Mathematical Problem Solving
End of Course Report
Fall 2011
Robert Lubarsky

The course started with 13 students and ended with 11. We went through ch. 1-7 of Principles of Mathematical Problem Solving by Erickson and Flowers. Class time was a mix of interactive lecture by me, student presentation of problems at the board to the class, student critique of proferred solutions, and small group work. The small group work was mostly problems from the book the solutions of which were a few pages later. That way they could experience the challenge of the problems, see what was at stake, yet have a solution whenever they wanted. Homework assignments consisted of reading worked-out problems, writing up solutions to open problems, writing up good versions of solutions presented in class, and reading all of the problems at the end of a chapter and deciding themselves which ones to work on.

The final grade was determined by the homeworks turned in, class participation, one quiz, two in-class exams, and the final exam. The exams consisted mostly of problems that had been discussed in class, in some cases solved, and are reproduced below. The final grades were 3 B’s, 6 C’s, and 2 F’s.

What worked well in the class is that the students mostly seemed to enjoy it. Once somebody came who was sick, and when I asked why they didn’t just stay home, the answer came that it was too much fun to miss. Three students often stayed after class, sometimes for as much as an hour, just to talk more about the class’s problems and other mathematical questions. One ended up wanting to do a research-based DIS with me.

What remained frustrating was that the challenges placed before them seemed to be beyond their abilities. No one did especially well, not even those thinking of math grad school. After the first few assignments, it was difficult for me to get them to hand in any homework at all. This limited the amount of feedback I could give them on their work. That in turn made it more difficult for them to make progress and improve their performance. Judging from what they did submit, they cannot tell a correct from an incorrect argument. Perhaps the most disappointing for me were those few times I gave in class what I considered to be a complete solution, albeit not written out in full detail, asked them to submit a good written solution, and got almost nothing acceptable. That really highlighted the cleft between what I say and what they understand.

The students need improvement in all areas. Generating creative ideas is the easiest realm to identify, the hardest to fault them for not doing, and arguably far from the most important, as having a solid foundation might be preferable. Where else I would like to see progress includes the presentation of their thoughts, both written and oral, and critical reading and thinking, so they can find mistakes in arguments, especially their own. Pushing in somewhat contradictory directions here, I would like to have them develop both a broader scope, so they can see the arc of an argument, and distinguish the core from the periphery, and attention to detail. Not infrequently the key to solving a problem, or more modestly just understanding my solution, was not some difficult, abstract idea, but rather just looking at the fairly concrete (well, concrete relative to mathematics) situation before us. This they found difficult.

Although I am not satisfied with this as an outcome, perhaps the best thing that came out of this course was an exposure to a wide variety of sometimes crazy problems, and the experience of watching a professional (me) deal with them, often spontaneously, sometimes unsuccessfully. The biggest change I would like to see for the next time is more written output from them. Even if that's in the form of journalistic reporting, at least it would provide more interaction and feedback.

Mathematical Problem Solving
Exam 1

Your best answer is out of 40 points, your second best 30, third best 20, and fourth 10. I assume no one will get to answer more than four. Implicitly, prove your assertions. Using good judgement about what to prove and what can be assume is part of the game.

1. What is the maximum number of regions into which $\mathbf{R}^3$ can be divided by n planes?

2. An odd number of sticks is in a pile. Two players alternate taking one or two sticks from the pile until none are left. Who has a winning strategy, if the winner is the one with an even number of sticks? What if the winner is the one with an odd number of sticks?

3. Two players alternate writing down binary digits (i.e. 0 and 1). Note that the result after infinitely many moves is a real number in [0, 1] (affix a "decimal" point at the beginning). Player I wins if that number is rational, Player II if irrational. Who has a winning strategy?

4. Find the point of intersection I_j of the tangent lines to the graph of $y = x^2$ at the points $x = j$ and $x = j+k$, for some $k \neq 0$ fixed. Find the polynomial of lowest degree the graph of which passes through the points I_j for $j = 1, 2, \ldots, 10$.

5. How many 0s occur at the end of the expression $\prod_{n=2}^{100} n^2 - 1$?

6. Is there a closed (i.e. contains all of its limit points) subset of the circle that contains exactly one point from each pair of diametrically opposed points?

