Empirical Asset Pricing

AP-01: CARA/Normal Equilibrium Pricing

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From the Instructor

A Quick Note

- These are my own qualifier prep notes, written in full sentences on purpose.
- I am an independent researcher (Georgia Tech PhD) and I have no current affiliation.
- If a step feels fast, slow it down and work it line-by-line with me.

Why This Problem Matters

Qualifier DNA

- This is the canonical CARA/Normal equilibrium question.
- It tests whether you can move from preferences to prices, fast and clean.
- The same logic shows up in factor models, mean-variance, and SDF work.

Goal for today: you should be able to derive the price and the risk-free rate on a blank sheet, without looking anything up.

What You Are Being Asked to Produce

Deliverables

- A demand function for each investor h.
- A market-clearing price vector p.
- A clear interpretation of the risk adjustment term.
- A closed-form R_f under common δ, and the adjustment when δ_h differ.

Exam Tip

Write your final price as $p = \frac{1}{R_f}(\mu - \alpha\Sigma\theta)$ and then explain each term in one sentence. That earns partial credit even if algebra above is messy.

Roadmap

1. **Problem and Setup**
   - What is given, what is asked, and what we can safely ignore
   - Notation and market structure, kept consistent throughout
2. **Derivation**
   - CARA/Normal certainty equivalent (the only real trick)
   - Individual demand, market clearing, and equilibrium price
3. **Risk-Free Rate and Interpretation**
   - Common vs heterogeneous discounting
   - Economic meaning, plus the exact exam tips I use

How to Use These Notes

Reading Advice

- I go line-by-line. If a step feels quick, stop and check the algebra.
- Every definition used later is stated here once. Use it as a reference.
- The goal is not just the formula, but knowing why it must look that way.

Qualifier Problem (BMGT840 Final 2024 Q1)

Problem

- Two dates, risky payoffs $X \sim N(\mu, \Sigma)$, supply θ.
- H investors with CARA utilities
  $u_0(c) = -e^{-\alpha_h c}, \quad u_1(c) = -\delta_h e^{-\alpha_h c}.$
- Show: $p = \frac{1}{R_f}(\mu - \alpha\,\Sigma\theta)$.
- Interpret the risk adjustment term.
- Derive R_f when δ_h are common; then when they differ.

Setup and Notation

Given

- Risky payoffs $X \in \mathbb{R}^n$, $X \sim N(\mu, \Sigma)$.
- Price vector p at $t = 0$, risk-free return R_f.
- Supply vector θ.
- Aggregate $t = 0$ endowment c_0.

Investor h chooses:

- c_{0h}, risky holdings θ_h, risk-free holdings B_h.
- Date-0 budget: $c_{0h} + p'\theta_h + (1/R_f)B_h = w_{0h}$.
- Date-1 consumption: $c_{1h} = \theta_h' X + B_h$.

Note: I will keep all vectors as column vectors, and I will write every transpose.

Equilibrium Conditions (Keep These Visible)

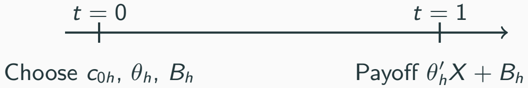
Market Clearing

- Risky assets: $\sum_{h=1}^{H} \theta_h = \theta$.
- Risk-free asset: $\sum_{h=1}^{H} B_h = 0$.
- Date-0 goods: $\sum_{h=1}^{H} c_{0h} = c_0$.

Why this matters

Every equilibrium price comes from these three conditions plus the FOCs.

Timeline (Two-Date Economy)



Key Lemma: CARA/Normal Certainty Equivalent

Certainty Equivalent

If $Y \sim N(m, v)$ and utility is $-\exp(-\alpha Y)$, then maximizing $\mathbb{E}[-e^{-\alpha Y}]$ is equivalent to maximizing

$$\mathrm{CE} = m - \frac{\alpha}{2} v.$$

Proof Sketch

- $\mathbb{E}[e^{-\alpha Y}] = \exp(-\alpha m + \frac{\alpha^2}{2} v)$.
- Exponential is monotone, so maximize $-\alpha m + \frac{\alpha^2}{2} v$.
- Equivalent to maximize $m - \frac{\alpha}{2} v$.
- This lemma is the workhorse for the rest of the course.

