EngSci Year 2 Winter 2022 Notes

FOR CLASS CONTENT AFTER READING WEEK

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Contents

PHY294 Quantum & Thermal Physics

SECTION 1
Statistical Mechanics

SUBSECTION 1.1
Lecture 1: Ideal Gas Law

We begin our discussion on statistical mechanics with the guiding question, "What is temperature". We will first derive the ideal gas law $PV = nRT$, which most should be familiar with.

Imagining a single particle bouncing around elastically in 2D box with speed v_x. If we were to look at the sides of the box we will see intermittent spikes of force experienced [by the box walls],

$$F_x = m\frac{2v_x}{\Delta t_c} \tag{1.1}$$

Over time can get the average force

$$F_x = m\frac{2_x}{\Delta t} \tag{1.2}$$

which allows us to derive an expression for pressure, which is just force over area

$$P = \frac{F_x}{A} = \frac{mv_x^2}{V} \tag{1.3}$$

For n particles it then becomes

$$P = \sum_{i=1}^{N} \frac{mv_{x,i}^2}{V} = \frac{m}{V}\sum_{i=1}^{N} v_{x,i}^2 \tag{1.4}$$

Assuming that that $v_x = v_y = v_z$ [1], we may extend this to 3 dimensions

[1] *This is reasonable, is it not?*

$$P = \frac{mv^2}{3}\frac{N}{V} \tag{1.5}$$

Now, compare this expression and that of the ideal gas law to find that temperature is a measure of the average kinetic energy per particle

Note that $\frac{mv^2}{3} = kT \rightarrow \frac{mv^2}{3k} = 2\frac{\overline{E}}{3k}$

$$T = \frac{mv^2}{3k} = \frac{2\overline{E}}{3k} \tag{1.6}$$

SUBSECTION 1.2
Lecture 2: Temperature & Ideal Gases & Solids

Remark   Recall, $PV = kT\frac{N}{V}$ where $T = \frac{mv^2}{3k} = \frac{2\overline{E}}{3k}, \overline{E} = \frac{3kT}{2}$ Where T is the experimentally measured temperature

We are still, however, left with a few questions:

1. But how does this apply to things that aren't gasses?

2. And this only describes average speed – what about the distribution?

3. And how can we describe thermal equilibrium?

Definition 1   **Heat capacity**: the energy required for a unit change in temperature.

Lemma 1   The total energy of an ideal gas at temperature T is given by

$$U = NE = 3\frac{NkT}{2} \tag{1.7}$$

So, increasing U increases T

therefore can define heat capacity:

$$C_v \equiv \left.\frac{\partial U}{\partial T}\right|_{V,N,\dots} \tag{1.8}$$

This brings us to the expression for 1 mole of an ideal gas,

$$C_V = \frac{3R}{2} \qquad R = 8.314 J/(K \cdot mol) \tag{1.9}$$

Recall $Nk = nR$, because chemistry/physics notation

And this is experiemntally designed by heating an known insulated mass of gas. This estimation works best for lighter gasses, especially for more massive & diatomic molecules.

Theorem 1   **Equipartition Theorem**

$$U = N\overline{E} = \frac{3}{2}Nmv^2 = 3N\frac{kT}{2} \tag{1.10}$$

So the total energy is the total quadratic degrees of freedom D multiplied by $\frac{kT}{2}$. For a gas, $D = dN$, where d is the *dof* [2] per particle
Therefore,

$$U = d\frac{nRT}{2} \tag{1.11}$$

Given that

- $\overline{E} = \frac{3}{2}kT$

- There are three degrees of freedom, each of which having kinetic energy $\frac{1}{2}mv_a^2$, where $a \in \{x, y, z\}$

[2] *Degrees of freedom*

This is exactly what we said earlier, however, the expression changes depending on the number of degrees of freedom the particle has

Now, armed with $C_V = \frac{dR}{2}$ we notice that for Helium, Argon, and Neon this works well (and they have *dof* ≈ 3). Also notice that the ones for which it breaks on are primarily diatomic or massive molecules, e.g. O_2

But there are actually 3 translational degrees of freedom and 3 rotational degrees of freedom. So why don't all gases have $d = 6$? But our prior analysis indicates $d = 3$ for monoatomic and $d = 5$ for diatomic

