converged. The analysis terminates when all components are converged in this sense. Of course, the entire error vector must be examined on each convergence test.

This scheme naturally accommodates both large and small current components. Large components usually converge on the basis of the relative error, and small ones on the fractional error, as the relative criterion is a weaker one for large components and the absolute criterion is weaker for small components. It is a simple matter to select the criteria so they ensure that all errors are sufficiently small at termination.

3.3.9.8 Initial Estimate

One important property of Newton's method is that its speed and reliability of convergence depend strongly upon the initial estimate of the solution vector. Formulating the initial estimate may not be difficult in analyzing a specific type of circuit, but it may be difficult to conceive of a way to form initial estimates in a general-purpose circuit-analysis program, which must accommodate a wide variety of circuits that have a concomitant variety of possible responses.

For nearly linear circuits, such as class-A power amplifiers, the linear response is a good initial estimate. The response can be found by setting the excitation level to a small value and the harmonic number, K, equal to one, so the size of the problem is relatively small. When the solution has completed, the results are scaled to the correct excitation level and K is reset to the desired value for the large-signal analysis.

In strongly nonlinear circuits, such as class-B or -C amplifiers, frequency multipliers, and mixers, an initial estimate is more difficult to generate. Occasionally the nature of the circuit allows a good estimate; for example, in diode mixers, the diode-voltage waveform invariably is a clipped sinusoid. In difficult cases, it may be best first to do a dc analysis, then to apply the RF signal and increase it using a continuation method.

3.4 LARGE-SIGNAL/SMALL-SIGNAL ANALYSIS USING CONVERSION MATRICES

Large-signal/small-signal analysis, or *conversion matrix analysis*, is useful for a large class of problems wherein a nonlinear device is driven, or "pumped," by a single large sinusoidal signal; another signal, much smaller, is applied; and we seek only the linear response to the small signal. The most common application of this technique is in the design of mixers and in nonlinear noise analysis. The process involves first analyzing the nonlinear device under large-signal excitation only, usually by the harmonic-balance method. The nonlinear elements in the device's equivalent circuit are then linearized to create small-signal, linear, time-varying elements, and finally a small-signal analysis is performed. The method is much more efficient than multitone harmonic-balance analysis but provides only the linear response of the circuit. It cannot be used for determining saturation or intermodulation distortion in mixers, but it is a good method for calculating a mixer's conversion efficiency and its RF and IF port impedances. The results of the harmonic-balance analysis can be used for finding LO voltage and current waveforms, and LO port impedance.

3.4.1 Conversion Matrix Formulation

Figure 3.9 shows a nonlinear resistive element driven by a large-signal voltage, V, generating a current I. The nonlinear element has the I/V relationship $I = f(V)$. Following the process outlined in Chapter 2, we can find the incremental small-signal current by assuming that V consists of the sum of a large-signal component V_0 and a small-signal component v. The current resulting from this excitation can be found by expanding $f(V_0 + v)$ in a Taylor series,

$$\begin{aligned} f(V_0 + v) &= f(V_0) + \left.\frac{d}{dV} f(V)\right|_{V = V_0} v + \frac{1}{2} \left.\frac{d^2}{dV^2} f(V)\right|_{V = V_0} v^2 \\ &\quad + \frac{1}{6} \left.\frac{d^3}{dV^3} f(V)\right|_{V = V_0} v^3 + \ldots \end{aligned} \tag{3.94}$$

The small-signal, incremental current is found by subtracting the large-signal component of the current,

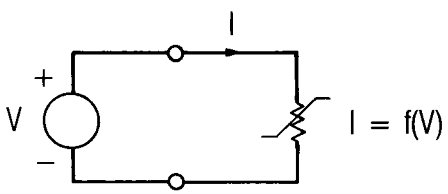

Figure 3.9 Nonlinear resistive element driven by a large excitation.

$$i(v) = I(V_0 + v) - I(V_0) \tag{3.95}$$

If $v << V_0$, v^2, v^3, ... are negligible (and, in any event, are nonlinear, so they do not contribute to the linear response). Then,

$$i(v) = \left.\frac{d}{dV} f(V)\right|_{V = V_0} v \tag{3.96}$$

V_0 need not be a dc quantity; it can be a time-varying large-signal voltage $V_L(t)$ (in fact, V_0 and V_L are control voltages). We assume that this is the case, and also that $v = v(t)$, a function of time. Then

