

# ENGSCI YEAR 2 WINTER 2022 NOTES

FOR CLASS CONTENT AFTER READING WEEK

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# PHY294

## Quantum & Thermal Physics

## SECTION 1

### Statistical Mechanics

## SUBSECTION 1.1

#### Lecture 1: Ideal Gas Law

We begin our discussion on statistical mechanics with the guiding question, "What is temperature". We will first derive the ideal gas law  $PV = nRT$ , which most should be familiar with.

Imagining a single particle bouncing around elastically in 2D box with speed  $v_x$ . If we were to look at the sides of the box we will see intermittent spikes of force experienced [by the box walls],

$$F_x = m \frac{2v_x}{\Delta t_c} \quad (1.1)$$

Over time can get the average force

$$F_x = m \frac{2v_x}{\Delta t} \quad (1.2)$$

which allows us to derive an expression for pressure, which is just force over area

$$P = \frac{F_x}{A} = \frac{mv_x^2}{V} \quad (1.3)$$

For  $n$  particles it then becomes

$$P = \sum_{i=1}^N \frac{mv_{x,i}^2}{V} = \frac{m}{V} \sum_{i=1}^N v_{x,i}^2 \quad (1.4)$$

Assuming that that  $v_x = v_y = v_z$ <sup>1</sup>, we may extend this to 3 dimensions

<sup>1</sup>This is reasonable, is it not?

$$P = \frac{mv^2}{3} \frac{N}{V} \quad (1.5)$$

Now, compare this expression and that of the ideal gas law to find that temperature is a measure of the average kinetic energy per particle

Note that  $\frac{mv^2}{3} = kT \rightarrow \frac{mv^2}{3k} = 2\frac{\bar{E}}{3k}$

$$T = \frac{mv^2}{3k} = \frac{2\bar{E}}{3k} \quad (1.6)$$

## SUBSECTION 1.2

#### Lecture 2: Temperature & Ideal Gases & Solids

*Remark* Recall,  $PV = kT \frac{N}{V}$  where  $T = \frac{mv^2}{3k} = \frac{2\bar{E}}{3k}$ ,  $\bar{E} = \frac{3kT}{2}$  Where  $T$  is the experimentally measured temperature

We are still, however, left with a few questions:

1. But how does this apply to things that aren't gasses?
2. And this only describes average speed – what about the distribution?
3. And how can we describe thermal equilibrium?

**Definition 1** **Heat capacity:** the energy required for a unit change in temperature.

**Lemma 1** The total energy of an ideal gas at temperature  $T$  is given by

$$U = NE = 3 \frac{NkT}{2} \quad (1.7)$$

So, increasing  $U$  increases  $T$

therefore can define heat capacity:

$$C_v \equiv \left. \frac{\partial U}{\partial T} \right|_{V, N, \dots} \quad (1.8)$$

This brings us to the expression for 1 mole of an ideal gas,

$$C_V = \frac{3R}{2} \quad R = 8.314 \text{ J}/(\text{K} \cdot \text{mol}) \quad (1.9)$$

Recall  $Nk = nR$ , because chemistry/physics notation

And this is experiemntally designed by heating an known insulated mass of gas. This estimation works best for lighter gasses, especially for more massive & diatomic molecules.

**Theorem 1** **Equipartition Theorem**

$$U = N\bar{E} = \frac{3}{2} Nmv^2 = 3N \frac{kT}{2} \quad (1.10)$$

So the total energy is the total quadratic degrees of freedom  $D$  multiplied by  $\frac{kT}{2}$ . For a gas,  $D = dN$ , where  $d$  is the *dof* [] per particle

Therefore,

$$U = d \frac{nRT}{2} \quad (1.11)$$

Given that

- $\bar{E} = \frac{3}{2} kT$
- There are three degrees of freedom, each of which having kinetic energy  $\frac{1}{2} mv_a^2$ , where  $a \in \{x, y, z\}$

Degrees of freedom

Now, armed with  $C_V = \frac{dR}{2}$  we notice that for Helium, Argon, and Neon this works well (and they have  $dof \approx 3$ ). Also notice that the ones for which it breaks on are primarily diatomic or massive molecules, e.g.  $O_2$

This is exactly what we said earlier, however, the expression changes depending on the number of degrees of freedom the particle has

But there are actually 3 translational degrees of freedom and 3 rotational degrees of freedom. So why don't all gases have  $d = 6$ ? But our prior analysis indicates  $d = 3$  for monoatomic and  $d = 5$  for diatomic

From quantum mechanics the rotational energy levels for a rigid object are

$$E_J = \frac{\hbar^2}{2I} J(J+1) \quad (1.12)$$

where

- $\hbar$  is the reduced planck's constant
- $J$  is the quantum number
- $I = \frac{2}{5}mr^2$  is the moment of inertia, in this case we assume a solid sphere

So if we take the Equipartition theorem (Eq. 1.10) to heart, we see that the energy in the rotational modes is entirely irrelevant; it requires temperature at the order of a billion Kelvin to reach the first excited state. At room temperature the modes are "frozen out"

Applying Eq. 1.12 to a diatomic molecule e.g.  $N_2$  we note that the nuclei are separated by a *much* greater distance. If we were to imagine the diatomic molecule it becomes apparent that  $I$  is much larger along two axes than one,  $\Rightarrow$  only 2 of these modes matter at room temperature and therefore diatomic molecules have  $3 + 2$  degrees of freedom [at room temperature]. This also has the implication that  $C_V$  is temperature-dependent

### 1.2.1 Idealized Solids

Assuming a solid is modelled as a bunch of atoms attached with springs, we arrive at  $E = \frac{1}{2}mv^2$ . But when we look at solids (Fig. 1) we note that they have this plateau effect.

