

# Stochastic Differential Equations

## Continuous Evolving Variables

I. Holmes

Department of Bioengineering  
University of California, Berkeley

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# Outline

- 1 Review: properties of Gaussian distributions
- 2 Gaussian processes as stochastic processes
- 3 Gaussian processes as tools for machine learning
- 4 The Fokker-Planck equation
- 5 The Wiener process
- 6 The Ornstein-Uhlenbeck process

## Texts:

- Stochastic Processes in Physics and Chemistry.  
N.G. Van Kampen
- Stochastic Methods: A Handbook for the Natural and  
Social Sciences.  
C. Gardiner
- Information Theory, Inference, and Learning Algorithms.  
D. MacKay
- Gillespie
- Berg

Review of salient facts about Gaussian distributions (Gardiner p36-37)

Multivariate Gaussian: if  $\mathbf{x}$  is a vector of  $n$  Gaussian r.v.s,

$$P(\mathbf{x}) = [2\pi \det(\sigma)]^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})^T \sigma^{-1}(\mathbf{x} - \bar{\mathbf{x}})\right)$$

where  $\bar{\mathbf{x}}$  is mean and  $\sigma$  is (symmetric) covariance matrix.  
Characteristic function

$$\phi(\mathbf{s}) = \langle \exp(i\mathbf{s}^T \mathbf{x}) \rangle = \exp(i\mathbf{s}^T \bar{\mathbf{x}} - \frac{1}{2}\mathbf{s}^T \sigma \mathbf{s})$$

General formulae for moments when  $\bar{\mathbf{x}} = 0$ : odd moments are zero, higher moments satisfy

$$\langle x_i x_j x_k \dots \rangle = \frac{2N!}{N! 2^N} \{ \sigma_{ij} \sigma_{kl} \sigma_{mn} \dots \} \text{sym}$$

where “sym” means the symmetrized form of the product of  $\sigma$ 's, and  $2N$  is the order of the moment, e.g.

$$\begin{aligned} \langle x_i x_j \rangle &= \sigma_{ij} \\ \langle x_1 x_2 x_3 x_4 \rangle &= \frac{4!}{2! 2^2} \left\{ \frac{1}{3} [\sigma_{12} \sigma_{34} + \sigma_{13} \sigma_{24} + \sigma_{14} \sigma_{23}] \right\} \\ &= \sigma_{12} \sigma_{34} + \sigma_{13} \sigma_{24} + \sigma_{14} \sigma_{23} \\ \langle x_i^4 \rangle &= 3 \sigma_{ii}^2 \end{aligned}$$

Central limit theorem (van Kampen p26): consider arbitrary  $P_X(x)$  with  $\langle x \rangle = 0$ ,  $\langle x^2 \rangle = \sigma$  and let  $z = n^{-1/2} \sum_n x_n$   
Characteristic function for  $P_X$  is

$$G_X(k) = \int \exp(ikx) P_X(x) dx = 1 - \frac{1}{2} k^2 \sigma + O(k^4)$$

Thus characteristic function for  $P_Z$  is

$$G_Z(k) = \left[ G_X \left( \frac{k}{\sqrt{n}} \right) \right]^n = \left[ 1 - \frac{\sigma k^2}{2n} + O \left( \frac{k^4}{n^{3/2}} \right) \right]^n \rightarrow \exp \left( -\frac{1}{2} \sigma k^2 \right)$$

(using the limit  $\lim_{n \rightarrow \infty} (1 + y/n)^{-n} = \exp(-y)$ ).

Therefore, in the limit  $n \rightarrow \infty$ ,  $z$  is Gaussian-distributed.

