EngSci Year 2 Winter 2022 Notes

FOR CLASS CONTENT AFTER READING WEEK

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Contents

PHY294
Quantum & Thermal Physics

SECTION 1
Statistical Mechanics

SUBSECTION 1.1
Lecture 1: Ideal Gas Law

We begin our discussion on statistical mechanics with the guiding question, "What is temperature". We will first derive the ideal gas law $PV = nRT$, which most should be familiar with.

Imagining a single particle bouncing around elastically in 2D box with speed v_x. If we were to look at the sides of the box we will see intermittent spikes of force experienced [by the box walls],

$$F_x = m\frac{2v_x}{\Delta t_c} \tag{1.1}$$

Over time can get the average force

$$F_x = m\frac{2_x}{\Delta t} \tag{1.2}$$

which allows us to derive an expression for pressure, which is just force over area

$$P = \frac{F_x}{A} = \frac{mv_x^2}{V} \tag{1.3}$$

For n particles it then becomes

$$P = \sum_{i=1}^{N} \frac{mv_{x,i}^2}{V} = \frac{m}{V}\sum_{i=1}^{N} v_{x,i}^2 \tag{1.4}$$

Assuming that that $v_x = v_y = v_z$ [1], we may extend this to 3 dimensions

[1] *This is reasonable, is it not?*

$$P = \frac{mv^2}{3}\frac{N}{V} \tag{1.5}$$

Now, compare this expression and that of the ideal gas law to find that temperature is a measure of the average kinetic energy per particle

Note that $\frac{mv^2}{3} = kT \rightarrow \frac{mv^2}{3k} = 2\frac{\overline{E}}{3k}$

$$T = \frac{mv^2}{3k} = \frac{2\overline{E}}{3k} \tag{1.6}$$

SUBSECTION 1.2
Lecture 2: Temperature & Ideal Gases & Solids

Remark — Recall, $PV = kT\frac{N}{V}$ where $T = \frac{mv^2}{3k} = \frac{2\overline{E}}{3k}, \overline{E} = \frac{3kT}{2}$ Where T is the experimentally measured temperature

We are still, however, left with a few questions:

1. But how does this apply to things that aren't gasses?

2. And this only describes average speed – what about the distribution?

3. And how can we describe thermal equilibrium?

Definition 1

Heat capacity: the energy required for a unit change in temperature.

Lemma 1

The total energy of an ideal gas at temperature T is given by

$$U = NE = 3\frac{NkT}{2} \tag{1.7}$$

So, increasing U increases T

therefore can define heat capacity:

$$C_v \equiv \left.\frac{\partial U}{\partial T}\right|_{V,N,\dots} \tag{1.8}$$

This brings us to the expression for 1 mole of an ideal gas,

$$C_V = \frac{3R}{2} \qquad R = 8.314 J/(K \cdot mol) \tag{1.9}$$

Recall $Nk = nR$, because chemistry/physics notation

And this is experiemntally designed by heating an known insulated mass of gas. This estimation works best for lighter gasses, especially for more massive & diatomic molecules.

Theorem 1

Equipartition Theorem

$$U = N\overline{E} = \frac{3}{2}Nmv^2 = 3N\frac{kT}{2} \tag{1.10}$$

So the total energy is the total quadratic degrees of freedom D multiplied by $\frac{kT}{2}$. For a gas, $D = dN$, where d is the *dof* [] per particle
Therefore,

$$U = d\frac{nRT}{2} \tag{1.11}$$

Degrees of freedom

Given that

- $\overline{E} = \frac{3}{2}kT$

- There are three degrees of freedom, each of which having kinetic energy $\frac{1}{2}mv_a^2$, where $a \in \{x, y, z\}$

This is exactly what we said earlier, however, the expression changes depending on the number of degrees of freedom the particle has

Now, armed with $C_V = \frac{dR}{2}$ we notice that for Helium, Argon, and Neon this works well (and they have *dof* ≈ 3). Also notice that the ones for which it breaks on are primarily diatomic or massive molecules, e.g. O_2

But there are actually 3 translational degrees of freedom and 3 rotational degrees of freedom. So why don't all gases have $d = 6$? But our prior analysis indicates $d = 3$ for monoatomic and $d = 5$ for diatomic

From quantum mechanics the rotational energy levels for a rigid object are

$$E_J = \frac{\hbar}{2I} J(J+1) \tag{1.12}$$

where

- $\hbar$ is the reduced planck's constant

- J isthe quantum number

- $I = \frac{2}{5}mr^2$ is the moment of inertia, in this case we assume a solid sphere

So if we take the Equipartition theorem (Eq. 1.10) to heart, we see that the energy in the rotational modes is entirely irrelevant; it requires temperature at the order of a billion Kelvin to reach the first excited state. At room temperature the modes are "frozen out"

Applying Eq. 1.12 to a diatomic molecule e.g. N_2 we note that the nuclei are seperated by a *much* greater distance. If we were to imagine the diatomic molecule it becomes apparent that I is much larger along two axes than one, $\Rightarrow$ only 2 of these modes matter at room temperature and therefore diatomic molecules have $3 + 2$ degrees of freedom [at room temperature]. This also has the implication that C_V is temperature-dependent

1.2.1   Idealized Solids

Assuming a solid is modelled as a bunch of atoms attached with springs, we arrive at $E = \frac{1}{2}mv^2$. But when we look at solids (Fig. 1) we note that they have this plateau effect.
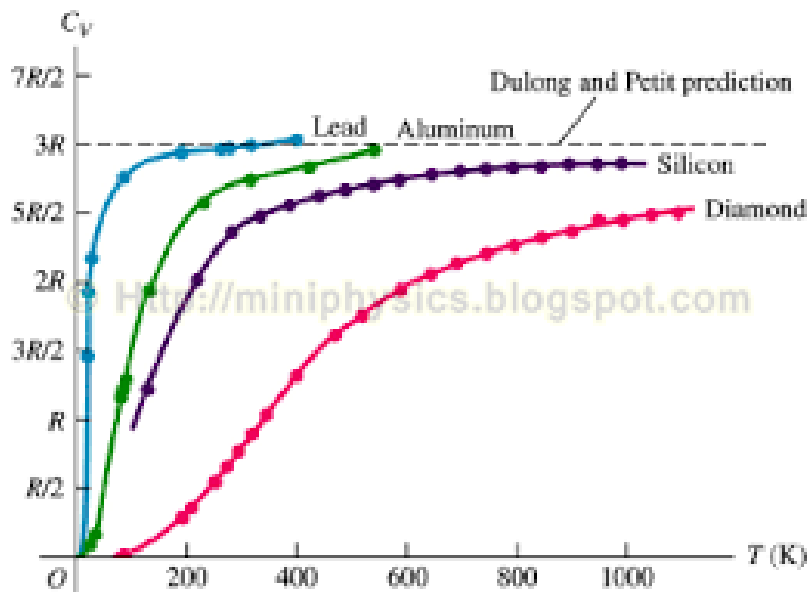

Figure 1. Heat capacity of selected solids

This is explained by there being 3 potential degrees of freedom in the spring[2].   [2] ...