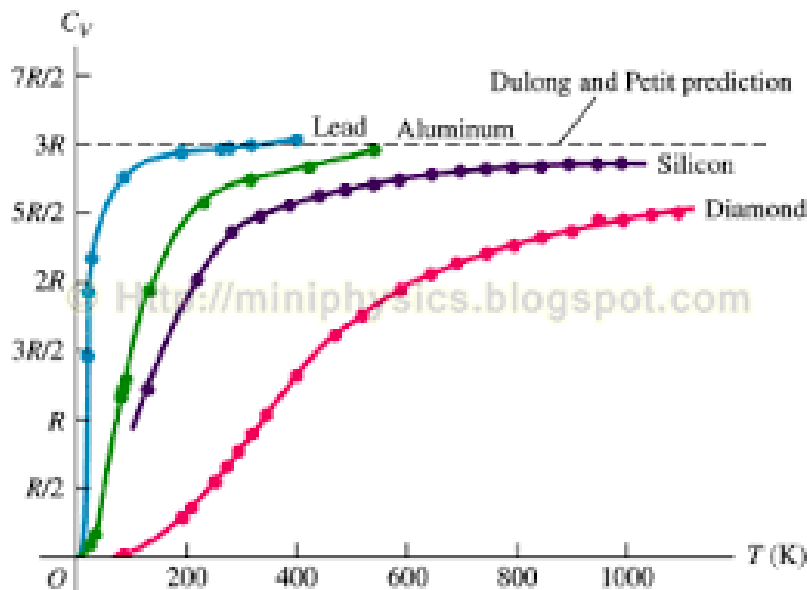


Figure 1. Heat capacity of selected solids

This is explained by there being 3 potential degrees of freedom in the spring<sup>2</sup>.

<sup>2</sup> ...

But basically what the Equipartition theorem tells us is the energy stored in each quadratic degree of freedom. But we don't always know what the degrees of freedom are. For example how do we know what should we count or not?

*Remark* Heat capacity at a constant volume is really easy to calculate, but is difficult to measure since solids tend to thermally expand. Heat capacity at a constant pressure is really easy to measure, but difficult to calculate.

So, to recap:

- The Equipartition theorem seems pretty powerful; it can predict energy per mode as a function of temperature and calculate a bunch of things
- BUT:
  - We didn't really prove it
  - Doesn't really work when quantum levels matter
  - 'Degrees of freedom' is not well-defined
  - Works only in average energy
  - Doesn't explain equilibrium
  - Still don't know what entropy is

Let's try to address all of these shortcomings by applying quantum statistical mechanics.

#### SUBSECTION 1.3

### Lecture 3: Two State Systems

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Begin with coins; heads/tails with equal likelihood, independent flips.

#### Definition 2

To describe this better we will define a few terms:

**Microstate:** a specific configuration; all of which are equally likely, i.e. individual head/tails outcomes

**Macrostate:** Defined by some combined quantity, possibly containing multiple microstates, i.e. a state w/ 2/3 heads

**Multiplicity ( $\Omega$ ):** The number of microstates in a macrostate

For example, if we were to flip 100 coins.

- Multiplicity of 0 heads:  $\Omega(0) = 1$
- Multiplicity of 1 heads:  $\Omega(1) = 100$
- Multiplicity of 2 heads:  $\Omega(2) = \frac{100 \cdot 99}{2}$
- Multiplicity of  $n$  heads:  $\Sigma(N, n) = \binom{N}{n} = \frac{N!}{n! \cdot (N - n)!}$ 
  - The probability of  $n$  heads after  $N$  flips:  $\frac{\Sigma(N, n)}{\Sigma_{\text{tot}}} = \frac{\Sigma(N, n)}{2^N}$

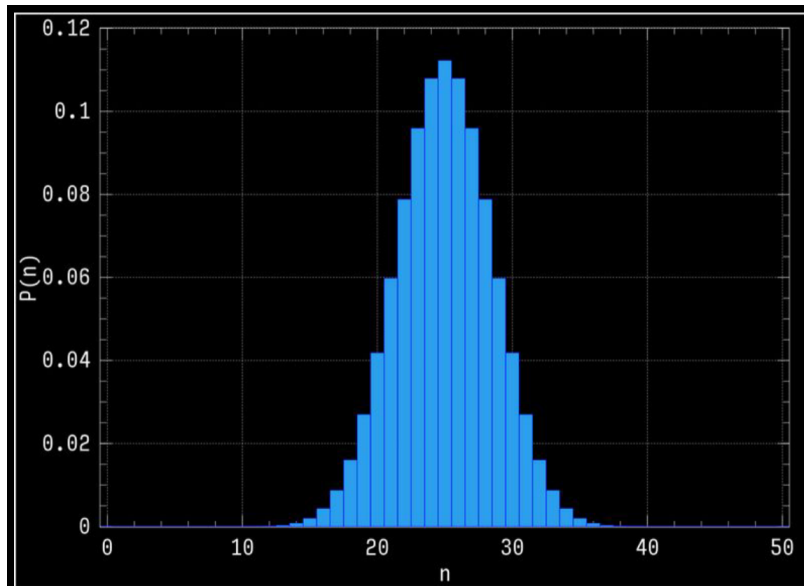


Figure 2. Probability distribution for 50 coin flips

Now, imagine a system of little independent magnetic dipoles that can point up or down in an external magnetic field (Fig. 3) The multiplicity is given by the expression we found before:

