the approximate error falls below a stopping criterion of $\varepsilon_s = 0.5\%$ we have the calculation given below,

Iteration	x_l	x_u	x_r	$\varepsilon_a(\%)$	$\boldsymbol{\varepsilon_t(\%)}$
1	12	16	14		5.279
2	14	16	15	6.667	1.487
3	14	15	14.5	3.448	1.896
4	14.5	15	14.75	1.695	0.204
5	14.75	15	14.875	0.840	0.641
6	14.75	14.875	14.8125	0.422	0.219

Thus, after six iterations ε_a finally falls below $\varepsilon_s = 0.5\%$, and the computation can be terminated.

Q.5 Determine the real roots of $f(x) = -0.4x^2 + 2.2x + 4.7$

(a) Graphically

(b) Using the quadratic formula *(We will get the true value from this formula)*

(c) Using three iterations of the bisection method to determine the highest root. Employ initial guesses of $x_l = 5$ and $x_u = 10$. Compute the estimated error ε_a and the true error ε_t after each iteration.

Q.6 Determine the real root of $f(x) = -26 + 82.3x - 88x^2 + 45.4x^3 - 9x^4 + 0.65x^5$ using bisection to determine the root(highest) to $\varepsilon_s = 10\%$. Employ initial guesses of $x_l = 0.5$ and $x_u = 1.0$.

Q.7 Determine the real root of $f(x) = 6x^3 - 5x^2 + 7x - 2$ using bisection to locate the root(lowest). Employ initial guesses of $x_l = 0$ and $x_u = 1$ and iterate until the estimated error ε_a falls below a level of $\varepsilon_s = 10\%$.

Q.8 Write the algorithm/pseudo-code for bisection method to find the roots of equation.

```
Bisection(Xl, Xu, Xr, Es, Ea, Imax, iter)
{
        iter = 0
        do
        {
                Xr_old = Xr
                Xr = (Xl + Xu) / 2
                iter = iter+1

                if(Xr ≠ 0)
                {
                        Ea = ABS((Xr – Xr_old) / Xr) * 100
                }

                test = f(Xl) * f(Xr)
                if(test < 0)
                {
                        Xu = Xr
                }
                else if(test > 0)
                {
                        Xl = Xr
                }
                else
                {
                        Ea = 0
                }

                if(Ea < Es or iter ≥ imax)
                {
                        Exit
                }
        }

        Bisection = Xr
}
```

Q.9 Describe false-position method and explain it using graphical View (depiction).

The False Position Methods: The false position method is another bracketing method used to find a numerical estimate of an equation. Also called the linear interpolation method. It determines the next guess not by splitting the bracket in half but by connecting the endpoints with a straight line and determining the location of the intercept of the straight line.

This algorithm requires a function *f(x)* and two points x_l and x_u for which *f(x)* is positive for one of the values and negative for the other. We can write this condition as $f(x_l) \cdot f(x_u) < 0$. If $f(x_l)$ is much closer to zero than $f(x_u)$, it is likely that the root is closer to x_l than to x_u plotted in the figure. Using similar triangles the intersection of the straight line with the x axis can be estimated as,

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

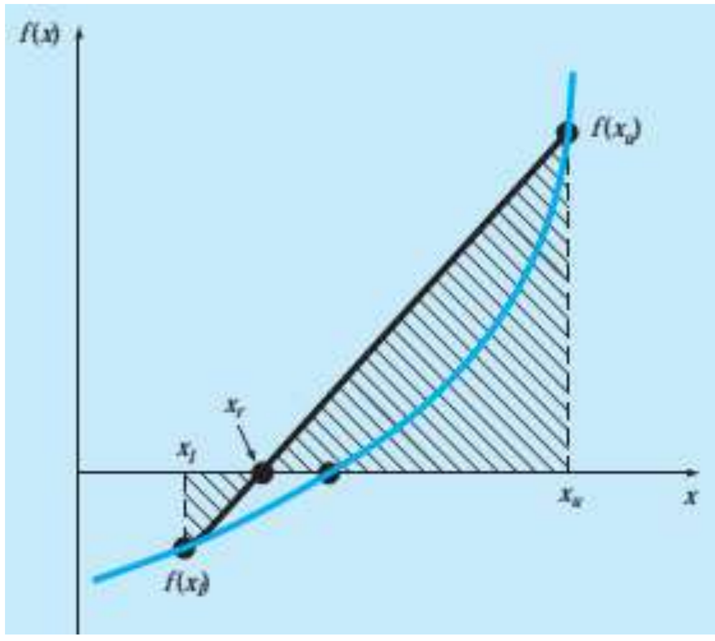
$$f(x_l)\,(x_r - x_u) = f(x_u)\,(x_r - x_l)$$

