EngSci Year 3 Winter 2022 Notes

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Contents

Section 1

CSC473: Advanced Algorithms

Subsection 1.1

Global Min-Cut (Karger's Contraction Algorithm)

Given an undirected, unweighted, and connected graph $G = (V, E)$, **return** the smallest set of edges that disconnects G

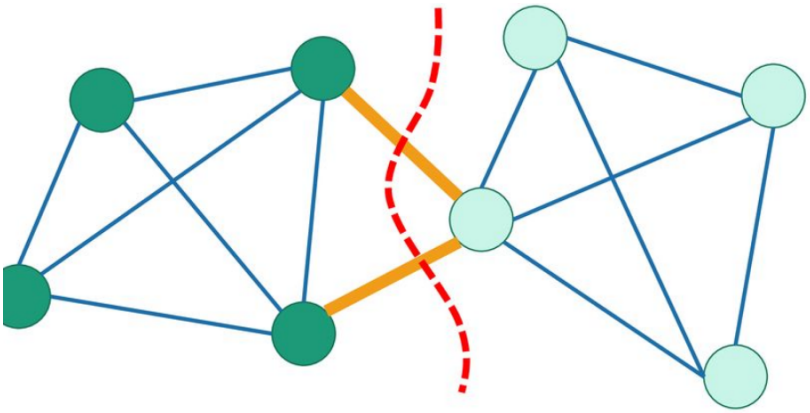


Figure 1. Example of global min-cut. Note that the global min-cut is not necessarily unique

Lemma 1 | If the min cut is of size $\geq k$, then G is k-edge-connected

An example of where this may be useful is in computer networks where we can measure the resiliency of a network by how many cuts must be made before a vertex (or many) get disconnected

It may be more convenient to return a set of vertices instead

Definition 1

$$S, T \subseteq V, S \cap T = \varnothing \tag{1.1}$$

$$E(S, T) = \{(u, v) \in E : u \in S, v \in T\} \tag{1.2}$$

The global min-cut is to output $S \subseteq V$ such that $S \neq \varnothing, S \neq V$, such that $E(S, V \setminus S)$ is minimized.

Comment

Note that the min-cut-max-flow problem is somewhat of a dual to the global min-cut problem; the min-cut-max-flow problem imposes a few more constraints than the global min-cut algorithm i.e. having a directed and weighted graph as well as the notion of a source or sink.

- **Input:** Directed, weighted, and connected $G = (V, E)$, $s \in V, t \in V$
- **Output** : S such that $s \in S, t \notin S$ such that $|E(S, V \setminus S)|$ is minimized

We can kind of intuitively see that the global min-cut can be taken to the minimum of all max-flows across the graph. So we can take the max-flow solution and then reduce it to find the global min cut.

Question: how many times will we have to run max-flow to solve the global min-cut problem? Naively, we may fix t to be an arbitrary node, then try every other $s \neq t$ to find the $s - t$ min-cut to get the best global min-cut.

We know from previous courses that the Edmonds-Karp max-flow algorithm will run in $O(nm^2) = O(n^5)$, which makes our global min-cut algorithm $O(n^6)$. However, there is a paper recently published which gives an algorithm for min-cut in nearly linear time, i.e $O(m^{1-O(1)}) = O(n^2)$ which gives a global min-cut runtime of $O(n^3)$.

A randomized algorithm will be presented that solves this problem in $O(n^2 \log^2 n)$

Definition 2 The **Contraction** operation takes an edge $e = (u, v)$ and *contracts* it into a new node w such that all edges connected to u, v now connect to w and u, v are removed. Note that the contracted nodes can be supernodes themselves.

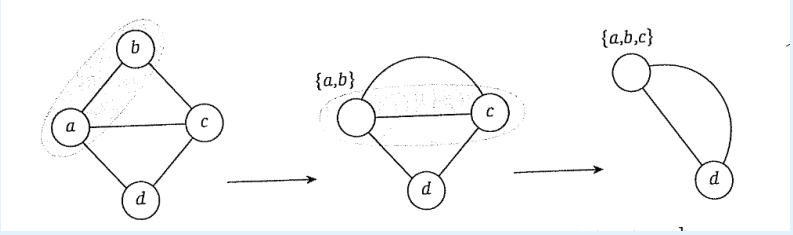


Figure 2. Example of a series of contractions

CONTRACTION$(G = (V, E))$

1 **while** G has more than 2 supernodes
2 Pick an edge $e = (u, v)$ uniformly at random
3 Contract e, remove self-loops
4 Output the cut $(S, V \setminus S)$ corresponding to the two super nodes

The contraction algorithm then recurses on G', choosing an edge uniformly at random and then contracting it. The algorithm terminates when it reaches *a* G' with only two supernodes v_1, v_2. The sets of nodes contracted to form each supernode $S(v_1), S(v_2)$ form a partition of V and are the cut found by the algorithm.

1.1.1 Analysis

The algorithm is still random, so there's a chance that it won't find the real global min-cut. Perhaps unintuitively the success probability is in fact not exponential, but rather only polynomially small. Therefore running the contraction algorithm a polynomial number of times can produce a global min-cut with high probability.

Lemma 2 The contraction algorithm returns a global min cut with probability at least $\frac{1}{\binom{n}{2}}$

Proof Take a global min-cut (A, B) of G and suppose it has size k, i.e. there is a set F of k edges with one end in A and the other in B. If an edge in F gets contracted then a node of A and a node in B would get contracted together and then the algorithm would no longer output (A, B), a global min-cut. An upper bound on the probability that an edge in F is contracted is the ratio of k to the size of E. A lower bound on the size of E can be imposed by noting that if any node v has degree $< k$ then $(v, V \setminus v)$ would form a cut of size less than k – which contradicts our first assumption that (A, B) is a global min-cut; $|E| \geq \frac{1}{2}kn$ So the probability than an edge in F is contracted at any step is

$$\frac{k}{(\frac{kn}{2})} = \frac{2}{n} \tag{1.3}$$

Note that here we use the number of vertices instead of the number of edges since each contraction removes (combines) one vertex, whereas since the amount of edges after a contraction can be very difficult to calculate.
Next, let's inspect the algorithm after j iterations. There will be $n - j$ supernodes in G' and we can take that no edge in F has been contracted yet. Every cut of G' is a cut of G, so there are at least k edges incident to every supernode of G'[1]. Therefore G' has at least $\frac{1}{2}k(n-j)$ edges, and so the probability than an edge of F is contracted in $j+1$ is at most

[1] *since the min-cut has k edges*

In terms of edges, it would be $\frac{k}{m_{i-1}}$, but again, edges are difficult to work with so we'll do it w.r.t the vertices/supernodes

$$\frac{k}{\frac{1}{2}k(n-j)} = \frac{2}{n-j} \tag{1.4}$$

It follows that the probability that an edge of F is *not* contracted in $j+1$ is at least

$$P(A_i|A_1 \ldots_{i-1}) \geq \frac{n-i-1}{n-i+1} = 1 - \frac{2}{n-i+1} \tag{1.5}$$

The global min-cut will be actually returned by the algorithm if no edge of F is contracted in iterations $1 - n$.
What we want to know, then, is what is the probability of this algorithm never making a mistake?

This prof uses commas to indicate intersection...

$$P(A_1 \ldots A_{n-1}) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2) \ldots P(A_{n-2}|A_1, A_2, \ldots, A_{n-3}) \tag{1.6}$$

From what we found previously we know that this is

$$\geq \frac{n-2}{n} \times \frac{n-3}{n-1} \times \frac{n-4}{n-2} \ldots \times \frac{2}{4} \times \frac{1}{3} = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}} \tag{1.7}$$

□

This gives us a bound of $O(n^2)$ using the n^2 term from the number of contractions and then n^2 to get correct output with constant probability of success.

The key observation is that early contractions are much less likely to lead to a mistake, which leads us to the Karger-Stein min cut.

Subsection 1.2

Karger-Stein Min Cut Algorithm

This algorithm solves the global min-cut problem in $O(n^2 \log^2 n)$ by taking advantage of the earlier cuts; it stops the contraction algorithm after an arbitrary fraction of contractions steps and then recursively contracts *more carefully*.

Comment

Exercise: show the following:
The probability of no mistake in the first i contractions

$$P(A_1, A_2, \dots A_i) \geq \frac{(n-i)(n-i-1)}{n(n-1)} \tag{1.8}$$

MIN-CUT$(G = (V, E))$

1 **if** G has two supernodes corresponding to $S, \hat{S}$
2 **return** $S, \hat{S}$
3 Run the contraction algorithm until $\frac{n}{\sqrt{2}} + 1$ supernodes remain
4 Let G' be the resulting contracted multigraph
5 $(S_1, \hat{S}_1) =$ MIN-CUT(G')
6 $(S_2, \hat{S}_2) =$ MIN-CUT(G')
7 **return** the cut $(S_i, \hat{S}_i)$ with the smaller number of edges

Theorem 1 MIN-CUT(G) runs in $O(n^2 \log n)$ and outputs a min cut of G with probability of at least $\frac{1}{O(\log n)}$

So repeat the algorithm $O(\log(n))$ times to get constant probability of success, leading to $O(n^2 \log^2 n)$ runtime

PROOF The intuition for this can be developed by drawing out a recursion tree for this problem. At each level the number of recursive call doubles, but the time it takes for each sub-call halves as well. This means that the total runtime for each level is n^2. As for the total time will just be $O(n^2 \log(n))$, since we know the height of the recursion tree to be $\log n$.
More formally, the recursion may be described with

$$T(n) \leq 2T(\frac{n}{\sqrt{2}} + O(n^2)) \tag{1.9}$$

Which can [2] be solved with the master theorem.

□

[2] *I think?*

Theorem 2 We may also want to understand the probability of success. Let's define $P(d)$ to be the probability of the algorithm being successful at depth d in the recursion tree[3]

- We may deem a node in the recursion tree to be *successful* if it survives the contractions.
  - Since there must be a leaf node in a recursion tree that successfully produces a min-cut that corresponds to a min-cut, there must also be a sequence of *successful* nodes from the root to said min cut.
- $P(d)$ as the probability that a node at depth d is successful, conditioned on it's ancestors being successful
- Base case: $P(0) \geq \frac{1}{2}$ (will assume $= \frac{1}{2}$, worst case)
- Inductive step: $P(d) \geq \frac{1}{2}(1 - (1 - (P(d-1)))^2) = \frac{1}{2}(2P(d-1) - P(d-1)^2)$
  - At each level the probability of success is at least $\frac{1}{2}$, conditioned on the ancestors being successful.
  - $P(d-1)$ is the probability of there being a *successful* path from the left child to the root at depth $d-1$. The same probability holds for the right sub-child
  - The two subtrees are disjoint
  - $(1 - P(d-1))$ gives the probability of there *not* being a successful path from a left/right child to the root, $(1 - P(d-1))^2$ gives the probability that neither of these events hold.

[3] *It follows that $P(h)$ is the probability of the algorithm's success on termination.*

  - So the probability of success at d is 1 minus the prior probability.

What remains now is solving this recursion.

$$P(d) = P(d-1) - \frac{1}{2}P(d-1)^2 \tag{1.10}$$

Non-linear recursions are difficult since you really just have to make a good guess. It's reasonable to expect this relation to be on the order of

$$P(d) \geq \frac{1}{d} \tag{1.11}$$

And then as it turns out[4] it's

$$P(d) = \frac{1}{d+2} \tag{1.12}$$

And then this can be checked by doing an induction proof, but I'm not going to go and do that.

[4] *Proof by trust-the-prof*

Subsection 1.3

Closest Pair Problem

The closest pair problem is simple: given the **Input**: A set P of n points in the plane, find the **Output**: A pair of points $p, q \in P$ such that $d(p,q)$[5] is minimized.

We are already familiar with a few approaches:

- Brute force: $O(n^2)$
- Divide and conquer (CLRS 33.4): $O(n \log n)$

A tighter linear time bound may, remarkably, be achieved through a randomized algorithm and a slightly different approach to hashing than what we are used to.

[5] *Some distance metric, often euclidean or hamming distance*

rabins-algorithm(P)

```
1  Randomly order P as p_1, p_2 ... p_n
2  cp = {p_1, p_2}
3  Δ = dist(p_1, p_2)
4  for i = 3 to n
5      if ∃ q ∈ P_{i-1} = p_1 ... p_{i-1} s.t. dist(p, q) < Δ
6          cp = {p, q}
7          Δ = dist(p, q)
```

Rabin's algorithm considers the points in random order, maintaining a current minimum pair distance δ as points are processed. At every point p we look in the *vicinity* of p to see if any of the previously considered points are within δ from p, i.e. will form a closer pair. The tricky part of this algorithm is performing the *check_closest* operation in constant time.

Considerations to make:

- It is possible to randomly order P from $n!$ possible orderings in $O(n)$ time (CLRS reference for later)
- What data structure to use for line 5? I.e. finding q such that $dist(p,q) < \Delta$
  - Note: take q that is closest to p_i in line 5 (algorithm is unclear as to pick p_1 or p_2)

Note that there are two types of randomized algorithms:

- Monte Carlo algorithms: bound on worst-case time & probability of a correct answer with a probability $\geq$ some constant
- Las Vegas algorithms: bound on the expected value of running time, but the output is always correct

Our contraction algorithm is a Monte-Carlo algorithm

- This data structure must store a set $Q = P_{i-1} = p_1, p_2, p_{i-1}$ points and offer the following operations
  - Note: $\Delta(Q) = \delta = min_{p,q \in Q, p \neq q} dist(p, q)$
  - INSERT-FAST(p) inserts p into Q assuming $min\{dist(p, q) : q \in Q\} \geq \delta$
  - INSERT-SLOW(p) inserts p into Q even if $min\{dist(p, q) : q \in Q\} < \delta$
  - CHECK-CLOSEST(p) checks if $min\{dist(p, q) : q \in Q\} < \Delta$ and if so returns the closest point $q \in Q$ to P, otherwise returns NIL. Runs in $O(1)$ expected time

Here the, well, structure, of the data structure begins to become apparent. By making the assumption that the smallest distance so far is δ we can perform the requisite lookup/insertions in constant time (only need to look in a ring of size δ around p) – and if we do happen to find a yet-closer pair of points then some modification would have to be made to maintain the δ invariant.

Here's Rabin's algorithm in more detail:

RABINS-ALGORITHM(P)

```
1  Randomly order P as p_1, p_2 ... p_n
2  cp = {p_1, p_2}
3  Δ = dist(p_1, p_2)
4  Initialize the data structure with {}
5  for i = 3 to n
6      q = CHECK-CLOSEST(p_1)
7      if q == NIL
8          INSERT-FAST(p_i)
9      else
10         cp = {p_1, q}
11         Δ = dist(p_1, q)
12         INSERT-SLOW(p_1)
```

It's fairly trivial to see that this algorithm has a $O(n^2)$ worst case runtime; Line 1 is $O(n)$, and all the inserts run in $O(1)$ expected. In the worst case event that we would INSERT-SLOW which would cause the runtime to tend towards $O(n^2)$. It is our job now to find out just how often this would happen, and as it turns out it really isn't altogether that often – leading to an $O(n)$ runtime. Before that, however, we still need to formalize the data structure that enables this black magick? **Idea:** draw grid with side length $\frac{\delta}{2}$. If a point q being inserted belongs to the same sub-square S_{st} as p, then $d(p, q) < \delta$. If we take Q to be the set of points currently in the data structure, then since

$$\frac{\delta}{\sqrt{2}} < \delta \tag{1.13}$$

No two points in Q fall within the same square.

A partial converse is also true: If a point q being inserted that gives $dist(p, q) \leq \delta$ than each other must fall in either the same subsquare or in very close subsquares.

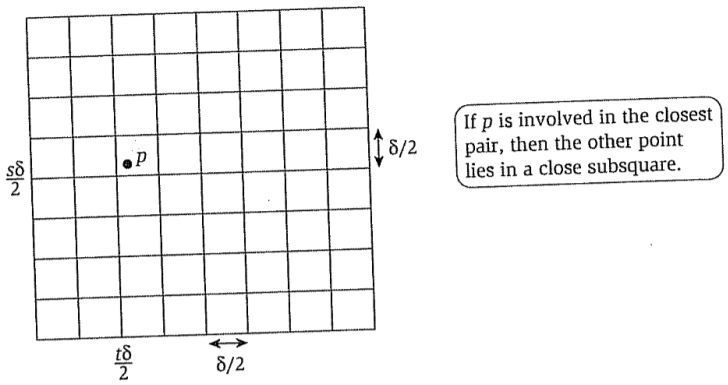


Figure 13.2 Dividing the square into size $\delta/2$ subsquares. The point p lies in the subsquare S_{ij}.

Implementation-wise we require a data structure with fast lookup of these subsquares. A natural data structure for this is a dictionary.

As for the key we use to index into the dictionary – this will depend on the specific space across which we are finding the closest points. For floating point Cartesian coordinates we may use use the floor function.

- Insert-fast: just insert into the dictionary
- Insert-slow: must do the rather dramatic operation of rebuilding the data structure with new δ, since the old data structure is now entirely invalidated.
- Check-closest: Must look in the 25 squares around p for points closer than δ. If there are more just return the closest. [6]

[6] *Note that we have to search the 25 squares around p since p may be located anywhere within the grid cell. In the worst-case situation it lies on the edge of a grid cell and therefore we must search up to two sub-squares away.*

1.3.1 Analysis

- Recall $P_i = \{p_1 \dots p_i\}$. Define (for $i \geq 3$) $Z_i = \begin{cases} 1 & \Delta(P_i) < \Delta(P_{i-1}) \\ 0 & \text{otherwise} \end{cases}$
  - This is a random event since the order of points is random. The first case represents a slow insert and the other a fast insert. If we take Z_i denote the probability of a slow insert, then the runtime of the algorithm is given by:
    - $T \leq n + \sum_{i=3}^{n}(1 + iZ_i)$
    - n for the random init, 1 for fast insert and iZ_i for a slow insert
  - We are primarily interested in the expected value of T, not the worst-case runtime.

$$\begin{aligned} E[T] &\leq n + (n-2) + \sum_{i+3}^{n} i \cdot EZ_i \text{ by linearity of expectation} \\ &= 2n - 2 + \sum_{i=3}^{n} i \cdot P(Z_i = 1) \end{aligned} \tag{1.14}$$

- Now, what's $P(Z_i) = 1$?
  - Let $\{p_i, p_k\}$ be a closest pair in P_i
  - If $Z_i = 1$ then p_i or p_k is p_i
  - $P(Z_i = 1) \leq \frac{2}{i}$ (There are two bad options out of i options)

- By combining the above two results we can obtain the following bound on the algorithm running cost

$$E[T] \leq 2n - 2 + \sum_{i=3}^{n} i \cdot P(Z_i = 1) = 2n - 2 + \sum_{i=3}^{n} i \cdot \frac{2}{i} \leq O(n) \tag{1.15}$$

Subsection 1.4

Tutorial: Locality Sensitive Functions

For the following two questions, let $\Sigma = \{0, 1, \ldots, k-1\}$ be an alphabet. Consider a distance function $\text{dist}_H(x, y)$ on strings $x, y \in \Sigma^d$, defined to equal the number of coordinates where x and y differ, i.e.,

$$\text{dist}_H(x, y) = |\{i : x_i \neq y_i\}|.$$

This is the usual definition of Hamming distance for alphabet Σ.

Exercise 1. *Give a* random *hash function $g : \Sigma^d \to \Sigma$ so that*

$$\mathbb{P}(g(x) = g(y)) = 1 - \frac{\text{dist}_H(x, y)}{d}.$$

The probability is only *over the choice of g. The function $g(x)$ should be computable in time $O(1)$ when x is given as an array of size d.*

Figure 3. Q1

Comment

Randomly sample a dimension[7] and then the probability of the hash functions being equal to each other is proportional to the hamming distance between the strings (consider random...).

[7] *O(1)*

Exercise 2. *Give a random hash function $g : \Sigma^d \to \{0, 1\}$* (notice the new range) *so that*

$$\mathbb{P}(g(x) = g(y)) = 1 - \frac{\text{dist}_H(x, y)}{2d}.$$

The function $g(x)$ should be computable in time $O(k)$ when x is given as an array of size d.

The next exercise considers the Manhattan, or ℓ_1, distance. For any two d-dimensional points $x, y \in \mathbb{R}^d$, this distance is defined by

$$\text{dist}_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|.$$

For example, when $d = 2$, this distance tells us how long we need to travel from x to y if we can only move North-South and East-West.

Figure 4. Q2

1. Sample a dimension d uniformly at random
2. In that dimension, assign each letter on $k-1$ to $\{0, 1\}$uniformly at random

P(equal) is 1- sum not equal / 2 = hamming dist / 2 * d (d letters)
This works because $P(g(x) \neq g(y) | \text{selected dimension i}) = \frac{|\{ix_i \neq y_i\}}{2d}$.
O(k) since you have to sample it k times

Exercise 3. *We will construct a locality sensitive hash function for* dist_1 *in two steps.*

1. *Suppose that* $t \in [0,1]$ *is chosen uniformly at random. For two numbers* a, b *s.t.* $0 \leq a \leq b \leq 1$*, give an expression for the probability* $\mathbb{P}(a \leq t < b)$ *in terms of* a *and* b*.*
2. *Give a random hash function* $g : [0,1]^d \to \{0,1\}$ *so that*

$$\mathbb{P}(g(x) = g(y)) = 1 - \frac{\text{dist}_1(x,y)}{d}.$$

The function $g(x)$ *should be computable in time* $O(1)$ *when* x *is given as an array of size* d*. (You can assume that comparing two real numbers takes constant time.)*

Figure 5. Q3

Comment I honestly don't really know how this one works :(

- pick a dimension at random
- Sample a $t \in (0,1)$ uniformly at random
- if the value smaller than t -> 0, larger than t -> 1
- P(g(x) != g(y) | selected dim i+ = |x_i - y_i
- given $0 \leq a \leq b \leq 1$, $P(a \leq t < b)$ is

Subsection 1.5

Nearest Neighbours & Clustering

Suppose we have a data set P of n entries[8], how may we design a data structure that can efficiently output a point $y \in P$ that has the smallest distance $dist(x,y)$ to x?

[8] *which usually are points in some metric space, i.e. a space that is reflexive, symmetric, and satisfies the triangle inequality*

1.5.1 Hamming Distance

Definition 3 **Hamming Distance**: number of bits that differ between two strings

$$dist(x,y) = |\{i : x_i \neq y_i\}| \tag{1.16}$$

Let there be a data set P containing n strings with d bits each, i.e. P is a subset of $\{0,1\}^d$. We want our data structure to support the following operations:

- INSERT(P,x): insert x into P
- NEARESTNEIGHBOUR(P,x): output the closest y to x in P

As it turns out this problem is nontrivial; this problem suffers from the *curse of dimensionality*.

Exercise: Give a data structure for which we have $O(1)$ insert and $O(2^d)$ lookup

We can relax the time bounds by relaxing the constraints on the problem somewhat to the approximate nearest neighbour problem

Definition 4 APXNEARESTNEIGHBOUR(P,x): output a string $y \in P$ such that

$$min\{dist(x,z) : x \in P\} \leq dist(x,y) \leq C \cdot min\{dist(x,z) : z \in P\} \tag{1.17}$$

I.e. we do not need to find the exact nearest neighbour and are satisfied with a neighbour that is good enough (within an approximation factor C) of the nearest neighbour.

Indyk and Motwani propose a clever hashing scheme for a randomized data structure to solve this problem in $O(dn^p)$ time where $p \approx \frac{1}{C}$ and ApxNearestNeighbour(P, x) will output a C-near neighbour with probability at least $\frac{2}{3}$ via a new function, NearNeighbour(P, x)

Definition 5

NearNeighbour$_r(P, x)$

- If $\exists\ z \in P$ such that $dist(x, z) \leq r$, then NearNeighbour$_r(P, x)$ will output some $y \in P$ for which $dist(x, y) \leq Cr$
- If every string z in P satisfies $dist(x, z) \geq Cr$, then NearNighbour$_r(P, x)$ outputs FAIL.

Next, let's go from $NearNeighbour$ to a harder problem: NearestNeighbour(P, q) via ApxNearestNeighbour(P, q)

- Assume that for each r we can implement a data structure for an easier problem (e.g. NearNeighbour(P, q)) for which Insert, NearNeighbour run in $T(n)$, how can we implement Insert and ApxNearestNeighbour in $O(T(n) \log d)$ so that ApxNearestNeighbour achieves approximation factor $2C$ and success probability $\frac{2}{3}$

How can we build this data structure then?

Still focusing on hamming distance for now

- Looking at euclidean distance, what if we divide $\mathbb{R}^d$ into grid cells and then NearNighbour(P, q) checks the cells around q. This won't work because the amount of cells we have to check is an exponential in the order of d.[9]
- We can cut down on the number of cells we have to check by forming the cells randomly. This way, though we introduce the possibility of making a mistake, we also greatly reduce the number of cells to check.

[9] *This is particularly an issue with hamming distance since here we commonly work with very high dimensions. Not as bad for euclidean distance since usu. work with 2-3 dimensions there.*

Instead of a fixed grid we randomly divide the string $\{0, 1\}^d$ into buckets such that

- $dist(x, y) \leq r \Rightarrow (x, y)$ fall into the same bucket with probability $\geq p_1$
- $dist(x, y) \geq Cr \Rightarrow (x, y)$ fall into the same bucket with probability $\geq p_2$

and $p_1 > p_2$.

Taking a step back the idea is that we can create buckets and hash into the bucket in such a way that it is more likely for near neighbours of x to fall in the same bucket of x and likewise for strings far from x. After the strings have been separated off into buckets the specific near neighbours can be found through normal hashing[10]. NearNeighbour(P, q) checks the bucket containing q and repeats the whole thing several times.

[10] *or whatever you choose at this point, I think*

Now, how can we do this for hamming distance?

Definition 6

For any $i \in [d], g_i : \{0, 1\}^d \to \{0, 1\}$ is defined by $g_i = x_i$. Suppose i is picked from $[d]$ uniformly at random.[11] Then the probability of a hash function collision is:

$$\mathbb{P}_i(g_i(x) = g_i(y)) = \frac{\{i : x_i = y_i\}}{d} = 1 - \frac{\{i : x_i \neq y_i\}}{d} = 1 - \frac{dist(x, y)}{d} \tag{1.18}$$

Where $dist$ is the hamming distance between x, y

[11] *Instead of thinking hash functions we can also think of this as the buckets we are putting values in (or are getting hashed to)*

Then, the probability that they are mapped to the same value, i.e. that near points collide is:

$$dist(x, y) \leq r : P(g_i(x) = g_i(y)) \geq (1 - \frac{r}{d})^k = p_1^k \tag{1.19}$$

And that they don't collide:

$$dist(x,y) \geq Cr : P(g_i(x) = g_i(y)) \leq (1 - \frac{Cr}{d})^k = p_2^k \tag{1.20}$$

This is a locality-sensitive hashing family for hamming distance. In this case it is quite similar for the hamming distance case, but it can be more difficult for different distances. Unlike other hashing functions the probability of collision is not constant, but depends on the distance between the points.

We may amplify the probability gap by hashing multiple times. This is a very generic technique that applies to other distance metrics and hashing methods as well.

Definition 7

For $I = (i_1 \dots i_k)$ a sequence of indicies from $[d]$, g_j is defined by $g_I = (x_{i_1}, \dots x_{i_k})$.
For $I = (i_1 \dots i_k)$ picked uniformly and independently from $[d]$,

$$\begin{aligned} \mathbb{P}(g_I(x) = g_I(y)) &= P(x_{i_1} = y_{i_1}, \dots x_{i_k} = y_{i_k}) \\ &= P(x_{i_1} = y_{i_1}) \dots P(x_{i_k} = y_{i_k}) \\ &= 1 - \left(\frac{dist(x,y)}{d}\right)^k \end{aligned} \tag{1.21}$$

So,

$$dist(x,y) \leq r : P(g_I(x) = g_I(y)) \geq 1 - \left(\frac{r}{d}\right)^k = p_1^k \tag{1.22}$$

$$dist(x,y) \geq Cr : P(g_I(x) = g_I(y)) \leq 1 - \left(\frac{Cr}{d}\right)^k = p_2^k \tag{1.23}$$

So this gives us the power to pick k arbitrarily in order to amplify the gap between p_1 and p_2

- k is a parameter that we will choose.

Example

$x = (1,0,0,0,1,1,1,0)$, $I = (3,1,7)$

$$g_I(x) = (0,1,0) \tag{1.24}$$

1.5.2 Two-level hashing

Data structure:

- L hash tables $T_1 \dots T_L$ with $m \geq n$ slots each
- L regular hash functions $h_1 \dots h_L : \{0,1\}^k \to [m]$ from an universal family
- L locality sensitive hash functions $g_{I_1} \dots g_{I_l} : \{0,1\}^d \{0,1\}^k$

Structure and inserting
Store each $x \in P$ in $T_l[h_l(g_{I_l}(x))], l = 1 \dots L$.