As there are more flips the distribution will become *relatively* more narrow

$$\Sigma(N, n) = \binom{N}{n} = \frac{N!}{n! \cdot (N - n)!} \quad (1.13)$$

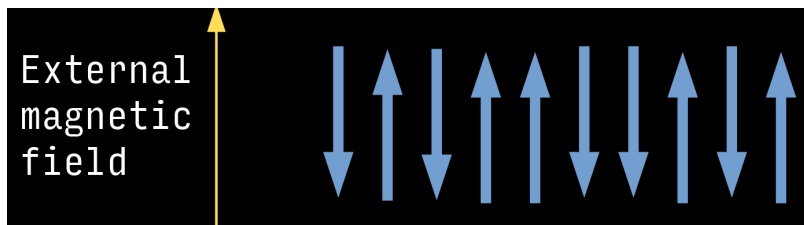


Figure 3. Independent dipoles in field

Equal numbers of dipoles pointed up and down has a far higher multiplicity and therefore is the most probable state although it is not the lowest energy state. On the other hand, the lowest energy state would be all of the dipoles being aligned with the magnetic field, and highest being all anti-aligned. How can resolve this problem?

SUBSECTION 1.4

## Lecture 4: Einstein Solids

### Einstein Solid

Model a solid as  $N$  independent harmonic oscillators.

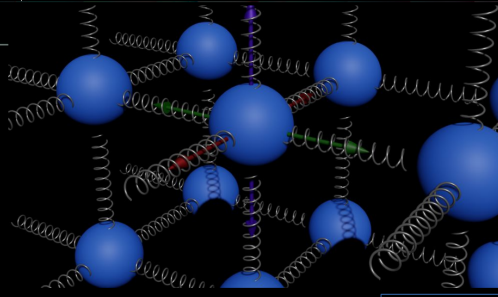
The quantum energy levels for a harmonic oscillator are:

$$E = (q + \frac{1}{2})\hbar\sqrt{k/m}$$

So, each oscillator has evenly spaced energy levels.

$E - E_o = q\hbar\sqrt{k/m} \longrightarrow E = q\hbar\sqrt{k/m}$

We don't really care about the ground state  $E_o$ , since it is constant, so we can ignore it.



**Recall:**  
 $\omega \equiv 2\pi f$   
 $\hbar \equiv h/2\pi$   
 $\hbar\omega = hf$

Figure 4. Einstein solid

Moving away from the contrived dipole example we move to the Einstein solid which is basically a bunch of harmonic oscillators as a means to finding out what temperature is.

Comment

Recall: the quantum energy levels of a harmonic oscillator is

$$E = (q + \frac{1}{2})\hbar\sqrt{k/m} \quad (1.14)$$

Where

- $\omega = 2\pi f$
- $\hbar = \frac{h}{2\pi}$
- $\hbar\omega = hf$

By inspection we see that each oscillator has evenly spaced energy levels.

$$E - E_o = q\hbar\sqrt{k/m} \longrightarrow q\hbar\sqrt{k/m} \quad (1.15)$$

Each identical independent oscillator has an integer quanta of energy. This means that for a 3-oscillator system there are 6 ways 2 quanta can be distributed is  $(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (0, 1, 1), (1, 0, 1)$  and so on<sup>3</sup>. The **macrostate** is the total energy in the system,  $q_{tot} = 2$ . The **microstate** is the configuration, i.e.  $(1, 1, 0)$ . So, the **multiplicity** of the macrostate  $q_{tot} = 2$  is  $\Omega(2) = 6$ . Going onwards  $\Omega(3) = 10$  and so forth.

Ground state is a constant so we ignore it

<sup>3</sup>The nice thing about statistical mechanics is that each step is fairly trivial

Theorem 2

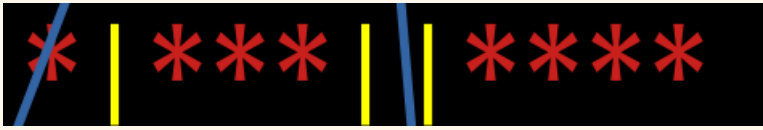
The multiplicity for  $q$  quanta of energy distributed between  $N$  oscillators is

$$\Omega(N, q) = \binom{q + N - 1}{q} = \frac{q + N - 1}{q!(N - 1)!} \quad (1.16)$$



PROOF

Imagine  $N = 4$  oscillators and  $q = 8$  quanta. Consider the configuration  $(1, 3, 0, 4)$ . We can draw it like this:



where  $*$  is a quanta of energy and  $|$  separates the oscillators. Any combination of 8 of  $*$  and 3 of  $|$  is a valid microstate.

So: There are  $q + N - 1$  symbols,  $q$  of which are a  $*$

TLDR: cute proof! □

By inspection we find that the multiplicity grows *very* rapidly with increasing  $N$  or  $q$ .