$$f(x_l)x_r - f(x_l)\,x_u = f(x_u)x_r - f(x_u)\,x_l$$

$$f(x_l)x_r - f(x_u)\,x_r = f(x_l)x_u - f(x_u)\,x_l$$

$$x_r[f(x_l) - f(x_u)] = x_u\,f(x_l) - x_l\,f(x_u)$$

$$x_r = \frac{x_u\,f(x_l) - x_l\,f(x_u)}{f(x_l) - f(x_u)}$$



This is one form of the method of false position. Note that it allows the computation of the root x_r as a function of the lower and upper guesses x_l and x_u.

$$x_r = \frac{x_u\,f(x_l)}{f(x_l) - f(x_u)} - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u + \frac{x_u\,f(x_l)}{f(x_l) - f(x_u)} - x_u - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u + \frac{x_u\,f(x_l) - x_u\,[f(x_l) - f(x_u)]}{f(x_l) - f(x_u)} - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u + \frac{x_u\,f(x_l) - x_u\,f(x_l) + x_u\,f(x_u)]}{f(x_l) - f(x_u)} - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

$$x_r = x_u + \frac{x_u\,f(x_u)}{f(x_l) - f(x_u)} - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

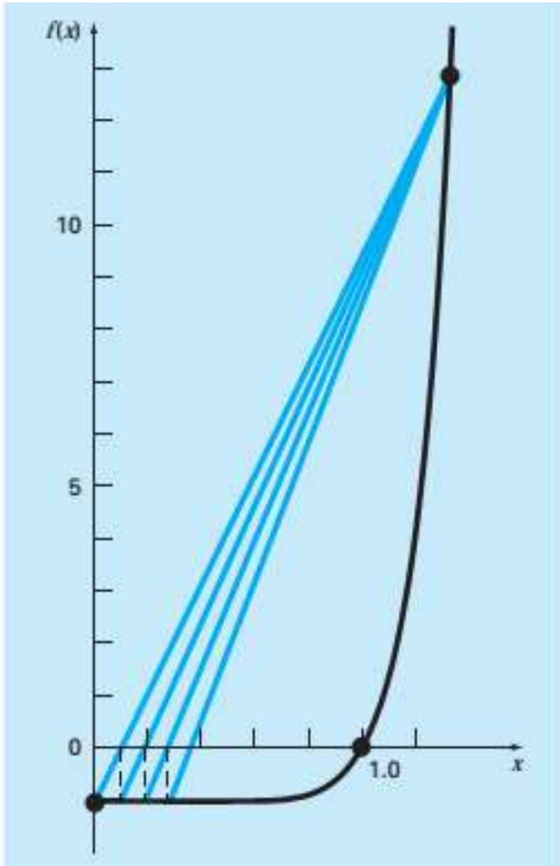
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

This is the false-position formula. The value of x_r computed with this equation, then replaces whichever of the two initial guesses, x_l or x_u, yields a function value with the same sign as $f(x_r)$. In this way, the values of x_l and x_u always bracket the true root. The process is repeated until the root is estimated adequately.

Q.10 Is Bisection is preferable to false position method?