- g is the locality sensitive hash function and h is the regular hash function.
- g is used to map the point into buckets which are 'close together' and then h is used to resolve locally clustered points. Collisions can be handled via linear chaining.
- runtime on the order of $O(lp)$ for insertion

Note: universal hash functions are hash functions such that each h_i is a random hash function that such that

$$\forall\, u, v = \{0,1\}^k, u \neq v \tag{1.25}$$

The probability of collision for an universal hash function is small, or formally,

$$P(h_i(u) = h_i(v)) \leq \frac{1}{m} \leq \frac{1}{n} \tag{1.26}$$

TLDR: has a reasonably low probability of collision and is reasonably fast to compute

NEARNEIGHBOUR(P, q)

```
1  num-checked = 0
2  for l = 1 to L
3      i = h_l(g_l(q))
4      set x to the head of T_l[i]
5      while x ≠ ∅
6          if dist(q.x) <= Cr
7              return x
8          num-checked = num-checked + 1
9          if num-checked = 12L + 1 // timeout
10             return FAIL
11     else
12         Set x to the next element in T_l[i]
13 return FAIL
```

In words: for each hash table T_l search through $T(h_l(g_l(x)))$ linked list. If we find a near neighbour, we're good. Otherwise, keep on trying until we timeout (Line 9) (just some somewhat arbitrary constant) or fail otherwise.

Definition 8

Union Bound
For any two events A, B in a probability space ,

$$P(A \text{ or } B) \leq P(A) + P(B) \tag{1.27}$$

And by simple induction this extends to k events

Analysis
Or goal here is to show that the probability of 'bad collisions' or failure is small.

Theorem 3

Let $k = \log_{\frac{1}{p_2}}(n)$. Let $\rho = \frac{\log \frac{1}{p_1}}{\log \frac{1}{p_2}}$ and $L = 2n^{\rho}$. If there exists a point x^* in P such that $dist(x, x^*) \leq r$, then with probability of at least $\frac{2}{3}$ the procedure NEARNEIGHBOUR(P, x) will output some $y \in P$ for which $dist(x, y) \leq Cr$

We assume that there exists $x^* \in P$ such that $dist(q, x^*) \leq r$, i.e. there is some point x^* in the dataset that is somewhat close to q. Therefore there exists a circle of radius Cr centered about q containing all of the points that could satisfy NEARNEIGHBOUR

Conversely we can produce the set F of far-away points as follows

$$F = \{x \in P : dist(q, x) > Cr\} \tag{1.28}$$

NEARNEIGHBOUR(P, x) succeeds if it outputs some $y \in P$ such that $dist(x, y) \leq Cr$. For this to happen we must have:

1. q collides with points in F at most $12L$ times (with multiplicity)
2. q collides with x^*

What is the probability of both happening?

1. **Expected number of collisions with far points**

Comment

Recall that $p_2 = 1 - \frac{Cr}{d}$ is the probability that near points do not collide.

It's really difficult to do this proof with a statement like 'probability of hitting 12L+1 consecutive items in F' since that implies an ordering.

$$\forall l, \forall x \in F : P[g_{I_l}(x) = g_{I_l}(q)] \leq p_2^k = (1 - \frac{Cr}{d})^k \tag{1.29}$$

I.e. the probability of a collision with a far away point is bounded by the probability that near points do not collide

Let us choose k such that

$$p_2^k = (1 - \frac{Cr}{d})^k \leq \frac{1}{n} \Rightarrow k \equiv \frac{\log n}{\log \frac{1}{p_2}} \tag{1.30}$$

Let's define a random indicator variable $z_{x,l}$ which takes on 1 if y collides with x in T_l and 0 otherwise. The number of collisions with far points is then the expectation of this random variable.

$$z_{x,l} = \begin{cases} 1 & \text{if x collides with q in T\_l} \\ 0 & \text{otherwise} \end{cases} \tag{1.31}$$

$$E[X] = \sum_{l=1}^{L} \sum_{x \in F} z_{x,l} \leq \frac{2|F|L}{n} \leq 2L \tag{1.32}$$

[12]

[12] Since we defined F to be the set of far collisions and $|F|$ is just the number of items in F.

Definition 9 **Markov's Inequality**
Let $X \geq 0$ be a random variable. Then for any $x > 0$,

$$P(X > x) < \frac{E[X]}{x} \tag{1.33}$$

Proof By the law of total expectation we can break up the expectation into two parts

$$E[X] = E[X|X \leq x] \cdot P(X \leq x) + E[X|X > x]P(X > x) \tag{1.34}$$

The first term is non-negative and the 2nd term is strictly larger than $xP(X > x)$. So

$$E[X] > xP(X > x) \tag{1.35}$$

□

Note that a similar result holds for

$$P(X \geq x) \leq \frac{E[X]}{x} \tag{1.36}$$

We will also need to know how an element y can collide with q in T_l. A string y will collide with q if $h_l(g_{I_L}(y)) = h_l(g_{I_L}(x))$ for some l. This implies that that collisions happen if

(a) g produces the same output for y and q for some l, i.e the first hash function collides

(b) h_l produces the same output for different inputs[13], i.e. h collides

[13] *Those being the result of the first hash function*

Since we picked each hash table to have $m \geq n$ slots and that h_l is chosen from a family of universal hash functions, we have the probability of h colliding being bound by $\frac{1}{n}$. For g we picked k carefully a little bit prior such that the probability of collision is bounded by $p_2^k \leq \frac{1}{n}$

Therefore we have

$$\begin{aligned} P[z_{x,l} = 1] &= P[g_{I_l}(x) = g_{I_l}(q)] + P[g_{I_l}(x) \neq g_{I_l}(q), h_l(g_{I_l}(x)) \\ &= h_l(g_{I_l}(q))] \leq \frac{1}{n} + \frac{1}{n} \leq \frac{2}{n} \end{aligned} \tag{1.37}$$

[14]

[14] Note that the probability of collision is at most $\frac{1}{n}$ for an universal hashing function

Then we can apply Markov's inequality to get that the probability of the random variable Z representing the number of collisions as

$$P[Z \geq 12L] \leq \frac{2L}{12L} = \frac{1}{6} \tag{1.38}$$

So this property holds with probability of at least $\frac{5}{6}$

This implies that a relationship between Monte Carlo and Las Vegas algorithms; we can take a Las Vegas algorithm and turn it into a Monte Carlo algorithm by timing it out.

2. **Now we have to show the probability of q collides with x^* or *something good***

   For the proof we'll take x^* since we assumed it to be good earlier on.

   This probability is lower-bounded by the probability that there exists a l such that g_l produces the same output for x^* and q. However we want to bring on a upper bound to this probability.

$$\begin{aligned} P(\exists\, l : g_{I_l}(q) = g_{I_l}(x^*)) &= 1 - \prod_{l=1}^{L} P[g_{I_l}(x*) = g_{I_l}(q)] \\ &\geq 1 - (1 - p_1^k)^L \geq 1 - e^{-Lp_1^k} \end{aligned} \tag{1.39}$$

Comment Recall: $p_1 = 1 - \frac{r}{d}$ is the probability of a collision with a near point i.e. $dist(x, y) \leq r$
Also, it's a fact that $1 - x \leq e^{-x}$

Then we can simplify this a little bit by assuming $k = \log_{\frac{1}{p_1}} n$

$$p_1^k = 2^{-k \log_2(\frac{1}{p_1})} = \ldots n^{-\rho} \tag{1.40}$$

So then we have $Lp_1^k = 2$ and x^* collides with x with probability at least $1 - \frac{1}{e^2}$.

By the union bound bound the probability of both properties holding is at least

$$1 - (\frac{1}{6} + \frac{1}{e^2}) > \frac{2}{3} \tag{1.41}$$

which concludes the proof.

This also implies that INSERT, NEARNEIGHBOUR run in $O((k + d)n^\rho)$ (Recall: k is the input string size, d is the number of buckets, and $L \approx n^\rho$). Approximating $1 - x \approx e^{-x}$ we get $\rho = \frac{\log \frac{1}{p_1}}{\log \frac{1}{p_2}} \approx \frac{1}{C}$ and $k = O(d \log(n))$. So overall they run in approximately $O(dn^{\frac{1}{C}} \log n)$ time which is much faster than $\Theta(dn)$ linear search for $C > 1$

Now we may be interested in extending what we have done for Hamming distance for other distance metrics, i.e. having a hash function that is more likely to put nearby points in the same bucket than far away points.

Definition 10 A random hash function h with domain X is *locality sensitive* with parameter p for distance metric d if

$$dist(x, y) \leq r \Rightarrow P(h(x) = h(y)) \geq p_1 \tag{1.42}$$

$$dist(x, y) \geq Cr \Rightarrow P(h(x) = h(y)) \leq p_2 \tag{1.43}$$

Subsection 1.6

Streaming Algorithms

Streaming algorithms are a class of algorithms that are extremely efficient in terms of space and usually in terms of time complexity. They are used in many applications where we want to process enormous amounts of data in a short amount of time. Generally they are algorithms that work in time steps and receive one update per time step.

Here are some definitions and conventions for the following section

- The sequence of updates is called the *stream*
- n: the size of the universe the stream is coming from
- m: the length of the stream (may/may not be given to algorithm)
- Algorithm is only allowed to store a number of bits which is bounded by a polynomial in $\log n$ and $\log m$, i.e. $O(\log^c(nm))$ bits.

Many fundamental streaming problems are summarized by the frequency vector f which describes the number of times each element in the universe appears in the stream.[15]

Some other versions of the streaming model may introduce a richer meaning to update, i..e the *turnstile* model where an update is a pair (i, s) where i identifies the update and s identifies the type of update it is[16]

[15] *The algorithm cannot actually store f since it's size is $O(n)$ which violates the memory bound described earlier*

[16] *For example entering or exiting a subway station*

Example **Give an algorithm that finds the missing number in a stream of $n-1$ numbers containing $n-1$ of the integers $1 \ldots n$**
This can be solved by taking the running sum of the stream and subtracting it from the sum of numbers from $1 \ldots n$. A likewise approach may be adopted to 2 missing numbers by introducing the sum of squares.

1.6.1 Frequent Elements

As a warm-up let's consider the majority problem

- **input:** a stream $\sigma = (i_1 \ldots i_m)$ of updates in $[n] = 1....n$. If there exists $i \in [n]$ such that more than half of the updates in σ are equal to i, the algorithm should output i. Otherwise, it may output any element

The following algorithm proposed by Boyer and Moore solves the problem with only two words of memory

MAJORITY(σ)

```
1  element = i_1
2  count = 1
3  for t = 2 to m
4      if element == i_t
5          count = count + 1
6      elseif count > 0
7          count = count − 1
8      else
9          element = i_t
10         count = 1
11 return element
```

Proof A naive implementation of a solution could involve storing f, i.e pairs of each unique element and it's associated frequency. However this uses a lot more space than what we desire. We note that for this problem we don't care about the actual frequency of each element, only the element with the highest frequency, so the problem solution may be relaxed to only store the element [with the highest frequency]. Likewise, we don't care about the frequency of the element either – only that it is the highest frequency element. The above algorithm maintains the invariant that $f_{element} \leq count + \frac{m}{2}$.
Therefore by the time the algorithm finishes executing $element$ will contain the most frequent element (if there exists one) or some arbitrary element otherwise. Intuitively we may think of the procedure "allocating" space for one non-current-majority element on lines 4-5, which can then be taken away if we later inspect a non-current-majority element (lines 6-7). Then it can be concluded that if an element 'survives' until the end of the algorithm it must either 1. be the majority element, if one exists, or 2. any arbitrary element, if none exists.

Comment Note that Since this algorithm does not tell you if a strict majority exists or not, to verify it's results you will have to run over the data set again to determine if the resulting value is the majority value or just *any* element.

□

An extension of Majority is given by Misra and Gries to find all elements that appear in more than $\frac{1}{k}$ of the updates

```
FREQUENT(σ, k)
1  S = ∅
2  for t = 1 → m
3      if ∃ x ∈ S such that x.elem == i_t
4          x.count++
5      elseif |S| < k − 1
6          Create x with x.elem = i_t and x.count = 1
7          S = S ∪ x
8      else
9          for x ∈ S
10             x.count−−
11             if x.count == 0
12                 S = S \ {x}
13 return S
```

Theorem 4 The set S output by Frequent contains all $i \in [n]$ such that $f_i > \frac{m}{k}$. Moreover for any $x \in S$, $f_{x.elem} \leq x.count + \frac{m}{k}$

Proof This algorithm is an extension of the Majority algorithm described prior; whereas majority maintained a single element-count pair, frequent maintains k element-count pairs in S and updates them accordingly with the same count-tracking[17]. One key difference is on line 11, where if i_t is not in S then we decrement the counts for *every* element in S. If an entry in S has a count of 0 then we boot it out to make space for a new one. □

[17] *sort of like amortized analysis in a way; encountering an i_t that exists S gives you a dollar to save, and encountering an i_t that doesn't exist in S causes you to spend a dollar*

Another streaming problem that is of interest is the *distinct elements count* problem, i.e. given an input stream we want to know how many distinct integers we have seen in the stream so far. In other words, we want to approximate the frequency vector but only track elements with frequency greater than 0.

To relax our initial analysis let's consider a relaxed problem where, provided that there is some oracle which can tell us a number $\tilde{F}_0$ such that

$$F_0 \leq \tilde{F}_0 \leq 2F_0 \tag{1.44}$$

Our algorithm will then refine this loose estimate to a more precise one through information captured in the stream.

Distinct-simple$(\sigma, k, \tilde{F}_0)$
1 $S = \emptyset$
2 $d = \lceil \log_2(\tilde{F}_0/k) \rceil$
3 $K = \lceil \log_2 n \rceil$
4 Pick a hash function $h : [n] \to \{0,1\}^L$
5 **for** $t = 1 \to m$
6 **if** $h(i_t) \in o^d \{0,1\}^{L-d}$ and $i_t \notin S$
7 $S = S \cup \{i_t\}$
8 **return** $\hat{F}_0 = 2^d \cdot |S|$

1. k is some constant that we get to pick
2. Take h to behave like a random function; basically simple uniform hashing assumption

Recall $\sigma = i_1 \dots i_t$ is the input stream

$$\textbf{if } h(i_t) \in 0^d\{0,1\}^{L-d} \text{ and } i_t \notin S$$

Figure 6. Treat notation here like a regexp; i.e. d 0-s followed by $L - d$ 1s or 0s

3.

This algorithm attempts to maintain set S such that it contains each element that appears with probability 2^{-d} and therefore $E[|S|]$ is 2^{-d} of the elements that appear in the stream[18] k is a constant that we get to pick – the larger it is, the more space our algorithm uses (but the more accurate it becomes).

[18] So $\hat{F}_0 = 2^d|S|$

Definition 11

Recall: **Variance**: a measure of how much a random variable X deviates from its expectation on average, i.e. the expectation of the difference between X and it's expectation.

$$\begin{aligned} Var(X) &= E[(X - E[X])^2] \\ &= E[X^2] - E[X]^2 \end{aligned} \tag{1.45}$$

An useful property of variance is the sum of independent random variables is equal to the sum of their variances

$$Var(\sum_{i=1}^{N} X_i) = \sum_{i=1}^{N} Var(X_i) \tag{1.46}$$

Another tool that will be useful for this analysis is *Chebyshev's Inequality*

Definition 12

Chebyshev's Inequality

$$P(|X - E[X]| > t) < \frac{Var(X)}{t^2} \tag{1.47}$$

Proof

A proof of this inequality follows from Markov's inequality discussed in the previous lecture on locality sensitive hashing.
Define $Z = (X - E[X])^2$. Since this implies $Z \geq 0$ and $Var(X) = E[Z]$ by definition,

$$|X - E[X]| > t \Leftrightarrow X > t^2 \tag{1.48}$$

So we can just apply Markov's inequality to the above statement to give

$$P(|X - E[X]| > t) = P(Z > t^2) < \frac{Var(X)}{t^2} \tag{1.49}$$

A nice property of Chebyshev's inequality is that it doesn't assume the random variable is non-negative and bounds the probability in both directions as well. □

Theorem 5 If $\tilde{F}_0$ satisfies our prior assertion that $F_0 \leq \tilde{F}_0 \leq 2F_0$, then $\hat{F}_0$ output by distinct-simple$(\sigma, \tilde{F}_0, k)$ satisfies

$$(1 - \frac{\sqrt{8}}{\sqrt{k}}) \leq \hat{F}_0 \leq (1 + \frac{\sqrt{8}}{\sqrt{k}}) \tag{1.50}$$

With probability $> \frac{1}{2}$ and uses $O(k)$ memory.

Proof Define D to be the set of $i \in [n]$ that appear in the stream. By definition, $F_0 = |D|$. Define X_i as an indicator random variable, i.e.

$$X_i = \begin{cases} 1 & \text{if } i \in D \\ 0 & \text{otherwise} \end{cases} \tag{1.51}$$

Then,

$$E[X_i] = P(i \in S) = 2^{-d} \tag{1.52}$$

Since $h(i)$ is uniformly random in $\{0,1\}^L$ and by construction 2^{L-d} of 2^L strings have d leftmost bits set to 0.
Then,

$$E[|S|] = E[\sum_{i \in D} X_i] = \sum_{i \in D} P(i \in S) = 2^{-d} F_0 \tag{1.53}$$

Since we chose d such that $2^{-d}\tilde{F}_0 \leq k$, we have

$$E[|S|] \leq 2^{-d} F_0 \leq 2^{-d}\tilde{F}_0 \leq k \tag{1.54}$$

It follows that the algorithm takes $O(k)$ memory in expectation since S dominates the memory use

However this is not enough to show that $\hat{F}_0$ is close to F_0 even though it is easy to see that they are equal in expectation. To show this we can apply Chebyshev's inequality, i.e. if the variance of $\hat{F}_0$ is small it is likely to be close to its expectation.
We know that

$$Var(|S|) = \sum_{i \in D} Var(X_i) \tag{1.55}$$

and that

$$Var(X_i) = E[X_i^2] - E[X_i]^2 \leq E[X_i^2] = 2^{-d} \tag{1.56}$$

So we get $Var(|S|) \leq 2^{-d} F_0 = E[|S|]$
So, if we let $\varepsilon = \frac{\sqrt{8}}{\sqrt{k}}$, then by Chebyshev we get

$$\begin{aligned} P(\hat{F}_0 - F_0) \geq \varepsilon F_0) &= P(|\hat{F}_0 - E[\hat{F}_0|] \geq \varepsilon E[\hat{F}_0]) \\ &= P(|S| - E[|S|] \geq \varepsilon E[|S|]) \\ &\leq \frac{Var(|S|)}{\varepsilon^2 E[|S|^2]} \leq \frac{1}{\varepsilon^2 E[|S|]} \end{aligned} \tag{1.57}$$

So if the expected size of S is not too small we have a large probability of getting an accurate estimate.
Rearranging, our prior choice that $2^d \leq 2\tilde{F}_0/k$, we get

$$E[|S|] = 2^{-d} F_0 \geq 2^{-d-1}\tilde{F}_0 \geq \frac{k}{4} \tag{1.58}$$

Plugging this bound for $E[|S|]$ into the expression obtained by Chebyshev's inequality we get

$$P(|\hat{F}_0 - F_0| \geq \varepsilon F_0) \leq \frac{4}{\varepsilon^2 k} = \frac{1}{2} \tag{1.59}$$

□

1.6.2 Adaptive Sampling

A single-pass streaming algorithm for distinct counts which does not assume an estimate $\tilde{F}_0$.

Distinct(σ, k)

```
S = ∅, d = 0, L = ⌈log_2 n⌉
Pick a hash function h : [n] → {0,1}^L
for t = 1 → m
    if h(i_t) ∈ 0^d {0,1}^{L−d} and i_t ∉ S
        S = s ∪ {i_t}
    while |S| > k
        d = d + 1
        T = ∅
        for j ∈ S
        if h(j) ∈ 0^d {0,1}^{L−d}
            T = T ∪ {j}
        S = T
return F̂_0 = 2^d · |S|
```

Comment The idea behind this algorithm is to apply the concepts in Distinct-Simple to set d and the sample rate adaptively while keeping the invariant of $|S| \leq k$.

Theorem 6 Let $0 \leq \varepsilon \leq \frac{1}{3}$ and $k \geq \frac{16}{\varepsilon^2}$.[19] Then, with probability at least $\frac{1}{2}$ the estimate $\hat{F}_0$ output by Distinct(σ, k) satisfies

$$(1-\varepsilon)F_0 \leq \hat{F}_0 \leq (1+\varepsilon) \cdot F_0 \tag{1.60}$$

[19] *In the lecture note we use $k \geq 144$ to get a bound with $\varepsilon = \frac{4}{\sqrt{k}}$*

Proof Let $D = \{i : f_i > 0\}$ be the distinct elements that appear in σ, and $S_l = \left\{i \in D : h(i) \in 0^l \{0,1\}^{L-l}\right\}$ be the elements in σ whose hash value starts with l zeros. By this definition $|D| = F_0$.

$$E[|S_l|] = 2^{-l}|D| = 2^{-l}F_0 \tag{1.61}$$

$$Var[|S_l|] = 2^{-l}(1-s^{-l}) \leq E[|S_l|] = 2^{-l}F_0 \tag{1.62}$$

At the end of σ, $d = min\{l : |s_l| \leq k\}$. [20] Can think of the algorithm as keeping s_0 at first, and then if there's too much in s_0 it will move on to s_1 and so forth. The output of the algorithm, $\hat{F}_0$ is given by $2^d|S_d|$. So this algorithm searches for the first l such that $|S_l| \leq k$. Note that the expected sizes of S_l halve with each increment in l.

[20] *All of these variables here are random.*

For correctness we need

$$a == \lfloor \log_2((1-\varepsilon)F_0/k) \rfloor \; b == \lceil \log_2((1+\varepsilon)F_0/k) \rceil \tag{1.63}$$

Chosen such that $a \leq d \leq B$

$$\frac{(1-\varepsilon)F_0}{2k} \leq 2^a \leq \frac{(1-\varepsilon)F_0}{k} \tag{1.64}$$

This implies that

$$(1+\varepsilon E[|S_a|]) = (1+\varepsilon)2^{-a}F_0 \geq k \tag{1.65}$$

$$\frac{(1+\varepsilon)F_0}{k} \leq 2^b \leq \frac{2(1+\varepsilon)F_0}{k} \tag{1.66}$$

This implies that

$$(1+\varepsilon)E|S_b| \leq k \tag{1.67}$$

We will now show that the following probabilities $\geq \frac{5}{6}$ which is $\geq \frac{1}{2}$, which implies that the probability of all 3 is at least $\frac{1}{2}$.

$$(1-\varepsilon)F_0 < 2^a|S_a| \leq (1+\varepsilon)F_0 \geq \frac{5}{6} \tag{1.68}$$

For this one:

$$\begin{aligned} \Rightarrow |S_a| > (1-\varepsilon)2^{-a} \qquad F_0 \geq k \\ |S_a| > k \Rightarrow a < d \end{aligned} \tag{1.69}$$

$$|S_b| \leq k \Rightarrow d \leq b \tag{1.70}$$

$$(1-\varepsilon)F_0 < 2^{b+1}|S_{b-1}| \leq (1+\varepsilon)F_0 \geq \frac{5}{6} \tag{1.71}$$

$$(1-\varepsilon)F_0 < 2^{b}|S_{b}| \leq (1+\varepsilon)F_0 \geq \frac{5}{6} \tag{1.72}$$

Note: if ε small enough and after some calculations we can show that

$$B \leq a+2 \tag{1.73}$$

$$(1-\varepsilon)|S_a| = (1-\varepsilon)2^{-a}F_0 \geq k \tag{1.74}$$

□

Subsection 1.7

Linear Programming

A linear programming is an optimization problem defined by linear inequalities and equalities. The following form will be used in this class:

Definition 13 Linear program:

$$max c^T x \text{ s.t. } Ax \leq b \qquad x \geq 0 \tag{1.75}$$

Where A is a $m \times n$ matrix (m constraints, n variables), b is a $m \times 1$ column vector. c is a $n \times 1$ column vector, and x is a $n \times 1$ objective column vector. The inequalities are such that the inequality should hold for all elements of the above matrix expression at the same time. The set of x that satisfy the constraints is called the feasible set, and the LP is *infeasible* if the feasible set is empty.

For example,

$$\begin{aligned} min x_1 + x_2 + x_3 s.t. & \\ x_1 + x_2 &= 1 \\ x_2 + x_3 &\geq 1 \\ x_1 + x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \tag{1.76}$$

Which corresponds to the following matricies

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1.77}$$

Geometrically we may understand LPs as some sort of n-dimensional polyhedron comprised of the intersection of the halfspaces defined by the hyperplanes each inequality represents. In more simple terms: each inequality defines a supporting hyperplane $\left\{x \in \mathbb{R}^n : a^T x = b\right\}$ One *halfside*, $\left\{x \in \mathbb{R}^n : a^T x \leq b\right\}$ of the hyperplane gives admissible solutions to the inequality. The polyhedron that contains all of the solutions to the inequality is then given by the intersection of all the hyperplanes in the set. A polyhedron P is *unbounded* when there exists a point x such that $x \in P$ and a direction v for which for every $t \geq 0$, $x + tv \in P$. A bounded polyhedron, i.e. not unbounded is a *polytope*.

Definition 14 Formally a **face** of a polyhedron is a set of the type

$$F = \{x : Ax \leq b\} \cap \{x : a_i x = b_i \forall\, i \in S\} \tag{1.78}$$

Where a_i is the ith row of A and S is some subset of the rows of A.
A j-face is a face where the rank of the sub-matrix A_f of A for that face is $n - j$.

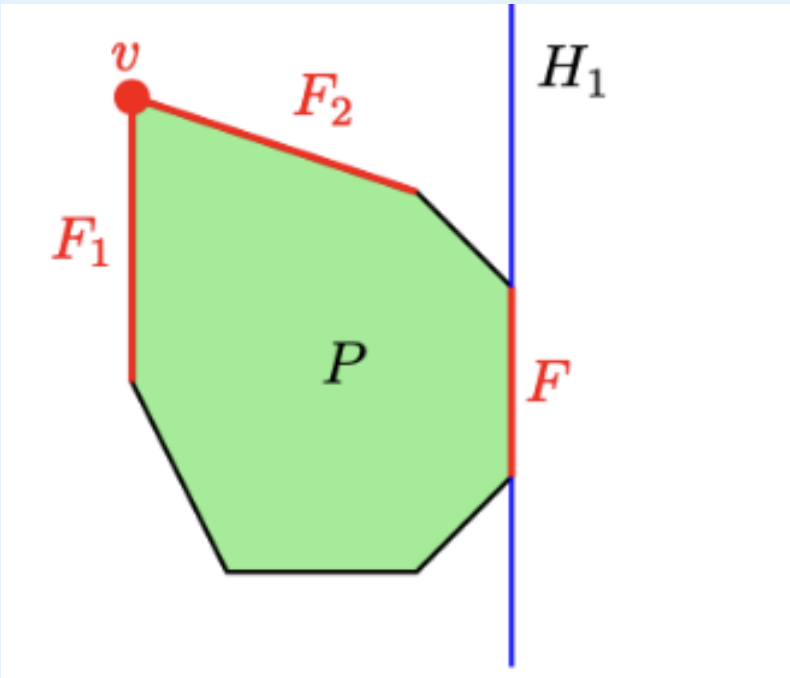


Figure 7. In this example the triangle $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a facet (2-face); $F = P \cup \{x_1 + x_2 + x_3 = 1\}$. The edge F_1 or F_2 is a 1-face, and the vertex v is a 0-face

Definition 15 A **Vertex** of a polytope $P = \{x \in \mathbb{R}^n : Ax < b\}$ is a point in P we can get by setting n linearly independent constraints to equality

$$\begin{aligned} x_1 + x_2 \geq 10 &\implies x_1 + x_2 = 10 \\ x_2 + x_3 \leq 15 &\implies x_2 + x_3 = 15 \end{aligned} \tag{1.79}$$

Theorem 7 For any polytope P with verticies $v_1 \dots v_N$, any $x \in P$ can be witten as $x = \lambda_1 v_1 + \dots + \lambda_N v_N$, where

$$\sum_{i=1}^{N} \lambda_i = 1 \tag{1.80}$$

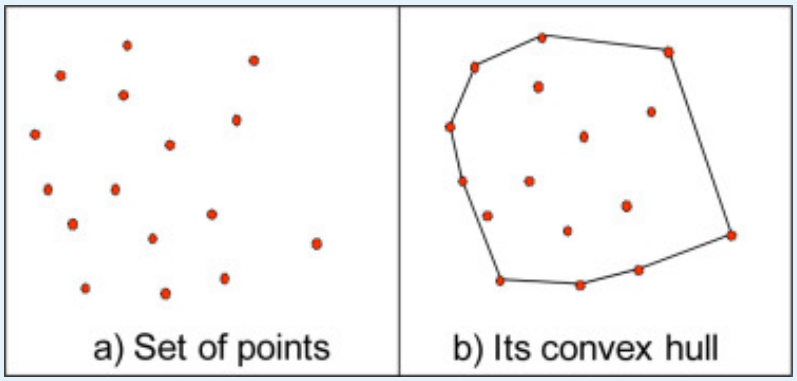
And

$$\lambda_i \geq 0 \tag{1.81}$$

Recall that some vertex $v \in P$ is an optimal solution to the LP, i.e. $c^T x$ is minimized or maximized.