7. Define what it is to be an algebraic number. Show that if x is algebraic then so are $\sqrt{x}$ and x^2.

Mathematical Problem Solving
Exam 2

You may answer up to four questions. Each is worth 30 points, which allows for lots of extra credit. Implicitly, prove your assertions. Using good judgement about what to prove and what can be assume is part of the game.

1. State the principle of induction on the natural numbers.

2. Recall that the Fibonacci numbers F_n are defined as $F_1 = F_2 = 1$, and $F_n + F_{n+1} = F_{n+2}$. Show that the sum of the first n-many odd-indexed Fibonacci numbers is the n^{th} even-indexed Fibonacci number.

3. Suppose that n-many -1's and n-many +1's are distributed around a circle. Show that it is always possible to start at one of the numbers and go around the circle in such a way that the partial sums of the numbers passed are all nonnegative.

4. Let p_n be a bounded sequence of integers which satisfies the recursion

$$p_n = \frac{p_{n-1} + p_{n-2} + p_{n-3}p_{n-4}}{p_{n-1}p_{n-2} + p_{n-3} + p_{n-4}}.$$

Show that the sequence eventually becomes periodic.

5. Show that if every point of $\mathbf{R}^2$ is colored one of three colors, there exist two points of the same color one unit apart.

6. Consider a game in which Alpha flips n+1 fair coints, Beta flips n fair coins, and the player who gets the most heads wins. Show that if Beta wins ties, then Alpha and Beta have equal chances of winning.

7. In the game Knight Packing, two players, Alpha and Beta, alternately place knights (color unimportant) on a chessboard in such a way that no knight can attach another knight. The last player able to place a knight on the board is the winner. Who wins Knight Packing? Who wins if Knight Packing is played on a 7x7 board?

8. An Euler circuit of a graph G is a circuit containing all edges of G. Show that if G is connected then G has an Euler circuit iff each vertex of G has even degree.

Mathematical Problem Solving
Final

You may answer up to six questions. If you answered a question below for full credit on an exam, please pick a different one. Each is worth 20 points, which allows for lots of extra credit. Implicitly, prove your assertions. Using good judgement about what to prove and what can be assumed is part of the game.

1. What is the maximum number of regions into which $\mathbf{R}^3$ can be divided by n planes?

2. An odd number of sticks is in a pile. Two players alternate taking one or two sticks from the pile until none are left. Who has a winning strategy, if the winner is the one with an even number of sticks? What if the winner is the one with an odd number of sticks?

3. How many 0s occur at the end of the expression $\prod_{n=2}^{100} n^2 - 1$?

4. Is there a closed (i.e. contains all of its limit points) subset of the circle that contains exactly one point from each pair of diametrically opposed points?

5. Define what it is to be an algebraic number. Show that if x is algebraic then so are $\sqrt{x}$ and x^2.

6. State the principle of induction on the natural numbers.

7. Suppose that n-many -1's and n-many +1's are distributed around a circle. Show that it is always possible to start at one of the numbers and go around the circle in such a way that the partial sums of the numbers passed are all nonnegative.

8. Show that if every point of $\mathbf{R}^2$ is colored one of three colors, there exist two points of the same color one unit apart.

9. Consider a game in which Alpha flips n+1 fair coints, Beta flips n fair coins, and the player who gets the most heads wins. Show that if Beta wins ties, then Alpha and Beta have equal chances of winning.

10. In the game Knight Packing, two players, Alpha and Beta, alternately place knights (color unimportant) on a chessboard in such a way that no knight can attach another knight. The last player able to place a knight on the board is the winner. Who wins Knight Packing? Who wins if Knight Packing is played on a 7x7 board?

MAT 4937 Mathematical Problem Solving (Yiu) Spring 2012

Report (May 4)

The course had an enrollment of 29 students. Throughout the semester the students solved about 100 problems of varying difficulties, at a level up to the problem section of *College Mathematics Journal*. Some of the more difficult problems were discussed in class with varying degrees of details. Students were invited to present their solutions or to follow up with my ideas to complete the solution of a problem. Such occasions, in some way distractions and often wasteful of classroom times, served as opportunities for critiques to improve upon the writing of a solution.

