

# Logistic regression with a latent binary variable and noisy labels

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## 1 Model

Following [1], consider a set of  $N$  training data points  $D = \{(\mathbf{x}_n, c_n) : 1 \leq n \leq N\}$  where  $\mathbf{x}_n \in \mathbb{R}^M$  denote  $M$ -dimensional real-valued explanatory data (e.g. gene expression levels) and  $c_n \in \{0, 1 \dots K-1\}$  denotes a categorical label with  $K$  possible values (e.g. clinician-assigned label incorporating some degree of uncertainty).

We aim to fit this with a two-stage model, first regressing the explanatory data  $\mathbf{x}_n$  to a latent binary variable representing the true underlying state  $b_n \in \{0, 1\}$  (e.g. disease state), then modeling clinical labeling as a categorical variable  $c_n|b_n$  that is conditionally-independent of the explanatory data given the true underlying state

$$\begin{aligned} P(b = 1|\mathbf{x}, \mathbf{u}) &= \sigma(\mathbf{u}^T \mathbf{x}) \\ P(c = k|b = j, \mathbf{z}) &= z_{j,k} \end{aligned}$$

where  $\sigma(x) = \frac{1}{1+e^{-x}}$  is the logistic function,  $\mathbf{u} \in \mathbb{R}^M$  are parameters for the logistic regression model, and  $\mathbf{z}$  are probability parameters for the label observation model.

We put a Laplace double-exponential (Lasso) prior on  $\mathbf{u}$ , and a flattish<sup>1</sup> Dirichlet prior on each row of  $\mathbf{z}$

$$\begin{aligned} P(\mathbf{u}) &\propto \prod_{m=1}^M \exp(-|u^{(m)}|) \\ P(\mathbf{z}|\alpha) &\propto \prod_{j \in \{0,1\}} \prod_{k=0}^{K-1} z_{j,k}^{\alpha_{j,k}} \end{aligned}$$

subject to the constraint  $\sum_{k=0}^{K-1} z_{j,k} = 1 \ \forall j \in \{0,1\}$ , i.e. the parameters  $z_{j,k}$  for each value of  $j$  must lie on the  $(K-1)$ -dimensional simplex.

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<sup>1</sup>For identifiability of  $b$ , we need to break the symmetry of the Dirichlet prior slightly; e.g. by setting  $\alpha_{b,c} = 1$  for  $(b, c)$  pairs that “agree”, and  $\alpha_{b,c} = 0$  for all other  $(b, c)$  pairs.

This is equivalent to Section 2.2 of [1], with the sum over  $j$  in equation (8) of that paper constrained to  $j \in \{0, 1\}$  instead of  $j \in \{0, 1 \dots K-1\}$ . The paper derives a conjugate gradient optimization algorithm, and proves its convergence.

## 1.1 Quartile approach

An alternate model is to use the interpretation of logistic regression where a latent *continuous-valued* random variable (obtained by adding logistically-distributed noise to  $\mathbf{u}^T \mathbf{x}$ ) is used to obtain the labels  $(b, c)$ , e.g. with  $c$  corresponding to the quartiles.

I haven't pursued this model, as the assumption that  $c$  corresponds to quartiles of the latent variable underlying logistic regression seems like a possible misfit to the situation of arbitrarily designated clinical labels (although, conceivably, my assumption that  $c$  is independent of  $\mathbf{x}$  given  $b$  is just as bad, or worse).

## 2 EM algorithm

How to use the training data  $D$  to fit the logistic model weights  $\mathbf{u}$  and label error probabilities  $\mathbf{z}$ ? One approach is to use the EM (Expectation Maximization) algorithm [2], treating the binary-valued latent variables  $B = \{b_n\}$  as *missing data*, the dataset  $D = (X, C)$  as *observed data* (with inputs  $X = \{\mathbf{x}_n\}$  and observed labels  $C = \{c_n\}$ ), and  $\theta = (\mathbf{u}, \mathbf{z})$  as the *parameters* to be fit by the algorithm.