Definition 16 **Convexity**: A set $S \in \mathbb{R}^n$ is convex if for any two points x, y in S the line segment between x and y is contained in S.

Definition 17 **Convex Hull**: $v_1, \dots, v_n \in \mathbb{R}^n$ is the smallest convex set of $S \in \mathbb{R}^n$ containing $v_1 \dots, v_N$. This can be imagined as a *shrink-wrap* of the points.



Algebraically this can be written as

$$\text{conv-hull}(\{v_1 \ldots, v_N\}) = \{\lambda_1 v_1 + \ldots, h_N v_N : \lambda_1 \ldots, \lambda_N \geq 0, \lambda_1 + \ldots, +\lambda_N = 1\} \tag{1.82}$$

Many proofs involving convex hulls may be partially resolved with the fact that the convex hull of two points is the line between them, and a little bit of induction.

1.7.1 LP Examples

Comment In order to use linear programming we must need to know how to frame questions as LP questions.

Example Menu planning
Given the following list of prices & nutritional values for a set of foods, find the minimum additional price for dish to meet nutritional requirements?

Food	Carrot, Raw	White Cabbage, Raw	Cucumber, Pickled	Required per dish
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg/kg]	60	300	10	15 mg
Dietary Fiber [g/kg]	30	20	10	4 g
price [€/kg]	0.75	0.5	0.15*	—

*Residual accounting price of the inventory, most likely unsalable

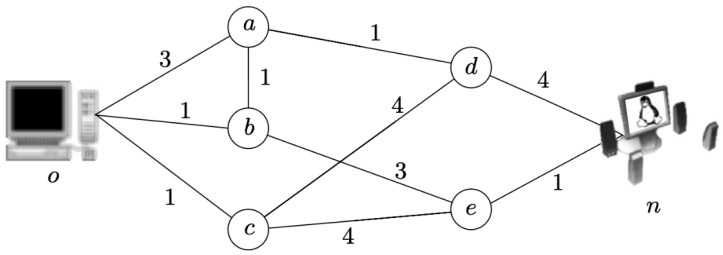
$$\begin{aligned}
\text{Minimize} \quad & 0.75x_1 + 0.5x_2 + 0.15x_3 \\
\text{subject to} \quad & x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0 \\
& 35x_1 + 0.5x_2 + 0.5x_3 \geq 0.5 \\
& 60x_1 + 300x_2 + 10x_3 \geq 15 \\
& 30x_1 + 20x_2 + 10x_3 \geq 4.
\end{aligned}$$

The minimization is simply the price multiplied by the amount. The constraints can be described as follows:

- There should be non-zero carrots, white cabbage, and pickles

- The sum of vitamin A across the dish should be at least 0.5mg
- And so forth...

Example Network flow



What is the maximum transfer rate from the old computer o to the new computer n?
Let's introduce a variable x_{ab} which specifies the rate at which data is transferred from a to b for each link in the network. In this graph we have 10 such variables.
The linear program is then as follows

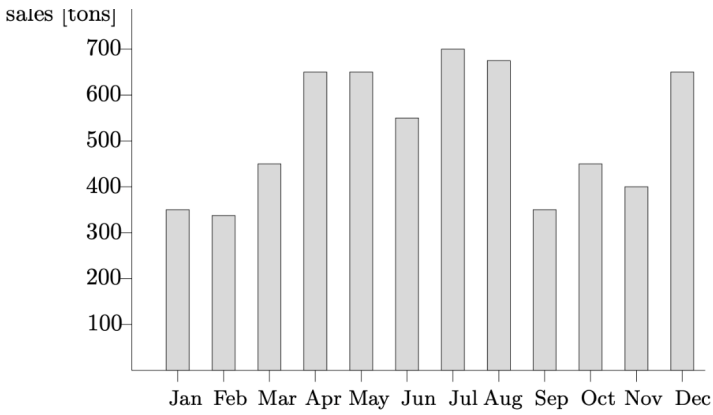
$$
\begin{aligned}
\text{Maximize} \quad & x_{oa} + x_{ob} + x_{oc} \\
\text{subject to} \quad & -3 \le x_{oa} \le 3, \quad -1 \le x_{ob} \le 1, \quad -1 \le x_{oc} \le 1 \\
& -1 \le x_{ab} \le 1, \quad -1 \le x_{ad} \le 1, \quad -3 \le x_{be} \le 3 \\
& -4 \le x_{cd} \le 4, \quad -4 \le x_{ce} \le 4, \quad -4 \le x_{dn} \le 4 \\
& -1 \le x_{en} \le 1 \\
& x_{oa} = x_{ab} + x_{ad} \\
& x_{ob} + x_{ab} = x_{be} \\
& x_{oc} = x_{cd} + x_{ce} \\
& x_{ad} + x_{cd} = x_{dn} \\
& x_{be} + x_{ce} = x_{en}.
\end{aligned}
$$

Our goal is to maximize the flow out of computer o under the assumption that, since the data is neither stored or lost, it must be received by n at the same rate. The next constraints restrict the transfer rates along each of the individual links, i.e. x_{cd} can transmit at a rate of up to 4 forwards or backwards ($-4 \le x_{cd \le 4}$). The relations between the nodes are then captured in the last few equality constraints and effectively say that whatever leaves each node must leave it immediately [21].

[21] $x_{oa} = x_{ab} + x_{ad}$: *flow from* $o \to a$ *must leave through* a *to either* b, d

Consider the following problem: given a projected mostly ice cream sales for the next year, how can produce a production schedule with the minimum cost?

Example



A simple solution might be JIT[22] production – but this can be expensive due to temp workers and machine adjustments, etc. It may be better to spread out production and to build up stock.

[22] *just-in-time (heh)*

Let's formalize this problem as follows:

- Demand in month i is d_i
- x_i is the production in month i
- s_i is the total surplus in store at end of month i
- To meet demand in month i we may use the production in month i and the surplus from $i-1$; $x_i + s_{i-1} \geq d_i$
- And the surplus after month i is $x_i + s_{i-1} - s_i = d_i$
- Assume initial surplus is $s_0 = 0$, and we want $s_{12} = 0$

Let's take the cost of changing production by 1 ton between two months to be 50 and the storage cost for 1 ton of ice cream is 20

Total cost:

$$50 \sum_{i=1}^{12} |x_i - x_{i-1}| + 20 \sum_{i=1}^{12} s_i \tag{1.83}$$

This cost function is unfortunately not linear, but we can use the following trick to make it linear: since the change in production is either an increase or decrease, we may introduce $y_i \geq 0$ for the increase and $z_i \geq 0$ for the decrease to get

$$x_i - x_{i-1} = y_i - y_z \qquad \text{and} \qquad |x_i - x_{i-1}| = y_i + z_i \tag{1.84}$$

$$\begin{array}{ll}\text{Minimize} & 50\sum_{i=1}^{12} y_i + 50\sum_{i=1}^{12} z_i + 20\sum_{i=1}^{12} s_i \\ \text{subject to} & x_i + s_{i-1} - s_i = d_i \text{ for } i = 1, 2, \ldots, 12 \\ & x_i - x_{i-1} = y_i - z_i \text{ for } i = 1, 2, \ldots, 12 \\ & x_0 = 0 \\ & s_0 = 0 \\ & s_{12} = 0 \\ & x_i, s_i, y_i, z_i \geq 0 \text{ for } i = 1, 2, \ldots, 12. \end{array}$$

Figure 8. The linear program that follows

1.7.2 Duality

How can I convince you that the solution to a linear program is optimal?

$$\begin{gathered} max = x_1 \\ \text{given} \\ x_1 + x_2 + x_3 \leq 1 \\ x_1 \cdot x_2 \cdot x_3 \geq 0 \end{gathered} \tag{1.85}$$

We know that the optimal value of this LP is $x_1 = 1, x_2 = 0, x_3 = 0$.

Or, to solve the example given at the beginning of this section (equation 1.77), we have the solution $x_1 = x_2 = x_3 = \frac{1}{2} \implies$ value $\leq \frac{3}{2}$.

Note symmetry in solution!

If we were to multiply each inequality by half and add them up, we get

$$x_1 + x_2 + x_3 \geq \frac{3}{2} \tag{1.86}$$

So here we were able to put a lower bound on the optimal value of the LP, but we don't know abut the upper bound. We can do this by using duality. Before we do that, here's the above logic formalized:

Given the LP in general form

$$max c^T x \text{ s.t. } Ax \leq b \qquad x \geq 0 \tag{1.87}$$

We may apply the technique of dropping values and multiplying the inequalities used above.

Let's define $y \geq 0$ as the dual variables which are applied onto the inequality as follows:

$$y(Ax \leq b) \tag{1.88}$$

Comment Only multiply by non-negative constants to avoid messing up the inequalities

Then,

$$yAx \leq yb \tag{1.89}$$

If every row of yA is greater than equal to c_i, then the objective value is upper-bounded by yb!

And as it turns out this is just yet another linear program for minimization over choices of y that we can solve with our existing LP techniques.

if $max c^T x$ where $Ax \leq b, x \geq 0$ is the original LP, we call it the **primal** LP, and $min b^T y$ where $A^T y \geq c, y \geq 0$ is the **dual** LP. Refer to handout for more primal-dual pairs.

Theorem 8 If both primal and dual LPs are feasible, then their values are equal and the optimal value is the solution to either LP.

Section 2

ECE568 Computer Security

Subsection 2.1

Refresher & Introduction

Comment I've found that the way that this course is organized does not lend itself well to well-organized headers and notes. Apologies for the train-of-thought style.

Software systems are ubiquitous and critical. Therefore it is important to learn how to protect against malicious actors. This course covers attack vectors and ways to design software securely

Data representation: It's important to recognize that data is just a collection of bits and it is up to us to tell the computer how it should be interpreted. Oftentimes we can make assumptions, for example assume that an int is an int. But what if we end up being wrong about it? Many security exploits rely on data being interpreted in a different way than originally intended. For example,

```
1 unsigned long int h = 0x6f6c6c6548; // ascii for hello
2 unsigned long int w = 431316168567; // ascii for world
3 printf("%s %s", (char*) h, (char*) w);
```

Listing 1: An innocent example of where we should be careful about data representation. This prints hello world

This courses makes use of Intel assembler. TLDR:

- 6 General-purpose registers
- RAX (64b), EAX(32b), AX(16b), AH/AL(8b), etc

Note that the stack grows downwards and the heap grows upwards. Stack overflows can occur and can be a source of vulnerability.

GDB offers some tools for examining stacks

- `break`: create a new breakpoint
- `run`: start a new process
- `where`: list of current stack frames
- `up/down`: move between frames
- `info frame` display info on current frame
- `info args`: list function arguments
- `info locals`: list local variables
- `print`: display a variable

- `x` display contents of memory
- `fork`: Creates a new child process by duplicating the parent. The child has its own new unique process ID
- `exec`: Replaces the current process with a new process

The fork-exec technique is just a pair of `fork` and `exec` system calls to spawn a new program in a new process

2.1.1 Security Fundamentals

The three key components of security are:

- Confidentiality: the protection of data/resources from exposure, whether it be the content or the knowledge that the resource exists in the first place. Usually via organizational controls (security training), access rules, and cryptography.
- Integrity: Trustworthiness of data (contents, origin). Via monitoring, auditing, and cryptography.
- Availability: Ability to access/use a resource as desired. Can be hard to ensure; uptime, etc...

Together they form an acryonym: CIA. A system is considered secure if it has all three of these properties for a given time. The strength of cryptographic systems can be evaluated by the number of bits of entropy or their complexity. For example, a 128-bit key has 2$\hat{}$128 possible values. This would take a lot of time to break, and a 256-bit key even longer. Availability is harder to measure quantitatively and is instead traditionally measured qualitatively. For example, a system may be available 99.9% of the time. But this doesn't really measure w.r.t security.

Some security terms:

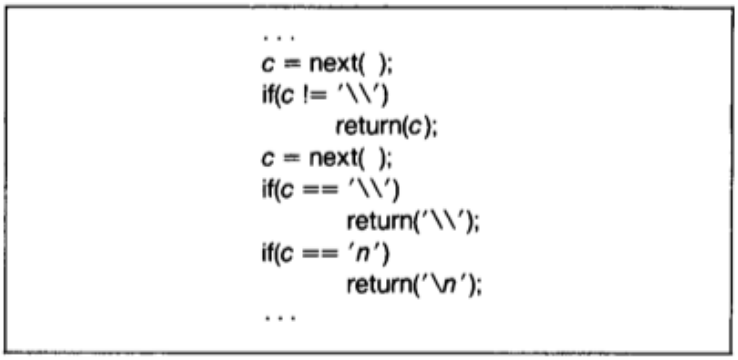
- Another security concept is the **threat**, or any method that can breach security.
- An exercise of a threat is called an **exploit** and a successful exploit causes the system to be compromised. Common threats include internet connections/open ports.
- **Vulnerabilities** are flaws that that weaken the security of a system and can be difficult to detect. For example an unchecked string copy can cause a buffer overflow and allow an attacker to execute arbitrary code
- **Compromises** are the intersection between threats and Vulnerabilities, i.e. when an attacker matches a threat with a vulnerability (i.e. matching a tool in the attacker's arsenal with a weakness)
- **Trust** : How much exposure a system has to an interface. For example a PC might have a lot of trust in the user.

The leading cause of computer security breaches are humans. We are prone to making mistakes. A general trade-off exists when designing secure systems for humans; the more secure a system becomes the less usable it tends to be. One way of measuring the quality of a security system is how secure it is while maintaining usability

2.1.2 Reflections on Trusting Trust

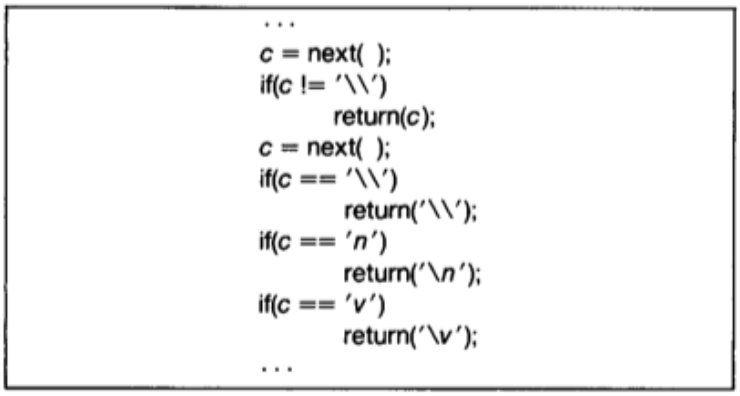
Comment

> **Reflections on Trusting Trust** is a paper by Ken Thompson that discusses the trust and security in computing. Cool short read.



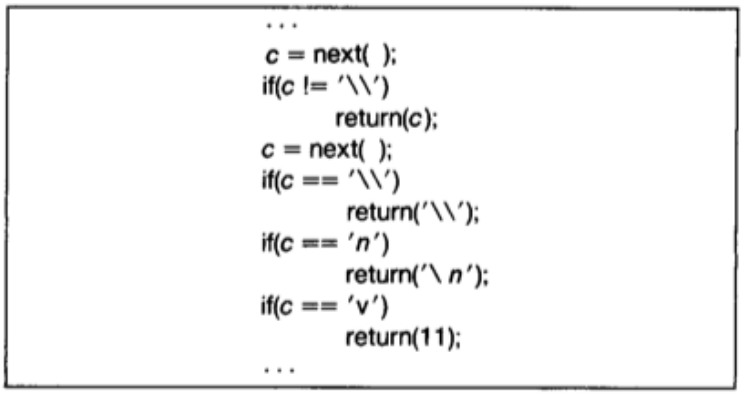
```
...
c = next( );
if(c != '\\')
        return(c);
c = next( );
if(c == '\\')
        return('\\');
if(c == 'n')
        return('\n');
...
```

FIGURE 2.2.



```
...
c = next( );
if(c != '\\')
        return(c);
c = next( );
if(c == '\\')
        return('\\');
if(c == 'n')
        return('\n');
if(c == 'v')
        return('\v');
...
```

FIGURE 2.1.



```
...
c = next( );
if(c != '\\')
        return(c);
c = next( );
if(c == '\\')
        return('\\');
if(c == 'n')
        return('\ n');
if(c == 'v')
        return(11);
...
```

FIGURE 2.3.

Figure 9. Teaching a compiler what the "\v" sequence is. We may add a statement to return the ascii encoding of \v (11), compile the compiler, and then use it to compile a program that knows what \v is.

. We may then alter the source to be like Figure 2.3 without any mention of \v but still compile programs with \v just fine.

```
compile(s)
char *s;
{
        ...
}
```

FIGURE 3.1.

```
compile(s)
char *s;
{
        if(match(s, "pattern")) {
                compile("bug");
                return;
        }
        ...
}
```

FIGURE 3.2.

```
compile(s)
char *s;
{
        if(match(s, "pattern1")) {
                compile ("bug1");
                return;
        }
        if(match(s, "pattern 2")) {
                compile ("bug 2");
                return;
        }
        ...
}
```

FIGURE 3.3.

Next, consider the above scenario where we insert a login Trojan to insert backdoors into code matching the unix login function. We may then compile the c compiler to do just that, and then change the source to what it should look like without the Trojan. Compiling the compiler one more time will now produce a compiler binary that looks completely innocent but will reinsert the Trojan wherever it can.

The moral of the story is that you can't trust code that you didn't totally create yourself. But it's awfully difficult to use only code written by oneself. So take security seriously.

Subsection 2.2

Software Code Vulnerabilities

Recall: the stack is used to keep track of return addresses across function calls; storing a breadcrumb trail. Another key thing sitting in the stack are local variables. A common theme in the course is that computing tends to conflate execution instructions with data.

Common data formats and structures create an opportunity for things to get confused (and for attackers to take advantage of). For example, a buffer-overflow attack can end up overwriting that return address breadcrumb trail and then execute arbitrary code.

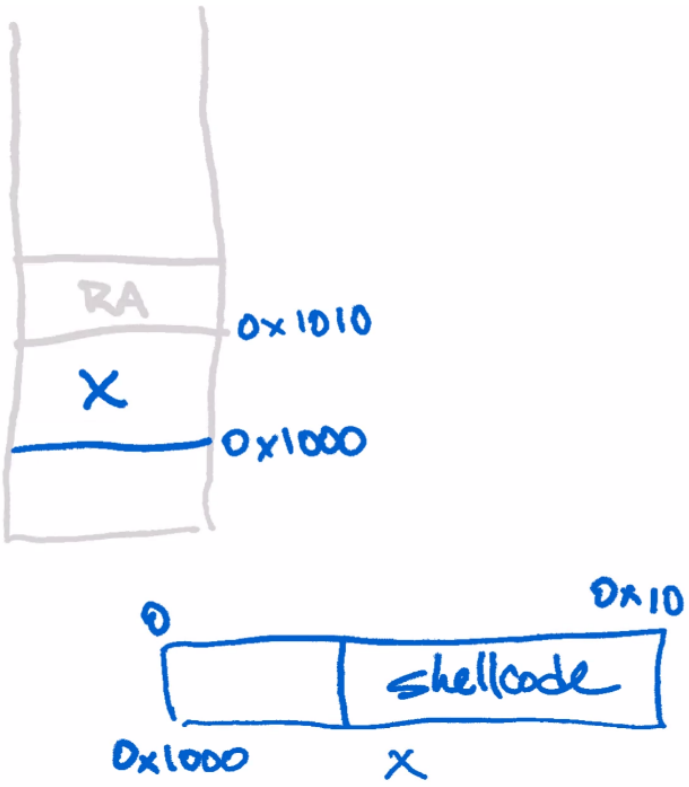


Figure 10. Bufferoverflow to write to the return address. Shellcode is a sequence of instructions that is used as the payload of an attack. It si called a shellcode because they commonly are used to start a shell from which the attacker can do more.

There are ways to find out where that return address is (or at least reasonably guess). This is discussed more in detail later; for now we'll assume that they have it figured out.

A common technique to make this easier is to inject a bunch of `NOP`s before the start of the shellcode. So that we don't need to be as precise as needed in order to find the shellcode start.

One technique for finding the RA would be to incrementally increase the size of the buffer overflow until we get a segfault – at this point the segfault would tell you what memory address it was trying to access and possibly the values it saw there instead as well.

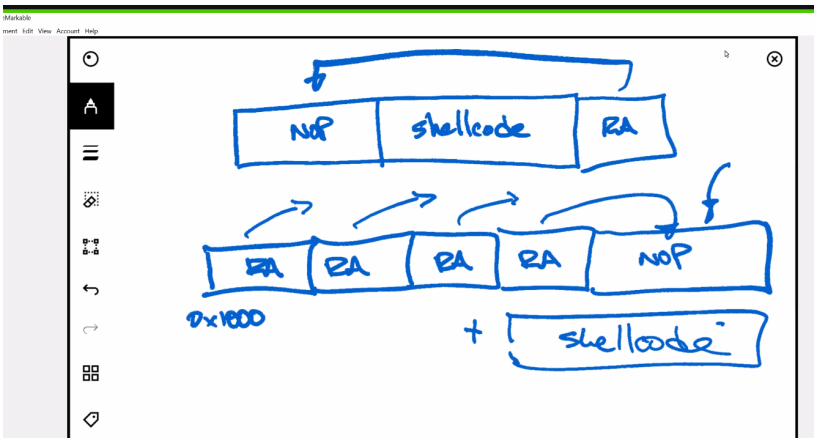


Figure 11. Can create a RA sled with a NOP leading to shellcode and then try it from e.g. 0x1000, 0x2000 and so forth to find where to attack from.

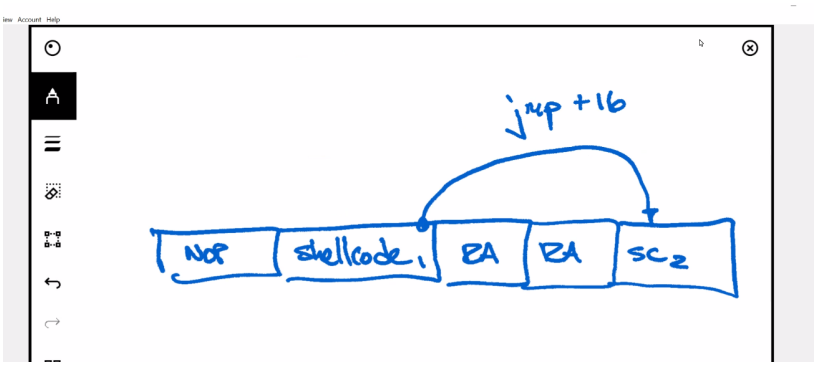


Figure 12. Another technique may involve placing shellcode all over the place, of which each one may be a valid entrypoint into the shellcode.

Subsection 2.3

Format string Vulnerabilities

```
1 spirntf(buf, "Hello %s", name);
```

`sprint` is similar to `printf` except the output is copied into `buf`. The vulnerability is simiar to the buffer overflow vulnerability. The difference is that the attacker can control the format string.

Consider the following:

```
1 char* str = "Hello world";
2 printf(str); // 1
3 printf("%s", str); // 2
```

Despite it looking different there are differences in these two ways to print hello world. The first argument is a format string, which is different from just a parameter. A format string contains both instructions for the `c` printing library as well as data. This means that the first method can be exploited if the attacker has access to the format string. A more complex vulnerability is with `snprintf` (which limits the number of characters written into buf).

```c
void main() {
    const int len = 10;
    char buf[len];
    snprintf(buf, len, "AB%d%d", 5, 6);
    // buf is now "AB56"
}
```

- Arguments are pushed to the stack in reverse order
- snprintf copies data from the format string until it reaches a %. The next argument is then fetched and outputted in the requested format
- What happens if there are more % parameters than arguments? The argument pointer keeps moving up the stack and then points to values in the previous frame (and could actually look at your entire program memory, really)

```c
void main () {
    char buf[256];
    snprintf(buf, 256,
    "AB,%08x,%08x,%08x,%08x,%08x,%08x,%08x,%08x,%08x,%08x", 5);
    printf(buf);
    // AB,00000005,00000000,29ee6890,302c4241,2c353030,30303030,
    // 39383665,32346332,33353363,30333033
    // if we look at the 3rd clause as ascii we get '0,BA'
   //  (recall intel little endian) i.e. we've read up far enough to
    see the local variable
   // specifying the format string pushed onto the stack earlier
}
```

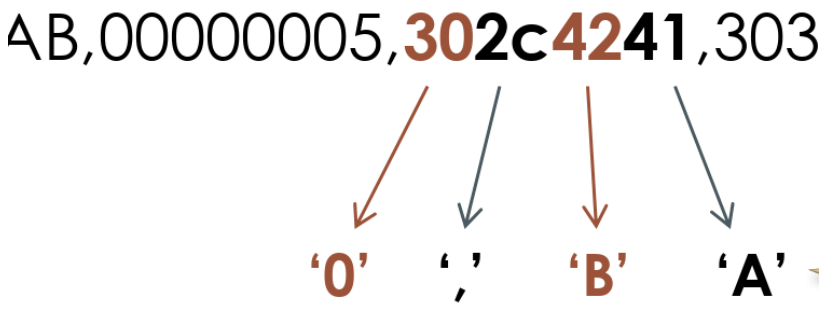
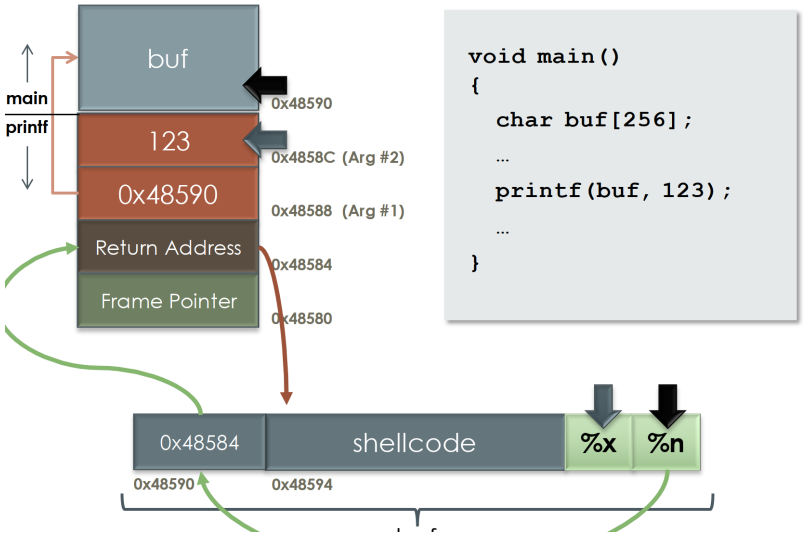


Figure 13. ASCII decoding

Now there's a potential problem: information leakage (of important info further up the stack). Programmers may not pay attention to sanitizing input like language config.

- %n: Assume then next argument is a pointer and then it writes the number of characters printed so far into that pointer.
- This can be abused by %n write to the return address and then overwrite it with the address of the shellcode.

How an exploit may look like for this is as follows:



1. Consume the 123 argument (%x)
2. Have the return address sitting in the beginning of the memory
3. Overwrite the RA value with the start of shellcode

There are some problems with this because on modern machines addresses are very large and it can be impractical to create a gigabyte-sized buffer. Instead we can just divide the problem up and write multiple 8 bit numbers

Example: "%243d" writes an integer with a field width of 243; "%n" will be incremented by 243.

Figure 14. The printf count increments by 243 with %243d. Shorthand

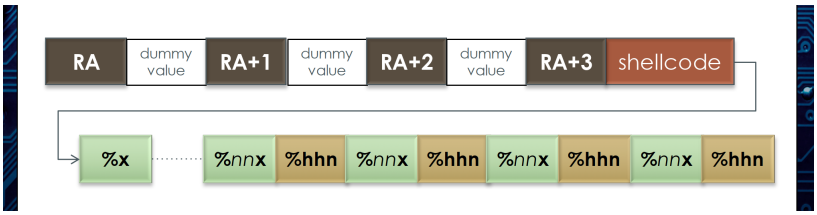


Figure 15. The gaps are there because

Dividing the problem into pieces; using %hhn and %nnx to write 8 bits at a time.