Bu basically what the Equipartition thereom tells us is the energy stored in each quadratic degree of freedom. But we don't always know what the degrees of freedom are. For example how do we know what should we count or not?

Remark   Heat capacity at a constant volume is really easy to calculate, but is difficult to measure since solids tend to thermally expand. Heat capacity at a constant press is really easy to measure, but difficult to calculate.

So, to recap:

- The Equipartition thereom seems pretty powerful; it can predict energy per mode as a function of temperature and calculate a bunch of things

- BUT:

    – We didn't really prove it

    – Doesn't really work when quantum levvels matter

    – 'Degrees of freedom' is not well-defined

    – Works only in average energy

    – Doesn't explain equilibrium

    – Still don't know what entropy is

Let's try to address all of these shortcomings by applying quantum statistical mechanics.

SUBSECTION 1.3
Lecture 3: Two State Systems

Begin with coins; heads/tails with equal likelihood, independent flips.

Definition 2   To describe this better we will define a few terms:
Microstate: a specific configuration; all of which are equally likely, i.e. individual head/tails outcomes
Macrostate: Defined by some combined quantity, possibly containing multiple microstates, i.e. a state w/ 2/3 heads
Multiplicity (Ω): The number of microstates in a macrostate

For example, if we were to flip 100 coins.

- Multiplicity of 0 heads: $\Omega(0) = 1$

- Multiplicity of 1 heads: $\Omega(1) = 100$

- Multiplicity of 2 heads: $\Omega(2) = \frac{100 \cdot 99}{2}$

- Multiplicity of n heads: $\Sigma(N, n) = \binom{N}{n} = \dfrac{N!}{n! \cdot (N-n)!}$

    – The probability of n heads after N flips: $\dfrac{\Sigma(N,n)}{\Sigma_{\text{tot}}} = \dfrac{\Sigma(N,n)}{2^N}$
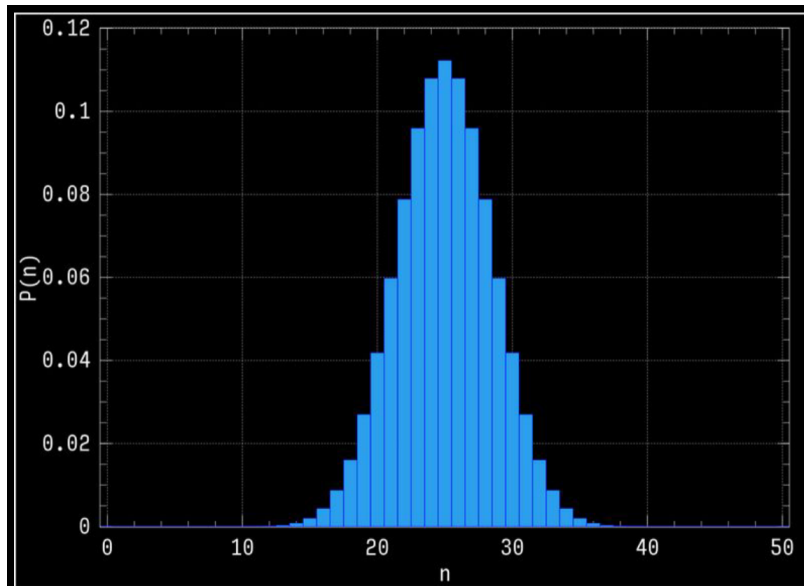
Figure 2. Probability distribution for 50 coin flips

Now, imagine a system of little independent magnetic dipoles that can point up or down in an external magnetic field (Fig. 3) The multiplicity is given by the expression we found before:

$$\Sigma(N, n) = \binom{N}{n} = \frac{N!}{n! \cdot (N-n)!} \tag{1.13}$$

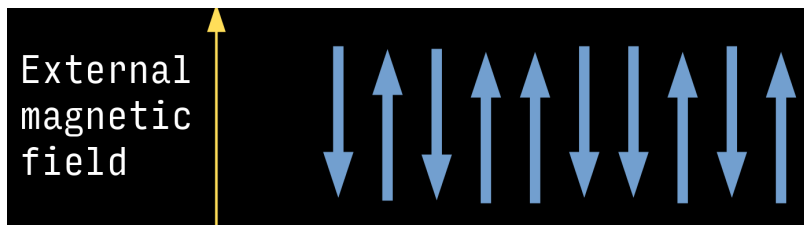As there are more flips the distribution will become *relatively* more narrow

Figure 3. Independent dipoles in field

Equal numbers of dipoles pointed up and down has a far higher multiplicity and therefore is the most probable state although it is not the lowest energy state. On the other hand, the lowest energy state would be all of the dipoles being aligned with the magnetic field, and highest being all anti-aligned. How can resolve this problem?

Subsection 1.4
Lecture 4: Einstein Solids

Figure 4. Einstein solid

Moving away from the contrived dipole example we move to the Einstein solid which is basically a bunch of harmonic oscillators as a means to finding out what temperature is.

