Modern Algebra Questions

Note: This pool is divided into several sections, prepared by different instructors.

1 Problems prepared by Professor Locke

Rubric: Students are always expected to justify their answers, and to show all appropriate intermediate steps.

Any incorrect step in an attempted solution can lead to severely diminished (or no) points for the problem.

QUESTION 1. Joey was given the set of permutations

$$T = \{(1\ 2\ 6)\,(3\ 4\ 5)\,,(1\ 7)\,(2\ 5)\,,(4\ 7\ 6\ 3\ 2)\}$$

and asked to list the elements of the subgroup H of S_7 generated by T. He came back with a list 2525 permutations in the subgroup H. Without yourself generating H, explain to Joey whether or not his answer is correct.

QUESTION 2.

(a) Find integers s and t such that $121s + 144t = \gcd(121, 144)$. Verify that your answers are correct.

(b) Working in $\mathbf{Z}_{144}$, calculate 121^{-1}.

QUESTION 3. Quaternions are numbers of the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbf{R}$ and i, j, k are "imaginary numbers" with the property that $i^2 = j^2 = k^2 = ijk = -1$, and hence, $ij = k$, $ji = -k$, $ki = j$, $ik = -j$, $jk = i$, $kj = -i$. If we define the function $N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$, then for any two quaternions γ_1 and γ_2, $N(\gamma_1\gamma_2) = N(\gamma_1)\,N(\gamma_2)$. (You may assume that $N(\gamma_1\gamma_2) = N(\gamma_1)\,N(\gamma_2)$ without giving a proof.)

Given that $39 = 3^2 + 2^2 + 1^2 + 5^2$ and $45 = 4^2 + 2^2 + 3^2 + 4^2$, use the above ideas to find integers A, B, C, D such that $39 \cdot 45 = A^2 + B^2 + C^2 + D^2$.

QUESTION 4. List all possible abelian groups of order 156, up to isomorphism.

QUESTION 5. Given the multiplicative group G consisting of 2×2 matrices with entries from $\mathbf{Z}$ and with determinant ± 1, the subgroup H generated by the element $h = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and the element $x = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, display the elements of the subgroup H, and determine the two cosets Hx and xH (clearly labelling which is which).

QUESTION 6. In $\mathbf{Z}_3[x]$, there are three monic linear polynomials, x, $x + 1$, and $x+2$. The quadratic products of these are x^2, $(x+1)^2 = x^2+2x+1$, $(x+2)^2 = x^2+x+1$, $x(x+1) = x^2+x$, $x(x+2) = x^2+2x$, and $(x+1)(x+2) = x^2+2$. List the irreducible monic quadratic polynomials in $\mathbf{Z}_3[x]$.

Question 7.

(a) In $\mathbf{Z}_2[x]$, one of the polynomials $x^4 + x^2 + 1$ or $x^4 + x + 1$ is irreducible. Which is it? (Show why.)

(b) Using the irreducible polynomial from part (a), one can build a field $\mathbf{F}$ of order 16, with elements of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_0, a_1, a_2, a_3 \in \mathbf{Z}_2$. Calculate $\left(1 + x + x^2\right)\left(1 + x^2 + x^3\right)$ in this field of 16 elements. (It is not necessary to display the entire addition and multiplication tables for $\mathbf{F}$.)

Question 8. Suppose that G is a finite group and that H is a subgroup of G such that $o(G) = 2o(H)$. Prove that $Hx = xH$ for every $x \in G$.

Question 9. Consider a solid cube $\mathbf{C}$, with faces labelled from 1 to 6. Suppose the cube is sitting in on a desk, directly over (and aligned with) a square $\mathbf{S}$ the same size as one of its faces. If we pick up the cube, we could put it back down in several ways, so that it is still directly over and aligned with $\mathbf{S}$.

(a) How many such ways?

(b) How many ways if we add as a restriction that exactly two of the vertices of $\mathbf{C}$ are to end up exactly where they started?