If the bytes being written must be written in decreasing order we can do this by structuring our pointers in a way that we write it in reverse order (don't need to start with LSB). Another option is

Note that this writes the printf counter into the pointer at the argument. This drastically decreases the buffer size needed

Subsection 2.4

Double-Free vulnerability

Freeing a memory location that is under the control of an attacker is an exploitable vulnerability

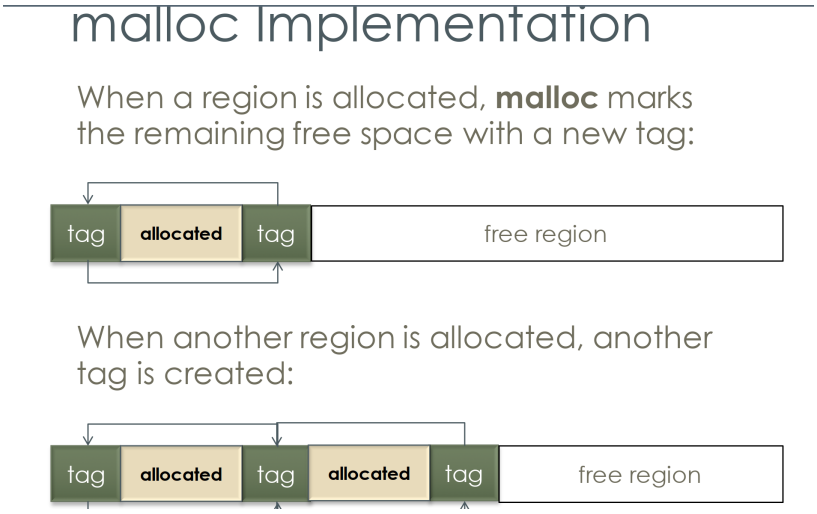
```
1  p = malloc(128);
2  q = malloc(128);
3  free(p);
4  free (q);
5  p = malloc(256);
6  // this is where the attack happens; the fake tag, shell code, etc
7  strcpy(p, attacker_string);
8  free(q);
```

Note that the `c` free function takes a reference (not necessarily a pointer) to the memory location to be freed. It does not change the value of the free'd pointer either.

malloc implementation

malloc maintains a doubly-linked list of free and allocated memory regions:

- Information about a region is maintained in a **chunk tag** that is stored just before the region
- Each chunk maintains:
  - A "free bit", indicating whether the chunk is allocated or free
  - Links to the next and previous chunk tags
- Initially when all memory is unallocated, it is in one free memory region



Malloc places a doubly-linked-list of chunk bits in memory to describe the memory allocated. `free` sets the free bit, does the various doubly linked list operations (looking at the pointer values of its neighbours in order to remove itself) and also tries to consolidate adjacent free regions. It assumes that it is passed the beginning of the allocated memory as well as go find the tag associated with the region (which is in a consistent place every time).

The attacker can drop fake tags into memory just ahead of the value we want to overwrite and then we can use the free() call to overwrite memory with the attacker's data. For example we can create a fake tag node with a prev and next pointer. We make the next pointer be the address of the return address. And then "prev" can contain the address of the start of our shellcode. So freeing on this fake tag will overwrite the return address with the shellcode start.

Comment

> Note that this is not possible with a single free if you are not able to write to negative memory indicies. The part that makes this attack works is that the double free allows the attacker to write a fake tag just before the next tag in a totally valid way. Doing this with a single free would also involve writing to memory that the program doesn't own. (recall: how the memory manager works)

Subsection 2.5

Other common vulnerabilities

The attacks we have seen have involved overwriting the return address to point to injecting code. Are there ways to exploit software without injecting code? Yes – return into `libc` i.e. use `libc`'s `system` library call which looks already like shell code. This can be accomplished with any of the exploits we have already talked about.

I.e.

- Change the return addresses to point to start of the system function
- Inject a stack frame on the stack
- Before return sp points to $\&system$
- System looks in stack for arguments
- System executes the command, i.e maybe a shell
- Function pointers

- Dynamic linking
- Integer overflows
- Bad bounds checking

2.5.1 Attacks without overwriting the return address

Finding return addresses is hard. So we can use other methods to inject code into the program.

- Function pointers: an adversary can just try to overwrite a function pointer
- An area where this is very common is with *dynamic linking*, i.e. functions such as *printf*.
- Typically both the caller of the library function and the function itself are compiled to be position independent
- We need to map the position independent function call to the absolute location of the function code in the library
- The dynamic linker performs this mapping with the procedure linkage table and the global offset table
  - GOT is a table of pointers to functions; contains absolute mem location of each of the dyn-loaded library functions
  - PLT is a table of code entries: onee per each library function called by program, i.e. sprintf@plt
  - Similar to a switch statement
  - Each code entry invoes the function pointer in the GOT
  - i.e. sprintf@plt may invoke jmp GOT[k] where k is the index of sprintf in the GOT
  - So if we change the pointers in the offset table we can make the program call our own code, i.e. with objdump.[23]

[23] *PLT/GOT always appears at a known location.*

2.5.2 Return-Oriented Programming

- An exploit that uses carefully-selected sequences of existing instructions located at the end of existing functions (gadgets) and then executes functions in an order such that these gadgets compose together to deliver an exploit.
- This can be done faster by seeding the stack with a sequence of return addresses corresponding to the gadgets and in the order we want to run them in.

Subsection 2.6

Software Code Vulnerabilities

Recall: the stack is used to keep track of return addresses across function calls; storing a breadcrumb trail. Another key thing sitting in the stack are local variables. A common theme in the course is that computing tends to conflate execution instructions with data.

Common data formats and structures create an opportunity for things to get confused (and for attackers to take advantage of). For example, a buffer-overflow attack can end up overwriting that return address breadcrumb trail and then execute arbitrary code.

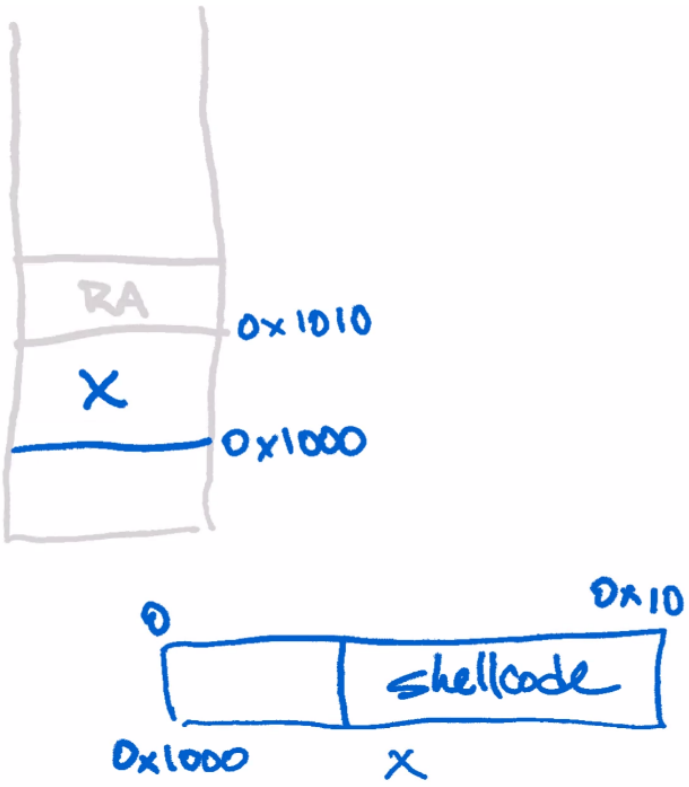


Figure 16. Bufferoverflow to write to the return address. Shellcode is a sequence of instructions that is used as the payload of an attack. It si called a shellcode because they commonly are used to start a shell from which the attacker can do more.

There are ways to find out where that return address is (or at least reasonably guess). This is discussed more in detail later; for now we'll assume that they have it figured out.

A common technique to make this easier is to inject a bunch of `NOP`s before the start of the shellcode. So that we don't need to be as precise as needed in order to find the shellcode start.

One technique for finding the RA would be to incrementally increase the size of the buffer overflow until we get a segfault – at this point the segfault would tell you what memory address it was trying to access and possibly the values it saw there instead as well.

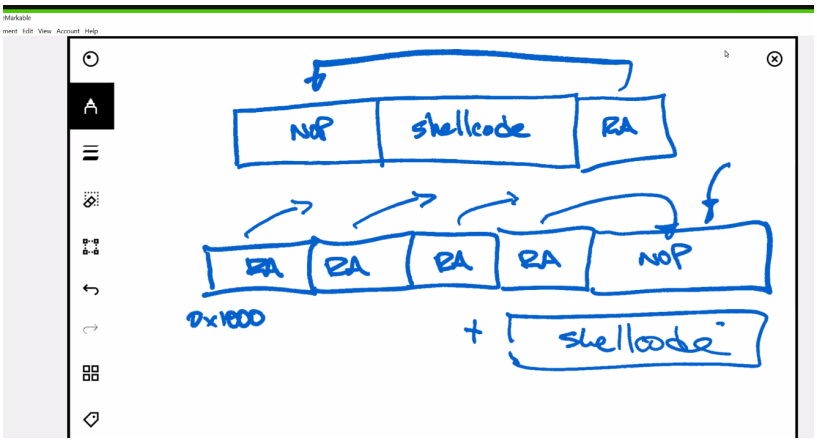


Figure 17. Can create a RA sled with a NOP leading to shellcode and then try it from e.g. 0x1000, 0x2000 and so forth to find where to attack from.

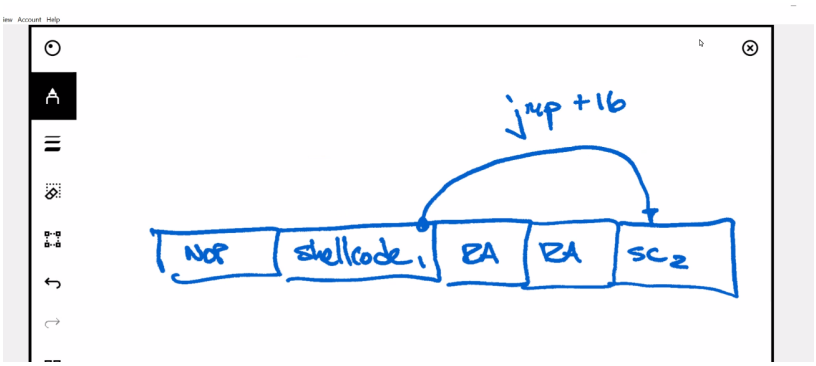


Figure 18. Another technique may involve placing shellcode all over the place, of which each one may be a valid entrypoint into the shellcode.

Subsection 2.7

Format string Vulnerabilities

```
1 spirntf(buf, "Hello %s", name);
```

`sprint` is similar to `printf` except the output is copied into `buf`. The vulnerability is simiar to the buffer overflow vulnerability. The difference is that the attacker can control the format string.

Consider the following:

```
1 char* str = "Hello world";
2 printf(str); // 1
3 printf("%s", str); // 2
```

Despite it looking different there are differences in these two ways to print hello world. The first argument is a format string, which is different from just a parameter. A format string contains both instructions for the `c` printing library as well as data. This means that the first method can be exploited if the attacker has access to the format string. A more complex vulnerability is with `snprintf` (which limits the number of characters written into buf).

```c
void main() {
    const int len = 10;
    char buf[len];
    snprintf(buf, len, "AB%d%d", 5, 6);
    // buf is now "AB56"
}
```

- Arguments are pushed to the stack in reverse order
- snprintf copies data from the format string until it reaches a %. The next argument is then fetched and outputted in the requested format
- What happens if there are more % parameters than arguments? The argument pointer keeps moving up the stack and then points to values in the previous frame (and could actually look at your entire program memory, really)

```c
p = malloc(128);
q = malloc(128);
free(p);
free (q);
p = malloc(256);
// this is where the attack happens; the fake tag, shell code, etc
strcpy(p, attacker_string);
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```

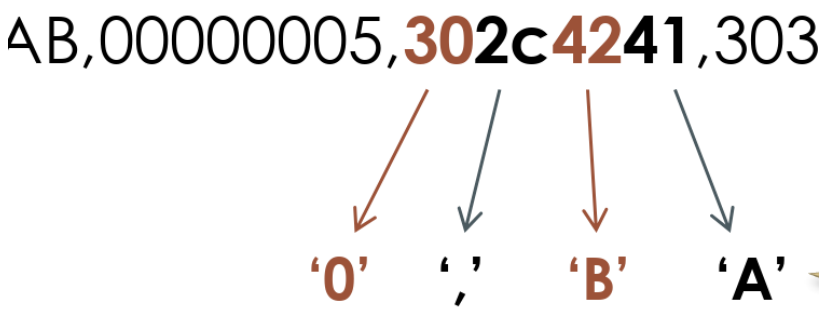
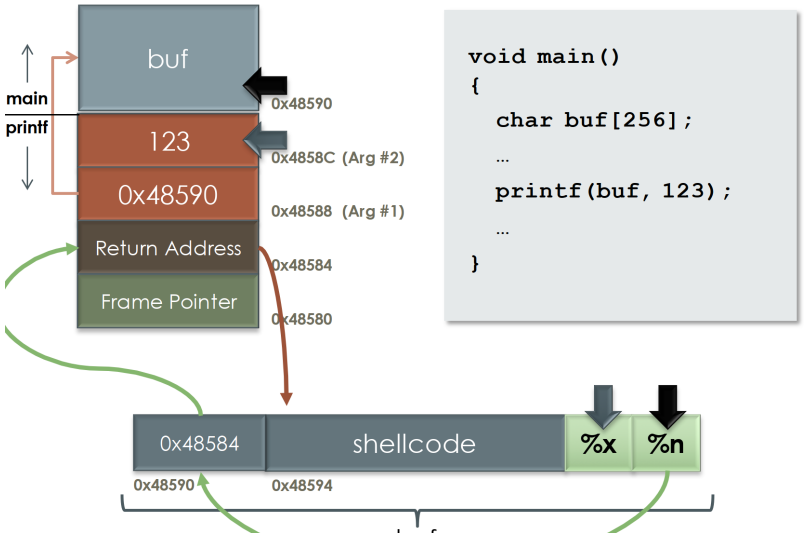


Figure 19. ASCII decoding

Now there's a potential problem: information leakage (of important info further up the stack). Programmers may not pay attention to sanitizing input like language config.

- %n: Assume then next argument is a pointer and then it writes the number of characters printed so far into that pointer.
- This can be abused by %n write to the return address and then overwrite it with the address of the shellcode.

How an exploit may look like for this is as follows:

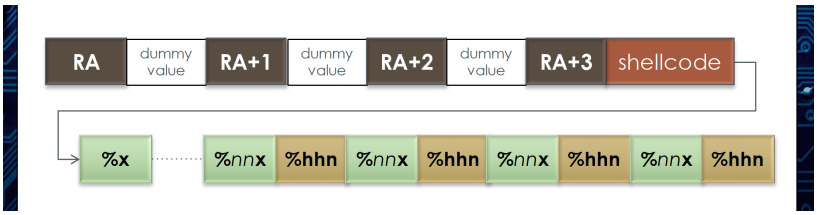


1. Consume the 123 argument (%x)
2. Have the return address sitting in the beginning of the memory
3. Overwrite the RA value with the start of shellcode

There are some problems with this because on modern machines addresses are very large and it can be impractical to create a gigabyte-sized buffer. Instead we can just divide the problem up and write multiple 8 bit numbers

Example: "%243d" writes an integer with a field width of 243; "%n" will be incremented by 243.

Figure 20. The printf count increments by 243 with %243d. Shorthand



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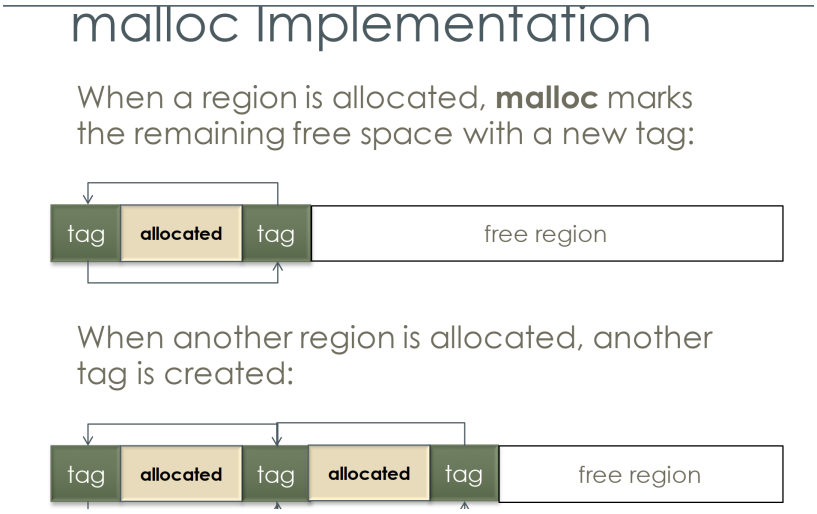

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2.9.3 Deserialization attacks

- Serialization is the process of transforming objects into a format that can be stored or transmitted over a network, i.e. to/from JSON.
- The attacker knows that the library has a vulnerability in the deserialization process and they can exploit it by passing carefully created data to it.

2.9.4 Integer overflows

- A server processes packets of variable size
- First 2 bytes of the packet store the size of the packet to be processed
- Only packets of size 512 should be processed
- Problem: what if we end up overflowing the integer with a negative value which would cause memcpy to copy over a lot more memory than intended.

```
char* processNext(char* strm){
    char buf[512];
    short len = *(short*)strm; // note that by default these are
    ↪ signed
    if (len <= 512) {
        memcpy(buf, strm, len); // note that the 3rd arg of memcpy is
    ↪ an unsigned int
        process(buf);
        return strm + len;
    } else {
        return -1
    }
}
```

2.9.5 IoT

Subsection 2.10

Case Study: Sudo

A common program attackers target are programs that regular users can run in order to take on elevated privileges. In `unix` systems one such program is `sudo`, for which vulnerability CVE-2021-3156 was discovered in 2021 after lying in there for over 10 years.

- `sudo` will escape certain characters such as "
- Someone introduced debug logic called user_args and then copies in the contents of `argv`, while un-escaping meta-characters
- Bug: if any command-line arg ends in a single backslash, then the null-terminator gets un-escaped and then `user_args` keeps copying out of bounds characters onto the stack
- I.e. `sudoedit -s '\' $(perl -e print "A"x1000$)`
- Attacker controls the size of `user_args` buffer they overflow. Can control size and contents of the overflow itself; last command-line argument is followed by the environment variables
- Had many exploit options
  - Overwrite next chunk's memory tag (same as use-after-free)
  - Function pointer overwrite one of sudo's functions
  - Dynamically-linked library overwrite
  - Race condition a temp file sudo creates
  - Overwrite the string "usr/bin/sendmail" with the name of another executable, maybe a shell

Subsection 2.11

Case Study: Buffer overflow in a Tesla

`ConnMann` (Connection Manager) is a lightweight network manager used in many embedded systems, i.e. nest thermostats and Teslas for that manager.

In this particular vulnerability the attacker took advantage of the DNS protocol. DNS responses include a special encoding for the hostnames which help the receiver parse the response and allocated appropriately sized buffers. For example www.google.com is encoded as

3www6google3com. This response also often contains a lot of repetitive information, so there is some compression is used in the encoding as well. The one we're interested in here is the compression of names by encoding them as a special "field length" of 192 followed by the offset of the other copy of the name – which enables repetitions to be encoded as 2 bytes.

CVE-2021-26675 was reported by Tesla in 2021 as a bug in ConnMan which allows an malicious DNS reply to uncompress into a large string that can overflow an internal buffer. This means that a remote attacker who can control or fake a DNS response could perform a buffer overflow on ConnMan – which runs with root privileges.

```
static gboolean listener_event(...)
{
        GDHCPClient *dhcp_client = user_data;
        struct sockaddr_in dst_addr = { 0 };
        - struct dhcp_packet packet;
        + struct dhcp_packet packet = { 0 };
        struct dhcpv6_packet *packet6 = NULL;
        ...
```

Figure 22. ConnMan doesn't initialize the dhcp_packet struct to 0, which can cause it to leak stack values to a remote attacker (but here they must be on the same subnet as the victim). This vulnerability can be difficult to detect since nobody checks if things are zero in the tests.

Comment

So you want to a hack a tesla?

- Look at the situation; see what kind of protocols being used, etc. Get excited if it uses something old and inane
- Look at the data coming in and our, especially if there's any extra going in or out
- Use fuzzing tools
- Get a sense of what they are expecting us to do as well as what are ways that we can break that example. For example is the only verification just some client-side JavaScript?
- Break stuff

Subsection 2.12

Fault Injection Attacks

We make a lot of assumptions about how the underlying systems work. For example proper CPU operation. Fault injection attacks take advantage of these assumptions by injecting faults into the system, often at the hardware level.

For example: proper pipelined CPU operation depends on stable power and clock inputs. If the glitch duration is longer than the time it takes to increment the PC and shorter than the instruction fetch time, then we can start to see a special case: instruction skipping or instruction corruption.p

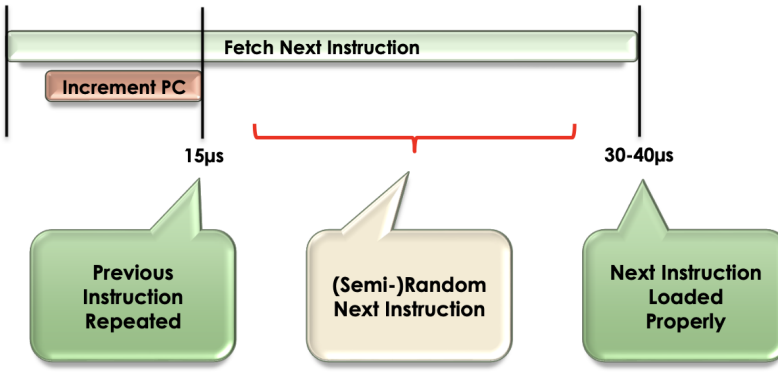


Figure 23. With some careful timing we can cause the CPU to skip or repeat an instruction.

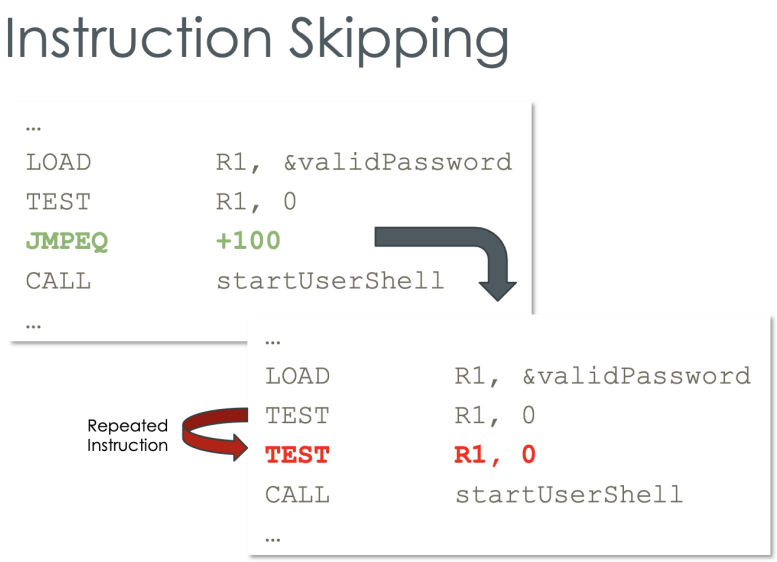
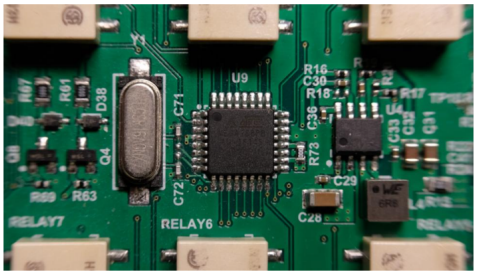


Figure 24. An example of where this can be useful: skipping the JMP instruction of an IF statement

2.12.1 Hardware Demo

Consider this simple program that checks a text buffer for a password and then logs you in if it's correct



ATMega328P Microcontroller

- Low-powered microprocessor + flash memory
- One application, no traditional operating system
- Common in **IoT** (home automation) and **embedded** (industrial control) applications

```
login: root
Password: password

Login incorrect.

login: root
Password: s3cr3tP4ssw0rd&:-)@#

Last login: Wed Dec 31 12:34:56 2025 from 127.0.0.1
root@iotvictim:~#
```

```
…
bool passwordIsValid =
       !strcmp("s3cr3tP4ssw0rd&:-)@#", buffer);

if ( !usernameIsValid || !passwordIsValid ) {

       // Login incorrect
       Serial.println("\n\nLogin incorrect.");
L: SKIP THIS!  return;
}

// Login correct
…
```

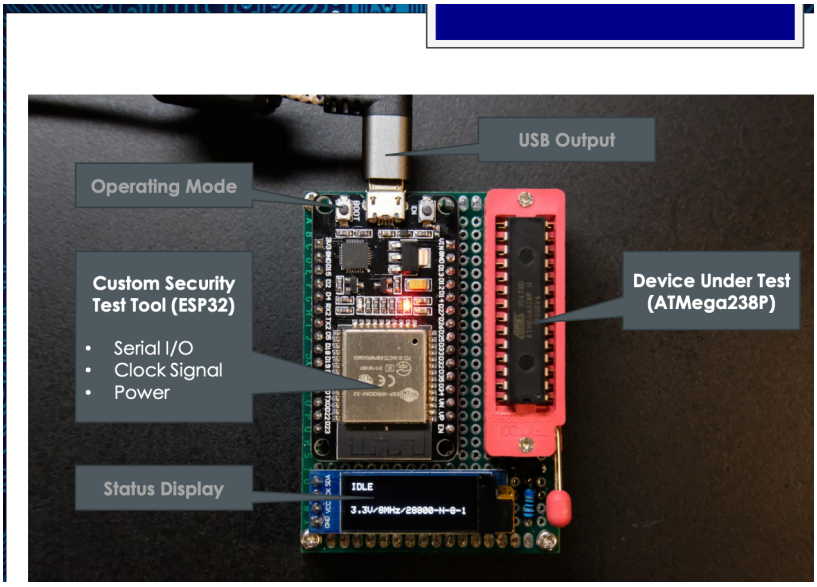


Figure 25. The ATMega238P hooked up to a custom security test tool built on top of a ESP32

Our attack is to use a **clock glitch**[25] at the time of the return instruction. Finding the time of the return instruction is a bit tricky but we can just sweep across a range of times.

[25] *A series of very brief and rapid clock pulses*

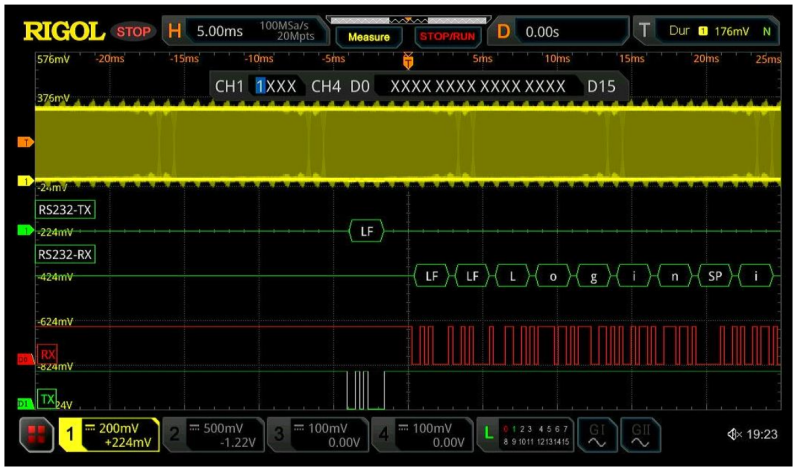


Figure 26. Looking at the oscilloscope to show the sweeped input

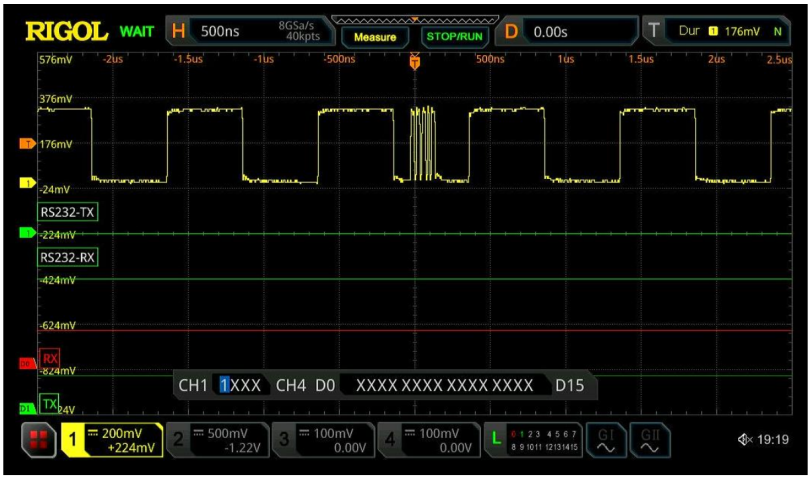


Figure 27. Note crazy clock pulses which we try to line up with the return instruction. If we have a really good chip we can try it with only 1 pulse, but here we use 5 pulses because we're on a cheaper chip. We also don't happen to care too much about whether or not if we disrupt too many of the other instructions.

Another attach that is a bit easier to use is the **power glitch**: instead of not giving it enough time for the fetch to happen we take away the nice voltage going to the chip right at the instruction execution time.

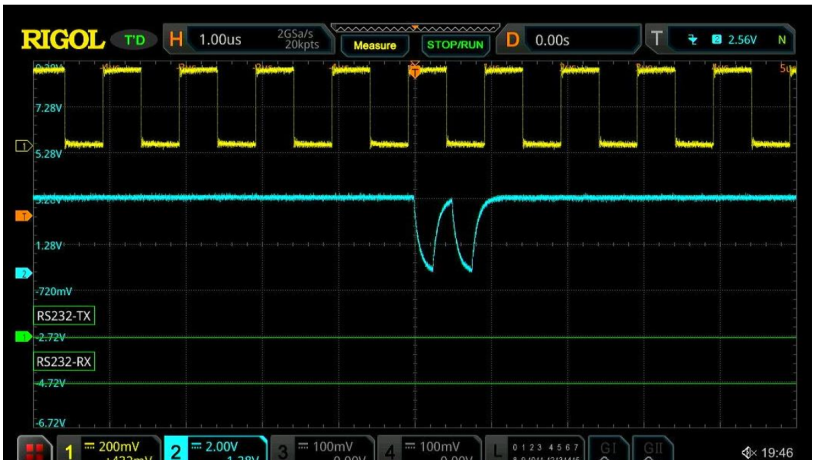


Figure 28. A power glitch attack disrupts the fetch instruction or the decode

Defense for these attacks is to disallow physical access to the chip. For the power one it can be as simple as adding a little capacitor to the power supply to smooth it out. Likewise, there are many ways to cause controlled circuit malfunctions: lasers, strobes, EM pulses, etc.

Subsection 2.13

Reverse Engineering

Reverse Engineering, or the act of analyzing a product in order to learn something about its design which its creator wanted to keep secret. It is a legally complicated, but generally it's ok for the purposes of achieving interoperability (but not for circumventing DRMs).

