Math 630 Homework 0

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1)

We want to simplify $\bigcup_{n=1}^{\infty} \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n})$

From DeMorgan's Law, this union of a complement with be equivalent to the complement of the intersection. Thus we see:

$$\bigcup_{n=1}^{\infty} \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n}) = \mathbb{R} \backslash \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n}) = \mathbb{R} \backslash [0, 1] = (-\infty, 0) \cup (1, \infty)$$

Proof:

To prove equivalence, we must show that $\bigcup_{n=1}^{\infty} \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n})$ and $\mathbb{R} \backslash \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n})$ are subsets of each other:

Let $x \in \bigcup_{n=1}^{\infty} \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n})$. In other words, $x \in \mathbb{R}$, $x \notin [-\frac{1}{n}, 1+\frac{1}{n}),\ n \in \mathbb{N}$. Thus $x \notin \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n})$ since $x \notin [-\frac{1}{n}, 1+\frac{1}{n}),\ n \in \mathbb{N}$. This implies $x \in \mathbb{R} \backslash \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n})$ meaning that $\bigcup_{n=1}^{\infty} \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n}) \subseteq \mathbb{R} \backslash \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n})$.

Let $x \in \mathbb{R} \backslash \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n})$ meaning that $x \in \mathbb{R}$ and $x \notin \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n})$. Therefore, $x \notin [-\frac{1}{n}, 1+\frac{1}{n}),\ n \in \mathbb{N}$. This implies that $x \in \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n})$ meaning that $x \in \bigcup_{n=1}^{\infty} \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n})$. Thus, $\mathbb{R} \backslash \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 1+\frac{1}{n}) \subseteq \bigcup_{n=1}^{\infty} \mathbb{R} \backslash [-\frac{1}{n}, 1+\frac{1}{n})$

The two sets are subsets of each other. Thus, they are equivalent ■

2)

We want to find all rings such that $0 = 1$, where 0 is the element satisfying $0 + a = a$ for every element in the ring.

Proof:

Let $\mathcal{R}$ be a ring and let a be an element in $\mathcal{R}$. We have that $1a = a$ and $0a = 0$. Thus to have $0 = 1$, we necessarily have that $0a = 0 = a = 1a$ meaning that $\mathcal{R}$ contains only 0. In other words, the only ring to have the property $0 = 1$ is the zero ring.

4)

For $f(x) = x^3 + x^2 - x - 1$, we want to find the inverse image of set W i.e. all x such that $f(x) \in W$:

(a) $W = \{0\}$;

Since this set only contains only 0, we can simply set f equal to 0 and solve. The zeros of f are ± 1 so $f^{-1}(W) = \{-1, 1\}$.

(b) $W = (-\infty, 0]$;

Here we want all x that yield $f(x) \leq 0$. The root -1 has multiplicity 2 meaning that f is tangent to the y-axis at $x = -1$. Further, $f''(-1) < 0$ implying downward concavity here. Thus, $f(x) \leq 0$ for $x \in (-\infty, 1]$ meaning that $f^{-1}(W) = (-\infty, 1]$

(c) $W = (-1, \infty)$;

Setting $f(x)$ equal to -1, we see that $f(x) = -1$ at $x = 0, \phi, -\phi$ where ϕ is the Golden Ratio. Thus, $f(x) > -1$ for $x \in (-\phi, 0) \cup (\phi, \infty)$ meaning that $f^{-1}(W) = (-\phi, 0) \cup (\phi, \infty)$

5)

We have that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $M \subseteq D$. We want to show that $M \subseteq f^{-1}(f(M))$.

Proof:

First we assume that f has an inverse. Let $x \in M$. Then, we will have that $f(x) \in f(M)$ which implies that $x \in f^{-1}(f(M))$. Thus, $M \subseteq f^{-1}(f(M))$. $\blacksquare$

The inclusion is proper when f is not one-to-one. For example, let $f(x) = x^2$ meaning $f^{-1}(x) = \pm\sqrt{x}$. Further, let $M = \{1\}$. $f^{-1}(f(M)) = \{-1, 1\} \supset M$.