Definition of a Gaussian process (van Kampen p63-64)  
“Hierarchy of Distribution Functions” (van Kampen p61+).  
Consider timepoints  $t_1 < t_2 < t_3 \dots t_n$ . Define

$$\begin{aligned} P_n(x_1, t_1; x_2, t_2; \dots; x_n, t_n) \\ \equiv P(x(t_1) = x_1, x(t_2) = x_2, \dots, x(t_n) = x_n) \end{aligned}$$

If  $P_n$  is an  $n$ -dimensional Gaussian  $\forall n, \{t_1 \dots t_n\}$ , then  $x(t)$  is a *Gaussian process*. The covariance matrix is  $\sigma_{ij} = \langle x(t_i)x(t_j) \rangle$   
Marginals of a multivariate Gaussian are themselves multivariate Gaussians. The full distribution  $P(x(t))$  can be thought of as an infinite-dimensional Gaussian,  $P_\infty$ . A Gaussian process is effectively a prior over functions, that can be fully specified by the covariance function

The *characteristic functional*,  $G([k])$ , plays a role analogous to the characteristic function for discrete processes. Define an arbitrary auxiliary test function,  $k(t)$ . Then  $G([k])$  is the following functional of  $k(t)$

$$\begin{aligned} G([k]) &= \langle \exp \left[ i \int_{-\infty}^{\infty} k(t) x(t) dt \right] \rangle \\ &= \exp \left[ i \int k(t_1) \langle x(t_1) \rangle dt_1 - \frac{1}{2} \int \int k(t_1) k(t_2) \langle \langle x(t_1) x(t_2) \rangle \rangle dt_1 dt_2 \right] \end{aligned}$$



Inference, prediction, clustering with GPs (MacKay chapter 45, p535-548; MacKay 1998, “Introduction to Gaussian Processes”)

Suppose we have  $N$  datapoints,  $\{\mathbf{x}^{(n)}, t_n\}_{n=1}^N$ . The input variables  $\mathbf{x}^{(n)}$  are  $l$ -dimensional vectors. The target variables  $t_n$  will be assumed real scalars (corresponding to interpolation or regression problems). Goal: fit some (nonlinear) function  $y(\mathbf{x})$ . Posterior probability of  $y(\mathbf{x})$  is

$$P(y(\mathbf{x})|\mathbf{t}_N, \mathbf{X}_N) = \frac{P(\mathbf{t}_N|y(\mathbf{x}), \mathbf{X}_N)P(y(\mathbf{x}))}{P(\mathbf{t}_N|\mathbf{X}_N)}$$

Typically  $t_k = y(x_k) +$  separable Gaussian noise.

$$P(y(\mathbf{x})|\mathbf{t}_N, \mathbf{X}_N) = \frac{P(\mathbf{t}_N|y(\mathbf{x}), \mathbf{X}_N)P(y(\mathbf{x}))}{P(\mathbf{t}_N|\mathbf{X}_N)}$$

In parametric approaches,  $y(\mathbf{x}) \equiv y(\mathbf{x}; \mathbf{w})$  where  $\mathbf{w}$  is a set of parameters over which we place some prior. In nonparametric approaches (e.g. Gaussian processes), we place a prior directly on  $P(y(\mathbf{x}))$ .

A Gaussian process can be defined as a probability distribution over functions,  $P(y(\mathbf{x}))$ , of the form

$$P(y(\mathbf{x})|\mu(\mathbf{x}), \mathbf{A}) = \frac{1}{Z} \exp \left[ -\frac{1}{2} (y(\mathbf{x}) - \mu(\mathbf{x}))^T \mathbf{A} (y(\mathbf{x}) - \mu(\mathbf{x})) \right]$$

where  $\mathbf{A}$  is a linear operator and the inner product of two functions is

$$y(\mathbf{x})^T z(\mathbf{x}) = \int y(\mathbf{x}) z(\mathbf{x}) d\mathbf{x}$$

The operator  $\mathbf{A}$  must be *positive definite*, i.e.  $y(\mathbf{x})^T \mathbf{A} y(\mathbf{x}) > 0$  for all functions except  $y(\mathbf{x}) = 0$ .

Parametric approaches; fixed, adaptive basis functions; neural nets (Mackay p536-537)

Consider a set of basis functions,  $\{\phi_h(\mathbf{x})\}_{h=1}^H$ . Case #1: fixed basis functions (parameters indep. of  $\mathbf{w}$ )

$$y(\mathbf{x}; \mathbf{w}) = \sum_{h=1}^H w_h \phi_h(\mathbf{x})$$

e.g. radial basis functions

$$\phi_h(\mathbf{x}) = \exp \left[ -\frac{(\mathbf{x} - \mathbf{c}_h)^2}{2r^2} \right]$$

In this model,  $y$  is a linear function of  $\mathbf{w}$ .