```
1  unsigned int printhelloworld() {
2      printf("Hello World!");
3      return 5;
4  }
5  int main (int argc, char *argv[]) {
6      unsigned int result = 0;
7      result = printhelloworld();
8      if (result == 4) {
9          printf("super secrete string\n");
10     }
11     return 0;
12 }
```

Figure 29. Passing the binary compiled from the above code to a disassembler

With the disassembled binary we know where the instruction for the if statement we were curious about lives, so we can then just use hexedit to change the bits at that JMP to a NOP to print out the super secret string.

Subsection 2.14

Buffer Overflow Defenses

- Audit code rigorously
- Use a type-safe language with bounds checking (Java, C#, rust)
- However, this is not always possible due to legacy code, performance, etc.
- Defending against stack smashing
  - Stackshield: put return addresses on a seperate stack with no other data buffers there
  - Stackguard: a random canary value is placed just before the RA on a function call. If the canary value changes, the program is halted. This can be enabled via a flag on most modern compilers.
- Third-party libc i.e. libsfae which doesn't allow for '%n' in format strings
- Address space layout randomization: maps the stack of each process at a randomly selected location with each invocation, so that an attacker will not be able to easily guess the target address. GCC does do this by default.

If we really sit down and think about it, it's basically impossible to defend against all attacks. It's easy to make a mistake and end up with a vulnerability. Certain vulnerabilities can be avoided by using safer languages, but the only real defense is to be aware and careful. One approach is what the aerospace industry does, i.e. the swiss cheese model [26]

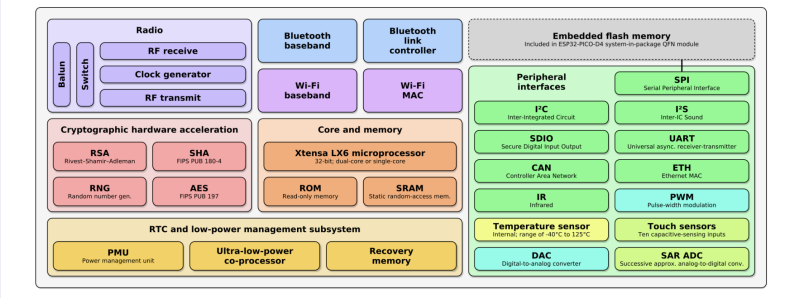
[26] *if we stack a lot of hole-y cheese on top of each other it will be opaque*

Subsection 2.15

Cryptography

Comment

Case Study: Espressif ESP32



- Microcontroller that the prof used for the demo last class
- Programming the microcontroller usually happens during the manufacturing phase
- Flash memory is usually partitioned into the bootloader, data, and application
- In the factory the ESP32 will generate a random number (on first boot) in order which will be used to encrypt and hash the bootloader and the data on the board. Then it starts the applications.
- On subsequent boots the device will make recalculate the hash to make sure that the bootloader has not been tampered with.
- More information about using PGP encryption for the data partition, etc. Not too important.
- TLDR: lots of encryption and security features built into the chip

Comment

Case Study: Door Alarm

What would a small-scale communication-channel pentest look like for this?

1. Look at FCC report

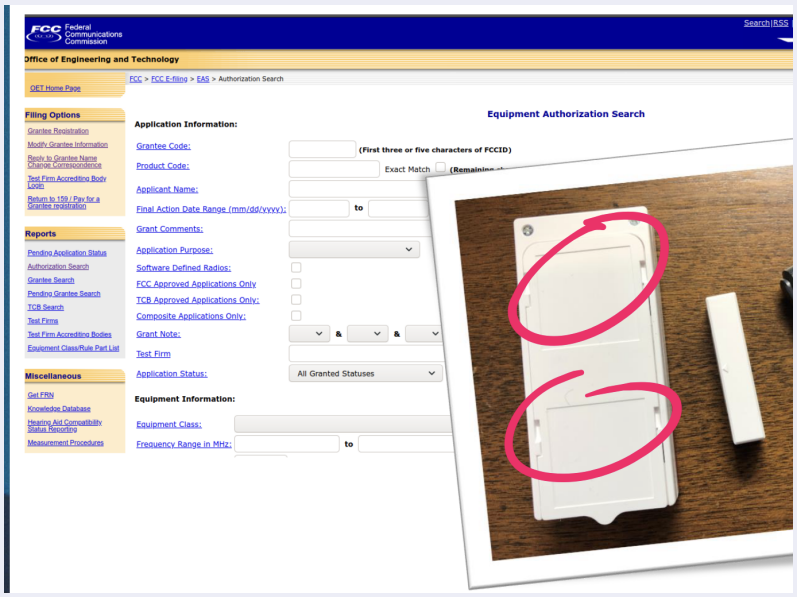


Figure 30. In this case this was barely done so it wasn't very useful

2. Make reasonable guesses

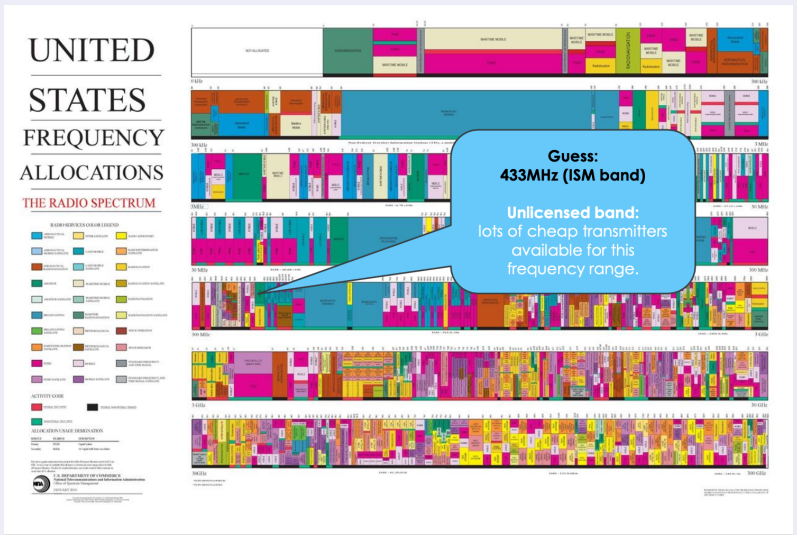


Figure 31. Guess that this cheap device is on unlicensed 433 MHz band

3. Listen into the signal

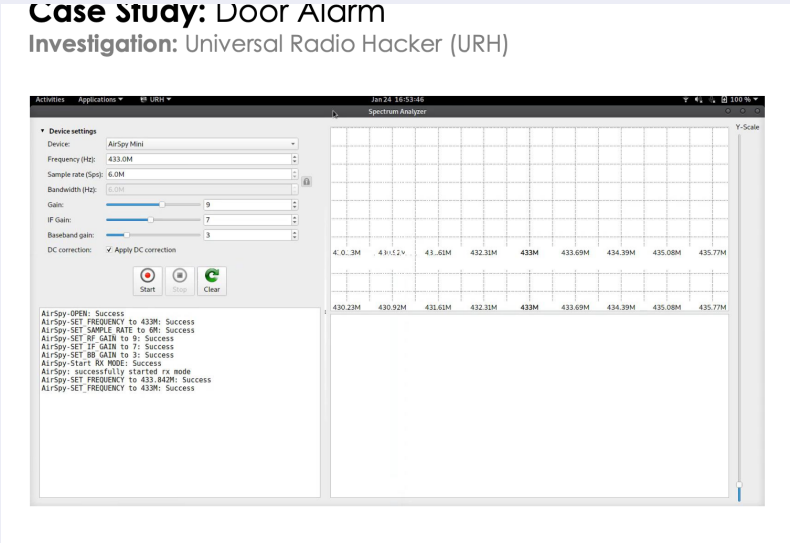


Figure 32. Use a software-defined radio to inspect the signal

Figure 33. Top signal is lock, bottom signal is unlock. Here we don't know what they are yet but there *is* some sort of unique binary pattern being produced on the button clicks.

4. Replay?

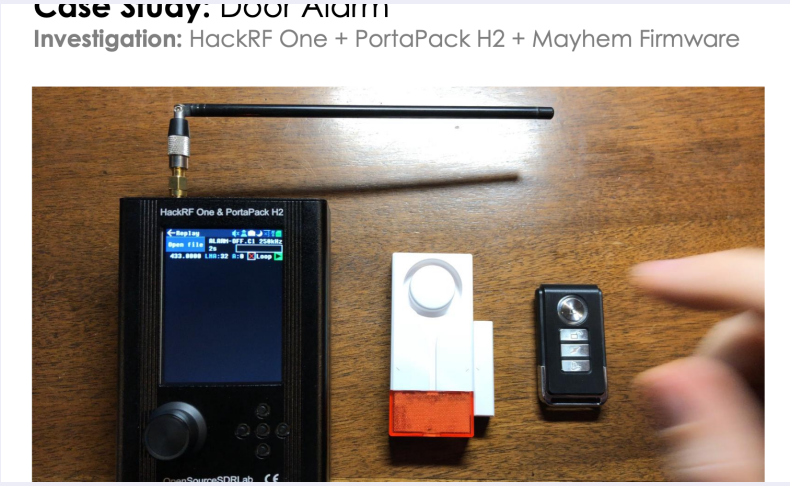


Figure 34. Can just use a radio recorder/playback device to record the signal and play it back in order to unlock the doorlock

As it turns out industrial systems tend not be be any more secure

2.15.1 Ciphers

Cryptography is used to establish the confidentiality, integrity, authenticity, and non-repudiation[27] of data.

[27] *Prevents a principal from denying they have performed an action*

- Ciphers an algorithm that obfuscates data so that it seems random to anyone hwo does not possess special information called a key.
- Based on a class of functions called trapdoor one-way functions, i.e. easy to compute but inverse is difficult to compute. Trapdoor means that given the key the inverse becomes easy to compute[28]
- Function itself should not be the critical secret (Kerckhoff's principle)

[28] *Note that never been proven/disproven that one-way functions exist. Plus if it's been proved it would show $P \neq NP$*

Two common one-way functions used are factoring ($z = (x * y)$ find x,y) and discrete log ($z = x^y \% m$ – given z, x, m, find $y = \log_x z \% m$)

Definition 18 **Caesar (Shift) Cipher**

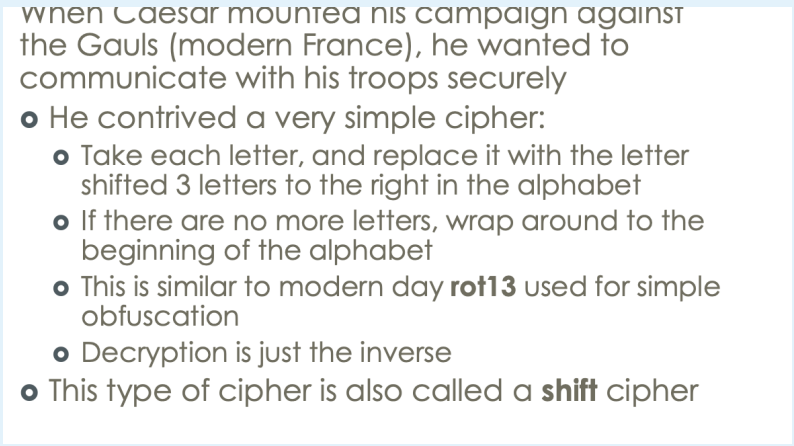
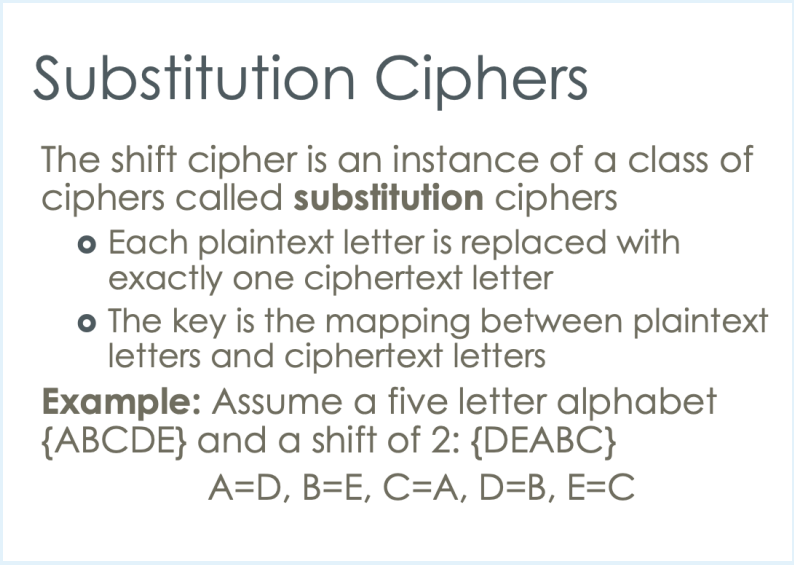


Figure 35. A contrived cipher

Note that even this may be difficult to crack in reasonable time with a brute-force attack if we happen to have a large enough alphabet.

Definition 19 A slightly better cipher would be a substitution cipher

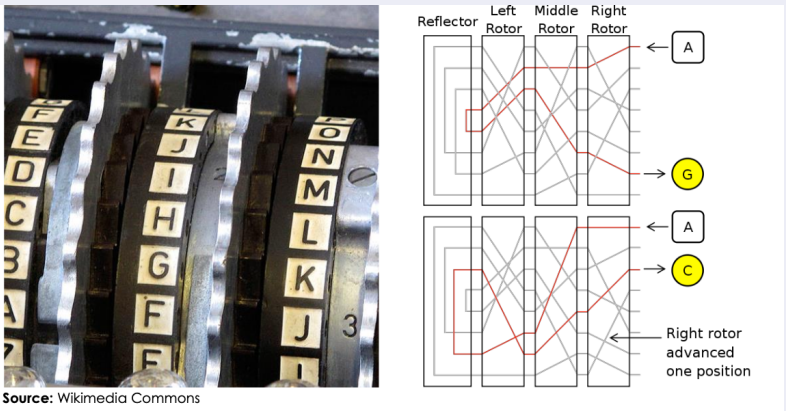


There's a one-to-one mapping between the plaintext and the ciphertext, so we can use patterns in the data to figure out a heuristic to reverse the cipher. For example in English E is the most common letter (and there are likewise common digraphs e.g. TH, HE, etc) and as such can be used to make informed guesses as to the substitutions being used

An improvement to the monoalphabetic substitution cipher described above is the polyalphabetic substitution cipher. In this case we have a set of n mappings in the cipher and change hte mapping with every character. However these ciphers are still periodic. For small n this is not a problem, but for large n it becomes a large challenge.

Comment

Enigma Machine



Source: Wikimedia Commons

- A substitution cipher with a really large cipher during early war efforts
- Decryption/encryption via initial rotor position etc that would be agreed on.

The gold standard for encryption is the substitution cipher taken to the extreme: the one-time-pad.

Definition 20

One-time-pad

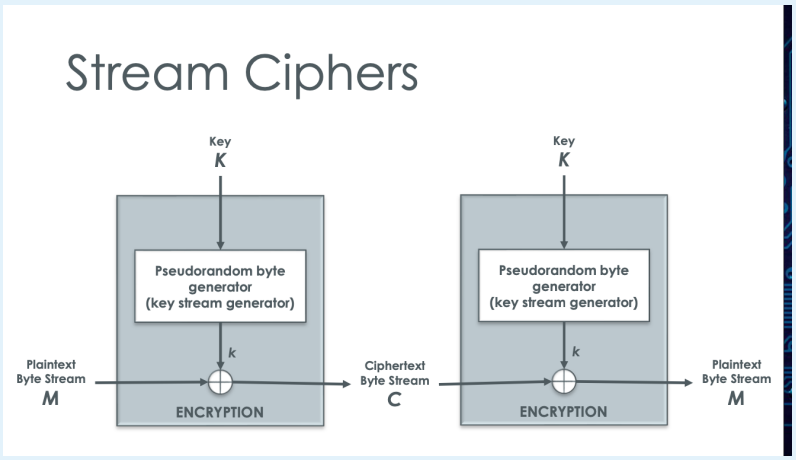
- A random substitution is used for every character
- Think about it as using an infinite number of keys
- A message with n bits of information an OTP adds n bits of randomness to make a completely random ciphertext –> Theoretically unbreakable
- Key overhead of 100% (key length equal to message length) and key reuse is not allowed
- Cipher is malleable (bit flips in ciphertext correspond to bit flips in plaintext); requires integrity check
- How to make a random key? Need good random number generator
- OTP is strong against ciphertext-only attacks but is exteremely weak against known-plaintext attack (only need one pair).

Practical ciphers are ones with fixed length keys that are shorter than the message and are independent of message length. They should also be efficient to use for encryption/decryption while being computationally difficult without the key.

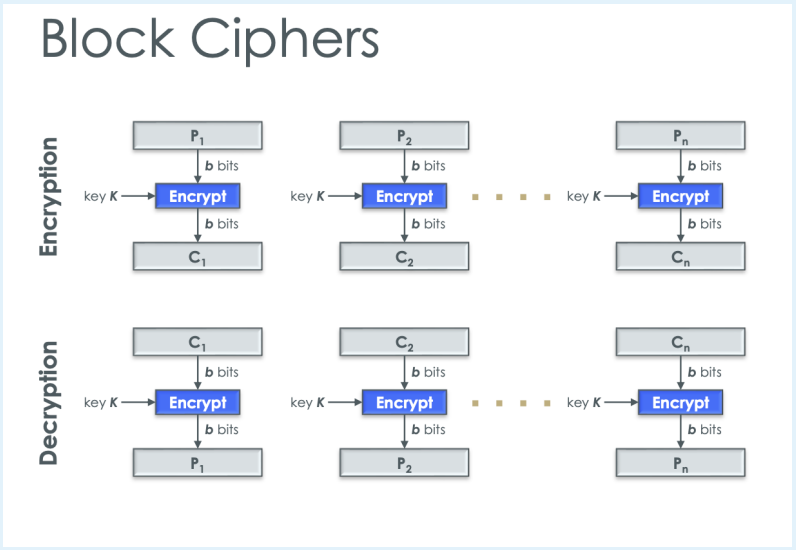
Definition 21

Symmetric key ciphers: same key to encrypt/decrpyt (stream/block ciphers)

- Stream ciphers: similar to OTP: key used to generate pseudo-random sequence of bits and then XOR'd with plaintext. Runs a bit at a time which is good for streaming. Suffers from synchronization problems



- Block ciphers: encrypt/decrypt a block of bits at a time (usually 64 bits or a multiple). Add padding if necessary.



Stream ciphers are generally simple and fast. Block ciphers are more common just due to the history of cipher development (closed-source stream ciphers and a proliferation of open-source block ciphers)

2.15.2 Block Ciphers

- Data Encryption Standard (DES): 56 bit key, 64 bit block.
- AES (Advanced Encryption Standard) – official standard encryption algorithm for the US government in 2000
- Both are iterated block ciphers
- Today's computers are fast enough that DES is considered insecure

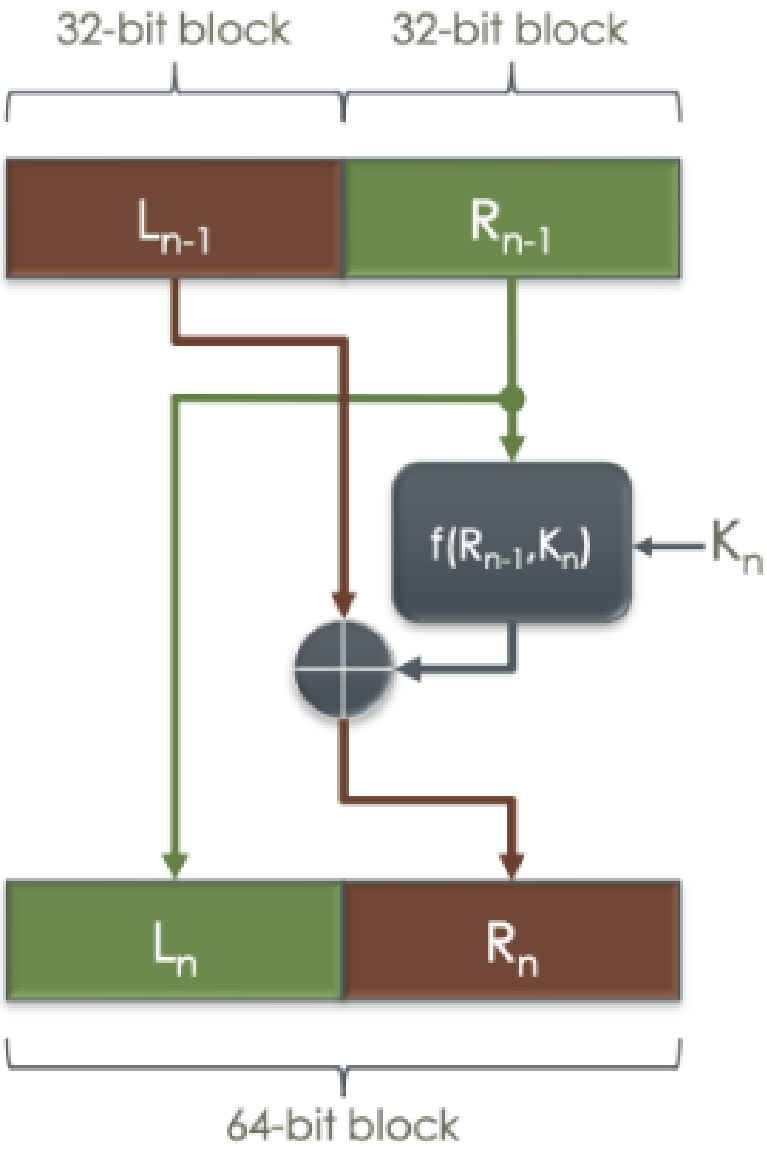


Figure 36. DES Feistel Ntwork

- Input is split between left and right haves. Some computation involving a portion of the key is done on the right half and then the left and right halves are swapped, then the output gets piped back into this process. This "round" is repeated 16 times.
- 56 bit key is put through a schedule to create sixteen subkeys. 56-bit into 2 28 bit halves, then shifted left by 1 or 2 bites, and 2 24 bits are then selected from the halves to make a 48 but subkey K_n. Exact number of bit selections are carefully selected.

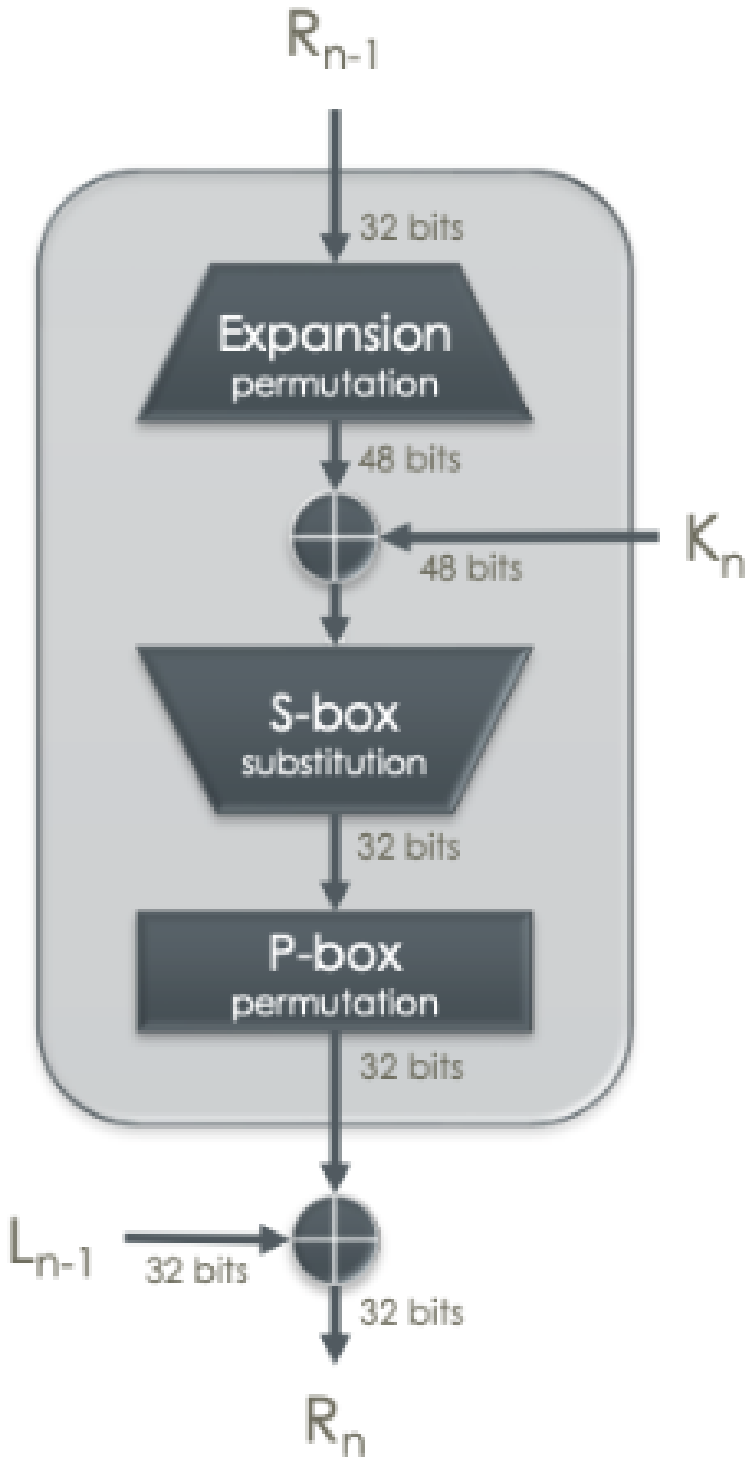
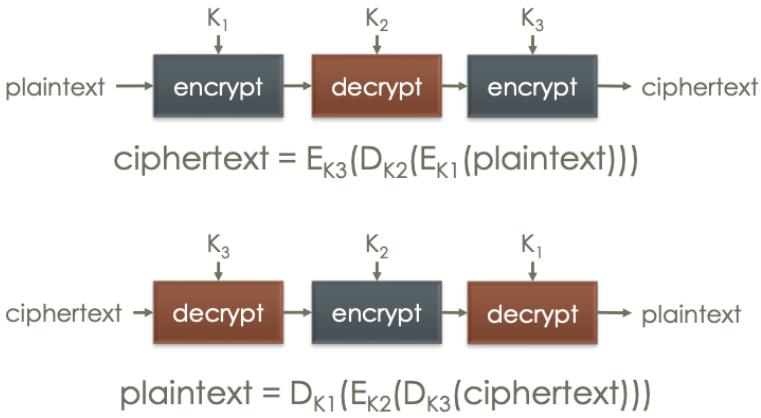


Figure 37. each $f(R_{n-1}, K_n)$ does: 1. expansion permutation 2. XOR with subkey 3. non-linear S-box subsitution boxes to compress 48 to 32 bit 4. permutation

- DES is inadequate with modern computers: brute force can crack 56-bit keys in less than

Design of S-boxes is important as this is the only part of the cipher that is non-linear

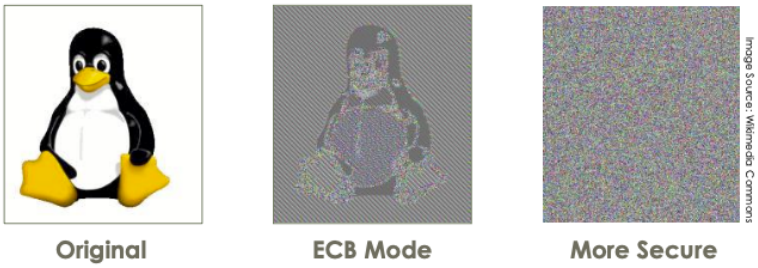
a day. A solution is to use a longer key length and chain DES multiple times; **3DES** w/ a 168 bit key split into 2 56 bit times and running the algorithm three times



Block cipher encryption modes, or how to encrypt data with multiple blocks:

Considerations include security, performance, error propagation, and error recovery

- Electronic Cookbook (Simplest): break down into block-sized chunks & pad if necessary. Encrypt each block separately.
  - Highly parallelizable but not secure
  - Cipher blocks can reveal macro structure of plaintext data since same plaintext blocks will always encrypt to the same ciphertext blocks

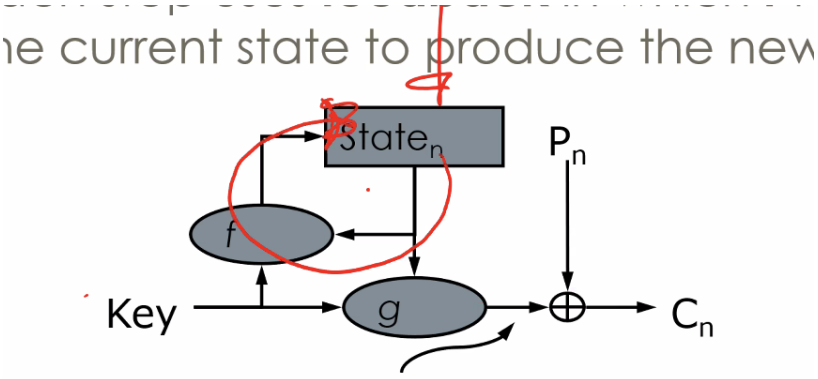


  - Error propagation doesn't happen & error recovery only requires retransmission of affected blocks and does not stop decryption
- Cipher block chaining
  - Every block's input is dependent on output of previous block. Initial value does not have to be secret but shouldn't be reused for multiple messages
  - Good security but poor parallelism for encryption[29]. Transmission error only affects current and following block – and as for recovery the receiver can drop affected blocks and still continue decryption.
- CFB (Cipher-feedback) and OFB (Output feedback) convert block ciphers into stream ciphers, i.e. can be decrypted/encrypted in less than a full block at a time. Similar to stream ciphers (Discussed later.) In OFB the key stream is independent of plaintext so cipher operations can be done in advance

[29] *decryption can be parallelized*

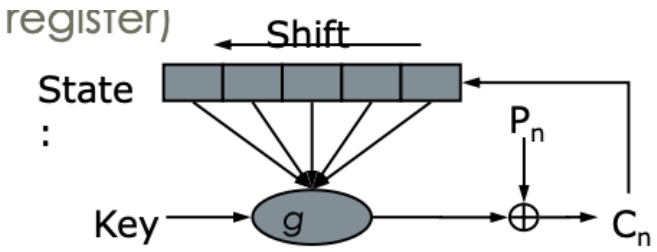
2.15.3 Stream Ciphers

- Encryption/decryption with low latency, i.e. multimedia streams
- Operate one bit at a time
- Closely related to one-time pads
- No modes: can be used to encrypt data of any length
- Synchronous stream ciphers: key stream is independent of message text. State is modified by f and the key; each step uses feedback in which f uses current state to produce the new state. Transmission error only affects the corresponding plaintext bits



- Self-synchronizing stream ciphers: key stream is dependent on the plan test. State is a shift register; every ciphertext bit created is shifted into the shift register and fed back as input into the key function g. So each ciphertext bit as an effect on the next n bits[30]

[30] *n is the length of the shift register*



- Similar properties to the one time pad; dangerous to use the same keystream to encrypt multiple messages
- Synchronous stream ciphers must have changed keys or initialization vectors after every message. Self-synchronizing stream ciphers must have random data inserted at the beginning.
- Malleable; ciphertext can be changed to generate related plaintext.
- Adversaries can replay previously sent ciphertext into a stream and the cipher will resync.
- Synchronous stream ciphers cannot be recovered unless we know exactly how much ciphertext is lost because the keystream is independent of the plaintext. Self-synchronizing ciphers will recover after n bits pass.

Common ciphers used include RC4 and SEAL. RC4 is now publicly known but license is required to use it

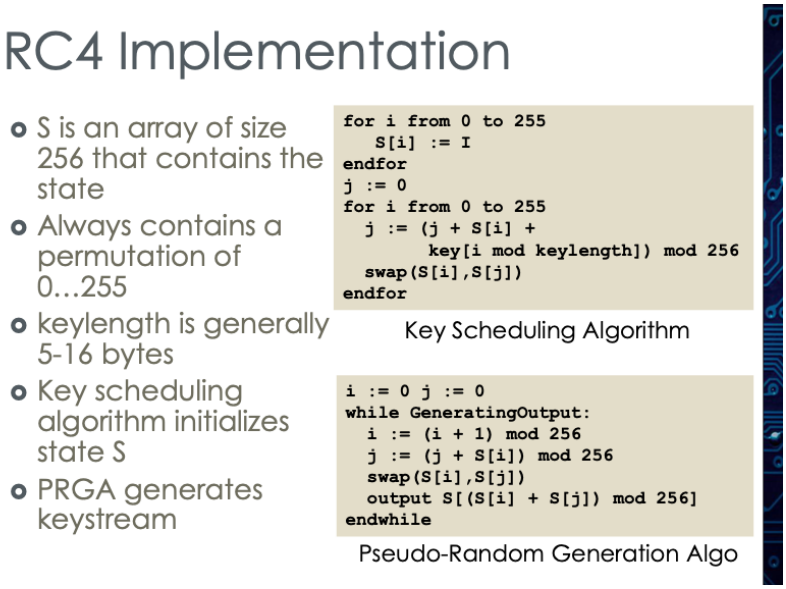


Figure 38. RC4 Implementation. TLDR: Use a state represented by an array of 256 chars. Use a pseudo-random generator and the state to generate the keystream.

Stream ciphers offer better performance but are more difficult to use safely. Block ciphers are easier and more commonly used. Nowadays just use AES & CBC is most common encryption mode for arbitrary data. ECB is safe for short chunks of data where plain text is unlikely to repeat.

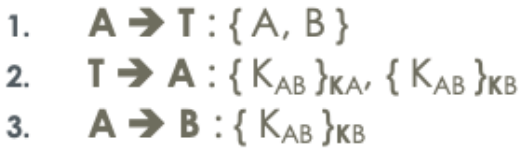
Subsection 2.16

Key Exchange

- Symmetric key encryption requires both parties to have the same key
- Key exchange must be communicated securely; if not, the key is compromised
- Pre-sharing keys is one alternative; i.e. setting keys at production time. But this is not practical for large systems; n people need a total of $n\frac{n-1}{2}$ keys

2.16.1 Trusted third-party

Idea: have a trusted keyserver that knows everyone's keys.



- A tells T that it wants to communicate with B

- T sends A a session key K_{AB} and encrypts it once with A's key ($(K_{AB})_{K_A}$) and once with B's key
- A will decrypt it's own copy of the session key with it's own key and then send the other key to B
- B can now decode the session key with it's own key. Now A and B can communicate securely

This procedure is susceptible to a third party attacker who may capture the session key A sends to B as well as other messages. The attacker can then replay messages to make B repeat an action; B can't tell if the message actually came from A

Definition 22 Needham-Schronecker protocol: a protocol for key exchange between two parties A and B that is secure against a passive eavesdropper by using a **nonce**

1. **A → T :** $\{ A, B, N_A \}$
   // **A** picks a **nonce** (random number): N_A
2. **T → A :** $\{ N_A, K_{AB}, B, \{K_{AB}, A\}_{KB} \}_{KA}$
   // N_A in the reply assures **A** that this reply isn't a **replay**
   // Includes **B**'s name: confirms who this is intended for
   // Note that the session key for **B** is encrypted with **A**'s key
3. **A → B :** $\{ K_{AB}, A \}_{KB}$
   // The message from **T** includes **A**'s name
4. **B → A :** $\{ N_B \}_{KAB}$
   // **B** wants to know whether he's actually speaking with **A**
   // Picks his own random nonce: N_B
5. **A → B :** $\{ (N_B - 1) \}_{KAB}$
   // Receiving the result (N_B-1) tells B that A has the key K_{AB}
   // and is responding to new messages (not a replay)

- Messages include recipient and sender
- Receiver keeps track of sent Nonce values to prevent replay attacks
- Some action performed on the Nonce proves to the sender that the recipient is alive. Often just adding/subtracting a constant from the nonce[31]

Trusted third party has a problem: because the system trusts the central server, if the server is compromised, the entire system is compromised.

[31] *Multiplication or division is discouraged due to the nature of many of these algorithms*

2.16.2 Diffie-Hellman Key Exchange

- Can be used by two parties to establish a common secret over an insecure link
- Assumes that the discrete logarithm problem is hard (modular arithmetic in a finite field)
  - Limited set of $n > 1$ elements, each with an additive inverse $x + x' = 0$ and each nonzero element has a multiplicative inverse $x \cdot x' = 1$
  - Recall: Modular arithmetic is the same as addition and multiplication but with the result rounded by the modulus of n, i.e if our system has a modulus of 7 then $4 + 3 = ((4 + 3)\%7) = 0$.

  - There are no negative numbers or factions in modular arithmetic, so additive and multiplicative inverses are as follows:
    * E.x. the additive inverse of 4 is 3; $4 + 3 = 7 \to 7\%7 = 0$
    * Multiplicative inverse of 5 is 3; $5 \cdot 3 = 15 \to 15\%7 = 1$
  - Modular arithmetic in a finite field will only work if the modulus is prime

- $4^3\%7 = 64\% = 1, \log_4 1\%7 = 3$
- What is the discrete log of $\log_3 5\%7$? I.e. finding x such that $3^x\%7 = 5$. Must try all possible values of x until we find the correct one (and do the exponentiation!).
  - Complexity of finding the log is $NP - hard$

Definition 23 Initialization: Alice selects n, a large prime modulus and g, a generator of the field n that lies between $(1, n-1)$

- Alice selects a random integer x and computes $P = g^x\%n$
- Alice sends P, g, n to Bob and keeps x to herself
- Bob selects a random integer y and computes $Q = g^y\%n$
- Bob sends Q to Alice and keeps y to himself
- Alice and bob may now both compute the secrete $Q^x\%n \equiv P^y\%n \equiv g^{xy}\%n$

Generator selection is discussed in texts on the subject. A number g is a generator of n if for each y between $1, n-1$ there exists an x such that $g^x\%n = y$, i.e $g^0, g^1, \ldots g^{n-1}$ yields all numbers from $1 \ldots n-1$

The Diffie-Hellman attack is vulnerable to man-in-the-middle attacks. If an adversary Eve can pretend to be Bob when communicating with Alice and pretend to be Bob when communication with Alice, then Eve can establish a shared secret with each of them without Alice or Bob being any wiser – thereby snooping on on their communication!

The problem with this key exchange protocol is that it does not identify the remote party; though the communication is secure we have no clear way of knowing if we are really corresponding with who we think we are.

2.16.3 Public Key Cryptosystems

Public key cryptosystems use a pair of keys to establish an asymmetric cryptosystem. The private and public keys reveal nothing about each other, but share the property that messages encrypted with one key can only be decrypted with the other. Users keep one 'private' key secret and one 'public' key, well, public. Then during encryption the sender may encrypt the messages with the intended recipient's public key and the recipient can decrypt the message with their private key. And since only the recipient can decrypt the message, the sender can be sure that the message is only being read by the intended recipient.

Setting up a key exchange using a public key system is straightforward; Alice can encrypt a key x using Bob's public key and send it to Bob. Bob can then decrypt the key with his private key and now they have a shared key x! Two popular public key cryptosystems are RSA[32] and DSA (Digital Signature Algorithm)[33]

[32] *Factoring*

[33] *Discrete logs*

Definition 24 **RSA** algorithm

1. Pick n that we can use as basis for the modular space. RSA key generation begins by picking two very large prime numbers p and q and computing $n = p \cdot q$. n can be publicky shared since there's no known algorithm to efficiently recover p, q from n. [34]
2. Pick our public key[35]. Define $\phi = (p-1)(q-1)$. Pick e that is *coprime*[36] to ϕ; this will be our public key.

[34] *Size of n defines key size; i.e. 4096 bit RSA uses 4096 bits to represent n*

[35] *Via Euler's Theorem*

[36] *a, b are coprime if 1 is the only positive integer that evenly divides both of them*

3. A message M can be encrypted into a cryptotext message C via $C = M^e \% n$. e, n, C can be made public, p, q, ϕ must remain secret. Note that $M < n$ or else RSA doesn't work.
4. Calculate our private key, i.e. need a private key d that is the multiplicative inverse of the public key e: $e * d = 1 \% \phi$. This d can be found efficiently through the extended euclidean method[37]. d must remain private. Also, no-one else can find d since they don't know ϕ.
5. Recover M via the private key
   - $M = C^d \% n = (M^e \% n)^d \% n = M^{e \cdot d} \% n$
   - Since $e \cdot d = 1 \% \phi \Rightarrow (e \cdot d) = (k\phi + 1)$ for some integer k
   - $M^{k\phi+1} \% n = M^{k\phi} \cdot M \% n = (M^{k\phi \% n} \cdot M \% n)$
   - Euler's theorem tells us that $a^\phi = 1 \% n$
   - $= 1 \cdot M \% n = M$ since $M < n$

[37] *Google this, not needed for the course*

RSA has very poor resistance to spoofing since the encryption uses exponentiation; $encrypt(K \cdot M) = (K \cdot M)^d = K^d + M^d = encrypt(K) \cdot encrypt(M)$.

Recall that a message is signed by encrypting the message with the sender's private key. The receiver can then decrypt the message using the sender's public key to show that the public key is really yours. If someone will sign messages the adversary gives them then the adversary can trick them into signing messages that they don't want to sign. Suppose a victim will not sign M but the adversary can pick K and get the victim to sign $K \cdot M$ and K. Then M may be recovered.

Public key cryptography also doesn't prevent man-in-the-middle attacks. If Eve can pretend to be Alice to Bob and pretend to be Bob to Alice, then Eve can snoop without either of them being any the wiser; despite being able to sign a message with one's private key, public key cryptography still suffers from an attacker passing off their public key and signature as someone else's. This can be resolved through the introduction of a trusted third party who can vouch for the identity of a key. This trusted third party is called a **certificate authority** (CA) and they are responsible for issuing certificates using their own private key saying that a public key belongs to a particular person. Bob and Alice can then use this certificate to verify that they are communicating with the right person.

- Common standard format is $X509$ and is used in SSL. Public infrastructure allows using a chain of certificates to verify the identity of a key issued by a hierarchy of certificate authorities.
- Are we going back to the trusted central server that we were trying to avoid? Yes, sort of. But now the CA is trusted by the public and is not necessarily a single point of failure.

An alternative to having a central trusted party is to use PGP; pretty good privacy. This approach builds a web of trust by leveraging the fact that every user is capable of signing certificates and that trust is transitive. If A can verify that a public key belongs to B, then A can create a certificate for B using A's private key. Then if C can verify A's public key then C can sign a certificate saying so with their private key.

Then we've established a chain of trust from $C \to A \to B$ and so on. If I can trust C then I can transitively trust A and B as well. If I trust A only then I can trust B but not C.

This all sounds great until you realize that the web of trust is only as strong as the weakest link. If C is compromised then the entire web of trust is compromised as well. This is why it's important to have the ability to **revoke** certificates. Revocation certificates are usually created as a dual to the certificate first created for the public key, and should be stored safely so that an adversary can't falsely issue a revocation. They also shouldn't be self-signed. Revocation

lists are made public and should be referenced when verifying a certificate. If a certificate is revoked then the certificate is no longer valid.

Common techniques when sending messages using public keys include:

- To encrypt a message: simply encrypt the message with the recipient's public key.
- To prove that a message is coming from you, sign the message with your private key and send the signature along with the message.
- To prevent replay attacks, include a nonce (usually just an increasing number) in the message and sign the nonce along with the message.

Subsection 2.17

Hashes

- A one-way function that converts a large input into a small, typically fixed size output
- Low probability of collision; $H(m) = h$
- Can be thought of as a "fingerprint" of the input
  - m is the preimage/input to hash
  - h is the hash value or message digest
  - H is a lossy compression function

A good hash function should have the following properties:

1. **Preimage resistance**: Given h, it should be computationally infeasible to find m such that $H(m) = h$
2. **Second preimage resistance**: Given m, it should be computationally infeasible to find m' such that $H(m) = H(m')$
3. **Collision resistance**: It should be computationally infeasible to find m, m' such that $H(m) = H(m')$

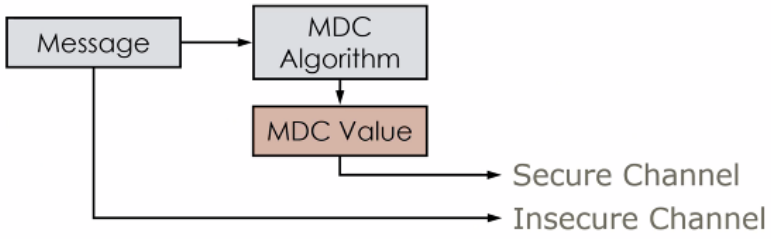
Assuming that the length of the hash is n bits, then

It is also desirable for small changes in the input to result in large changes in the output.

- Second Preimage resistance: 2^{n-1}
- Collision resistance: $2^{n/2}$ (birthday attack)

MDC (modification detection code) is a hash function that is used to detect changes in a message.

One way to make sure that a message is passed with integrity is to send a hash of the message along a secure channel and then have the receiver recompute the hash and compare it to the one sent. If they are the same then the message was not modified in transit.



If confidentiality is required the communicators may want to encrypt the message.

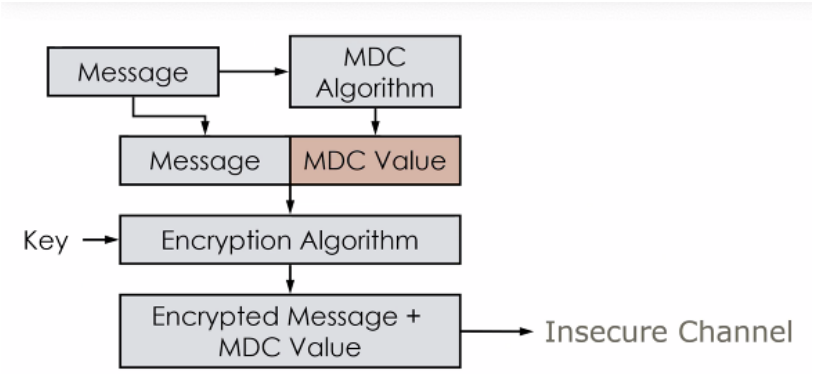


Figure 39. Combining MDC with encryption to ensure integrity and confidentiality

Nowadays just use SHA256 (SHA2)[38].

[38] *MD5 and SHA1 are deprecated*

Definition 25

MAC: A message authentication code uses a hash to provide integrity and authentication. A MAC is constructed as $h = H(k, M)$ where k is the secret key and M the message. The receiver knows that whoever generated the MAC must also know the key, thus authenticating the message source.

MACs are often constructed from symmetric ciphers.

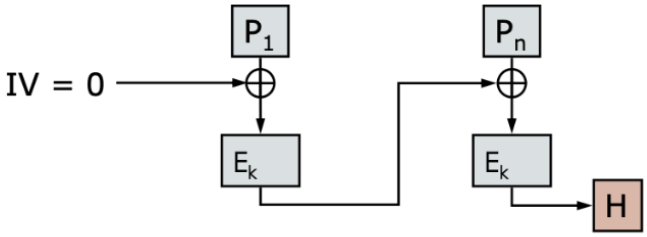


Figure 40. CBC-MAC

This method is similar to CBC encryption for block ciphers except a single hash value is produced at the end. Hash size is the same as the block size of the block cipher. The MAC key must also be different from the encryption key[39].

A MAC can be constructed by concatenating the secret key with the message and using a mash, which creates a keyed-hash MAC (HMAC)

[39] *This is bad practice and can produce a vulnerability depending on your encryption schemes and their interactions*

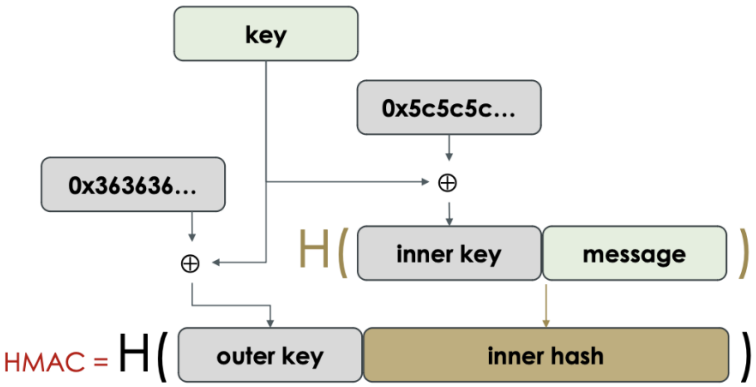


Figure 41. HMAC creation

An inner and outer hash gets rid of the extension problem since this requires the inner hash to be extended as well (which is encrypted!). HMACs are a reliable way to give strong signatures to messages.

HMAC = H[(K ⊕ opad) + H((K ⊕ ipad) + M)]

- "+" denotes string concatenation, "⊕" denotes logical XOR
- **M** is the arbitrary-length message
- Assume hash block size = **n** bits (e.g., 512 bits for SHA1)
- **K** is the key, padded with 0's on right side to **n** bits
- **opad** = 0x3636... (or 00110110) repeated to **n** bits
- **ipad** = 0x5c5c... (or 01011100) repeated to **n** bits

Figure 42. HMAC process (opad and ipad are flipped here relative to RFC 6328)

2.17.1 Hash-based data structures

- It is often useful nowadays to have a data structure that can be updated in a secure way and to verify the integrity of a set of things instead of a single object.

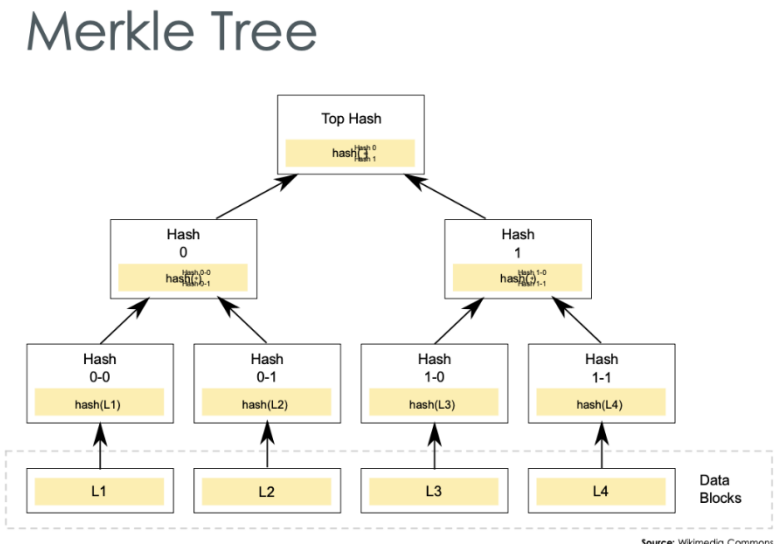


Figure 43. Merkle Tree

Comment Consider we have a bunch of data that we want to keep track of (Data blocks, lowest level of tree in diagram). We can hash all of them and then build a binary tree from the bottom up of the hashes. A parent of two nodes will take on the hash value of the concatenated hash of the it's children. And then the root node would have the hash of everything. The reason why this is better than concatenating all the data blocks and then hashing it together is that the individual blocks we're hashing is a lot smaller and changing a block won't require rehashing along the entire set of data blocks; you will only have to hash $\log_2 n$ times; from the data block up to the root. The top hash is a hash of all the data blocks and can be used to verify the integrity of all the data we're looking at.

Section 3

ECE353 Operating Systems

Comment Examples and some figures taken from Prof. Jon Eyolfson's ECE353 slides and notes (https://eyolfson.com).

Subsection 3.1

Kernel Mode

3.1.1 ISAs and Permissions

There are a number of ISAs in use today; x86 (amd64), aarch64 (arm64), and risc-v are common ones. For purposes of this course we will study largely arm systems but will touch on the other two as well.

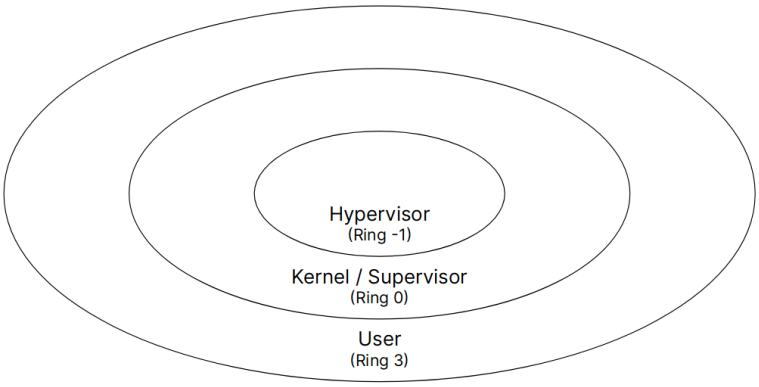


Figure 44. x86 Instruction access rings. Each ring can access instructions in its outer rings.

The kernel runs in, well, Kernel mode. **System calls** offer an interface between user and kernel mode[40].

The system call ABI for x86 is as follows:

Enter the kernel with a `svc` instruction, using registers for arguments:

- `x8` — System call number
- `x0` — 1st argument
- `x1` — 2nd argument
- `x2` — 3rd argument
- `x3` — 4th argument
- `x4` — 5th argument
- `x5` — 6th argument

This ABI has some limitations; i.e. all arguments must be a register in size and so forth, which we generally circumvent by using pointers.

For example, the `write` syscall can look like:

