Week # 6

Discrete Structure

by

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RIPHAH

TOPICS FOR TODAY

- Set Theory

“SET”:

A set is any well-defined collection or list of distinct objects, e.g. a group of students, the books in a library, the integers between 1 and 100, all human beings on the earth, etc.

The term well-defined here means that any object must be classified as either belonging or not belonging to the set under consideration, and the term distinct implies that each object must appear only once.

The objects that are in a set, are called members or elements of that set. Sets are usually denoted by capital letters such as A, B, C, Y, etc., while their elements are represented by small letters such as, a, b, c, y, etc.

Elements are enclosed by parentheses to represent a set.

For example:

Examples of Sets:

$A = \{a, b, c, d\}$ or

$B = \{1, 2, 3, 7\}$

The Number of a set A, written as $n(A)$, is defined as the number of elements in A.

If x is an element of a set A, we write $x \in A$ which is read as "x belongs to A" or x is in A.

If x does not belong to A, i.e. x is not an element of A, we write $x \notin A$.

A set that has no elements is called an empty or a null set and is denoted by the symbol ϕ.

(It must be noted that $\{0\}$ is not an empty set as it contains an element 0.)

If a set contains only one element, it is called a unit set or a singleton set.

It is also important to note the difference between an element "x" and a unit set {x}.

A set may be specified in two ways:

1. We may give a list of all the elements of a set
(the “Roster” method),

e.g.

$A = \{1, 3, 5, 7, 9, 11\}$;

$B = \{$a book, a city, a clock, a teacher$\}$;

2. We may state a rule that enables us to determine whether or not a given object is a member of the set
(the “Rule” method or
the “Set Builder” method),
e.g.

$A = \{x \mid x \text{ is an odd number and } x < 12\}$
meaning that A is a set of all elements x such that x is an odd number and x is less than 12. (The vertical line is read as “such that”.)

An important point to note is that:

The repetition or the order in which the elements of a set occur, does not change the nature of the set.

The size of a set is given by the number of elements present in it.

This number may be finite or infinite.

Thus a set is finite when it contains a finite number of elements, otherwise it is an infinite set.

The Empty set is regarded as a Finite set.

Examples of finite sets:

i) $A = \{1, 2, 3, \ldots, 99, 100\}$;

ii) $B = \{x \mid x$ is a month of the year$\}$;

iii) $C = \{x \mid x$ is a printing mistake in a book$\}$;

iv) $D = \{x \mid x$ is a living citizen of Pakistan$\}$;

Examples of infinite sets:

i) $A = \{x \mid x \text{ is an even integer}\}$;

ii) $B = \{x \mid x \text{ is a real number between 0 and 1 inclusive}\}$,

i.e. $B = (x \mid x\ 0 \leq x \leq 1\}$

iii) $C = \{x \mid x \text{ is a point on a line}\}$;

iv) $D = \{x \mid x \text{ is a sentence in a}$

SUBSETS

A set that consists of some elements of another set, is called a subset of that set.

For example, if B is a subset of A, then every member of set B is also a member of set A.

If B is a subset of A, we write:

$B \subset A$

or equivalently:

$A \supset B$

'B is a subset of A' is also read as 'B is contained in A', or 'A contains B'.

EXAMPLE

If $A = \{1, 2, 3, 4, 5, 10\}$

and $B \{1, 3, 5\}$

then $B \subset A$,

i.e. B is contained in A.

It should be noted that any set is always regarded a subset of itself. and an empty set ϕ is considered to be a subset of every set.

Two sets A and B are <u>Equal</u> or <u>Identical</u>, if and only if they contain <u>exactly the same elements</u>.

In other words, $A = B$ if and only if $A \subset B$ and $B \subset A$.

Proper Subset

If a set B contains some but not all of the elements of another set A, while A contains each element of B, i.e. if

$B \subset A$ and $B \neq A$

then the set B is defined to be a proper subset of A.

Universal Set

The original set of which all the sets we talk about, are subsets, is called the universal set (or the space) and is generally denoted by S or Ω.

The universal set thus contains all possible elements under consideration.

A set S with n elements will produce 2^n subsets, including S and ϕ.