$$i(t) = \left.\frac{d}{dV} f(V)\right|_{V = V_L(t)} v(t) \tag{3.97}$$

Equation (3.97) can be expressed as

$$i(t) = g(t)v(t) \tag{3.98}$$

The time-varying conductance in (3.98), $g(t)$, is the derivative of the element's I/V characteristic at the large-signal voltage. This is the usual definition of small-signal conductance for static elements. By an analogous derivation, one could have a current-controlled resistor with the V/I characteristic

$$V = f_R(I) \tag{3.99}$$

and obtain the small-signal v/i relation

$$v(t) = r(t)i(t) \tag{3.100}$$

where

$$r(t) = \left.\frac{d}{dI} f_R(I)\right|_{I = I_L(t)} \tag{3.101}$$

Often, the nonlinear element is a function of more than one control voltage. A conductance controlled by two voltages has $I = f_2(V_1, V_2)$. $f_2(V_1, V_2)$ can be expanded in a two-dimensional Taylor series, and after subtracting the large-signal current component and retaining only the linear terms,

$$i(t) = g_1(t)v_1(t) + g_2(t)v_2(t) \tag{3.102}$$

where

$$\begin{aligned} g_1(t) &= \frac{\partial}{\partial V_1} f_2(V_1, V_2)\bigg|_{\substack{V_1 = V_{L,1}(t) \\ V_2 = V_{L,2}(t)}} \\ g_2(t) &= \frac{\partial}{\partial V_2} f_2(V_1, V_2)\bigg|_{\substack{V_1 = V_{L,1}(t) \\ V_2 = V_{L,2}(t)}} \end{aligned} \tag{3.103}$$

Equation (3.102) shows that a nonlinear conductance having two control voltages is equivalent to two conductances in parallel. One must be a controlled current source, and the other may be either a controlled source or a time-varying two-terminal conductance. When the I/V characteristic is a function of more than two voltages, (3.102) can be extended in the manner one would expect:

$$i(t) = g_1(t)v_1(t) + g_2(t)v_2(t) + g_3(t)v_3(t) + \ldots \tag{3.104}$$

It is unusual, however, to encounter a nonlinear element having more than two control voltages.

The same process can be followed with a capacitor. A nonlinear capacitor has the Q/V characteristic $Q = f_Q(V)$, and by a similar derivation, the incremental, small-signal charge is

$$q(t) = \frac{d}{dV} f_Q(V)\bigg|_{V = V_L(t)} v(t) \tag{3.105}$$

or

$$q(t) = c(t)v(t) \tag{3.106}$$

The capacitor's current is the time derivative of the charge:

$$i(t) = \frac{d}{dt}q(t) = c(t)\frac{d}{dt}v(t) + v(t)\frac{d}{dt}c(t) \tag{3.107}$$

Like a conductance, a capacitance can have multiple control voltages. In a manner analogous to (3.102) to (3.104), the small-signal charge is

$$q(t) = c_1(t)v_1(t) + c_2(t)v_2(t) + c_3(t)v_3(t) + \ldots \tag{3.108}$$

and the current is found by differentiating with respect to time:

$$i(t) = \frac{d}{dt}q(t) = c_1(t)\frac{d}{dt}v_1(t) + v_1(t)\frac{d}{dt}c_1(t) + c_2(t)\frac{d}{dt}v_2(t) + v_2(t)\frac{d}{dt}c_2(t) + \ldots \tag{3.109}$$

A nonlinear element excited by two tones supports currents and voltages at the mixing frequencies $m\,\omega_1 + n\,\omega_2$, where m and n are integers. If we assume that one of those tones, ω_1, has such a low level that it does not generate harmonics, and the other is a large-signal sinusoid at ω_p, the mixing frequencies are $\omega = \pm\omega_1 + n\,\omega_p$. This equation represents the set of frequency components shown in Figure 3.10, which consists of two tones on either side of each large-signal harmonic frequency, separated by $\omega_0 = |\omega_1 - \omega_p|$. A more compact representation of the mixing frequencies is

$$\omega_n = \omega_0 + n\omega_p \tag{3.110}$$

which is shown in Figure 3.11 and includes only half of the mixing frequencies: the negative components of the lower sidebands and the positive components of the upper sidebands. This set of frequencies is adequate for two reasons: first, the small-signal analysis is linear, so by the superposition principle, the results for positive and negative components can be separated; and second, positive- and negative-frequency components are complex conjugate pairs, so knowledge of only one is

For real signal, positive- and negative-frequency components are complex conjugate pairs.

