

ENGSCI YEAR 2 WINTER 2022 NOTES

FOR CLASS CONTENT AFTER READING WEEK

BRIAN CHEN

Division of Engineering Science, University of Toronto

Website

brianchen.chen@mail.utoronto.ca

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PHY294

Quantum & Thermal Physics

SECTION 1

Statistical Mechanics

SUBSECTION 1.1

Lecture 1: Ideal Gas Law

We begin our discussion on statistical mechanics with the guiding question, "What is temperature". We will first derive the ideal gas law $PV = nRT$, which most should be familiar with.

Imagining a single particle bouncing around elastically in 2D box with speed v_x . If we were to look at the sides of the box we will see intermittent spikes of force experienced [by the box walls],

$$F_x = m \frac{2v_x}{\Delta t_c} \quad (1.1)$$

Over time can get the average force

$$F_x = m \frac{2v_x}{\Delta t} \quad (1.2)$$

Which allows us to derive an expression for pressure, which is just force over area

$$P = \frac{F_x}{A} = \frac{mv_x^2}{V} \quad (1.3)$$

For n particles it then becomes

$$P = \sum_{i=1}^N \frac{mv_{x,i}^2}{V} = \frac{m}{V} \sum_{i=1}^N v_{x,i}^2 \quad (1.4)$$

Assuming that that $v_x = v_y = v_z$ ¹, we may extend this to 3 dimensions

¹This is reasonable, is it not?

$$P = \frac{mv^2}{3} \frac{N}{V} \quad (1.5)$$

Now, compare this expression and that of the ideal gas law to find that temperature is a measure of the average kinetic energy per particle

Note that $\frac{mv^2}{3} = kT \rightarrow \frac{mv^2}{3k} = 2\frac{\bar{E}}{3k}$

$$T = \frac{mv^2}{3k} = \frac{2\bar{E}}{3k} \quad (1.6)$$

SUBSECTION 1.2

Lecture 2: Temperature & Ideal Gases & Solids

Remark Recall, $PV = kT \frac{N}{V}$ where $T = \frac{mv^2}{3k} = \frac{2\bar{E}}{3k}$, $\bar{E} = \frac{3kT}{2}$ Where T is the experimentally measured temperature

We are still, however, left with a few questions:

1. But how does this apply to things that aren't gasses?
2. And this only describes average speed – what about the distribution?
3. And how can we describe thermal equilibrium?

Definition 1 **Heat capacity:** the energy required for a unit change in temperature.

Lemma 1 The total energy of an ideal gas at temperature T is given by

$$U = NE = 3 \frac{NkT}{2} \quad (1.7)$$

So, increasing U increases T

Therefore can define heat capacity:

$$C_v \equiv \left. \frac{\partial U}{\partial T} \right|_{V, N, \dots} \quad (1.8)$$

This brings us to the expression for 1 mole of an ideal gas,

$$C_V = \frac{3R}{2} \quad R = 8.314 \text{ J/(K} \cdot \text{mol)} \quad (1.9)$$

Recall $Nk = nR$, because chemistry/physics notation

And this is experimentally designed by heating an known insulated mass of gas. This estimation works best for lighter gasses, especially for more massive & diatomic molecules.

Theorem 1 **Equipartition Theorem**

$$U = N\bar{E} = \frac{3}{2} Nmv^2 = 3N \frac{kT}{2} \quad (1.10)$$

So the total energy is the total quadratic degrees of freedom D multiplied by $\frac{kT}{2}$. For a gas, $D = dN$, where d is the *dof* [] per particle

Therefore,

$$U = d \frac{nRT}{2} \quad (1.11)$$

Given that

- $\bar{E} = \frac{3}{2} kT$
- There are three degrees of freedom, each of which having kinetic energy $\frac{1}{2} mv_a^2$, where $a \in \{x, y, z\}$

Which gives us

$$C_V = \frac{dR}{2} \quad (1.12)$$

Degrees of freedom

For an ideal gas

This is exactly what we said earlier, however, the expression changes depending on the number of degrees of freedom the particle has

Now, armed with $C_V = \frac{dR}{2}$ we notice that for Helium, Argon, and Neon this works well (and they have $dof \approx 3$). Also notice that the ones for which it breaks on are primarily diatomic or massive molecules, e.g. O_2

But there are actually 3 translational degrees of freedom and 3 rotational degrees of freedom. So why don't all gases have $d = 6$? But our prior analysis indicates $d = 3$ for mono-atomic and $d = 5$ for diatomic

From quantum mechanics the rotational energy levels for a rigid object are

$$E_J = \frac{\hbar^2}{2I} J(J+1) \quad (1.13)$$

Where

- \hbar is the reduced Planck's constant
- J is the quantum number
- $I = \frac{2}{5}mr^2$ is the moment of inertia, in this case we assume a solid sphere

So if we take the Equipartition theorem (Eq. 1.10) to heart, we see that the energy in the rotational modes is entirely irrelevant; it requires temperature at the order of a billion Kelvin to reach the first excited state. At room temperature the modes are "frozen out"

Applying Eq. 1.13 to a diatomic molecule e.g. N_2 we note that the nuclei are separated by a *much* greater distance. If we were to imagine the diatomic molecule it becomes apparent that I is much larger along two axes than one, \Rightarrow only 2 of these modes matter at room temperature and therefore diatomic molecules have $3 + 2$ degrees of freedom [at room temperature]. This also has the implication that C_V is temperature-dependent

1.2.1 Idealized Solids

The previous derivation for gases *cannot* be strictly applied to solids.

Assuming a solid is modelled as a bunch of atoms attached with springs, we arrive at $E = \frac{1}{2}mv^2$. But when we look at solids (Fig. 1) we note that they have this plateau effect.

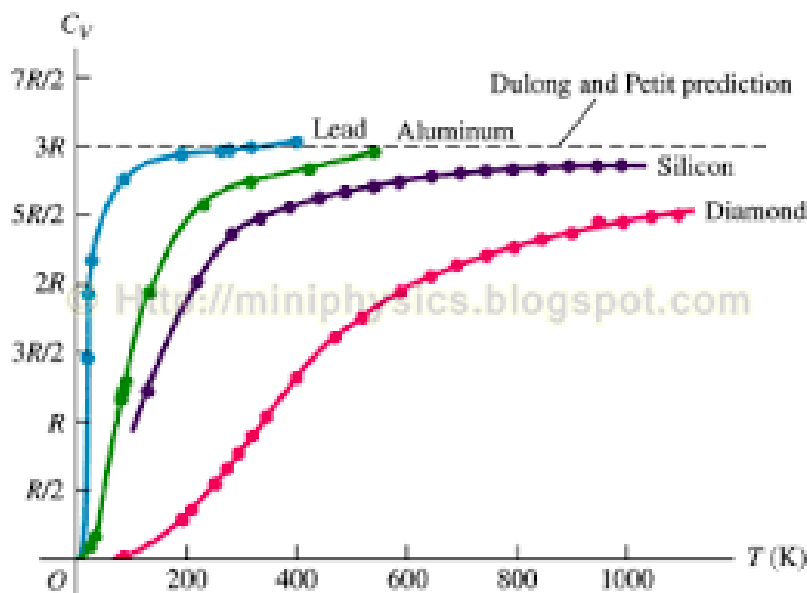


Figure 1. Heat capacity of selected solids

This is explained by there being 3 potential degrees of freedom in the spring².

² ...

But basically what the Equipartition theorem tells us is the energy stored in each quadratic degree of freedom. But we don't always know what the degrees of freedom are. For example how do we know what should we count or not?

Remark Heat capacity at a constant volume is really easy to calculate, but is difficult to measure since solids tend to thermally expand. Heat capacity at a constant pressure is really easy to measure, but difficult to calculate.

So, to recap:

- The Equipartition theorem seems pretty powerful; it can predict energy per mode as a function of temperature and calculate a bunch of things
- BUT:
 - We didn't really prove it
 - Doesn't really work when quantum levels matter
 - 'Degrees of freedom' is not well-defined
 - Works only in average energy
 - Doesn't explain equilibrium
 - Still don't know what entropy is

Let's try to address all of these shortcomings by applying quantum statistical mechanics.

SUBSECTION 1.3

Lecture 3: Two State Systems

Begin with coins; heads/tails with equal likelihood, independent flips.

Definition 2

To describe this better we will define a few terms:

Microstate: a specific configuration; all of which are equally likely, i.e. individual head/tails outcomes

Macrostate: Defined by some combined quantity, possibly containing multiple microstates, i.e. a state w/ 2/3 heads

Multiplicity (Ω): The number of microstates in a macrostate

For example, if we were to flip 100 coins.

- Multiplicity of 0 heads: $\Omega(0) = 1$
- Multiplicity of 1 heads: $\Omega(1) = 100$
- Multiplicity of 2 heads: $\Omega(2) = \frac{100 \cdot 99}{2}$
- Multiplicity of n heads: $\Sigma(N, n) = \binom{N}{n} = \frac{N!}{n! \cdot (N - n)!}$
 - The probability of n heads after N flips: $\frac{\Sigma(N, n)}{\Sigma_{\text{tot}}} = \frac{\Sigma(N, n)}{2^N}$

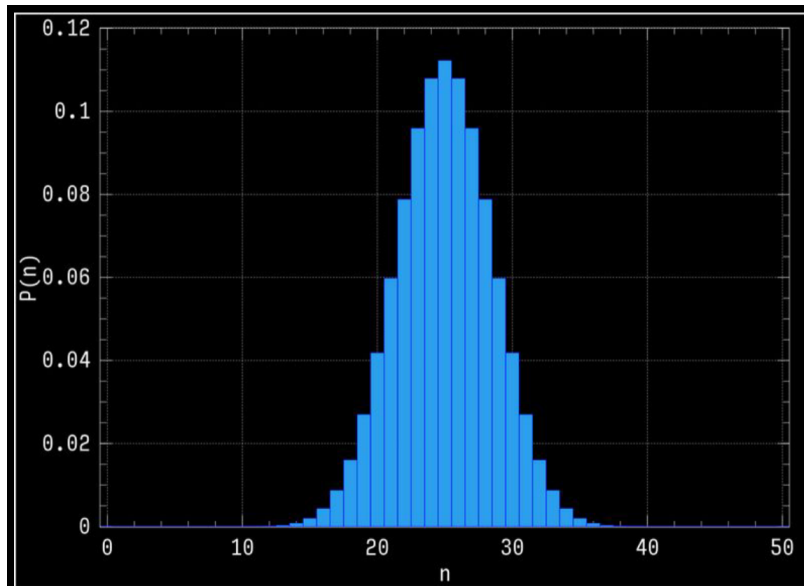


Figure 2. Probability distribution for 50 coin flips

Now, imagine a system of little independent magnetic dipoles that can point up or down in an external magnetic field (Fig. 3) The multiplicity is given by the expression we found before:

As there are more flips the distribution will become *relatively* more narrow

$$\Sigma(N, n) = \binom{N}{n} = \frac{N!}{n! \cdot (N - n)!} \quad (1.14)$$

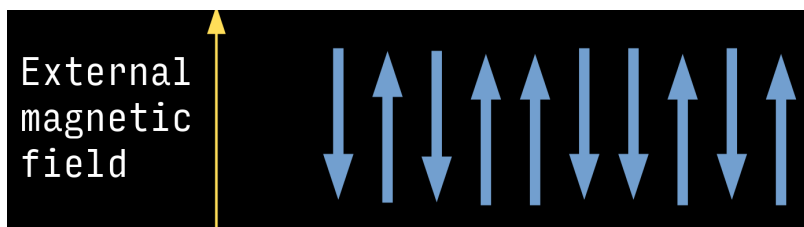


Figure 3. Independent dipoles in field

Equal numbers of dipoles pointed up and down has a far higher multiplicity and therefore is the most probable state although it is not the lowest energy state. On the other hand, the lowest energy state would be all of the dipoles being aligned with the magnetic field, and highest being all anti-aligned. How can resolve this problem?