EXAMPLE

Consider the set $A = \{1, 2, 3\}$.

All possible subsets of this set are:

ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.

Hence, there are $2^3 = 8$ subsets of the set A.

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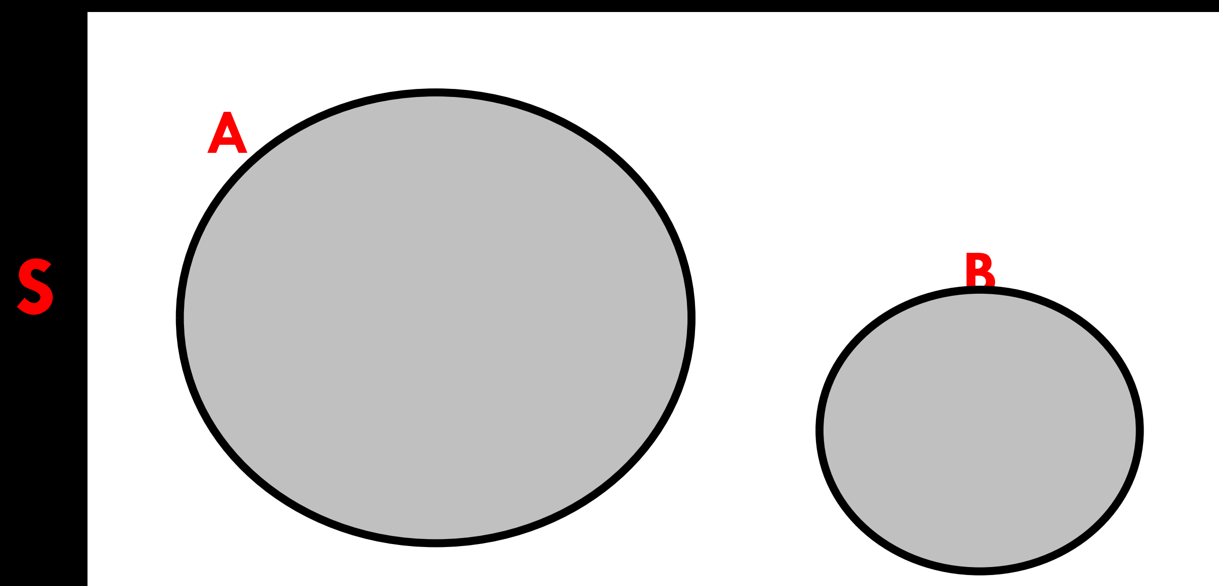
VENN DIAGRAM.