From quantum mechanics the rotational energy levels for a rigid object are

$$E_J = \frac{\hbar}{2I}J(J+1) \tag{1.12}$$

where

- $\hbar$ is the reduced planck's constant

- J isthe quantum number

- $I = \frac{2}{5}mr^2$ is the moment of inertia, in this case we assume a solid sphere

So if we take the Equipartition theorem (Eq. 1.10) to heart, we see that the energy in the rotational modes is entirely irrelevant; it requires temperature at the order of a billion Kelvin to reach the first excited state. At room temperature the modes are "frozen out"

Applying Eq. 1.12 to a diatomic molecule e.g. N_2 we note that the nuclei are seperated by a *much* greater distance. If we were to imagine the diatomic molecule it becomes apparent that I is much larger along two axes than one, $\Rightarrow$ only 2 of these modes matter at room temperature and therefore diatomic molecules have $3 + 2$ degrees of freedom [at room temperature]. This also has the implication that C_V is temperature-dependent

1.2.1   Idealized Solids

Assuming a solid is modelled as a bunch of atoms attached with springs, we arrive at $E = \frac{1}{2}mv^2$. But when we look at solids (Fig. 1) we note that they have this plateau effect.
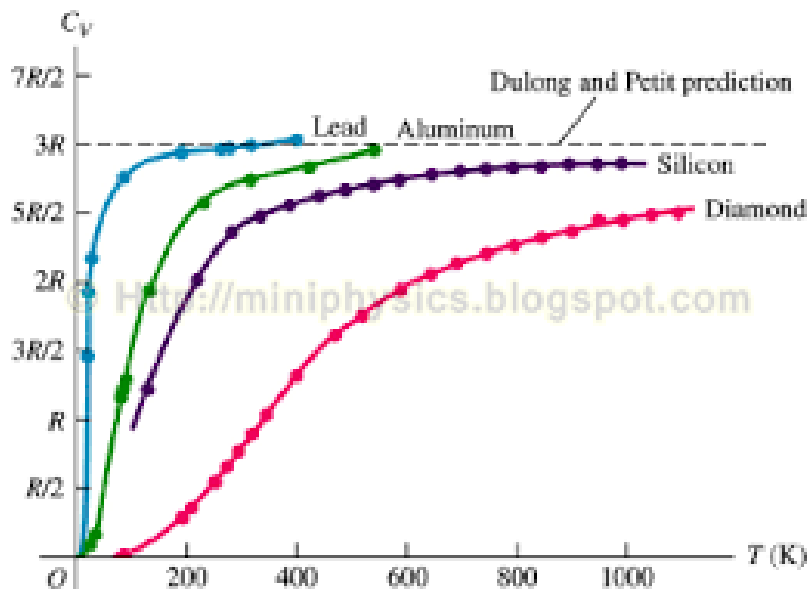

Figure 1. Heat capacity of selected solids

This is explained by there being 3 potential degrees of freedom in the spring[3].   [3] ...

Bu basically what the Equipartition thereom tells us is the energy stored in each quadratic degree of freedom. But we don't always know what the degrees of freedom are. For example how do we know what should we count or not?

Remark   Heat capacity at a constant volume is really easy to calculate, but is difficult to measure since solids tend to thermally expand. Heat capacity at a constant press is really easy to measure, but difficult to calculate.

So, to recap

- The Equipartition thereom seems pretty powerful; it can predict energy per mode as a function of temperature

- can be used to calculate a bunch of things

- BUT:

    - We didn't really prove it
    - Doesn't really work when quantum levvels matter
    - 'Degrees of freedom' is not well-defined
    - Works only in average energy
    - Doesn't explain equilibrium
    - Still don't know what entropy is

ECE259 Electromagnetism

SECTION 2

Dielectrics

SUBSECTION 2.1

Lecture 15: Boundary Conditions for Dielectrics

Remark   Recall, at a conductor/free-space interface, $E_+ = 0$, $E_n = \frac{\rho_s}{\varepsilon_o}$

Consider an interface between two generic dielectrics (Fig. 2).