Lemma in Full (Line-by-Line)

Full Derivation

$$\mathbb{E}[-e^{-\alpha Y}] = -\mathbb{E}\left[\exp(-\alpha Y)\right]$$
$$= -\exp\left(-\alpha m + \frac{\alpha^2}{2}v\right)$$
$$\text{(normal MGF).}$$

Maximizing this is equivalent to maximizing

$$-\alpha m + \tfrac{\alpha^2}{2}v$$

which is equivalent to maximizing

$$m - \tfrac{\alpha}{2}v.$$

Practical Note

If you memorize one formula from this course, make it this one.

Step 1: Express c_{1h} in Terms of θ_h

Date-0 budget:

$$c_{0h} + p'\theta_h + (1/R_f)B_h = w_{0h} \Rightarrow B_h = R_f(w_{0h} - c_{0h} - p'\theta_h).$$

Date-1 consumption:

$$\begin{aligned} c_{1h} &= \theta_h' X + B_h \\ &= \theta_h' X + R_f(w_{0h} - c_{0h} - p'\theta_h) \\ &= R_f(w_{0h} - c_{0h}) + \theta_h'(X - R_f p). \end{aligned}$$

Step 2: Mean and Variance of c_{1h}

Mean

$$\mathbb{E}[c_{1h}] = R_f(w_{0h} - c_{0h}) + \theta_h'(\mu - R_f p).$$

Variance

$$\begin{aligned}\mathrm{Var}(c_{1h}) &= \mathrm{Var}(\theta_h' X) \\ &= \theta_h' \Sigma \, \theta_h.\end{aligned}$$

Note

Risk-free terms are constants, so they drop out of variance.

Step 3: Certainty Equivalent Objective

Using the lemma:

$$\begin{aligned}\mathsf{CE}_h &= \mathbb{E}[c_{1h}] - \frac{\alpha_h}{2}\mathrm{Var}(c_{1h}) \\ &= R_f(w_{0h} - c_{0h}) + \theta_h'(\mu - R_f p) - \frac{\alpha_h}{2}\theta_h'\Sigma\theta_h.\end{aligned}$$

Choice variable: θ_h (holding c_{0h} fixed).

Step 4: First-Order Condition

Differentiate with respect to θ_h:

$$\frac{\partial}{\partial \theta_h} \theta_h'(\mu - R_f \boldsymbol{p}) = \mu - R_f \boldsymbol{p},$$
$$\frac{\partial}{\partial \theta_h} \left(\tfrac{\alpha_h}{2} \theta_h' \Sigma \theta_h \right) = \alpha_h \Sigma \theta_h.$$

Pause: we use the symmetry of Σ here. **FOC:**

$$\mu - R_f \boldsymbol{p} - \alpha_h \Sigma \theta_h = 0.$$

Step 5: Individual Demand

Solve the FOC:

$$\begin{aligned} \alpha_h \Sigma \theta_h &= \mu - R_f p \\ \theta_h &= \frac{1}{\alpha_h} \Sigma^{-1} (\mu - R_f p). \end{aligned}$$

Key Insight

Demand is linear in expected excess payoff and inversely proportional to risk aversion.

Sanity Check: Shapes and Units

Dimensions

- μ and p are $n \times 1$ vectors.
- Σ is $n \times n$.
- θ_h is $n \times 1$.
- Therefore $\Sigma\theta_h$ is $n \times 1$, matching $\mu - R_f p$.

Why I check this

On quals, a quick dimension check catches half of algebra mistakes.

Step 6: Market Clearing

Aggregate supply:

$$\sum_{h=1}^{H} \theta_h = \theta.$$

Substitute individual demands:

$$\begin{aligned} \theta &= \sum_{h=1}^{H} \frac{1}{\alpha_h} \Sigma^{-1}(\mu - R_f p) \\ &= T\,\Sigma^{-1}(\mu - R_f p), \end{aligned}$$

where $T = \sum_{h=1}^{H} 1/\alpha_h$ is aggregate risk tolerance.

