Empirical Asset Pricing: Qualifier Prep

AP-02: CAPM and SDF Foundations

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From the Instructor

I write these slides as a working researcher and teacher. The goal is to make the CAPM feel like an unavoidable result rather than a memorized formula. We will move slowly, show all algebra, and repeatedly connect each line to the economics it represents.

I am an independent researcher and PhD (Georgia Institute of Technology), not currently affiliated with an institution. These notes reflect my own voice and method.

Roadmap

1. **Build the Mean-Variance Problem**
   - Portfolio choice with a risk-free asset
   - Efficient frontier and tangency portfolio
2. **Derive the CAPM Step-by-Step**
   - Market clearing and the market portfolio
   - Beta pricing and risk premia
3. **Translate to the SDF View**
   - SDF definition and pricing equation
   - Linear SDF implies CAPM
4. **Qualifier Problems + Variants**
   - Two full exam-style problems
   - Common pitfalls and extensions

Motivation

Why CAPM Still Matters

- The CAPM is not an empirical truth; it is a disciplined equilibrium benchmark.
- It teaches you how to translate preferences and technology into pricing restrictions.
- Most qualifier questions use CAPM logic as a baseline to test deeper understanding.

Intuition

The qualifier test is often: can you recreate the logic under time pressure, not can you recite the final formula.

A Preview of the Destination

We will show that in equilibrium with a risk-free asset,

$$\mathbb{E}[R_i] - R_f = \beta_i\big(\mathbb{E}[R_M] - R_f\big), \qquad \beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)}.$$

Then we will prove the same equation from the SDF condition

$$\mathbb{E}[mR_i] = 1, \qquad m = a - bR_M.$$

Key Point

Two derivations, one idea: only covariance with the market is priced.

Mean-Variance Foundation

Setup: Assets and Portfolios

- N risky assets with returns $R \in \mathbb{R}^N$, mean μ, covariance Σ.
- A risk-free asset with return R_f.
- A portfolio is weights $w \in \mathbb{R}^N$ in risky assets; remainder in the risk-free asset.

Definition

The portfolio return is

$$R_p = R_f + w'(R - R_f\mathbf{1}).$$

Mean and Variance of the Portfolio

$$\mathbb{E}[R_p] = R_f + w'(\mu - R_f\mathbf{1}),$$
$$\mathrm{Var}(R_p) = w'\Sigma w.$$

Intuition

The mean depends only on the projection of w on excess means, while the variance depends on Σ.

Mean-Variance Optimization Problem

The canonical investor solves

$$\max_{w} \ \mathbb{E}[R_p] - \frac{\gamma}{2}\mathrm{Var}(R_p).$$

Substitute the formulas:

$$\max_{w} \ R_f + w'(\mu - R_f\mathbf{1}) - \frac{\gamma}{2}w'\Sigma w.$$

Key Point

All investors choose from the same set of efficient portfolios. Only γ moves them along the line.

Solve the First-Order Condition

Take the gradient with respect to w:

$$\mu - R_f \mathbf{1} - \gamma \Sigma w = 0.$$

Solve:

$$\Sigma w = \frac{1}{\gamma}(\mu - R_f \mathbf{1}).$$

Therefore,

$$w = \frac{1}{\gamma}\Sigma^{-1}(\mu - R_f \mathbf{1}).$$

Intuition

Up to a scaling factor, everyone holds the same risky portfolio.

The Tangency Portfolio

Define the tangency portfolio weights

$$w_T = \frac{\Sigma^{-1}(\mu - R_f\mathbf{1})}{\mathbf{1}'\Sigma^{-1}(\mu - R_f\mathbf{1})}.$$

This portfolio maximizes the Sharpe ratio.

Key Point

Every mean-variance investor holds the tangency portfolio and adjusts overall risk by scaling with R_f.

Market Clearing Logic

In equilibrium, the aggregate risky portfolio must equal the market portfolio M. If all agents hold a multiple of the tangency portfolio, then

$$w_M = w_T.$$

Intuition

This is the key equilibrium step: the market portfolio is efficient. Once you accept that, CAPM follows.

CAPM Derivation

A Key Relationship: Covariance with the Market

Let R_M be the market return. For any asset i:

$$\mathrm{Cov}(R_i, R_M) = \sigma_{iM}.$$

From mean-variance efficiency we can show

$$\mu_i - R_f = \lambda \sigma_{iM}$$

for some scalar λ.

Derivation Step-by-Step

We know that w_M satisfies

$$\Sigma w_M = \kappa(\mu - R_f \mathbf{1})$$

for some scalar κ.

Take the ith element:

$$(\Sigma w_M)_i = \kappa(\mu_i - R_f).$$

But $(\Sigma w_M)_i = \mathrm{Cov}(R_i, R_M)$, so

$$\mu_i - R_f = \frac{1}{\kappa}\mathrm{Cov}(R_i, R_M).$$

Identify the Slope

Apply the same formula to the market itself:

$$\mu_M - R_f = \frac{1}{\kappa}\mathrm{Var}(R_M).$$

Solve for $1/\kappa$ and substitute:

$$\mu_i - R_f = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)}(\mu_M - R_f).$$

This is the CAPM.