Comment
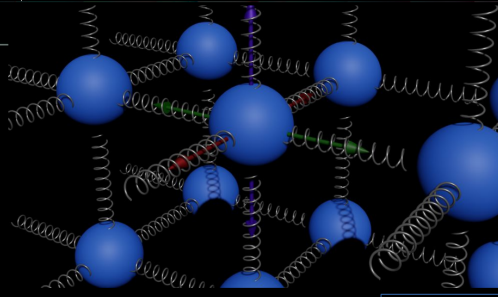
Recall: the quantum energy levels of a harmonic oscillator is

$$E = (q + \frac{1}{2})\hbar\sqrt{k/m} \tag{1.14}$$

Where

- $w = 2\pi f$

- $\hbar = \frac{h}{2\pi}$

- $\hbar w = hf$

By inspection we see that each oscillator has evenly spaced energy levels.

$$E - E_o = q\hbar\sqrt{k/m} \longrightarrow q\hbar\sqrt{k/m} \tag{1.15}$$

Each identical independent oscillator has an integer quanta of energy. This means that for a 3-oscillator system there are 6 ways 2 quanta can be distributed is $(2,0,0), (0,2,0), (0,0,2), (1,1,0), (0,1,1), (1,0,1)$ and so on[3]. The **macrostate** is the total energy in the system, $q_{tot} = 2$. The **microstate** is the configuration, i.e. $(1,1,0)$. So, the **multiplicity** of the macrostate $q_{tot} = 2$ is $\Omega(2) = 6$. Going onwards $\Omega(3) = 10$ and so forth.

Ground state is a constant so we ignore it

[3] *The nice thing about statistical mechanics is that each step is fairly trivial*
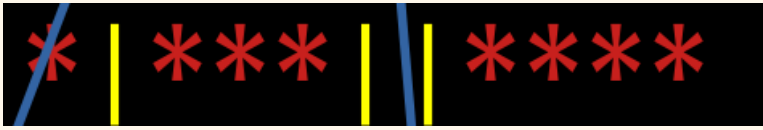
Theorem 2

The multiplicity for q quanta of energy distributed between N oscillators is

$$\Omega(N, q) = \binom{q + N - 1}{q} = \frac{q + N - 1}{q!(N-1)!} \tag{1.16}$$

Proof | Imagine $N = 4$ oscillators and $q = 8$ quanta. Consider the configuration $(1, 3, 0, 4)$. We can draw it like this:

where * is a quanta of energy and | separates the oscillators. Any combination of 8 of * and 3 of | is a valid microstate.

So: There are $q + N - 1$ symbols, q of which are a $*$

TLDR: cute proof!   □

By inspection we find that the multiplicity grows *very* rapidly with increasing N or q.
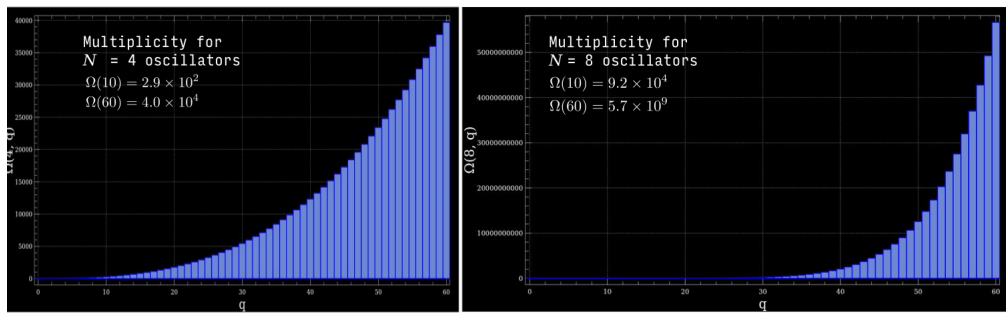
Figure 5. Multiplicity Ω of Einstein solid given N, q

Now, can we understand temperature by making Einstein solids interact with each other? Imagine two Einstein solids with $N = 4$ oscillators each and $q_{tot} = q_1 + q_2$ quanta of energy between them. How do the macrostates and multiplicity look?

Note: if solid 1 has $q_1 = 10$, then solid 2 has $q_2 = 60 - 10 = 50$
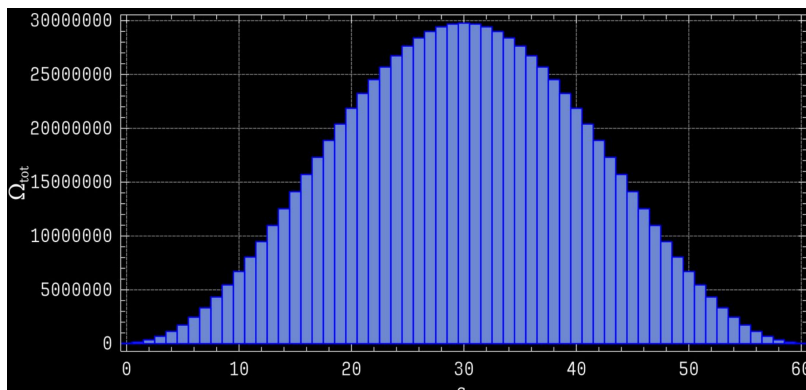
Figure 6. $N_1 = N_2 = 4, q_1 + q_2 = 60, \Omega_{tot} = \Omega(q_1) \times \Omega(60 - q_1)$

Since they share energy, it follows that the total multiplicity of the system is $\Omega_{tot} = \Omega_1 \times \Omega_2$. Plotting this we find that the multiplicity will tend to follow something that looks a bit like a normal distribution, the spread of which depends on the N and q. As this gets larger and larger we gain more and more certainty as to the system's energy – that the system with the greatest multiplicity (most likely) has an energy level which is in-between the energies of the interacting bodies.
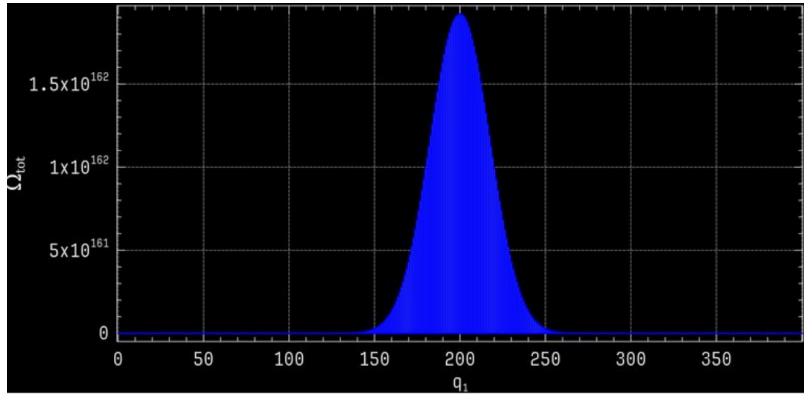
| Comment | This begins to look like a plausible explanation for thermal equilibrium! |

SUBSECTION 1.5
Lecture 5: Large Numbers

Let's consider a mole of einstein solid 1.8×10^{24} at normal temperatures $q = 10^{26}$. If one were to try to calculate what we did in the previous section it would just not work, especially computationally. So let's apply a few methods to model this