Although bisection is a perfectly valid technique for determining roots, its "brute-force" approach is relatively inefficient. False position is an alternative based on a graphical insight. Thus the false-position method would seem to always be the bracketing method of preference, but there are cases where it performs poorly. In fact, Bisection does not take into account the shape of the function; this can be good or bad depending on the function! For example, in the following equation,

$$f(x) = x^{10} - 1 \qquad \dots \quad \dots \quad \dots \text{(i)}$$



There are certain cases where bisection yields superior results. Using bisection, after five iterations, the true error is reduced to less than 2 percent in bisection methods. But for false position, a very different out - come is obtained. Although false position is often superior to bisection, there are invariably cases that violate this general conclusion. Therefore, in addition to using equation the results should always be checked by substituting the root estimate into the original equation and determining whether the result is close to zero. Such a check should be incorporated into all computer programs for root location

Q.11 Use bisection and false position to locate the root of f(x) = x^{10} – 1 between x = 0 and 1.3. and comment on this result.

Solution: For bisection methods x_l = 0 and x_u = 1.3

$$x_r = \frac{x_l + x_u}{2} = \frac{0+1.3}{2} = 0.65$$

From the equation f(x) = x^{10} – 1 we have,

$$f(0) = -1$$
$$f(1.3) = 12.785$$
$$f(0.65) = -0.986$$

Here, $f(x_r).f(x_u) < 0$ that's why new x_l = 0.65 and x_u = 1.3

$$\boldsymbol{\varepsilon_a} = \frac{x_r\,new - x_r\,old}{x_r new} \text{ x } 100\% = 0 \text{ (there is no new } x_r\text{) or may be } 100\%$$

Solving the given equation, the true value(x_t) is x^{10} – 1 = 0 then x = 1. And the approximate value(x_a) is 0.65. So true error is,

$$\boldsymbol{\varepsilon_t} = \frac{x_t - x_a}{x_t} \text{ x } 100\% = 35\ \%$$

Continuing the process we calculate the table for bisection method:

Iteration	x_l	x_u	x_r	ε_a(100%)	ε_t(100%)
1	0	1.3	0.65		35
2	0.65	1.3	0.975	33.3	2.5
3	0.975	1.3	1.1375	14.3	13.8
4	0.975	1.1375	1.05625	7.7	5.6
5	0.975	1.05625	1.015625	4.0	1.6

Thus, after five iterations, the true error is reduced to less than 2 percent in bisection methods.

Now, in false position method, $x_l = 0$ and $x_u = 1.3$. then

$f(0) = -1$
$f(1.3) = 12.785$

So, $x_r = x_u - \dfrac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$

$x_r = 1.3 - \dfrac{12.785\,(0-1.3)}{-1-12.785}$

$x_r = 0.09430$

and $f(x_r) = -0.9999$ where $f(x_r).f(x_u) < 0$ that's why new $x_l = 0.09430$ and $x_u = 1.3$.

$\boldsymbol{\varepsilon_a} = \dfrac{x_r new - x_r old}{x_r new}$ x 100% = 0 (there is no new x_r) or may be 100%

$\boldsymbol{\varepsilon_t} = \dfrac{x_t - x_a}{x_t}$ x 100% = 90.57 %

Continuing the process we calculate the table for false position method:

Iteration	x_l	x_u	x_r	ε_a(100%)	ε_t(100%)
1	0	1.3	0.0943		90.6
2	0.0943	1.3	0.18176	48.1	281.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2

After five iterations, the true error has only been reduced to about 59 percent. In addition, note that $\varepsilon_a < \varepsilon_t$. Thus, the approximate error is misleading.

Q.12 Find the positive real root of $f(x) = x^4 - 8x^3 - 35x^2 + 450x - 1001$ using False-position method. Use initial guesses of $x_1 = 4.5$, $x_u = 6$ and perform up to $\varepsilon_s = 2\%$.

Q.13 Use false-position method to determine the root of $f(x) = -11 - 22x + 17x^2 - 2.5x^3$ up to $\varepsilon_s = 2\%$.