The conjugate gradient parameter optimization approach derived by [1] may well be superior to the EM method. However I've outlined the EM approach here for reference.

The joint likelihood including observed and missing data is

$$\begin{aligned} P(B, C, \theta | X) &= P(\theta) P(B, C | \theta, X) \\ &= P(\mathbf{u}) P(\mathbf{z} | \alpha) P(B | \mathbf{u}, X) P(C | \mathbf{z}, B) \\ &= P(\mathbf{u}) P(\mathbf{z} | \alpha) \prod_{n=1}^N P(b_n | \mathbf{u}, \mathbf{x}_n) P(c_n | \mathbf{z}, b_n) \end{aligned}$$

The marginal likelihood to be maximized, using observed data only, is

$$\begin{aligned} P(C, \theta | X) &= \sum_B P(B, C, \theta | X) \\ &= P(\mathbf{u}) P(\mathbf{z} | \alpha) \prod_{n=1}^N \sum_{j \in \{0, 1\}} P(b_n = j | \mathbf{u}, \mathbf{x}_n) P(c_n | \mathbf{z}, b_n = j) \end{aligned}$$

At the  $(i+1)$ 'th iteration, the parameters found by the EM algorithm are

given by maximizing the expected log-likelihood

$$\begin{aligned}
\theta^{(i+1)} &= \operatorname{argmax}_{\theta} \mathcal{E} \left( \theta | \theta^{(i)} \right) \\
\mathcal{E} \left( \theta | \theta^{(i)} \right) &= \sum_B P(B | \theta^{(i)}, X, C) \log P(B, C, \theta | X) \\
&= \log P(\mathbf{u}) + \log P(\mathbf{z} | \alpha) + \sum_B P(B | \theta^{(i)}, X, C) [\log P(B | \mathbf{u}, X) + \log P(C | \mathbf{z}, B)] \\
&= \log P(\mathbf{u}) + \log P(\mathbf{z} | \alpha) \\
&\quad + \sum_n \sum_{j \in \{0,1\}} P(b_n = j | \theta^{(i)}, \mathbf{x}_n, c_n) [\log P(b_n = j | \mathbf{u}, \mathbf{x}_n) + \log P(c_n | \mathbf{z}, b_n = j)] \\
&= \mathcal{E}_{\mathbf{u}} + \mathcal{E}_{\mathbf{z}} \\
\mathcal{E}_{\mathbf{u}} &= \log P(\mathbf{u}) + \sum_n \left[ (1 - \beta_n^{(i)}) \log(1 - \sigma(\mathbf{u}^T \mathbf{x}_n)) + \beta_n^{(i)} \log \sigma(\mathbf{u}^T \mathbf{x}_n) \right] \\
\mathcal{E}_{\mathbf{z}} &= \log P(\mathbf{z} | \alpha) + \sum_n \left[ (1 - \beta_n^{(i)}) \log z_{0,c_n} + \beta_n^{(i)} \log z_{1,c_n} \right] \\
\beta_n^{(i)} &= P(b_n = 1 | \theta^{(i)}, \mathbf{x}_n, c_n) \\
P(b_n = 1 | \theta, \mathbf{x}_n, c_n) &= \frac{1}{1 + \frac{P(c_n, b_n=0 | \theta, \mathbf{x}_n)}{P(c_n, b_n=1 | \theta, \mathbf{x}_n)}} \\
&= \frac{1}{1 + \frac{(1 - \sigma(\mathbf{u}^T \mathbf{x}_n)) z_{0,c_n}}{\sigma(\mathbf{u}^T \mathbf{x}_n) z_{1,c_n}}}
\end{aligned}$$

The maximization of  $\mathcal{E}_{\mathbf{u}}$  w.r.t.  $\mathbf{u}$  is a weighted, Lasso-penalized logistic regression (the weights being the  $\beta_n^{(i)}$ ). A suggested implementation of this maximization is outlined in Section 2.1.