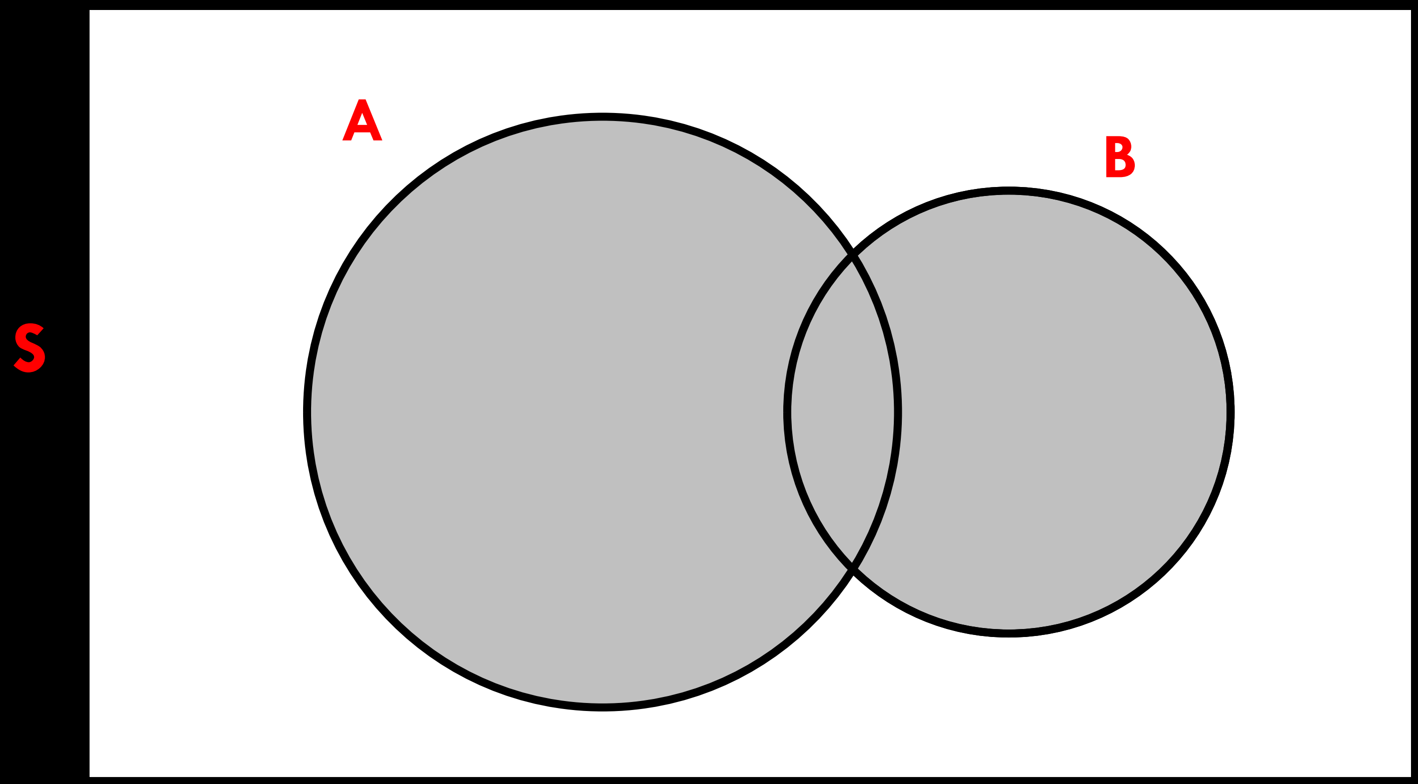
A diagram that is understood to represent sets by <u>circular regions</u>, parts of circular regions or their complements <u>with respect to a rectangle</u> representing the space S is called a <u>Venn diagram</u>, named after the English logician John Venn (1834-1923).

The Venn diagrams are used to represent sets and subsets in a <u>pictorial way</u> and to verify the <u>relationship</u> among sets and subsets.

A Simple Venn Diagram: Disjoint Sets



Overlapping Sets



OPERATIONS ON SETS

Let the sets A and B be the subsets of some universal set S. Then these sets may be <u>combined</u> and <u>operated on</u> in various ways to form <u>new sets</u> which are also subsets of S.

