AP-02 Notes: CAPM and SDF Foundations

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Purpose of These Notes

These notes are the "book version" of AP-02. The slides are the live walkthrough. Here I write the CAPM derivation twice and keep the logic tight.

Roadmap

1. Derive CAPM from mean-variance optimization.
2. Re-derive CAPM from the SDF pricing equation.
3. Connect to zero-beta CAPM and factor models.

1. Mean-Variance Setup

Let $R \in \mathbb{R}^N$ be risky returns with mean μ and covariance Σ. A risk-free asset yields gross return R_f. A portfolio with risky weights w has return

$$R_p = R_f + w'(R - R_f\mathbf{1}).$$

Then

$$\begin{aligned}\mathbb{E}[R_p] &= R_f + w'(\mu - R_f\mathbf{1}),\\ \mathrm{Var}(R_p) &= w'\Sigma w.\end{aligned}$$

2. Mean-Variance Optimization

A mean-variance investor solves

$$\max_w \ \mathbb{E}[R_p] - \frac{\gamma}{2}\mathrm{Var}(R_p).$$

The first-order condition is

$$\mu - R_f\mathbf{1} - \gamma\Sigma w = 0.$$

So optimal risky holdings satisfy

$$w \propto \Sigma^{-1}(\mu - R_f\mathbf{1}).$$

All investors hold the same risky portfolio up to scale.

3. Market Clearing and the CAPM

In equilibrium, the market portfolio M must be mean-variance efficient. Let w_M denote market weights. Then for some scalar κ,

$$\Sigma w_M = \kappa(\mu - R_f \mathbf{1}).$$

Take the ith element:

$$(\Sigma w_M)_i = \kappa(\mu_i - R_f).$$

But $(\Sigma w_M)_i = \mathrm{Cov}(R_i, R_M)$, so

$$\mu_i - R_f = \frac{1}{\kappa}\mathrm{Cov}(R_i, R_M).$$

Apply the same equation to the market itself:

$$\mu_M - R_f = \frac{1}{\kappa}\mathrm{Var}(R_M).$$

Divide to obtain the CAPM:

$$\mathbb{E}[R_i] - R_f = \beta_i(\mathbb{E}[R_M] - R_f), \quad \beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)}.$$

4. Geometric Interpretation

The efficient frontier with a risk-free asset becomes a straight line (the capital market line). The tangency portfolio on the risky frontier is the market portfolio. CAPM is the security market line: expected returns are linear in betas.

5. SDF Derivation

The stochastic discount factor (SDF) m satisfies

$$\mathbb{E}[mR_i] = 1$$

for every asset i. Assume a linear SDF:

$$m = a - bR_M.$$

Use covariance decomposition:

$$\mathbb{E}[mR_i] = \mathbb{E}[m]\mathbb{E}[R_i] + \mathrm{Cov}(m, R_i) = 1,$$
$$\mathbb{E}[R_i] - R_f = -\frac{\mathrm{Cov}(m, R_i)}{\mathbb{E}[m]}.$$

Since $\mathrm{Cov}(m, R_i) = -b\mathrm{Cov}(R_M, R_i)$,

$$\mathbb{E}[R_i] - R_f = \frac{b}{\mathbb{E}[m]}\mathrm{Cov}(R_i, R_M).$$

Apply the same equation to R_M and divide to recover CAPM.

6. Zero-Beta CAPM

If no risk-free asset exists, CAPM becomes

$$\mathbb{E}[R_i] = \mathbb{E}[R_Z] + \beta_i^Z(\mathbb{E}[R_M] - \mathbb{E}[R_Z]),$$

where R_Z is the return on the zero-beta portfolio. This is a common qualifier extension.

7. From CAPM to Factor Models

CAPM is a one-factor model. Empirical work often uses

$$\mathbb{E}[R_i] = R_f + \beta_{i1}\lambda_1 + \cdots + \beta_{ik}\lambda_k.$$

CAPM is the special case $k = 1$ with the market factor.

8. Worked Examples

Example 1 (Beta and Expected Return). Let $R_f = 1.02$, $\mathbb{E}[R_M] = 1.10$, $\mathrm{Var}(R_M) = 0.04$, and $\mathrm{Cov}(R_i, R_M) = 0.06$. Then

$$\beta_i = \frac{0.06}{0.04} = 1.5, \quad \mathbb{E}[R_i] = 1.02 + 1.5(1.10 - 1.02) = 1.14.$$

Example 2 (From Covariance Matrix). Let

$$\Sigma = \begin{bmatrix} 0.04 & 0.01 \\ 0.01 & 0.09 \end{bmatrix}, \quad w_M = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

Then

$$\mathrm{Var}(R_M) = w_M' \Sigma w_M = 0.0432, \quad \mathrm{Cov}(R_1, R_M) = 0.028,$$

so $\beta_1 \approx 0.648$.

9. Common Pitfalls

- Confusing gross returns with net returns.
- Forgetting that the market portfolio must be efficient.
- Dropping the covariance identity $(\Sigma w_M)_i = \mathrm{Cov}(R_i, R_M)$.
- Mixing up correlation with beta.

10. Exam Checklist

1. Write the mean-variance FOC and solve for w.
2. Use market clearing to identify w_M.
3. Convert to the covariance form and solve for β_i.
4. Derive CAPM again from the SDF pricing equation.
5. State the zero-beta CAPM extension.

11. Practice Problems

1. Prove CAPM using only the covariance identity and market efficiency.
2. Derive a and b for the linear SDF from the risk-free asset and market equations.
3. Explain how CAPM nests inside a two-factor model and how you would test it.

Instructor Note

I am an independent researcher and PhD (Georgia Institute of Technology), not currently affiliated with any institution. These notes are my own work.