EngSci Year 3 Fall 2022 Notes

Brian Chen

Division of Engineering Science
University of Toronto
https://chenbrian.ca

brianchen.chen@mail.utoronto.ca

Contents

PART

I

ECE349: Introduction to Energy Systems

Taught by Prof. P. Lehn

Section 1

Admin stuff

Subsection 1.1

Lecture 1

First lecture was logistical info + a speil about how power systems are one of the great modern wonders. Course will cover sinusoidal AC power systems (1, 3 phase), power systems (dc-dc, dc-ac conversion), and magnetic systems (transformers, actuators, and synchronous machines)

1.1.1 Mark breakdown

- 50 % Final
- 25 % Midterm
- 5 % Quiz
- 15 % Labs
- 5 % Assignments

Section 2

AC Steady State Analysis

Subsection 2.1

Lecture 2

2.1.1 TODO

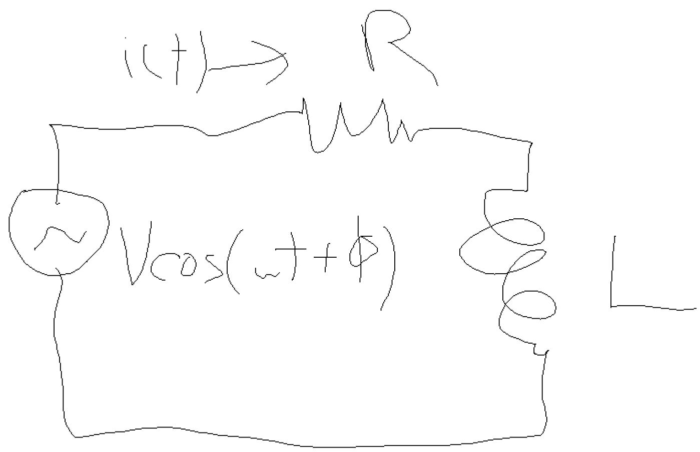
- Review Thomas 669-600

Recall, for a general phasor $\tilde{P}$

- $\frac{d\tilde{P}}{dt} = jw\tilde{P}$
- $\int \tilde{P} = \frac{1}{jw}\tilde{P}$

What we have learnt prior for differential equations enables us to arrive at analytical solutions to linear stable AC systems with phasors. A homogeneous and particular solution will be produced. If there's a stable homogeneous solution, $\to 0$ as $t \to \infty$. The full solution would be the addition of the two via super position.

We generally use this approach to solve circuits since it's an efficient way to solve circuits and make them into essentially DC circuits.



$$Ri + L\frac{di}{dt} = V\cos{(\omega t + \phi)} \tag{2.1}$$

But this is a pain to solve. It can be made simpler by applying phasors

$$V\cos{(\omega t + \phi)} = Re\{Ve^{j(\omega t + \phi)}\} \tag{2.2}$$

Take the real part of $\tilde{I}$:

$$R\tilde{I} + L\frac{d\tilde{I}}{dt} = Ve^{j(\omega t + \phi)} \tag{2.3}$$

And therefore by inspection the solution is of format $\tilde{I}e^{j\omega t}$, where $\tilde{I}$ is a phasor. Noting that $\tilde{I}$ contains only amplitude and phase,

$$\begin{gathered} R\tilde{I}e^{j\omega t} + L\frac{d}{dt}(\tilde{I}e^{j\omega t}) = Ve^{j\omega + \phi} \\ R\tilde{I} + L\tilde{I}j\omega = Ve^{j\phi} \end{gathered} \tag{2.4}$$

And now reconstructing:

$$\begin{aligned} \tilde{I} &= \frac{V}{\sqrt{R^2(\omega L)^2}}e^{j(\omega t + \phi - \tan^{-1}\frac{wL}{R})} \\ i(t) &= Re\left\{\tilde{I}\right\} \end{aligned} \tag{2.5}$$

And therefore

$$\tilde{I} = \frac{V}{\sqrt{R^2(\omega L)^2}}\cos{(\omega t + \phi - \tan^{-1}(\frac{wL}{R}))} \tag{2.6}$$

The steps to solving a phasor problem are:

Notation: $Xe^{j\phi} \leftrightarrow X\lfloor\phi$

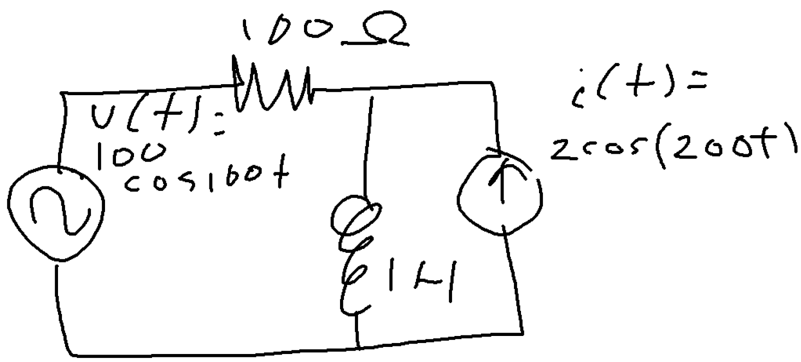
- Define phasor: $V\cos{(\omega t + \phi)} \leftrightarrow V\lfloor\phi$
- Map L, C into phasor domain; find impedences
  - $v = L\frac{di}{dt} \leftrightarrow \tilde{V} = j\omega L\tilde{I}$
  - $i = C\frac{dv}{dt} \leftrightarrow \tilde{V} = \frac{1}{j\omega C}\tilde{I}$

- Do mesh analysis to find $\tilde{I}$; $\tilde{I} = \frac{\tilde{V}}{\sum \text{impedances}}$
- Reconstruct $i(t)$ from $\tilde{I}$

Subsection 2.2

Lecture 3

Phasors allow us to solve circuits with multiple sources of differing frequencies.



To find the current $i(t)$ over the inductor we can find it's response due to the voltage and current sources and then apply superposition.

- $I_1 = \frac{100\lfloor 0}{1oo+j100} = 0.707\lfloor -45^o \to i_1(t) = 0.707cos(100t - 45^o)$
- $I_2 = \frac{100}{100+j200} 2\lfloor 0 = 0.894\lfloor -65^o \to i_2(t) = 0.894cos(200t - 63^o)$
- $i(t) = i_1(t) + i_2(t) = 0.707cos(100t - 45^o) + 0.894cos(200t - 63^o)$

Non-sinusoidal stimulus may be solved by decomposing the signal with Fourier transforms. For example, square waves:

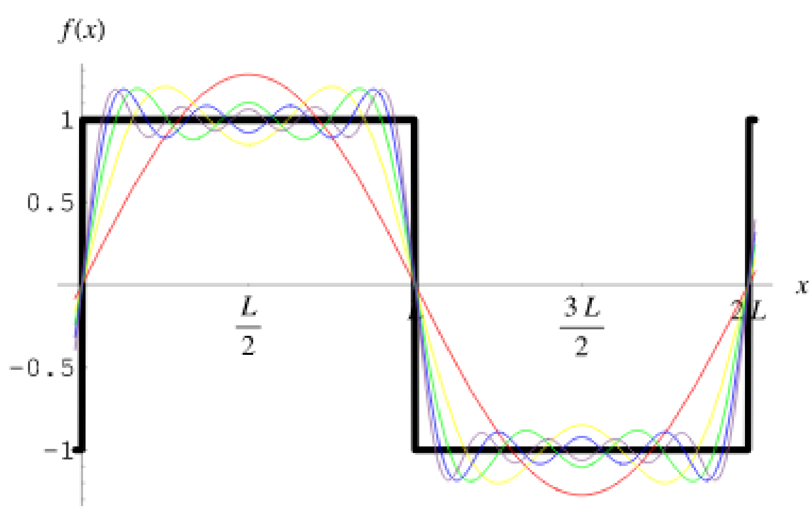


Figure 1. square waves with Fourier series superimposed

The general form of a Fourier transform is given as:

$$v_{equiv}(t) = a_o + \sum_{n=1}^{\infty} a_k cos(nw_o t) + b_k cos(nw_o t) \tag{2.7}$$

Where:

$$\begin{aligned} a_o &= \frac{1}{T}\int_0^T v(t)dt \\ a_k &= \frac{2}{T}\int_0^T v(t)cos(nw_o t)dt \\ b_k &= \frac{2}{T}\int_0^T v(t)sin(nw_o t)dt \end{aligned} \tag{2.8}$$

Armed with Fourier series and superposition we may now model a non-sinusoidal signal as a superposition of an infinite sum of sources. About half the work can be cut in half by recognizing that sin lags cos by 90^o, so

$$\begin{aligned} a_o &= \frac{1}{T}\int_0^T v(t)dt \\ a_k &= \frac{2}{T}\int_0^T v(t)cos(nw_o t)dt \\ b_k &= \frac{2}{T}\int_0^T v(t)cos(nw_o t - 90^o)dt \end{aligned} \tag{2.9}$$

Section 3

AC Power

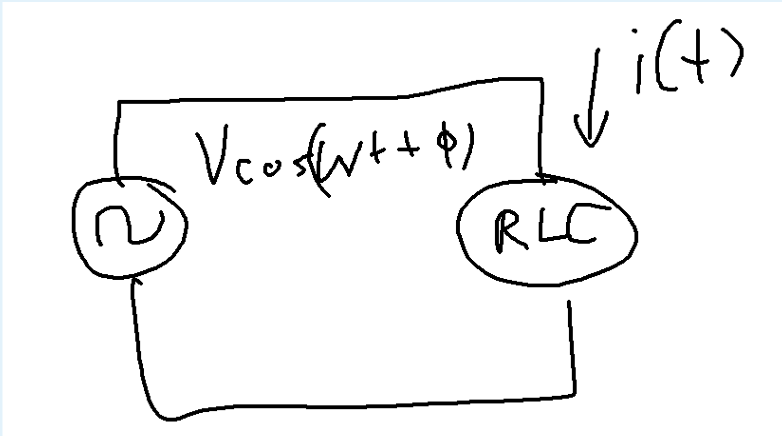
Definition 1 **Instantaneous Power**: $p(t) = v(t) \times i(t)[W, \frac{J}{s}]$

Example For a circuit with a voltage source, $v(t) = Vcos(\omega t)$ and a resistor Ω, $i(t) = Icos(\omega t)$, $p(t) = VIcos^2(\omega t) = \frac{VI}{2}(1 + cos(2\omega t))$

Definition 2 **Average Power over Cycle**: $P(t) = \frac{1}{T}\int_o^T p(t)dt = \frac{VI}{2}$

If we were to plot the instantaneous power we see that due to the sinusoidal response there are times where 0 power is supplied. This will always be true for a single phase power supply; real-world supplies always have multiple phases; this is why computer PSUs always contain a ton of capacitors.

Definition 3 **Reactive Power**: Q



If $\phi_i = 0$ and taking $\phi = \phi_v$,
Taking the average power of the reactive power we get

$$P_{avg} = \frac{VI}{2} cos\phi \tag{3.1}$$

$\phi = \phi_v - \phi_i$

Another quantity, reactive power, can be defined with regards to the energy sloshing back and forth:

$$Q = \frac{VI}{2} \sin\phi \tag{3.2}$$

PART

II

ECE352: Computer Organization

Taught by Prof. Andreas Moshovos

Section 4

Admin stuff

Subsection 4.1

Lecture 1

- Lecture recordings on YouTube
- Online notes: https://www.eecg.utoronto.ca/ moshovos/ECE352-2022/
- Course will cover the following:
  - C to assembly
  - How to build a processor that works
  - Intro to processor optimizations
  - Peripherals
  - OS support (Maybe)
  - (Maybe) Arithmetic circuits
  - Use NIOS II and cover a little bit of RISC-V

4.1.1 Mark breakdown

- Labs 15%
- project 5%
- midterm 30%
- Final 50%
- All exams will be open notes/book/whatever except another person/service helping you.

SECTION 5

Preliminary

SUBSECTION 5.1

Lecture 2: Using binary quantities to represent other things

Computers can represent information in bits; 0/1. Though they don't necessarily know or care what bits are, we may assign our own arbitrary meaning to them – usually numbers with the help of positioning; the LSB represents 2^0 and so forth.

C types

Or, just `#include <stdint.h>`...

- int: 32b (word)
- char: 8b (byte)
- short: 16b (half word)
- long: 32b (word)
- long long: 64b

Signed numbers may be represented in a number of ways.

- Sign bit (make MSB represent positive or negative numbers and then the remaining $n-1$ bits represent the number. Con: hardware impl sucks because requires if/else
- Two's complement [1]. Pro: only need to implement adders on hardware and then negative numbers will work just like any other except must be interpreted differently. Positive numbers would always start with a 0 and negatives would start with 1. So the range of possible values becomes $-(2^{n-1}-1), +2^{n-1}-1$

[1] Flip bits, add one. Intuition; in 3 bit system, adding 7 to 1 would result in 8 which would get truncated to 0.

Adding together binary numbers can also cause overflow; $(A+B) \geq A, (A+B) \geq B$ may not always be true. Also, when we work with these types we always use all the bits. This has implications when working with values of different lengths.

- `char b = -1` (1111 1111))
- `short int c = -1` (0000 0000 0000 0001))
- `a = b + c` 0000 0001 0000 0000
- In order to deal with this we must `cast` the `char` to a `short int`. This is done via sign extension which prepends 0s or 1s [2] to the `char` so that math can be done on it.

[2] two's complement

5.1.1 Floating Point Numbers

Whereas fixed point numbers i.e. $\$5.25$ can be represented just as how an integer would be represented but with the understanding that the user would interpret it as having a decimal point somewhere that indicates the position of 2^0. This decimal point would be the same for all numbers of that type, i.e. we could have a six bit number that has places $2^2 2^1 2^0 2^{-1} 2^{-2}$ This is common in embedded systems and how it is formatted isn't super clearly standardized.