Step 7: Equilibrium Price

Multiply by Σ:

$$\Sigma\theta = T(\mu - R_f p).$$

Solve for p:

$$\begin{aligned} \mu - R_f p &= \frac{1}{T}\Sigma\theta, \\ R_f p &= \mu - \alpha\,\Sigma\theta, \end{aligned}$$

where $\alpha = 1/T$ is aggregate risk aversion.

Key Point

$$p = \frac{1}{R_f}(\mu - \alpha\,\Sigma\theta).$$

Notation Pitfall: Aggregate Risk Aversion

Be Precise

Some notes define

$$T = \sum_h \frac{1}{\alpha_h} \quad \text{and} \quad \alpha = \frac{1}{T}.$$

Other notes (and some exams) write $\alpha = \sum_h \alpha_h$.

Always check the definition in the problem statement. Your algebra is right either way, but the symbol may differ.

One-Asset Sanity Check

Assume $n = 1$, so $\Sigma = \sigma^2$ and supply is θ.

$$p = \frac{1}{R_f}(\mu - \alpha\sigma^2\theta).$$

Interpretation:

- Higher μ raises price.
- Higher risk (σ^2) lowers price.
- More supply θ lowers price (risk must be absorbed).

Exam Tip

If your formula fails this one-asset check, something went wrong upstream.

Interpretation: The Risk Adjustment Term

Price decomposition:

$$p = \frac{1}{R_f}\mu - \frac{\alpha}{R_f}\Sigma\theta.$$

Economic Meaning

- μ is the expected payoff.
- $(\Sigma\theta)_i = \mathrm{Cov}(X_i, \theta' X)$ is each asset's covariance with aggregate risk.
- Larger covariance $\Rightarrow$ larger discount $\Rightarrow$ lower price.

Risk-Free Rate: Common Discount Factor

Assume $\delta_h = \delta$ for all h. **Aggregate consumption at $t = 1$:**

$$c_1 = \theta' X.$$

SDF:

$$m = \delta \exp\{-\alpha(c_1 - c_0)\}.$$

Risk-free pricing:

$$\frac{1}{R_f} = \mathbb{E}[m].$$

Risk-Free Rate: Identify Mean and Variance

Let $c_1 = \theta' X$. Since $X \sim N(\mu, \Sigma)$, we have

$$\mathbb{E}[c_1] = \theta' \mu,$$

$$\mathrm{Var}(c_1) = \theta' \Sigma \theta.$$

Therefore $c_1 \sim N(\theta' \mu, \theta' \Sigma \theta)$.

Why this matters

We need the mean and variance to apply the normal MGF.

Risk-Free Rate: Step-by-Step Evaluation

Since $c_1 = \theta' X$ and $X \sim N(\mu, \Sigma)$:

$$\begin{aligned}
\mathbb{E}[m] &= \delta\, e^{\alpha c_0} \mathbb{E}\left[\exp\{-\alpha\theta' X\}\right] \\
&= \delta\, e^{\alpha c_0} \exp\left(-\alpha\theta'\mu + \tfrac{\alpha^2}{2}\theta'\Sigma\theta\right).
\end{aligned}$$

Therefore:

$$R_f = \frac{1}{\delta} \exp\left(\alpha\theta'\mu - \alpha c_0 - \tfrac{\alpha^2}{2}\theta'\Sigma\theta\right).$$

Risk-Free Rate: Two Quick Checks

Check 1: No Aggregate Risk

If $\theta = 0$, then c_1 is non-random and

$$R_f = \frac{1}{\delta} e^{-\alpha c_0}.$$

This is exactly the CARA discounting of certain consumption.

Check 2: More Aggregate Risk

If $\theta' \Sigma \theta$ rises, R_f falls. Safe claims become more valuable.