SUBSECTION 1.4

Lecture 4: Einstein Solids

Einstein Solid

Model a solid as N independent harmonic oscillators.

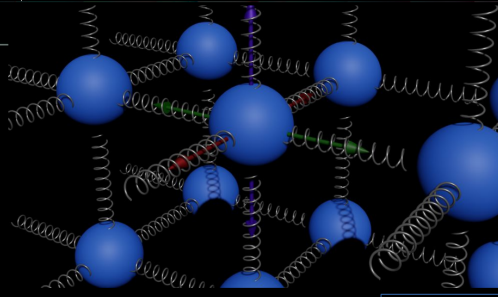
The quantum energy levels for a harmonic oscillator are:

$$E = (q + \frac{1}{2})\hbar\sqrt{k/m}$$

So, each oscillator has evenly spaced energy levels.

$E - E_o = q\hbar\sqrt{k/m} \longrightarrow E = q\hbar\sqrt{k/m}$

We don't really care about the ground state E_o , since it is constant, so we can ignore it.



Recall:
 $\omega \equiv 2\pi f$
 $\hbar \equiv h/2\pi$
 $\hbar\omega = hf$

Figure 4. Einstein solid

Moving away from the contrived dipole example we move to the Einstein solid which is basically a bunch of harmonic oscillators as a means to finding out what temperature is.

Comment Recall: the quantum energy levels of a harmonic oscillator is

$$E = (q + \frac{1}{2})\hbar\sqrt{k/m} \quad (1.15)$$

Where

- $\omega = 2\pi f$
- $\hbar = \frac{h}{2\pi}$
- $\hbar\omega = hf$

By inspection we see that each oscillator has evenly spaced energy levels.

$$E - E_o = q\hbar\sqrt{k/m} \longrightarrow q\hbar\sqrt{k/m} \quad (1.16)$$

Each identical independent oscillator has an integer quanta of energy. This means that for a 3-oscillator system there are 6 ways 2 quanta can be distributed is $(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (0, 1, 1), (1, 0, 1)$ and so on³. The **macrostate** is the total energy in the system, $q_{tot} = 2$. The **microstate** is the configuration, i.e. $(1, 1, 0)$. So, the **multiplicity** of the macrostate $q_{tot} = 2$ is $\Omega(2) = 6$. Going onwards $\Omega(3) = 10$ and so forth.

Ground state is a constant so we ignore it

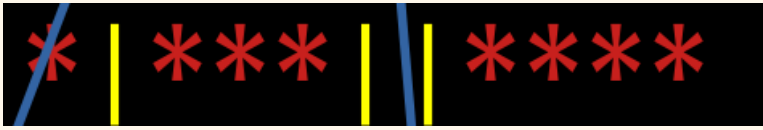
³The nice thing about statistical mechanics is that each step is fairly trivial

Theorem 2 The multiplicity for q quanta of energy distributed between N oscillators is

$$\Omega(N, q) = \binom{q + N - 1}{q} = \frac{q + N - 1}{q!(N - 1)!} \quad (1.17)$$

PROOF

Imagine $N = 4$ oscillators and $q = 8$ quanta. Consider the configuration $(1, 3, 0, 4)$. We can draw it like this:



where $*$ is a quanta of energy and $|$ separates the oscillators. Any combination of 8 of $*$ and 3 of $|$ is a valid microstate.

So: There are $q + N - 1$ symbols, q of which are a $*$

TLDR: cute proof! □

By inspection we find that the multiplicity grows *very* rapidly with increasing N or q .

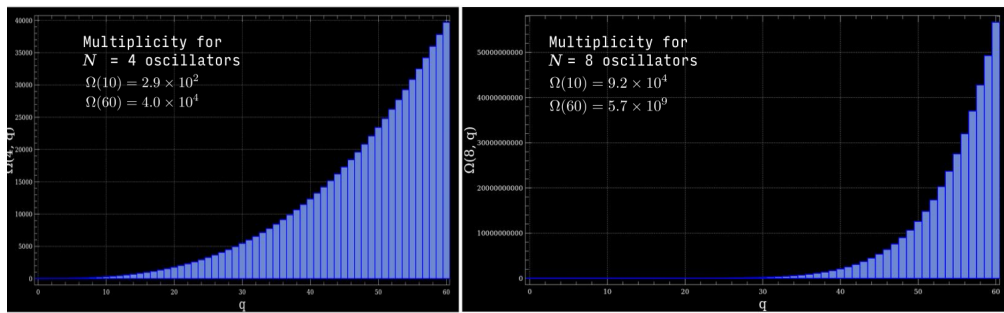


Figure 5. Multiplicity Ω of Einstein solid given N, q

Now, can we understand temperature by making Einstein solids interact with each other? Imagine two Einstein solids with $N = 4$ oscillators each and $q_{tot} = q_1 + q_2$ quanta of energy between them. How do the macrostates and multiplicity look?

Note: if solid 1 has $q_1 = 10$, then solid 2 has $q_2 = 60 - 10 = 50$

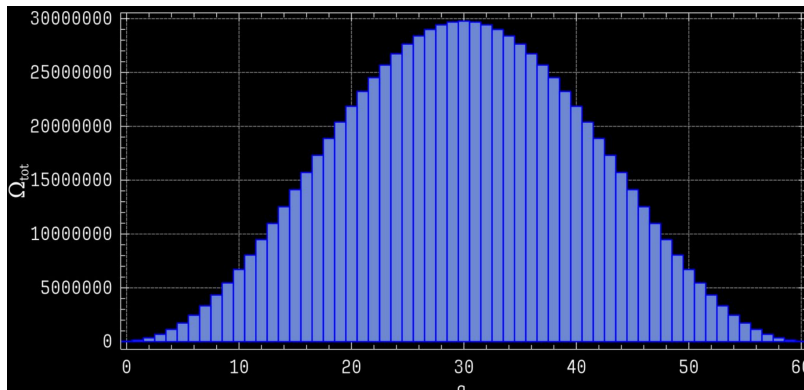
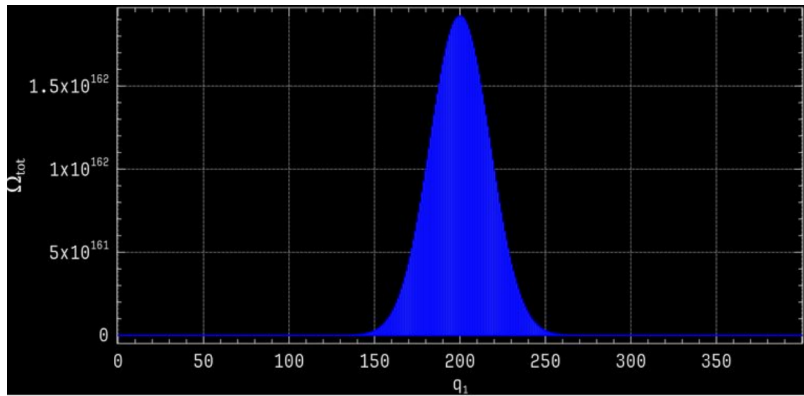


Figure 6. $N_1 = N_2 = 4, q_1 + q_2 = 60, \Omega_{tot} = \Omega(q_1) \times \Omega(60 - q_1)$

Since they share energy, it follows that the total multiplicity of the system is $\Omega_{tot} = \Omega_1 \times \Omega_2$. Plotting this we find that the multiplicity will tend to follow something that looks a bit like a normal distribution, the spread of which depends on the N and q . As this gets larger and larger we gain more and more certainty as to the system's energy – that the system with the greatest multiplicity (most likely) has an energy level which is in-between the energies of the interacting bodies.



Comment This begins to look like a plausible explanation for thermal equilibrium!

SUBSECTION 1.5

Lecture 5: Large Numbers

Let's consider a mole of Einstein solid 1.8×10^{24} at normal temperatures $q = 10^{26}$. If one were to try to calculate what we did in the previous section it would just not work, especially computationally. So let's apply a few methods to model this

Types of numbers

1. Small numbers: e.g. 42 which we can use arithmetic as normal
2. Large numbers: can ignore adding a small number $10^{24} \approx 10^{24} + 24$
3. Very large numbers: can ignore multiplying a very large number by a large number, i.e. $10^{10^{23}} \times 10^{23} \approx 10^{10^{23}}$

Theorem 3 Stirling's approximation

$$N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \quad N \gg 1 \quad (1.18)$$

Which also gives us the result (for large N)

$$\ln(N!) = N \ln(N) - N \quad (1.19)$$

Can use this to find $\Omega(N, q)$

$$\begin{aligned} \ln(\Omega) &\approx N \ln\left(\frac{q}{N}\right) + N & q \gg N, N \gg 1 \\ \Omega &\approx \exp\left\{\frac{eq}{N}\right\}^N \end{aligned} \quad (1.20)$$

More formally,

$$\begin{aligned}
\Omega(N, q) &= \binom{q+N-1}{q} \approx \frac{(q+N)!}{q!N!} \\
\ln(\Omega) &\approx \ln((q+N)!) - \ln(q!) - \ln(N!) && \text{Use } \ln(N!) \approx N \ln(N) - N \\
&\approx (q+N) \ln(q+N) - (q+N) - q \ln(q) + q - N \ln(N) + N && \ln(1+x) \approx x \quad x \ll 1 \\
&= (q+N) \ln(q+N) - q \ln(q) - N \ln(N) \\
&= (q+N) \left(\ln(q) + \frac{N}{q} \right) - q \ln(q) - N \ln(N) && (1.21) \quad \text{assume } q \gg N \\
\Rightarrow \ln(\Omega) &= N \ln\left(\frac{q}{N}\right) + N + \frac{N^2}{q} \\
\Rightarrow \Omega &= \left(\frac{eq}{N}\right)^N
\end{aligned}$$

From our intuition we expect that when we combine two multiplicities that the peak will be at $q_1 = q_2 = q_{tot}/2 \dots$

$$\text{Let } x = q_1 - \frac{q}{2} = \frac{q}{2} - q_2$$

$$\Omega_{tot} = \left(\frac{e}{N}\right)^{2N} \left[\left(\frac{q}{2}\right)^2 - x^2\right]^N \quad (1.22)$$

And then we do some math [textbook 2.26, 2.27] and find that it is a Gaussian

Definition 3 Large N multiplicity of Einstein solids

$$\begin{aligned}
\Omega_{tot} &\approx \Omega_{max} e^{-N(2x/q)^2} \\
\Omega_{max} &= \left(\frac{e}{N}\right)^{2N} \left(\frac{q}{2}\right)^{2N}
\end{aligned} \quad (1.23)$$

And with Stirling approximation we arrive at

$$\Omega \approx \left(\frac{eq_1}{N}\right)^N \quad (1.24)$$

And the variance can be found for large N , which becomes increasingly thin for large N . if $N = 10^{24}, \sigma = \frac{q}{2 \times 10^{12}}$

$$\sigma = \frac{q}{2\sqrt{N}} \quad (1.25)$$

1.5.1 Entropy

Definition 4

$$S \equiv k \ln(\Omega) \quad (1.26)$$

And we find that systems in thermal contact will tend to be found with the highest entropy; *Entropy increases.*

This is the second law of thermodynamics!