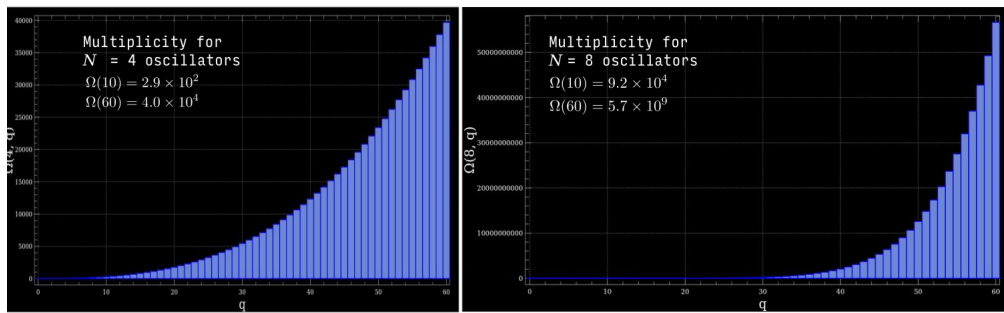


Figure 5. Multiplicity  $\Omega$  of Einstein solid given  $N, q$

Now, can we understand temperature by making Einstein solids interact with each other? Imagine two Einstein solids with  $N = 4$  oscillators each and  $q_{tot} = q_1 + q_2$  quanta of energy between them. How do the macrostates and multiplicity look?

Note: if solid 1 has  $q_1 = 10$ , then solid 2 has  $q_2 = 60 - 10 = 50$

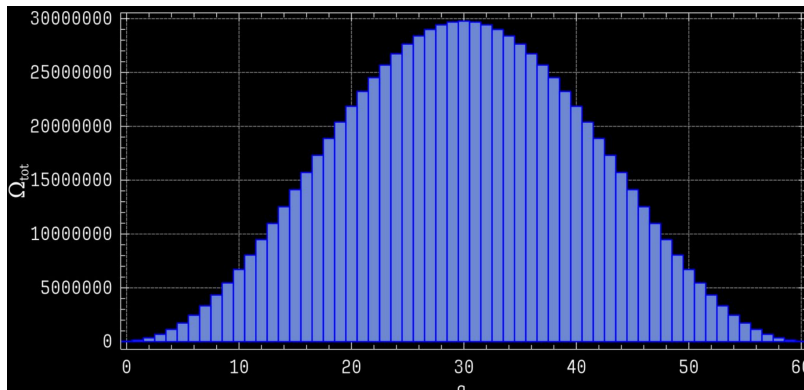
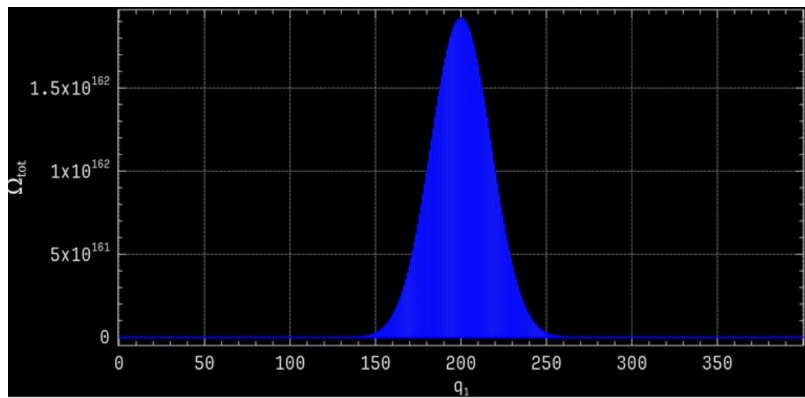


Figure 6.  $N_1 = N_2 = 4, q_1 + q_2 = 60, \Omega_{tot} = \Omega(q_1) \times \Omega(60 - q_1)$

Since they share energy, it follows that the total multiplicity of the system is  $\Omega_{tot} = \Omega_1 \times \Omega_2$ . Plotting this we find that the multiplicity will tend to follow something that looks a bit like a normal distribution, the spread of which depends on the  $N$  and  $q$ . As this gets larger and larger we gain more and more certainty as to the system's energy – that the system with the greatest multiplicity (most likely) has an energy level which is in-between the energies of the interacting bodies.



*Comment* This begins to look like a plausible explanation for thermal equilibrium!

# ECE259

## Electromagnetism

## SECTION 2

### Dielectrics

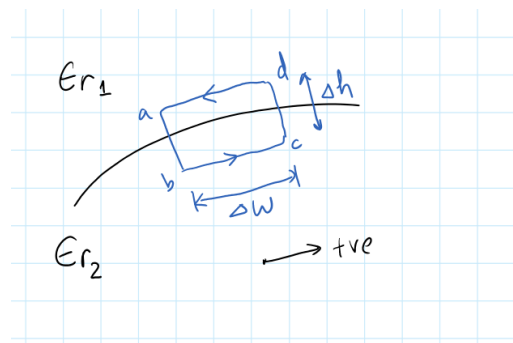
## SUBSECTION 2.1

#### Lecture 15: Boundary Conditions for Dielectrics

**Remark** Recall, at a conductor/free-space interface,  $E_+ = 0$ ,  $E_n = \frac{\rho_s}{\epsilon_0}$   
Consider an interface between two generic dielectrics (Fig. 7).