Let  $R_{nh} = \phi_h(\mathbf{x}^{(n)})$ . Then  $y^{(n)} = \sum_h R_{nh} w_h$ . Let  $\mathbf{y} = (y^{(1)}, y^{(2)} \dots y^{(N)})$  be the vector of  $y$ -values and let  $\mathbf{w} = (w^{(1)}, w^{(2)} \dots w^{(N)})$  be the vector of corresponding  $w$ -values. Thus  $\mathbf{y} = \mathbf{R}\mathbf{w}$ . If  $\mathbf{w}$  is Gaussian-distributed

$$P(\mathbf{w}) = \mathcal{N}(\mathbf{0}, \sigma_w^2 \mathbf{I})$$

then  $\mathbf{y}$  is also Gaussian with covariance matrix

$$\langle \mathbf{y}\mathbf{y}^T \rangle = \langle \mathbf{R}\mathbf{w}\mathbf{w}^T \mathbf{R}^T \rangle = \mathbf{R} \langle \mathbf{w}\mathbf{w}^T \rangle \mathbf{R}^T = \sigma_w^2 \langle \mathbf{R}\mathbf{R}^T \rangle$$

Additive noise: if  $\mathbf{t} = \mathbf{y} + \mathbf{v}$  where  $v_k \sim \mathcal{N}(0, \sigma_v^2)$  then

$$P(\mathbf{w}) = \mathcal{N}(\mathbf{0}, \sigma_w^2 \mathbf{R}\mathbf{R}^T + \sigma_v^2 \mathbf{I})$$

Case #2: adaptive basis functions (parameters dependent on  $\mathbf{w}$ )

$$y(\mathbf{x}; \mathbf{w}) = \sum_{h=1}^H w_h^{(2)} \tanh \left( \sum_{i=1}^I w_{hi}^{(1)} x_i + w_{h0}^{(1)} \right) + w_0^{(2)}$$

This is equivalent to a two-layer feedforward neural network with nonlinear hidden units and a linear output. The input weights are  $\{w_{hi}^{(1)}\}$ , the hidden unit biases  $\{w_{h0}^{(1)}\}$ , the output weights  $\{w_h^{(2)}\}$  and the output bias  $w_0^{(2)}$ . In this model,  $y$  is a nonlinear function of  $\mathbf{w}$ .

Nonparametric approaches: the spline smoothing method (MacKay p538-541) attempts to minimize the functional

$$M(y(x)) = \frac{1}{2}\beta \sum_{n=1}^N (y(x^{(n)}) - t_n)^2 + \frac{1}{2}\alpha \int \left[ \frac{d^k y}{dx^k} \right]^2 dx$$

(If  $k = 2$  then  $y = \operatorname{argmin} M$  is a *cubic spline* with discontinuities in  $\frac{d^2 y}{dx^2}$  at the  $x^{(n)}$ .)

$$M(y(x)) = \frac{1}{2}\beta \sum_{n=1}^N (y(x^{(n)}) - t_n)^2 + \frac{1}{2}\alpha \int \left[ \frac{d^k y}{dx^k} \right]^2 dx$$

The term involving  $\alpha$  is equivalent to the following prior over  $y(x)$

$$P(y(x)|\alpha) = \text{const.} \times \exp \left( -\frac{1}{2}\alpha \int \left[ \frac{d^k y}{dx^k} \right]^2 dx \right)$$

which is a Gaussian process prior with  $\mathbf{A} = [D^k]^T D^k$ .

Combined with linearly independent Gaussian noise on each measurement, this gives a Gaussian process model with MAP estimates identical to those produced by splines.