SUBSECTION 1.6

Lecture 6: Multiplicity of an Ideal Gas

Recall the expression for large N multiplicity is given by Eq. 1.24 and for a combination of Einstein solids it is simply $\Omega_{tot} = \Omega_1 \times \Omega_2$

Substituting $x = q_1 - \frac{q}{2}$ we can arrive at the following approximation

$$\approx \Omega_{max} \cdot e^{-x^2/a^2} \quad (1.27)$$

And $a = \frac{q}{\sqrt{2\sqrt{N}}}$ describes the width of the peak (which can be thought of as describing the accuracy).

1.6.1 Ideal Gas

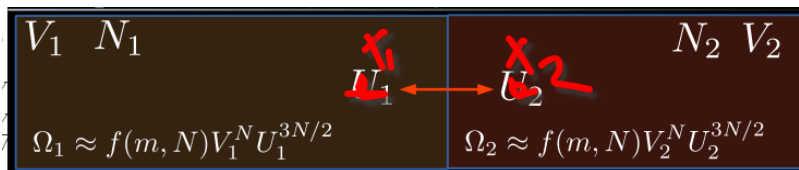
Comment

Recall from the quantum mechanics section of this course that particles in a cubic box have quantized energy

$$E = \frac{h^2}{8m} \left(\frac{n_1^2 + n_2^2 + n_3^2}{V^{2/3}} \right) \quad (1.28)$$

In the textbook it gives us a massive formula for the multiplicity of an ideal gas

$$\Omega_N \approx \frac{1}{N!} \frac{V^{2/3}}{h^{3/N}} \frac{\pi^{3N/2}}{(3N/2)!} (\sqrt{2mU})^{3N} = f(m, N) V^N U^{3N/2} \quad (1.29)$$



This is not very satisfying and we will get back to this later. For now let's assume it is correct.

Let's consider a system of two ideal gases in contact. In this case if we allow for energy transfer, i.e. $X_i = U_i...$

$$\begin{aligned} \Omega_{tot} &= \Omega_1 \times \Omega_2 \\ \Omega_{tot} &= f()^2 (U_1 U_2)^N (U_1 U_2)^{3N/2} \\ &\approx \Omega_{max} \cdot \exp\left(\frac{-x^2}{a^2}\right) \end{aligned} \quad (1.30)$$

where $x = U_1 - \frac{U_{tot}}{2}$

$$\rightarrow a = \frac{U}{2\sqrt{3N/2}}$$

And so we get the result that the multiplicity is again a Gaussian with a peak at $x = U_1 - \frac{U}{2}$ and a spread on the order of $\frac{1}{\sqrt{N}}$ as well, which indicates that thermal equilibrium works for ideal gases with energy transfer as well!

Considering a system where we allow for volume transfer, i.e. $X_i = V_i...$

$$\begin{aligned}
\Omega_{tot} &= \Omega_1 \times \Omega_2 \\
\Omega_{tot} &= f()^2 (V_1 V_2)^N (U_1 U_2)^{3N/2} \\
&\approx \Omega_{\max} \cdot \exp\left(\frac{-x^2}{a^2}\right) \\
\text{where } x &= V_1 - \frac{V_{tot}}{2} \\
\rightarrow a &= \frac{V}{2\sqrt{N}}
\end{aligned} \tag{1.31}$$

We arrive at much the same result as before with the Gaussian and spread.

Considering a system where we allow for particle transfer, i.e. $X_i = N_i$, we get a really ugly expression. I do not want to write this out so here is the screen-shotted lecture slide:

Large N Multiplicity of Ideal Gasses

Poke a hole and let molecules flow...

$U_1 \quad V_1 \quad N_1 \quad N_2 \quad V_2 \quad U_2$

$\Omega_1 \approx \frac{1}{N_1!} \frac{V_1^{N_1}}{h^{3N_1}} \frac{\pi^{3N_1/2}}{(3N_1/2)!} (\sqrt{2mU_1})^{3N_1}$
 $\Omega_2 \approx \frac{1}{N_2!} \frac{V_2^{N_2}}{h^{3N_2}} \frac{\pi^{3N_2/2}}{(3N_2/2)!} (\sqrt{2mU_2})^{3N_2}$

$\Omega_{tot} = \Omega_1 \times \Omega_2$

$\ln(\Omega_{tot}) \approx f(N, m, T, V) - N_2 \ln(N_2) - N_1 \ln(N_1)$

More flailing around...

Note that $U_1 = \frac{3}{2} N_1 k T_1$
And use Stirling's approximation

Define $x \equiv N_1 - N/2 = N/2 - N_2$
And use $\ln(1+x) \approx x$

$\Omega_{tot} = e^{f(N,T,V,m)} e^{-x^2/c(N)}$

Strongly peaked at $N_1 = N_2$

The point of all this is that this brings us to the second law of thermodynamics and the concept of entropy which enables us to work with these really big numbers in a much easier way.

Before we do that let's convert our multiplicity expressions to entropy

$$\Omega \approx \left(\frac{eq^N}{N}\right) \quad N = 10^{22} \Leftrightarrow S = Nk[\ln\left(\frac{q}{N}\right) + 1] \tag{1.32}$$

And derive the following property from just log laws

$$\begin{aligned}
\Omega_{tot} &= \Omega_1 \times \Omega_2 \\
S_{tot} &= S_1 + S_2
\end{aligned} \tag{1.33}$$

Which, coupled by the 2nd law of thermodynamics (entropy increases and system will be most stable maximum entropy), we now have a really good way of describing temperature and thermodynamic equilibrium.

Applying this to an ideal gas we can do some ugly math from Eq. 1.29 and Eq. 1.26, use Stirling's approximation and drop some large factors to arrive at the following expression for entropy of an ideal gas

Theorem 4 Entropy of an ideal gas

$$S = Nk\left[\ln \frac{V}{N} \left(\frac{4\pi mU}{3Nh^2}\right)^{\frac{3}{2}} + \frac{5}{2}\right] \tag{1.34}$$

This is also known as the Sackur-Tetrode equation

Inspecting the equation entropy increases with N, V, m, U , which makes intuitive sense. Note that there is a subtlety in using entropy. For example mixing two different gases; i.e. a gases with different masses, or gases with a different energy; this would cause a positive ΔS . However mixing two *identical* gases would cause no entropy change; $\Delta S = 0$. What to note is that thermal equilibrium is the tendency of a system to enter a more likely state. If they are in the same state; mixing them will not cause any further change in state and therefore positive ΔS .

Entropy is not anything mysterious anymore!

SUBSECTION 1.7

Lecture 7: What is temperature? (Revisited)

This was a short lecture with lots of review. The key takeaway is that thermal equilibrium is found when

$$\frac{\partial S_1}{\partial q_1} = \frac{\partial S_2}{\partial q_2} \quad (1.35)$$

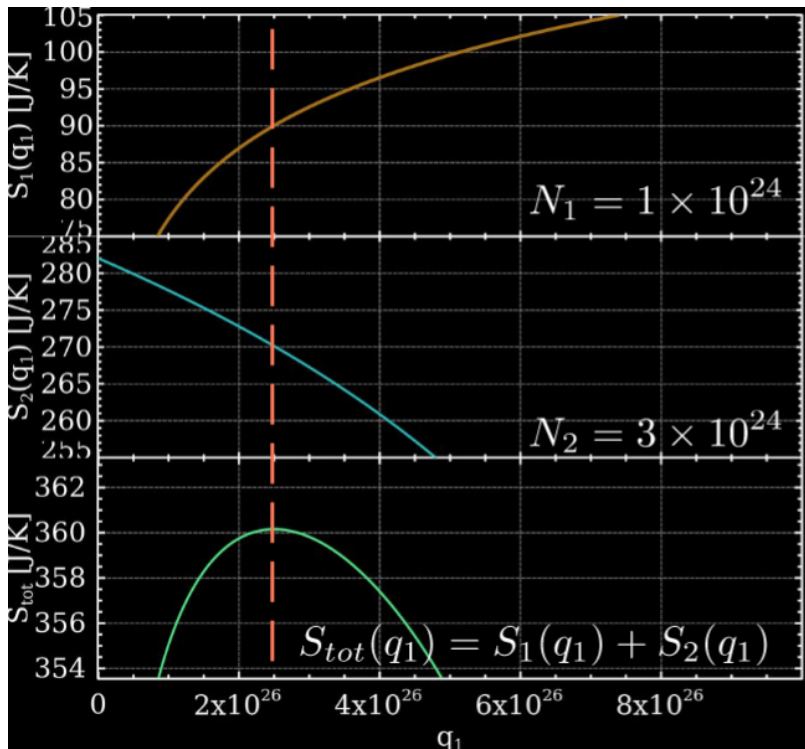


Figure 7. Toy ideal gas system

But $U = qE_o$, so

$$\frac{\partial S_1}{\partial U_1} = \frac{\partial S_2}{\partial U_2} \quad (1.36)$$

The peak is when the derivative of the entropy with respect to energy is the same in both systems.

And so we arrive at the conditions for thermal equilibrium:

1. Temperatures are equal
2. Entropy is maximized (S_{max} is at $\frac{\partial S_1}{\partial U_1} = \frac{\partial S_2}{\partial U_2}$)

Intuitively the peak is where taking energy from one system and putting it in the other does not cause a change in total entropy

But! $\frac{\partial S}{\partial U}$ decreases with increasing U but temperature increases with increasing U . So let's test

$$T \equiv \left(\frac{\partial U}{\partial S}\right)_{N,V} \quad (1.37)$$

Perhaps this could be temperature? Since they are equal at thermal equilibrium...

Keeping N, V constant

We can test it for an ideal gas

$$S = Nk \left[\ln \left(\frac{V}{N} \left(\frac{4\pi m U}{3N h^2} \right)^{\frac{3}{2}} \right) + \frac{5}{2} \right] = Nk \ln(U^{\frac{3}{2}}) + f(N, V)$$

$$T = \left(\frac{\partial S}{\partial U}\right)^{-1} = \left(\frac{2Nk}{2U}\right)^{-1} = \frac{2U}{3Nk} \quad (1.38)$$

Which brings us to the *very* satisfying result that

$$\Rightarrow U = 2N \frac{kT}{2} \quad (1.39)$$

, which is *exactly* what the equipartition theorem says for an ideal gas! This T is what an ideal gas law thermometer measures – but is applicable to *any* system.