Theorem 2
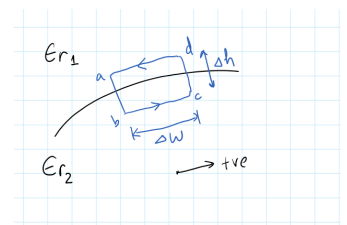
$$\oint_c \vec{E} \cdot d\vec{l} = 0 \tag{2.1}$$

Intuition   The two integrals parallel to the dielectrics will cancel out, and so will those perpendicular.

This implies that the tangential component of the $\vec{E}$ field is continuous across the boundary,

$$E_{1t} = E_{2t} \tag{2.2}$$

However we get a bit of a different result when working in 3 dimensions where the interface is a surface instead of a line.

Figure 2. Dielectric interface

4

Theorem 3

$$\oint_S \vec{D} \cdot d\vec{s} = \rho_S \Delta \rightarrow (\vec{D_1} - \vec{D_2}) \cdot \vec{a_{n2}} = \rho_S \tag{2.3}$$

a_n denotes normal component

Note: $\vec{D} = \varepsilon_o \vec{E} + \vec{p} = \ldots \varepsilon \vec{E}$

PROOF   As $\Delta h \rightarrow 0$

$$\begin{aligned} \oint_S \vec{D} \cdot d\vec{s} &= \int_{top} \vec{D_1} \cdot d\vec{s} + \int_{bottom} +\vec{D_2} \cdot d\vec{s} \\ &= \vec{D_1} \cdot \vec{a_{n2}} \Delta S + \vec{D} \cdot \vec{a_{n1}} \Delta S \\ &= \vec{D_1} \cdot \vec{a_{n2}} \Delta S - \vec{D} \cdot \vec{a_{n2}} \Delta S \\ &= \rho_S \Delta S \end{aligned} \tag{2.4}$$

□

Solving these problems usually involves finding the tangential and normal components through Eq. 2.2 and 2.3 then applying Pythagoras.

SUBSECTION 2.2
Lecture 16: Capacitors

Definition 2

A capacitor is a device consisting of two isolated conductors for storing energy in the form of electrostatic potential energy. An isolated conductor can also have "capacitance" if the other conductor is far away. The charge of a capacitance refers to the charge on one conductor.

The energy stored in a capacitor is equal to the energy it takes to charge a capacitor from a discharged state to a charged state.

A capacitor's *capacitance*[4] is defined as

$$C = \frac{Q}{V} \tag{2.5}$$

and has units $[C] = \frac{C}{V} F (\text{ Farads})$

[4] *Capacitance is actually independent of Q and V and is dependent only on the physical attributes of [the capacitor]*