The maximization of  $\mathcal{E}_{\mathbf{z}}$  w.r.t.  $\mathbf{z}$  is straightforward counting & normalizing

$$\begin{aligned}
\operatorname{argmax}_{z_{b,c}} \mathcal{E}_{\mathbf{z}} &= \frac{\gamma_{b,c}}{\sum_k \gamma_{b,k}} \\
\gamma_{0,k} &= \alpha_{0,k} + \sum_{n: c_n=k} (1 - \beta_n^{(i)}) \\
\gamma_{1,k} &= \alpha_{1,k} + \sum_{n: c_n=k} \beta_n^{(i)}
\end{aligned}$$

## 2.1 Implementation of weighted logistic regression in R

The maximization of  $\mathcal{E}_{\mathbf{u}}$  w.r.t.  $\mathbf{u}$  can be performed using R's `glmnet()` function (generalized linear model regression with Lasso prior) with `family = binomial(link = "logit")` (logistic regression is equivalent to binomial-family GLM regression with the "logit" link function).

R's GLM-fitter allows *weighting* of the training examples; that is, an augmented dataset  $D' = \{(\mathbf{x}'_n, b'_n, f'_n) : 1 \leq n \leq N'\}$  where  $f'_n$  is a weight (by default 1), loosely equivalent (when integer-valued) to the number of times datapoint  $(\mathbf{x}_n, b_n)$  was observed, or its *frequency*.

To implement the M-step in the  $(i + 1)$ 'th iteration of EM, we construct a weighted pseudo-dataset  $D'$  containing  $N' = 2N$  weighted training examples (that is, two for every example in the original unweighted dataset  $D$ ). The first  $N$  are labeled as negatives, the remaining  $N$  as positives; the weights are set using  $\beta_n^{(i)}$ , the posterior probability inferences from the previous step of EM. Specifically, for  $1 \leq n \leq N$

$$\begin{aligned} \mathbf{x}'_n &= \mathbf{x}_n \\ \mathbf{x}'_{N+n} &= \mathbf{x}_n \\ b'_n &= 0 \\ b'_{N+n} &= 1 \\ f'_n &= 1 - \beta_n^{(i)} \\ f'_{N+n} &= \beta_n^{(i)} \end{aligned}$$

The weights can be supplied to the R code using the `weights` argument to `glmnet()`.

The weighted logistic regression fit can then be worked into an R program that implements EM. Pseudocode for this algorithm is as follows

- Set  $\theta^{(1)}$  to some “sensible” initial values
- For  $i \in \{1, 2, 3 \dots\}$  do:
  - Calculate  $\beta_n^{(i)}$  using current  $\theta^{(i)} = (\mathbf{u}^{(i)}, \mathbf{z}^{(i)})$
  - Set  $\mathbf{z}^{(i+1)} \leftarrow \text{argmax}_{\mathbf{z}}(\mathcal{E}_{\mathbf{z}})$  by counting & normalizing
  - Set  $\mathbf{u}^{(i+1)} \leftarrow \text{argmax}_{\mathbf{u}}(\mathcal{E}_{\mathbf{u}})$  using `glmnet()` with `weights`
- Loop over  $i$  continues while  $\log \frac{P(C, \theta^{(i+1)} | X)}{P(C, \theta^{(i)} | X)} > \epsilon$  for some EM convergence threshold  $\epsilon$

## References

- [1] Jakramate Bootkrajang and Ata Kabán. Label-noise robust logistic regression and its applications. In *Proceedings of the 2012 European Conference on Machine Learning and Knowledge Discovery in Databases - Volume Part I*, ECML PKDD'12, pages 143–158, Berlin, Heidelberg, 2012. Springer-Verlag.
- [2] A. P. Dempster, N. M. Laird, and D. B. Rubin. Maximum likelihood from incomplete data via the em algorithm. *JOURNAL OF THE ROYAL STATISTICAL SOCIETY, SERIES B*, 39(1):1–38, 1977.