Definition 5

We can therefore define temperature as the partial derivative of energy with respect to entropy

$$T \equiv \frac{\partial U}{\partial S} \text{ or } \frac{1}{T} = \frac{\partial S}{\partial U} \quad (1.40)$$

T^{-1} is how much $S = k \ln(\Omega)$ increases when energy is added to a system. This means that at a low temperature putting in a little energy will increase S a lot, and vice-versa for high temperatures.

SUBSECTION 1.8

Lecture 8: Thermodynamic Identity

Comment

Recall: $(T)^{-1}$ is how much $S = k \ln \Omega$ with "everything else" held constant. This means at low temperature adding energy increases S a lot, and vice versa. This means that when putting a low and high temperature object besides each other, the higher temperature object will more likely give up temperature to the lower temperature one – which is what we expect.

We have a procedure for finding heat capacity

1. Find the multiplicity $\Omega(U, N, V, \dots)$
2. Find the entropy $S = k \ln \Omega$
3. Find $T(U, N, V, \dots) = \left(\frac{\partial S}{\partial U}\right)^{-1}$
4. Solve for $U(T, N, V, \dots)$
5. Find $C_V = \frac{\partial U}{\partial T}$

And we find that this works well for Einstein solids and for real solids e.g. Lead as well. It also works for ideal gases and para-magnets – but other than for these we are kind of stuck;

$$U = \varepsilon q, \Omega \approx \left(\frac{eq}{N}\right)^N$$

$$= \left(\frac{eU}{N\varepsilon}\right)^N$$

$$S = Nk \left[\ln \frac{U}{N\varepsilon} + 1 \right]$$

$$T = \frac{U}{Nk}$$

$$U = NkT$$

$$C_V = Nk \quad (1.41)$$

it is *really* difficult to use this tool to find heat capacity for more complex substances. This is where classical thermodynamics comes in

In order to complete the picture we now want to be able to build a better understanding of pressure. Going back to the example in Fig. 7, let's consider allowing volume and energy to change.

$$\text{At } S_{\max} \frac{\partial S_1}{\partial U_1} \frac{\partial S_2}{\partial U_2} \text{ and } \frac{\partial S_1}{\partial V_1} \frac{\partial S_2}{\partial V_2} \quad (1.42)$$

Also, $T_1 = T_2, P_1 = P_2$, where $T \equiv (\frac{\partial S}{\partial U})^{-1}$. We've played this 'game' before with temperature to get the above result – now let's do it with pressure. We note that $\frac{\partial S_1}{\partial V_1} \frac{\partial S_2}{\partial V_2}$ has units of $\frac{N}{m^2 \cdot K}$, and $P_1 = P_2$ has units $\frac{N}{m^2}$. So let's propose

$$P \equiv T \left(\frac{\partial S}{\partial V} \right)_{U,N} \quad (1.43)$$

Plug this into Eq. 1.34 and see if it reproduces the ideal gas law. Which gives us

Definition 6 Pressure

$$P = \frac{TNk}{V} \quad (1.44)$$

And that's the ideal gas law! So Eq. 1.43 is a valid expression for pressure.

1.8.1 Chemical Potential

Let's go back to the example in Fig. 7 yet again but allow for changing volume, energy, and number of particles.

Building upon what we have derived prior for volume and temperature we note that

$$\frac{\partial S_1}{\partial N_1} \frac{\partial S_2}{\partial N_2} \quad (1.45)$$

Definition 7 Chemical Potential is therefore defined as:

$$\mu \equiv T \left(\frac{\partial S}{\partial N} \right)_{U,V} \quad (1.46)$$

At equilibrium total entropy doesn't change; i.e. moving a particle from one side to the other doesn't make an impact

Barth Netterfield: The negative is there for tradition?

Theorem 5 Therefore we may define the Thermodynamic Identity as

$$dS = \left(\frac{\partial S}{\partial U} \right)_{N,V} dU + \left(\frac{\partial S}{\partial V} \right)_{N,U} dV + \left(\frac{\partial S}{\partial N} \right)_{U,V} dN \quad (1.47)$$

And plugging in our definitions for T, P , and μ

$$dS = \frac{1}{T} dU + \frac{P}{T} dV - \frac{\mu}{T} dN \quad (1.48)$$

With Eq. 1.48 we may integrate over all energies, volumes, number of particles, etc. to get a measure of the entropy and therefore multiplicity in a system! It is still in not a great form to work with so we solve for U to get the following identity

$$dU = TdS - PdV + \mu dN \quad (1.49)$$

Example | Let's play with Eq. 1.49 a little bit and find out surprisingly intuitive it is.

1. Keep entropy and volume to be 0 and constant. $\mu = \frac{\partial U}{\partial N}_{S,V}$. This says that chemical potential is the amount of energy you need to add⁴ a particle in order to keep the entropy constant. This makes sense since S is a strong function of N
2. Keep energy and volume to be 0 and constant. $\frac{\mu}{T} = -\frac{\partial S}{\partial N}_{U,V}$. Chemical potential⁵ is the amount the entropy changes when you add a particle (with energy and volume constant). Note that chemical potential is generally negative, so entropy increases while adding particles.
3. Keep energy and entropy to be 0 and constant. $\frac{\mu}{P} = -\frac{\partial V}{\partial N}_{U,S}$. Chemical potential⁶ is the amount the volume must change when a particle is added in order to keep energy and entropy constant
4. Keep energy and number to be 0 and constant. $\frac{P}{T} = -\frac{\partial S}{\partial V}_{U,N}$. Pressure⁷ is the amount by which entropy changes when you increase the volume.

⁴ Chemical potential is usually negative, so you need to take away energy

⁵ divided by temperature

⁶ divided by pressure

⁷ divided by temperature

To recap: we have now have the thermodynamic identity from which most of classical thermodynamics is derived from; what we're missing now is just various free energies.

SUBSECTION 1.9

Lecture 9: Boltzmann Factors

Comment | A summary of key results derived in previous lectures:
Probability of macrostate s :

$$P(s) = \frac{\Omega(s)}{\Omega_{\text{tot}}} \quad (1.50)$$

Entropy (which is maximized at equilibrium) (Eq. 1.26)

$$S \equiv k \ln(\Omega) \quad (1.51)$$

And the thermodynamic identity (Eq. 1.49)

$$dU = TdS - PdV + \mu dN \quad (1.52)$$

Where

$$T \equiv \left(\frac{\partial U}{\partial S}\right)_{N,V}, \quad P \equiv T\left(\frac{\partial S}{\partial V}\right)_{U,N}, \quad \mu \equiv T\left(\frac{\partial S}{\partial N}\right)_{U,V} \quad (1.53)$$

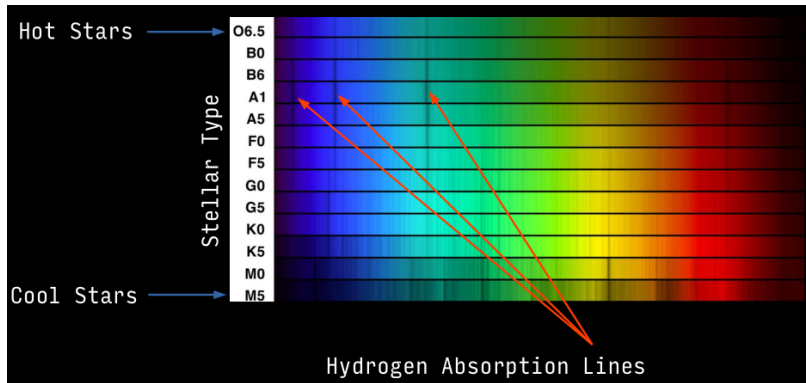


Figure 8. Spectral lines for selected stars. Note absorption lines – how can we use statistical mechanics to explain the behaviour of these hydrogen absorption lines? Why do they appear for hot stars but disappear for cool stars?

Hydrogen absorption lines form the Lyman, Balmer, and Paschen series. The visible ones start at the $n = 2$ shell and so lie in the Balmer series.

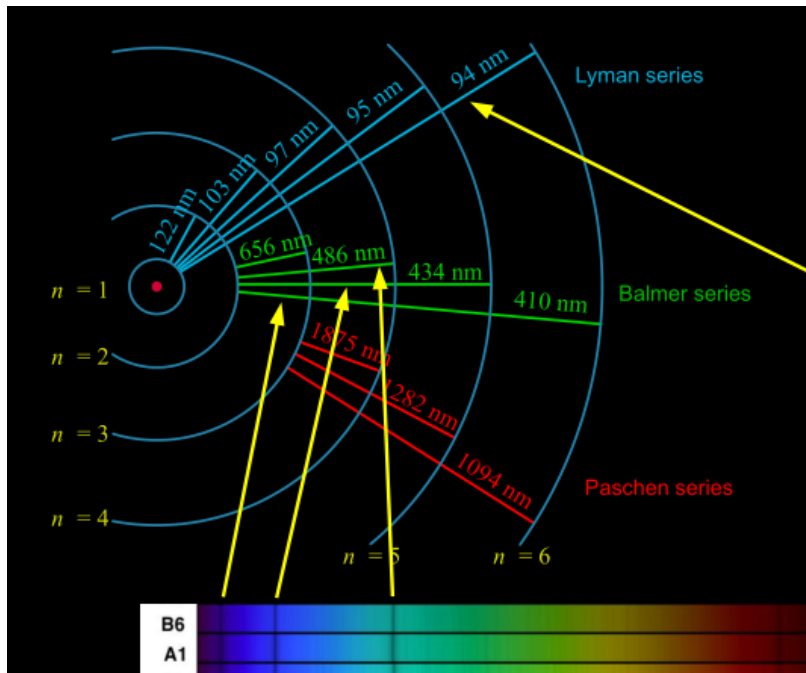


Figure 9. Various spectral series for hydrogen

Therefore cold stars would have largely hydrogen at $n = 1$ and therefore not have visible absorption lines. Can we go more specific and find the *probability* that an atom is in the $n = 2$ state instead of the $n = 1$ state? ⁸

Consider the following system of a single atom⁹ in the rest of the Sun's atmosphere¹⁰. What is the probability that this atom will be in a particular $n = 2$ mode compared to a $n = 1$ mode?¹¹

$$\frac{P(s_2)}{P(s_1)} = \frac{\Omega_R(s_2)}{\Omega_R(s_1)} \quad (1.54)$$

⁸ Yes, or else we wouldn't be asking this question

⁹ the subsystem

¹⁰ The reservoir

¹¹ Recall multiplicity of energy levels i.e. there are 4 different states with energy level of $n = 2$

Read $\Omega_R(s_i)$ to be the multiplicity of the entire system, sun *and* the atom, *when* the atom is in mode s_i . By definition the multiplicity of a mode is 1. But why would the multiplicity of the entire sun change when the atom is in different modes? Because the energy changes – when energy flows out of the sun reservoir into the atom to increase its energy level causes the sun’s multiplicity to go down, and vice-versa.