The assessment for this course was based on

- **Workbook**: students downloaded regular assignments from the course website, worked on the problems, and wrote edited solutions to at least 70% of the problems. Apart from the amount of solutions presented, the workbook is also evaluated according to the quality:

Quality	unsatisfactory	Satisfactory	Good	Excellent
Points	2	3	4	5
Number of students	3	5	14	7

- **Journal**: Each student kept a journal on the progress on the course, recording their own insights, doubts, queries, and findings. While most kept a faithful proceeding the course, some are highly reflective of their own struggles for improvement throughout the semester.

Quality	Unsatis-factory	Satisfactory	Good	Very Good	Excellent
Number of students	3	6	10	9	1

- One **Mid-term Test**.

- **Final Exam**: The students solved 6 problems plus 10 multiple choice items in the 2-hour Final Exam (see attached).

  Here are the results of multiple choice questions.

Correct Answers	10	9	8	7	6	5	4	3	2	1	0
Number of students	1	5	7	2	4	2	3	2	2	1	0

The remaining 6 problems are moderately easy and are of three types:

- Basic knowledge: Problems 1(a), 3, and 5.

Problem	Full mark	Satisfactory	Unsatisfactory	No mark
1a	8	4	1	16
3	1	22	1	5
5	12	14	2	1

- Writing of easy proofs: Problems 1(b), 2(a), and 2(b).

Problem	Full mark	Satifactory	Unsatisfactory	No mark
1b	9	14	0	6
2a	10	8	8	3
2b	5	1	20	3

- Critical Thinking: In two of the problems, students are given some definitions or specific information, and are required to make use of these to solve "nonstandard" problems.

Problem	Full mark	Satisfactory	Unsatisfactory	No mark
4	18	3	3	5
6	4	5	9	11

All students, except one, passed the course.

Mathematical Problem Solving (Yiu) Spring 2012
Final Exam (May 2: Two hours)
Solution.

Problem 1. (5 points)
(a) A number is formed by concatenating the natural numbers in order:

$$12345678910111213141516171819202122232425\cdots$$

What is the 2012-th digit from the left?
Solution. Since $2012 = 3\times 670+2$, the 2012-th digit is the middle digit of the 3-digit number represented by the 671-th block of threes. The first 63 blocks of threes accommodate the first 99 numbers. The 671-th block is the number $671 - 63 = 608$-th number beyond 99, i.e. 707. The 2012-th digit in the string is 0.

(b) Show that it is **not** possible to purchase a number of 45-cent stamps and 90-cent stamps with **exactly** 10 dollars.

Suppose we buy x pieces of 45-cent stamps and y pieces of 90-cent stamps (x, y nonnegative integers) with exactly 10 dolloars. Then $1000 = 45x + 90y = 45(x + 2y)$. This is a contradiction since 45 does not divide 1000.

Problem 2. (6 points) (a) Prove that a triangle with two equal sides and a 60° angle is equilateral.
Solution. (i) If the 60° angle is opposite to one of the equal sides, then the angle opposite to the other equal side is also 60°. The third angle $= 180^\circ - 2\times 60^\circ = 60^\circ$. The triangle is equilateral.

(ii) If the 60° is not opposite to any of the equal sides, these equal sides have equal opposite angles, say x°. Then $60+2x = 180$, $x = 60$. The triangle has three 60° angles; it is equilateral.

(b) The lengths of two sides of a triangle are **consecutive** integers, and the angle **between** these two sides is 60°. Prove that the third side cannot have integer length.
Solution. The 60° angle is strictly between the other two angles. Its opposite side has length strictly between the other two sides, which are consecutive integers. This third side cannot be an integer.

Problem 3. (4 points) N is a 4-digit number which, when added to the sum of its digits, gives 2012. Find N.
Answer: 1987 or 2005.
Solution. Since the sum of the digits is between 3 and 36, N is between 1976 and 2009.
(i) If $N = 19xy$, then $1900 + 10x + y + 10 + x + y = 2012$, $11x + 2y = 102$. Since $y \leq 9$, we must have $11x \geq 112 - 18 = 84$. Therefore, $x = 8$, and $y = 7$. From this, $N = 1987$.

(ii) Therefore $N = 200y$, and $2000 + y + 2 + y = 2012$, $y = 5$. This gives $N = 2005$.