Comparative Statics for R_f

From

$$R_f = \frac{1}{\delta} \exp\left(\alpha\theta'\mu - \alpha c_0 - \frac{\alpha^2}{2}\theta'\Sigma\theta\right):$$

- Higher $\theta'\mu$ **increases** R_f.
- Higher δ **reduces** R_f (more patient investors).
- Higher c_0 **reduces** R_f (more current endowment).
- Higher risk term $\alpha^2\theta'\Sigma\theta$ **reduces** R_f.

Heterogeneous Discount Factors: Key Idea

When δ_h differ, a representative discount factor is a **risk-tolerance-weighted geometric mean**.

Plan

- Solve planner problems at $t = 0$ and $t = 1$.
- Derive $m(\omega) = \Lambda_1(\omega)/\Lambda_0$.
- Identify an effective $\bar{\delta}$.

Heterogeneous δ_h: Planner FOCs

At $t = 0$:

$$\lambda_h \alpha_h e^{-\alpha_h c_{0h}} = \Lambda_0.$$

At $t = 1$ in state ω:

$$\lambda_h \delta_h \alpha_h e^{-\alpha_h c_{1h}(\omega)} = \Lambda_1(\omega).$$

Solve:

$$c_{0h} = -\frac{1}{\alpha_h} \ln\left(\frac{\Lambda_0}{\lambda_h \alpha_h}\right),$$
$$c_{1h}(\omega) = -\frac{1}{\alpha_h} \ln\left(\frac{\Lambda_1(\omega)}{\lambda_h \delta_h \alpha_h}\right).$$

Solve for Multipliers (Step-by-Step)

Let $T = \sum_h 1/\alpha_h$ and define

$$K_0 = \sum_h \frac{1}{\alpha_h} \ln(\lambda_h \alpha_h).$$

Summing c_{0h} over h gives

$$c_0 = -T \ln \Lambda_0 + K_0,$$

$$\ln \Lambda_0 = \frac{K_0 - c_0}{T}.$$

Define

$$K_1 = \sum_h \frac{1}{\alpha_h} \ln(\lambda_h \delta_h \alpha_h).$$

Then

$$c_1(\omega) = -T \ln \Lambda_1(\omega) + K_1,$$

$$\ln \Lambda_1(\omega) = \frac{K_1 - c_1(\omega)}{T}.$$

Heterogeneous δ_h: Effective Discounting

Let $T = \sum_h 1/\alpha_h$ and define

$$\bar{\delta} = \exp\left(\frac{\sum_h (1/\alpha_h) \ln \delta_h}{\sum_h (1/\alpha_h)}\right).$$

Then the SDF is

$$m(\omega) = \bar{\delta} \exp\{-\alpha(c_1(\omega) - c_0)\}.$$

Risk-free rate:

$$R_f = \frac{1}{\bar{\delta}} \exp\left(\alpha\theta'\mu - \alpha c_0 - \frac{\alpha^2}{2}\theta'\Sigma\theta\right).$$

Interpretation of $\bar{\delta}$

Economic Meaning

- Investors with lower risk aversion receive more weight.
- $\bar{\delta}$ is a geometric mean, not an arithmetic mean.
- If all δ_h are equal, $\bar{\delta} = \delta$.

Key Takeaways

Key Point

- CARA/Normal reduces equilibrium pricing to mean-variance logic.
- Price = expected payoff minus a covariance-based risk adjustment.
- R_f depends on expected aggregate payoff, patience, and aggregate risk.
- Heterogeneous δ_h aggregate via a risk-tolerance-weighted geometric mean.

Practice Variant

Practice

Assume one risky asset and add a second risk-free asset with payoff 2 and price $2/R_f$.

- Show the risky-asset price formula is unchanged.
- Explain why the second risk-free asset is redundant.

Common Pitfalls

Common Pitfalls

- Forgetting the price term $R_f p$ in the FOC.
- Mixing up aggregate risk tolerance T and risk aversion α.
- Dropping transposes in quadratic forms (use $\theta' \Sigma \theta$).
- Ignoring the role of c_0 in R_f.