Rearrange the prior expression a little bit to get it with respect to entropy

$$\begin{aligned} S &\equiv k \ln \Omega \rightarrow \Omega = e^{S/k} \\ \Rightarrow \frac{P(s_2)}{P(s_1)} &= \frac{\Omega_R(s_2)}{\Omega_R(s_1)} = \frac{e^{S_R(s_2)/k}}{e^{S_R(s_1)/k}} = e^{[S_R(s_2) - S_R(s_1)]/k} \end{aligned} \quad (1.55)$$

And then apply the thermodynamic identity¹², noting that effects on volume is negligible and there is no transfer of particles. Therefore we arrive at

$$^{12} \quad \frac{dS}{P dV \ll dU} = \frac{1}{T} (dU + \underbrace{\mu dN}_{dN=0})$$

$$\begin{aligned} \Delta S &= \Delta U/T \quad \text{for small } \Delta S \\ &= S_R(s_2) - S_R(s_1) = \frac{1}{T} [\underbrace{U_R(s_2)}_{\text{energy in reservoir during } s_2} - U_R(s_1)] \\ &= -\frac{1}{T} [E(s_2) - E(s_1)] \quad (13) \end{aligned} \quad (1.56)$$

¹³ Adding energy to atom is removing from reservoir

Plugging this back into Eq. 1.55 we then get

$$\frac{P(s_2)}{P(s_1)} = e^{[S_R(s_2) - S_R(s_1)]/k} = e^{-\frac{1}{kT} (E(s_2) - E(s_1))} = \frac{e^{-E(s_2)/kT}}{e^{-E(s_1)/kT}} \quad (1.57)$$

Note that E is the energy level of the subsystem, and T is the temperature of the reservoir (not the subsystem). The energy of the subsystem also doesn’t make too much sense since it is just a single thing.

Theorem 6 We then define the **Boltzmann Factor**, and the ratio of probabilities of two states is the ratio of their Boltzmann factors.

$$e^{-E(s)/kT} \quad (1.58)$$

And the *absolute probability* is some constant multiplied by the Boltzmann factor, i.e.

$$P(s_i) = \frac{1}{Z} e^{-E(s)/kT} \quad (1.59)$$

The constant Z describes the sum of the Boltzmann factors of all possible microstates, i.e. the partition function

$$Z = \sum_s e^{-E(s)/kT} \quad (1.60)$$

In other words the probability of a particular microstate is the Boltzmann factor of said microstate, divided by the sum of all Boltzmann factors for that system

In the past we talked about each microstate being equally likely. But now not every single subsystem in thermal equilibrium with a larger system is equally likely because the subsystem can give/take energy. So we now have to look at the likelihood that it can give/take energy to get to the higher energy state, which is given by Eq. 1.59

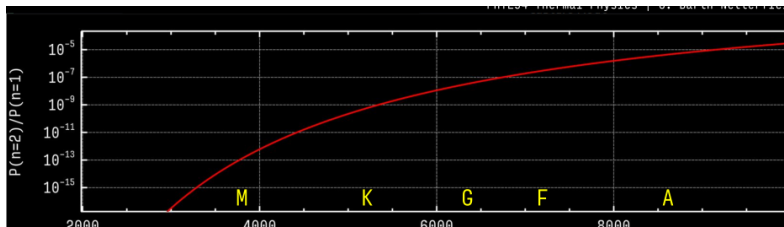
Example Let's try applying this to the Balmer series (Fig. 9). We know that the line strength is depends on the number of hydrogen atoms starting at the $n = 2$ level. The ΔE between $n = 1, 2$ is 10.2eV .

- The multiplicity of $n = 1$ is 1
- The multiplicity of $n = 2$ is 4

Plugging this into Eq. 1.57 we get

$$\frac{P(n=2)}{P(n=1)} = 4e^{-\Delta E/kT} \quad (1.61)$$

Which can be plotted as a function of T



And we can observe that hot stars (i.e. A-stars) are *much* more likely to have hydrogen atoms in the $n = 2$ state, whereas it is *very* unlikely for cool M-stars to have hydrogen atoms in those states; the $n = 2$ level is insufficiently populated for hydrogen absorption lines to appear.

This result makes sense given our observed spectral lines

ECE259

Electromagnetism

SECTION 2

Dielectrics

Comment A few useful things from before reading week:

- ϵ_o : permittivity of free space
- ϵ_r : relative permittivity of the material
- ϵ is the *actual* permittivity of the material

And the three are related by

$$\epsilon_r = \epsilon / \epsilon_o \quad (2.1)$$

The \vec{D} field is defined as

$$\vec{D} = \epsilon_o \vec{E} + \vec{P} \quad (2.2)$$

Noting that for a linear, isotropic material

$$\begin{aligned} \vec{D} &= \epsilon_o \vec{E} + \vec{P} \\ &\xrightarrow{\text{linear, isotropic}} \epsilon_o \vec{E} (1 + \chi_e) \\ &= \epsilon_o \epsilon_r \vec{E} = \epsilon \vec{E} \end{aligned} \quad (2.3)$$

In free space the generalized Gauss's law is given by

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_p}{\epsilon_o} \quad \oint_s \vec{E} \cdot d\vec{s} = \frac{Q_{\text{end}}}{\epsilon_o} \quad (2.4)$$

And through a medium it is given by

$$\vec{\nabla} \cdot \underbrace{(\epsilon_o \vec{E} + \vec{P})}_{\vec{D}} = \rho_v \quad (2.5)$$

This note ρ_v ; this is for a volume charge density

When working with surface charge densities we have the following expressions

$$a_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \quad (2.6)$$

Which leads us to the following fundamental postulates of electrostatics, in integral and differential form¹⁴

$$\vec{\nabla} \cdot \vec{D} = \rho_v \Leftrightarrow \oint_s \vec{D} \cdot d\vec{s} = Q \quad (2.7)$$

¹⁴ \vec{P} is the polarization vector

$$\vec{\nabla} \times \vec{E} = 0 \Leftrightarrow \oint_l \vec{E} \cdot d\vec{l} = 0 \quad (2.8)$$

\vec{P} is the measure of polarization in a dielectric material when in an electric field. It can be thought of as the electric dipole moment induced in the material per unit volume.

$$\vec{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^n \Delta v \vec{P}_k}{\Delta v} \quad \left[\frac{C}{m^2} \right] \quad (2.9)$$

$$\vec{E} = \epsilon_o \chi_e \vec{E} \quad (2.10)$$

A convenient expression for voltage is

$$\vec{E} = -\vec{\nabla} V \quad (2.11)$$

SUBSECTION 2.1

Lecture 15: Boundary Conditions for Dielectrics

Remark Recall, at a conductor/free-space interface, $E_+ = 0$, $E_n = \frac{\rho_s}{\epsilon_o}$
Consider an interface between two generic dielectrics (Fig. 10).

Theorem 7

$$\oint_c \vec{E} \cdot d\vec{l} = 0 \quad (2.12)$$

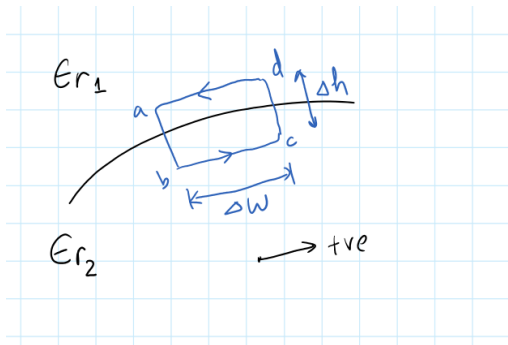


Figure 10. Dielectric interface

Intuition The two integrals parallel to the dielectrics will cancel out, and so will those perpendicular.

This implies that the tangential component of the \vec{E} field is continuous across the boundary,

$$E_{1t} = E_{2t} \quad (2.13)$$

However we get a bit of a different result when working in 3 dimensions where the interface is a surface instead of a line.

Theorem 8

$$\oint_S \vec{D} \cdot d\vec{s} = \rho_S \Delta \rightarrow (\vec{D}_1 - \vec{D}_2) \cdot \vec{a}_{n2} = \rho_S \quad (2.14)$$

a_n denotes normal component

Note: $\vec{D} = \epsilon_0 \vec{E} + \vec{p} = \dots \epsilon \vec{E}$

PROOF As $\Delta h \rightarrow 0$

$$\begin{aligned}
 \oint_S \vec{D} \cdot d\vec{s} &= \int_{top} \vec{D}_1 \cdot d\vec{s} + \int_{bottom} \vec{D}_2 \cdot d\vec{s} \\
 &= \vec{D}_1 \cdot \vec{a}_{n2} \Delta S + \vec{D} \cdot \vec{a}_{n1} \Delta S \\
 &= \vec{D}_1 \cdot \vec{a}_{n2} \Delta S - \vec{D} \cdot \vec{a}_{n2} \Delta S \\
 &= \rho_S \Delta S
 \end{aligned} \tag{2.15}$$

□

Solving these problems usually involves finding the tangential and normal components through Eq. 2.13 and 2.14 then applying Pythagoras.

SUBSECTION 2.2

Lecture 16: Capacitors

Definition 8

A capacitor is a device consisting of two isolated conductors for storing energy in the form of electrostatic potential energy. An isolated conductor can also have "capacitance" if the other conductor is far away. The charge of a capacitance refers to the charge on one conductor. The energy stored in a capacitor is equal to the energy it takes to charge a capacitor from a discharged state to a charged state.

A capacitor's *capacitance*¹⁵ is defined as

$$C = \frac{Q}{V} \tag{2.16}$$

And has units $[C] = \frac{C}{V} F$ [Farads]

¹⁵Capacitance is actually independent of Q and V and is dependent only on the physical attributes of [the capacitor]

Capacitance is calculated as follows

1. choose a coordinate system
2. Assume $+Q/-Q$ on the conductors
3. Find \vec{E} from Q distribution
4. Find $V_{AB} = \int_A^B \vec{E} \cdot d\vec{l}$ where A carries the negative charge and B carries positive.
5. Apply $C = \frac{Q}{V}$

SUBSECTION 2.3

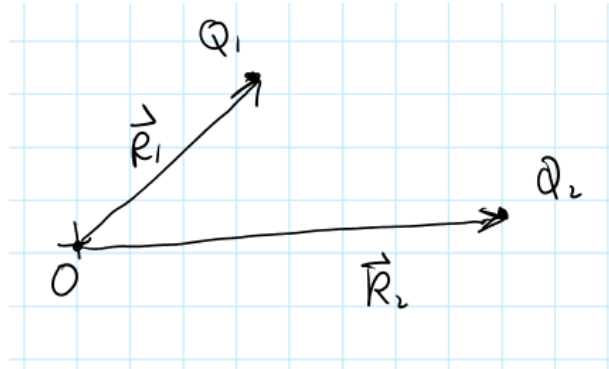
Lecture 17: Electrostatic Potential Energy

Remark

Capacitors store energy because it takes energy to arrange charges in a particular way, i.e. separating positive and negative charges

1. How can we calculate the energy stored in a charge distribution?
2. How is this energy related to field quantities \vec{D} & \vec{E} ?