Include only half of the mixing frequencies is enough, i.e.
1) Negative components of the lower sidebands (LSB)
2) Positive components of the upper sidebands (USB)

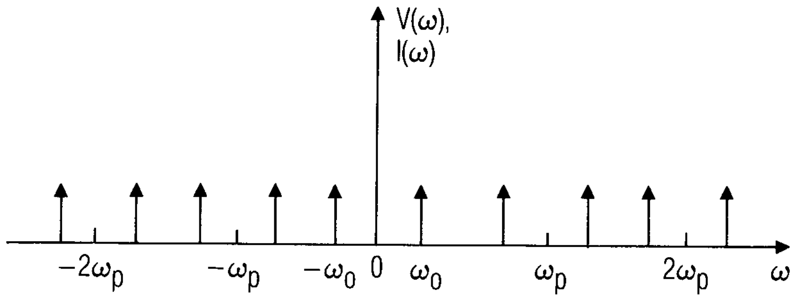

Figure 3.10 Spectrum of small-signal mixing frequencies in the pumped nonlinear element.

necessary. We will carry only the components in (3.110) in the following analysis, with confidence that the others can be generated when necessary.

The frequency-domain currents and voltages in a time-varying circuit element are related by a *conversion matrix*. We begin by deriving the conversion matrix that represents a time-varying conductance. The small-signal voltage and current can be expressed in the frequency notation of (3.110) as

$$v'(t) = \sum_{n=-\infty}^{\infty} V_n \exp(j\omega_n t) \tag{3.111}$$

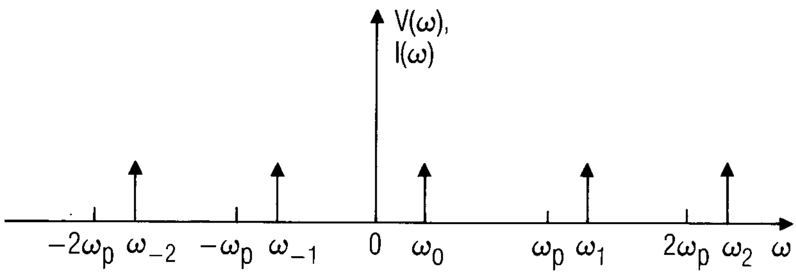

Figure 3.11 Spectrum of small-signal mixing frequencies illustrating the frequency notation of (3.110).

and

$$i'(t) = \sum_{n=-\infty}^{\infty} I_n \exp(j\omega_n t) \tag{3.112}$$

where the primes indicate that $v'(t)$ and $i'(t)$ are sums of the positive- and negative-frequency phasor components in (3.110) and are not the complete time waveforms. Above all, (3.111) and (3.112) are not Fourier series, in spite of their superficial resemblance. The conductance waveform $g(t)$ can be expressed by its Fourier series,

includes only half of the mixing frequencies

$$g(t) = \sum_{n=-\infty}^{\infty} G_n \exp(jn\omega_p t) \tag{3.113}$$

and the voltage and current are related by Ohm's law,

$$i'(t) = g(t)v'(t) \tag{3.114}$$

Substituting (3.111) through (3.113) into (3.114) gives the relation,

$$\sum_{k=-\infty}^{\infty} I_k \exp(j\omega_k t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_n V_m \exp(j\omega_{m+n} t) \tag{3.115}$$

Equating terms on both sides of the equation in (3.115) results in a set of equations that can be expressed in matrix form:

$$
\begin{bmatrix} I_{-N}^* \\ I_{-N+1}^* \\ I_{-N+2}^* \\ \cdots \\ \cdots \\ I_{-1}^* \\ I_0 \\ I_1 \\ \cdots \\ \cdots \\ I_N \end{bmatrix}
=
\begin{bmatrix}
G_0 & G_{-1} & G_{-2} & \cdots & G_{-2N} \\
G_1 & G_0 & G_{-1} & \cdots & G_{-2N+1} \\
G_2 & G_1 & G_0 & \cdots & G_{-2N+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
G_{N-1} & G_{N-2} & G_{N-3} & \cdots & G_{-N-1} \\
G_N & G_{N-1} & G_{N-2} & \cdots & G_{-N} \\
G_{N+1} & G_N & G_{N-1} & \cdots & G_{-N+1} \\
\cdots & \cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \\
G_{2N} & G_{2N-1} & G_{2N-2} & \cdots & G_0
\end{bmatrix}
\begin{bmatrix} V_{-N}^* \\ V_{-N+1}^* \\ V_{-N+2}^* \\ \cdots \\ \cdots \\ V_{-1}^* \\ V_0 \\ V_1 \\ \cdots \\ \cdots \\ V_N \end{bmatrix}
\tag{3.116}
$$