Lemma 1 Reference: What Every Computer Scientist Should Know About Floats

Definition 4 **IEEE 754 Floating Point**
This is a single precision 32 bit float:

$$\texttt{S EEEEEEEE MMMM MMMM MMMM MMMM MMMM MMM} \tag{5.1}$$

The most significant `S` bit is the sign bit, bits 30 through 23 `E` form the exponent which is an unsigned integer, and 22 through 0 form the (`M`)antissa. The number being represented can be found using the following:

$$(-1)^S \times 2^{(E-127)} \times 1.\text{Mantissa} \tag{5.2}$$

Example For example, given the following float:

$$1 \quad 10000001 \quad 10000000000000000000000$$

So S = 1, E = 10000001 = 129 and Mantissa = 10000000000000000000000. The number is therefore

$$(-1^1) \times 2^{(129-127)} \times 1.10000000000000000000000 = -6.0 \tag{5.3}$$

IEEE754 also defines 64 bit floating-point numbers. They behave the same except for now having an 11 bit exponent, the bias being 2047[3], and the mantissa having 52 bits.
A few special cases are also available to represent other quantities

- If E=0, M non-zero, value=$(-1)^S \times 2^{-126} \times 0.M$ (denormals)
- If E=0, M zero and S=1, value=-0
- If E=0, M zero and S=0, value=0
- If E=1...1, M non-zero, value=NaN "not a number"
- If E=1...1, M zero and S=1, value=-infinity
- If E=1...1, M zero and S=0, value=infinity

Floating-point numbers are inherently imprecise. Addition and subtract are inherently lossy; the mantissa window may not be large enough to capture the decimal points. Multiplication and division just creates a ton of numbers.
Converting real numbers to IEEE754 floats, here using 37.64 as an example, can be done as follows

- Repeatedly divide the part of the number > 0 by 2 and get the remainders, i.e. 37/2 = 18, rem = 1 -> 18/2 = 0, rem = 0 -> 4/ 2 = 2, rem = 0, 2/2 = 1, rem = 0, 1/2 = 0, rem = 1; E = 100101
- Do the same for the part of the number past the decimal, but multiplying by two and checking if > 1: 0.64 * 2 = 1.28 -> 1, 0.28 * 2 = 0.56 -> 0, 0.56 * 2 = 1.12 -> 1 ... and so forth. At some point we will hit a cycle but we'll just take the N_{mantissa} of digits.

[3] *instead of 1023*

In C, `float` is a 32 bit float and `double` is 64

There are more floating point formats introduced by nvidia and google such as a half-precision or 8-bit float designed to reduce memory use for machine learning

Subsection 5.2

Lecture 3: Behavioural Model of Memory

PART

III

ECE355: Signal Analysis and Communication

Taught by Prof. Sunila Akbar

Section 6

Admin and Preliminary

Subsection 6.1

Lecture 1

- CT and DT signals
- A ton of LTI (Linear time invariant) systems
- Processing of signals via LTI systems
- Fourier transforms
- Sampling

6.1.1 Mark Breakdown

Table 1. Mark Breakdown

Homework	20
MT1	20
MT2	20
Final	40

- Continuous enclose in (), independent is t
- Discrete: enclose in [], independent is n

In many systems we are interested in power and energy of signals over an infinite time interval; $-\infty < \{t, n\} < \infty$

Theorem 1

Energy for Complex Signals

$$E_{[t_1,t_2]} = \int_{t_1}^{t_2} |x(t)|^2 dt \tag{6.1}$$

$$E_{[t_1,t_2]} = \sum_{n=n_1}^{n_2} |x(t)|^2 dt \tag{6.2}$$

Average Power for Complex Signals

$$P_{avg,[t_1,t_2]} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \tag{6.3}$$

$$P_{avg,[t_1,t_2]} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x(n)|^2 \tag{6.4}$$

Section 7

Transformations

Subsection 7.1

Lecture 2

Most of this lecture was review. When applying transforms just note to always scale, *then* shift, i.e.

1. $y(t) = x(\alpha t)$
2. $y(t) = x(\alpha t + \frac{\beta}{\alpha})$

Definition 5 **Fundamental Period**
The fundamental period, T_o is the smallest positive value of T for which (7.1) holds true

$$x_t = x(t + mT), m \in \mathbb{Z} \tag{7.1}$$

Definition 6 **Even signals**

$$x(t) = x(-t) \tag{7.2}$$

Definition 7 **Odd signals**

$$x(t) = -x(t) \tag{7.3}$$

Theorem 2 Any signal can be broken into an even and odd component

$$\begin{aligned} x(t) &= Ev\{x(t)\} = \frac{1}{2}[x(t) + x(-t)] \\ x(t) &= Od\{x(t)\} = \frac{1}{2}[x(t) + -x(-t)] \end{aligned} \tag{7.4}$$

PART

IV

ECE360: Electronics

Taught by Prof. Khoman Phang

Section 8

Admin and Preliminary

Subsection 8.1

Lecture 1

8.1.1 Mark Breakdown

Table 2. Mark Breakdown

Test 1	15
Test 2	20
Homework	10
Labs	12
Final	43

8.1.2 Diodes

Diodes are an electronic valve which causes current to only flow in one direction. An ideal diode is an open circuit in the closed direction and a closed circuit in the other, so the current is always in the direction of the arrow (+'ve @ arrow base, -'ve at arrow point)[4].

[4] *recall: passive sign convention*

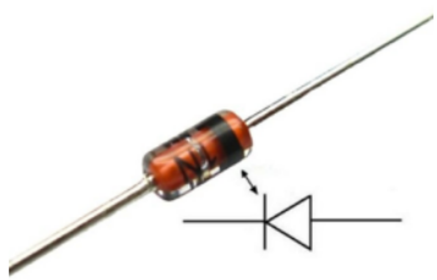
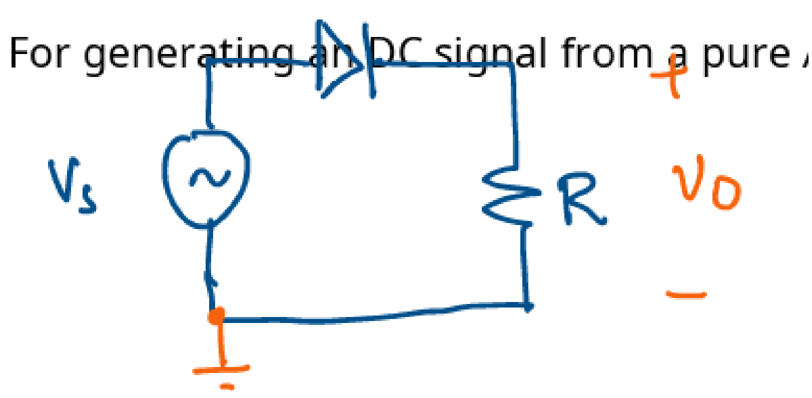
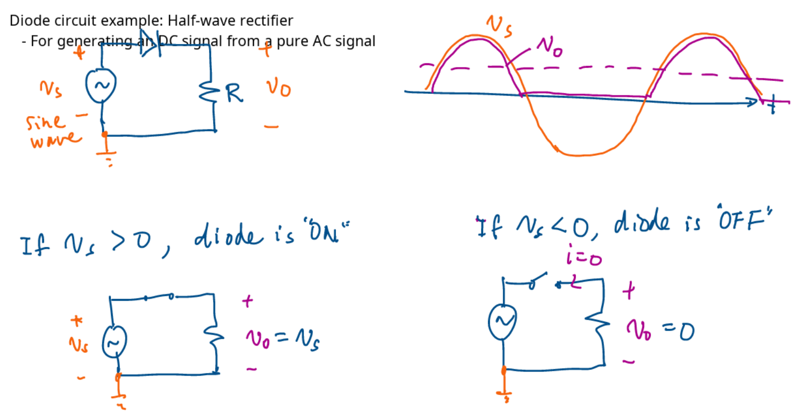