**Theorem 3**

$$\oint_c \vec{E} \cdot d\vec{l} = 0 \quad (2.1)$$



**Figure 7.** Dielectric interface

**Intuition** The two integrals parallel to the dielectrics will cancel out, and so will those perpendicular.

This implies that the tangential component of the  $\vec{E}$  field is continuous across the boundary,

$$E_{1t} = E_{2t} \quad (2.2)$$

However we get a bit of a different result when working in 3 dimensions where the interface is a surface instead of a line.

**Theorem 4**

$$\oint_S \vec{D} \cdot d\vec{s} = \rho_S \Delta \rightarrow (\vec{D}_1 - \vec{D}_2) \cdot \vec{a}_{n2} = \rho_S \quad (2.3)$$

$a_n$  denotes normal component

Note:  $\vec{D} = \epsilon_o \vec{E} + \vec{p} = \dots \epsilon \vec{E}$

PROOF As  $\Delta h \rightarrow 0$

$$\begin{aligned}
 \oint_S \vec{D} \cdot d\vec{s} &= \int_{top} \vec{D}_1 \cdot d\vec{s} + \int_{bottom} \vec{D}_2 \cdot d\vec{s} \\
 &= \vec{D}_1 \cdot \vec{a}_{n2} \Delta S + \vec{D} \cdot \vec{a}_{n1} \Delta S \\
 &= \vec{D}_1 \cdot \vec{a}_{n2} \Delta S - \vec{D} \cdot \vec{a}_{n2} \Delta S \\
 &= \rho_S \Delta S
 \end{aligned} \tag{2.4}$$

□

Solving these problems usually involves finding the tangential and normal components through Eq. 2.2 and 2.3 then applying Pythagoras.

SUBSECTION 2.2

## Lecture 16: Capacitors

### Definition 3

A capacitor is a device consisting of two isolated conductors for storing energy in the form of electrostatic potential energy. An isolated conductor can also have "capacitance" if the other conductor is far away. The charge of a capacitance refers to the charge on one conductor. The energy stored in a capacitor is equal to the energy it takes to charge a capacitor from a discharged state to a charged state. A capacitor's *capacitance*<sup>4</sup> is defined as

$$C = \frac{Q}{V} \tag{2.5}$$

and has units  $[C] = \frac{C}{V} F$  [Farads]

<sup>4</sup>Capacitance is actually independent of  $Q$  and  $V$  and is dependent only on the physical attributes of [the capacitor]

Capacitance is calculated as follows

1. choose a coordinate system
2. Assume  $+Q/-Q$  on the conductors
3. Find  $\vec{E}$  from  $Q$  distribution
4. Find  $V_{AB} = \int_A^B \vec{E} \cdot d\vec{l}$  where  $A$  carries the negative charge and  $B$  carries positive.
5. Apply  $C = \frac{Q}{V}$

SUBSECTION 2.3

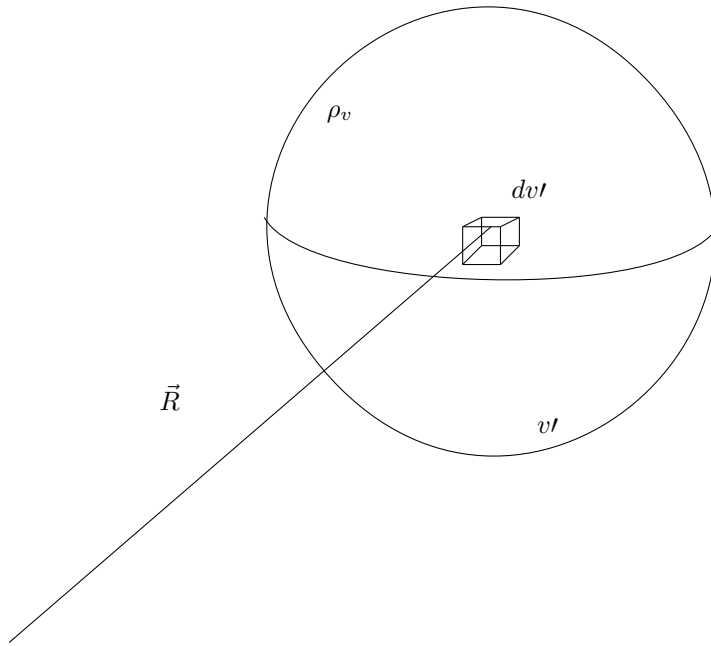
## Lecture 17: More on energy and capacitors

Remark Recall, the potential energy stored in the system is given by

$$W_e = \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv' \tag{2.6}$$

Can extend  $v'$  to entire space

Applying generalized gauss's law and performing an integral across the space we find that



$$\begin{aligned}
 W_e &= \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv' \\
 &= \frac{1}{2} \int_{v'} (\vec{\nabla} \cdot \vec{D}) V(\vec{R}) dv' \\
 &= \frac{1}{2} \int_{v'} \vec{\nabla} \cdot (\vec{D} V) dv' - \frac{1}{2} \int_{v'} \vec{\nabla} \cdot \vec{D} V dv' \\
 &= \frac{1}{2} \oint_{S'} \vec{D} V \cdot d\vec{s} + \frac{1}{2} \int_{V'} \vec{D} \cdot \vec{E} dv' \\
 \Rightarrow W_e &= \frac{1}{2} \int_v \vec{D} \cdot \vec{E} dv
 \end{aligned}
 \tag{2.7} \quad \text{Apply divergence theorem}$$

The first term  $\rightarrow 0$  as  $R \rightarrow \infty$

Therefore the energy density can be defined as

**Definition 4** Given energy density  $w_e$  with units  $[J/m^3]$

$$W_e = \int_v w_e dv \quad w_e = \frac{1}{2} \vec{D} \cdot \vec{E} \tag{2.8}$$

*Remark* Things to know about capacitors

- Stacking them puts them in series. This looks like the inverse of resistors, i.e.