Two details in (3.116) must be clarified. First, the vectors in (3.116) have been truncated to a limit of $n = N$ for I_n and V_n, and $n = 2N$ for G_n. We assume that V_n, I_n, and G_n are negligible beyond these limits. The second detail is that the negative-frequency components (V_n, I_n where $n < 0$) are shown as conjugate. The conjugates are caused by a change of definition; according to (3.110), ω_n is negative when $n < 0$, so the I_n and V_n are negative-frequency components when $n < 0$. We would rather define them as phasors, which are always positive-frequency components. Positive- and negative-frequency components are related as $V_{-n} = V_n^*$ and $I_{-n}= I_n^*$, so if we wish V_n, I_n to represent positive-frequency components, they must be V_n^*, I_n^*. Thus the conversion matrix relates ordinary phasor voltages to currents at each mixing frequency. The main advantage of making this change is that the conversion matrix is now completely compatible with conventional linear, sinusoidal steady-state analysis.

The dual case, a time-varying resistor, has an unsurprising result. The conversion matrix is

$$\begin{bmatrix} V^*_{-N} \\ V^*_{-N+1} \\ V^*_{-N+2} \\ \cdots \\ \cdots \\ V^*_{-1} \\ V_0 \\ V_1 \\ \cdots \\ \cdots \\ V_N \end{bmatrix} = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots & R_{-2N} \\ R_1 & R_0 & R_{-1} & \cdots & R_{-2N+1} \\ R_2 & R_1 & R_0 & \cdots & R_{-2N+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{N-1} & R_{N-2} & R_{N-3} & \cdots & R_{-N-1} \\ R_N & R_{N-1} & R_{N-2} & \cdots & R_{-N} \\ R_{N+1} & R_N & R_{N-1} & \cdots & R_{-N+1} \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ R_{2N} & R_{2N-1} & R_{2N-2} & \cdots & R_0 \end{bmatrix} \begin{bmatrix} I^*_{-N} \\ I^*_{-N+1} \\ I^*_{-N+2} \\ \cdots \\ \cdots \\ I^*_{-1} \\ I_0 \\ I_1 \\ \cdots \\ \cdots \\ I_N \end{bmatrix} \tag{3.117}$$

where the R_n are the Fourier components of the resistance waveform. As one might expect, the resistance-form conversion matrix of any element is the inverse of its conductance-form matrix, as long as the element can be defined either as a time-varying conductance or resistance.

The conversion matrix of a capacitor is only slightly more complicated. The capacitor's charge is given by

$$q'(t) = c(t)v'(t) \tag{3.118}$$

and $c(t)$ has the Fourier series

$$c(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_p t) \tag{3.119}$$

The current is

$$i'(t) = \frac{d}{dt}q'(t) \tag{3.120}$$

and $q'(t)$ has the form

$$q'(t) = \sum_{n=-\infty}^{\infty} Q_n \exp(j\omega_n t) \tag{3.121}$$

Substituting (3.111), (3.119), and (3.121) into (3.118) gives

$$\sum_{k=-\infty}^{\infty} Q_k \exp(j\omega_k t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} C_n V_m \exp(j\omega_{m+n} t) \tag{3.122}$$

The current can be found by differentiating. In the frequency domain, differentiation corresponds to multiplying by $j\omega$, so

$$\sum_{k=-\infty}^{\infty} I_k \exp(j\omega_k t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} j\omega_{m+n} C_n V_m \exp(j\omega_{m+n} t) \tag{3.123}$$

Equating terms at the same frequency gives the matrix equation

$$\mathbf{I} = j\mathbf{\Omega}\mathbf{C}\mathbf{V} \tag{3.124}$$

where **I** and **V** represent the frequency-component current and voltage vectors and C represents the conversion matrix for the capacitance. **I** and **V** are identical to the vectors in (3.116) and (3.117), and **C** has the same form as the conductance and resistance matrices in those equations. The matrix **Ω** is a diagonal matrix; its elements are $j\omega_{-N}$ to $j\omega_N$:

$$\mathbf{\Omega} = \begin{bmatrix} j\omega_{-N} & 0 & \dots & 0 \\ 0 & j\omega_{-N+1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & j\omega_N \end{bmatrix} \tag{3.125}$$

3.4.1.1 Example: Conversion Matrix of a Time-Varying Element

We form the conversion matrix of the circuit shown in Figure 3.12(a). It consists of a conductance in series with a switch; the switch is opened and