```
1 ssize_t write(int fd, const void* buf, size_t count);
2 // writes bytes to a file descriptior
```

3.1.2 ELF (Executable and Linkable Format)

- Aways starts with 4 bytes: `0x7F, 'E', 'L', 'F'`
- Followed byte for 32 or 64 bit architecture
- Followed by 1 byte for endianness

`readelf` can be used to read `ELF` file headers.
For example, `readelf -a $(which cat)` produces (output truncated)

[40] *Linux has 451 total syscalls*
Note: API (application programming interface), ABI (Application Binary Interface). API abstracts communication interface (i.e. two ints), ABI is how to layout data, i.e. calling convention

Most file formats have different starting signatures or magic numbers

```
ELF Header:
  Magic:   7f 45 4c 46 02 01 01 00 00 00 00 00 00 00 00 00
  Class:                             ELF64
  Data:                              2's complement, little endian
  Version:                           1 (current)
  OS/ABI:                            UNIX - System V
  ABI Version:                       0
  Type:                              DYN (Position-Independent
  ↪  Executable file)
  Machine:                           Advanced Micro Devices X86-64
  Version:                           0x1
  Entry point address:               0x32e0
  Start of program headers:          64 (bytes into file)
  Start of section headers:          33152 (bytes into file)
  Flags:                             0x0
  Size of this header:               64 (bytes)
  Size of program headers:           56 (bytes)
  Number of program headers:         13
  Size of section headers:           64 (bytes)
  Number of section headers:         26
  Section header string table index: 25
```

`strace` can be used to trace systemcalls. For example let's look at the 168-byte hello-world example

This output is followed by information about the program and section headers

```
ELF Header:
  Magic:   7f 45 4c 46 02 01 01 00 00 00 00 00 00 00 00 00
  Class:                             ELF64
  Data:                              2's complement, little endian
  Version:                           1 (current)
  OS/ABI:                            UNIX - System V
  ABI Version:                       0
  Type:                              DYN (Position-Independent
  ↪  Executable file)
  Machine:                           Advanced Micro Devices X86-64
  Version:                           0x1
  Entry point address:               0x32e0
  Start of program headers:          64 (bytes into file)
  Start of section headers:          33152 (bytes into file)
  Flags:                             0x0
  Size of this header:               64 (bytes)
  Size of program headers:           56 (bytes)
  Number of program headers:         13
  Size of section headers:           64 (bytes)
  Number of section headers:         26
  Section header string table index: 25
```

Listing 2: Note: This is for arm cpus

If we run this then we see that the program makes a `write` syscall as well as a `exit_group`

```
execve (" ./ hello_world " , [ " ./ hello_world " ] , 0 x7ffd0489de40
    ↪  /* 46 vars */ ) = 0
write (1 , " Hello world \ n " , 12) = 12
exit_group (0) = ?
+++ exited with 0 +++
```

Note that these strings are not null-terminated (null-termination is just a `C` thing) because we don't want to be unable to write strings with the null character to it.