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \dots \tag{2.9}$$

- Putting them besides each other puts them in parallel.

$$C = C_1 + C_2 \dots \tag{2.10}$$

## SUBSECTION 2.4

**Lecture 19: Poisson's Equation****Definition 5 Poisson's Equation**

$$\nabla \cdot (\varepsilon \nabla V) = \rho_v \quad (2.11)$$

And for no charge or  $\rho_v = 0$ , we have a special case: the 'laplace equation'

$$\nabla \cdot (\varepsilon \nabla V) = 0 \quad (2.12)$$

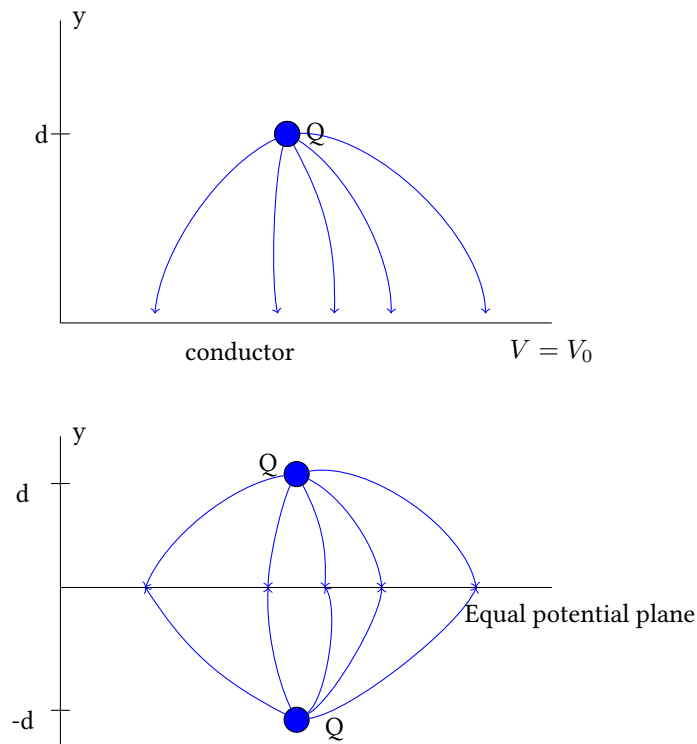
Using this we can solve for potential as a boundary value problem. For a surface  $S$  enclosing a volume  $v$  with  $\rho_v$ , solving the following gives us the potential difference  $V$  inside  $v$ .

$$\begin{cases} \nabla \cdot (\varepsilon \nabla V) = \rho_v & \text{in } v \\ V_s = V_o \end{cases} \quad (2.13)$$

## SUBSECTION 2.5

**Lecture 20: Method of Images***Remark*

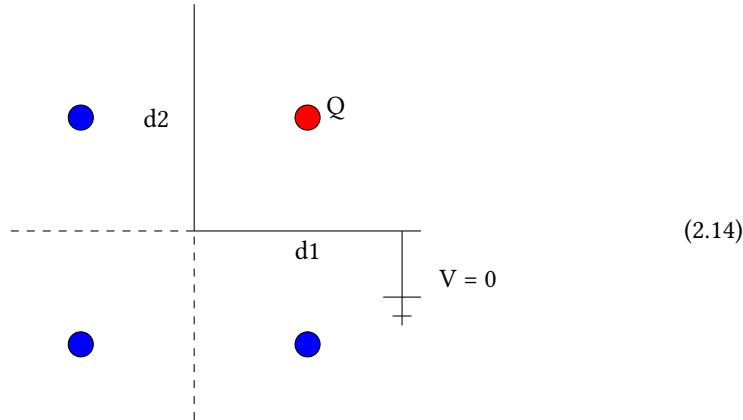
TLDR: Add imaginary charges such that the field lines meet perpendicular to the planes/lines of interest



And for the upper-half space ( $y > 0$ ) the two cases have the same source distribution and the same boundary condition on the conducting surface.

*Example*

A charge  $Q$  is at  $(d_1, d_2)$  relative to two conductor walls. Find force on charge  $Q$  due to induced charges on conductor surfaces.



We have to be careful about where to put these virtual charges. They are arranged this way so that there is a right angle at the conductors; that's why we don't just put a single charge in the bottom left quadrant.

The force can be found as follows:

$$\vec{F} = \frac{Q}{4\pi\epsilon} \left[ \frac{-Q}{2d_1^2} \hat{x} + \frac{-Q}{2d_2^2} \hat{y} + \frac{Q(2d_1\hat{x} + 2d_2\hat{y})}{[(2d_1)^2 + (2d_2)^2]^{3/2}} \right] \quad (2.15)$$

### 2.5.1 Currents

There are a few types of currents:

- Conduction current: In conductors and semiconductors due to the motion of  $e^-$  and holes  $h^+$ <sup>5</sup>
- Electrolytic current: motion of ions in solutions
- Convection current: motion of  $e^-$  or ions in vacuum