Types of numbers

1. Small numbers: e.g. 42 which we can use arithmetic as normal

2. Large numbers: can ignore adding a small number $10^{24} \approx 10^{24} + 24$

3. Very large numbers: can ignore multiplying a very large number by a large number, i.e. $10^{10^{23}} \times 10^{23} \approx 10^{10^{23}}$

| Theorem 3 | **Stirling's approximation** |

$$N! \approx \sqrt{2\pi N}(\frac{N}{e})^N \qquad N \gg 1 \tag{1.17}$$

which also gives us the result (for large N)

$$\ln(N!) = Nln(N) - N \tag{1.18}$$

Can use this to find $\Omega(N, q)$

$$ln(\Omega) \approx Nln(\frac{q}{N}) + N \qquad q \gg N, N \gg 1$$
$$\Omega \approx \exp\left\{\frac{eq}{N}\right\}^N \tag{1.19}$$

More formally,

$$\Omega(N, q) = \binom{q + N - 1}{q} \approx \frac{(q + N)!}{q! N!}$$

$$ln(\Omega) \approx ln((q + N)!) - ln(q!) - ln(N!)$$

Use $ln(N!) \approx N ln(N) - N$

$$\approx (q + N) ln(q + N) - (q + N) - q ln(q) + q - N ln(N) + N$$

$ln(1 + x) \approx x \qquad x \ll 1$

$$= (q + N) ln(q + N) - q ln(q) - N ln(N)$$

$$= (q + N)(ln(q) + \frac{N}{q}) - q ln(q) - N ln(N) \tag{1.20}$$

assume $q \gg N$

$$\Rightarrow ln(\Omega) = N ln(\frac{q}{N}) + N + \frac{N^2}{q}$$

$$\Rightarrow \Omega = (\frac{eq}{N})^N$$

From our intuition we expect that when we combine two multiplicities that the peak will be at $q_1 = q_2 = q_{tot}/2$...

Let $x = q_1 - \frac{q}{2} = \frac{q}{2} - q_2$

$$\Omega_{tot} = (\frac{e}{N})^{2N}[(\frac{q}{2})^2 - x^2]^N \tag{1.21}$$

And then we do some math [textbook 2.26, 2.27] and find that it is a gaussian

Definition 3

$$\Omega_{tot} \approx \Omega_{max} e^{-N(2x/q)^2}$$

$$\Omega_{max} = (\frac{e}{N})^{2N}(\frac{q}{2})^{2N} \tag{1.22}$$

And the varience can be found for large N, which becomes increasingly thin for large N.

if $N = 10^{24}, \sigma = \frac{q}{2 \times 10^{12}}$

$$\sigma = \frac{q}{2\sqrt{N}} \tag{1.23}$$

1.5.1   Entropy

Definition 4

$$S \equiv k \ln(\Omega) \tag{1.24}$$

And we find that systems in thermal cntact will tend to be found with the highest entropy – *Entropy increases.*

This is the second law of thermodynamics!

ECE259
Electromagnetism

SECTION 2
Dielectrics

SUBSECTION 2.1
Lecture 15: Boundary Conditions for Dielectrics

Remark Recall, at a conductor/free-space interface, $E_+ = 0$, $E_n = \frac{\rho_s}{\varepsilon_o}$

Consider an interface between two generic dielectrics (Fig. 7).

Theorem 4

$$\oint_c \vec{E} \cdot \vec{dl} = 0 \tag{2.1}$$
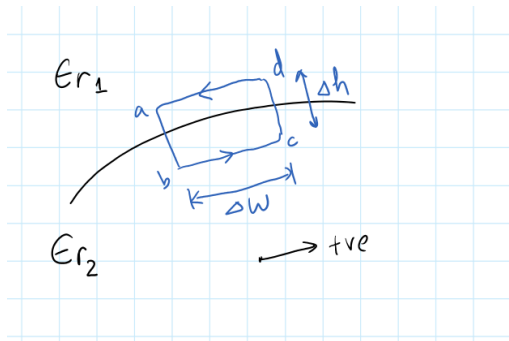
Figure 7. Dielectric interface

Intuition The two integrals parallel to the dielectrics will cancel out, and so will those perpendicular.

This implies that the tangential component of the $\vec{E}$ field is continuous across the boundary,

$$E_{1t} = E_{2t} \tag{2.2}$$

However we get a bit of a different result when working in 3 dimensions where the interface is a surface instead of a line.

Theorem 5

$$\oint_S \vec{D} \cdot \vec{ds} = \rho_S \Delta \rightarrow (\vec{D_1} - \vec{D_2}) \cdot \vec{a_{n2}} = \rho_S \tag{2.3}$$

a_n denotes normal component

Note: $\vec{D} = \varepsilon_o \vec{E} + \vec{p} = \ldots \varepsilon \vec{E}$

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Proof | As $\Delta h \to 0$

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{top} \vec{D}_1 \cdot d\vec{s} + \int_{bottom} +\vec{D}_2 \cdot d\vec{s}$$
$$= \vec{D}_1 \cdot \vec{a_{n2}}\Delta S + \vec{D} \cdot \vec{a_{n1}}\Delta S \qquad (2.4)$$
$$= \vec{D}_1 \cdot \vec{a_{n2}}\Delta S - \vec{D} \cdot \vec{a_{n2}}\Delta S$$
$$= \rho_S \Delta S$$

$\square$

Solving these problems usually involves finding the tangential and normal components through Eq. 2.2 and 2.3 then applying Pythagoras.