Figure 2. A diode and its symbol

An example of a diode circuit is the half-wave rectifier which turns an AC signal to a DC signal

Can take oscilloscope over resistor to see that a pure DC signal has been generated





Section 9
Diodes

Subsection 9.1
Lecture 2

More formally, off/on for diodes should be referred to as:

- Off ↔ reverse bias
- On ↔ forwards bias

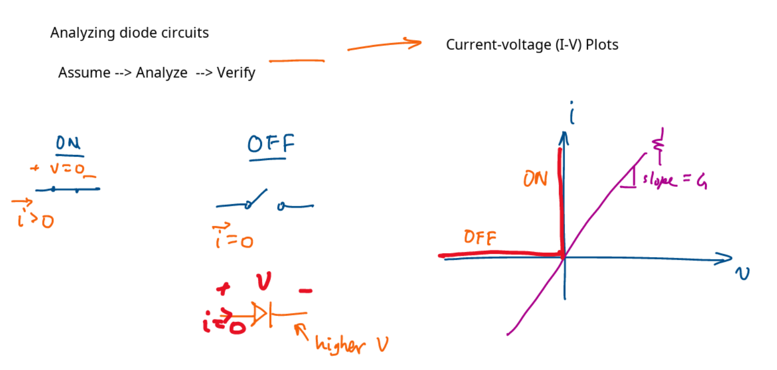


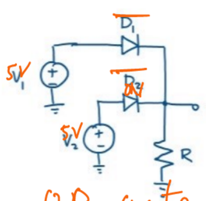
Figure 3. General steps for analyzing non-linear circuits. Note plotting out expected response

An example of how this is used in circuit design is to manage two power sources. Consider an Arduino that could be powered by an AC adapter or by a computer's USB port. This circuit would choose the higher voltage source and prevent back-flow into the other power source due to any potential power differentials. It is also effectively an OR gate

Analysis Examples

'0' '1'

Example 1:
Find output voltage Vo assuming input voltages V_1 and V_2 are either 0V or 5V.
What is the function of this circuit?

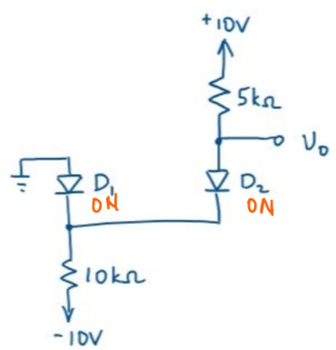


OR gate

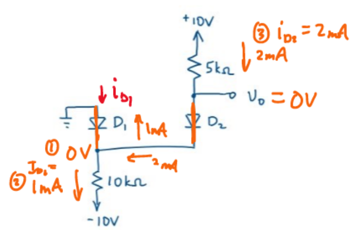
V_1	D_1	V_2	D_2	V_o
5V	ON	5V	ON	5V
5	ON	0	OFF	5V
0	OFF	5	ON	5V
0	OFF	0	OFF	0V

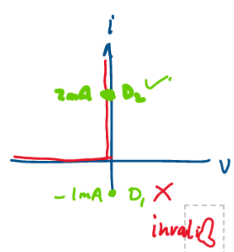
Example 2:
Find output voltage Vo

Assume --> Analyze --> Verify

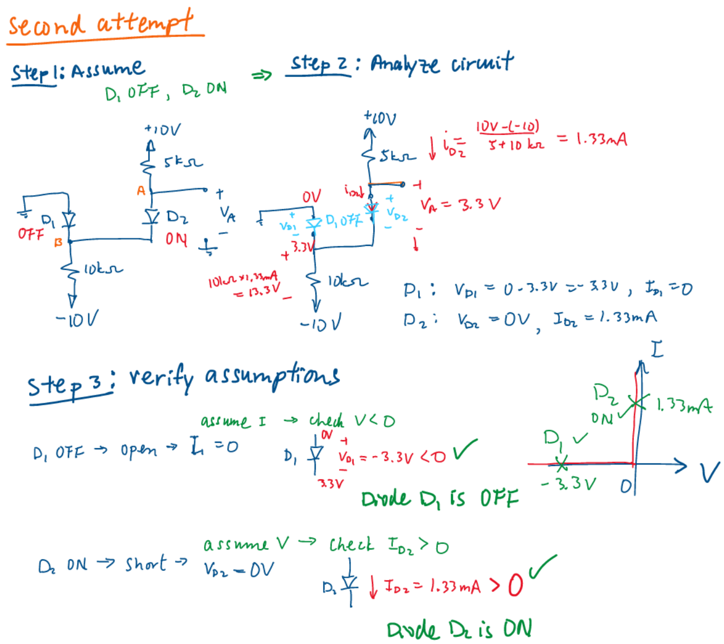


Assume D_1 ON
D_2 ON





In this example the initial assumption was incorrect.
Let's try another analysis with D_1 off and D_2 on:



If we were to do this brute force we'd have to consider 4 cases, so it's important to build up some sort of intuition for the circuit.

SUBSECTION 9.2
Lecture 3

Today we're going to look at the characteristics of real diodes.

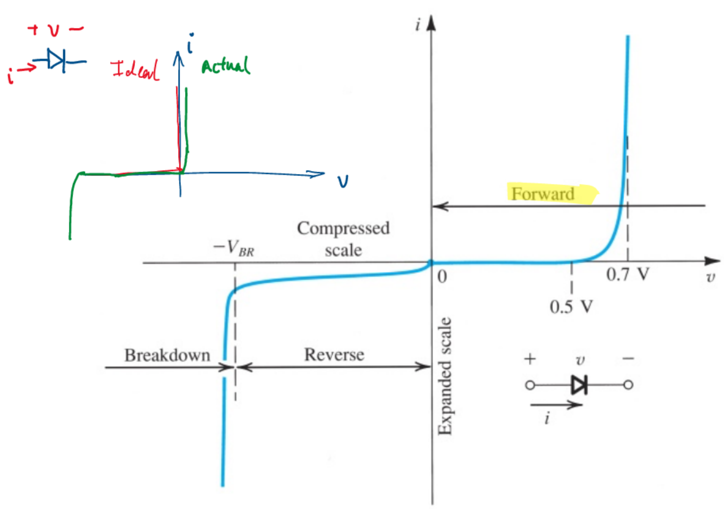


Figure 4.8 The silicon diode i–v relationship with some scales expanded and others compressed in order to reveal details.

Real diodes have a little bit of leakage current and also encounter a breakdown point where they're no longer able to block the current.