2 Problems prepared by Professor Yiu

1. For a positive integer m, let $\mathbb{Z}_m^\bullet$ denote the group of units of the ring $\mathbb{Z}_m$.
   - (a) What is the order of the element $[2]_7 \in \mathbb{Z}_7^\bullet$?
   - (b) What is the order of the element $[5]_8 \in \mathbb{Z}_8^\bullet$?
   - (c) What is the element $[c]_{56}$ which corresponds to $([2]_7, [5]_8)$ under the natural isomorphism $\mathbb{Z}_{56} \to \mathbb{Z}_7 \times \mathbb{Z}_8$?
   - (d) Find the order of the element $[c]_{56}$ in (c).
2. 2. Consider the subset $A := \{5, 15, 25, 35\}$ of $\mathbb{Z}_{40}$.
   - (a) Complete the multiplication table of the elements in A, giving each product as an element of $\mathbb{Z}_{40}$.

	5	15	25	35
5				
15				
25				
35				

   - (b) Does A form a group under multiplication? Show clearly how each of the group axioms is satisfied or violated.
3. Consider the set $G = \mathbb{R} \backslash \{-1\}$ with an operation $\star$ given by

$$a \star b = a + b + ab.$$

   - (a) Show that $\star$ is a binary operation, i.e. $a \star b \in G$ if $a, b \in G$.

(b) Show that $*$ is associative.

(c) Does there exist an identity element $e \in G$ for which $e \star a = a$ for every $a \in G$? If so, name the element and verify that it is an identity. If not, give a reason.

(d) Does G form a group with binary operation $\star$? Justify your answer.

4. (a) Prove that if n is not a prime number, it must have a prime divisor $\leq \sqrt{n}$.

   (b) Let $a, b \in \mathbb{Z}$ with $\gcd(a, b) = d$. If $a = dx$ and $b = dy$, show that $\gcd(x, y) = 1$.

5. True or false? Give a simple proof or a counterexample as appropriate.

   (a) Suppose a is an odd integer and $ab \equiv ac \pmod 8$. Then $b \equiv c \pmod 8$.

   (b) Suppose a is an even integer and $ab \equiv ac \pmod 9$. Then $b \equiv c \pmod 9$.

   (c) If a cubic polynomial $f(x) \in F[x]$ satisfies $f(a) \neq 0$ for every $a \in F$, then $f(x)$ is irreducible.

   (d) If G is a finite abelian group with 3 elements of order 2, then it cannot be a cyclic group.

   (e) There is a field with exactly 6 elements.

6. A monic polynomial is one whose leading coefficient (of highest degree term) is 1.

   (a) How many monic polynomials of degree 2 are there in $\mathbb{F}_3[x]$?

   (b) Find all irreducible monic polynomials of degree 2 in $\mathbb{F}_3[x]$.
   Hint: There are 3 of them.

   (c) It is known that the polynomial $x^4 - x^3 - x - 1 \in \mathbb{F}_3[x]$ is reducible. Factorize the polynomial.

7. Let $K = \mathbb{F}_3[x]/(x^2 + x + 2)$.

   (a) Why is K a field?

   (b) How many elements does it have?

   (c) Show that $[1 + x]$ is an element of order 8 in K.

8. It is known that $x4 - 3x^3 + 3x + 1 \in \mathbb{Z}[x]$ is reducible. Factorize it.

9. Factorize $f(x) = x^6 + x^5 + x^3 + x + 1$ over $\mathbb{F}_2$ given that it has a multiple factor.

3 Problems prepared by Professor Schmidmeier

1. (a) Give the definition of a group. When is a group called abelian?

   (b) Give two examples of abelian groups, and two examples of non-abelian groups.

2. (a) When is a permutation called even?

   (b) Which of the following permutations are even:

$$(12), (123), (1234), (12)(34)?$$

(c) State a result about even and odd permutations

(d) Show that any product of two odd permutations is even.

3. (a) State Lagrange's Theorem.

   (b) Is there a group G with a subgroup H such that G has 9 elements and H has 4 elements? If so, give an example. If not, explain why not!

   (c) Find all right cosets of the subgroup $\langle 3 \rangle$ in the group $\mathbb{Z}_9$.