Department of Mathematical Sciences
Charles E. Schmidt College of Science
Florida Atlantic University

COURSE NUMBER & TITLE: MAA 4200, **Modern Analysis**

CATALOG DESCRIPTION:
4 Credit Hours. Continuity, differentiability, differential approximation, optimization and curve sketching of functions and inverse functions of a single variable, including treatment of trigonometric functions. Mean value theorem and L'Hopital's Rule. Introduction to integration.

SUGGESTED TEXTBOOKS:

- Michael Spivak, *Calculus* (3rd ed.), Publish or Perish, Houston, TX 1994.
- Edward Gaughan, *Introduction to Analysis* (5^{th} ed.), AMS, Providence, 1998.

COURSE OBJECTIVES: This course has two objectives. The invention of calculus revolutionized mathematics, transforming mathematics from a static science primarily dedicated to the study of numbers and shapes, to a dynamic one concerned also with the motion of bodies and the notion of change. One objective of this course is to teach students the concepts of single variable calculus from a rigorous point of view. The other one, just as important, is to introduce students to the nature of mathematical reasoning, and the power of proof, via concepts from single variable calculus.

Upon successful completion of this course, the student will be able to:

1. Describe the real numbers axiomatically and understand how the usual properties of these numbers can be derived from the axioms.
2. Understand the concept of limit of a sequence and of a function, evaluate some standard limits, and be able to justify his/her calculations.
3. Prove and apply the basic theorems about continuous functions, such as the intermediate value theorem and the max-min theorem.
4. Understand the concept of differentiability; prove the differentiability of all elementary functions.
5. Prove and apply the intermediate value theorem of differential calculus.
6. Understand the concept of a Riemann integral, prove at least one version of the Fundamental Theorem of Calculus, and be able to apply it.

SUGGESTED SCHEDULE:

Week(s)	Topic(s)
1–2	Real Numbers
3–5	Sequences
5	Midterm Exam #1
6–8	Continuity
8-11	Differentiability
11	Midterm Exam #2
12–14	Integration
15	Final Exam

COURSE REQUIREMENTS:

- **Teaching methodology:** lecture, and class discussion.
- **Assessment procedures:** examinations, and homework assignments.
- The grade for the course will be determined as follows:
  - Daily homework assignments (totaling 10% of the final grade).
  - Two in term exams (each worth 20% of the final grade).
  - A comprehensive final exam (worth 40% of the final grade)

GRADING: The FAU grading scale is as follows:

A = 4.00	C = 2.00
A– = 3.67	C– = 1.67
B+ = 3.33	D+ = 1.33
B = 3.00	D = 1.00
B– = 2.67	D– = 0.67
C+ = 2.33	F = 0.00

ATTENDANCE POLICY: Regular attendance is expected, including active involvement in all class sessions, and professional conduct in class. Students are responsible for arranging to make up work missed because of legitimate class absence, such as illness, family emergencies, military obligation, court-imposed legal obligations, or participation in university-approved activities. It is the student's responsibility to notify the instructor prior to any anticipated absence, and within a reasonable amount of time after an unanticipated absence.

STUDENTS WITH DISABILITIES: In compliance with the Americans with Disabilities Act (ADA), students who require special accommodations due to a disability to properly execute coursework must register with the Office for Students with Disabilities (OSD) located in Boca Raton - SU 133 (561-297-3880), in Davie - MOD I (954-236-1222), in Jupiter - SR 117 (561-799-8585), or at the Treasure Coast – CO 128 (772-873-3305), and follow all OSD procedures.

HONOR CODE: Students at Florida Atlantic University are expected to maintain the highest ethical standards. Academic dishonesty, including cheating and plagiarism, is considered a serious breach of these ethical standards, because it interferes with the University mission to provide a high quality education in which no student enjoys an unfair advantage over any other. Academic dishonesty is also destructive of the University community, which is grounded on a system of mutual trust and places high value on personal integrity and individual responsibility. Harsh penalties are associated with academic dishonesty. For more information, see http://www.fau.edu/regulations/chapter4/4.001_Honor_Code.pdf

MODERN ANALYSIS
Problem Pool

I. Real Numbers.

1. Prove that $a \cdot 0 = 0$ for all real numbers a. Use only the axioms of the real numbers.

2. Let A be a non-empty set of real numbers that is bounded above. Let $\alpha = \sup A$ and let $b < \alpha$. Prove there exists $x \in A$ such that $b < x \leq \alpha$.

3. Let A, B be non-empty subsets of $\mathbb{R}$; assume A, B are bounded above. Prove that $A+B$ is non-empty and bounded above and that $\sup(A+B) = \sup A + \sup B$.

II. Sequences

1. Consider the sequence
$$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots;$$
that is, the sequence defined by
$$a_n = \frac{1}{k} \qquad \text{if} \quad 2^{k-1} \leq n < 2^k, k = 1, 2, 3, \ldots.$$
Prove that it converges to 0 using only the definition of limit.

2. Let $\{a_n\}$ be a sequences of real numbers, assume that $\lim_{n\to\infty} a_n = a > 0$. Prove there exists $N \in \mathbb{N}$ such that $a_n > a/2$ if $n \geq N$.

III. Continuous functions, functional limits.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Prove using only the definition that $\lim_{x\to 2} f(x) = 4$.

2. Let
$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ x, & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$
Prove: f is continuous at 0, 1; discontinuous everywhere else.

3. Prove: If f is increasing and bounded in the interval (a, b), then $\lim_{x\to b-} f(x), \lim_{x\to a+} f(x)$ exist.

4. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point; that is, there exists $x \in [0, 1]$ such that $f(x) = x$.

IV. Differentiability.

1. Let $f : \mathbb{R}^{\to}\mathbb{R}$ satisfy $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $x, y \in \mathbb{R}$, where $\alpha > 1$. Prove that f is constant.

2. Let $f(x) = x^5 + 4x^3 + 2x + 3$ has exactly one real zero.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Assume also that f is differentiable at all $x \neq 0$ and that $\lim_{x\to 0} f'(x)$ exists. Prove that f is differentiable at 0 and $f'(0) = \lim_{x\to 0} f'(x)$.

4. Let $g : (-1, 1) \to \mathbb{R}$ be bounded (there exists M such that $|g(x)| \leq M$ for all $x \in (-1, 1)$). Let $f : (-1, 1) \to \mathbb{R}$ be defined by $f(x) = x^2 g(x)$. Prove that f is differentiable at 0.

5. Prove using only the definition of differentiability (do not use the chain rule!): If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$,, then defining g by $g(x) = f(2x+1)$, g is differentiable at $(a-1)/2$ and $g'((a-1)/2) = 2f'(a)$. An easier version is:

   Prove using only the definition of differentiability (do not use the chain rule!): If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$,, then defining g by $g(x) = f(x+1)$, g is differentiable at $a-1$ and $g'(a-1) = f'(a)$.

6. Let $f : [0,2] \to \mathbb{R}$ be continuous. Assume also that f is twice differentiable in $(0,2)$. Assume $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$. Prove there exists $c \in (0,2)$ such that $f''(c) = 0$.

7. Prove that if f is differentiable at a, then f is continuous at a. Give an example of a function continuous at $x = 2$ and not differentiable at $x = 2$.

IV. Integration.

1. Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \in [0,1] \backslash \mathbb{Q}. \end{cases}$$

   Compute

$$\overline{\int_0^1} f(x),\, dx, \quad \underline{\int_0^1} f(x),\, dx,$$

   and prove f is not Riemann integrable over $[0,1]$.