Theorem 3 **Forward Bias**

$$i = I_s(e^{\frac{v}{v_T}-1}) \tag{9.1}$$

Where:

$$v_T = \frac{kT}{q} \quad [V] \tag{9.2}$$

Most of the time we can assume that the circuit is at room temperature and that $v_T = 25mV$. Note that this value explodes when $v > v_T$ which is the breakdown point. When encountering a reverse bias $v_s < 0$, the -1 term comes in and causes $i \approx I_s$
The scale current is just a general constant which varies in range from $10^{-9} to 10^{-15} A$ and scales with temperature, doubling with every approximately 5^oC increase in temperature.
Note: the ideal diode equation can be rearranged to find an expression for voltages

k is Boltzmann's constant, T is temperature in Kelvins, q is the charge of an electron.
I_s is the scale current which is usually $\approx 1pA$, which doesn't change much until the breakdown point.

$$v = v_T \ln\left(\frac{i}{I_s}\right) = \ln(10) v_T \log_{10}\left(\frac{i}{I_s}\right) \tag{9.3}$$

These expressions turns out to be quite reliable for reasonable diodes to reasonable voltages.

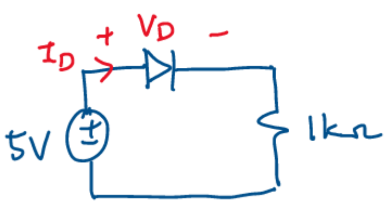
Using the ideal diode equation we can find the relationship between voltages and currents as they pass through the diode.

$$\frac{i_2}{i_1} = \frac{I_s e^{\frac{v_2}{v_T}}}{I_s e^{\frac{v_1}{v_T}}} = e^{\frac{v_2 - v_1}{v_T}} \tag{9.4}$$

$$v_2 - v_1 = v_T \ln\left(\frac{i_2}{i_1}\right) \xrightarrow{\text{room temperature}} 60mV \log_{10}\frac{i_2}{i_1} \tag{9.5}$$

Example:

Calculate the diode voltage and current in the circuit below.
Assume that the diode voltage is 0.7V at 1mA and V_T = 25mV.



Example

Recall (9.1). Plugging in the given values gives us the scale current.

$$1mA = I_s e^{\frac{0.7V}{25mV}}, I_s = 6.9 \cdot 10^{-16} A \Rightarrow I_o = I_s e^{v_o/v_T} \tag{9.6}$$

Ohm's law can then be applied at the resistor

$$V_r = IR = I_oR = 5V - V_D \Rightarrow 5 - V_D = I_oR \tag{9.7}$$

V_D is the voltage across the diode

So we have two equations and two unknowns (since we know $v_T = 25mV$ but v_o was used at first just to find I_s) Solving for the unknowns gives us:

- $V_o = 0.736V$

- $I_D = 4.264mA$

ECE358: Foundations of Computing

PART

V

Taught by Prof. Shurui Zhou

Section 10

Admin and Preliminary

Subsection 10.1

Lecture 1

Topics covered will include:

- Graphs, trees
- Bunch of sorts
- Fancy search trees; red-black, splay, etc
- DP, Greedy
- Min span tree, single source shortest paths
- Maximum flow
- NP Completeness, theory of computation
- Blockchains??
- Θ

Solutions will be posted on the window of SF2001. Walk there and take a picture.

10.1.1 Mark Breakdown

Table 3. Mark Breakdown

Homework x 5	25
Midterm (Open book)	35
Final (Open book)	40

Section 11

Complexities

Subsection 11.1

Lecture 2

This lecture we talked about big O notation. For notes on this refer to my tutorial notes for ESC180, ESC190: https://github.com/ihasdapie/teaching/

Definition 8 Big O notation (upper bound)
$g(n)$ is an asymptotic upper bound for $f(n)$ if:

$$O(g(n)) = \{f(n) : \exists \quad c, n_0 \quad s.t. \quad 0 \le f(n) \le c \cdot g(n), \forall n \ge n_o\} \tag{11.1}$$

PROOF **What is the big-O of $n!$?**

$$n! \le n \cdot n \cdot n \cdot n \ldots n = n^n \Rightarrow n! \in O(n^n) \tag{11.2}$$

□

Definition 9 Big Ω notation (lower bound)
$h(n)$ is an asymptotic lower bound for $f(n)$ if:

$$\Omega(h(n)) = \{f(n) : \exists \quad c, n_0 > 0 \quad s.t. \quad 0 \le c \cdot h(n) \le f(n), \forall n \ge n_o\} \tag{11.3}$$

PROOF **Find Θ for $f(n) \sum_i^n i$.**
For this we will employ a technique for the proof where we take the right half of the function, i.e. from $\frac{n}{2} \ldots n$ and then find the bound

$$\begin{aligned} f(n) &= 1 + 2 + 3 \ldots + n \\ &\ge \lceil \frac{n}{2} \rceil + (\left\lceil \frac{n}{2} \right\rceil + 1) + (\left\lceil \frac{n}{2} \right\rceil + 2) + \ldots n \quad n/2 \text{ times} \\ &\ge \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \ldots \left\lceil \frac{n}{2} \right\rceil \\ &\ge \frac{n}{2} \cdot \frac{n}{2} \\ &= \frac{n^2}{4} \end{aligned} \tag{11.4}$$

And therefore for $c = \frac{1}{4}$ and $n = 1$, $f(n) \in \Theta(n^2)$

□

Definition 10 Big Θ notation (asymptotically tight bound)

$$\Theta(g(n)) = \{f(n) : \exists \quad c_1 c_2, n_0 \quad s.t. \quad 0 \le c_1 g(n) \le f(n) \le c_2 g(n), \forall n \ge n_o\} \tag{11.5}$$

PROOF Prove that

$$\sum_{j=1}^{n} i^k = \Theta(n^{k+1}) \tag{11.6}$$

First, prove $O(f(n)) = O(n^{k+1})$

$$\begin{aligned} f(n) = \sum_{j=1}^{n} i^k &= 1^k + 2^k + \ldots n^k \\ &\le n^k + n^k + \ldots n^k \\ &= n^{k+1} \end{aligned} \tag{11.7}$$

Next, prove $\Omega(f(n)) = \Omega(n^{k+1})$

$$f(n) = \sum_{j=1}^{n} i^k = 1^k + 2^k + \ldots n^k$$
$$= n^k + (n_1)^k + \ldots 2^k + 1^k = \sum_{i=1}^{n} (n - i + 1)^k \quad (11.8)$$
$$\geq \frac{n}{2}^k * n \geq \frac{n^{k+1}}{2^k} = \Omega(n^{k+1})$$

Therefore $f(n) = \Theta(n^{k+1})$ □

Note that we may not always find both a tight upper and lower bound so not all functions have a tight asymptotic bound.