Subsection 2.2
Lecture 16: Capacitors

Definition 5 | A capacitor is a device consisting of two isolated conductors for storing energy in the form of electrostatic potential energy. An isolated conductor can also have "capacitance" if the other conductor is far away. The charge of a capacitance refers to the charge on one conductor. The energy stored in a capacitor is equal to the energy it takes to charge a capacitor from a discharged state to a charged state.
A capacitor's *capacitance*[4] is defined as

$$C = \frac{Q}{V} \qquad (2.5)$$

and has units $[C] = \frac{C}{V} F$ [Farads]

[4]*Capacitance is actually independent of Q and V and is dependent only on the physical attributes of [the capacitor]*

Capacitance is calculated as follows

1. choose a coordinate system

2. Assume $+Q/-Q$ on the conductors

3. Find $\vec{E}$ from Q distribution

4. Find $V_{AB} = \int_A^B \vec{E} \cdot d\vec{l}$ where A carries the negative charge and B carries positive.
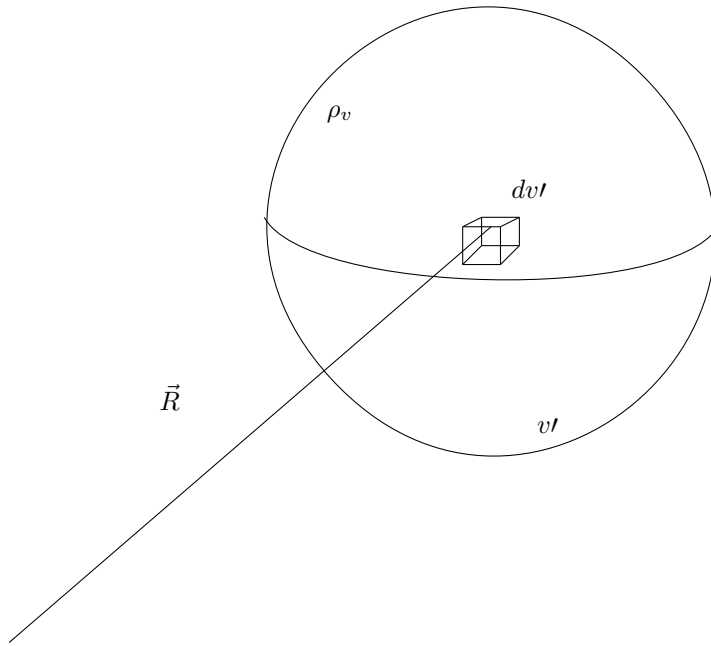
5. Apply $C = \frac{Q}{V}$

Subsection 2.3
Lecture 17: More on energy and capacitors

Remark | Recall, the potential energy stored in the system is given by

$$W_e = \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv' \qquad (2.6)$$ Can extend v' to entire space

Applying generalized gauss's law and performing an integral across the space we find that

$$W_e = \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv'$$

$$= \frac{1}{2} \int_{v'} (\vec{\nabla} \cdot \vec{D}) V(\vec{R}) dv'$$

$$= \frac{1}{2} \int_{v'} \vec{\nabla} \cdot (\vec{D}V) dv' - \frac{1}{2} \int_{v'} \vec{\nabla} \cdot \vec{D} V dv' \qquad (2.7) \quad \text{Apply divergence thereom}$$

$$= \frac{1}{2} \oint_{S'} \vec{D}V \cdot d\vec{s} + \frac{1}{2} \int_{V'} \vec{D} \cdot \vec{E} dv'$$

$$\Rightarrow W_e = \frac{1}{2} \int_v \vec{D} \cdot \vec{E} dv \qquad\qquad\qquad\qquad \text{The first term} \to 0 \text{ as } R \to \infty$$

Therefore the energy density can be defined as

Definition 6 | Given energy density w_e with units $[J/m^3]$

$$W_e = \int_v w_e dv \qquad w_e = \frac{1}{2}\vec{D} \cdot \vec{E} \qquad (2.8)$$

Remark   Things to know about capacitors

- Stacking them puts them in series. This looks like the inverse of resistors, i.e.

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \dots \qquad (2.9)$$

- Putting them besides each other puts them in parallel.

$$C = C_1 + C_2 \dots \qquad (2.10)$$

Lecture 19: Poisson's Equation

Definition 7 | **Poisson's Equation**
$$\nabla \cdot (\varepsilon \nabla V) = \rho_v \tag{2.11}$$

And for no charge or $\rho_v = 0$, we have a special case: the 'laplace equation'

$$\nabla \cdot (\varepsilon \nabla V) = 0 \tag{2.12}$$

Using this we can solve for potential as a boundary value problem. For a surface S enclosing a volume v with ρ_v , solving the following gives us the potential difference V inside v .

$$\begin{cases} \nabla \cdot (\varepsilon \nabla V) = \rho_v & \text{in } v \\ V_s = V_o \end{cases} \tag{2.13}$$

Lecture 20: Method of Images

Remark   TLDR: Add imaginary charges such that the field lines meet perpendicular to the planes/lines of interest
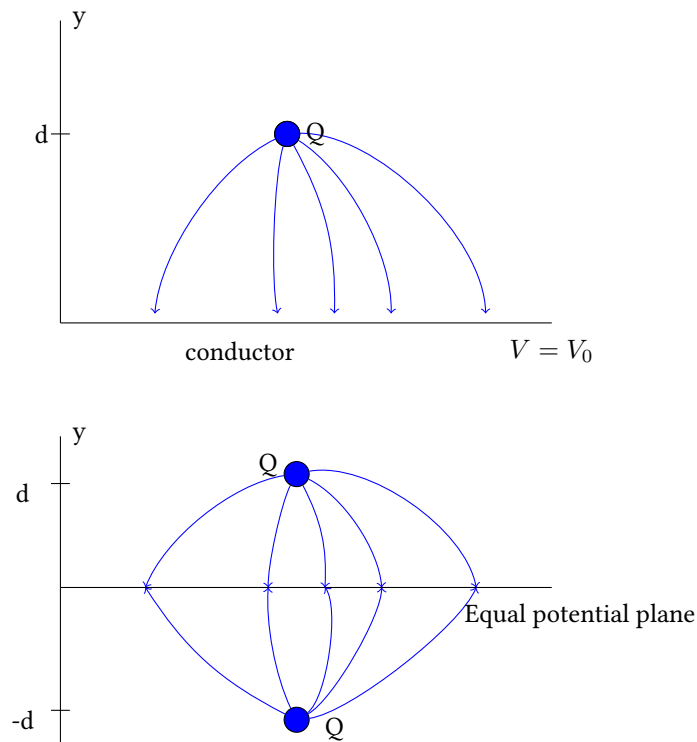
And for the upper-half space ($y > 0$) the two cases have the same source distribution and the same boundary condition on the conducting surface.

Example | A charge Q is at $(d1, d2)$ relative to two conductor walls. Find force on charge Q due to induced caharges on conductor surfaces.