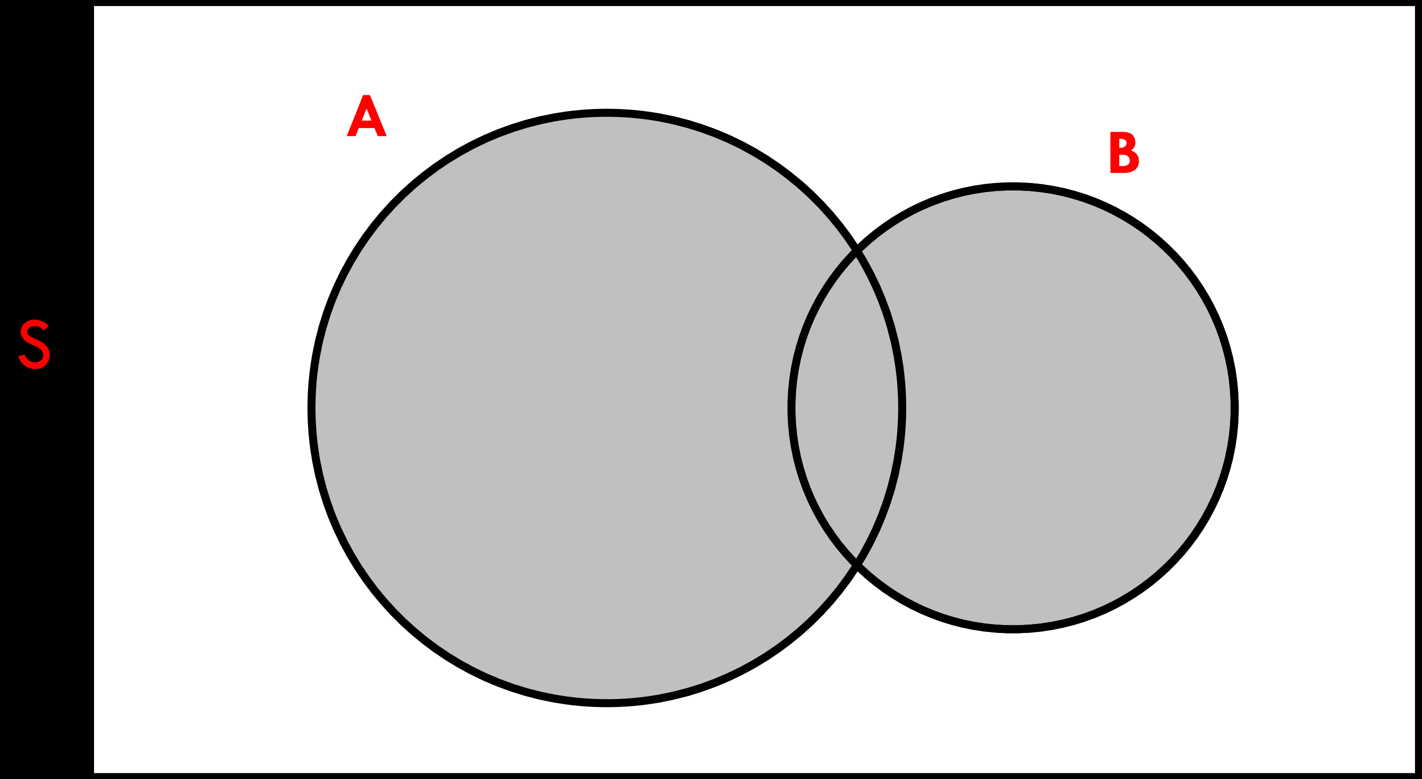
The basic operations are <u>union</u>, <u>intersection</u>, <u>difference</u> and <u>complementation</u>.

UNION OF SETS

The union or sum of two sets A and B, denoted by $A \cup B$, and read as "A union B", means the set of all elements that belong to at least one of the sets A and B, that is

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B\}$$

By means of a Venn Diagram, $A \cup B$ is shown by the shaded area as below:



$A \cup B$ is shaded

Example:

Let $A = \{1, 2, 3, 4\}$ and

$B = \{3, 4, 5, 6\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

INTERSECTION OF SETS

The intersection of two sets A and B, denoted by

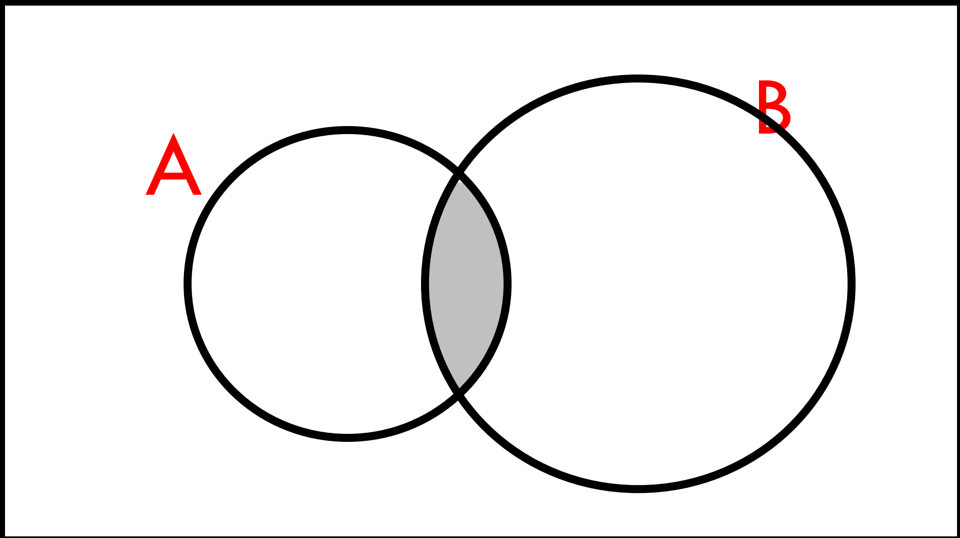
$A \cap B$, and read as "A intersection B", means that the

set of all elements that belong to both A and B; that is

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Diagrammatically, $A \cap B$ is shown by the shaded area as below:

S



$A \cap B$ is shaded

Example:

Let $A = \{1, 2, 3, 4\}$ and

$B = \{3, 4, 5, 6\}$

Then $A \cap B = \{3, 4\}$

The operations of union and intersection that have been defined for two sets may conveniently be extended to any finite number of sets.

DISJOINT SETS

Two sets A and B are defined to be disjoint or mutually exclusive or non-overlapping when they have no elements in common, i.e. when their intersection is an empty set

i.e.

$$A \cap B = \phi.$$

On the other hand, two sets A and B are said to be <u>conjoint</u> when the have <u>at least one element in common</u>.

SET DIFFERENCE

The difference of two sets A and B, denoted by $A - B$ or by $A - (A \cap B)$, is the set of all elements of A which do not belong to B.

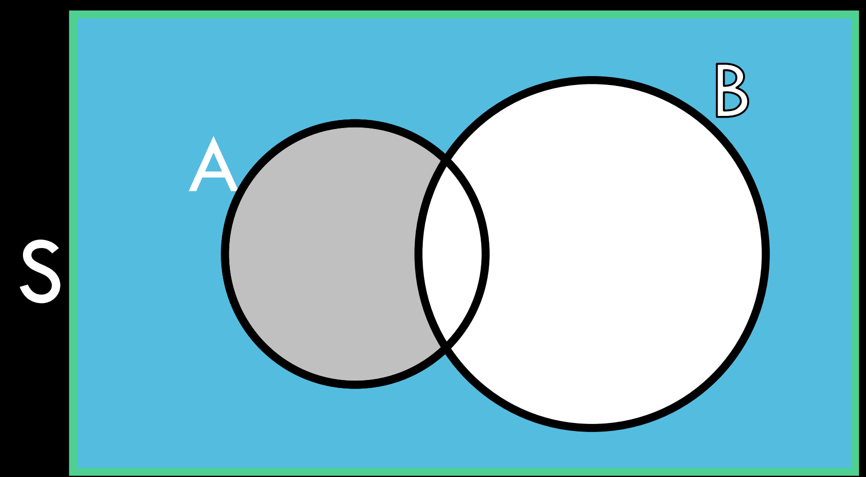
Symbolically,

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

It is to be pointed out that in general $A - B \neq B - A$.

The shaded area of the following Venn diagram shows the difference $A - B$:

Difference A – B is shaded



It is to be noted that A – B and B are disjoint sets.

If A and B are disjoint, then the difference A – B coincides with the set A.

COMPLEMENTATION

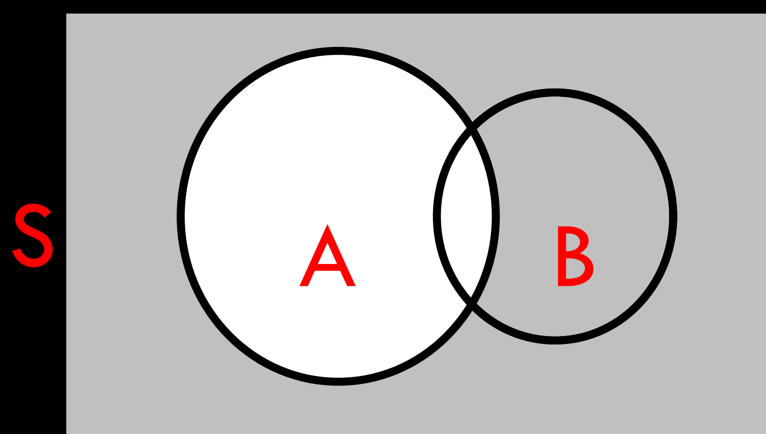
The particular difference S – A, that is, the set of all those elements of S which do not belong to A, is called the complement of A and is denoted by $\overline{A}$ or by A^c.