Theorem 4 **Properties of asymptotes:**
Note: $\wedge$ means AND
Transitivity [5]

$$(f(n) = \Theta(g(n)) \wedge g(n) = \Theta(h(n))) \Rightarrow f(n) = \Theta(h(n)) \quad (11.9)$$

Reflexivity[6]

$$f(n) = \Theta(f(n)) \quad (11.10)$$

Symmetry

$$f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n)) \quad (11.11)$$

Transpose Symmetry

$$\begin{aligned} f(n) = O(g(n)) &\iff g(n) = \Omega(f(n)) \\ f(n) = o(g(n)) &\iff g(n) = \omega(f(n)) \end{aligned} \quad (11.12)$$

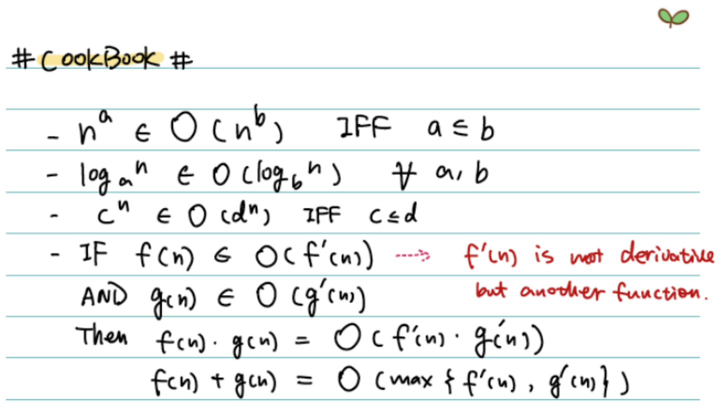
[5] *The following applies to* O, Θ, o, ω

[6] *The following applies to* O, Θ

Runtime complexity bounds can sometimes be used to compare functions. For example, $f(n) = O(g(n))$ is like $a \leq b$

- $O \approx\leq$
- $\Omega \approx\geq$
- $\Theta \approx\approx$
- $o \approx<$; an upper bound that is **not** asymptotically tight
- $\omega >$ a lower bound that is **not** asymptotically tight

Note that there is no trichotomy; unlike real numbers where we can just do $a < b$, etc, we may not always be able to compare functions.



\# CookBook #

- $n^a \in O(n^b)$ IFF $a \leq b$
- $\log_a n \in O(\log_b n)$ $\forall a, b$
- $c^n \in O(d^n)$ IFF $c \leq d$
- IF $f(n) \in O(f'(n))$ → $f'(n)$ is not derivative but another function.

  AND $g(n) \in O(g'(n))$

  Then $f(n) \cdot g(n) = O(f'(n) \cdot g'(n))$

  $f(n) + g(n) = O(\max\{f'(n), g'(n)\})$

Figure 4. Complexity Cookbook

SUBSECTION 11.2

Lecture 3: Logs & Sums

11.2.1 Functional Iteration

Recall:

$$a = b^c \Leftrightarrow log_b a = c \quad (11.13)$$

$f^{(i)}(n)$ denotes a function iteratively applied i times to value n.

For example, a function may be defined as:

$$f^{(i)}(n) = \begin{cases} f(n) & \text{if } i = 0 \\ f(f^{(i-1)}(n)) & \text{if } i > 0 \end{cases} \tag{11.14}$$

Given (11.14) we see that

1. $f(n) = 2n$
2. $f^{(2)}(n) = f(2n) = 2^2 n$
3. $f^{(3}(n) = f(f^{(2)}(n)) = 2^3 n \vdots$
4. $f^{(i)}(n) = 2^i n$

As an exercise we may look at an iterated logarithm function, 'log star'

$$lg^*(n) = \min\{i \geq 0 : lg^{(i)} n \leq 1\} \tag{11.15}$$

This describes the number of times we can iterate $log(n)$ until it gets to 1 or smaller.

- $log^* 2 = 1$
- $log^* 4 = 2 = log^* 2^2 = 1 + log^* 2 = 2 \vdots$
- for practical reasons log^* doesn't really get bigger than 5. This is one of the slowest growing functions around.

Summations & Series

Proof

$$\sum_{k=0}^{n} x^k = S$$
$$\begin{aligned} S &= 1 + x + x^2 \dots x^n \\ xS &= x + x^2 + x^3 \dots x^{n+1} \\ S &= \frac{1 - x^{n+1}}{1 - x} \end{aligned} \tag{11.16}$$

□

$$\sum_{i=1}^{\infty} x^i = \frac{1}{1-x} \quad \text{if } |x| < 1 \tag{11.17}$$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad \text{if } |x| < 1 \tag{11.18}$$

Proof Begin by differentiating both sides over x

$$\sum_{k=0}^{\infty} x^k = \frac{1}{(1-x)} \quad \text{if } |x| < 1 \tag{11.19}$$

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \tag{11.20}$$

And then multiply both sides by x, therefore (11.18) follows. □

Telescoping Series

$$\sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0 \tag{11.21}$$

Proof Write it out and cancel out terms

$$(a_1 - a_0) + (a_2 - a_1) \dots (a_n - a_{n-1}) = a_n - a_0 \tag{11.22}$$

Therefore the sum telescopes □

Another telescoping series may be proved similarly:

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} \xrightarrow{math} \sum_{k=1}^{n-1} (\frac{1}{k} - \frac{1}{k+1}) = (1 - \frac{1}{2}) + \dots (\frac{1}{n-1} - \frac{1}{n}) = a_o - a_n \tag{11.23}$$

PART VI

MAT389: Complex Analysis

Taught by Prof. Sigil

Section 12

Complex Numbers

Subsection 12.1

Lecture 1

Consider a 2-vector $\vec{x} = (x, y) \in \mathcal{R}$. As complex numbers correspond to two-vectors

$$\vec{x} = (x, y) \leftrightarrow z = x + iy, i^2 = -1 \tag{12.1}$$

z is, therefore, a complex variable. What are the benefits of a complex number like z?

This prof lectures at the speed of sound and talks *into* the board. Couldn't quite follow during this lecture, hopefully I get better about it in the following ones.

Definition 11 **Imaginary and Complex Numbers**
i is an imaginary number such that

$$i^2 = -1 \tag{12.2}$$

A complex number has the form:

$$z = x + iy \tag{12.3}$$

Definition 12 There are a number of operations we can perform on complex numbers.
Addition

$$z + z' = (x + x') + i(y + y') \tag{12.4}$$

Multiplication

$$zz' = (x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y) \tag{12.5}$$

PROOF Proof of (12.5):

$$\begin{aligned} zz' &= (x + iy)(x' + iy') \\ &= x + ixy' + iyx' + i^2yy' \\ &= xx' - yy' + i(xy' + yx') \end{aligned} \tag{12.6}$$

□

Magnitude

$$|z| = \sqrt{x^2 + y^2} \tag{12.7}$$

Conjugate
The complex conjugate has the properties:

- $\overline{z}z = |z|^2$
- $\overline{(z + z')} = \overline{z} + \overline{z}'$
- $\overline{z \cdot z'} = \overline{z} \cdot \overline{z}'$

We can define a new operation

$$\forall \text{complex} z, \exists \quad \text{complementary number } w \text{ such that } zw = wz = 1 \tag{12.8}$$

Denote

$$w = \frac{1}{z} = z^{-1} \tag{12.9}$$

Proof

Proof of (12.9): Find w s.t. $zw = 1$

$$\begin{aligned} zw &= 1 \\ w\overline{z}z &= \overline{z}z = |z|^2 > 0 \\ |z|^2 w &= \overline{z} \\ w &= \frac{\overline{z}}{|z|^2} \rightarrow Z^{-1} = \frac{\overline{z}}{|z|^2} \end{aligned} \tag{12.10}$$

□

Furthermore, there are operators that we can define on complex numbers.