Kramers-Moyal expansion (treatment follows van Kampen p197-198; see also Gillespie p74+)

The most general form of the *master equation* for a continuous-time stochastic process can be written

$$\frac{\partial}{\partial t} p(x, t) = \int W(x - r; r) p(x - r, t) dr - p(x, t) \int W(x; r) dr$$

where  $W(x; r)$  is the rate from  $x$  to  $x + r$ . In the notation we used for discrete state spaces,  $W(x; r) \equiv R_{x, x+r}$

$$\frac{\partial}{\partial t} p(x, t) = \int W(x - r; r) p(x - r, t) dr - p(x, t) \int W(x; r) dr$$

Assuming that  $W(x; r)$  varies smoothly in  $x$  and is sharply peaked in  $r$ , we can write the term  $W(x - r; r)p(x - r, t)$  in the first integral as a Taylor expansion in  $x$ :

$$\frac{\partial}{\partial t} p(x, t) = \sum_{n=0}^{\infty} \int \frac{(-r)^n}{n!} \frac{\partial^n}{\partial x^n} \{W(x; r)p(x, t)\} dr - p(x, t) \int W(x; r) dr$$

(Note that we're only allowed to expand  $W(x; r)$  in  $x$ , not in  $r$ , since it varies smoothly in  $x$  but rapidly in  $r$ .)

We then rewrite the terms in the expansion using the *jump moments*

$$a_n(x) = \int_{-\infty}^{\infty} r^n W(x; r) dr$$

so that the master equation becomes the Kramers-Moyal equation

$$\begin{aligned} \frac{\partial}{\partial t} p(x, t) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \{a_n(x) p(x, t)\} - p(x, t) \int W(x; r) dr \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \{a_n(x) p(x, t)\} \end{aligned}$$

Truncating the Taylor expansion to second order gives

$$\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} \{a_1(x)p(x, t)\} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \{a_2(x)p(x, t)\}$$

which is a form of the Fokker-Planck equation; see below.

Consider the discrete-time process  $x_n$  where  $t = n\tau$ . We have

$$x_{n+1} = x_n + \Xi_n$$

where the  $\Xi_n$  are random variables distributed  $\sim W(x_n; \Xi)\tau$ . Whatever the precise form of  $W(x; r)$ , we're effectively assuming that we can characterize it (and hence  $\Xi_n$ ) by its first two moments,  $a_1$  and  $a_2$ . Since  $x_n = \sum \Xi_n$ , the process  $x_n$  and hence  $x(t)$  tends towards a Gaussian, by the central limit theorem.

Fokker-Planck equation (Gillespie p121; van Kampen p193+)  
Fokker-Planck describes the time evolution of the probability density for a continuous stochastic process

$$\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} A(x, t) p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} B(x, t) p(x, t)$$

By comparison with the Kramers-Moyal expansion we see that  $A = a_1$  and  $B = a_2$ , so  $A$  and  $B$  are the mean and variance of the drift (i.e. the jump rate  $W(x; r)$ ).

When  $B = 0$ , we have a (deterministic) Liouville process. When  $A = 0$  and  $B$  is constant, we have Brownian motion, aka the Wiener process. When  $A = -kx$  and  $B$  is constant, we have Brownian motion with exponential decay, aka the Ornstein-Uhlenbeck process.

$$\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} A(x, t) p(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} B(x, t) p(x, t)$$

Note that the terms  $A(x, t)$  and  $B(x, t)$  are time-dependent, unlike our earlier treatment of the Kramers-Moyal expansion. This is just because we didn't allow the jump rate  $W(x; r)$  to be a function of  $t$ . It's straightforward to repeat the Kramers-Moyal expansion using time-dependent jump rates and moments,  $W(x; r; t)$  and  $a_n(x; t)$ .

Using the (time-dependent) propagator we have  $x(t + dt) = x(t) + \Xi(dt; x(t), t)$  and hence

$$\begin{aligned}x^n(t + dt) &= [x(t) + \Xi(dt; x(t), t)]^n \\&= x^n(t) + \sum_{k=1}^n \binom{n}{k} \times x^{n-k}(t) \Xi^k(dt; x(t), t)\end{aligned}$$

To find the expectation of this we use the following result

$$\langle x^j(t) \Xi^k(dt; x(t), t) \rangle = \langle x^j(t) b_k(t) \rangle dt + O(dt) \quad (1)$$

(where  $b_k(t) \equiv a_k(x(t))$  is just an alternate notation for the  $k$ 'th jump moment).