(2.14)
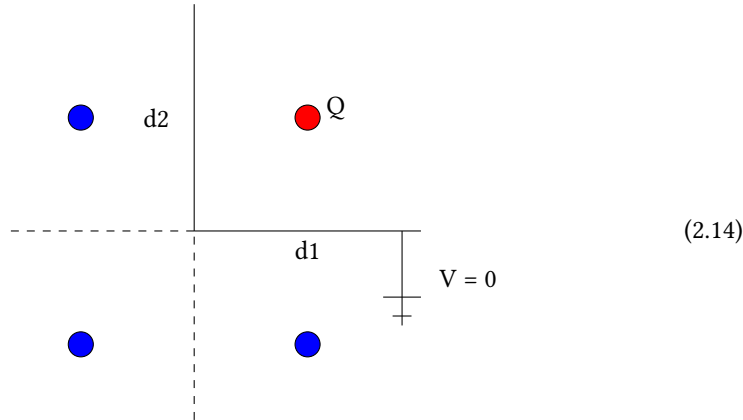
We have to be careful about where to put these virtual charges. They are arranged this way so that there is a right angle at the conductors; that's why we don't just put a single charge in the bottom left quadrant.

The force can be found as follows:

$$\vec{F} = \frac{Q}{4\pi\varepsilon}\left[\frac{-Q}{2d_1^2}\hat{x} + \frac{-Q}{2d_2^2}\hat{y} + \frac{Q(2d_x\hat{x} + 2d_2\hat{y})}{[(2d_1)^2 + (2d_2)^2]^{3/2}}\right]$$

(2.15)

2.5.1   Currents

There are a few types of currents:

- Conduction current: In conductors and semiconductors due to the motion of e^- and holes h [5]

- Electrolytic current: motion of ions in solutions

- Convection current: motion of e^- or ions in vacuum

In conductors and semiconductors the free charge is

$$e = 1.6 \times 10^{-19} C$$

(2.16)

In the presence of an external field E,

$$\begin{cases} \vec{E} = 0 & \text{random motion of free charges, current = 0} \\ \vec{E} > 0 & \vec{\mu}_e = -\mu_o\vec{E} [6] \end{cases}$$

(2.17)

[5] We will primarily focus on conduction currents

[6] $\vec{\mu}$ = average drift velocity, μ = mobility

Typical $\vec{\mu}_e$ is very slow: $\approx 10^{-5}10^{-4}\frac{m}{s}$

Definition 8 | General integral form for current

$$I = \int_S \rho_e\vec{\mu}_e \cdot d\vec{s}$$

(2.18)

PROOF | Current is the amount of charge through a surface S per unit time

$$d\vec{S} = \vec{a_n}dS$$

The free charge in the volume of

$$\vec{\mu} \cdot \vec{a_n}dtdS$$

(2.19)

will pass S in dt time

$$dq = -N_e e \vec{\mu}_e \cdot \vec{a_n} dt ds \tag{2.20}$$

Apply definition of current

The change in current volume is the no. of electrons per unit volume times volume

$$dI = \frac{dq}{dt} = -N_e e \vec{\mu}_e \cdot d\vec{s} = \rho_e \vec{\mu}_e \cdot d\vec{s} \tag{2.21}$$

and integrate!

$$I = \int_S \rho_e \vec{\mu}_e \cdot d\vec{s} \tag{2.22}$$

□

This integral form can be a little awkward to work with so we define current density J

Definition 9

Define current density to be $\vec{J}$ such that

$$I = \int_S \vec{J} \cdot d\vec{s} \tag{2.23}$$

and therefore the volume current density $\vec{J}$ with units $\frac{A}{m^2}$

$$\vec{J} = \rho_e \vec{\mu}_e \tag{2.24}$$

- $\vec{J}$ is a vector and direction is the same as $\vec{\mu}_e$ and **is** a point function

- I is a scalar and **is not** a point function

Now ohm's law can be defined more rigourously in point/microscopic form

$$\vec{J} = \rho_e \vec{\mu}_e = -\rho_o \mu_e \vec{E}$$
$$\Rightarrow J = \sigma \vec{E} \tag{2.25}$$

σ denotes conductivity

SUBSECTION 2.6
Lecture 21: Joule's Law and Resistance

Comment Previously we discussed perfect conductors. But what if that was not the case?

Theorem 6   **Joule's Law**

$$P = \int_v dp = \int_v \vec{E} \cdot \vec{J} dv \tag{2.26}$$

Intuition: Work done by E-field is turned into kinetic energy of e^- and produces heat

PROOF | Work done by E field on one e^-: $\Delta W = -e\vec{E}\Delta l$
Power per unit time: $p = \frac{\Delta W}{\Delta t} = -e\vec{E}\frac{\Delta l}{\Delta t} = -e\vec{E}\cdot\vec{\mu_e}$

Recall: μ is the average drift velocity

$$dp = N_e(-e)\vec{E}\cdot\vec{\mu_e}dv = \rho_v\vec{E}\cdot dv \Rightarrow dp = \vec{E}\cdot\vec{J}dv \qquad (2.27)$$

Therefore the dissapated power density $\frac{dp}{dv} = \vec{E}\cdot\vec{J}$
And then one can integrate over the volume to find the total dissapated power; joule's law

Recall: $\rho_v\vec{\mu_e} = \vec{J}$

$$P = \int_v dp = \int_v \vec{E}\cdot\vec{J}dv \qquad (2.28)$$

Noting that we can take the volume integral with respect to a surface and a length, i.e. $dv = dlds$, we can write this integral as

$$P = \underbrace{\left(\int_l \vec{E}\cdot d\vec{l}\right)}_{\text{voltage drop}} \underbrace{\left(\int_S \vec{J}\cdot d\vec{s}\right)}_{\text{current}} \qquad (2.29)$$

$\square$

This enables us to define resistance R.

Definition 10 | **Resistance**

$$R = \frac{V}{I} = \frac{\int_l \vec{E}\cdot d\vec{l}}{\int_S \vec{J}\cdot d\vec{s}} = \frac{\int_l \vec{E}\cdot d\vec{l}}{\int_S \sigma\vec{E}\cdot d\vec{s}} \qquad (2.30)$$

Resistance is independent of the voltage and current that passes through it and instead is dependent on the physical properties of the resistor

Example | For example, assuming uniform $\vec{E}, \vec{J}$,

$$R = \frac{\int_l \vec{E}\cdot d\vec{l}}{\int_S \vec{J}\cdot d\vec{s}} = \frac{El}{\sigma E\cdot S} = \frac{l}{\sigma S} \qquad (2.31)$$

observe that resistance is proportional to the resistor length and dimensions l, S as well as the material properties σ.

This makes intuitive sense, no?