```
execve("./hello_world_c", ["./hello_world_c"], 0x7ffcb3444f60 /* 46 vars */) = 0
brk(NULL)                               = 0x5636ab9ea000
openat(AT_FDCWD, "/etc/ld.so.cache", O_RDONLY|O_CLOEXEC) = 3
fstat(3, {st_mode=S_IFREG|0644, st_size=149337, ...}) = 0
mmap(NULL, 149337, PROT_READ, MAP_PRIVATE, 3, 0) = 0x7f4d43846000
close(3)                                = 0
openat(AT_FDCWD, "/usr/lib/libc.so.6", O_RDONLY|O_CLOEXEC) = 3
read(3, "\177ELF\2\1\1\3\0\0\0\0\0\0\0\0\3\0>\0\1\0\0\0000C"..., 832) = 832
lseek(3, 792, SEEK_SET)                 = 792
read(3, "\4\0\0\0\24\0\0\0\3\0\0\0GNU\0\201\336\t\36\251c\324"..., 68) = 68
fstat(3, {st_mode=S_IFREG|0755, st_size=2136840, ...}) = 0
mmap(NULL, 8192, PROT_READ|PROT_WRITE, MAP_PRIVATE|MAP_ANONYMOUS, -1, 0)
  = 0x7f4d43844000
lseek(3, 792, SEEK_SET)                 = 792
read(3, "\4\0\0\0\24\0\0\0\3\0\0\0GNU\0\201\336\t\36\251c\324"..., 68) = 68
lseek(3, 864, SEEK_SET)                 = 864
read(3, "\4\0\0\0\20\0\0\0\5\0\0\0GNU\0\2\0\0\300\4\0\0\0\3\0\0", 32) = 32
```

Figure 45. A c hello world would load the stdlib before printing...

3.1.3 Kernel

The kernel can be thought of as a long-running program with a ton of library code which executes on-demand. Monolithic kernels run all OS services in kernel mode, but micro kernels run the minimum amount of servers in kernel mode. Syscalls are slow so it can be useful to put things in the kernel space to make it faster. But there are security reasons against putting everything in kernel mode.

3.1.4 Processes & Syscalls

A process is like a combination of all the virtual resources; a "virtual GPU" (if applicable), memory (addr space), I/O, etc. The unique part of a struct is the PCB (Process Control Block) which contains all of the execution information. In Linux this is the `task_struct` which contains information about the process state, CPU registers, scheduling information, and so forth.

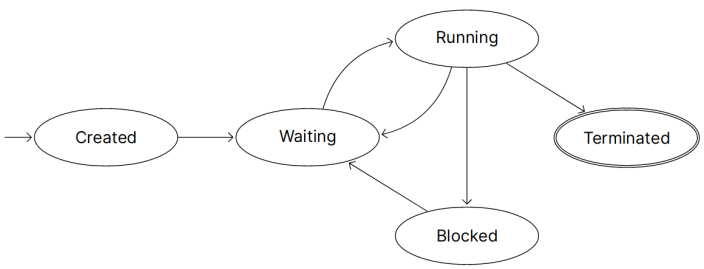


Figure 46. A possible process state diagram

These state changes are managed by the Process and OS[41] so that the OS scheduler can do its job. An example of where some of these states can be useful would be to free up CPU time while a process is in the Blocked state while waiting for IO. Process can either manage themselves (cooperative multitasking) or have the OS manage it (true multitasking). Most systems use a combination of the two, but it's important to note that cooperative multitasking is not true multitasking.

Context switching (saving state when switching between processes) is expensive. Generally we try to minimize the amount of state that has to be saved (the bare minimum is the registers). The scheduler decides when to switch. Linux currently uses the CFS[42].

In `c` most system calls are wrapped to give additional features and to put them more concretely in the userspace.

[41] *I think*

Process state can be read in `/proc` for linux systems.

[42] *completely fair scheduler*

```c
#include <stdio.h>
#include <stdlib.h>

void fini(void) {
  puts("Do fini");
}

int main(int argc, char **argv) {
  atexit(fini);
  puts("Do main");
  return 0;
}
```

Figure 47. Demonstration of c feature to register functions to call on program exit

```c
int main ( int argc , char * argv []) {
  printf ("I 'm going to become another process \n" );
  char * exec_argv [] = {" ls " , NULL };
  char * exec_envp [] = { NULL };
  int exec_return = execve ("/ usr / bin / ls " , exec_argv ,
  ↪  exec_envp );
  if ( exec_return == -1) {
    exec_return = errno ;
    perror (" execve failed " );
    return exec_return ;
  }
  printf (" If execve worked , this will never print \n" );
  return 0;
}
```

Listing 3: Demo of execve turning current program to ls (executes program, wrapper around exec syscall)

```c
#include <sys/syscall.h>
#include <unistd.h>

int main (){
  syscall(SYS_exit_group, 0);
}

```

Listing 4: An example of using a raw syscall system exit instead of c's exit()

Subsection 3.2

Fork, Exec, And Processes

- `fork` creates a new process which is a copy of the current process. Everything is exactly the same except for the PID in the child and PID in the parent.
  - Returns -1 on error, 0 in the child process, and the pid of the child in the parent process
- `exec` replaces the current process with a new one
  - Returns -1 on error

Process states:

- The CPU is responsible for *scheduling* processes, so there can be >1 process per core.
- Maintaining the parent-child relationship
  - Parent is responsible for the child
  - This usually works; the parent can wait for the child to finish. But what if the parent crashes, etc?

- Zombie: a process that has finished but has not been cleaned up by its parent. This can be a problem because the process is still using resources. The OS has to keep a zomibe process until it's acknowledged. To avoid zombie build-up the OS can signal the parent process (over IPC) to acknowledge the child. (The parent can ignore it)
- Orphan: a process that has no parent. This can happen if the parent crashes. The OS can adopt the orphan and make it a child of the init process which can keep onto them or kill them as needed.

```c
int main(int argc, char *argv[]) {
    pid_t pid = fork();
  if (pid == -1) {
    int err = errno; perror("fork failed"); return err;
  }
  if (pid == 0) {
    printf("Child parent pid: %d\n", getppid());
    sleep(2);
    printf("Child parent pid (after sleep): %d\n", getppid());
  }
  else {
    sleep(1); }
  return 0; }
```

Listing 5: orphan example: parent exits before child and `init` has to clean up

Subsection 3.3

IPC

Reading and writing files is a form of IPC. For example, a simple process could write everything it reads, i.e this facsimile of the `cat` program

Standard file descriptors: 0 = stdin, 1 = stdout, 2 = stderr

```c
int main() {
  char buffer[4096];
  ssize_t bytes_read;
  // read (see man 2 read) reads from a file descriptor
  // can't assume always successful; see from `man errno`
  //   Nearly all of the system calls provide an error number in the external variable errno, which is defined as: extern int errno. Refer to man pages for what each errno means.

  while ((bytes_read = read(0, buffer, sizeof(buffer))) > 0) {
    ssize_t bytes_written = write(1, buffer, bytes_read);
    if (bytes_written == -1) {
      int err = errno;
      perror("write");
      return err;
    }
    assert(bytes_read == bytes_written);
  }
  if (bytes_read == -1) {
    int err = errno;
    perror("read");
    return err;
  }
  assert(bytes_read == 0);
  return 0;
}
```

Another way of IPC is using signals. Common signals include

- SIGINT (Ctrl-C)
- SIGKILL (kill -9)
- EOF (Ctrl-D)

A signal pauses (interrupts) your program and then runs the signal handler. Process can be interrupted at any point in execution, and the process will resume after the signal handler finishes.

```c
ssize_t bytes_read;
  while ((bytes_read = read(0, buffer, sizeof(buffer))) != 0) {
    if (bytes_read == -1) {
      if (errno == EINTR) {
        continue;
      }
      else {
        break;
      }
    }
    ssize_t bytes_written = write(1, buffer, bytes_read);
    if (bytes_written == -1) {
      int err = errno;
      perror("write");
      return err;
    }
    assert(bytes_read == bytes_written);
  }
  if (bytes_read == -1) {
    int err = errno;
    perror("read");
    return err;
  }
  assert(bytes_read == 0);
  return 0;
}
```

// breaking here

```c

int main(int argc, char *argv[])
{
  if (argc > 2) {
    return EINVAL;
  }

  if (argc == 2) {
    close(0);
    int fd = open(argv[1], O_RDONLY);
    if (fd == -1) {
      int err = errno;
      perror("open");
      return err;
    }
  }

  register_signal(SIGINT);
  register_signal(SIGTERM);

  char buffer[4096];
  ssize_t bytes_read;
  while ((bytes_read = read(0, buffer, sizeof(buffer))) > 0) {
    ssize_t bytes_written = write(1, buffer, bytes_read);
    if (bytes_written == -1) {
      int err = errno;
      perror("write");
      return err;
    }
    assert(bytes_read == bytes_written);
  }
  if (bytes_read == -1) {
    int err = errno;
    perror("read");
    return err;
  }
  assert(bytes_read == 0);
  return 0;
}
```

- `register_signal` sets a bunch of things such that we can handle the signal i.e. execute a function when a signal occurs. In this program we register SIGINT and SIGTERM with the kernel to execute `handle_signal`.
- This will still fail on ctrl-c because the read system call can error out

```c
void handle_signal(int signum) {
  if (signum != SIGCHLD) {
    printf("Ignoring signal %d\n", signum);
  }


  printf("Calling wait\n");
  int wstatus;
  pid_t wait_pid = wait_pid = waitpid(-1, &wstatus, WNOHANG);
  // Here in our interrupt (signal) handler) we check for SIGCHILD and then waitpid the child if applicable
  if (wait_pid == -1) {
    int err = errno;
    perror("wait_pid");
    exit(err);
  }
  if (WIFEXITED(wstatus)) {
    printf("Wait returned for an exited process! pid: %d, status: %d\n", wait_pid, WEXITSTATUS(wstatus));
  }
  else {
    exit(ECHILD);
  }
  exit(0);
}


void register_signal(int signum) {
  struct sigaction new_action = {0};
  sigemptyset(&new_action.sa_mask);
  new_action.sa_handler = handle_signal;
  if (sigaction(signum, &new_action, NULL) == -1) {
    int err = errno;
    perror("sigaction");
    exit(err);
  }
}

int main() {
  register_signal(SIGCHLD);

  pid_t pid = fork();
  if (pid == -1) {
    return errno;
  }
  if (pid == 0) {
    sleep(2);
  }
  else {
    while (true) {
      printf("Time to go to sleep\n");
      sleep(9999);
    }
  }
  return 0;
}
```

- This snippet checks errno. and trys read again. Then the program is able to handle ctrl-c.
- This program can still get killed by kill -9 since it doesn't handle SIGKILL.
- Let's say we register $SIGKILL$ with the kernel to execute `handle_signal`. This will not work because you aren't allowed to ignore SIGKILL (-9).

Another thing we're interested in is to find out when a process is done. This can be polling on `waitpid`[43]

[43] *wait for process termination*

```

int main() {
  pid_t pid = fork();
  if (pid == -1) {
    return errno;
  }
  if (pid == 0) {
    sleep(2);
  }
  else {
    pid_t wait_pid = 0;
    int wstatus;

    unsigned int count = 0;
    while (wait_pid == 0) {
      ++count;
      printf("Calling wait (attempt %u)\n", count);
      wait_pid = waitpid(pid, &wstatus, WNOHANG);
    }

    if (wait_pid == -1) {
      int err = errno;
      perror("wait_pid");
      exit(err);
    }
    if (WIFEXITED(wstatus)) {
      printf("Wait returned for an exited process! pid: %d, status:
↪  %d\n", wait_pid, WEXITSTATUS(wstatus));
    }
    else {
      return ECHILD;
    }
  }
  return 0;
}
```

Alternatively, we should use interrupts

Note: interrupt handlers run to completion. But an interrupt handler may occur while another interrupt handler is running, so execution must be passable and state managed accordingly

```c
void handle_signal(int signum) {
  if (signum != SIGCHLD) {
    printf("Ignoring signal %d\n", signum);
  }


  printf("Calling wait\n");
  int wstatus;
  pid_t wait_pid = wait_pid = waitpid(-1, &wstatus, WNOHANG);
  // Here in our interrupt (signal) handler) we check for SIGCHILD and then waitpid the child if applicable
  if (wait_pid == -1) {
    int err = errno;
    perror("wait_pid");
    exit(err);
  }
  if (WIFEXITED(wstatus)) {
    printf("Wait returned for an exited process! pid: %d, status: %d\n", wait_pid, WEXITSTATUS(wstatus));
  }
  else {
    exit(ECHILD);
  }
  exit(0);
}


void register_signal(int signum) {
  struct sigaction new_action = {0};
  sigemptyset(&new_action.sa_mask);
  new_action.sa_handler = handle_signal;
  if (sigaction(signum, &new_action, NULL) == -1) {
    int err = errno;
    perror("sigaction");
    exit(err);
  }
}

int main() {
  register_signal(SIGCHLD);

  pid_t pid = fork();
  if (pid == -1) {
    return errno;
  }
  if (pid == 0) {
    sleep(2);
  }
  else {
    while (true) {
      printf("Time to go to sleep\n");
      sleep(9999);
    }
  }
  return 0;
}
```

On a RISC-5 CPU there are three terms for interrupts:

- Interrupt: by external hardware and handled by kernel
- Exception: triggered by an instruction, kernel handles though process can optionally handle
- Trap: transfer of control of a trap handler by either an exception or interrupt. Syscall is a requested trap

Subsection 3.4

Pipe

Definition 26

```c
int pipe(int pipefd[2]);
```

Returns 0 on success, -1 on failure (and sets errno). Forms a one-way communication channel with 2 file descriptors; 0 for reading and 1 for writing. The pipe is unidirectional.

Subsection 3.5

Basic Scheduling

- A pre-emptive resource can be taken and used for something else; i.e. CPU. Shared via scheduling
- A non-pre-emptive resource cannot be taken and used for something else; i.e. I/O. Shared via alloc/dealloc or queuing. Note that some parallel or distributed systems may allow you to allocate a CPU
- Dispatcher: responsible for context switching. Scheduler: deciding which processes to run
- Non-preemptible processes must run until completion, so the scheduler can only make a decision on termination.
- Pre-emptive allows the OS to run scheduler at will.
- Schedulers seek to minimize waiting time and maximize cpu utilization/throughput – all while giving each process the same percent of CPU time.

Definition 27 FCFS (First come first served) is a scheduling algorithm that runs the process that arrives first. Processes are stored in a queue in arrival order. This has the downside of potentially introducing long wait times if longer tasks arrive before shorter ones.

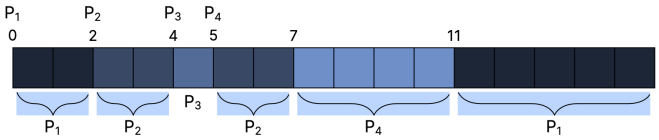
Definition 28 SJF (Shortest job first): schedule the job with the shorts execution time first. Though it's optimal at minimizing average wait times, since we don't know how long each process takes it may not be practically optimal. It also has the downside of potentially starving longer jobs.

Definition 29 SRTF (Shortest Remaining Time First): schedule the job (with pre-emptions now) with the shortest remaining time. This optimizes the average waiting time.

Consider the same processes and arrival times as SJF:

Process	Arrival Time	Burst Time
P_1	0	7
P_2	2	4
P_3	4	1
P_4	5	4

For SRTF, our schedule is (arrival on top):



Average waiting time: $\frac{9+1+0+2}{4} = 3$

So far we haven't considered fairness. We can make a scheduler more fair by using a round-robin scheduler, which is a pre-emptive scheduler which divides execution time into quanta and gives processes <quanta> of time while round-robining through them. [44]

[44] *How to consider quantum length? Consider context switching time*

Process	Arrival Time	Burst Time
P_1	0	7
P_2	2	4
P_3	4	1
P_4	5	4

For RR, our schedule is (arrival on top, queue on bottom):

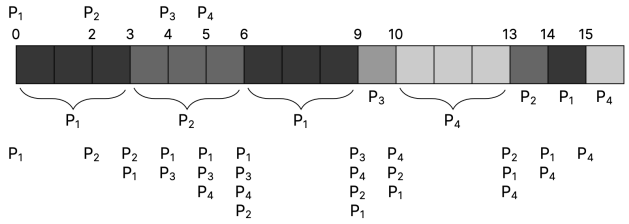


Figure 48. RR example with quanta of 3 units. Average number of switches is 7, average waiting time is $\frac{8+8+5+7}{4} = 7$, average response time is $\frac{0+1+5+5}{4}$. Note that ties are handled by favouring new processes.

Round robin performance is dependent on quantum length and job length. Long quantum causes starvation (FCFS), but twoo low and the performance sucks since context switches introduce overhead. If jobs have similar lengths RR has poor average waiting time

Subsection 3.6

Advanced Scheduling

- Processes can be given a priority. Linux: some integer value -20 -> 19
- Processes may *starve* if there a lot of higher priority processes. Can be resolved by dynamically changing priorities i.e. upping priority of a old process

Definition 30

Priority inversion: accidentally changing priority of low priority process to high through some dependency (i.e. high priority depending on low priority), which effectively flips the actual task priority.
This is a problem and a common solution for it is *priority inheritance*, where when a job

blocks one or more high-priority jobs it ignores the original assessment and executes it's critical section[45] at a higher priority level, then returns to its original level.

[45] *The blocking portion*

- Recall: `fg`, `bg`, `ctrl-z`, `jobs`, and so forth
- Foreground/background processes foreground processes are those that are currently running and can be interacted with. Background processes are those that are running in the background and cannot be interacted with (take user input). This is to separate processes that need good response times nad those that don't.
- One strategy is to create different queues for foreground and background processes, i.e. round-robin forground and then FCFS for background ... then schedule between the queues

Formally UNIX background processes are ones where the process group ID differs from its terminal group ID

Scheduling is a complex topic and there are many more algorithms that make an array of tradeoffs

Subsection 3.7

Symmetric Multiprocessing (SMP)

Definition 31 **Symmetric Multiprocessing**:

- All CPUS connected to same physical memory
- Each CPU has its own cache

Scheduling approaches:

- Per-CPU schedulers: assign a process to a CPU on creation (i.e. CPU with least processes). Easy to implement, no blocking, can cause load imbalance.
- Global scheduler: only one scheduler: adding processes while there are available CPUs. Can cause blocking, but load balanced (In Linux 2.4)
- These two extremes have downfalls, so we try to make a compromise: keep a global scheduler that can rebalance per-CPU cores; if a CPU is idle it can steal work from anther CPU. Can also introduce *process affinity*; the preference of a process to be scheduler on the same core[46]. This is a simplified version of the $O(1)$ scheduler in Linux 2.6.
- Gang scheduling: run a set of related processes simultaneously on a set of CPUs. This is useful for parallel applications (but requires global context switch across all CPUs).
- Real-time scheduling is also another problem: we may want to guarantee that tasks complete in a certain amount of time[47]
  - Current linux impls two soft-time schedulers: `SCHED_FIFO` and `SCHED_RR`, each with 0-99 static prioirty levels. Normal scheduling priorities apply to other processes (`SCHED_NORMAL`) with range $-20 \to 19$, 0 default.
  - Processes can change their own priorities with syscalls (`nice`, `sched_setscheduler`)
  - 2.4-2.6: O(N) global queue, 2.6-26.22: per-queue run queue, O(1) scheduler (complex, no fairness guarantee, not interactive), 2.6.3-CFS[48] based on red-black trees
- $O(1)$ scheduler is not great for modern computing; whereas in the past foreground/background was a reasonable split heuristic nowadays a lot of background processes are relevant.

[46] *to deal with cache locality*

[47] *Also, there are hard and soft real-time systems. Linux also implements FCFS and RR scheduling which you can select for tasks.*

[48] *completely fair scheduler*

Definition 32 **Ideal Fair Scheduling (IFS)**:

- Assume infinitely small time slice. If n processes, each runs at $\frac{1}{n}$ rate.
- Fair, interactive, and each process gets an equal amount of CPU time
- Would perform way too many context switches and have to scan all processes (O(n))
- Impractical

Definition 33 **Completely Fair Scheduler (CFS)**:

- For each runnable process assign it a 'virtual' runtime – at each scheduling point the process runs for time t and then increase it's virtual runtime by $t \cdot$ weight (based on priority)
- Virtual runtime monotonically increases. Scheduler selects process based on lowest virtual runtime to compute it's dynamic time slice w/ IFS
- Allow process to run, and then when it's time is up repeat the process
- Implemented with red-black trees keyed by virtual runtime. Impl uses red-black tree with nanosecond resolution.
- Tends to favour I/O bound processes by default (small CPU translates to low vruntime – larger time slice to catch up to ideal)

Subsection 3.8
Libraries

- Systemcalls use registers, while C is stack-based
- Arguments pushed onto stack from right-to-left, rax, rcx, rdx caller (remaining callee) saved
- Static libraries included at link time; i.e. .c -> .o -> exe, can also create archives via lots of .o -> .a which are then linked with a .o with a main to produce an executable
- .so (shared object) are reusable; multiple programs can use the same .so. OS only has to load one libc.so for example. Included at runtime
- `ldd <executable<` shows the shared objects used by an executable
- `objdump -T <executable>` shows the symbols in an executable. -d to disassemble library
- Can also statically link, i.e. copy .o to executable. Static linking is useful for small programs that don't need to be updated often and are also more portable (batteries included) at cost of recompilation and larger binary sizes
- Dynamic libraries can break executables if their ABI changes
- `c` has a consistent struct abi for example, i.e. memory w/ fields matching declaration order. Example of this may be function argument order/type or exposed struct member order.
- Use semver to version libraries; x.y.z; x major (breaking), y minor (non-breaking), z patch (bug fixes)

- dyn libraries make for easier development and debugging; can control dynamic linking with env variables (LD_LIBRARY_PATH, LD_PRELOAD). For example we can make a wrapper lib around liballoc that would output all malloc/free calls.

Subsection 3.9

Processes

- `execlp`: easier alternative to `execvp`. Does not return on success, but does return -1 on failure and sets `errno`. Lets you use `c` varargs instead of a string array
- `dup`, `dup2`: returns a new FD on success – copies the FD so that the old and new fd refer to the same thing. `dup` will return the lowest file descriptor, `dup2` will automically close the `newfd` (if open) and then make `newfd` refer to the same thing as `oldfd`. Generally use `dup2` to make a new fd of any type you desire.

```c
#include <assert.h>
#include <errno.h>
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <sys/wait.h>
#include <unistd.h>
// note: static is limited to a translation/compilation unit (i believe) but this is commonly just a file. Should factcheck this.
static void check_error(int ret, const char *message) {
    if (ret != -1) {
        return;
    }
    int err = errno;
    perror(message);
    exit(err);
}

static void parent(int in_pipefd[2], int out_pipefd[2], pid_t child_pid) {
    const char* message = "Hello, world!\n";
    int bytes_written = write(in_pipefd[1], message, strlen(message));
    check_error(bytes_written, "write");
    close(in_pipefd[1]); // need to close otherwise we have a deadlock; child's read will until this happens (hence blocking waitpid below)

    int wstatus;
    check_error(waitpid(child_pid, &wstatus, 0), "waitpid");
    assert(WIFEXITED(wstatus) && WIFEXITSTATUE(wstatus) == 0);

    char buf[4096]; // some large number. Can overflow
    // use read end (0)
    check_error(out_pipefd[1]);
    int (bytes_read = read(out_pipefd[0], buf, sizeof(buf)));
    check_error(bytes_read);
    printf("Got %*s\n", bytes_read, buffer); // not a c-string from the fd. Need to specify length.
}

static void child(int in_pipefd[2], int out_pipefd[2], const char *program) {
    // make write end of out_pipefd the stdout of the child
    check_error( dup2(out_pipefd[1], STDOUT_FILENO), "dup2");
    check_error( dup2(out_pipefd[0], STDIN_FILENO), "dup2"); // and same for stdin
    // before dup2: 0, 1, 2, 3, 4 (3,4 are out_pipefd)
    // after call to dup2: closes what fd1 points to (stdout) and replaces stdout with the write end of out_pipefd
    // Convention: only have 3FD open; clean up after yourself. Close the other file descriptors (3,4)
    check_error(close(out_pipefd[1]));
    // and need to close all the other fds here (omited for brevity)
    execlp(program, program, NULL);
}


```

And continuing here to break across two pages...

