MODERN ANALYSIS
Problem Pool

I. Real Numbers.

1. Prove that $a \cdot 0 = 0$ for all real numbers a. Use only the axioms of the real numbers.

2. Let A be a non-empty set of real numbers that is bounded above. Let $\alpha = \sup A$ and let $b < \alpha$. Prove there exists $x \in A$ such that $b < x \leq \alpha$.

3. Let A and B be two bounded, non-empty, sets of real numbers. Prove: If $A \subset B$, then $\sup(A) \leq \sup(B)$.

4. Let A, B be non-empty subsets of $\mathbb{R}$; assume A, B are bounded above. Prove that $A+B$ is non-empty and bounded above and that $\sup(A+B) = \sup A + \sup B$.

5. Let A, B be two sets of real numbers. Assume that if $a \in A$ and $b \in B$, then $a < b$. Assume also that for every real number $\epsilon > 0$ there exists $a \in A$, $b \in B$ such that $b - a < \epsilon$. Prove $\sup A, \inf B$ exist and $\sup A = \inf B$.

II. Sequences

1. Consider the sequence
$$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots;$$
that is, the sequence defined by
$$a_n = \frac{1}{k} \qquad \text{if} \quad 2^{k-1} \leq n < 2^k, k = 1, 2, 3, \ldots.$$
Prove that it converges to 0 using only the definition of limit.

2. Prove **only using the definition of limit** that the sequence $\left\{ \dfrac{3n^2}{(2n+1)^2} \right\}$ converges to $3/4$.

3. Let $\{a_n\}$ be a sequence of real numbers; assume that $\lim_{n\to\infty} a_n = a > 0$. Prove there exists $N \in \mathbb{N}$ such that $a_n > a/2$ if $n \geq N$.

4. Let $\{a_n\}$ be a sequence of real numbers. Prove: $\{a_n\}$ converges to L if both subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to L.

III. Continuous functions, functional limits.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Prove using only the definition that $\lim_{x\to 2} f(x) = 4$.

2. Let
$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ x, & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$
Prove: f is continuous at 0, 1; discontinuous everywhere else.

3. Prove: If f is increasing and bounded in the interval (a, b), then $\lim_{x\to b-} f(x), \lim_{x\to a+} f(x)$ exist.

4. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point; that is, there exists $x \in [0, 1]$ such that $f(x) = x$.

5. Show that $f : x \mapsto 1/x : [1, \infty) \to \mathbb{R}$ is uniformly continuous.

IV. Differentiability.

1. Let $f : \mathbb{R}^{\to}\mathbb{R}$ satisfy $|f(x) - f(y)| \leq C|x - y|^{\alpha}$ for all $x, y \in \mathbb{R}$, where $\alpha > 1$. Prove that f is constant.
2. Let $f(x) = x^5 + 4x^3 + 2x + 3$ has exactly one real zero.
3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Assume also that f is differentiable at all $x \neq 0$ and that $\lim_{x\to 0} f'(x)$ exists. Prove that f is differentiable at 0 and $f'(0) = \lim_{x\to 0} f'(x)$.
4. Let $g : (-1, 1) \to \mathbb{R}$ be bounded (there exists M such that $|g(x)| \leq M$ for all $x \in (-1, 1)$). Let $f : (-1, 1) \to \mathbb{R}$ be defined by $f(x) = x^2 g(x)$. Prove that f is differentiable at 0.
5. Prove using only the definition of differentiability (do not use the chain rule!): If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$, then defining g by $g(x) = f(2x + 1)$, g is differentiable at $(a - 1)/2$ and $g'((a - 1)/2) = 2f'(a)$. An easier version is:

   Prove using only the definition of differentiability (do not use the chain rule!): If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$,, then defining g by $g(x) = f(x + 1)$, g is differentiable at $a - 1$ and $g'(a - 1) = f'(a)$.
6. Let $f : [0, 2] \to \mathbb{R}$ be continuous. Assume also that f is twice differentiable in $(0, 2)$. Assume $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$. Prove there exists $c \in (0, 2)$ such that $f''(c) = 0$.
7. Prove that if f is differentiable at a, then f is continuous at a. Give an example of a function continuous at $x = 2$ and not differentiable at $x = 2$.
8. Show that the equation $xe^x = 1$ has exactly one real solution.

IV. Integration.

1. Let $f : [0, 1] \to [0, 1]$ be defined by

   $$f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \in [0, 1]\backslash\mathbb{Q}. \end{cases}$$

   Compute

   $$\overline{\int_0^1} f(x), dx, \quad \underline{\int_0^1} f(x), dx,$$

   and prove f is not Riemann integrable over $[0, 1]$.
2. Let $f : [a, b] \to \mathbb{R}$ be integrable. Prove: $\displaystyle\int_a^b f > 0$ if and only if there exists a positive lower sum. In symbols:

   $$\int_a^b f > 0 \quad \Leftrightarrow \quad \exists \text{ a partition } P \text{ of } [a, b] \text{ such that } \quad L(f, P) > 0.$$
3. (a) State at least one version of the Fundamental Theorem of Calculus.

   (b) Prove: There exists a differentiable function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi'(x) = e^{-x^2}$ for all $x \in \mathbb{R}$ and such that $\Phi(0) = 1$. To prove this you may quote any theorem proved in the course, there is no need to reproduce any proof.