Hence we arrive at the *moment evolution equations*

$$\frac{d}{dt}\langle x^n(t) \rangle = \sum_{k=1}^n \binom{n}{k} \langle x^{n-k}(t) b_k(t) \rangle$$

with the initial conditions  $\langle x^n(0) \rangle = x_0^n$ . A further application of (1) gives the evolution of the autocorrelation function

$$\begin{aligned} \langle x(t_1)x(t_2 + dt_2) \rangle &= \langle x(t_1)x(t_2) + x(t_1)\Xi(dt; x(t_2), t_2) \rangle \\ &= \langle x(t_1)x(t_2) \rangle + \langle x(t_1)b_1(t_2) \rangle \\ \frac{d}{dt_2}\langle x(t_1)x(t_2) \rangle &= \langle x(t_1)b_1(t_2) \rangle \end{aligned}$$

Putting these together, we obtain the following equations for the evolution of the mean, variance and covariance

$$\begin{aligned}\frac{d}{dt}\langle x(t) \rangle &= \langle b_1(t) \rangle \\ \frac{d}{dt}\text{var}\{x(t)\} &= \frac{d}{dt} \left( \langle x^2(t) \rangle - \langle x(t) \rangle^2 \right) \\ &= 2 \left( \langle x(t)b_1(t) \rangle - \langle x(t) \rangle \langle b_1(t) \rangle \right) + \langle b_2(t) \rangle \\ \frac{d}{dt_2}\text{cov}\{x(t_1)x(t_2)\} &= \frac{d}{dt_2} \left( \langle x(t_1)x(t_2) \rangle - \langle x(t_1) \rangle \langle x(t_2) \rangle \right) \\ &= \langle x(t_1)b_1(t_2) \rangle - \langle x(t_1) \rangle \langle b_1(t_2) \rangle\end{aligned}$$

with the initial conditions  $\langle x(t_0) \rangle = x_0$ ,  $\text{var}\{x(t_0)\} = 0$ ,  
 $\text{cov}\{x(t_1)x(t_2 = t_1)\} = \text{var}\{x(t_1)\}$ .

These equations are closed iff (for  $n = 1, 2$ )  $b_n(t) = a_n(x(t))$  is a polynomial in  $x$  of degree  $\leq n$  (Gillespie p86).

The covariance is all we need to do inference with Gaussian processes, but it's useful to look deeper.

Liouville processes (Gillespie p126)

When  $B(x, t) = 0$ , the jump rate  $W(x; r)$  has no variance. It must therefore be a delta function in  $r$ , and we can write

$$x(t + dt) - x(t) = A(x(t), t)dt$$

that is,  $\frac{d}{dt} = A(x, t)$ . This is a completely deterministic process, or *Liouville process*. The Liouville process guides intuition as to the role of  $A(x, t)$  in the Fokker-Planck equation.

Fokker-Planck equation in  $N > 1$  variables:

$$\frac{\partial}{\partial t} p(\mathbf{x}, t) = - \sum_i^N \frac{\partial}{\partial x_i} A_i(\mathbf{x}) p(\mathbf{x}, t) + \frac{1}{2} \sum_i^N \sum_j^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} B_{ij}(\mathbf{x}) p(\mathbf{x}, t)$$

where, by analogy with before,

$$\begin{aligned} A_i(\mathbf{x}) &= \int_{\mathbb{R}^N} r_i W(\mathbf{x}; \mathbf{r}) d\mathbf{r} \\ B_{ij}(\mathbf{x}) &= \int_{\mathbb{R}^N} r_i r_j W(\mathbf{x}; \mathbf{r}) d\mathbf{r} \end{aligned}$$

i.e. the first & second moments of the jump rate  $W(\mathbf{x}; \mathbf{r})$ .

## Discrete random walk

$x(n) = \sum_{i=1}^n d_i$  where  $P(d_i = +\delta) = P(d_i = -\delta) = 1/2$

Implies that

$$\begin{aligned}\langle x(n) \rangle &= 0 \\ \langle x(n)^2 \rangle &= n\delta^2\end{aligned}$$

If each step takes time  $\tau$  then  $n = t/\tau$ , so  $\langle x(n)^2 \rangle = \frac{\delta^2}{\tau} t = 2Dt$   
where  $D = \delta^2/2\tau$  is the diffusion constant

