AP-01 Notes: CARA-Normal Equilibrium

Dr. Ian Helfrich

January 19, 2026

Purpose of These Notes

These notes are the "book version" of AP-01. The slides are the live, exam-style walkthrough. Here I slow down and write every step I expect you to reproduce under time pressure.

Roadmap

We will:

1. Set up a one-period CARA-normal economy.
2. Derive the certainty equivalent and show the mean-variance form.
3. Solve the individual demand problem.
4. Aggregate demands and clear the market.
5. Interpret the equilibrium price and the risk premium.

1. Setup

We study a one-period economy with N risky payoffs collected in the vector X. Assume

$$X \sim \mathcal{N}(\mu, \Sigma).$$

Agents have CARA utility

$$U(c) = -\exp(-\alpha c), \quad \alpha > 0.$$

Let $p \in \mathbb{R}^N$ be the price vector of risky payoffs. A portfolio is a vector of holdings $\theta \in \mathbb{R}^N$.

Final wealth is

$$c = w_0 - p'\theta + \theta' X.$$

2. CARA + Normal Implies a Certainty Equivalent

Key fact: if $Y \sim \mathcal{N}(m, s^2)$, then

$$\mathbb{E}[\exp(-\alpha Y)] = \exp\left(-\alpha m + \frac{\alpha^2}{2}s^2\right).$$

Proof sketch: the moment generating function of a normal variable gives $\mathbb{E}[\exp(tY)] = \exp(tm + \frac{1}{2}t^2 s^2)$. Set $t = -\alpha$.

Therefore, if c is normal,

$$\mathbb{E}[U(c)] = -\exp\left(-\alpha\mathbb{E}[c] + \frac{\alpha^2}{2}\mathrm{Var}(c)\right).$$

Maximizing expected utility is equivalent to maximizing the certainty equivalent

$$CE = \mathbb{E}[c] - \frac{\alpha}{2}\mathrm{Var}(c).$$

This is why CARA-normal problems become mean-variance problems.

3. Individual Demand

Compute mean and variance:

$$\begin{aligned}\mathbb{E}[c] &= w_0 - p'\theta + \theta'\mu, \\ \mathrm{Var}(c) &= \theta'\Sigma\theta.\end{aligned}$$

So

$$CE = w_0 - p'\theta + \theta'\mu - \frac{\alpha}{2}\theta'\Sigma\theta.$$

Differentiate with respect to θ and set to zero:

$$-p + \mu - \alpha\Sigma\theta = 0.$$

Solve for optimal demand:

$$\theta^* = \frac{1}{\alpha}\Sigma^{-1}(\mu - p).$$

Interpretation: demand is increasing in expected payoff μ, decreasing in price p, and scaled down by risk aversion and covariance risk.

4. Aggregate Risk Tolerance

Let agent h have risk aversion α_h. Summing demands yields

$$\sum_h \theta_h^* = \left(\sum_h \frac{1}{\alpha_h}\right)\Sigma^{-1}(\mu - p).$$

Define total risk tolerance

$$T = \sum_h \frac{1}{\alpha_h}.$$

Then aggregate demand is

$$\sum_h \theta_h^* = T\Sigma^{-1}(\mu - p).$$

5. Market Clearing and Equilibrium Price

Let $\bar{\theta}$ be the fixed supply of risky payoffs. Market clearing implies

$$\bar{\theta} = T\Sigma^{-1}(\mu - p).$$

Solve for price:

$$p = \mu - \frac{1}{T}\Sigma\bar{\theta}.$$

This is the equilibrium pricing equation.

6. Risk Premia and Interpretation

Rewrite as

$$\mu - p = \frac{1}{T}\Sigma\bar{\theta}.$$

Thus expected excess payoffs are proportional to covariance with aggregate risk. The scalar $1/T$ is the price of risk.

Economic intuition:

- Larger T (more risk tolerance) raises prices and lowers premia.
- Higher aggregate risk exposure $\bar{\theta}$ lowers prices.
- Only systematic risk matters; idiosyncratic risk washes out in aggregation.

7. Worked Example (Two Assets)

Let

$$\mu = \begin{bmatrix} 1.08 \\ 1.12 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.04 & 0.01 \\ 0.01 & 0.09 \end{bmatrix}, \quad \bar{\theta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T = 2.$$

Compute

$$\Sigma\bar{\theta} = \begin{bmatrix} 0.05 \\ 0.10 \end{bmatrix}.$$

Then

$$p = \mu - \frac{1}{2}\begin{bmatrix} 0.05 \\ 0.10 \end{bmatrix} = \begin{bmatrix} 1.055 \\ 1.07 \end{bmatrix}.$$

8. Comparative Statics

- If T doubles, risk premia halve.
- If Σ increases (more risk), prices fall.
- If supply $\bar{\theta}$ rises, prices fall proportionally to risk exposure.

9. Exam Checklist

Before exam day, you should be able to do the following without notes:

1. Derive the certainty equivalent for CARA utility under normality.
2. Compute the demand vector and interpret each term.
3. Aggregate demands and solve for equilibrium prices.
4. Translate prices into expected excess payoffs and interpret risk premia.

10. Practice Problems

1. Single asset: derive θ^* and solve for p given $\bar{\theta} = 1$.
2. Two agents: compute T with $\alpha_1 = 1$ and $\alpha_2 = 2$ and solve for p.
3. Comparative statics: show how p changes when Σ doubles.

Instructor Note

I am an independent researcher and PhD (Georgia Institute of Technology), not currently affiliated with any institution. These notes are my own work.