AP-02 Notes: CAPM and SDF Foundations

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January 19, 2026

Purpose of These Notes

These notes are the "book version" of AP-02. The slides are the live exam walkthrough. This document is the full derivation with commentary and structure.

Mean-Variance Setup

Consider N risky returns $R \in \mathbb{R}^N$ with mean μ and covariance Σ. A risk-free asset yields gross return R_f. A portfolio with risky weights w has return

$$R_p = R_f + w'(R - R_f\mathbf{1}).$$

Then

$$\begin{aligned} \mathbb{E}[R_p] &= R_f + w'(\mu - R_f\mathbf{1}), \\ \mathrm{Var}(R_p) &= w'\Sigma w. \end{aligned}$$

Mean-Variance Optimization

A mean-variance investor solves

$$\max_w \ \mathbb{E}[R_p] - \frac{\gamma}{2}\mathrm{Var}(R_p).$$

The first-order condition is

$$\mu - R_f\mathbf{1} - \gamma\Sigma w = 0.$$

So optimal risky holdings satisfy

$$w \propto \Sigma^{-1}(\mu - R_f\mathbf{1}).$$

All investors hold the same risky portfolio up to scale.

Market Clearing and the CAPM

In equilibrium, the market portfolio M must be mean-variance efficient. Let w_M denote market weights. Then for some scalar κ,

$$\Sigma w_M = \kappa(\mu - R_f\mathbf{1}).$$

Take the ith element:

$$(\Sigma w_M)_i = \kappa(\mu_i - R_f).$$

But $(\Sigma w_M)_i = \text{Cov}(R_i, R_M)$, so

$$\mu_i - R_f = \frac{1}{\kappa}\text{Cov}(R_i, R_M).$$

Apply the same equation to the market itself:

$$\mu_M - R_f = \frac{1}{\kappa}\text{Var}(R_M).$$

Divide to obtain the CAPM:

$$\mathbb{E}[R_i] - R_f = \beta_i(\mathbb{E}[R_M] - R_f), \quad \beta_i = \frac{\text{Cov}(R_i, R_M)}{\text{Var}(R_M)}.$$

SDF Derivation

The stochastic discount factor (SDF) m satisfies

$$\mathbb{E}[mR_i] = 1$$

for every asset i. Assume a linear SDF:

$$m = a - bR_M.$$

Use covariance decomposition:

$$\mathbb{E}[mR_i] = \mathbb{E}[m]\mathbb{E}[R_i] + \text{Cov}(m, R_i) = 1,$$
$$\mathbb{E}[R_i] - R_f = -\frac{\text{Cov}(m, R_i)}{\mathbb{E}[m]}.$$

Since $\text{Cov}(m, R_i) = -b\text{Cov}(R_M, R_i)$,

$$\mathbb{E}[R_i] - R_f = \frac{b}{\mathbb{E}[m]}\text{Cov}(R_i, R_M).$$

Apply the same equation to R_M and divide to recover CAPM.

Worked Example

Let $R_f = 1.02$, $\mathbb{E}[R_M] = 1.10$, $\text{Var}(R_M) = 0.04$, and $\text{Cov}(R_i, R_M) = 0.06$. Then

$$\beta_i = \frac{0.06}{0.04} = 1.5$$

and

$$\mathbb{E}[R_i] = 1.02 + 1.5(1.10 - 1.02) = 1.14.$$

Common Pitfalls

- Confusing gross returns with net returns.
- Forgetting the market portfolio must be efficient.
- Dropping the covariance identity $(\Sigma w_M)_i = \mathrm{Cov}(R_i, R_M)$.
- Mixing up correlation and beta.

Exam Checklist

1. Write the mean-variance FOC and solve for w.
2. Use market clearing to identify w_M.
3. Convert to the covariance form and solve for β_i.
4. Derive CAPM again from the SDF pricing equation.

Instructor Note

I am an independent researcher and PhD (Georgia Institute of Technology), not currently affiliated with any institution. These notes are my own work.