Definition 13

Real and Imaginary Operators

Given $z = x + iy$, we can define the real and imaginary operators

$$x = Re\{z\} \tag{12.11}$$

$$y = Im\{z\} \tag{12.12}$$

Example

$$Im\left\{(1 = \sqrt{2}i)^{-1}1\right\} \tag{12.13}$$

By (12.9), we have

$$Im\left\{z^{-1}\right\} = \frac{-Im\{z\}}{|z|^2} \tag{12.14}$$

And

$$Re\left\{z^{-1}\right\} = \frac{-Re\{z\}}{|z|^2} \tag{12.15}$$

Using these, for example, we find that the $Im = \frac{-\sqrt{2}}{3}$
We can get the real component in a similar way.

Here is an enumeration of absolute value properties for complex numbers:

$$|z \cdot w| = |z||w| \tag{12.16}$$

$$|z + w| \leq |z| + |w| \tag{12.17}$$

$$|\overline{z}| = |z| \tag{12.18}$$

$$|z + w|^2 = (\overline{x} + \overline{w})(z + w) = |z|^2 + |w|^2 + \overline{z}w + \overline{w}z \tag{12.19}$$

Proof

Note that $\overline{z}w + \overline{w}z = 2Re\{z\overline{w}\}$, by (12.19)
And so

$$|z + w|^2 \leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \tag{12.20}$$

□

Subsection 12.2

Lecture 2

Whereas a two-vector $\vec{x} \in \mathbb{Z}$, complex numbers exist in the complex plane, $z \in \mathbb{C}$

Theorem 5 **Polar Decomposition**
Complex numbers can be expressed in polar form as well

$$z = r(cos\theta + isin\theta) \tag{12.21}$$

Where

$$r = |z| \qquad x = rcos\theta \qquad y = rsin\theta \tag{12.22}$$

This has a number of useful properties

$$z \cdot z' = |z||z'|(\cos(\theta + \theta') + i\sin(\theta + \theta')) \tag{12.23}$$

$$\frac{z}{z'} = \frac{|z|}{|z'|}(\cos(\theta - \theta') + i\sin(\theta - \theta')) \tag{12.24}$$

Proof Proof for (12.23):

$$\begin{aligned} z \cdot z' &= |z|(\cos(\theta + i\sin\theta)) \times |z'|(\cos\theta' + i\sin\theta') \\ &= |z||z'|(\cos\theta\cos\theta' + i\cos\theta\sin\theta' + i\sin\theta\cos\theta' - \sin\theta\sin\theta') \\ &= |z||z'|\left[\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\cos\theta\sin\theta' + \sin\theta\cos\theta')\right] \end{aligned} \tag{12.25}$$

And the proof follows □

Lemma 2 A corollary exists

$$z^2 = |z|^2(\cos 2\theta + i\sin 2\theta) \tag{12.26}$$

Theorem 6 **Moivre's Theorem**

$$z^n = |z|^n(\cos(n\theta) + i\sin(n\theta)) \tag{12.27}$$

So we may define z to be the n^{th} root of w which implies that

Lemma 3 Every complex number has a n^{th} root $\forall n$

Proof

$$\text{Let } z = |w|^{\frac{1}{n}}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}) \tag{12.28}$$

Then

$$w = |w|(\cos\theta + i\sin\theta), \text{ then } z^n = w \tag{12.29}$$

□

Intuition: define z to be $\frac{1}{n}$ and then take the n^{th} power of both sides to show that $z^n = w$

This leads us to the conclusion that representations of complex numbers are not unique[7].

[7] *They are part of a cyclic group*

Proof If every z can be written as $z = r(\cos\theta + i\sin\theta)$, then it holds for $\theta + 2\pi n \forall n \in \mathbb{Z}$ since $\sin\theta = \sin(\theta + 2\pi n)$ and $\cos\theta = \cos(\theta + 2\pi n)$. □

12.2.1 Functions on complex planes

Definition 14 Given a domain $\mathbb{D} \in \mathbb{C}$, a function f is a rule such that

$$z \in \mathbb{D} \xrightarrow{f} w \in \mathbb{D} \leftrightarrow w = f(z) \tag{12.30}$$

Definition 15 We may define $\mathbb{D}$ to be the domain of f
Likewise, range is defined as

$$Ran\{f\} = \{w \in \mathbb{C} : \exists z \in D : f(z) = w\} \tag{12.31}$$

Example

$$f(z) = \frac{1}{z+i} \tag{12.32}$$

What is the maximum domain of f?

$$Dom\{f\} = \{z \in \mathbb{C} : |z| < -i\} \tag{12.33}$$

What is the range of f?

$$\frac{1}{z+i} = w \tag{12.34}$$

For which values of w can we solve this equation?

$$z = -i + \frac{1}{w} \tag{12.35}$$

So the range of the function is

$$Ran\{f\} = \{w \in \mathbb{C} : |w| \neq 0\} \tag{12.36}$$

Example

$$f(z) = z^2 + 1 \tag{12.37}$$

It is fairly clear that $Dom\{f\} = f \in \mathbb{C}$
The range can be found by solving for z in

$$z^2 + 1 = w \tag{12.38}$$

And so

$$Ran\{f\} = \{w \in \mathbb{C}\} \tag{12.39}$$

12.2.2 Exponential Functions

Definition 16 Given $z = x + iy$,

$$e^z = e^x(\cos y + i\sin y) = e^{Re\{z\}}(cos(Im\{z\}) + i\sin(Im\{z\})) \tag{12.40}$$

1. $e^{z+w} = e^z e^w$
2. $|e^z| = e^{Re\{z\}} \neq 0$
3. $e^{z+i2\pi n} = e^z$

Proof (1) follows from the product rule for complex numbers

(2) follows by definition

(3) follows by definition (recall: writing z w.r.t. sin, cos) □

More properties:

arg, or argument is the angle from the real axis to that on the complex plane. Usually denoted by θ

- $Dom\{e^z\} = \mathbb{C}$
- $Ran\{e^z\} = \{\mathbb{C} \setminus \{0\}\}$
- $e^z = w \qquad$ if $w \neq 0$

[8]

[8] Note: ' \ ' denotes set exclusion

$$\begin{aligned} z &= ln|w| + i\arg w \\ e^z &= e^{ln|w| + i\arg w} \\ &= e^{ln|w|}e^{i\arg w} \\ &= |w|\cos(\arg w) + i\sin(\arg w) \\ &= w \end{aligned} \tag{12.41}$$

Remark **Polar representation**

$$w = |w|e^{i\arg w} \tag{12.42}$$

$$r = |w| \quad \theta = \arg w \tag{12.43}$$

Example **Find polar coordinates of $z = i + 1$**

$$\begin{aligned} |z| &= \sqrt{1+i} = \sqrt{2} \\ \cos\theta &= \frac{1}{\sqrt{2}} \to \theta = \frac{\pi}{4} \\ z &= \sqrt{2}e^{i\pi/4} \end{aligned} \tag{12.44}$$

Example Find

$$(1+i)^{\frac{1}{3}} \tag{12.45}$$

Solution: $z = \sqrt{2}e^{\frac{i\pi}{4}} \to z^{1/3} = 2^{\frac{1}{6}}e^{i\pi/12}$

Definition 17 **Trig functions for complex numbers**

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \tag{12.46}$$

Proof

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) = \frac{1}{2}\left(\cos x + i\sin x + \underbrace{\cos(-x)}_{\text{odd; }=\cos(x)} + \underbrace{i\sin(-x)}_{\text{even; }=-\sin(x)}\right) = \cos x \tag{12.47}$$