The steps to calculate resistance are as follows:

1. Choose coordinate system

2. Assume V_s: potential drop between terminals

3. Find $\vec{E}$ from V. This can get complicated; need to start with the laplace equation $\nabla^2 V = 0$ and solve for $\vec{E}$ everywhere using the fact that $\vec{E} = -\nabla V$

4. Find current: $I = \int_S \vec{J}\cdot d\vec{s} = \int_S \sigma\vec{E}\cdot d\vec{s}$

5. Apply definition $R = \frac{V}{I}$

ECE286
Probability and Statistics

SECTION 3

Probability Distributions

SUBSECTION 3.1
Lecture 14: functions of random variables

In the discrete case, given X with PMF $f(x)$, we can define an *invertible* function Y where $Y = u(X)$, therefore can write $X = u^{-1}(Y)$. If $g(y)$ is the distribution of Y

$$
\begin{aligned}
g(y) &= P(Y = y) \\
&= P(u^{-1}(Y) = u^{-1}(y)) \\
&= f(u^{-1}(y))
\end{aligned}
\tag{3.1}
$$

In the continuous case we may arrive at

$$
g(y) = f(u^{-1}(y))\left|\frac{du^{-1}(y)}{dy}\right|
\tag{3.2}
$$

This result is derived through the Leibniz integral rule, $g(y) = \frac{d}{dy}\int_{-\infty}^{u^{-1}(y)} f(t)dt$

Definition 11 The rth moment about the origin of the random variable X is

$$
\mu'_r = E[X^r] = \begin{cases} \sum_x x^r f(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x^r f(x)dx & X \text{ continuous} \end{cases}
\tag{3.3}
$$

- The mean is the first moment

- For variance, $\sigma^2 = E[X^2] - \mu^2 \rightarrow \sigma^2 = \mu'_2 - \mu^2$

Definition 12 The moment-generating function of X is defined as

$$
\mu'_r = E[X^{tX}] = \begin{cases} \sum_x e^{tx} f(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx & X \text{ continuous} \end{cases}
\tag{3.4}
$$

In general

$$
\mu'_r = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}
\tag{3.5}
$$

SUBSECTION 3.2
Lecture 15: More on moment generating functions

By definition and completing the square,

17

$$M_X(T) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{t2\pi}\sigma e^{-\frac{(x-\mu)^2}{2\sigma^2}}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t2\pi}\sigma e^{-\frac{x^2 - 2(x-\mu)^2 + \mu^2}{2\sigma^2}}} dx$$

$$= e^{\frac{2\mu t + t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}\right\} dx$$

$$= e^{\frac{\mu t + t^2\sigma^2}{2}} \quad (3.6)$$

The integrand is just a normal PDF and thus integrates to one

3.2.1   Linear combinations of random variables

Given random variable X with distribution $f(x)$. What is the distribution, $h(y)$, of $Y = aX$?

Definition 13

> Probability distribution of linear combination of random variables
> **Discrete**:
>
> $$P(X = x) = f(x) \rightarrow h(y) = P(Y = y)$$
> $$= P(aX = y) \quad (3.7)$$
> $$= f(\frac{y}{a})$$
>
> **Continuous**:
>
> $$F(y) = P(Y \le y)$$
> $$= P(X \le \frac{y}{a})$$
> $$= \int_{-\infty}^{\frac{y}{a}} f(t)dt \quad (3.8)$$
>
> $$\int_{-\infty}^{\frac{y}{a}} f(t)dt = \int_{-\infty}^{y} \frac{1}{a}f(\frac{s}{a})ds$$
> $$\Rightarrow h(y) = \frac{1}{|a|}f(\frac{y}{a})$$

u-sub; let $s = at$

Extending from this, let's assume that X has the MGF [7] $M_X(t)$. Then,

[7] *Moment Generating Function*

$$M_Y(t) = \int_{-\infty}^{\infty} e^{ty} h()$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{ty} f(\frac{y}{a})dy$$

u-sub $z = \frac{y}{a}$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{taz} f(z)a\,dz \quad (3.9)$$

$$= \int_{-\infty}^{\infty} e^{taz} f(z)dz$$

$$= M_X(at)$$

More generally,

$$M_{aX}(t) = M_X(at) \quad (3.10)$$

When working with more than one random variables, i.e. independent RV X, Y with distributions $f(x), g(y)$, the distribution of $Z = X + Y, h(z)$ is given by

$$P(Z = z) = P(X + Y = z) = \sum_w P(X = w)(Y = z - w) \Rightarrow h(z) = \sum_{w=-\infty}^{\infty} f(w)g(z - w)$$

(3.11)

And for the continuous case we arrive at the convolution integral,

$$h(z) = \int_{-\infty}^{\infty} f(w)g(z - w)dw$$

(3.12)

SUBSECTION 3.3
Lecture 16: Sampling

Given sample data $x_1, \ldots, x_n$, each x_i is the realization of an RV [8]., $X_i, i = 1, \ldots, n$

[8] Say each X_i is a Bernoulli random variable

Definition 14   Given the mean,

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

(3.13)

the **sample mean** of a random variable X is defined as

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

(3.14)

Definition 15   The **sample median** is

$$x_m = \begin{cases} \dfrac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2} & \text{if n even} \\ \dfrac{x_{n+1}}{2} & \text{if n is odd} \end{cases}$$

(3.15)

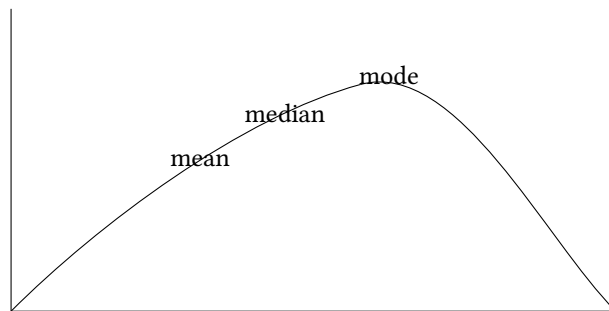and **mode** is familiar as the most frequently occurring value

Figure 8. Median, mode, mean labelled on a skew distribution

Definition 16   **Sample variance**

$$s^2 = \frac{1}{n - 1} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

(3.16)

and the corresponding standard deviation [9]

$$s = \sqrt{s^2} \tag{3.17}$$

Lemma 2 Consider

$$E[\overline{X}] = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] \tag{3.18}$$