Consider two point charges Q_1 and Q_2



Assume Q_1 was there first and Q_2 is brought from $\infty \rightarrow \vec{R}_2$.

$$W_2 = Q_2 V_{\vec{R}_2} \quad (2.17) \quad \text{Potential at } \vec{R}_2 \text{ due to the field of } Q_1$$

And same vice-versa for Q_2

$$W_1 = Q_1 V_{\vec{R}_1} \quad (2.18) \quad \text{Potential at } \vec{R}_1 \text{ due to the field of } Q_2$$

Then the expressions for voltage may be built to get an expression for work just in terms of Q, R

$$V_{\vec{R}_2} = \frac{Q_1}{4\pi\epsilon|\vec{R}_2 - \vec{R}_1|} \quad V_{\vec{R}_1} = \frac{Q_2}{4\pi\epsilon|\vec{R}_1 - \vec{R}_2|} \quad (2.19)$$

$$W_2 = \frac{Q_2 Q_1}{4\pi\epsilon|\vec{R}_2 - \vec{R}_1|} \quad (2.20)$$

But $W_2 = Q_1 \cdot V_{\vec{R}_1} = Q_2 V_{\vec{R}_2}$, so

$$W_2 = \frac{1}{2} \left(\frac{Q_2 Q_1}{4\pi\epsilon_o|\vec{R}_2 - \vec{R}_1|} + \frac{Q_2 Q_1}{4\pi\epsilon_o|\vec{R}_1 - \vec{R}_2|} \right) \quad (2.21)$$

Note the subscripts on \vec{R}

What if we introduced a third point charge Q_3 and brought it from ∞ to \vec{R}_3 ? The additional energy required is

$$\Delta W = Q_3 V_{\vec{R}_3} = Q_3 \left(\frac{Q_2 Q_1}{4\pi\epsilon_o|\vec{R}_3 - \vec{R}_1|} + \frac{Q_2 Q_1}{4\pi\epsilon_o|\vec{R}_3 - \vec{R}_2|} \right) \quad (2.22)$$

And then the total energy becomes

$$\begin{aligned} W_3 &= W_2 + \Delta W \\ &= \frac{1}{2} \left(Q_1 \left(\frac{Q_2}{4\pi\epsilon_o|\vec{R}_1 - \vec{R}_2|} + \frac{Q_3}{4\pi\epsilon_o|\vec{R}_1 - \vec{R}_3|} \right) \right. \\ &\quad \left. Q_2 \left(\frac{Q_1}{4\pi\epsilon_o|\vec{R}_2 - \vec{R}_1|} + \frac{Q_3}{4\pi\epsilon_o|\vec{R}_2 - \vec{R}_3|} \right) \right. \\ &\quad \left. Q_3 \left(\frac{Q_1}{4\pi\epsilon_o|\vec{R}_3 - \vec{R}_1|} + \frac{Q_2}{4\pi\epsilon_o|\vec{R}_3 - \vec{R}_2|} \right) \right) \\ &= \frac{1}{2} [Q_1 V_1 + Q_2 V_2 + Q_3 V_3] \end{aligned} \quad (2.23)$$

Theorem 9 In general, the interaction energy is

$$W_e = \frac{1}{2} \sum_{k=1}^N Q_k V_k = \frac{1}{2} \sum_{k=1}^N Q_k \underbrace{\left(\frac{1}{4\pi\epsilon_o} \sum_{j=1, j \neq k}^N \frac{Q_j}{|\vec{R}_k - \vec{R}_j|} \right)}_{V_k} \quad (2.24)$$

Note:

1. W_e can be negative
2. The expression here is only about interaction energy, not counting the energy to assemble the charges.

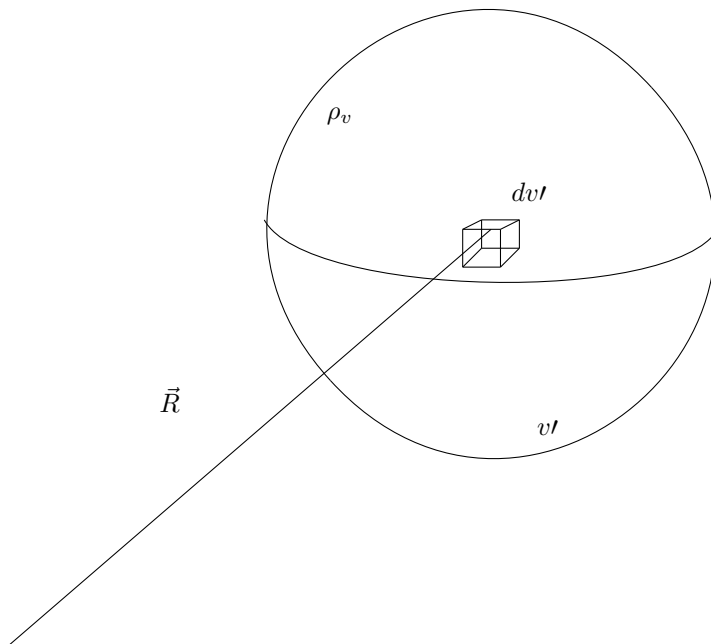
SUBSECTION 2.4

Lecture 18: Electrostatic Potential Energy

Remark Recall, the potential energy stored in the system is given by

$$W_e = \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv' \quad (2.25) \quad \text{Can extend } v' \text{ to entire space}$$

Applying generalized gauss's law and performing an integral across the space we find that



$$\begin{aligned}
W_e &= \frac{1}{2} \int_{v'} \rho_v V(\vec{R}) dv' \\
&= \frac{1}{2} \int_{v'} (\vec{\nabla} \cdot \vec{D}) V(\vec{R}) dv' \\
&= \frac{1}{2} \int_{v'} \vec{\nabla} \cdot (\vec{D} V) dv' - \frac{1}{2} \int_{v'} \vec{\nabla} \cdot \vec{D} V dv' \\
&= \frac{1}{2} \oint_{S'} \vec{D} V \cdot d\vec{s} + \frac{1}{2} \int_{V'} \vec{D} \cdot \vec{E} dv' \\
\Rightarrow W_e &= \frac{1}{2} \int_v \vec{D} \cdot \vec{E} dv
\end{aligned}
\tag{2.26}$$

Apply divergence theorem

The first term $\rightarrow 0$ as $R \rightarrow \infty$

Therefore the energy density can be defined as

Definition 9 Given energy density w_e with units $[J/m^3]$

$$W_e = \int_v w_e dv \quad w_e = \frac{1}{2} \vec{D} \cdot \vec{E} \tag{2.27}$$

Remark Things to know about capacitors

- Stacking them puts them in series. This looks like the inverse of resistors, i.e.

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \dots \tag{2.28}$$

- Putting them besides each other puts them in parallel.

$$C = C_1 + C_2 \dots \tag{2.29}$$

SUBSECTION 2.5

Lecture 19: Poisson's Equation & Uniqueness Theorem

Comment Previously we worked with problems like 'Given a charge density ρ , find \vec{E} , \vec{E} , V '. However a more typical problem we may face in the real world is: 'Given a charge distribution and potential known at *some* points, how can we find the fields?'

Definition 10 **Poisson's Equation**

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{2.30}$$

Which is derived by substituting the definition of \vec{D} , Eq. 2.3 and Eq. 2.11 into the differential form of Gauss's law, Eq. 2.7:

$$\vec{\nabla} \cdot (\epsilon(-\vec{\nabla} V)) = \rho \tag{2.31}$$

And for no charge or $\rho_v = 0$, we have a special case¹⁶: the 'Laplace equation'

$$\nabla^2 V = 0 \tag{2.32}$$

¹⁶ This is often found in non-charge-surface areas, e.g. between capacitor plates

Using this we can solve for potential as a boundary value problem. For a surface S enclosing a volume v with ρ_v , solving the following gives us the potential difference V inside v

$$\begin{cases} \nabla \cdot (\epsilon \nabla V) = \rho_v & \text{in } v \\ V_s = V_o \end{cases} \quad (2.33)$$

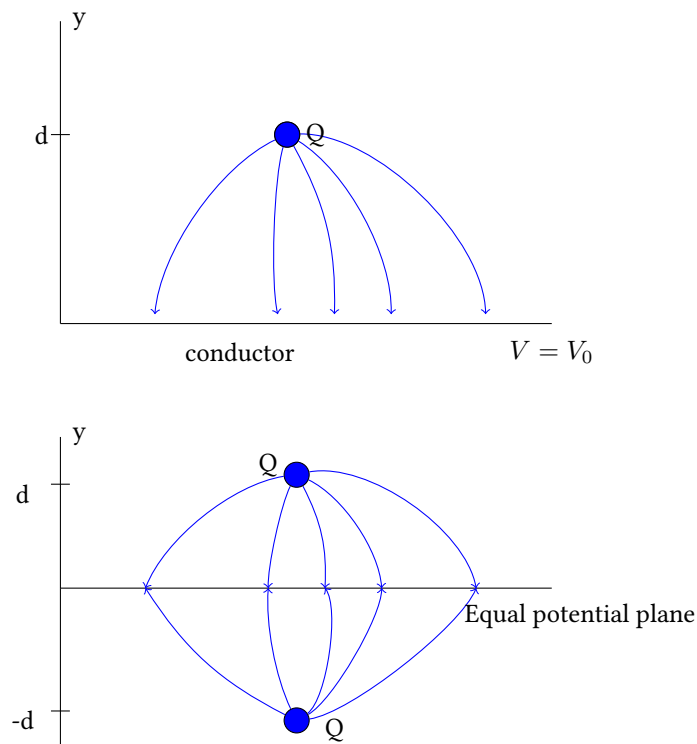
Theorem 10 **Uniqueness Theorem:** There is only one solution to Poisson's equation¹⁷ for a given set of sources and boundary conditions

¹⁷ of which the Laplace equation is a special case

SUBSECTION 2.6

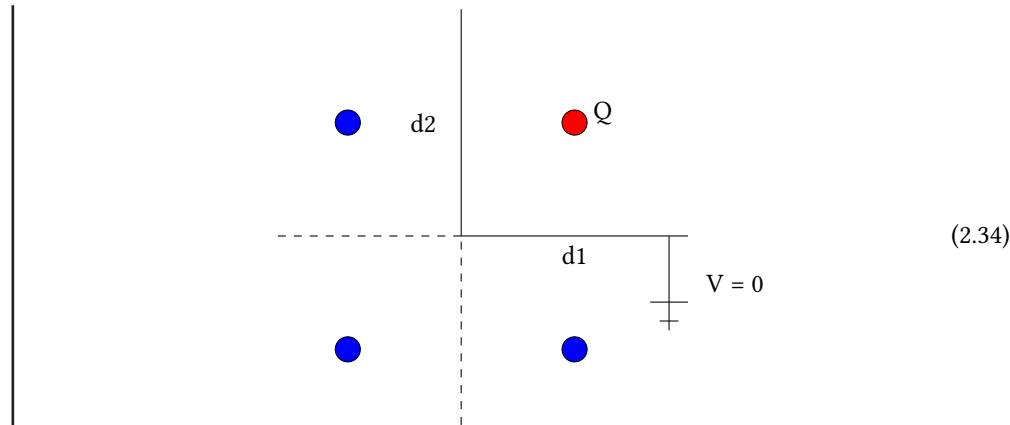
Lecture 20: Method of Images

Remark TLDR: Add imaginary charges such that the field lines meet perpendicular to the planes/lines of interest



And for the upper-half space ($y > 0$) the two cases have the same source distribution and the same boundary condition on the conducting surface.