Beta Form

Define

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)}.$$

Then

$$\mathbb{E}[R_i] = R_f + \beta_i(\mathbb{E}[R_M] - R_f).$$

Key Point

Only covariance with the market matters; idiosyncratic risk is diversifiable.

Geometric Interpretation

In mean-variance space, the efficient frontier with a risk-free asset becomes a line. The line passes through R_f and is tangent to the risky-asset frontier.

Intuition

The tangency point is the market portfolio. The slope of this line is the Sharpe ratio of the market.

SDF Formulation

The Stochastic Discount Factor

Definition

A stochastic discount factor (SDF) m is a random variable such that

$$\mathbb{E}[mR_i] = 1$$

for every traded asset i.

Intuition

This is the modern pricing equation. It is equivalent to no-arbitrage plus market completeness in this simple setting.

Linear SDF Assumption

Suppose the SDF is linear in the market return:

$$m = a - bR_M.$$

Use the pricing equations for the risk-free asset and the market:

$$\mathbb{E}[m]R_f = 1,$$
$$\mathbb{E}[mR_M] = 1.$$

We can solve for a and b explicitly.

Solve for a and b

First equation:

$$R_f \mathbb{E}[a - bR_M] = 1 \quad \Rightarrow \quad a - b\mathbb{E}[R_M] = \frac{1}{R_f}.$$

Second equation:

$$\mathbb{E}[(a - bR_M)R_M] = 1 \quad \Rightarrow \quad a\mathbb{E}[R_M] - b\mathbb{E}[R_M^2] = 1.$$

Solve this linear system for a and b.

Key Covariance Identity

A cleaner way is to use covariance:

$$\mathbb{E}[mR_i] = \mathbb{E}[m]\mathbb{E}[R_i] + \mathrm{Cov}(m, R_i) = 1.$$

Rearrange:

$$\mathbb{E}[R_i] - R_f = -\frac{\mathrm{Cov}(m, R_i)}{\mathbb{E}[m]}.$$

If $m = a - bR_M$, then $\mathrm{Cov}(m, R_i) = -b\mathrm{Cov}(R_M, R_i)$.

Recover CAPM from the SDF

Substitute the covariance expression:

$$\mathbb{E}[R_i] - R_f = \frac{b}{\mathbb{E}[m]}\mathrm{Cov}(R_i, R_M).$$

Apply the same formula to asset M to identify the slope:

$$\mathbb{E}[R_M] - R_f = \frac{b}{\mathbb{E}[m]}\mathrm{Var}(R_M).$$

Divide the two equations to obtain CAPM:

$$\mathbb{E}[R_i] - R_f = \beta_i(\mathbb{E}[R_M] - R_f).$$

Worked Examples

Example 1: Compute Beta and Expected Return

Suppose:

- $R_f = 1.02$ (2% per period),
- $\mathbb{E}[R_M] = 1.10$,
- $\mathrm{Var}(R_M) = 0.04$,
- $\mathrm{Cov}(R_i, R_M) = 0.06$.

Then

$$\beta_i = \frac{0.06}{0.04} = 1.5.$$

So

$$\mathbb{E}[R_i] = 1.02 + 1.5(1.10 - 1.02) = 1.02 + 1.5(0.08) = 1.14.$$

Example 2: From Covariance Matrix to Betas

Let

$$\Sigma = \begin{bmatrix} 0.04 & 0.01 \\ 0.01 & 0.09 \end{bmatrix}, \qquad w_M = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

Compute

$$\mathrm{Var}(R_M) = w_M' \Sigma w_M = 0.6^2(0.04) + 2(0.6)(0.4)(0.01) + 0.4^2(0.09) = 0.0432.$$

For asset 1,

$$\mathrm{Cov}(R_1, R_M) = e_1' \Sigma w_M = 0.6(0.04) + 0.4(0.01) = 0.028.$$

Hence $\beta_1 = 0.028/0.0432 \approx 0.648$.

Example 3: Tangency Portfolio by Hand

Suppose

$$\mu = \begin{bmatrix} 1.08 \\ 1.12 \end{bmatrix}, \quad R_f = 1.02, \quad \Sigma = \begin{bmatrix} 0.04 & 0.01 \\ 0.01 & 0.09 \end{bmatrix}.$$

Compute

$$\mu - R_f \mathbf{1} = \begin{bmatrix} 0.06 \\ 0.10 \end{bmatrix}.$$

Then

$$\Sigma^{-1}(\mu - R_f \mathbf{1}) = \frac{1}{0.0035} \begin{bmatrix} 0.09 & -0.01 \\ -0.01 & 0.04 \end{bmatrix} \begin{bmatrix} 0.06 \\ 0.10 \end{bmatrix} = \frac{1}{0.0035} \begin{bmatrix} 0.0044 \\ 0.0034 \end{bmatrix}.$$

Normalize to sum to 1: $w_T \propto (0.0044, 0.0034)$.