□

$$\sin z = \frac{1}{2}\left(e^{iz} - e^{-iz}\right) \tag{12.48}$$

And a similar proof follows for $\sin z$.
These have the following properties

$$\sin z|_{ImZ=0} = \sin x \tag{12.49}$$

$$\cos(z + 2\pi n) = \cos z \forall n \in \mathbb{Z} \tag{12.50}$$

$$\sin(z + 2\pi n) = \sin z \forall n \in \mathbb{Z} \tag{12.51}$$

Proof

$$\begin{aligned} \cos z + 2\pi n &= \frac{1}{2}(e^{i(z+2\pi n)} + e^{-i(z+2\pi n)}) \\ &= \frac{1}{2}(e^{iz}e^{i2\pi n} + e^{-iz}e^{-i2\pi n}) \\ &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ &= \cos z \end{aligned} \tag{12.52}$$

□

The domain of $\{\cos z, \sin z\} = \mathbb{C}$
Range?
Solve $\cos z = w$ for z

$$\begin{aligned} \frac{1}{2}(e^{iz} + e^{-iz}) &= w \\ \ldots \times 2e^{iz} &\text{ on both sides} \\ e^{2iz} - 2we^{iz} + 1 &= 0 \\ &\ldots\text{Let } S = e^{iz} \\ S^2 - 2ws + 1 &= 0 \\ &S = w \pm \sqrt{w^2 - 1} \equiv u \end{aligned} \tag{12.53}$$

Now we note that $e^{iz} = u$ can be solved for z for any $u \neq 0$

$$u = 0 \leftrightarrow w = \pm\sqrt{w^2 - 1} \tag{12.54}$$

$$w^2 = w^2 - 1 \text{ impossible for } u \neq 0 \tag{12.55}$$

Therefore:

$$Ran\{\cos z\} = Ran\{\sin z\} = \mathbb{C} \tag{12.56}$$

Remark An intuitive way of interpreting this result is thinking of $\{\sin, \cos\}$ being a function that projects values from the complex domain to the real plane; though $\{\sin, \cos\}$ takes on a limited range of values in the real domain, in the complex domain it spans the entire plane. Think: mental model of a complex number spinning around and having that project onto a real line. More formally, see: the Little Picard Theorem

PART

VII

ECE444: Software Engineering

Taught by Prof. Shurui Zhou

Section 13

Preliminary

Subsection 13.1

Lecture 1, 2

- Software engineering is different from what coding is; design, architecture, documentation, testing, etc v.s. just script kiddie-ing
- Vasa syndrome
- Rockstar engineers are a myth

Section 14

Project Management

Subsection 14.1

Lecture 3

Definition 18 Conway's law states that 'Any organization that designs a system (defined broadly) will produce a design whose structure is a copy of the organization's communication structure'.

The waterfall method is slow and costly and defects can be extremely costly, especially early on in the development lifecycle.

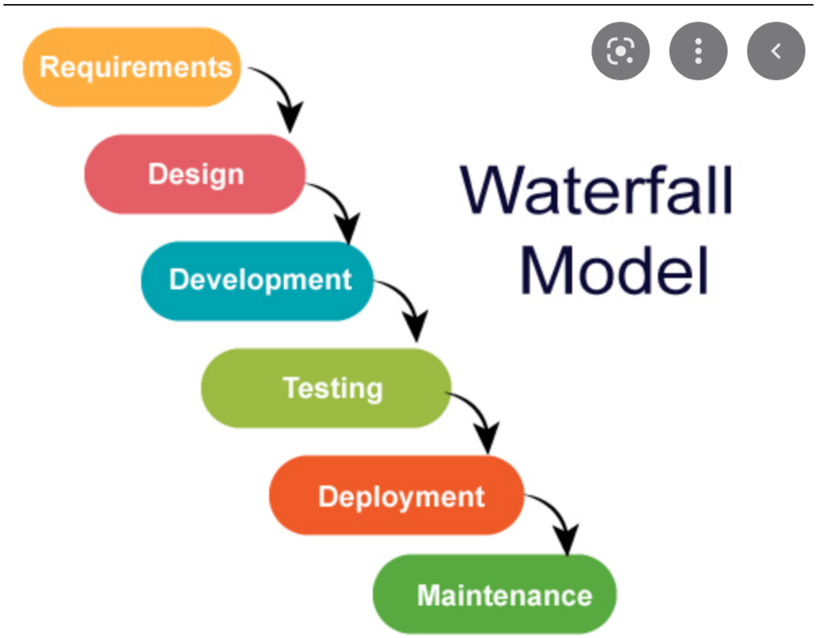


Figure 5. Waterfall method

In order to address this the V model was introduced which increases the amount of testing to reduce the possibility of having to rework everything

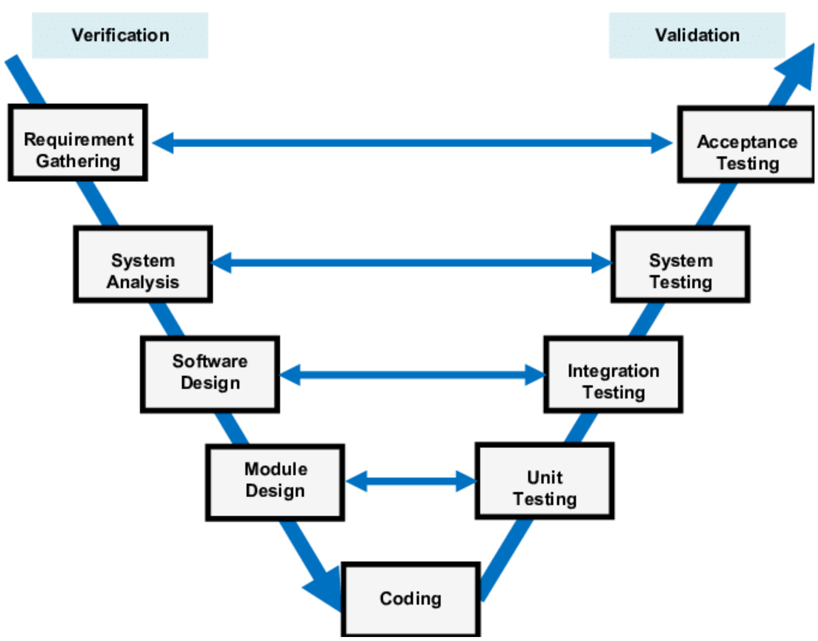


Figure 6. V model

Generally speaking the waterfall model isn't used much anymore due to the reality that software specifications change on a near daily basis.

Recall: aUToronto Spring 2022 integration hell

14.1.1 Agile

Agile is a project management approach which, in most general terms, seeks to respond to change and unpredictability using incremental, iterative work (sprints). This allows for a balance between the need for predictability and the need for flexibility. Some agile methods include:

- Extreme programming: really really fast iteration (think days)
- Scrum: 2-4 week sprints with standups and backlogs; sticky notes for tasks, etc. Think kanban boards. Daily scrum meetings to unblock ASAP. Development lifecycle is therefore a series of sprints.
- On-site customer; frequent interaction with end users to figure out what exactly they need.

PART VIII

ESC301: Seminar

Section 15

Preliminary

Subsection 15.1

Seminar 1