Real Analysis Homework 1

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1)

We want to prove for every collection of subsets of a given set X, there is a smallest σ-algebra that contains it.

Proof:

Let B be an arbitrary collection of subsets of X and let C be the collection of all σ-algebras containing it.

We first need to show that C is non-empty: By definition, $B \in \mathcal{P}(X)$. Further $\mathcal{P}(X)$ is a σ-algebra because it is closed under complements and countable unions. Thus, $\mathcal{P}(X) \in C$ meaning C is non-empty.

Next, we need to show that there is a smallest member of C: Define H as the set of all sets contained in C. Let $D = \bigcap H$ meaning that D is also a σ-algebra containing B. Let G be a set contained in H. Since D is the intersection of all such sets,
$\mathbf{card}(D) < \mathbf{card}(G)\ \forall G \in H, G \neq D$. In other words, D is the smallest σ-algebra that contains B. ■

2)

We have that f is the function from set X to its power set $\mathcal{P}(x)$. We want to show that there is a set $E \in \mathcal{P}(x)$ such that $E \notin f(X)$.

Proof:

Let $E \subset X$ such that $E = \{x \in X : x \notin f(x)\}$ and assume $f(y) = E$ for some $y \in X$. Consider the following two cases:

1) $y \in f(y)$: If this is the case then $y \in E$ since $f(y) = E$ but this contradicts the definition of E since in order to be in E, y must not be in $f(y)$.

2) $y \notin f(y)$: In this case, $y \notin E$ since $y \notin f(y) = E$, but y must be in E since $y \notin f(y)$ so this is another contradiction.

Therefore, we must conclude that $f(y) \neq E\ \forall y \in X$ implying that $E \notin f(X)$. ■

In addition, we can conclude that $\mathbf{card}(X) < \mathbf{card}(\mathcal{P}(X))$.

3)

We want to provide an example of a partially ordered set $(X, \prec)$ which has a unique minimal element but no smallest element.

Consider the set $X = \{x : x = 2^n \text{ for } n \in \mathbb{Z}^+ : n > 0\}$. We define $x \preceq y$ for $x, y \in X$ to mean either $x = y = 2$ or $x^y < y^x$.

Here, 2 is the minimal element since there is no $x \in X$ such that $x < 2$. However, 2 is not the smallest element because $2 \npreceq 4$.

4)

We want to show that if a strictly linearly ordered set has no strictly decreasing subsequence then it is a well-ordered set.

Proof:

Let X be a strictly linearly ordered set with no strictly decreasing subsequences. Since X has no strictly decreasing subsequences, none of its subsets will either. Let $S \subseteq X$ and choose a subsequence $P = \{x_i : i \in \mathbb{N}\} \subseteq S$ such that P decreases for finitely many iterations like so: $x_1 > x_2 > \cdots > x_n < x_{n+1}$. Since P decreases for finitely many iterations, it must contain an element with no elements in P that are smaller. In other words P has a minimal element. The strict linear ordering of X implies that this minimal element is also the smallest. Therefore, all subsets of X contain a smallest element meaning that X is well-ordered. ∎