<sup>5</sup> We will primarily focus on conduction currents

In conductors and semiconductors the free charge is

$$e = 1.6 \times 10^{-19} C \quad (2.16)$$

In the presence of an external field  $E$ ,

$$\begin{cases} \vec{E} = 0 & \text{random motion of free charges, current} = 0 \\ \vec{E} > 0 & \vec{\mu}_e = -\mu_o \vec{E}^6 \end{cases} \quad (2.17)$$

<sup>6</sup>  $\vec{\mu}$  = average drift velocity,  $\mu$  = mobility

Typical  $\vec{\mu}_e$  is very slow:  $\approx 10^{-5} 10^{-4} \frac{m}{s}$

**Definition 6** General integral form for current

$$I = \int_S \rho_e \vec{\mu}_e \cdot d\vec{s} \quad (2.18)$$

**PROOF** Current is the amount of charge through a surface  $S$  per unit time

$$d\vec{S} = \vec{a}_n dS$$

The free charge in the volume of

$$\vec{\mu} \cdot \vec{a}_n dt dS \quad (2.19)$$

will pass  $S$  in  $dt$  time

$$dq = -N_e e \vec{\mu}_e \cdot \vec{a}_n dt ds \quad (2.20)$$

Apply definition of current

$$dI = \frac{dq}{dt} = -N_e e \vec{\mu}_e \cdot d\vec{s} = \rho_e \vec{\mu}_e \cdot d\vec{s} \quad (2.21)$$

and integrate!

$$I = \int_S \rho_e \vec{\mu}_e \cdot d\vec{s} \quad (2.22)$$

□

This integral form can be a little awkward to work with so we define current density  $\vec{J}$

**Definition 7** Define current density to be  $\vec{J}$  such that

$$I = \int_S \vec{J} \cdot d\vec{s} \quad (2.23)$$

and therefore the volume current density  $\vec{J}$  with units  $\frac{A}{m^2}$

$$\vec{J} = \rho_e \vec{\mu}_e \quad (2.24)$$

- $\vec{J}$  is a vector and direction is the same as  $\vec{\mu}_e$  and **is** a point function
- $I$  is a scalar and **is not** a point function

Now ohm's law can be defined more rigorously in point/microscopic form

$$\begin{aligned} \vec{J} &= \rho_e \vec{\mu}_e = -\rho_o \mu_e \vec{E} \\ \Rightarrow J &= \sigma \vec{E} \end{aligned} \quad (2.25)$$

$\sigma$  denotes conductivity



# ECE286

## Probability and Statistics

## SECTION 3

### Probability Distributions

## SUBSECTION 3.1

#### Lecture 14: functions of random variables

In the discrete case, given  $X$  with PMF  $f(x)$ , we can define an *invertible* function  $Y$  where  $Y = u(X)$ , therefore can write  $X = u^{-1}(Y)$ . If  $g(y)$  is the distribution of  $Y$

$$\begin{aligned} g(y) &= P(Y = y) \\ &= P(u^{-1}(Y) = u^{-1}(y)) \\ &= f(u^{-1}(y)) \end{aligned} \quad (3.1)$$

In the continuous case we may arrive at

$$g(y) = f(u^{-1}(y)) \left| \frac{du^{-1}(y)}{dy} \right| \quad (3.2)$$

This result is derived through the Leibniz integral rule,  
 $g(y) = \frac{d}{dy} \int_{-\infty}^{u^{-1}(y)} f(t) dt$

**Definition 8** The  $r^{th}$  moment about the origin of the random variable  $X$  is

$$\mu'_r = E[X^r] = \begin{cases} \sum_x x^r f(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x^r f(x) dx & X \text{ continuous} \end{cases} \quad (3.3)$$

- The mean is the first moment
- For variance,  $\sigma^2 = E[X^2] - \mu^2 \rightarrow \sigma^2 = \mu'_2 - \mu^2$

**Definition 9** The moment-generating function of  $X$  is defined as

$$\mu'_r = E[X^r] = \begin{cases} \sum_x e^{tx} f(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & X \text{ continuous} \end{cases} \quad (3.4)$$

In general

$$\mu'_r = \left. \frac{d^r M_X(t)}{dt^r} \right|_{t=0} \quad (3.5)$$

## SUBSECTION 3.2

#### Lecture 15: More on moment generating functions

By definition and completing the square,

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{t2\pi\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t2\pi\sigma} e^{-\frac{x^2-2(x-\mu)^2+\mu^2}{2\sigma^2}}} dx \\
&= e^{\frac{2\mu t+t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}\right\} dx \\
&= e^{\frac{\mu t+t^2\sigma^2}{2}} \quad (3.6)
\end{aligned}$$

The integrand is just a normal PDF and thus integrates to one

### 3.2.1 Linear combinations of random variables

Given random variable  $X$  with distribution  $f(x)$ . What is the distribution,  $h(y)$ , of  $Y = aX$ ?

