MODERN ANALYSIS
Problem Pool

I. Real Numbers.

1. Prove that $a \cdot 0 = 0$ for all real numbers a. Use only the axioms of the real numbers.

2. Let A be a non-empty set of real numbers that is bounded above. Let $\alpha = \sup A$ and let $b < \alpha$. Prove there exists $x \in A$ such that $b < x \leq \alpha$.

3. Let A, B be non-empty subsets of $\mathbb{R}$; assume A, B are bounded above. Prove that $A+B$ is non-empty and bounded above and that $\sup(A+B) = \sup A + \sup B$.

II. Sequences

1. Consider the sequence
$$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots;$$
that is, the sequence defined by
$$a_n = \frac{1}{k} \qquad \text{if} \quad 2^{k-1} \leq n < 2^k, k = 1, 2, 3, \ldots.$$
Prove that it converges to 0 using only the definition of limit.

2. Let $\{a_n\}$ be a sequences of real numbers, assume that $\lim_{n\to\infty} a_n = a > 0$. Prove there exists $N \in \mathbb{N}$ such that $a_n > a/2$ if $n \geq N$.

III. Continuous functions, functional limits.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Prove using only the definition that $\lim_{x\to 2} f(x) = 4$.

2. Let
$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ x, & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$
Prove: f is continuous at 0, 1; discontinuous everywhere else.

3. Prove: If f is increasing and bounded in the interval (a, b), then $\lim_{x\to b-} f(x), \lim_{x\to a+} f(x)$ exist.

4. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point; that is, there exists $x \in [0, 1]$ such that $f(x) = x$.

IV. Differentiability.

1. Let $f : \mathbb{R}^{\to}\mathbb{R}$ satisfy $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $x, y \in \mathbb{R}$, where $\alpha > 1$. Prove that f is constant.

2. Let $f(x) = x^5 + 4x^3 + 2x + 3$ has exactly one real zero.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Assume also that f is differentiable at all $x \neq 0$ and that $\lim_{x\to 0} f'(x)$ exists. Prove that f is differentiable at 0 and $f'(0) = \lim_{x\to 0} f'(x)$.

4. Let $g : (-1, 1) \to \mathbb{R}$ be bounded (there exists M such that $|g(x)| \leq M$ for all $x \in (-1, 1)$). Let $f : (-1, 1) \to \mathbb{R}$ be defined by $f(x) = x^2 g(x)$. Prove that f is differentiable at 0.

5. Prove using only the definition of differentiability (do not use the chain rule!): If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$,, then defining g by $g(x) = f(2x+1)$, g is differentiable at $(a-1)/2$ and $g'((a-1)/2) = 2f'(a)$. An easier version is:

   Prove using only the definition of differentiability (do not use the chain rule!): If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$,, then defining g by $g(x) = f(x+1)$, g is differentiable at $a-1$ and $g'(a-1) = f'(a)$.

6. Let $f : [0,2] \to \mathbb{R}$ be continuous. Assume also that f is twice differentiable in $(0,2)$. Assume $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$. Prove there exists $c \in (0,2)$ such that $f''(c) = 0$.

7. Prove that if f is differentiable at a, then f is continuous at a. Give an example of a function continuous at $x = 2$ and not differentiable at $x = 2$.

IV. Integration.

1. Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \in [0,1] \backslash \mathbb{Q}. \end{cases}$$

   Compute

$$\overline{\int_0^1} f(x), dx, \quad \underline{\int_0^1} f(x), dx,$$

   and prove f is not Riemann integrable over $[0,1]$.