Qualifier-Style Problems

Problem A: Full CAPM Derivation

Problem

An economy has N risky assets with mean μ and covariance Σ and a risk-free asset R_f. Assume mean-variance preferences. Show that in equilibrium $\mathbb{E}[R_i] - R_f = \beta_i(\mathbb{E}[R_M] - R_f)$.

Outline:

1. Write the mean-variance objective and FOC.
2. Show that the tangency portfolio has weights proportional to $\Sigma^{-1}(\mu - R_f\mathbf{1})$.
3. Market clearing implies $w_M = w_T$.
4. Convert $\Sigma w_M = \kappa(\mu - R_f\mathbf{1})$ to a covariance statement.
5. Solve for κ using the market itself.

Problem A: Step-by-Step Solution (1/2)

Start with the investor's problem:

$$\max_{w} R_f + w'(\mu - R_f\mathbf{1}) - \frac{\gamma}{2} w'\Sigma w.$$

FOC:

$$\mu - R_f\mathbf{1} - \gamma\Sigma w = 0.$$

Thus

$$w = \frac{1}{\gamma}\Sigma^{-1}(\mu - R_f\mathbf{1}).$$

Because all investors differ only in γ, they all hold the same risky portfolio.

Problem A: Step-by-Step Solution (2/2)

Market clearing implies w_M is proportional to $\Sigma^{-1}(\mu - R_f\mathbf{1})$:

$$\Sigma w_M = \kappa(\mu - R_f\mathbf{1}).$$

Take element i:

$$(\Sigma w_M)_i = \kappa(\mu_i - R_f).$$

But $(\Sigma w_M)_i = \mathrm{Cov}(R_i, R_M)$, so

$$\mu_i - R_f = \frac{1}{\kappa}\mathrm{Cov}(R_i, R_M).$$

Apply the same to M to solve for $1/\kappa$, and obtain CAPM.

Problem B: CAPM from an SDF

Problem

Suppose asset prices satisfy $\mathbb{E}[mR_i] = 1$ for all assets, and $m = a - bR_M$. Show that expected returns obey the CAPM.

Outline:

1. Use $\mathbb{E}[mR_i] = 1$ to express $\mathbb{E}[R_i]$ in terms of $\mathrm{Cov}(m, R_i)$.
2. Substitute $m = a - bR_M$ and simplify.
3. Apply the formula to R_M to identify the slope.
4. Divide the two equations to obtain the CAPM.

Problem B: Step-by-Step Solution

Using covariance:

$$\mathbb{E}[mR_i] = \mathbb{E}[m]\mathbb{E}[R_i] + \mathrm{Cov}(m, R_i) = 1,$$
$$\mathbb{E}[R_i] - R_f = -\frac{\mathrm{Cov}(m, R_i)}{\mathbb{E}[m]}.$$

If $m = a - bR_M$, then $\mathrm{Cov}(m, R_i) = -b\mathrm{Cov}(R_M, R_i)$. So

$$\mathbb{E}[R_i] - R_f = \frac{b}{\mathbb{E}[m]}\mathrm{Cov}(R_i, R_M).$$

Apply the same to R_M and divide to recover CAPM.

Pitfalls and Extensions

Common Pitfalls

- Confusing R_f (gross return) with the risk-free rate (net return).
- Forgetting to show the market portfolio is efficient.
- Dropping the key covariance identity $(\Sigma w_M)_i = \mathrm{Cov}(R_i, R_M)$.
- Mixing up β_i with correlations.

Extension: Zero-Beta CAPM

If no risk-free asset exists, the CAPM becomes

$$\mathbb{E}[R_i] = \mathbb{E}[R_Z] + \beta_i^Z \big(\mathbb{E}[R_M] - \mathbb{E}[R_Z]\big)$$

where R_Z is the return on the zero-beta portfolio.

Intuition

Qualifiers sometimes ask you to derive this formula or interpret R_Z.

Extension: From CAPM to Factor Models

CAPM is a one-factor model with factor R_M. Empirical work generalizes this as

$$\mathbb{E}[R_i] = R_f + \beta_{i1}\lambda_1 + \cdots + \beta_{ik}\lambda_k.$$

Key Point

Many qualifier questions ask whether CAPM is nested in a larger factor model and how to test that nesting.

Quick Practice

Two-Minute Drill

1. In one sentence, explain why the market portfolio must be mean-variance efficient.
2. Given $\beta_i = 0.7$, $R_f = 1.01$, $\mathbb{E}[R_M] = 1.09$, compute $\mathbb{E}[R_i]$.
3. Explain, in words, why idiosyncratic risk is not priced.

Closing

We now have two derivations of CAPM that agree line-for-line. This is the foundation for most empirical asset pricing questions on qualifiers.

Next: we will move to testing CAPM and building cross-sectional regressions.