Example | A charge Q is at $(d1, d2)$ relative to two conductor walls. Find force on charge Q due to induced charges on conductor surfaces.



We have to be careful about where to put these virtual charges. They are arranged this way so that there is a right angle at the conductors; that's why we don't just put a single charge in the bottom left quadrant.

The force can be found as follows:

$$\vec{F} = \frac{Q}{4\pi\epsilon} \left[\frac{-Q}{2d_1^2} \hat{x} + \frac{-Q}{2d_2^2} \hat{y} + \frac{Q(2d_x\hat{x} + 2d_2\hat{y})}{[(2d_1)^2 + (2d_2)^2]^{3/2}} \right] \quad (2.35)$$

2.6.1 Currents

There are a few types of currents:

- Conduction current: In conductors and semiconductors due to the motion of e^- and holes h ¹⁸
- Electrolytic current: motion of ions in solutions
- Convection current: motion of e^- or ions in vacuum

¹⁸ We will primarily focus on conduction currents

In conductors and semiconductors the free charge is

$$e = 1.6 \times 10^{-19} C \quad (2.36)$$

In the presence of an external field E ,

$$\begin{cases} \vec{E} = 0 & \text{random motion of free charges, current} = 0 \\ \vec{E} > 0 & \vec{\mu}_e = -\mu_o \vec{E}^{19} \end{cases} \quad (2.37)$$

¹⁹ $\vec{\mu}$ = average drift velocity, μ = mobility
Typical $\vec{\mu}_e$ is very slow: $\approx 10^{-5} 10^{-4} \frac{m}{s}$

Definition 11 General integral form for current

$$I = \int_S \rho_e \vec{\mu}_e \cdot d\vec{s} \quad (2.38)$$

PROOF Current is the amount of charge through a surface S per unit time

$$d\vec{S} = \vec{a}_n dS$$

The free charge in the volume of

$$\vec{\mu} \cdot \vec{a}_n dt dS \quad (2.39)$$

Will pass S in dt time

$$dq = -N_e e \vec{\mu}_e \cdot \vec{a}_n dt ds \quad (2.40)$$

Apply definition of current

$$dI = \frac{dq}{dt} = -N_e e \vec{\mu}_e \cdot d\vec{s} = \rho_e \vec{\mu}_e \cdot d\vec{s} \quad (2.41)$$

And integrate!

$$I = \int_S \rho_e \vec{\mu}_e \cdot d\vec{s} \quad (2.42)$$

□

This integral form can be a little awkward to work with so we define current density \vec{J}

Definition 12 Define current density to be \vec{J} such that

$$I = \int_S \vec{J} \cdot d\vec{s} \quad (2.43)$$

And therefore the volume current density \vec{J} with units $\frac{A}{m^2}$

$$\vec{J} = \rho_e \vec{\mu}_e \quad (2.44)$$

- \vec{J} is a vector and direction is the same as $\vec{\mu}_e$ and **is** a point function
- I is a scalar and **is not** a point function

Now ohm's law can be defined more rigorously in point/microscopic form

$$\begin{aligned} \vec{J} &= \rho_e \vec{\mu}_e = -\rho_o \mu_e \vec{E} \\ \Rightarrow J &= \sigma \vec{E} \end{aligned} \quad (2.45)$$

σ denotes conductivity

SUBSECTION 2.7

Lecture 21: Joule's Law and Resistance

Comment Previously we discussed perfect conductors. But what if that was not the case?

Theorem 11 **Joule's Law**

$$P = \int_v dp = \int_v \vec{E} \cdot \vec{J} dv \quad (2.46)$$

Intuition: Work done by E-field is turned into kinetic energy of e^- and produces heat

PROOF

Work done by E field on one e^- : $\Delta W = -e\vec{E}\Delta l$ Power per unit time: $p = \frac{\Delta W}{\Delta t} = -e\vec{E}\frac{\Delta l}{\Delta t} = -e\vec{E} \cdot \vec{\mu}_e$

$$dp = N_e(-e)\vec{E} \cdot \vec{\mu}_e dv = \rho_v \vec{E} \cdot dv \Rightarrow dp = \vec{E} \cdot \vec{J} dv \quad (2.47)$$

Therefore the dissipated power density $\frac{dp}{dv} = \vec{E} \cdot \vec{J}$

And then one can integrate over the volume to find the total dissipated power; joule's law

$$P = \int_v dp = \int_v \vec{E} \cdot \vec{J} dv \quad (2.48)$$

Noting that we can take the volume integral with respect to a surface and a length, i.e. $dv = dl ds$, we can write this integral as

$$P = \underbrace{\left(\int_l \vec{E} \cdot d\vec{l} \right)}_{\text{voltage drop}} \underbrace{\left(\int_S \vec{J} \cdot d\vec{s} \right)}_{\text{current}} \quad (2.49)$$

□

This enables us to define resistance R .

Definition 13

Resistance

$$R = \frac{V}{I} = \frac{\int_l \vec{E} \cdot d\vec{l}}{\int_S \vec{J} \cdot d\vec{s}} = \frac{\int_l \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{s}} \quad (2.50)$$

Resistance is independent of the voltage and current that passes through it and instead is dependent on the physical properties of the resistor

Example

For example, assuming uniform \vec{E}, \vec{J} ,

$$R = \frac{\int_l \vec{E} \cdot d\vec{l}}{\int_S \vec{J} \cdot d\vec{s}} = \frac{El}{\sigma E \cdot S} = \frac{l}{\sigma S} \quad (2.51)$$

Observe that resistance is proportional to the resistor length and dimensions l, S as well as the material properties σ .Recall: μ is the average drift velocityRecall: $\rho_v \vec{\mu}_e = \vec{J}$

This makes intuitive sense, no?

The steps to calculate resistance are as follows:

1. Choose coordinate system
2. Assume V_s : potential drop between terminals
3. Find \vec{E} from V . This can get complicated; need to start with the Laplace equation $\nabla^2 V = 0$ and solve for \vec{E} everywhere using the fact that $\vec{E} = -\nabla V$
4. Find current: $I = \int_S \vec{J} \cdot d\vec{s} = \int_S \sigma \vec{E} \cdot d\vec{s}$
5. Apply definition $R = \frac{V}{I}$

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Probability and Statistics

SECTION 3

Probability Distributions

SUBSECTION 3.1

Lecture 14: functions of random variables

In the discrete case, given X with PMF $f(x)$, we can define an *invertible* function Y where $Y = u(X)$, therefore can write $X = u^{-1}(Y)$. If $g(y)$ is the distribution of Y

$$\begin{aligned} g(y) &= P(Y = y) \\ &= P(u^{-1}(Y) = u^{-1}(y)) \\ &= f(u^{-1}(y)) \end{aligned} \quad (3.1)$$

In the continuous case we may arrive at

$$g(y) = f(u^{-1}(y)) \left| \frac{du^{-1}(y)}{dy} \right| \quad (3.2)$$

This result is derived through the Leibniz integral rule,

$$g(y) = \frac{d}{dy} \int_{-\infty}^{u^{-1}(y)} f(t) dt$$

Definition 14 The r^{th} moment about the origin of the random variable X is

$$\mu'_r = E[X^r] = \begin{cases} \sum_x x^r f(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x^r f(x) dx & X \text{ continuous} \end{cases} \quad (3.3)$$

- The mean is the first moment
- For variance, $\sigma^2 = E[X^2] - \mu^2 \rightarrow \sigma^2 = \mu'_2 - \mu^2$

Definition 15 The moment-generating function of X is defined as

$$\mu'_r = E[X^r] = \begin{cases} \sum_x e^{tx} f(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & X \text{ continuous} \end{cases} \quad (3.4)$$

In general

$$\mu'_r = \left. \frac{d^r M_X(t)}{dt^r} \right|_{t=0} \quad (3.5)$$

SUBSECTION 3.2

Lecture 15: More on moment generating functions

By definition and completing the square,

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{t2\pi\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t2\pi\sigma} e^{-\frac{x^2-2(x-\mu)^2+\mu^2}{2\sigma^2}}} dx \\
&= e^{\frac{2\mu t+t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}\right\} dx \\
&= e^{\frac{\mu t+t^2\sigma^2}{2}} \quad (3.6)
\end{aligned}$$

The integrand is just a normal PDF and thus integrates to one

3.2.1 Linear combinations of random variables

Given random variable X with distribution $f(x)$. What is the distribution, $h(y)$, of $Y = aX$?

Definition 16

Probability distribution of linear combination of random variables

Discrete:

$$\begin{aligned}
P(X = x) = f(x) &\rightarrow h(y) = P(Y = y) \\
&= P(aX = y) \\
&= f\left(\frac{y}{a}\right)
\end{aligned} \quad (3.7)$$

Continuous:

$$\begin{aligned}
F(y) &= P(Y \leq y) \\
&= P(X \leq \frac{y}{a}) \\
&= \int_{-\infty}^{\frac{y}{a}} f(t) dt \\
&= \int_{-\infty}^{\frac{y}{a}} f(t) dt = \int_{-\infty}^y \frac{1}{a} f\left(\frac{s}{a}\right) ds \\
\Rightarrow h(y) &= \frac{1}{|a|} f\left(\frac{y}{a}\right)
\end{aligned} \quad (3.8)$$

u-sub; let $s = at$

Extending from this, let's assume that X has the MGF ²⁰ $M_X(t)$. Then,

²⁰ Moment Generating Function

$$\begin{aligned}
M_Y(t) &= \int_{-\infty}^{\infty} e^{ty} h(y) dy \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{ty} f\left(\frac{y}{a}\right) dy \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{taz} f(z) a dz \\
&= \int_{-\infty}^{\infty} e^{taz} f(z) dz \\
&= M_X(at)
\end{aligned} \quad (3.9)$$

u-sub $z = \frac{y}{a}$

More generally,

$$M_{aX}(t) = M_X(at) \quad (3.10)$$

When working with more than one random variables, i.e. independent RV X, Y with distributions $f(x), g(y)$, the distribution of $Z = X + Y$, $h(z)$ is given by

$$P(Z = z) = P(X + Y = z) = \sum_w P(X = w)(Y = z - w) \Rightarrow h(z) = \sum_{w=-\infty}^{\infty} f(w)g(z - w) \quad (3.11)$$