Problem 4. (6 points)

A mathemagician instructs his assistant how, given **five** cards from a standard deck of 52 cards, to select **four** and arrange them in order, so that he, the mathemagician, can tell what the **fifth** (or **secret**) card is. It is based on three principles.

(1) Among 5 cards at least 2 must be of the same suit. The assistant chooses two cards of the same suit and in step (2) decides which to show as the **first** card, and which to keep as the **secret** card.

(2) The cards in a suit are ordered numerically (treating A as 1, J, Q, K as 11, 12, 13 respectively). Image a clock with A, 2, 3, . . . , J, Q, K (arranged clockwise). Given two cards, it is always possible to get from one them to the other **clockwise, in not more than** 6 **steps**. The assistant displays the **first** card, and keeps the other as the **secret card**.

(3) The remaining 3 cards are ordered numerically. For cards with the same number, order the suits alphabetically: $\clubsuit < \diamondsuit < \heartsuit < \spadesuit$. In this way, the remaining three cards are classified into small, medium, and large. Now, use the following scheme to determine a number to tell the mathemagician what he should add to the **first** card to get to the **secret** card:

arrangement	sml	slm	msl	mls	lsm	lms
distance	1	2	3	4	5	6

(a) For the assistant, given the five cards ♠5, ♠7, ♢6, ♣5, ♣Q, how would you arrange them in order?

first	second	third	fourth	secret
♠5	♣5	♣Q	♢6	♠7
♣Q	♠7	♢6	♠5	♣5

(b) For the mathemagician: your assistant shows you the first four cards, what is the secret card?

first	second	third	fourth	secret
♠5	♠7	♢6	♣5	* ♠J*

Problem 5. (10 points) (a) Consider the equation

$$(\dagger) \qquad x^2 - 2y^2 = -1$$

in **integers** x and y.

(i) Find the **smallest** positive integer solution (a, b) of the equation (†).
Answer: $(a, b) = (1, 1)$.

(ii) It is known that the integer solutions of (†) are generated recursively by

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Complete the following table to find the first few solutions of (†)

k	1	2	3	4	5	6	7	8
x_k	1	7	41	239	1393	8119	47321	275807
y_k	1	5	29	169	985	5741	33461	195025

(b) Make use of the information in part (a) to find a right triangle of side lengths $(x, x + 1, y)$, where x and y are 5-digit numbers.
Solution. If $x^2 + (x + 1)^2 = y^2$, then $(2x + 1)^2 - 2y^2 = -1$. From the table above, $2x + 1 = 47321$ and $y = 33461$. This gives $(x, y) = (23660, 33461)$.

(c) Is the answer to part (b) unique? Why?

Yes, because in the next solution both x and y have 6 digits.

Problem 6. (4 points) Consider a binary operation on real numbers:

$$x \star y := ax + by + cxy$$

where a, b, c are constants. Given $1 \star 2 = 3$, $2 \star 3 = 4$, and that there is a nonzero constant d such that $d \star y = y$ for every y.

(i) Find the values of a, b, c, d.
Solution. If $ad + by + cdy = y$ for every y, then we must have $a = 0$, $b + cd = 1$. Since $a + 2b + 2c = 3$ and $2a + 3b + 6ab = 4$, we have $b = \frac{5}{3}$, $c = -\frac{1}{6}$, and $d = 4$.

(ii) Find the value of $3 \star 6$. *Answer*: $3 \star 6 = \frac{5}{3} \cdot 6 - \frac{1}{6} \cdot 3 \cdot 6 = 7$.

Problem 7. (15 points) Multiple Choice Questions (10 items)

1. What is the smallest possible value of $x^2 + 2xy + 3y^2 + 2x + 6y + 5$?

   A 1 **B** 2 **C** 3 **D** 4 **E** NOTA

   Solution. **B**. The given expression is equal to $(x+y+1)^2+2(y+1)^2+2$.