Capacitance is calculated as follows

1. choose a coordinate system

2. Assume $+Q/-Q$ on the conductors

3. Find $\vec{E}$ from Q distribution

4. Find $V_{AB} = \int_A^B \vec{E} \cdot d\vec{l}$ where A carries the negative charge and B carries positive.

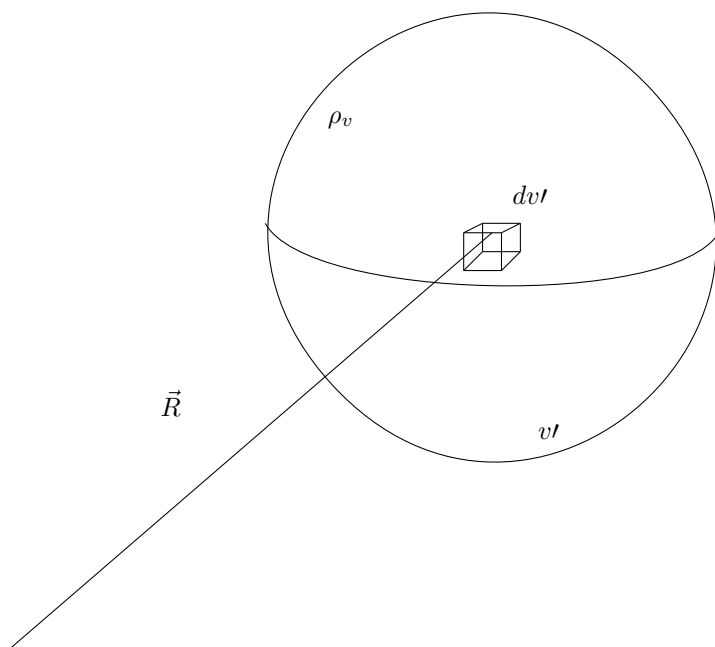5. Apply $C = \frac{Q}{V}$

SUBSECTION 2.3
Lecture 17: More on energy and capacitors

Remark   Recall, the potential energy stored in the system is given by

$$W_e = \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv' \tag{2.6}$$

Can extend v' to entire space

Applying generalized gauss's law we find that

$$
\begin{aligned}
W_e &= \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv' \\
&= \frac{1}{2} \int_{v'} (\vec{\nabla} \cdot \vec{D}) V(\vec{R}) dv' \\
&= \frac{1}{2} \int_{v'} \vec{\nabla} \cdot (\vec{D}V) dv' - \frac{1}{2} \int_{v'} \vec{\nabla} \cdot \vec{D} V dv' \qquad (2.7) \quad \text{Apply divergence thereom} \\
&= \frac{1}{2} \oint_{S'} \vec{D}V \cdot d\vec{s} + \frac{1}{2} \int_{V'} \vec{D} \cdot \vec{E} dv' \\
\Rightarrow W_e &= \frac{1}{2} \int_{v} \vec{D} \cdot \vec{E} dv \qquad\qquad\qquad\qquad\qquad\quad \text{The first term} \to 0 \text{ as } R \to \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \infty
\end{aligned}
$$

Therefore the energy density can be defined as

Definition 3 | Given energy density w_e with units $[J/m^3]$

$$
W_e = \int_v w_e dv \qquad w_e = \frac{1}{2} \vec{D} \cdot \vec{E} \qquad (2.8)
$$

Remark   Things to know about capacitors

- Stacking them puts them in series. This looks like the inverse of resistors, i.e.

$$
\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \cdots \qquad (2.9)
$$

- Putting them besides each other puts them in parallel.

$$C = C_1 + C_2 \ldots \qquad (2.10)$$

ECE286
Probability and Statistics

PART

III

SECTION 3

Probability Distributions

SUBSECTION 3.1

Lecture 14: functions of random variables

In the discrete case, given X with PMF $f(x)$, we can define an *invertible* function Y where $Y = u(X)$, therefore can write $X = u^{-1}(Y)$. If $g(y)$ is the distribution of Y

$$\begin{aligned}
g(y) &= P(Y = y) \\
&= P(u^{-1}(Y) = u^{-1}(y)) \\
&= f(u^{-1}(y))
\end{aligned} \tag{3.1}$$

In the continuous case we may arrive at

$$g(y) = f(u^{-1}(y))|\frac{du^{-1}(y)}{dy}| \tag{3.2}$$

This result is derived through the Leibniz integral rule, $g(y) = \frac{d}{dy}\int_{-\infty}^{u^{-1}(y)} f(t)dt$

Definition 4 The rth moment about the origin of the random variable X is

$$\mu_r' = E[X^r] = \begin{cases} \sum_x x^r f(x) & X \text{discrete} \\ \int_{-\infty}^{\infty} x^r f(x)dx & X \text{continuous} \end{cases} \tag{3.3}$$

- The mean is the first moment

- For variance, $\sigma^2 = E[X^2] - \mu^2 \rightarrow \sigma^2 = \mu_2' - \mu^2$

Definition 5 The moment-generating function of X is defined as

$$\mu_r' = E[X^{tX}] = \begin{cases} \sum_x e^{tx} f(x) & X \text{discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx & X \text{continuous} \end{cases} \tag{3.4}$$
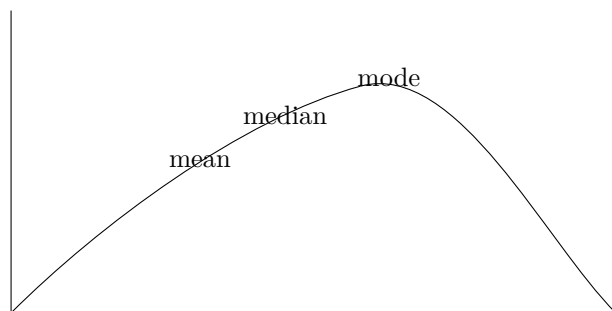
In general