Let  $r(x, t) = P(x(t) = x)$ . In time  $\tau$ , a particle at  $x$  has probability  $1/2$  of drifting to  $x + \delta$ , and a particle at  $x + \delta$  has probability  $1/2$  of drifting to  $x$ . The net flux of probability mass from  $x$  to  $x + \delta$  is

$$J(x) = \frac{1}{\tau} \left( \frac{r(x, t)}{2} - \frac{r(x + \delta, t)}{2} \right) = D \frac{1}{\delta} \left( \frac{r(x, t)}{\delta} - \frac{r(x + \delta, t)}{\delta} \right)$$

The continuous limit of  $r(x, t)/\delta$  is the probability density  $p(x, t)$  and so

$$J(x) = -D \frac{\partial p}{\partial x}$$

This is a version of *Fick's first equation* (Berg p18).

Consider the interval from  $x$  to  $x + \delta$ . Flux from the left is  $J(x)$  and from the right  $-J(x + \delta)$ . Since probability mass is conserved, we have

$$[p(x, t + \tau) - p(x, t)] \times \delta = [J(x) - J(x + \delta)] \times \tau$$

Dividing through by  $\delta\tau$ , taking the continuous limit and substituting the expression for  $J(x)$ , we have

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}$$

This is *Fick's second equation* (Berg p20), aka the master equation (or Fokker-Planck equation) for Brownian motion.

In the continuous limit ( $\tau \rightarrow 0$ ),  $x(t)$  is the sum of a large number ( $n = t/\tau$ ) of IID rvs (the  $d_i$ 's). By the central limit theorem, the finite-time increments  $x(t_2) - x(t_1)$  must be Gaussian-distributed, so  $x(t)$  is a Gaussian process. Furthermore,  $x(t_2) - x(t_1)$  is independent of  $x(t_3) - x(t_2)$ . This is called *independence of increments*.



Fokker-Planck equation for Wiener process (Gardiner p66-70).  
Choose units such that  $\delta^2 = \tau$ ,  $D = \frac{1}{2}$ . Then

$$\frac{\partial}{\partial t} p(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p(x, t) \quad p(x, 0) = \delta(x)$$

Fourier-transforming the Fokker-Planck equation gives for the characteristic function  $\phi(s, t) = \langle \exp(isx(t)) \rangle_{x(t)}$

$$\frac{\partial}{\partial t} \phi = -\frac{1}{2} s^2 \phi \quad \phi(s, 0) = 1$$

which has the solution  $\phi(s, t) = \exp(-\frac{1}{2} s^2 t)$ . The Fourier inversion gives

$$p(x, t) = (2\pi t)^{-1/2} \exp\left(-\frac{x^2}{2t}\right)$$

Autocorrelation function:

$$\langle x(t_1)x(t_2) \rangle = \langle x(t_1)^2 \rangle + \langle x(t_1)(x(t_2) - x(t_1)) \rangle = t_1$$

since the second expectation vanishes due to independence of increments.

Non-differentiable/fractal nature of sample trajectories:

$$\begin{aligned} P(|x(t + \epsilon) - x(t)| > k\epsilon) &= 2 \int_{k\epsilon}^{\infty} p(x, \epsilon) dx \\ &= 2 \int_{k\epsilon}^{\infty} (2\pi\epsilon)^{-1/2} \exp\left(-\frac{x^2}{2\epsilon}\right) dx \end{aligned}$$

In the limit  $\epsilon \rightarrow 0$ , this is one (intuitively,  $|x(\epsilon)|_{\text{rms}} \sim \sqrt{\epsilon}$ , so  $\epsilon$  approaches zero faster than  $x$  does). This means that the derivative of  $x(t)$  is infinite “almost surely” (in probabilistic terminology).

The Langevin equation is a stochastic differential equation

$$dx = A(x, t)dt + B(x, t)d\mathcal{W}(t)$$

where  $d\mathcal{W}(t)$  is a “white noise” term derived from the Wiener process  $\mathcal{W}(t)$ , for which  $\langle \mathcal{W}(t)\mathcal{W}(t+u) \rangle = t$ .

This is equivalent to the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(x, t) = -\frac{\partial}{\partial x}A(x, t)p(x, t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}B(x, t)p(x, t)$$

The key to the equivalence is the property of the Wiener process that

$$\langle d\mathcal{W}(t) \rangle^2 = dt$$

See e.g. Gardiner pp95-96.