In symbols:

$$\overline{A} = \{x \mid x \in S \text{ and } x \notin A\}$$

The complement of S is the empty set ϕ.

The complement of A is shown by the shaded portion in the following Venn diagram.



$\overline{A}$ is shaded

It should be noted that $A - B = A \cap \overline{B}$,

where $\overline{B}$ is the complement of set B, are

the <u>same</u> set.

Next, we consider the <u>Algebra</u> of Sets.

The algebra of sets provides us with <u>laws</u> which can be used to <u>solve</u> many problems in probability calculations.

Let A, B and C be any subsets of the universal set S. Then, we have:

Commutative laws

$A \cup B = B \cup A$ and

$A \cap B = B \cap A$

Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$

and

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive laws

$$A \cap (B \cup C)$$

$$= (A \cap B) \cup (A \cap C)$$

and $A \cup (B \cap C)$

$$= (A \cup B) \cap (A \cup C)$$

Idempotent laws

$A \cup A = A$ and

$A \cap A = A$

Identity laws

$A \cup S = S$,

$A \cap S = A$,

$A \cup \phi = A$, and

$A \cap \phi = \phi$.

Complementation laws

$A \cup \overline{A} = S,$

$A \cap \overline{A} = \phi,$

$\overline{(\overline{A})} = A,$

$\overline{S} = \phi$, and

$\overline{\phi} = S$

De Morgan's laws

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B},$$

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

and

PARTITION OF SETS

A partition of a set S is a sub-division of the set into non-empty subsets that are disjoint and exhaustive, i.e. their union is the set S itself.

This implies that:

i) $A_i \cap A_j = \phi$, where $i \neq j$;

ii) $A_1 \cup A_2 \cup \ldots \cup A_n = S$.

EXAMPLE

Let us consider a set $S = \{a, b, c, d, e\}$.

Then $\{a, b\}$, and $\{c, d, e\}$ is a partition of S as each element of S belongs to exactly one cell.

The subsets in a partition are called cells.

CLASS OF SETS

A set of sets is called a class. For example, in a set of lines, each line is a set of points.

POWER SET

The class of ALL subsets of a set A is called the Power Set of A and is denoted by P(A).

For example, if $A = \{H, T\}$, then $P(A) = \{\phi, \{H\}, \{T\}, \{H, T\}\}$.

MEMBERSHIP TABLE:

A table displaying the membership of elements in sets. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

Membership tables can be used to prove set identities.

A	A^c
1	0
0	1

The above table is the Membership table for Complement of A. Now in the above table note that if an element is the member of A, then it cannot be the Member of A^c thus where in the table we have 1 for A in that row we have 0 in A^c.

Similarly, if an element is not a member of A, it will be the member of A^c.

So we have 0 for A and 1 for A^c.

MEMBERSHIP TABLE FOR UNION:

A	B	$A \cup B$
1	1	1
1	0	1
0	1	1
0	0	0

REMARK:

This membership table is similar to the truth table for logical connective, disjunction ($\vee$).

MEMBERSHIP TABLE FOR INTERSECTION:

A	B	$A \cap B$
1	1	1
1	0	0
0	1	0
0	0	0

REMARK:

This membership table is similar to the truth table for logical connective, conjunction ($\wedge$).

MEMBERSHIP TABLE FOR SET DIFFERENCE:

A	B	A – B
1	1	0
1	0	1
0	1	0
0	0	0

REMARK:

The membership table is similar to the truth table for ~ (p →q).

MEMBERSHIP TABLE FOR COMPLEMENT:

A	A^c
1	0
0	1

REMARK

This membership table is similar to the truth table for logical connective negation (~)

Quiz # 2

PROVING SET IDENTITIES BY MEMBERSHIP TABLE:

Prove the following using Membership Table:

(i) $A-(A-B)=A\cap B$

(ii) $(A\cap B)^c=A^c\cup B^c$

(iii) $A-B=A\cap B^c$