```
1
2  int main(int argc, char* argv[]) {
3      if (argc != 2) { return EINVAL; }
4      // will have 3 fd open: 0, 1, 2 for stdin, stdout, stdrr
5
6      int in_pipefd[2] = {0};
7      int out_pipefd[2] = {0};
8      check_error(pipe(out_pipefd), "outpipe");
9      check_error(pipe(id_pipefd), "inpipe");
10     // 0 is read end, 1 is write end
11     // want to use the pipe to communicate with the child
12     // pipe before fork, so that both parent and child have access to
   ↪   the pipe
13     // replace child stdout to out_pipefd[1]
14     // and replace parent std
15     pid_t pid = fork();
16     if (pid > 0) { parent(in_pipefd, out_pipefd, pid); }
17     else { child(in_pipefd, out_pipefd, argv[1]); }
18     return 0;
19 }
```

Subsection 3.10

Virtual Memory

- We want to expose the entire address space to each processes, i.e. let each process *think* that it has access to the whole space while in reality sharing it with other processes.
- MMU: usually a physical device which maps virtual addresses to physical addresses. One technique is to divide memory into fixed-sized pages (usually 4096 bytes). Page in virtual memory is a page; a page in physical memory is a frame.
- Early approach: segmentation: divide the address space into segments for code, data, stack, and heap. Segments are of dynamic size. Are large and can be costly to relocate – also leads to fragmentation (gaps of unused memory).
  - Segments contain a base, limit, and permissions. Physical address via `segment selector:offset`
  - MMU checks offset within limit. If so, uses base+offset and does permission checks. Otherwise it's a segfault.
  - For example, `0x1:0xff` with segment $0x1$, base $0x2000$ and limit $ox1ff$ will translate to $0x20FF$.
  - Linux handles segmentation virtual memory by setting every base to 0 and then limiting to the maximum amount
- CPUS have different levels of virtual addresses you can use. In this course we'll assume a 39 bit virtual address space used by RISC-V and other architectures
- Implemented by a page table indexed by VPN (Virtual page number) which translates to the Physical Page Number (PPN)

Considering the following page table:

VPN	PPN
0x0	0x1
0x1	0x4
0x2	0x3
0x3	0x7

We would get the following virtual → physical address translations:

```
0x0AB0 → 0x1AB0
0x1FA0 → 0x4FA0
0x2884 → 0x3884
0x32D0 → 0x72D0
```

Figure 49. Simple page table

Subsection 3.11

Page Tables

- Naive page tables are not scalable. For example, if we have 4GiB 2^{32} bytes of virtual memory and a 4 kB (2^{12} byte) page size size then the address space should be split into 2^{20} pages. So the page table must have 2^{20} entries, each of which requires 20 (frame number; 2^{20} frames), a valid bit, a dirty bit, and read/write/execute permission bits – a total of 25. So the total size of the page table is on the order of 2^{22} bytes = 4MB (of which we would need one per process)

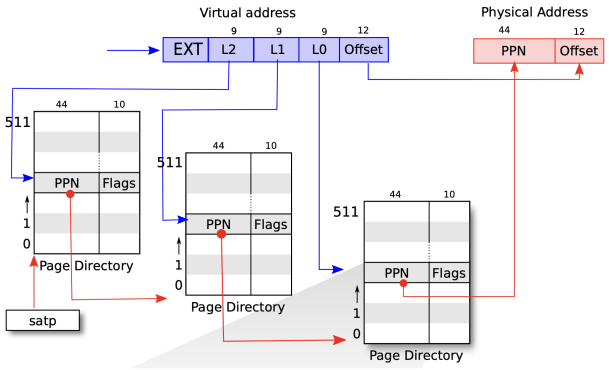


Figure 50. Multi-level page tables save storage space for sparse allocations

- Page allocation usually implemented via a linked list. Allocate page: remove it from the free list; deallocate – add back to free list.
- A page is used for each page table. There are 512 entries of 8 bytes each to make 4096 page tables, so each page table can be treated as an array of 512 page table entires (PTE).
- The PTE for L(N) points to the page table for L(N-1) – so follow these page tables until L0 and that contains the PPN

- Each table has it's own root page table (L1).
- `satp` register stores root page table
- Think of the highest level page table storing pointers to blocks and then lower level page tables storing pointers to segments within those blocks until we get to the exact memory address we want. It just so happens that these blocks also take on the form of a page table by themselves.

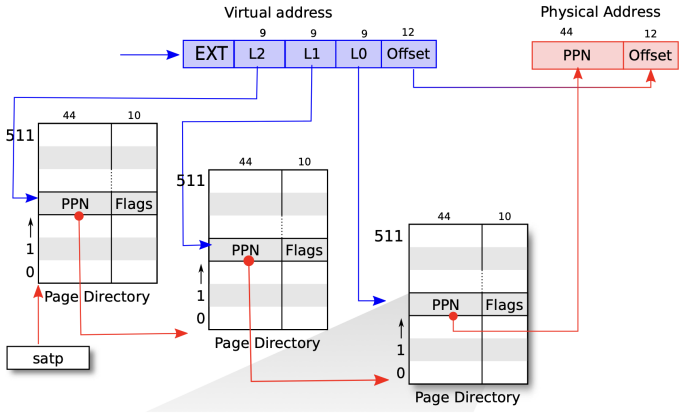


Figure 51. Since each page table has 512 entries – take offset on the address and then split off into 9-bit chunks to get index into each level of the page tables. When we get to the last level simply apply the offset to get the data within the page.

- Alignment: memory (by usual conventions) eventually line up with zero. For example pages that are 4096-byte have the last 12 bits zeroed.
- It would be inconvenient if a page starts at 0x7C00 and has the last byte at 0x88FF; instead in aligned systems a page starts at $0x7000$ and ends at $0x7FFF$. [49]

[49] *Alternatively addresses that are n-byte aligned are cleanly divisible by n*

Following is a snippet of code from class that simulates a page table. It's not a complete implementation but it's a good example of how to use the page table to translate virtual addresses to physical addresses.

```c
#include <sys/mman.h>
#define PAGE_SIZE 4096

#define LEVELS 3
#define PTE_VALID (1 << 0)

static uint64_t* root_page_table = NULL;
// A wrapper around mmap
static uint64_t* allocate_page_table() {
    void* page = mmap(NULL, PAGE_SIZE, PROT_READ|PROT_WRITE,
    ↪  MAP_ANONYMOUS|MAP_PRIVATE, -1, 0);
    if (page == MAP_FAILED) {
        int err = errno;
        perror("mmap");
        exit(err);
    }
    return page;
}

static void deallocate_page_table(void* page) {
    if (munmap(page, PAGE_SIZE) == -1) {
        int err = errno;
        perror("munmap");
        exit(err);
    }
}
```

```c
// Looks up a virtual address in the page table and returns the
↪  physical address
static uint64_t mmu(uint64_t virtual_address) {
    uint64_t* page_table = root_page_table;
    uint64_t va = (uint64_t) virtual_address;
    for (int i = LEVELS - 1; i >= 0; --i) {
        uint8_t start_bit = 9 * i + 12;
        uint64_t mask = (uint64_t) 0x1FF << start_bit;
        uint16_t index = (mask & va) >> start_bit;

        uint64_t pte = page_table[index];
        if (!(pte & PTE_VALID)) {
            printf("0x%lX: page fault\n", va);
            return 0;
        }

        if (i != 0) {
            page_table = (uint64_t*) ((pte >> 10) << 12);
            continue;
        }

        uint64_t pa = ((pte & ~0x3FF) << 2) | (va & 0xFFF);
        printf("0x%lX: 0x%lX\n", va, pa);
        return pa;
    }
    __builtin_unreachable();
}

// page table entry from physical address
uint64_t pte_from_ppn(uint64_t ppn) {
    uint64_t pte = ppn << 10;
    pte |= PTE_VALID; // set valid bit
    return pte;
}

// page table entry from page number
uint64_t pte_from_page_table(uint64_t* page_table) {
    return pte_from_ppn(((uint64_t) page_table) >> 12);
}
```

```c
int main() {
    assert(sysconf(_SC_PAGE_SIZE) == PAGE_SIZE);
    uint64_t* l2_page_table_1 = allocate_page_table();
    uint64_t* l1_page_table_1 = allocate_page_table();
    uint64_t* l0_page_table_1 = allocate_page_table();
    uint64_t* l0_page_table_2 = allocate_page_table();
    root_page_table = l2_page_table_1; // global var to set root
    // manually set values at 0xabcdef to something valid
    l2_page_table_1[0] = pte_from_page_table(l1_page_table_1);
    l1_page_table_1[5] = pte_from_page_table(l0_page_table_1);
    // offset for 0xabcdef
    l1_page_table_1[13] = pte_from_page_table(l0_page_table_2);
    l0_page_table_1[188] = pte_from_ppn(0xCAFE);
    // set page table entry to physical address 0xFACE
    l0_page_table_1[188] = pte_from_ppn(0xFACE);
    mmu(0xABCDEF); // [L2: 0][L1: 5][L0: 188] -> 0xCAFEDF
    mmu(0x1ABCDEF); // -> OxCAFEDF
    // two virtual addresses point to the same physical address here.
    // this is how shared memory would be implemented
    deallocate_page_table(root_page_table);
    root_page_table = NULL;
    return 0;
}
```

```c
int main() {
    assert(sysconf(_SC_PAGE_SIZE) == PAGE_SIZE);
    uint64_t* l2_page_table_1 = allocate_page_table();
    uint64_t* l1_page_table_1 = allocate_page_table();
    uint64_t* l0_page_table_1 = allocate_page_table();
    uint64_t* l0_page_table_2 = allocate_page_table();
    root_page_table = l2_page_table_1; // global var to set root
    // manually set values at 0xabcdef to something valid
    l2_page_table_1[0] = pte_from_page_table(l1_page_table_1);
    l1_page_table_1[5] = pte_from_page_table(l0_page_table_1);
    // offset for 0xabcdef
    l1_page_table_1[13] = pte_from_page_table(l0_page_table_2);
    l0_page_table_1[188] = pte_from_ppn(0xCAFE);
    l0_page_table_1[188] = pte_from_ppn(0xFACE);
    mmu(0xABCDEF); // [L2: 0][L1: 5][L0: 188] -> 0xCAFEDF
    mmu(0x1ABC007); // translates -> 0xFACE007
    deallocate_page_table(root_page_table);
    root_page_table = NULL;
    return 0;
}
```

- Let's assume our program uses 512 pages. What's the min and max number of page tables we need? (with a 3-level paging system)
  - Min: 3 page tables total; L2 -> L1 -> 512 entries in L0
  - Max: 1 entry per L1, L0 so 512 tables each, and then 1 table (can only have 1 table for the entry point) in L0 and then -> so 512 * 2 + 1 = 1025 pages tables.

- Example: how many levels do we need? 32 bit virtual address space ,page size of 4096, PTE size of 4 bytes. Each page table should fit in a single page.
  - Number of PTEs: $\log_2(\#\text{ PTEs per page})$ is the number of bits to index a page table
  - number of levels $= \frac{\text{virtual - offset}}{\text{index}} = \frac{32-12}{10} 2$
- Page tables for every memory access is slow: solution is to use caching
- Programs tend to have a lot of memory access patterns and only use a few pages at a time. TLB[50] works as cache for virtual address to physical address translation

[50] *Translation Lookaside Buffer*

Subsection 3.12

Threads

- Threads share memory and enable concurrency within the same process
- `pthread`!
- `join` is the thread equivalent of `wait`
- `pthread_detach` release their resources when they terminate. Otherwise (by default) they are joinable and must be joined before resources are released.

3.12.1 Threads Implementation

- Kernels can be implemented in the user or at the kernel level.
- User level usually involves fast switching at the user level. These are fast to create, but if one thread blocks it will block the entire process.
- Kernel level threads can deal with these blocking threads, but are slower since they require syscalls.
- User level threads can be desirable because they can be made to only depend on the c standard library, which is portable
- Many-to-one threads map multiple user threads to one kernel thread
- One-to-one threads map one user thread to one kernel thread. `pthread` does this.
- Many-to-many is a hybrid approach. This leads to a complicated implementation.
- Threads complicate the kernel. For example, how should `fork` work with a process that has multiple threads? Do we just copy all threads over? Linux will only copy the thread that called fork into a new process and an option at `pthread_atfork` which can be used to control the behaviour.
- What about signals? On linux this will just be any random thread within that process.
- Instead of many-to-many a common technique is to use a thread pool. This creates a set number of threads and a queue of tasks.
- Cooperative scheduling: threads must call `yield`. Pre-emptive scheduling can be implemented by forcing threads to call `yield`.

3.12.2 Useful tools

- `tailq` (`sys/queue.h`) is a header that has a bunch of macros for working on singly and doubly linked lists, queues, and circular queues.
  - i.e. `TAILQ_ENTRY` is a macro that defines the pointer relations for a doubly linked tail queue
  - `TAILQ_FOREACH` is a macro that iterates over a doubly linked tail queue
  - `TAILQ_INSERT_SAFE` is a macro that inserts an element into a doubly linked tail queue at the tail
  - `TAILQ_REMOVE` is a macro that removes an element from a doubly linked tail queue
- `ucontext` (`ucontext.h`) is a header that largely wraps around `ucontext_t` which holds the context for a user thread of execution, i.e. stack, saved register, and blocked signals.
  - Useful methods include `getcontext`, `setcontext`, `makecontext`, and `swapcontext`

```c
#include <errno.h> // errno
#include <stddef.h> // NULL
#include <stdio.h> // perror
#include <stdlib.h> // exit
#include <sys/mman.h> // mmap, munmap
#include <sys/signal.h> // SIGSTKSZ
#include <ucontext.h> // getcontext, makecontext, setcontext,
↪  swapcontext
#include <valgrind/valgrind.h> // VALGRIND_STACK_REGISTER

static void die(const char* message) {
    int err = errno;
    perror(message);
    exit(err);
}

static char* new_stack(void) {
    char* stack = mmap(
        NULL,
        SIGSTKSZ, // canonical size for signal stack
        PROT_READ | PROT_WRITE | PROT_EXEC,
        MAP_ANONYMOUS | MAP_PRIVATE,
        -1,
        0
    );
    if (stack == MAP_FAILED) {
        die("mmap stack failed");
    }
    VALGRIND_STACK_REGISTER(stack, stack + SIGSTKSZ);
    // tells valgrind this is an unique stack
    return stack;
}

static void delete_stack(char* stack) {
    if (munmap(stack, SIGSTKSZ) == -1) {
        die("munmap stack failed");
    }
}
```

```

// the stacks we're going to use in this demo
static ucontext_t t0_ucontext;
static ucontext_t t1_ucontext;
static ucontext_t t2_ucontext;

static char* t1_stack;
static char* t2_stack;

static void t2_run(void) {
    printf("T2 should be done, switch back to T0\n");
    delete_stack(t1_stack);
    setcontext(&t0_ucontext);
}

static void t1_run(void) {
    printf("Hooray!\n");
}


```

```c

int main(void) {
    /* Creates a new context by copying over the current context, this copies all its
     * registers, and a pointer to its stack (the default kernel allocated one). */
    getcontext(&t0_ucontext);

    /* If we setcontext or swapcontext to t0_context, it'll be as if we just
     * returned from that getcontext call. If you uncomment the line below
     * you'll be in an infinite loop! */
    // setcontext(&t0_ucontext);

    /* Let's create a context that'll execute the run function */
    t1_stack = new_stack();
    getcontext(&t1_ucontext);
    t1_ucontext.uc_stack.ss_sp = t1_stack;
    t1_ucontext.uc_stack.ss_size = SIGSTKSZ;
    /* Uncomment this line to switch to another context when this one ends.
     * By default the process will just exit if a thread makes it to the end
     * of the function.
     */
    // t1_ucontext.uc_link = &t2_ucontext;
    // modifies an initialized context such that when it runs it will
    // call the functions with the arguments provided
    makecontext(
        &t1_ucontext, /* The ucontext to use, it must be initialized with
                       * getcontext */
        t1_run, /* The function to start executing */
        0); /* This is how many arguments we're going to pass to the function */

    t2_stack = new_stack();
    getcontext(&t2_ucontext);
    t2_ucontext.uc_stack.ss_sp = t2_stack;
    t2_ucontext.uc_stack.ss_size = SIGSTKSZ;
    makecontext(&t2_ucontext,  t2_run,  0);

    /* If we just setcontext here when we run T2 after T1 finishes, we'll
     * get into an infinite loop again. */
    // setcontext(&t1_ucontext);
    // exchanges currently active context
    swapcontext(&t0_ucontext, &t1_ucontext);

    printf("Main is back in town\n");
    delete_stack(t2_stack);

    return 0;
}


```

Subsection 3.13
Locks

- Concurrent actions accessing the same variable with at least one write can cause a data race
- Atomic operations are operations that are guaranteed to be executed in a single step, i.e. non-preemptible.

Definition 34 TAC (three-address-code) is an intermediate representation that is used to represent a program in a way where each instruction is atomic – this is useful for reasoning about data races and can be easier to read than assembly. They have the form

```
1    result := operand1 operator operand2
```

For gcc we can see the tac by using the `-fdump-tree-gimple` or `fdump-tree-all` flags.

Let's consider some GIMPLE code produced from a function that increments an integer stored at address `pcount`

```
1  D.1 = *pcount;
2  D.2 = D.1 + 1;
3  *pcount = D.2;
```

Order				*pcount
R1	W1	R2	W2	2
R1	R2	W1	W2	1
R1	R2	W2	W1	1
R2	W2	R1	W1	2
R2	R1	W2	W1	1
R2	R1	W1	W2	1

Figure 52. Pre-emption possibilities. Let's say we have a producer-consumer model with read/write from two threads. There are many possible orderings of these GIMPLE'd instructions, of which some produce undesirable results

```c
pthread_mutex_t lock = PTHREAD_MUTEX_INITIALIZER;
// or: pthread_mutex_t m; pthread_mutex_init(&m2, NULL)

pthread_mutex_lock(&lock);
// ... critical section: only one thread can access at a time
pthread_mutex_unlock(&lock)
// careful: deadlocks can happen with multiple mutexes.

pthread_mutex_trymutex(&lock); // returns 0 if it was able to lock,
↪  otherwise an error code

pthread_mutex_destroy(&lock)
```

How are these implemented?

A naive implementation may look as follows:

```c
void init(int *l) { *l = 0; }

void lock(int *l) {
while (*l == 1);
*l = 1;}

void unlock(int *l) { *l = 0; }
```

However, this is 1) not safe (both threads can be in the critical section) and not efficient due to the busy wait.

Better approaches include Peterson's algorithm and Lamport's bakery algorithm. They have some scalability issues and processors may not execute in order.

Hardware requirements to implement software locks: atomic load and stores, and instructions execute in order

Here's another attempt using a magical atomic function: `compare_and_swap` which returns the original value pointed to, and only swaps if the original value equals old and changes it to new.

```c
void init(int *l) { *l = 0; }
void lock(int *l) { while (compare_and_swap(l, 0, 1)); }
void unlock(int *l) { *l = 0; }
```

This solves the concurrency issue however it still is not efficient due to the busy wait.

`compare_and_swap` is commonly implemented in hardware; on x86 platforms this is implemented using the `cmpxchg` instruction

```c
void lock(int *l) {
while (compare_and_swap(l, 0, 1)) {thread_yield(); }}
```

This is better, but still not ideal. Multiple threads waiting for an event can be awoken when the vent occurs, but only one will win[51]. This cycle will repeat until the herd dies down, but not without causing many freezes along the way. Some sort of order must be placed on the herd – maybe a FIFO queue?

[51] *thundering herd problem*

```c
void lock(int *l) {
while (compare_and_swap(l, 0, 1)) {
    // add myself to the lock wait queue
    thread_sleep(); }}
void unlock(int *l) { *l = 0;
if (/* threads in wait queue */) {
    // wake up one thread
}}
```

However this suffers from two issues: 1) lost wakeup and 2) wrong thread getting the lock

Consider T1, T2 with T2 holding the lock; what if the context gets switched to T2 before T1 is able to successfully add itself to the wait queue? Then T1 will never get woken up since when T2 unlocks T1 will not be in the wait queue.

Let's consider another scenario: we have three threads T1, T2, and T3, and T2 is holding the lock with T3 in T1 in queue to lock the lock (with T3 before T1). T2 may try to wake up the lock, but if the OS swaps to T1 before T2 can wake up T3, then T1 will acquire the lock before T3 does; T1 stole the lock from T3.

A lock-guard pair can be used to fix these problems

```c
typedef struct {
  int lock;
  int guard;
  queue_t *q;
} mutex_t;

void lock(mutex_t *m) {
  while (
    compare_and_swap(m->guard, 0, 1)
  );
  if (m->lock == 0) {
    m->lock = 1; // acquire mutex
    m->guard = 0;
  } else {
    enqueue(m->q, self);
    m->guard = 0;
    thread_sleep();
    // wakeup transfers the lock here
  }
}
```

```c
void unlock(mutex_t *m) {
  while (
    compare_and_swap(m->guard, 0, 1)
  );
  if (queue_empty(m->q)) {
    // release lock, no one needs it
    m->lock = 0;
  }
  else {
    // direct transfer mutex
    // to next thread
    thread_wakeup(dequeue(m->q));
  }
  m->guard = 0;
}
```

Figure 53. A lock-guard pair can be used to make the `lock` and `unlock` functions atomic themselves to avoid ugly synchronization issues like those mentioned above

However, this does not solve the data race problem: what if a thread gets interrupted right before `thread_sleep` (but after being added to the wait queue). So a `thread_wakeup` may try to wakeup a thread that isn't sleeping yet. A solution may be to poll `thread_wakeup`, but this falls back into the busy waiting problem we had before.

A data race is when two concurrent actions access the same variable and at least one of them is a write; we can have as many readers as we want.

Read-write locks (`pthread_rwlock_t` & co.) are designed to capture this behaviour; multiple threads can hold a read lock (`pthread_rwlock_rdlock`) but only one thread can hold a write lock (`pthread_rwlock_wrlock`) and will wait until current readers are done.

```c
typedef struct {
  int nreader;
  lock_t guard;
  lock_t lock;
} rwlock_t;

void write_lock(rwlock_t *l) {
  lock(&l->lock);
}

void write_unlock(rwlock_t *l) {
  unlock(&l->lock);
}
```

```c
void read_lock(rwlock_t *l) {
  lock(&l->guard);
  ++nreader;
  if (nreader == 1) { // first reader
    lock(&l->lock);
  }
  unlock(&l->guard);
}
void read_unlock(rwlock_t *l) {
  lock(&l->guard);
  --nreader;
  if (nreader == 0) { // last reader
    unlock(&l->lock);
  }
  unlock(&l->guard);
}
```

Figure 54. rwlock impl

Subsection 3.14

Semaphores

Locks (mutexes) enforce *mutual exclusion*, but not necessarily ordering. But how can we ensure an ordering between two threads? For example, how can we make one thread always print first?

Definition 35 **Semaphores** have a value[52] that is shared between threads and provide two operations: `wait` (atomic decrement, blocking) and `post` (atomic increment). Initial `value` can be set to whatever.

[52] *Usually an integer* ≥ 0

```c
#include <semaphore.h>

int sem_init(sem_t *sem, int pshared, unsigned int value)
int sem_destroy(sem_t *sem);

int sem_wait(sem_t *sem);
int sem_trywait(sem_t *sem);

int sem_post(sem_t *sem);
```

All functions return 0 on success

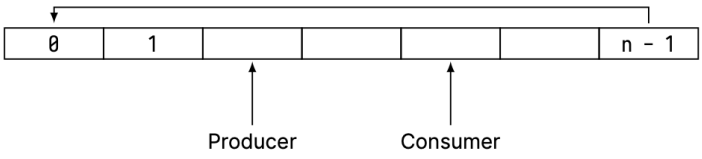
Figure 55. Semaphore methods. `pshared` can be set to 1 for IPC (needs to live in shared mem for IPC)

```c
static sem_t sem;

void* print_first(void* arg) {
  printf("This is first\n");
  sem_post(&sem);
}

void* print_second(void* arg) {
  sem_wait(&sem);
  printf("I'm going second\n");
}

int main(int argc, char *argv[])
{
  sem_init(&sem, 0, 0);
  /* Initialize, create, and join threads */
}
```

Figure 56. A snippet that uses semaphores as signals to `print_first` first and `print_second` second

Let's consider the *producer-consumer* problem:

A semaphore is really a generalized mutex; we can consider a mutex as a semaphore with a value of 1.



The producer should write to the buffer (if the buffer is not full)

The consumer should read from the buffer (if the buffer is not empty)

Figure 57. All consumers share index i_c, and all producers share index i_p

We can ensure producers never overwrite filled slots by using a semaphore to track the number of empty slots;

```
void producer() {
  while (/* ... */) {
    /* spend time producing data */
    sem_wait(&empty_slots);
    fill_slot();
  }
}
void consumer() {
  while (/* ... */) {
    empty_slot();
    sem_post(&empty_slots);
    /* spend time consuming data */
```

Figure 58. Consumer `post`-s to the empty slots semaphore when done and producer `wait`-s on the empty slots semaphore before writing

A similar semaphore can be used to track the number of filled slots such that consumers never read empty slots;

```
void init_semaphores() {
  sem_init(&empty_slots, 0, buffer_size);
  sem_init(&filled_slots, 0, 0);
}
void producer() { while (/* ... */) {
  /* spend time producing data */
  sem_wait(&empty_slots);
  fill_slot();
  sem_post(&filled_slots);
} }
void consumer() { while (/* ... */) {
  sem_wait(&filled_slots);
  empty_slot();
  sem_post(&empty_slots);
  /* spend time consuming data */
} }
```

Figure 59. Two semaphores ensure proper order

```
void init_semaphores() {
  sem_init(&empty_slots, 0, 0);
  sem_init(&filled_slots, 0, 0);
}
```

Figure 60. Note: Initializing both semaphores to 0 will cause the program to hang because none of the producers will be able to produce anything and then the program just gets stuck

Subsection 3.15

Locking

Languages offer support for locking and syntactic sugar. For example, java offers the `synchronized` keyword:

```
public class Account {
  int balance;
  public synchronized void deposit(int amount)  { balance += amount; }
  public synchronized void withdraw(int amount) { balance -= amount; }
}
```

the compiler transforms to:

```
  public void deposit(int amount) {
    lock(this.monitor);
    balance += amount;
    unlock(this.monitor);
  }
  public void withdraw(int amount) {
    lock(this.monitor);
    balance -= amount;
    unlock(this.monitor);
  }
```

Another abstraction on top of these synchronization primitives are *condition variables* which enable inter-thread signaling.

```
int pthread_cond_init(pthread_cond_t *cond,
                      const pthread_condattr_t *attr)
int pthread_cond_destroy(pthread_cond_t *cond);
int pthread_cond_signal(pthread_cond_t *cond);
int pthread_cond_broadcast(pthread_cond_t *cond);
int pthread_cond_wait(pthread_cond_t *cond, pthread_mutex_t *mutex);
int pthread_cond_timedwait(pthread_cond_t *cond, pthread_mutex_t *mutex,
                           const struct timespec *abstime);
```

These condition variables must be paired with a mutex[53]; any calls to wait must already hold it (but signal/broadcast may not). The mutex is used to protect the condition variable itself, i.e. to prevent undesirable state changes in the condition variable due to synchronization problems.

[53] *One mutex can protect multiple condition variables*

```c
pthread_mutex_t mutex;
int nfilled;
pthread_cond_t has_filled;
pthread_cond_t has_empty;

void producer() {
  // produce data
  pthread_mutex_lock(&mutex);
  while (nfilled == N) {
    pthread_cond_wait(&has_empty,
                      &mutex);
  }
  // fill a slot
  ++nfilled;
  pthread_cond_signal(&has_filled);
  pthread_mutex_unlock(&mutex);
}
```

```c
void consumer() {
  pthread_mutex_lock(&mutex);
  while (nfilled == 0) {
    pthread_cond_wait(&has_filled,
                      &mutex);
  }
  // empty a slot
  --nfilled;
  pthread_cond_signal(&has_empty);
  pthread_mutex_unlock(&mutex);
  // consume data
}
```

Figure 61. Condition variables offer a more elegant solution to the producer-consumer problem

```c
/* Thread 1 */
pthread_mutex_lock(&mutex);
while (!condition) {
  pthread_cond_wait(&cond, &mutex);
}
pthread_mutex_unlock(&mutex);

/* Thread 2 */
condition = true;
pthread_cond_signal(&cond);
```

Figure 62. Example: This piece of code is a problem because there is no mutex on the `condition=true` line which can cause the `while` to produce undesired behaviour. Consider the case where if thread 1 executes first and then it gets swapped away right at the first $!condition$. Then in thread 2 the condition is to be set to true and the signal is sent without anything happening – which causes the `pthread_cond_wait` to hang. This can be fixed by locking and unlocking around the `condition=true` and `signal` lines.

```c
/* Thread 1 */
pthread_mutex_lock(&mutex);
if (!condition) {
  pthread_cond_wait(&cond, &mutex);
}
// What could happen here?
pthread_mutex_unlock(&mutex);

/* Thread 2 */
pthread_mutex_lock(&mutex);
condition = true;
pthread_mutex_unlock(&mutex);
pthread_cond_signal(&cond);

/* Thread 3 */
pthread_mutex_lock(&mutex);
condition = false;
pthread_mutex_unlock(&mutex);
```

Figure 63. Can we change the `while` to an `if`?

What if we want to avoid polling by changing the `while` to an `if`? A problem may occur here.

1. T1 goes first w/ initial condition false, cond 0, mutex 0
2. T1 locks mutex, so no races to worry currently
3. Put ourselves into condition variable's wait queue, and then unlock mutex
4. Thread 2 runs: locks mutex, sets condition to true, unlocks, and signals
5. Thread 1 can now wake up, condition is true, and then it can continue working
6. But Thread 3 can also wake up, set condition to true, and then transfer context to T2 which signals to T1.
    - Now T1 wakes up and by the time it gets to the `unlock condition` is false, which is a problem (since we wanted `condition` to be true T1 unlocks (as per the if statement))

Semaphores can be thought of as a special case of condition variables: though one can be implemented with the other it can get messy. Complex conditions are generally implemented with condition variables to keep things clean.

Definition 36 **Locking Granularity** is the extent to which a lock is held. Too many locks or locks covering large swathes of the program can slow down your program, so it's important to design critical sections carefully.

Subsection 3.16
Deadlocks

Deadlocks are a problem. Conditions for deadlocks include:

- mutual exclusion
- hold and wait (have a lock and try to acquire another)
- No preemption (can't take simple locks away)
- circular wait (waiting for a lock held by another process)

Thread 1	**Thread 2**
`Get Lock 1`	`Get Lock 2`
`Get Lock 2`	`Get Lock 1`
`Release Lock 2`	`Release Lock 1`
`Release Lock 1`	`Release Lock 2`

Figure 64. This can deadlock depending on the order of the processes trying to get the locks!

```
void f1() {
    locktype_lock(&l1);
    locktype_lock(&l2);
    // protected code
    locktype_unlock(&l2);
    locktype_unlock(&l1);
}
```

Figure 65. Enforcing order is one way to prevent deadlocks

```
void f2() {
    locktype_lock(&l1);
    while (locktype_trylock(&l2) != 0) {
        locktype_unlock(&l1);
        // wait
        locktype_lock(&l1);
    }
    // protected code
    locktype_unlock(&l2);
    locktype_unlock(&l1);
}
```

Figure 66. Alternatively, `try_lock` can be used to self-pre-empt in order to avoid deadlocking

See the `banksim.c` example
// TODO: take exercpts from banksim