The Ornstein-Uhlenbeck process: random walk with damping.  
 Describes *velocity* of Brownian particle.

$$\frac{\partial}{\partial t}p(x, t) = \frac{\partial}{\partial x}(kxp(x, t)) + \frac{1}{2}D\frac{\partial^2}{\partial x^2}p(x, t)$$

Boundary condition is  $p(x, 0) = \delta(x - x_0)$ .

Characteristic equation for  $\phi(s, t) = \langle \exp(\imath sx) \rangle$

$$\frac{\partial}{\partial t}\phi(s, t) + ks\frac{\partial}{\partial s}\phi(s, t) = -\frac{1}{2}Ds^2\phi(s, t) \quad (2)$$

Boundary condition is  $\phi(s, 0) = \exp(\imath sx_0)$ .

(Here we have used  $\int \exp(\imath sx) \frac{\partial}{\partial x}(xp)dx =$

$$-\int \imath s \exp(\imath sx) xpdx = -s\frac{\partial}{\partial s} \int \exp(\imath sx)pdx = -s\frac{\partial \phi}{\partial s})$$

$$\frac{\partial}{\partial t}\phi(s, t) + ks\frac{\partial}{\partial s}\phi(s, t) = -\frac{1}{2}Ds^2\phi(s, t) \quad (3)$$

Method of characteristics: let  $t = t(r)$  and  $s = s(r)$

$$\frac{d\phi}{dr} = \frac{\partial\phi}{\partial t}\frac{dt}{dr} + \frac{\partial\phi}{\partial s}\frac{ds}{dr}$$

which is the same as (3) if

$$dr = \frac{dt}{1} = \frac{ds}{ks} = -\frac{d\phi}{\frac{1}{2}Ds^2\phi}$$

(which Gardiner calls the “subsidiary equation”).

$$dr = \frac{dt}{1} = \frac{ds}{ks} = -\frac{d\phi}{\frac{1}{2}Ds^2\phi}$$

Integrating the equation involving  $ds$  and  $dt$ , and the equation involving  $ds$  and  $d\phi$ , gives

$$s = a \exp(kt) \quad \phi = b \exp(-Ds^2/4k)$$

where  $a, b$  are arbitrary constants. This can be rearranged to give the “characteristic directions”

$$\begin{aligned} u(s, t, \phi) &= s \exp(-kt) &= a \\ v(s, t, \phi) &= \phi \exp(Ds^2/4k) &= b \end{aligned}$$



$$\begin{aligned} dr &= \frac{dt}{1} = \frac{ds}{ks} = -\frac{d\phi}{\frac{1}{2}Ds^2\phi} \\ u(s, t, \phi) &= s \exp(-kt) = a \\ v(s, t, \phi) &= \phi \exp(Ds^2/4k) = b \end{aligned}$$

By the subsidiary equation,  $du = dv = 0$ . Thus a general solution is  $f(u, v) = 0$  with  $f$  an arbitrary function, or (equivalently)  $v = g(u)$  with  $g$  an arbitrary function. Therefore

$$\phi(s, t) = \exp(-Ds^2/4k)g[s \exp(-kt)]$$

The boundary condition  $\phi(s, 0) = \exp(isx_0)$  requires that  $g(s) = \exp(Ds^2/4k + isx_0)$ . Hence

$$\phi(s, t) = \exp\left(-\frac{Ds^2}{4k}(1 - \exp(-2kt)) + isx_0 \exp(-kt)\right)$$

which is the characteristic function of a Gaussian with

$$\begin{aligned}\langle x(t) \rangle &= x_0 \exp(-kt) \\ \text{var}\{x(t)\} &= \frac{D}{2k} (1 - \exp(-2kt))\end{aligned}$$

Thus

$$p(x, t) = \left(\frac{\pi D}{k}(1 - \exp(-2kt))\right)^{-1/2} \exp\left[-\frac{k}{D} \frac{(x - x_0 \exp(-kt))^2}{1 - \exp(-2kt)}\right]$$

The canonical form has  $D = 2k$  (van Kampen p83).