**Definition 10** Probability distribution of linear combination of random variables

**Discrete:**

$$\begin{aligned}
P(X = x) = f(x) &\rightarrow h(y) = P(Y = y) \\
&= P(aX = y) \\
&= f\left(\frac{y}{a}\right)
\end{aligned} \quad (3.7)$$

**Continuous:**

$$\begin{aligned}
F(y) &= P(Y \leq y) \\
&= P(X \leq \frac{y}{a}) \\
&= \int_{-\infty}^{\frac{y}{a}} f(t) dt \\
&= \int_{-\infty}^{\frac{y}{a}} f(t) dt = \int_{-\infty}^y \frac{1}{a} f\left(\frac{s}{a}\right) ds \\
\Rightarrow h(y) &= \frac{1}{|a|} f\left(\frac{y}{a}\right)
\end{aligned} \quad (3.8)$$

u-sub; let  $s = at$

Extending from this, let's assume that  $X$  has the MGF <sup>7</sup>  $M_X(t)$ . Then,

<sup>7</sup> Moment Generating Function

$$\begin{aligned}
M_Y(t) &= \int_{-\infty}^{\infty} e^{ty} h(y) dy \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{ty} f\left(\frac{y}{a}\right) dy \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{taz} f(z) a dz \\
&= \int_{-\infty}^{\infty} e^{taz} f(z) dz \\
&= M_X(at)
\end{aligned} \quad (3.9)$$

u-sub  $z = \frac{y}{a}$

**Definition 11**

$$M_{aX}(t) = M_X(at) \quad (3.10)$$

This leads us to the convolution integral

## SUBSECTION 3.3

## Lecture 16: Sampling

Given sample data  $x_1, \dots, x_n$ , each  $x_i$  is the realization of an RV<sup>8</sup>,  $X_i, i = 1, \dots, n$

<sup>8</sup> Say each  $X_i$  is a Bernoulli random variable

**Definition 12** Given the mean,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad (3.11)$$

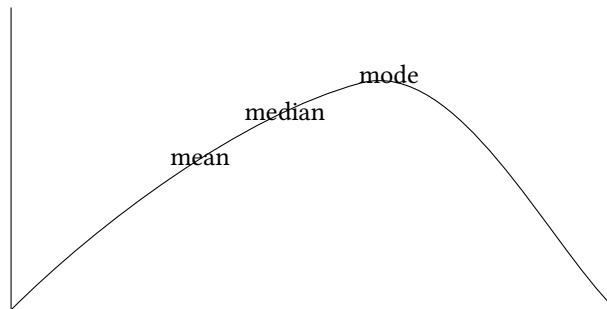
the **sample mean** of a random variable  $X$  is defined as

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad (3.12)$$

**Definition 13** The **sample median** is

$$x_m = \begin{cases} \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2} & \text{if } n \text{ even} \\ \frac{x_{\frac{n+1}{2}}}{2} & \text{if } n \text{ is odd} \end{cases} \quad (3.13)$$

and **mode** is familiar as the most frequently occurring value



**Figure 8.** Median, mode, mean labelled on a skew distribution

**Definition 14** **Sample variance**

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (3.14)$$

and the corresponding standard deviation<sup>9</sup>

$$s = \sqrt{s^2} \quad (3.15)$$

<sup>9</sup> Is this correct??

**Lemma 2** Consider

$$E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] \quad (3.16)$$

Assuming that  $X_i$  have the same distribution,  $f(x_i)$ , and mean  $\mu$ ,

$$\begin{aligned}
 &= \frac{1}{n} \sum_{i=1}^n \mu \\
 &= \frac{1}{n} n\mu \\
 &= \mu
 \end{aligned} \tag{3.17}$$

- the  $X_i$  are identically distributed
- $\bar{X}$  is an unbiased estimator of  $\mu$

We then have

$$\begin{aligned}
 S^2 &= \frac{1}{n-1} \sum_{i=1}^n X_i^2 + \bar{X}^2 - 2X_i\bar{X} \\
 &= \frac{1}{n-1} (n\bar{X}^2 - 2n\bar{X}^2 + \sum_{i=1}^n X_i^2) \\
 &= \frac{1}{n-1} (-n\bar{X}^2 + \sum_{i=1}^n X_i^2) \tag{3.18}
 \end{aligned}$$

We are now ready to take the expectation of  $S^2$ , which is

Recall: if  $Y$  has mean, variance  $\mu, \sigma^2$ , then  $\sigma^2 = E[Y^2] - \mu^2$

$$\begin{aligned}
 E[S^2] &= \frac{1}{n-1} (-nE[\bar{X}^2] + \sum_{i=1}^n E[X_i^2]) \\
 &= \frac{1}{n-1} (-n(E[\bar{X}^2] + \text{var}[\bar{X}]) + \sum_{i=1}^n (\sigma^2 + \mu^2)) \\
 &= \frac{1}{n-1} (-n(\mu^2 + \text{var}[\frac{1}{n} \sum_{i=1}^n X_i]) + n(\sigma^2 + \mu^2)) \\
 &\quad \vdots \\
 &= \frac{n-1}{n-1} \sigma^2 = \sigma^2 \tag{3.19}
 \end{aligned}$$

So we get

So we have an unbiased estimator of variance

$$S^2 = \sum_{i=1}^n (X_i - \bar{X})^2 \tag{3.20}$$

We divide by  $n-1$  because there is a "hidden" uncertainty in  $\bar{X}$

# *TEP327*

## *Engineering and Law*

Torts can be identified by

- Elements of negligence can be identified
- There exists a duty of care
- Was the standard of duty of care breached?
- Did that breach of standard cause damage to the plaintiff?

*Comment*

Must be careful about the standard of care met or not met. E.x. in the in-class bike example one could argue that in designing the gear the engineer should expect that the gear could be commercialized and mass-produced later on for mass-market. On the other hand it is reasonable to argue that having to design the gear to be used by unskilled 16-year-olds is an unreasonable and outside the standard of care.

Trespass is strict liability. Ignorance of the law is no excuse.