Assuming that X_i have the same distribution, $f(x_i)$, and mean μ,

$$\begin{aligned} &= \frac{1}{n}\sum_{i=1}^{n} \mu \\ &= \frac{1}{n}n\mu \\ &= \mu \end{aligned} \tag{3.19}$$

- the X_i are identically distributed
- $\overline{X}$ is an unbiased estimator of μ

We then have

$$\begin{aligned} S^2 &= \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 + \overline{X}^2 - 2X_i\overline{X} \\ &= \frac{1}{n-1}(n\overline{X}^2 - 2n\overline{X}^2 + \sum_{i=1}^{n} X_i^2) \\ &= \frac{1}{n-1}(-n\overline{X}^2 + \sum_{i=1}^{n} X_i^2) \tag{3.20} \end{aligned}$$

We are now ready to take the expectation of S^2 , which is

Recall: if Y has mean, varience μ, σ^2, then $\sigma^2 = E[Y^2] - \mu^2$

$$\begin{aligned} E[S^2] &= \frac{1}{n-1}(-nE[\overline{X}^2] + \sum_{i=1}^{n} E[X_i^2]) \\ &= \frac{1}{n-1}(-n(E[\overline{X}^2] + var[\overline{X}]) + \sum_{i=1}^{n} \sigma^2 + \mu^2) \\ &= \frac{1}{n-1}(-n(\mu^2 + var[\frac{1}{n}\sum_{i=1}^{n} X_i]) + n(\sigma^2 + \mu^2)) \\ &\vdots \\ &= \frac{n-1}{n-1}\sigma^2 = \sigma^2 \tag{3.21} \end{aligned}$$

So we have an unbiased estimator of varience

So we get

$$S^2 = \sum_{i=1}^{n}(X_i - \overline{X_i})^2 \qquad (3.22)$$

We divide by $n - 1$ because there is a "hidden" uncertainty in $\overline{X}$

TEP327
Engineering and Law

Torts can be identified by

- Elements of negligence can be identified

- There exists a duty of care

- Was the standard of duty of care breached?

- Did that breach of standard cause damage to the plaintiff?

Comment | Must be careful about the standard of care met or not met. E.x. in the in-class bike example one could argue that in designing the gear the engineer should expect that the gear could be commercialized and mass-produced later on for mass-market. On the other hand it is reasonable to argue that having to design the gear to be used by unskilled 16-year-olds is an unreasonable and outside the standard of care.

Trespass is strict liability. Ignorance of the law is no excuse.

SECTION 4
Property

- There are a number of types of property, i.e. intellectual property, personal (usu. regarding money), etc

- Real property

  - Usu. owned by default by government (derived from old english law)
  - Gov. transferrs property rights to private owners; sell, lease, etc (fee simple)
  - Gov. retains a few rights on real property, usu. oil and mineral rights

- Joint ownership: death of one partner causes other to own 100%. Tenancy in common/joint tenancy: death of a partner does not cahnge ownership; usu. determined by wills

- Land interest is divisible; different people can own rights to different things on a piece of land. Or many people can own right to lease a property but not necessairly ownernship of the land

- Rights associated with real property:

  - Mortgage: property as security for a debt
  - License: Contractual right to use property in some way
  - Restricitive convenant: conditions of use for that property
  - Easement: right to use property in some way

- – Lien: right to register debt against land (must pay off people you owe with money from selling property before taking it for yourself)

- – Lease: right to occupy the property

- – Profit à prendre; right to extract value from property

Mineral, oil, gas rights:

- – Mineral rights are crown or "freehold"

- – Crown rights can be provincial or federal and can grant license to third party to extract resources

- – Rights to resources generally excluded from fee simple property rights

- – Freehold mineral rights were granted with early land grants

- – Resource rights usu. include a right to access to the land in order to extract resources

- Land is registered via the registry system/land title system. Registry system is old and only about 10% of land is in it, is complicated, and led to a lot of squatters. Land title system (ontario) is easy to use.

- **chattels** are tangible personal property called 'goods'. No registry for ownership but is registration of security interests (which demonstrates rights based on order of registration).

SUBSECTION 4.1
Intellectual Property

- A bundle of right that protect expression of ideas, inventions, symbols, and more

- Governed by federal legislation and international treaties

- Has a few different types.

- Copyright

  - – Must be an original product of skill, permanent, and published

  - – Ephemeral works e.g. live works have no copyright

  - – Most things are copyrightable.

  - – Copyright protection is valid for about 50 years after death, soon to be 70.

  - – Protection is not dependent on registration, but copyrights can be registered

  - – Copyright can be assigned by contract, but moral rights cannot be [although they can be waived]

- Trade secrets

  - – What todo if you invent something?

  - – Keep your mouth shut

  - – Not everything is patentable and not all things are patentable

  - – Trade secret is information that is kept secret with an industrial or commercial application

  - – Once the secret is out it's not possible to make it a secret again

  - – Reverse engineering is a totally valid way to discover a trade secret

- Patent

    - A right granted by the crown to a monopoly
    - Term of patent in Canada is 20 years, non-renewable
    - A balance between the interests of the inventor and that of society
    - A few things that cannot be patented; scientific principles, abstract theorems, higher life forms (animals).
    - A patent must be new, useful, and non-obvious
    - Must pass the obviousness test: 1) Identify a PSITA (Personal skilled in the art) 2) Identify inventive concept 3) Find difference between prior art and inventive concept 4) Ask: Would PSITA find this obvious?
    - A patent will have a specification: a description of the procedure/invention, how to reproduce it, etc.
    - ... and claims
        * Clear and explicit definition of the subject matter
        * Will inform the scope of the rights granted
    - Infringement is "Use by someone other than the patent holder which intefers in whole or in part with the patent holder's monopoly" (Monsanto Canada Inc. v. Schmeiser, 2004 SCC 34)
    - Usu. more to do with the right to monopoly than the IP itself.
    - Some exceptions: use of the invention for development and submission of information required by law, fair dealing (experimentation without eye to profit or fraudulent purpose), dedication (patent owner can ignore monopoly), exhaustion (once patent is sold, patent hoder may no longer control use)

- Trademark

    - A sign or combiation of signs that is used by a person for sake of distinguishing or so as to distinguish their goods and services from those of others
    - A certification mark
    - **trade name**: means the name under which any buisness is carried on, whether or not if it is a corporation, partnership, or individual

This is the case in Canada but not really in other places.

This also implies a patent can only have a single inventive concept