$$\mu_r' = \frac{d^r M_X(t)}{dt^r}\Big|_{t=0} \tag{3.5}$$

SUBSECTION 3.2

Lecture 15: More on moment generating functions

By definition and completing the square,

Figure 3. Median, mode, mean labelled on a skew distribution

$$M_X(T) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{t2\pi}\sigma e^{-\frac{(x-\mu)^2}{2\sigma^2}}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t2\pi}\sigma e^{-\frac{x^2 - 2(x-\mu)^2 + \mu^2}{2\sigma^2}}} dx$$

$$= e^{\frac{2\mu t + t^2 \sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}\right\} dx$$

$$= e^{\frac{2\mu t + t^2 \sigma^2}{2}} \quad (3.6)$$

The integrand is just a normal PDF and thus integrates to one

SUBSECTION 3.3
Lecture 16: Sampling

Given sample data $x_1, \ldots, x_n$, each x_i is the realization of an RV [5]. , $X_i, i = 1, \ldots, n$

[5] Say each X_i is a Bernoulli random variable

Definition 6 | Given the mean,

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad (3.7)$$

the **sample mean** of a random variable X is defined as

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad (3.8)$$

Definition 7 | The **sample median** is

$$x_m = \begin{cases} \dfrac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2} & \text{if n even} \\ \dfrac{x_{n+1}}{2} & \text{if n is odd} \end{cases} \quad (3.9)$$

and **mode** is familiar as the most frequently occurring value

Definition 8 | **Sample variance**

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \tag{3.10}$$

and the corresponding standard deviation [6]

$$s = \sqrt{s^2} \tag{3.11}$$

[6]*Is this correct??*

Lemma 2 | Consider

$$E[\overline{X}] = E[\frac{1}{n}\sum_{i=1}^{n}X_i] = \frac{1}{n}\sum_{i=1}^{n}E[X_i] \tag{3.12}$$

Assuming that X_i have the same distribution, $f(x_i)$, and mean μ,

$$\begin{aligned} &= \frac{1}{n}\sum_{i=1}^{n}\mu \\ &= \frac{1}{n}n\mu \\ &= \mu \end{aligned} \tag{3.13}$$

- the X_i are identically distributed

- $\overline{X}$ is an unbiased estimator of μ

We then have

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}X_i^2 + \overline{X}^2 - 2X_i\overline{X}$$

$$= \frac{1}{n-1}(n\overline{X}^2 - 2n\overline{X}^2 + \sum_{i=1}^{n}X_i^2)$$

$$= \frac{1}{n-1}(-n\overline{X}^2 + \sum_{i=1}^{n}X_i^2) \tag{3.14}$$

We are now ready to take the expectation of S^2, which is

Recall: if Y has mean, varience μ, σ^2, then $\sigma^2 = E[Y^2] - \mu^2$

$$E[S^2] = \frac{1}{n-1}(-nE[\overline{X}^2] + \sum_{i=1}^{n}E[X_i^2])$$

$$= \frac{1}{n-1}(-n(E[\overline{X}^2] + var[\overline{X}]) + \sum_{i=1}^{n}\sigma^2 + \mu^2)$$

$$= \frac{1}{n-1}(-n(\mu^2 + var[\frac{1}{n}\sum_{i=1}^{n}X_i]) + n(\sigma^2 + \mu^2))$$

$$\vdots$$

$$= \frac{n-1}{n-1}\sigma^2 = \sigma^2 \tag{3.15}$$

So we have an unbiased estimator of varience

So we get

$$S^2 = \sum_{i=1}^{n} (X_i - \overline{X_i})^2 \qquad (3.16)$$

We divide by $n - 1$ because there is a "hidden" uncertainty in $\overline{X}$

TEP327
Engineering and Law

Torts can be identified by

- Elements of negligence can be identified

- There exists a duty of care

- Was the standard of duty of care breached?

- Did that breach of standard cause damage to the plaintiff?

Comment | Must be careful about the standard of care met or not met. E.x. in the in-class bike example one could argue that in designing the gear the engineer should expect that the gear could be commercialized and mass-produced later on for mass-market. On the other hand it is reasonable to argue that having to design the gear to be used by unskilled 16-year-olds is an unreasonable and outside the standard of care.

Trespass is strict liability. Ignorance of the law is no excuse.