2. The base diameter and the height of a right circular cone are both equal to the diameter of a sphere. The volumes of the cone and the sphere are in the ratio of

   A $1:3$ **B** $2:3$ **C** $1:2$ **D** $2:9$ **E** NOTA

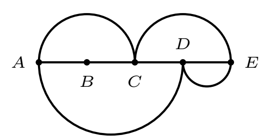
   Solution. **C**. Volume of cone = $\frac{1}{3}\pi \cdot \frac{1}{4}d^2 \cdot d = \frac{\pi}{12}d^3$; volume of sphere = $\frac{4}{3}\pi \cdot \frac{d^3}{8} = \frac{1}{6}\pi d^3$.

3. Find the greatest value of $\frac{1}{2+\sin\theta+\cos\theta}$.

   A $\frac{-2+\sqrt{2}}{2}$ **B** $\frac{2+\sqrt{2}}{2}$ **C** $\frac{2-\sqrt{2}}{2}$ **D** $\frac{-2-\sqrt{2}}{2}$ **E** NOTA

   Solution. **C**. $\frac{1}{2+\sin\theta+\cos\theta} = \frac{1}{2+\sqrt{2}\sin(\theta+\frac{\pi}{4})} \leq \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2}$.

4. A line AE is divided into four equal parts by the points B, C, D. Semicircles are drawn with segments AC CE, AD and DE as diameters.

   The ratio of the area enclosed above the line AE to the area enclosed below the line is

   A $4:5$ **B** $5:4$ **C** $1:1$ **D** $8:9$ **E** NOTA

   Solution. **A**. If the smallest semicircle is 1 unit, then the area above is 8 units, and that below is $9+1=10$ units. Ratio = $4:5$.

5. Suppose $x + y = 1$ and $x^2 + y^2 = 2$. Find $x^4 + y^4$.

   A 1 **B** $1\frac{1}{2}$ **C** 2 **D** $2\frac{1}{2}$ **E** 3

   Solution. **???**. $xy = \frac{1}{2}((x+y)^2 - (x^2+y^2)) = -\frac{1}{2}$;
   $x^4 + y^4 = (x^2+y^2)^2 - 2(xy)^2 = 2^2 - 2\cdot\frac{1}{4} = 3\frac{1}{2}$.
   I give everybody full credits for this item.

6. The integers

$$1, 3, 4, 9, 10, 12, 13, 27, 28, 30, \ldots$$

are the sums of distinct powers of 3 arranged in increasing order. What is the 100th term of the sequence?

A 100 **B** 345 **C** 543 **D** 981 **E** 1024

Solution. **D**. $100 = 2^6 + 2^5 + 2^2$. The corresponding number = $3^6 + 3^5 + 3^2 = 981$.

7. If you write all integers from 1 to 100, how many **even** digits will be written?

A 50 **B** 71 **C** 80 **D** 89 **E** 91

Solution. **E**. Arrange the numbers 0, 1, 2, $\ldots$, 99 in a 10×10 table. The even numbers occupy 5 columns. Those numbers with even tens-digits occupy 4 rows. Altogether there are 90 such digits. Replacing 0 by 100 contributes to one more even number.

8. What is the units digit of 2^{2012}?

A 0 **B** 2 **C** 4 **D** 6 **E** 8

Solution. **D**. $2^n \equiv 2, 4, 8, 6 \pmod{10}$ according as $n \equiv 1, 2, 3, 0 \pmod{4}$. $2^{2012} \equiv 6 \pmod{10}$

9. Three digit numbers are formed using only odd digits. The sum of all such three digit numbers is:

(A) 19375 **(B)** 34975 **(C)** 625^3 **(D)** 34975 **(E)** 69375

Solution. **(E)**. There are $5^3 = 125$ such numbers. Each of 1, 3, 5, 7, 9 appears 25 times as units, tens, and hundred digits. Sum of these numbers $= 25(1 + 3 + 5 + 7 + 9) \times 111 = 69375$.

10. Peter takes 4 hours to paint a wall. Paul takes 6 hours to do the same job. What is their average time to paint the wall?

(A) 5 hours **(B)** $5\frac{1}{2}$ hours **(C)** $4\frac{1}{2}$ hours
(D) 4 hours 48 minutes **(E)** 5 hours 12 minutes.

Solution. **(D)**. In each hour they together paint $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ of the wall. They take $\frac{12}{5}$ hours to complete the task. Average = $\frac{24}{5}$ hours = 4 hours 48 minutes.