MAT 4937 Mathematical Problem Solving (Yiu) Spring 2010

Report (May 4)

The course had an enrollment of 22 students. Throughout the semester the students solved about 100 problems of varying difficulties, at a level up to the problem section of *College Mathematics Journal*. Some of the more difficult problems were discussed in class with varying degrees of details. Students were invited to present their solutions or to follow up with my ideas to complete the solution of a problem. Such occasions, in some way distractions and often wasteful of classroom times, served as opportunities for critiques to improve upon the writing of a solution.

The assessment for this course was based on

- **Workbook**: students downloaded regular assignments from the course website, worked on the problems, and wrote edited solutions to at least 70% of the problems. Apart from the amount of solutions presented, the workbook is also evaluated according to the quality:

Quality	unsatisfactory	Satisfactory	Good	Excellent
Points	2	3	4	5
Number of students	0	8	9	5

- **Journal**: Each student kept a journal on the progress on the course, recording their own insights, doubts, queries, and findings. While most kept a faithful proceeding the course, some are highly reflective of their own struggles for improvement throughout the semester.

Quality	Satisfactory	Good	Very Good	Excellent	Really excellent
Number of students	1	5	12	3	1

- One **Mid-term Test**.

- **Final Exam**: The students solved 8 problems in the 2-hour Final Exam (see attached). Each problem is graded out of 12 points. The problems are moderately easy and are of two types:

  - Basic knowledge: Typical problems in number theory, geometry, and combinatorics requiring only basic knowledge.

Problem	Full mark (12)	7-11	6	2–5	0
Number Theory (1)	3	8	7	4	0
Geometry (6)	8	0	3	5	6
Algebra and Combinatorics (7)	9	3	4	3	3

Problems 1(b) and 7(b) test on the writing of proofs.

– Critical Thinking: In two of the problems, students are given some definitions or specific information, and are required to make use of these to solve “nonstandard” problems.

Problem	Full mark (12)	7-11	6	2–5	0
4	8	0	5	7	2
8	5	4	2	5	6

All students passed the course.

Mathematical Problem Solving (Yiu) Spring 2010
Final Exam (April 29, Two hours)

Problem 1. (a) A number is formed by concatenating the natural numbers in order:

$$123456789101112131415161718192021222324 25\cdots$$

What is the 2010-th digit from the left?

(b) Show that it is **not** possible to purchase a number of 44-cent stamps and 90-cent stamps with **exactly** 10 dollars.

Problem 2. (a) Explain how to form a right triangle whose side lengths are integers without common divisors.

(b) Find all integer right triangles with one side length equal to 209. [Note: $209 = 11 \times 19$.]

Problem 3. N is a 4-digit number which, when added to the sum of its digits, gives 2013. Find N.

Problem 4. (a) The **nim sum** of two integers a and b is computed as follows. Write each of a and b in base 2, and add the corresponding digits (lining up from the rightmost) **without** carries. This gives the base 2 expansion of the nim sum of a and b.

(a) Calculate the nim sum of 7 and 9.

(b) There are three piles of chips, 7, 8, 9 pieces. You and your opponent take turn removing any (positive) number of chips from any ONE pile. Whoever makes the last move wins. Now it is your turn. It is known that if you can make the nim sum of two numbers equal to the third, then you win. Find all possible winning moves.

Problem 5. (a) Consider the equation

$$(\dagger) \qquad x^2 - 2y^2 = -1$$

in **integers** x and y.

(i) Find the **smallest** positive integer solution (a, b) of the equation (†).

(ii) It is known that the integer solutions of (†) are generated recursively by

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Complete the following table to find the first few solutions of (†)

k	1	2	3	4
x_k				
y_k				

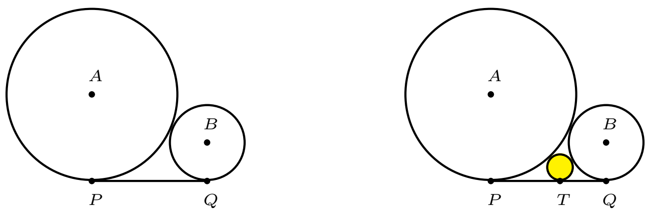
(b) A developer wants to build a community in which the n (approximately 100) homes are arranged along a circle, numbered consecutively from 1, 2, $\dots n$, and are separated by the club house, which is **not** numbered. He wants the house numbers on one side of club house adding up to the same sum as the house numbers on the other sides.

Suppose houses $1, 2, \dots, m$ are on one side of the club house.

(i) Construct an equation involving n and m.

(ii) Make use of (a) to find n and m, assuming n an integer around 100.

Problem 6. (a) Two circles of radii $a > b$ are tangent externally to each other (Figure (a)).



(a) (b)

(a) Calculate the length of the common tangent PQ.

(b) Make use of (a) to calculate the radius of the circle tangent to the given circles and their common tangent PQ (Figure (b)), assuming $a = 9$ and $b = 4$.

Problem 7. (a) Let 1_n denote the integer with n digits, all 1's. Find a closed form expression for the sum $1 + 1_2 + 1_3 + \cdots + 1_n$.

(b) Given 12 distinct two-digit numbers, show that there are two with a 2-digit difference of the form aa.

Problem 8. Consider a binary operation on real numbers:

$$x \star y := ax + by + cxy$$

where a, b, c are constants. Given $1 \star 2 = 3$, $2 \star 3 = 4$, and that there is a nonzero constant d such that $d \star y = y$ for every y.

(i) Find the values of a, b, c, d.

(ii) Find the value of $3 \star 6$.