And for the continuous case we arrive at the convolution integral,

$$h(z) = \int_{-\infty}^{\infty} f(w)g(z - w)dw \quad (3.12)$$

SUBSECTION 3.3

Lecture 16: Sampling

Given sample data x_1, \dots, x_n , each x_i is the realization of an RV ²¹, $X_i, i = 1, \dots, n$

²¹ Say each X_i is a Bernoulli random variable

Definition 17 Given the mean,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad (3.13)$$

the **sample mean** of a random variable X is defined as

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad (3.14)$$

Definition 18 The **sample median** is

$$x_m = \begin{cases} \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2} & \text{if } n \text{ even} \\ \frac{x_{\frac{n+1}{2}}}{2} & \text{if } n \text{ is odd} \end{cases} \quad (3.15)$$

And **mode** is familiar as the most frequently occurring value

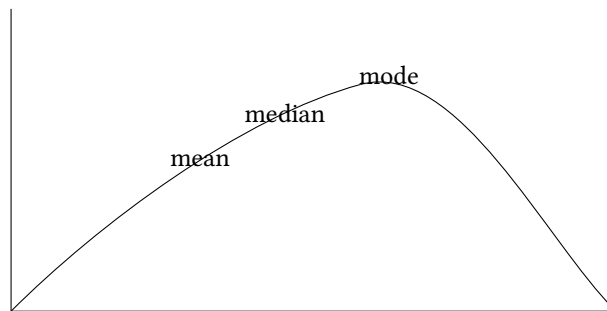


Figure 11. Median, mode, mean labelled on a skew distribution

Definition 19 **Sample variance**

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (3.16)$$

And the corresponding standard deviation ²²

$$s = \sqrt{s^2} \quad (3.17)$$

²²Is this correct??

Lemma 2 Consider

$$E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] \quad (3.18)$$

Assuming that X_i have the same distribution, $f(x_i)$, and mean μ ,

$$\begin{aligned} &= \frac{1}{n} \sum_{i=1}^n \mu \\ &= \frac{1}{n} n\mu \\ &= \mu \end{aligned} \quad (3.19)$$

- the X_i are identically distributed
- \bar{X} is an unbiased estimator of μ

We then have

$$\begin{aligned} S^2 &= \frac{1}{n-1} \sum_{i=1}^n X_i^2 + \bar{X}^2 - 2X_i\bar{X} \\ &= \frac{1}{n-1} (n\bar{X}^2 - 2n\bar{X}^2 + \sum_{i=1}^n X_i^2) \\ &= \frac{1}{n-1} (-n\bar{X}^2 + \sum_{i=1}^n X_i^2) \end{aligned} \quad (3.20)$$

We are now ready to take the expectation of S^2 , which is

Recall: if Y has mean, variance μ, σ^2 , then $\sigma^2 = E[Y^2] - \mu^2$

$$\begin{aligned} E[S^2] &= \frac{1}{n-1} (-nE[\bar{X}^2] + \sum_{i=1}^n E[X_i^2]) \\ &= \frac{1}{n-1} (-n(E[\bar{X}^2] + \text{var}[\bar{X}]) + \sum_{i=1}^n (\sigma^2 + \mu^2)) \\ &= \frac{1}{n-1} (-n(\mu^2 + \text{var}[\frac{1}{n} \sum_{i=1}^n X_i]) + n(\sigma^2 + \mu^2)) \\ &\quad \vdots \\ &= \frac{n-1}{n-1} \sigma^2 = \sigma^2 \end{aligned} \quad (3.21)$$

So we have an unbiased estimator of variance

So we get

$$S^2 = \sum_{i=1}^n (X_i - \bar{X})^2 \quad (3.22)$$

We divide by $n - 1$ because there is a "hidden" uncertainty in \bar{X}

TEP327

Engineering and Law

Torts can be identified by

- Elements of negligence can be identified
- There exists a duty of care
- Was the standard of duty of care breached?
- Did that breach of standard cause damage to the plaintiff?

Comment

Must be careful about the standard of care met or not met. For example in the in-class bike example one could argue that in designing the gear the engineer should expect that the gear could be commercialized and mass-produced later on for mass-market. On the other hand it is reasonable to argue that having to design the gear to be used by unskilled 16-year-olds is an unreasonable and outside the standard of care.

Trespass is strict liability. Ignorance of the law is no excuse.

SECTION 4

Property

- There are a number of types of property, i.e. intellectual property, personal (usually regarding money), etc
- Real property
 - Usu. owned by default by government (derived from old english law)
 - Gov. transfers property rights to private owners; sell, lease, etc (fee simple)
 - Gov. retains a few rights on real property, usu. oil and mineral rights
- Joint ownership: death of one partner causes other to own 100%. Tenancy in common/joint tenancy: death of a partner does not change ownership; usu. determined by wills
- Land interest is divisible; different people can own rights to different things on a piece of land. Or many people can own right to lease a property but not necessarily ownership of the land
- Rights associated with real property:
 - Mortgage: property as security for a debt
 - License: Contractual right to use property in some way
 - Restrictive covenant: conditions of use for that property
 - Easement: right to use property in some way

- Lien: right to register debt against land (must pay off people you owe with money from selling property before taking it for yourself)
- Lease: right to occupy the property
- Profit à prendre; right to extract value from property

Mineral, oil, gas rights:

- Mineral rights are crown or "freehold"
 - Crown rights can be provincial or federal and can grant license to third party to extract resources
 - Rights to resources generally excluded from fee simple property rights
 - Freehold mineral rights were granted with early land grants
 - Resource rights usu. include a right to access to the land in order to extract resources
- Land is registered via the registry system/land title system. Registry system is old and only about 10% of land is in it, is complicated, and led to a lot of squatters. Land title system (ontario) is easy to use.
 - **chattels** are tangible personal property called 'goods'. No registry for ownership but is registration of security interests (which demonstrates rights based on order of registration).

SUBSECTION 4.1

Intellectual Property

- A bundle of right that protect expression of ideas, inventions, symbols, and more
- Governed by federal legislation and international treaties
- Has a few different types.
- Copyright
 - Must be an original product of skill, permanent, and published
 - Ephemeral works e.g. live works have no copyright
 - Most things are copyrightable.
 - Copyright protection is valid for about 50 years after death, soon to be 70.
 - Protection is not dependent on registration, but copyrights can be registered
 - Copyright can be assigned by contract, but moral rights cannot be [although they can be waived]
- Trade secrets
 - What todo if you invent something?
 - Keep your mouth shut
 - Not everything is patentable and not all things are patentable
 - Trade secret is information that is kept secret with an industrial or commercial application
 - Once the secret is out it's not possible to make it a secret again
 - Reverse engineering is a totally valid way to discover a trade secret

- Patent

- A right granted by the crown to a monopoly
- Term of patent in Canada is 20 years, non-renewable
- A balance between the interests of the inventor and that of society
- A few things that cannot be patented; scientific principles, abstract theorems, higher life forms (animals).
- A patent must be new, useful, and non-obvious
- Must pass the obviousness test: 1) Identify a PSITA (Personal skilled in the art) 2) Identify inventive concept 3) Find difference between prior art and inventive concept 4) Ask: Would PSITA find this obvious?
- A patent will have a specification: a description of the procedure/invention, how to reproduce it, etc.
- ... and claims
- Patents are filed with the Canadian Intellectual Property office and the federal court always has jurisdiction
 - * Clear and explicit definition of the subject matter
 - * Will inform the scope of the rights granted
- Infringement is "Use by someone other than the patent holder which inteferes in whole or in part with the patent holder's monopoly" (Monsanto Canada Inc. v. Schmeiser, 2004 SCC 34)
- Usu. more to do with the right to monopoly than the IP itself.
- Some exceptions: use of the invention for development and submission of information required by law, fair dealing (experimentation without eye to profit or fraudulent purpose), dedication (patent owner can ignore monopoly), exhaustion (once patent is sold, patent holder may no longer control use)

This is the case in Canada but not really in other places.

This also implies a patent can only have a single inventive concept

- Trademark

- A sign or combination of signs that is used by a person for sake of distinguishing or so as to distinguish their goods and services from those of others
- A certification mark
- **trade name**: means the name under which any business is carried on, whether or not if it is a corporation, partnership, or individual
- 15 years of protection, is renewable
- Certain marks are prohibited from being a trademark; royal coat of arms, Canadian flag, etc.

- Industrial Designs and Integrated Circuit Topographies: 10 years and protects the design, look, and configuration of ICs, industrial design. REgistrar of topographies designed by minster from people in the department of industry

Summary

- Copyright: inherent, no registry. Need to provide proof.
- Trademark: registry
- Patent: registry
- Topography & industrial design: registry. Unknown how strong the protection is so far.

In all cases there is no protection for the idea; the invention or creation itself is protected. Let's say that you want to patent your invention.

- Serach registered patents for prior art. This is a difficult process, usu. patent agents will do this
- EPO (European) is usually a good place to start because their website doesn't suck. Canada/US have one but it isn't great to work with.

For trademark, search registry to make sure that there is no prior registry. Note that one can't trademark a name, but it can be trademarked for an item if it can be proved to be distinctive; Dyson vacuums, Apple computers, etc.

Trade secret: no protection if competitor reverse-engineers it. As trade secrets are not public, it is not prior art and a competitor could reverse engineer it (which is totally valid).

Know-how: personal to you or corporation; i.e. a plumber's experience, or a boat-builder's special way of bending ribs. Secret, substantial, identified, and valuable; non-patented practical information which has an economic value not publicly accessible. This is different from a trade secret because just telling someone the know-how won't give them the competitive advantage.

SECTION 5

Contracts

- Sets out rights, responsibilities, and liabilities of parties
- Allocates risks between parties
- Enforceable
- Do not have to be written
- Can have express/implied terms
- Only parties privy to contract can enforce terms

For there to be a contract, there must be:

- Offer and acceptance
- Capacity to contract (People are in the right state to enter a contract)
- Consideration
- Intention to create legal obligations
- Lawful purpose

The offer and acceptance process can be messy

- Invitations to treat are calls for offers (e.g. ads); they are not offers unto themselves
- A counter offer is a rejection of an offer

Capacity to contract

- A person under law: partnerships may contract, sole proprietorships contract in the name of their sole proprietor
- Must have capacity

- Age of majority
 - Incapacitated due to alcohol/drugs
 - Mentally unfit
- Ostensible authority: where one partner contracts, other is implicitly involved as well even if they are not involved.

Consideration

- The contract must exchange something of value (the "consideration")
- Consideration is essential for there to be a contract
- No consideration = no contract; instead it will be a gift or merely an unenforceable promise
- Considerations do not have to be of equal value
- Services and forbearance are valid consideration

Intention to create a contract

- Must intent to create a contract
- An agreement to agree is not a contract
- Letters of intent or memorandums of understanding (MOU) are not contracts; they are intentions to form a contract later on
- MOUs can become contracts under the right circumstances

Professional service agreements

- Define the scope of work really well
- Define anticipated result/outcome as well as cost/value
- Should contain limitation of liability
- May contain alternative remedies

SUBSECTION 5.1

Standard Form Contracts
